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History of PME

The International Group for the Psychology of Mathematics Education came into existence at the Third International Congress on Mathematical Education (ICME-3) in Karlsruhe, Germany in 1976. It is affiliated with the International Commission for Mathematical Instruction.

Goals of PME-NA

The major goals of the North American Chapter of the International Group for the Psychology of Mathematics Education are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education.

2. To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers.

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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Preface

These Proceedings are the product of the 32nd Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education held in Columbus, Ohio, October 28-31, 2010. They are a written record of the research presented at the conference.

The theme of the conference, *Optimizing Student Understanding in Mathematics*, has been at the fore-front of the mathematics education as a professional community since its institution in late 1890s in North America. Indeed, it has also been the motive behind a majority of inquiry activities within mathematics education as a research community since its conception in mid 20th century. With introduction of new theories of learning and teaching mathematics, conceptualization of new methodologies for studying the problems of mathematics learning and teaching, development of new curricular materials for advancement of mathematical knowledge of teachers and learners either implicitly or explicitly the agenda of the community has focused on defining ways and constructs that might help attain this long-standing goal and target of our discipline.

This year's conference brings together the voices of scholars from various genres of research in mathematics education. Different perspectives offered by this diverse group can help build a list of issues that need further contemplation as we, collectively, continue to conceptualize and operationalize how Student Understanding in Mathematics might be initiated, nurtured and monitored.

Azita Manouchehi & Douglas Owens
Conference Co-Chairs

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Plenary sessions

THE POWER OF NATURAL THINKING: APPLICATIONS OF COGNITIVE PSYCHOLOGY TO MATHEMATICS EDUCATION

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The talk begins with an inquiry into the relationship between people's natural thinking *– the suit of skills that is acquired by all people spontaneously and successfully under normal developmental conditions – and mathematical thinking. More specifically, when do these two thinking modes go together and when do they clash? The influential* dual-process theory *from cognitive psychology is applied to shed some light on this issue. A remarkable conclusion is that many of the recurring errors we make come from the* strength *of our mind rather than its weakness. The talk then proceeds to address a crucial* design issue*: In cases where natural and mathematical thinking clash, what can we as math educators do to help students create peaceful coexistence between the two? The extensively-researched* medical diagnosis problem *will be used to demonstrate how theory, design and experiment collaborate in the pursuit of this goal.*

Introduction

In the background of this talk lurks the momentous *rationality debate*: Are humans rational beings or not? Or, better, how rational are human beings? Or, still better, what kind of rationality (or irrationality) is invoked under what conditions? This question had been endlessly debated by the great philosophers through the millennia, but has become an empirical issue for cognitive psychologists in the second half of the 20th century, culminating with the 2002 Nobel Prize in economy to Daniel Kahneman for his work with Tversky on "intuitive judgment and choice" (Kahneman, 2002).

In this talk I will focus on a narrower (and more immediately relevant) facet of the rationality debate: What is the relation between people's natural thinking – the suit of skills that is acquired by all people spontaneously and successfully under normal developmental conditions – and mathematical thinking. More specifically, when do these two modes of thinking go together and when do they clash? Or, even more specifically, when can we as math educators build on the strength of students' natural thinking, and when do we need to devise ways to overcome it (For a comprehensive discussion of the rationality debate see Gigerenzer, 2005; Samuels *et al*., 2004; Saunders and Over, 2009; Stanovich and West, 2000; Stanovich, 2004; Stein, 1996).

To begin our one-thousand-mile journey with a small step, consider the following puzzle:

> *A baseball bat and ball cost together one dollar and 10 cents. The bat costs one dollar more than the ball. How much does the ball cost?*

This simple arithmetical puzzle would be totally devoid of interest, if it were not for the fact that it poses what I will call a *cognitive challenge*, best summarized in Kahneman's (2002) Nobel Prize lecture:

> *Almost everyone reports an initial tendency to answer '10 cents' because the sum $1.10 separates naturally into $1 and 10 cents, and 10 cents is about the right magnitude. Frederick found that many intelligent people yield to this immediate*

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> *impulse: 50% (47/93) of Princeton students, and 56% (164/293) of students at the University of Michigan gave the wrong answer* (p. 451).

The trivial arithmetical challenge has thus turned into a non-trivial challenge for cognitive psychologists: What is it about the workings of our mind that causes so many intelligent people to err on such a simple problem, when they surely possess the necessary mathematical knowledge to solve it correctly?

Complicating this cognitive challenge even further, research in cognitive psychology has revealed that *harder* versions of the task may result in *better* performance by the subjects. For example, we can enhance the subjects' performance by making the numbers more messy (let the bat and ball cost together $1.12 and the bat cost 94 cents more than the ball), or by displaying the puzzle via hard-to-read font on a computer screen (Song & Schwarz, 2008).

This challenge, and many others like it, have led to one of the most influential theories in current cognitive psychology, *Dual Process Theory* (DPT), roughly positing the existence of "two minds in one brain". These two thinking modes – intuitive and analytic – mostly work together to yield useful and adaptive behavior, yet, as the long list of cognitive challenges demonstrate, they can also fail in their respective roles, yielding non-normative answers to mathematical, logical or statistical tasks. A corollary of particular interest for mathematics education is that many recurring and prevalent mathematical errors originate from general mechanisms of our mind and not from faulty mathematical knowledge. Significantly, such errors often result from the *strengths* of our mind rather than its weaknesses (hence *the power of natural thinking* in the title). This paper is organized in two main parts. In the first part (based on Leron & Hazzan 2006, 2009), I introduce the dual process theory and demonstrate its explanatory power in math education research. In the second part, which is based on work in progress with Abraham Arcavi and with Lisser Rye Ejersbo, I address the educational challenge of bridging the gap between intuitive and analytical thinking. This is treated as a design issue. That is, given a problem with counter-intuitive solution (in our case, the famous and extensively-researched *medical diagnosis problem*), design a variation of the problem that brings the solution closer to intuition (or, alternatively, stretches the intuition towards the solution).

I hope that by focusing on the power of students' natural thinking, this talk might contribute to the goal of this conference: ***Optimizing** student understanding in mathematics*!

"Doin' what comes natur'lly": Dual-process theory (DPT)

Annie Oakley's phrase "doin' what comes natur'lly", from Irving Berlin's musical *Annie get your Gun*, touches charmingly on the ancient distinction between intuitive and analytical modes of thinking. This distinction has achieved a new level of specifity and rigor in what cognitive psychologists call *dual-process theory* (DPT). In fact, there are several such theories, but since the differences are not significant for the present discussion, we will ignore the nuances and will adopt the generic framework presented in Stanovich and West (2000), Kahneman and Frederick (2005) and Kahneman (2002). For state of the art thinking on DPT – history, empirical support, applications, criticism, adaptations, new developments – see Evans and Frankish (2009). The present concise – and much oversimplified – introduction to DPT and its applications in mathematics education is based on Leron and Hazzan (2006, 2009).

According to dual-process theory, our cognition and behavior operate in parallel in two quite different modes, called *System 1* (S1) and *System 2* (S2), roughly corresponding to our common sense notions of intuitive and analytical thinking. These modes operate in different ways, are activated by different parts of the brain, and have different evolutionary origins (S2 being evolutionarily more recent and, in fact, largely reflecting *cultural* evolution). The distinction between perception and cognition is ancient and well known, but the introduction of S1, which sits midway between perception and (analytical) cognition is relatively new, and has important consequences for how empirical findings in cognitive psychology are interpreted, including applications to the rationality debate and to mathematics education research.

Like perceptions, S1 processes are characterized as being fast, automatic, effortless, non-conscious and inflexible (hard to change or overcome); unlike perceptions, S1 processes can be language-mediated and relate to events not in the here-and-now. S2 processes are slow, conscious, effortful, computationally expensive (drawing heavily on working memory resources), and relatively flexible. In most situations, S1 and S2 work in concert to produce adaptive responses, but in some cases (such as the ones concocted in the heuristics-and-biases and in the reasoning research), S1 may generate quick automatic *non-normative* responses, while S2 may or may not intervene in its role as monitor and critic to correct or override S1's response. The relation of this framework to the concepts of intuition, cognition and meta-cognition as used in the mathematics education research literature (e.g., Fischbein, 1987; Stavy & Tirosh, 2000; Vinner, 1997) is elaborated in Leron and Hazzan (2006).

Many of the non-normative answers people give in psychological experiments – and to mathematics education tasks, for that matter – can be explained by the quick and automatic responses of S1, and the frequent failure of S2 to intervene in its role as critic of S1. Significantly, according to this framework, some of the ubiquitous mathematical misconceptions may have their origins in general mechanisms of the human mind, and not in faulty mathematical knowledge.

The bat-and-ball task is a typical example for the tendency of the insuppressible and fast-reacting S1 to "hijack" the subject's attention and lead to a non-normative answer. Specifically, the salient features of the problem cause S1 to jump automatically and immediately with the answer of 10 cents, since the numbers one dollar and 10 cents are salient, and since the orders of magnitude are roughly appropriate. For many people, the effortful and slow moving S2 is not alerted, and they accept S1's output uncritically, thus in a sense "behave irrationally" (Stanovich, 2004). For others, S1 also immediately had jumped with this answer, but in the next stage, their S2 interfered critically and made the necessary adjustments to give the correct answer (5 cents). Evolutionary psychologists, who study the ancient evolutionary origins of universal human nature, stress that the way S1 worked here, namely coming up with a very quick decision based on salient features of the problem and of rough sense of what's appropriate in the given situation, would be adaptive behaviour under the natural conditions of our ancestors, such as searching for food or avoiding predators (Buss, 2005; Cosmides & Tooby, 1997; Tooby & Cosmides, 2005). Gigerenzer (2005; Gigerenzer *et al*., 1999) claims that this is a case of *ecological rationality* being fooled by a tricky task, rather than a case of irrationality.

Evans (2009) offers a slightly different view – called *default-interventionist* approach – of the relations between the two systems. According to this approach, applied to the bat-and-ball data, only S1 has access to all the incoming data, and its role is to filter it and submit its "suggestions" for S2's scrutiny, analysis and final decision. This is a particularly efficient way to operate in view of the huge amount of incoming information the brain constantly needs to process, because it saves the scarce working memory resources that S2 depends on. On the other hand, it is error-prone, because the features that S1 selects are the most accessible but not always the most essential. In the bat-and-ball phenomenon, according to this model, the features that S1 has selected and submitted to S2 were the salient numbers 10 cents and 1 dollar, but the condition about the difference has remained below consciousness level. Even though S2 has the authority to override S1's decision, it may not do it due to lack of access to all the pertinent data.

The seemingly paradoxical phenomena that more difficult task formulations actually enhance performance is also well explained by DPT. Making the task more difficult in the above-mentioned sense, has the effect of suppressing the automatic response of S1, thus forcing S2 to participate. Since the subjects' S2 does possess the necessary mathematical knowledge, all that is required to solve the problem correctly is suppressing S1 and activating S2, which is exactly the effect of these added complications.

It is important to note that skills can migrate between the two systems. When a person becomes an expert in some skill, perhaps after a prolonged training, this skill may become S1 for this person. For example, driving is an effortful S2 behavior for beginners, requiring deep concentration and full engagement of working memory processing. For experienced drivers, in contrast, driving becomes an S1 skill which they can perform automatically while their working memory is engaged in other tasks, such as a deep intellectual or emotional conversation. Conversely, many S1 skills (such as walking straight or talking in a familiar but non-native language), deteriorate with advancing age, or when just being tired or drunk, all of a sudden requiring conscious effort to perform successfully (behaving in effect like S2).

We have now moved well along our one-thousand-mile journey, contemplating the power of natural thinking (roughly the psychologists' S1), and its uneasy relation with analytical thinking (S2). The psychological research literature on dual-process theory is immense, and we could barely touch the surface here. Many interesting and important questions remain open, such as what are the mechanisms that cause (or fail to cause) S2 to intervene and override S1's output. Much more is known to psychologists about such questions, but equally much still remains unanswered (see Evans and Frankish, 2009). In the rest of this paper we will delve more deeply into the educational relevance of the foregoing theoretical framework.

Bridging intuitive and analytical thinking: A design approach

Our second one-thousand-mile journey begins again with a small step – this time the famous string-around-the-earth puzzle (dating back to 1702).

> *Imagine you have a string tightly encircling the equator of a basketball. How much extra string would you need for it to be moved one foot from the surface at all points? Hold that thought, and now think about a string tightly encircling the Earth – making it around 25,000 miles long. Same question: how much extra string would you need for it to be one foot from the surface at all points?*

Everybody seems to feel strongly that the Earth would need *a lot* more extra string than the basketball. The surprising answer is that they both need the same amount: 2π, or approximately 6.28 feet (If R is the radius of any of them, then the extra string is calculated by the formula $2\pi(R+1) - 2\pi R = 2\pi$).

As with the bat-and-ball puzzle, *this surprise is what we are after*, for it tells us something important about how the mind works, which is why cognitive psychologists are so interested in such puzzles. This time, however, alongside with the cognitive challenge, there is also an important *educational challenge*, to which we now turn. Suppose you present this puzzle to your math class. Being a seasoned math teacher, you first let them be surprised; that is, you first let them do some guessing, bringing out the strong intuitive feeling that the required additional string is small for the basketball but huge for the earth. Then you have them carry out the easy calculation (as above) showing that – contrary to their intuition – the additional string is actually quite small, and is in fact independent of the size of the ball.

Now, given the classroom situation just described, here is the educational challenge: *As teachers and math educators, what do we do next?* I haven't conducted a survey, but my guess is that most teachers would leave it at that, or at best discuss with the students the clash they have just experienced between the intuitive and analytical solutions. But is this the best we can do?

Taking my clue from Seymour Papert (of Logo fame), I claim that in fact we can do better. We want to avoid the "default" conclusion that students should not trust their intuition, and should abandon it in the face of conflicting analytical solution. We also want to help students deal with the uncomfortable situation, whereby their mind harbours two conflicting solutions, one intuitive but now declared illegitimate, the other correct but counter-intuitive.

Papert's (1993/1980, 146-150) response to this educational challenge is simple but ingenious: Just imagine a cubic earth instead of a spherical one! Now follow in your mind's eye the huge square equator with two strings, one snug around its perimeter and the other running in parallel 1 foot away (Fig. 1). Then you can actually *see* that the two strings have the same length along the sides of the square (the size doesn't matter!), and that the only additional length is at the corners. In addition, you can now *see* why the extra string should be 2π: It is equal to the perimeter of the small circle (of radius 1 foot) that we get by joining together the 4 circular sectors at the corners.

The final step in Papert's ingenious construction is to bridge the gap between the square and the circle with a chain of perfect polygons, doubling the number of sides at each step. The next polygon in the chain after the square would be an octagon (Fig. 1, right). Here we have 8 circular sections at the corners, each half the size of those in the square case, so that they again can be joined to form a circle of exactly the same size as before. This demonstrates that doubling the number of sides (and getting closer to a circle) leaves us with the same length of extra string.

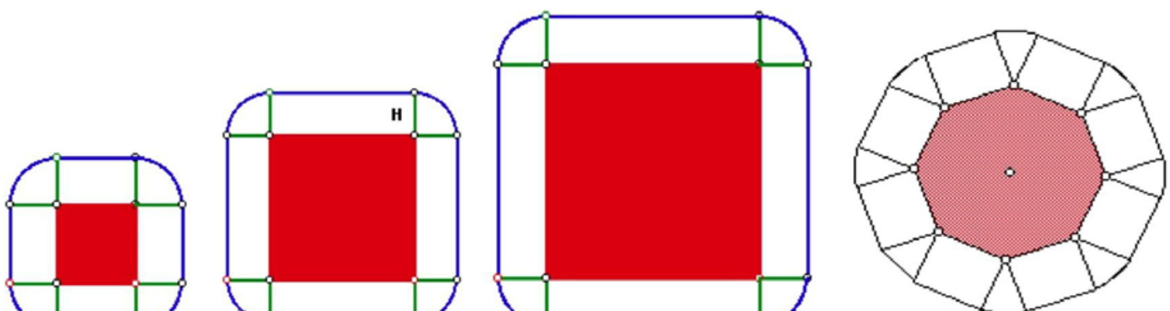

Fig. 1: A string around a square Earth – and around an octagonal one.

Can this beautiful example be generalized? When intuitive and analytical thinking clash, can we always design such "bridging tasks" that will help draw them closer? How should theory, design and experiment be put together in this search? We go more deeply into these questions in the next section.

Theory, design, experiment: The medical diagnosis problem (MDP)

Drawing inspiration from Papert's approach, and prompted by the questions closing up the last section, Lisser Rye Ejersbo and I set ourselves the challenge of designing an analogous treatment for the more advanced, relevant, and extensively-researched task from cognitive psychology: *the medical diagnosis problem* (MDP).

MDP background. Here is a standard formulation of the MDP task and data, taken from Samuels *et al*. (2004):

> *Before leaving the topic of base-rate neglect, we want to offer one further example illustrating the way in which the phenomenon might well have serious practical consequences. Here is a problem that Casscells et. al. (1978) presented to a group of faculty, staff and fourth-year students at the Harvard Medical School.*
>
> > *[**MDP**:] If a test to detect a disease whose prevalence is 1/1000 has a false positive rate of 5%, what is the chance that a person found to have a positive result actually has the disease, assuming that you know nothing about the person's symptoms or signs? ___%*
>
> *Under the most plausible interpretation of the problem, the correct Bayesian answer is 2%. But only eighteen percent of the Harvard audience gave an answer close to 2%. Forty-five percent of this distinguished group completely ignored the base-rate information and said that the answer was 95%* (p.136).

This task is intended to test what is usually called *Bayesian thinking*: how people update their initial statistical estimates (the *base rate*) in the face of new evidence (the *diagnostic information*). In this case, the base rate is 1/1000, the diagnostic information is that the patient has tested positive, and the task is intended to discover how the subjects will update their estimate of the chance that the patient actually has the disease. The meaning of "5% false positive rate" is that 5% of the healthy people taking the test would test positive. *Base-rate neglect* reflects the widespread tendency among subjects to ignore the base rate, instead simply subtracting the false positive rate of 5% from 100%. Indeed, it is not at all intuitively clear why the base rate should matter, and how it could be taken into the calculation.

A formal solution to the task is based on Bayes' theorem, but there are many complications and controversies involving mathematics, psychology and philosophy, concerning the interpretation of that theorem. Indeed, this debate, "Are humans good intuitive statisticians after all?" (Cosmides & Tooby, 1996) is a central issue in the great rationality debate. See Barbey and Sloman (2007) for a comprehensive discussion, and a glimpse of the controversy.

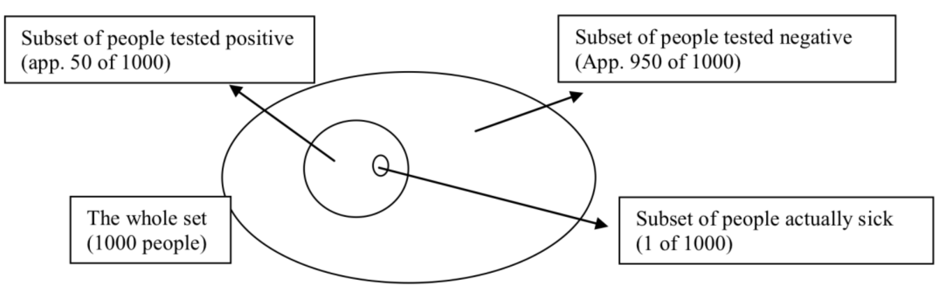

Fig. 2: Nested subsets in the medical diagnosis problem (for 1000 people)

Here is a simple intuitive solution for the MDP, bypassing Bayes' theorem. Assume that the population consists of 1,000 people and that all have taken the test (see Fig. 2). We know that one person will have the disease (because of the base rate) and will test positive (because no false negative rate is indicated). In addition, 5% of the remaining 999 healthy people (approximately 50) will test false-positive – a total of 51 positive results. Thus, the probability that a person who tests positive actually has the disease is 1/51, which is about 2%.

Researchers with evolutionary and ecological orientation (Cosmides & Tooby, 1996; Gigerenzer *et al*., 1999) claim that people are "good statisticians after all" if only the input and output is given in "natural frequencies" (integers instead of fractions or percentages):

> *In this article, we will explore what we will call the "frequentist hypothesis" – the hypothesis that some of our inductive reasoning mechanisms do embody aspects of a calculus of probability, but they are designed to take frequency information as input and produce frequencies as output* (Cosmides & Tooby, 1996, p. 3).

Evolutionary psychologists theorize that the brains of our hunter-gatherer ancestors developed such a module because it was vital for survival and reproduction, and because this is the statistical format that people would naturally encounter under those conditions. The statistical formats of today, in contrast, are the result of the huge amount of information that is collected, processed and shared by modern societies with modern technologies and mass media. Indeed, Cosmides and Tooby (1996) have replicated the Casscells *et al*. (1978) experiment, but with natural frequencies replacing the original fractional formats, and the base-rate neglect has all but disappeared:

> *Although the original, non-frequentist version of Casscells et al.'s medical diagnosis problem elicited the correct bayesian answer of "2%" from only 12% of subjects tested, pure frequentist versions of the same problem elicited very high levels of bayesian performance: an average of 76% correct for purely verbal frequentist problems and 92% correct for a problem that requires subjects to construct a concrete, visual frequentist representation* (Cosmides & Tooby, 1996, p. 58).

These results, and the evolutionary claims accompanying them, have been consequently challenged by other researchers (Evans, 2006; Barbie & Sloman, 2007). In particular, Evans (2006) claims that what makes the subjects in these experiments achieve such a high success rate is not the frequency format per se, but rather a problem structure that cues explicit mental models of nested-set relationships (see below). However, the fresh perspective offered by evolutionary psychology has been seminal in re-invigorating the discussion of statistical thinking in particular, and of cognitive biases in general. The very idea of the frequentist hypothesis, and the exciting and fertile experiments that it has engendered by supporters and opponents alike, would not have been possible without the novel evolutionary framework. Here is how Samuels *et al*. (1999) summarize the debate:

> *But despite the polemical fireworks, there is actually a fair amount of agreement between the evolutionary psychologists and their critics. Both sides agree that people do have mental mechanisms which can do a good job at bayesian*

> *reasoning, and that presenting problems in a way that makes frequency information salient can play an important role in activating these mechanisms* (p. 101).

The educational challenge as design issue

The extensive data on base rate neglect in the MDP (leading to the 95% answer) demonstrates the counter-intuitive nature of the analytical solution, as in the case of the string around the earth. As math educators, we are interested in helping students build bridges between the intuitive and analytical perspectives, hopefully establishing peaceful co-existence between these two modes of thought. As we have seen in Papert's example, achieving such reconciliation involves a design issue: Design a new *bridging task*, which is logically equivalent to, but psychologically much easier than the given task. (Compare Clements' (1993) "bridging analogies" and "anchoring intuitions" in physics education.)

From the extensive experimental and theoretical research in psychology on the MDP, we were especially influenced in our design efforts by the *nested subsets hypothesis* (Fig. 2):

> *All this research suggests that what makes Bayesian inference easy are problems that provide direct cues to the nested set relationships involved […]*
> *It appears that heuristic [S1] processes cannot lead to correct integration of diagnostic and base rate information, and so Bayesian problems can only be solved analytically [i.e., by S2]. This being the case, problem formats that cue construction of a single mental model that integrates the information in the form of nested sets appears to be critical* (Evans, 2006, p. 391).

Indeed, it is not easy to form a mental representation of the subsets of sick and healthy people, and even less so for the results of the medical test. Mental images of people all look basically the same, whether they are sick or healthy or tested positive or negative. The task of finding a more intuitive version of the MDP has thus been operationalized to finding a task which will "cue construction of a single mental model that integrates the information in the form of nested sets" (*ibid*).

Based on this theoretical background, we formulated three *design criteria* for the new task (which would also serve as testable predictions):

1. *Intuitively accessible*: The bridging task we will design will be easier ("more intuitive") than the original MDP (i.e., significantly more people – the term is used here in a qualitative sense –will succeed in solving it correctly).
2. *Bridging function*: Significantly more people will solve the MDP correctly, without any instruction, after having solved the new task.
3. *Nested subsets hypothesis*: Base rate neglect will be significantly reduced.

Note that the first two design criteria pull the new task in opposite directions. Criterion 1 (turning the hard task into an easy one) requires a task that is sufficiently different from the original one, while criterion 2 (the bridging function) requires a task that is sufficiently similar to the original one. The new task, then, should be an *equilibrium point* in the "design space" – sufficiently different from the original task but not *too* different.

Armed with these criteria, we set out on the search for the new bridging task. After a long process of trial and error, intermediate versions, partial successes and failures, we have finally come up with the *Robot-and-Marbles Problem* (RMP), which we felt had a good chance of satisfying the design criteria and withstanding the empirical test. The RMP is based on the idea of replacing sick and healthy people in the population by red and green marbles in a box. The medical test is then replaced by a colour-detecting robot, which can distinguish between red and green marbles via a colour sensor. The sensor is not perfect, however, and 5% of the green marbles are falsely identified as red, corresponding to the 5% healthy people in the MDP who are falsely diagnosed as sick. We also decided to make the action of the robot on the marbles more vividly imaginable by actually describing the process, not just the result. A final step in the design of the new problem was to slightly change the numbers from the original MDP, in order to make the connection less obvious. According to the bridging criterion, our subjects who solved the RMP first, should then solve more successfully the MDP. For this to happen, they would first need to recognize the *similarity* between the two problems, and we didn't want to make this too obvious by using the same numbers. (For a more detailed and nuanced description of the design process, as well as the experiments that followed, see Ejersbo and Leron, submitted).

Here then is the final product of our design process, the version that would actually be put to the empirical test to see whether the design criteria have been satisfied.

> **RMP:** In a box of red and green marbles, 2/1000 of the marbles are red. A robot equipped with green-marble detector with a 10% error rate (10% green marbles are identified as red), throws out all the marbles which it identifies as green, and then you are to pick a marble at random from the box. What is the probability that the marble you have picked would be red?

The experiment

The participants in the experiment were 128 students studying towards M.A degree in Educational Psychology at a Danish university, with no special background in mathematics or statistics. All the participants were assigned the two tasks – the medical diagnosis problem (MDP) and the robot-and-marbles problem (RMP) – and were given 5 minutes to complete each task. (In a pilot experiment we found that 5 minutes were enough both for those who could solve the problem and those who couldn't.) The subjects were assigned randomly into two groups of 64 students each. The order of the tasks was MDP first and RMP second for one group (called here the *MR group*), and the reverse order for the second group (the *RM group*). The results of the RM group were clearly our main interest, the MR group serving mainly as control.

	Group 1: Robot first		Group 2: Medical diagnosis first	
	RMP 1st	MDP 2nd	MDP 1st	RMP 2nd
Correct	31	17	8	20
Base-rate neglect	1	12	22	4
Incorrect other	32	35	34	40
Total	64	64	64	64

Table 1: Numbers of responses in the various categories

The results are summarized in Table 1 above, and it can be seen that the design criteria have been validated and the predictions confirmed. A brief summary of the results for the RM group (with comparative notes in parentheses) follows.

- The RMP succeeded in its role as bridge between intuitive and analytical thinking: 48% (31/64) of the subjects in the RM group solved it correctly. (compared to 18% success on the MDP in the original Harvard experiment and 12% (8/64) in our MR group.)
- The RMP succeeded in its role as stepping stone for the MDP: More than 25% (17/64) solved the MDP, without any instruction, when it followed the RMP. (Again compared to 18% in the original Harvard experiment and 12% in our MR group.)
- The notorious base-rate neglect has all but disappeared in the RMP: it was exhibited by only 1 student out of 64 in the RM group and 4 out of 64 in the MR group. (Compared to 45% on the MDP in the original Harvard group and 34% on the MDP in our MR group.)
- Remarkably, the MDP, when given first, does not at all help in solving the RMP that follows. Worse, the MDP gets in the way: The table shows 48% success on the RMP alone, vs. 31% success on the RMP when given after the MDP.
- Even though the performance on the RMP and the MDP has greatly improved in the RM group, still the largest number of participants appear in the "incorrect other" category. This category consists of diverse errors which do not directly relate to the MDP, including (somewhat surprisingly for this population) many errors concerning misuse of percentages.

Conclusion

In this article I illustrated how the dual-process theory from cognitive psychology highlights and helps explain the power of natural thinking. I used the medical diagnosis problem to discuss the gap between intuition (S1) and analytical thinking (S2), and to develop design principles for bridging this gap. It is my belief that bridging the gap between intuition and analytical thinking (in research, curriculum planning, learning environments, teaching methods, work with teachers and students) is a major step towards "optimizing student understanding in mathematics".

Based on the above examples and analysis, and indulging in a bit of over-optimism, here are some of the developments in the educational system I'd like to see happen in the future. I make no claim of originality of these suggestions since many related ideas have previously appeared in the math education literature in various forms.

- Map out the high school curriculum (and beyond) for components that could build on natural thinking and parts that would need to overcome it. For example, which aspects of functions (or fractions, or proofs) are consonant or dissonant with natural thinking?
- Design curricula, learning environments, teaching methods, that build from the power of natural thinking.
- Build a stock of puzzles and problems which challenge the intuition, and develop ways to work profitably with teachers and students on these challenges.

I wish to conclude with an even bigger educational challenge. If you ask mathematicians for examples of beautiful theorems, you will discover that many of them are *counter-intuitive*; indeed, that they are beautiful *because* they are counter-intuitive, *because* they challenge our

natural thinking. Like a good joke, the beauty is in the surprise, the unexpected, the unbelievable. Like a world-class performance in classical ballet, sports, or a soprano coloratura, the beauty is (partly) in overcoming the limitations of human nature. Examples abound in the history of mathematics: The infinity of primes, the irrationality of $\sqrt{2}$, the equi-numerosity of even and natural numbers, the impossibility theorems (trisecting angles by ruler and compass, solving 5^{th}-degree equations by radicals, enumerating the real numbers, Gödel's theorems). Recall, too, the joy of discovering that – contrary to your intuition – the extra length in the string-around-the-earth puzzle is quite small, and the beauty of Papert's cubic earth thought experiment.
Here, then, is the challenge: By all means, let us build on the power of natural thinking, but let us also look for ways to help our students feel the joy and see the beauty of going beyond it, or even against it. We thus arrive at the closing slogan – a variation on the title of the talk:
The power of natural thinking, the challenge of stretching it, the beauty of overcoming it.

Acknowledgement

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A Critique and Reaction to

THE POWER OF NATURAL THINKING: APPLICATIONS OF COGNITIVE PSYCHOLOGY TO MATHEMATICS EDUCATION

HOW MAY CONCEPTUAL *LEARNING* IN MATHEMATICS BENEFIT FROM DUAL PROCESSING THEORIES OF *THINKING*?

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Concurring with Uri Leron's cross-disciplinary approach to distinct modes of mathematical thinking, intuitive and analytic, I discuss his elaboration and adaptation to our field of the cognitive psychology dual-processing theory (DPT). I reflect on (a) the problem significance, (b) aspects of the theory he adapts, and (c) elegance of presentation. Then, I further discuss DPT in light of a constructivist stance on the inseparability of thinking and learning. I link DPT to accounts of (i) brain-based conceptual learning and (ii) how mathematics teaching may promote such learning—and discuss advantages of those accounts.

Introduction

I thank the PME-NA organizers for a learning opportunity they created for me in discussing Uri Leron's plenary address. It re-acquainted me with the inspiring work that he and his colleagues were conducting in the two decades since I last studied with him. It also provided me with a window into literature outside mathematics education that I found thought provoking and relevant to our field. Finally, via his paper(s) I realized how naturally his approach linked with my recent efforts to relate math education with brain studies. I concur with him that "bridging the gap between intuition and analytical thinking … is a major step towards 'optimizing student understanding in mathematics'," and am delighted to provide my reflections on this endeavor.

In itself, the thesis that human thinking and rationality consist of two distinct modes is not new to math education. Skemp (1979) articulated and linked both, termed *intuitive* and *reflective* intelligences. As far as I know, his constructivist theory evolved independently of the 'heuristic and bias' approach (Kahneman, Slovic, & Tversky, 1982; Kahneman & Tversky, 1973; Tversky & Kahneman, 1973). Moreover, in our field it distinction can be traced back to Dewey's (1933) notion of *reflective* thought (contrasted with *unconscious* mental processes), and to Vygotsky's (1986) notion of ZPD and his related distinction between *spontaneous* and *scientific* concepts.

However, two novelties in Uri's contribution seem useful for math education. First, his review of cognitive psychology literature reveals studies in which a dual view of thinking has been elaborated on (Evans, 2006; Kahneman & Frederick, 2002; Stanovich, 2008) and 'mapped' onto corresponding brain regions (Lieberman, 2003). Thus, a timely direction, of linking math education with brain studies, is supported by relevant findings from cognitive psychology. Second, he reports on studies (Leron & Hazzan, 2006, 2009) informed by DPT that demonstrated applicability to our field, including articulation of instructional goals and design criteria.

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Significant Problem/Questions, Useful Theory, Elegant Exposition

Significance

Like many math teachers, Uri and his colleagues noticed what seemed to puzzle researchers in other fields. Often, observers of people's task solutions framed them as recurring faulty judgments. Examples abound in the above papers; I shall add 3 of my own. Studies of such examples fueled a debate on human rationality that conjoined epistemology and psychology (Nisbett & Ross, 1994; Quine, 1994). For example, alluding to computational complexity, Cherniak (1994) saw 'ideal' (normative) rationality as intractable. He proposed 'minimal' rationality, in which using 'quick and dirty' heuristics that evade mental paralysis.

Addressing this puzzling and significant problem in math education is more pressing and weighty than in other fields, in which solidly explaining why/how the human mind produces erroneous judgments will suffice. Uri's work indicates that for us this is but a start, while making two key contributions: (a) *clarifying a goal* for student and teacher learning—closing the gap between intuitive and analytic reasoning, and (b) *explicating our duty* to find ways of designing and implementing teaching that fosters student development of and disposition toward analytic reasoning. To these ends, Uri identifies four vital questions for mathematics educators:

i) What differentiates among those who solve problems correctly and incorrectly, that is, why do the latter fail to use analytic reasoning whereas the former successfully do so?
ii) How do problem formats cue for correctly solving a problem, and what does it entail for task design?
iii) When using puzzling problems in our teaching (e.g., string-around-earth), what follow-up strategies can be used to effectively capitalize on students' "Aha" moments regarding those puzzlements?
iv) How can teaching promote (a) students' *awareness* of improper intuitions and (b) disposition toward activating analytic reasoning to override the faulty intuitions (i.e., resist + critique)?

Useful Theory?

I use 4 examples to present key features of and articulate purposes DPT can serve in math education (to be brief, language does not precisely replicate the problems).

A. Adults with college education are asked: Two items cost $1.10; the difference in price is $1. How much does each cost? (Over 50% submit to impulse and respond: $1 & $0.10)
B. In the elevator, the 9th floor button is already lit. A person (you?) who also wants that floor gets on the elevator and, though seeing the lit button, presses it again.
C. Third graders were asked what to bet on next (Head/Tail), after 4 'Heads' were flipped in a row. About 50% said 'Head', because it's always been so; the rest said 'Tail', because it cannot always be 'Head'. No one said 50-50 and that previous results are irrelevant.
D. A Sudoku expert solves a 'black-belt' puzzle and makes two careless errors (Fig. 1a-1b).

The key insight and tenet of DPT is that responses to diverse problems, faulty or correct, all share a common root: two different modes of brain processing are at work (Evans, 2006; Stanovich, 2008; Stanovich & West, 2000). One mode, *intuitive* reasoning (or heuristic), is evolutionary more ancient, shared with animals, automatic (reflexive, sub-conscious), rapid, and parallel in nature, with only its final product available to consciousness. The other, *analytic*, is

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evolutionary recent, unique to humans, intentional (reflective, conscious), relatively slow, and sequential in nature. The second mode monitors, critics, and corrects judgments of the first. It suppresses (inhibits) default responses, serving as a failure prevention+correction mental device. As Uri points out, some DPT proponents refer to these modes as System-1 (S1) and System-2 (S2) respectively, stating that often both work mostly in tandem (i.e., S1 judgment agreed by S2).

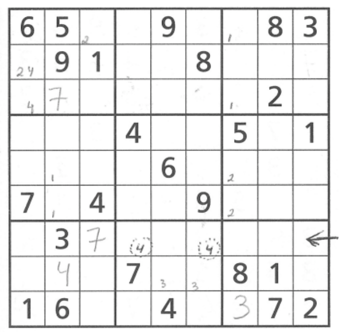

Figure 1a. Processing error almost committed—placing '4' in mid-lower left cell (transposed row, ignored vertical)

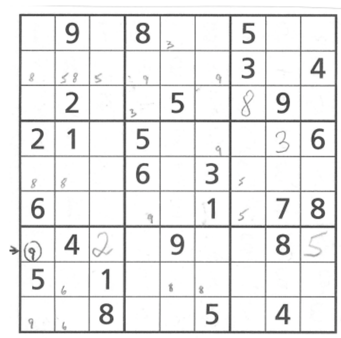

Figure 1b. Same error repeated & committed; '9' in left-lower cell (checked for vertical only)

A second tenet of DPT is that faulty responses reflect failure of analytic processes to prevent-and-correct intuitive outputs. A key corresponding assumption, indicated by the notion of *rational* judgment, is that at any given problem (e.g., economic benefit, academic success) a person intends for a correct solution that serves one's purposes. In the examples above, a person tries solving the problems correctly but, as DPT explains, the "tendency of the insuppressible and fast-reacting S1 to "hijack" the subject's attention [leads] to a non-normative answer" (Leron, this volume). In Example *A*, S1 'falls prey' to one item's cost ($1) being equal to the difference; in Example *B*, S1 activates a planned action (enter elevator, identify + press 9th floor) before S2 re-evaluates circumstantial necessity; In Example *D* (Figure 1b), S1 directed my actions to place digits with only partial checking before S2 detected that partiality. (Note: it occurred *soon after* I actually *thought* of placing the '4' where it's shown in Figure 1a, but then avoided this error.) Example *C* highlights a few hurdles with DPT, particularly the effect of solvers' cognitive abilities (Stanovich & West, 2000). What an observer considers *non-normative* responses seemed a proper response to children—a case of S1 and S2 working in tandem for the *reasoner*.

A few points before turning to hindrances I find in DPT. First, Evans (2006) distinguished between dual *processes* and dual *systems*. This is important for us particularly because, as he asserted, dual system views are too broad. He suggested specifying dual-reasoning accounts at an intermediate level that explain solutions to *particular tasks*. To me, his goal (particular task) seems primary whereas the means (dual accounts, or singular, or triple) seems secondary.

This leads to my second point—the need to pay attention to solution *processes* and kinds of tasks—in which the analytic successfully monitored and corrected the intuitive **before** the latter reached its judgment. For example, upon reading Example *A* in Uri's paper, I immediately identified the task as 'inviting' a faulty conclusion, as well as my conscious, proactive 'flagging' of this tendency. Consequently, I used a reflective process. This mental adjustment happened before I calculated the faulty difference (90 cents), precisely the desired state of affairs indicated in Uri's question #4 above. This shows the need to precisely analyze how intuitive and analytic processes interact. Initial DPT assumed sequential operation—outcomes of intuitive processes

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(or S1) served as input for and triggered the analytic only if S2 identified S1 output as faulty. Recently, parallel processing of both modes was postulated, including possibly competing for the immediate/final judgment to a given task (Evans, 2006). To further theorize S1-S2 interaction, he suggested 3 principles: (a) **singularity—**epistemic mental models are generated and judged one-at-a-time, (b) **relevance—**the intuitive contextualizes problems to exploit relevance to a person's goals, and (c) **satisficing—**the analytic accepts intuitive judgments unless there is a good reason to override them. These crucial principles fall short of accounting for how I solved Example *A*.

My last point refers to factors that make a difference in ways people solve particular problems (see Stanovich, 2008; Stanovich & West, 2000 for review). Here, I refer to a key factor for math education highlighted in Uri's address—the impact of problem format ('packaging') on suppression of intuitive judgments. A substantial part of Uri's work, and a major contribution to our field, focused on the design of bridging tasks for triggering solvers' analytic processes and enabling solution of congruent tasks that seemed 'unpackable' without bridging. This indirect allusion to assimilatory conceptions of those for whom bridging is required points to a hindrance.

As theoretical and practical hindrance I find in DPT is the unproblematic application of an observer's frame of reference ('normative') to the evaluation of people's responses ('rational' or not). If people of varied cognitive abilities solve the same task differently, and if many who failed on a congruent task solved a bridging task, then the mental toolbox solvers and observers bring to the task must be distinguished. Simply put, the use of two cognitive frames of reference is glossed over by DPT's equating of the normative with rational (see Nisbett & Ross, 1994).

Theoretically, and crucial for math education, this lack of distinction fails to acknowledge different interpretation(s) of a task and different mental activities available to the observed for solving it. That is, it overlooks *assimilation* (Piaget, 1985; von Glasersfeld, 1995). Recent cognitive psychology studies pointed out to solvers' different interpretations (Stanovich & West, 2000). But the implication of addressing two frames of reference at once did not seem to follow. In my view, distinguishing the observer and using assimilation as a starting point are necessary in our field to move beyond cognitive psychology's focus on thinking to accounts of learning as a conceptual advance in someone else's mind (Steffe, 1995). As Skemp (1979) and Thompson (2010) asserted, at the core of a math education theory of teaching one must articulate learning as a process of cognitive transformation in what the learner already knows toward intended math.

Practically, overlooking learners' extant conceptions when analyzing their solutions, correct or faulty, hinders the design of bridging tasks shown by Uri. Indirectly, specific features of those tasks (e.g., cueing for a nested sub-set or for the invariant length of string when shapes increase) and the rationale/criteria he provided for using those features (e.g., make the task accessible to the solver's intuition), draw on conjectured inferences about how a person may interpret and/or solve the alternative tasks. This nicely leads to the discussion of Uri's elegant and effective exposition, which I shall follow by elaborating on DPT's core hindrance (Section 3).

Elegant, Effective Exposition

I found Uri's presentation of DPT's contribution to math education to be elegant and effective. Due to space limitations, I focus on two main features: (a) examples used to portray DPT and (b) applying DPT in his empirical studies. To write this paper, I read a few articles about DPT. My non-exhaustive sample revealed a complex set of constructs, as well as subtle distinctions and heavy debates. Yet, via strategically chosen and lucid examples Uri's paper successfully conveyed the essence of DPT. Those examples depicted for me, as naïve reader, (a) the problems addressed by DPT, (b) key assumptions and explanations of phenomena studied, and (c) possible ways in which DPT can promote solution of these problems in math education. Those of us whose interest is sparked by his exposition may inquire further (a great sign of reach!); as is, it clearly portrays DPT and the potential contributions to math education. Further, his examples draw on studies he conducted with colleagues to test how useful DPT can be. The examples from those studies accomplish two key purposes of scholarly exchange of ideas. They demonstrate a commitment for learning through authentic experimentation—walking the walk ***and*** talking the talk. They also provide a critical look into adaptations needed for DPT to become useful. In this sense, his elaboration of (bridging) tasks was powerful as it enabled many students to 'see' a solution for problems they couldn't unpack otherwise, while indicating general criteria for designing such instructional tools. As one who's struggling with writing scholarly papers, especially with drawing on examples, I marvel at Uri's writing.

A Constructivist Lens on DPT: 'Brainy' Mathematics Learning and Teaching

Taking Issue with DPT

I adhere to a core premise common to Piaget's (1970), Dewey's (1902), and Vygotsky's (1978) grand theories: knowing (thinking) cannot be understood apart from how one's knowing evolved. This premise entails my twofold thesis about hurdles in adapting DPT to math education. First, contrasting normative and faulty 'snapshot' *reasoning* in math (or cognitive psychology) falls short of accounts needed to intentionally foster optimal student understandings. Second, although DPT can inform our work, math education already has models that interweave accounts of knowing, coming to know (learning), and teaching. As I shall discuss below, one such model seems to (a) singularly resolve issues of faulty/normative *reasoning* **and** of *conceptual learning* (with or without teaching) and (b) explain different modes of thinking without alluding to 2 systems (or distinct processes). Moreover, this model is supported by and gives support to cognitive neuroscience models of the brain. Due to space limitations, the brief exposition below makes wide use of references to comprehensive versions.

Uri's work, and accounts of DPT I read, raised 7 critical questions for math education:

1. Why/how does the mental system of some people make an error (e.g., selects $1 and 10 cents in Example A) whereas other people focus also on the difference? Unless one also considers a problem solver's assimilatory conceptions, this question (and #2, #3, & #4 below) cannot be resolved by DPT assumptions that S2 has no direct access to the perceived information, or that it selects accessible instead of relevant information.
2. When a response is not normative, is it due to (a) having the required conceptions but failing to trigger them (e.g., Sudoku and elevator examples), (b) having a rudimentary form of those conceptions that require some prompting (e.g., nested subset in Uri's bridging task, pointing out the difference feature in Example *A* and/or making the numbers more 'difficult'), or (c) lacking a conception for S2 to monitor S1 (e.g., next coin-flip and DMP base-rate examples)? And how can we distinguish among these three cases?
3. How does S2, which failed to monitor S1 in a specific task, become capable of doing so? Is the process of learning, and required teaching, different for each of the three cases above?
4. How do new monitoring capacities learned by S2 'migrate' to S1 (become automatic)?
5. What is the source of learners' surprise (e.g., string-around-earth example), how may it be linked to learning, and how might teaching capitalize on this (one of Uri's key pleas)?
6. What role do specifically designed examples/illustrations play in learning (by S2 and/or S1)?

7. Can we explain why particular bridging tasks promote some students' learning but not others', and provide explicit ideas for changing them in the latter case?

A Brain-Based Model of Knowing and Learning

In recent years, a few interdisciplinary meetings of cognitive neuroscientists with math educators took place. One of those (Vanderbilt, 2006) dealt with the design of tasks that (a) reveal difficult milestones and (b) can be examined at the brain level. Using the reflection on activity-effect relationship (*Ref**AER*) account of knowing and learning (Simon & Tzur, 2004; Simon, Tzur, Heinz, & Kinzel, 2004; Tzur, 2007; Tzur & Simon, 2004), I presented fraction tasks. This presentation, and the fertile dialogue with brain researchers that ensued, led to an elaborated, brain-based *Ref**AER* account (Tzur, accepted for publication), found highly consistent with DPT studies of the brain (Lieberman, 2003).

Briefly, knowing (having a conception) is explained as *anticipating* and *justifying* an *invariant relationship* between a goal-directed activity-sequence the mental system executes at any given moment (Evans' *Singularity* principle), potentially or actually, and the effect it must bring forth. Learning is explained as transformation in such anticipation via two basic types of reflection. *Type-I* consists of continual, automatic comparison the mental system executes *between the goal* it sets for the activity-sequence and *subsequent effects* produced. As Piaget (1985) asserted, the internal global goal (anticipated effect) regulates the execution and detects interim effects and the final one (*Relevance* principle) (see also Stich, 1994). The effects either match the anticipation or not (*Satisficing* principle). By default, the mental system runs the activity-sequence to its completion as determined by the goal (e.g., elevator example). Yet, the execution may stop earlier if the goal detects unanticipated sub-effects (e.g., Sudoku–1a) or if a different goal became the regulator, including a sub-goal of the activity-sequence overriding the global goal. *Type-II* consists of comparison across records of experiences, each containing a linked, re-presented 'run' of the activity and its effect, sorted as match/no-match. Type-II is not automatic—it may or may not be executed. Recurring, invariant *AER* across those experiences are linked with the situation(s) in which they were found anticipatory of the proper goal and are registered as a new conception. Constructing a new conception proceeds through 2 stages. The first, *participatory* stage requires reflection Type-I and is marked by anticipation that a solver can access only if somehow prompted for the novel, provisional *AER* (In a forthcoming paper with Lambert I linked this stage with ZPD). The second, *anticipatory* stage requires reflection Type-II and is characterized by spontaneously activating and justifying the novel anticipation. It should be noted that this model, although developed independently, is consistent with Skemp's (1979) foundational theory; the reflection types and stage distinctions extend his work.

To link *Ref**AER* to the brain, I separated and 'distributed' von Glasersfeld's (1995) tripartite notion of scheme—situation, activity, and result—across 3 major neuronal systems in which they are postulated to be processed. The assumption about both knowing and learning is that the basic unit of analysis in the brain is not one synaptic connection or a neuron (Hebb, 1949, cited in Baars & Gage, 2007; Fuster, 1997). Rather, and to stress neuronal 'firing' in the brain and growth/change/decay of neuronal networks, I introduced the term Synapse inhibition-Excitation Constellation (*SIEC*)—any-size aggregate of synapses of connected neurons that, once 'firing' and updating, forms a pattern of activity. The roles and functions of *SIECs* are described below in terms of three neuronal networks in which they may be activated (Baars & Gage, 2007): a 'Recognition System' (RecSys) with the sensory input/buffer and long-term memories, a

'Strategic System' (StrSys) with the Central Executive, and an 'Engagement-Emotive System' (EngSys). Within these, solving a problem is postulated as follows (indices match Figure 2):

1. Solving a problem begins with assimilating it via one's sensory modalities into the **Situation** part of an extant scheme in the RecSys. This *SIEC* is firing and updating until reaching its activity pattern (recognizing state), and activates firing and updating of a Goal *SIEC* in the StrSys.
2. A Goal *SIEC* is set in the StrSys, setting a desired inhibition-excitation state that will regulate the execution and termination of an activity sequence. The goal *SIEC* also triggers:
   a. Corresponding *SIECs* in the EngSys, which set the desirability of the experience and the sense of control the learner has over the activity (McGaugh, 2002; Tzur, 1996; Zull, 2002); This emotional component was linked to activity in the anterior cingulate cortex (Bush, Luu, & Posner, 2000; Lieberman, 2003).
   b. A temporary auxiliary *SIEC* checks if an activity has already been partly executed and can thus be resumed. If its output is 'Yes', it re-triggers the *AER's* execution in the StrSys from the stopping point (go to #4); if 'No,' it triggers the Goal *SIEC* to trigger #3 below.
3. A *SIEC* responsible for searching/selecting an available *AER* is triggered by the Goal *SIEC*. The search operates on three different long-term memory 'storages' of *SIECs*. Using a metaphor of 'road-map', Skemp (1979) explained that, within every universe of discourse (e.g., math, economy), the 'path' from a present state to a goal state may consist of multiple activity-sequences from which the one eventually executed is selected (see also multiple-trace theory in Nadel, Samsonovich, Ryan, & Moscovitch, 2000). The 3 *SIECs* are:
   a. Anticipatory *AER* of the mental process to be carried out;
   b. Participatory *AER* that a learner is currently forming and can be called up only if prompted (dotted arrow);
   c. Mathematical 'objects,' which are essentially anticipatory *AER* established and encapsulated previously (e.g., 'number' is an encapsulated, anticipated effect of a counting operation).
4. Once an operation and 'object' *AER* were selected, the brain executes them while monitoring progress to the goal via a meta-cognitive *SIEC* in the StrSys responsible for Type-I reflections. Skemp's (1979, see ch. 11) model articulates this component in great details, including how it can be carried out automatically (intuitive) and/or reflectively (analytic). This component seems compatible with Norman and Shallice's (2000) model of schema activation and Corbetta and Shulman's (2002) notion of 'circuit breaker'.
5. The execution of the selected *AER* is constantly monitored by ***Type-I* reflection** to determine 3 features:
   a. Was the learner's goal, as set in *SIEC* 2a, met?
   b. Is the *AER* execution moving toward or away from the goal (see McGovern, 2007, for relevant emotions)?
   c. Is the final effect of the executed portion of the *AER* different from the anticipated, set goal. Goldberg and Bougakov (2007) suggested that this is mainly a prefrontal cortex (PFC) function.

   Each feature (5a, 5b, 5c) can stop the executed *AER*. If the output of 5c is 'No', that 'run' of the *AER* is registered as another record of experience of the existing scheme (see Zull, 2002). Symbolically, such no-novelty can be written: Situation$_0$-Goal$_0$-AER_0. If the output is 'Yes', symbolized as Situation$_0$-Goal$_0$-AER_1, a new conceptualization may commence (see next).

This perturbing state of the mental system (von Glasersfeld, 1995), has recently been linked to anticorrelations of brain networks (Fox, et al., 2005).

6. ***Type-II*** reflective comparisons **may** then operate on the output records of *Type-I* reflection. Whenever the output of *Type-I* question 5c is 'Yes,' the brain *updates a new SIEC* for that recently run *AER* and stores it in a temporary auxiliary *SIEC* in the *RecSys* (symbolized A_0-E_1, or AER_1). Each repetition of the solution process for which the output of 5c is 'Yes' adds another such record to the temporary auxiliary.
7. The accruing records of temporary AER_1 (novel) compounds are continually monitored by the *Type-II* reflective comparison *SIEC* in terms of two features:
   a. Is the effect of the new *AER* (E_1) closer to or further away from the *Goal*?
   b. How is the new AER_1 similar to or different from the extant anticipatory and/or participatory *AER* in the *RecSys*? This aspect of *Type-II* reflection is consistent with Moscovitch et al.'s (2007) articulation of the constant interchanges between the Medial Temporal Lobe (MTL) and PFC.

   The output of recurring *Type-II* reflections is a new *SIEC* (AER_1). As the anticipatory-participatory stage distinction implies, initially the Search/Select *SIEC* (#3) may only access a new *SIEC* if the learner is prompted for the activity (A_0), which generates the effect (E_1) and thus 'opens' a neuronal path to using AER_1 in response to the triggering situation (Situation$_0$). In time, *Type-II* comparisons of the repeated use of AER_1 for Situation$_0$ produces a new neuronal pathway from the Situation$_0$ *SIEC* to the newly formed AER_1, that is, to the construction of a directly retrievable, anticipatory *SIEC* (new scheme symbolized as Situation$_1$-Goal$_1$-AER_1).

*Comparing DPT with Brain-Based Ref*AER.*

I contend that *Ref*AER*, with its brain-based elaboration, simultaneously resolves not only the reasoning puzzlement addressed by DPT, but also central problems of math learning and teaching. Concerning *normative* solutions, *Ref*AER* **explains and predicts** their production as the outcome of either an anticipatory conception, which can run automatically and/or reflectively, or a compatible participatory conception that was accessed via a prompt—self/internal (e.g., Soduku-1a) or external (e.g., Uri's bridging task, apple falling on Newton's head). Accordingly, faulty solutions are the outcome of (a) partial/inefficient/flawed execution of a suitable anticipatory conception (e.g., Soduku-1b, elevator), (b) prompt-dependent inability to access a proper participatory conception (e.g., incorrectly solving the $1.10 when difference=$1 but correctly with other amounts), and, often, (c) lack of suitable conception for correctly solving the given task (e.g., 3rd graders facing the next coin flip problem; students in Uri's study who could not solve the bridging task).

I further contend that, for cognitive psychology and math education purposes, *Ref*AER* resolves DPT problems better. Instead of postulating two systems (or processes), it explains how the brain gives rise to a multi-part **single thought process** by which a problem solver may get at a normative or a faulty *answer*. Furthermore, it stresses that a '*solution*' must encompass not only the answer, but also the (inferred) *solver's* reasoning processes used for producing it. *Ref*AER* makes such inferences via analyzing the solver's: (i) goal and sub-goals (see Stanovich & West, 2000, for differing researcher/subject goals), (ii) entire or partial activity-sequence selected and executed (see Kahneman & Frederick, 2002, for Attribute Substitution), (iii) suitability of objects operated on (see Uri's explication of objects, such as length *gap* in string-around-earth and *nested sub-set* in his RMP bridging task), (iv) sub- and final effects noticed and successful/failed reflections (both Types). Last but not least, *Ref*AER* analyses distinguish between two frames of reference in the evaluation of solvers' judgments—observer's advanced/ justified frames and observeds' evolving and sensible relative to her or his extant conceptions.

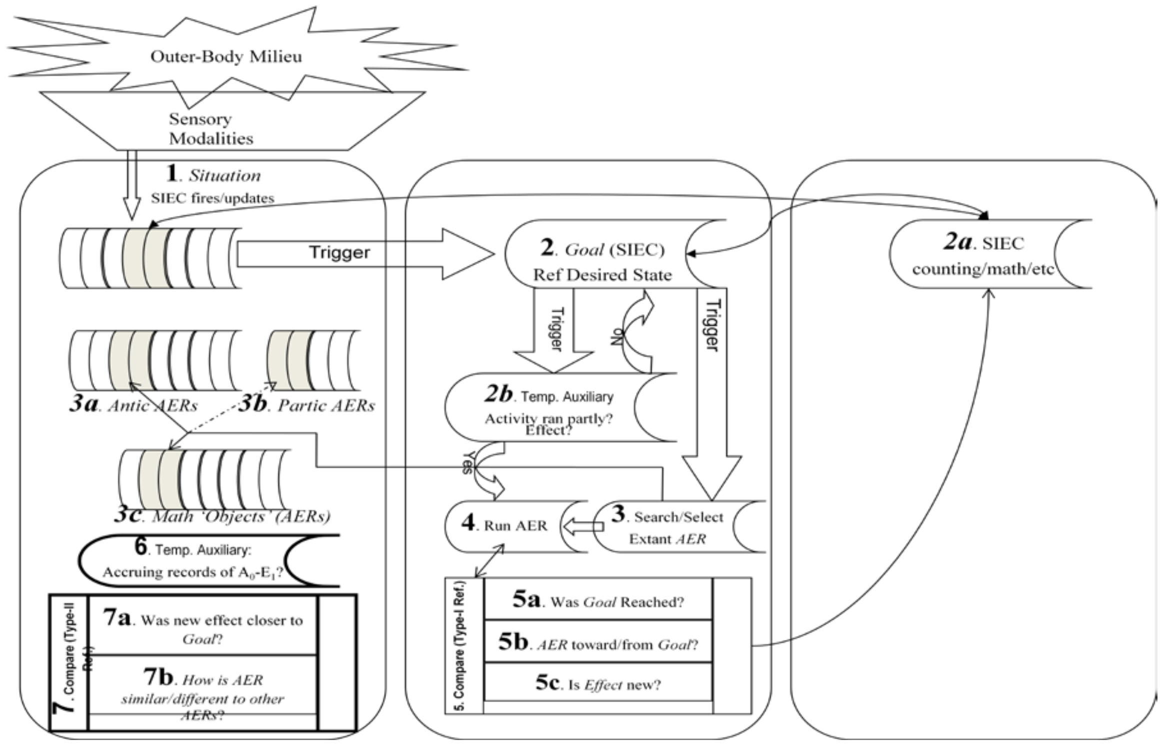

Figure 2. Brain problem solving and learning processes

Thus, consistent with Stich's (1994) assertion that cognitive systems serve an organism's goals and not absolute truths, *Ref*AER* evades the pitfalls of equating normative with rational. Instead, it clarifies that upon a solver's assimilation of a task and setting her/his goal(s), one path is selected among multiple, extant activity-sequences (spontaneous or prompted), executed, and being monitored by the *solver's* goal. By default, the brain runs the sequence to its completion, which is signaled via Type-I comparison (goal *SIEC*), and may thus be portrayed by an observer as intuitive/automatic. However, at any given moment during the execution or after completion, the system's regulator (goal *SIEC*) may notice effects that require interruption and/or correction to the run and/or even to the goal (portrayed as analytic/reflective). Instead of "I think, therefore I err," (Gigerenzer, 2005), we say: "I (learn to) think, therefore I adjust erroneous anticipations."

Most importantly, I contend that *Ref*AER* also contributes to resolving two problems that, while not addressed by DPT, are vital for a foundation of math education (Thompson, 2010), namely, explaining (a) how *learning* to reason—intuitively and analytically—may occur and (b) how can *teaching* capitalize on it and foster (optimize) students' maths. The former has been articulated above; the latter exceeds the scope of this discussion and was articulated elsewhere (Tzur, 2008, 2010) as a 7-step cycle that proceeds from analysis of students' extant conceptions.

To briefly convey the potential of this teaching cycle, I return to Uri's RMP bridging task. In that task, a 2-phase activity-sequence of considering base-rate (1/1000) **and** diagnostic info (5% false positive) was made explicit as linked sub-goals. What's more, 'objects' on which this alternative sequence would operate were replaced, from multiplicatively related quantities (fractions, percents) to whole number frequencies considered additively until the final multiplicative calculation (51/1000). In terms of *Ref*AER*, these alterations shed light on why some of those who incorrectly solved the DMP problem could correctly solve the RMP problem. The alteration was more likely to orient a solver to (a) clearly coordinate sub-goals (**specifying** each of the nested sub-sets) of the task's global goal and (b) select and operate on **accessible quantities**—anticipatory *AER* ('objects')—in place of quantities that are notoriously prompt-dependent (or lacking) in youngsters and adults and, not surprisingly, were '*neglected*'. These insightful alterations also explain the educative power of the RMP task as a 'bridge'. It brought forth an anticipatory *AER* that, I infer, could serve as an internal prompt for correctly selecting/executing the entire activity-sequence when, later, operating similarly on the more difficult-to-grasp multiplicative quantities/relationships. A novel dissertation study of my PhD student, Xianyan Jin, provides a penetrating examination of how *Bridging* (*xianjie*) tasks are uniformly fitted within a 4-component lesson structure in Chinese mathematics teaching. Her work includes 'mapping' the 7-step cycle onto the fourfold lesson structure, emphasizing the role that bridging tasks, like those designed by Uri et al., can play in activating students' extant assimilatory conceptions. Alluding to Uri's closing slogan, while **not** positing thinking dualities, I believe that such brain-based, *Ref*AER*-informed teaching can nurture *"the power of natural thinking,"* address *"the challenge of stretching it,"* and inform *"the beauty of overcoming it."*

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PROMOTING STUDENT UNDERSTANDING THROUGH COMPLEX LEARNING

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The world's increasing complexity, competitiveness, interconnectivity, and dependence on technology generate new challenges for nations and individuals that cannot be met by "continuing education as usual" (The National Academies, 2009). With the proliferation of complex systems have come new technologies for communication, collaboration, and conceptualization. These technologies have led to significant changes in the forms of mathematical thinking that are required beyond the classroom. This paper argues for the need to incorporate future-oriented understandings and competencies within the mathematics curriculum, through intellectually stimulating activities that draw upon multidisciplinary content and contexts. The paper also argues for greater recognition of children's learning potential, as increasingly complex learners capable of dealing with cognitively demanding tasks.

> Although reformers have disagreed on many issues, there is a widely shared concern for enhancing opportunities for students to *learn mathematics with understanding* and thus a strong interest in promoting *teaching mathematics for understanding* (Silver, Mesa, Morris, Star, & Benken, 2009, p.503).

Introduction

In recent decades our global community has rapidly become a knowledge driven society, one that is increasingly dependent on the distribution and exchange of services and commodities (van Oers, 2009), and one that has become highly inventive where creativity, imagination, and innovation are key players. At the same time, the world has become governed by complex systems—financial corporations, the World Wide Web, education and health systems, traffic jams, and classrooms are just some of the complex systems we deal with on a regular basis. For all citizens, an appreciation and understanding of the world as interlocked complex systems is critical for making effective decisions about one's life as both an individual and as a community member (Bar-Yam, 2004; Jacobson & Wilensky, 2006; Lesh, 2006).

Complexity—the study of systems of interconnected components whose behavior cannot be explained solely by the properties of their parts but from the behavior that arises from their interconnectedness—is a field that has led to significant scientific methodological advances. With the proliferation of complex systems have come new technologies for communication, collaboration, and conceptualization. These technologies have led to significant changes in the forms of mathematical thinking that are needed beyond the classroom. For example, technology can ease the thinking needed in information storage, retrieval, representation, and transformation, but places increased demands on the complex thinking required for the interpretation of data and communication of results. Computational skills alone are inadequate here—the ability to interpret, describe, and explain data and communicate results of data analyses is essential (Hamilton, 2007; Lesh, 2007a; Lesh, Middleton, Caylor & Gupta, 2008).

The rapid increase in complex systems cannot be ignored in mathematics education. Indeed, educational leaders from different walks of life are emphasizing the importance of developing students' abilities to deal with complex systems for success beyond school. Such abilities include:

constructing, describing, explaining, manipulating, and predicting complex systems; working on multi-phase and multi-component component projects in which planning, monitoring, and communicating are critical for success; and adapting rapidly to ever-evolving conceptual tools (or complex artifacts) and resources (Gainsburg, 2006; Lesh & Doerr, 2003; Lesh & Zawojewski, 2007).

In this paper I first consider future-oriented learning and then address some of the understandings and competencies needed for success beyond the classroom, which I argue need to be incorporated within the mathematics curriculum. A discussion on complex learners and complex learning, with mathematical modeling as an example, is presented in the remaining section.

Future-oriented learning

> *Every advanced industrial country knows that falling behind in science and mathematics means falling behind in commerce and property* (Brown, 2006).

Many nations are highlighting the need for a renaissance in the mathematical sciences as essential to the well-being of all citizens (e.g., Australian Academy of Science, 2006; The National Academies, 2009). Indeed, the first recommendation of The National Academies' *Rising above the Gathering Storm* (2007) was to vastly improve K-12 science and mathematics education. Likewise the Australian Academy of Science has indicated the need to address the "critical nature" of the mathematical sciences in schools and universities, especially given the unprecedented, worldwide demand for new mathematical solutions to complex problems. In addressing such demands, the Australian Academy emphasizes the importance of interdisciplinary research, given that the mathematical sciences underpin many areas of society including financial services, the arts, humanities, and social sciences.

The interdisciplinary nature of the mathematical sciences is further evident in the rapid changes in the nature of the problem solving and reasoning needed beyond the school years (Lesh, 2007b). Indeed, numerous researchers and employer groups have expressed concerns that schools are not giving adequate attention to the understandings and abilities that are needed for success beyond school. For example, potential employees most in demand in the mathematical sciences are those that can (a) interpret and work effectively with complex systems, (b) function efficiently and communicate meaningfully within diverse teams of specialists, (c) plan, monitor, and assess progress within complex, multi-stage projects, and (d) adapt quickly to continually developing technologies (Lesh, 2008). Research indicates that such employees draw effectively on interdisciplinary knowledge in solving problems and communicating their findings. Furthermore, although such employees draw upon their school learning, they do so in a flexible and creative manner, often generating or reconstructing mathematical knowledge to suit the problem situation (unlike the way in which they experienced mathematics in school; Gainsburg 2006; Hamilton 2007; Zawojewski, Hjalmarson, Bowman, & Lesh, 2008). Indeed, such employees might not even recognize the relationship between their school mathematics and the mathematics they apply in solving problems in their daily work activities. We thus need to rethink the nature of the mathematical learning experiences we provide students, especially those experiences we classify as "problem solving;" we also need to recognize the increased capabilities of students in today's era.

In his preface to the book, *Foundations for the Future in Mathematics Education*, Lesh (2007b) pointed out that the kinds of mathematical understandings and competencies that are targeted in textbooks and tests tend to "represent only a shallow, narrow, and often non-central subset of those that are needed for success when the relevant ideas should be useful in 'real life" situations" (p. viii). Lesh's argument raises a number of issues, including:

What kinds of understandings and competencies should be emphasized to reduce the gap between the mathematics addressed in the classroom (and in standardized testing), and the mathematics needed for success beyond the classroom?

How might we address the increasing complexity of learning and learners to advance their mathematical understanding within and beyond the classroom?

Understandings and competencies for success beyond the classroom

The advent of digital technologies changes the world of work for our students. As Clayton (1999) and others (e.g., Jenkins, Clinton, Purushotma, Robinson & Weigel, 2006; Lombardi & Lombardi, 2007; Roschelle, Kaput, & Stroup, 2000) have stressed, the availability of increasingly sophisticated technology has led to changes in the way mathematics is being used in work place settings; these technological changes have led to both the addition of new mathematical competencies and the elimination of existing mathematical skills that were once part of the worker's toolkit.

Studies of the nature and role of mathematics used in the workplace and other everyday settings (e.g., nursing, engineering, grocery shopping, dieting, architecture, fish hatcheries) are important in helping us identify some of the key understandings and competencies for the 21st century (e.g., de Abreu, 2008; Gainsburg, 2006; Roth, 2005). A major finding of the 2002 report on workplace mathematics by Hoyles, Wolf, Molyneux-Hodgson and Kent was that basic numeracy is being displaced as the minimum required mathematical competence by an ability to apply a much wider range of mathematical concepts in using technological tools as part of working practice. Although we cannot simply list a number of mathematical competencies and assume these can be automatically applied to the workplace setting, there are several that employers generally consider to be essential to productive outcomes (e.g., Doerr & English, 2003; English, 2008; Gainsburg, 2006; Lesh & Zawojewski, 2007). In particular, the following are some of the core competencies that have been identified as key elements of productive and innovative work place practices (English, Jones, Bartolini Bussi, Lesh, Tirosh, & Sriraman, 2008). I believe these competencies need to be embedded within our mathematics curricula: Problem solving, including working collaboratively on complex problems where planning, overseeing, moderating, and communicating are essential elements for success;

- Applying numerical and algebraic reasoning in an efficient, flexible, and creative manner;
- Generating, analyzing, operating on, and transforming complex data sets;
- Applying an understanding of core ideas from ratio and proportion, probability, rate, change, accumulation, continuity, and limit;
- Constructing, describing, explaining, manipulating, and predicting complex systems;
- Thinking critically and being able to make sound judgments, including being able to distinguish reliable from unreliable information sources;
- Synthesizing, where an extended argument is followed across multiple modalities;
- Engaging in research activity involving the investigation, discovery, and dissemination of pertinent information in a credible manner;
- Flexibility in working across disciplines to generate innovative and effective solutions.

Although a good deal of research has been conducted on the relationship between the learning and application of mathematics in and out of the classroom (e.g., de Abreu 2008; Nunes & Bryant 1996; Saxe 1991), we still know comparatively little about students' mathematical capabilities, especially problem solving, beyond the classroom. We need further knowledge on why students have difficulties in applying the mathematical concepts and abilities (that they presumably have learned in school) outside of school—or in classes in other disciplines.

A prevailing explanation for these difficulties is the context-specific nature of learning and problem solving, that is, competencies that are learned in one situation take on features of that situation; transferring them to a new problem situation in a new context poses challenges (Lobato 2003). This suggests we need to reassess the nature of the typical mathematical problem-solving experiences we give our students, with respect to the nature of the content and how it is presented, the problem contexts and the extent of their real-world links, the reasoning processes likely to be fostered, and the problem-solving tools that are available to the learner (English & Sriraman, 2010). This reassessment is especially needed, given that "problems themselves change as rapidly as the professions and social structures in which they are embedded change" (Hamilton, 2007, p. 2). The nature of learners and learning changes likewise. With the increasing availability of technology and exposure to a range of complex systems, children are different types of learners today, with a potential for learning that cannot be underestimated.

Complex learners, complex learning

Winn (2006) warned of the "dangers of simplification" when researching the complexity of learning, noting that learning is naturally confronted by three forms of complexity—the complexity of the learner, the complexity of the learning material, and the complexity of the learning environment (p. 237). We cannot underestimate these complexities. In particular, we need to give greater recognition to the complex learning that children are capable of—they have greater learning potential than they are often given credit for by their teachers and families (English, 2004; Lee & Ginsburg, 2007; Perry & Dockett, 2008; *Curious Minds*, 2008). They have access to a range of powerful ideas and processes and can use these effectively to solve many of the mathematical problems they meet in daily life. Yet their mathematical curiosity and talent appear to wane as they progress through school, with current educational practice missing the goal of cultivating students' capacities (National Research Council, 2005; *Curious Minds*, 2008). The words of Johan van Benthem and Robert Dijkgraaf, the initiators of *Curious Minds* (2008), are worth quoting here:

> What people say about children is: "They can't do this yet."
> We turn it around and say: "Look, they can already do this."
> And maybe it should be: "They can still do this now."

As Perry and Dockett (2008) noted, one of our main challenges here is to find ways to utilize the powerful mathematical competencies developed in the early years as a springboard for further mathematical power as students progress through the grade levels. I offer three interrelated suggestions for addressing this challenge:

1. Recognize that learning is based within contexts and environments that we, as educators shape, rather than within children's maturation (Lehrer & Schauble, 2007).

2. Promote active processing rather than just static knowledge (*Curious Minds*, 2008).
3. Create learning activities that are of a high cognitive demand (Silver et al., 2009).

In the remainder of this paper I give brief consideration to these suggestions. In doing so, I argue for fostering complex learning through activities that encourage knowledge generation and active processing. While complex learning can take many forms and involve numerous factors, there are four features that I consider especially important in advancing students' mathematical learning. These appear in Figure 1.

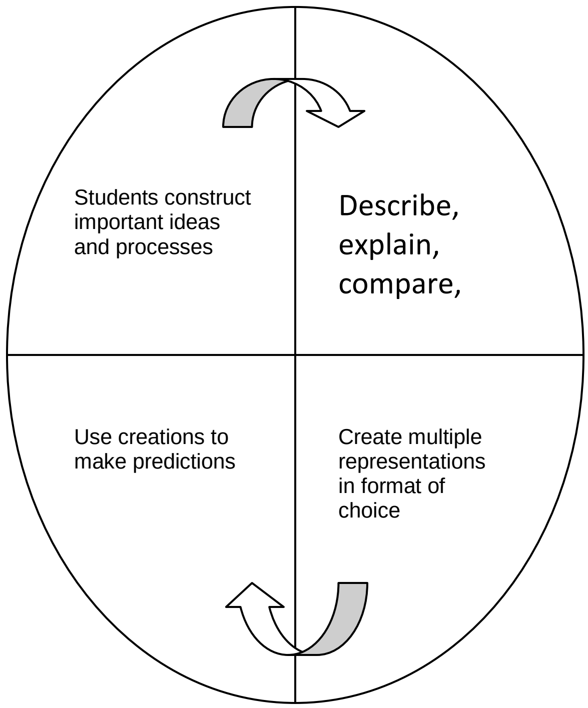

Figure 1. Key Features of Complex Learning

Research in the elementary and middle school indicates that, with carefully designed and implemented learning experiences, we can capitalize on children's conceptual resources and bootstrap them towards advanced forms of reasoning not typically observed in the regular classroom (e.g., English & Watters, 2005; Ginsburg, Cannon, Eisenband, & Pappas, 2006; Lehrer & Schauble, 2007). Most research on young students' mathematical learning has been restricted to an analysis of their actual developmental level, which has failed to illuminate their potential for learning under stimulating conditions that challenge their thinking—"Research on children's current knowledge is not sufficient" (Ginsburg et al., 2006, p.224). We need to redress this situation by exploring effective ways of fashioning learning environments and experiences that challenge and advance students' mathematical reasoning and optimize their mathematical understanding.

Recent research has argued for students to be exposed to learning situations in which they are *not* given all of the required mathematical tools, but rather, are required to create their own versions of the tools as they determine what is needed (e.g., English & Sriraman, 2010; Hamilton, 2007; Lesh, Hamilton, & Kaput, 2007). For example, long-standing perspectives on classroom problem solving have treated it as an isolated topic, with problem-solving abilities assumed to develop through the initial learning of basic concepts and procedures that are then practised in solving word ("story") problems. In solving such word problems, students generally engage in a one- or two-step process of mapping problem information onto arithmetic quantities and operations. These traditional word problems restrict problem-solving contexts to those that often artificially house and highlight the relevant concept (Hamilton, 2007). These problems thus preclude students from creating their own mathematical constructs. More opportunities are needed for students to generate important concepts and processes in their own mathematical learning as they solve thought-provoking, authentic problems. Unfortunately, such opportunities appear scarce in many classrooms, despite repeated calls over the years for engaging students in tasks that promote high-level mathematical thinking and reasoning (e.g., Henningsen & Stein, 1997; Silver et al., 2009; Stein & Lane, 1996).

Silver et al.'s recent research (2009) analyzing portfolios of "showcase" mathematics lessons submitted by teachers seeking certification of highly accomplished teaching, showed that activities were not consistently intellectually challenging across topics. About half of the teachers in the sample (N=32) failed to include a single activity that was cognitively demanding, such as those that call for reasoning about ideas, linking ideas, solving complex problems, and explaining and justifying solutions. Furthermore, the teachers were more likely to use cognitively demanding tasks for assessment purposes than for teaching to develop student understanding. While Silver et al.'s research revealed positive features of the teachers' lessons, it also indicated that the use of cognitively demanding tasks in promoting mathematical understanding needs systematic attention.

Modeling Activities

One approach to promoting complex learning through intellectually challenging tasks is mathematical modeling. Mathematical models and modeling have been interpreted variously in the literature (e.g., Romberg, Carpenter, & Kwako, 2005; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; English & Sriraman, 2010; Greer, 1997; Lesh & Doerr, 2003). It is beyond the scope of this paper to address these various interpretations, however, but the perspective of Lesh and Doerr (e.g., Doerr & English, 2003; Lesh & Doerr, 2003) is frequently adopted, that is, models are "systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behavior of some other familiar system" (Doerr & English, 2003, p.112). From this perspective, modeling problems are realistically complex situations where the problem solver engages in mathematical thinking beyond the usual school experience and where the products to be generated often include complex artifacts or conceptual tools that are needed for some purpose, or to accomplish some goal (Lesh & Zawojewski, 2007).

In one such activity, the *Water Shortage Problem*, two classes of 11-year-old students in Cyprus were presented with an interdisciplinary modeling activity that was set within an engineering context (English & Mousoulides, in press). In the *Water Shortage Problem*, constructed according to a number of design principles, students are given background information on the water shortage in Cyprus and are sent a letter from a client, the Ministry of Transportation, who needs a means of (model for) selecting a country that can supply Cyprus with water during the coming summer period. The letter asks students to develop such a model using the data given, as well as the Web. The quantitative and qualitative data provided for each country include water supply per week, water price, tanker capacity, and ports' facilities. Students can also obtain data from the Web about distance between countries, major ports in each country, and tanker oil consumption. After students have developed their model, they write a letter to the client detailing how their model selects the best country for supplying water. An extension of this problem gives students the opportunity to review their model and apply it to an expanded set of data. That is, students receive a second letter from the client including data for two more countries and are asked to test their model on the expanded data and improve their model, if needed.

Modeling problems of this nature provide students with opportunities to repeatedly express, test, and refine or revise their current ways of thinking as they endeavor to create a structurally significant product—structural in the sense of generating powerful mathematical (and scientific) constructs. The problems are designed so that multiple solutions of varying mathematical and scientific sophistication are possible and students with a range of personal experiences and knowledge can participate. The products students create are documented, shareable, reusable, and modifiable models that provide teachers with a window into their students' conceptual understanding. Furthermore, these modeling problems build communication (oral and written) and teamwork skills, both of which are essential to success beyond the classroom.

Concluding Points

The world's increasing complexity, competitiveness, interconnectivity, and dependence on technology generate new challenges for nations and individuals that cannot be met by "continuing education as usual" (The National Academies, 2009). In this paper I have emphasized the need to incorporate future-oriented understandings and competencies within the mathematics curriculum, through intellectually stimulating activities that draw upon multidisciplinary content and contexts. I have also argued for greater recognition of children's learning capabilities, as increasingly complex learners able to deal with cognitively demanding tasks.

The need for more intellectually stimulating and challenging activities within the mathematics curriculum has also been highlighted. It is worth citing the words of Greer and Mukhopadhyay (2003) here, who commented that "the most salient features of most documents that lay out a K-12 program for mathematics education is that they make an intellectually exciting program boring," a feature they refer to as "intellectual child abuse" (p. 4). Clearly, we need to make the mathematical experiences we include for our students more challenging, authentic, and meaningful. Developing students' abilities to work creatively with and generate mathematical knowledge, as distinct from working creatively on tasks that provide the required knowledge (Bereiter & Scardamalia, 2006) is especially important in preparing our students for success in a knowledge-based economy. Furthermore, establishing collaborative, knowledge-building communities in the mathematics classroom is a significant and challenging goal for the advancement of students' mathematical learning (Scardamalia, 2002).

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A Critique and Reaction to

PROMOTING STUDENT UNDERSTANDING THROUGH COMPLEX LEARNING

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The plenary by Lyn English addresses "the world's increasing complexity, competitiveness, interconnectivity, and dependence on technology" which education and changes hitherto to education fails to meet. In her talk, future oriented competencies and multidisciplinary activities are proposed as ways in which complex learning as opposed to simplistic (rote) learning can be promoted. In this critique I weigh the strengths and weaknesses of her proposals in light of existing research.

The times we live in have often been characterized as the nexus of the information age and globalization in which society is increasingly driven by industry, economies, innovation, research and development that need not occur at a local level. Numerous mathematics and science educators as well as position documents from the national academy of science have called for promoting a better understanding of the world in which students are situated by making use of complex adaptive systems in the curriculum and in day to day lessons (The National Academies, 2007, 2009). The plenary paper by English focuses on three main areas, namely:

(1) What is future oriented learning and why it is relevant?
(2) What are the competencies necessary for success beyond the classroom, and
(3) What are complex systems and examples of complex learning activities?

By surveying the recommendations of existing research within models and modeling, complex adaptive learning, and different types of workplace situated learning, English asks us to recognize that contextual and complex learning within an idea rich environment should be strived for in the classroom. Examples of learning activities are then given which place high cognitive demands on students (Silver et al., 2009) as well as active knowledge processing (Curious Minds, 2008).

Future oriented learning is described as learning that takes into account that disciplinary boundaries of science today are not as rigid as they were, say even a decade ago, and promotes competencies which takes this into account. Indeed new fields of study such as mathematical biology, neuroeconomics, bioinformatics, ethnobotany, and other professions that have emerged are often both interdisciplinary and transdisciplinary, and call for competencies related to understanding complex real world phenomena, team work, communication and technological skills.

Numerous definitions of complex systems are found in the literature. One commonly used definition by different non-linear dynamic groups is as follows:

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

Complex systems are spatially and/or temporally extended nonlinear systems characterized by collective properties associated with the system as a whole—and that are different from the characteristic behaviors of the constituent parts.

Viewing learning in a classroom as a complex adaptive system is a non-trivial enterprise. The examples of research which (attempt to) "operationalize" complexity theory and examine learning as a complex adaptive system in students occur in lessons involving simulations of a real world phenomena which a classroom as a whole or in groups can simultaneously observe and manipulate (Wilensky & Stroup, 1999, 2000), and those involved in models and modeling research which make use of model eliciting activities (Lesh et al., 2007; Lesh & Sriraman, 2010). In learning situations which make use of model eliciting activities, complex systems are understood as: (a) "real life" systems that occur (or are created) in everyday situations, (b) conceptual systems that humans develop in order to design, model, or make sense of the preceding "real life" systems, and (c) models that researchers develop to describe and explain students' modeling abilities. Models for designing or making sense of such complex systems are, in themselves, important "pieces of knowledge" that should be emphasized in teaching and learning – especially for students preparing for success in future-oriented fields that are heavy users of mathematics, science, and technology (English & Sriraman, 2010; Lesh, 2006; Lesh et al., 2007; Lesh & Sriraman, 2005).

Having summarized some of the major themes in the plenary paper by Lyn English, I now turn my attention to the problem of researching learning in complex adaptive situations. The assumptions in such research are that individual learners, groups and classrooms are complex adaptive systems (Hurford, 2010), and that learners adapt to the classroom situation involving a model eliciting activity, or a simulation involving a CAS. With these assumptions in mind, the goal of researchers is to identify the constituent components of learning without losing sight of the big picture. In other words, if an individual learner is considered to be a meta-agent whose activity is the observable pattern emerging from other interacting agents (Lesh & Yoon, 2004, Hurford, 2010), then how are the internal conceptual models that learners are dynamically forming reflected in the rules, actions and artifacts in external artifacts and representations. The question I am posing is whether there is a match in the internal and external representations produced by such learning activities, and if so, what implication does this have for a coherent theory of learning that goes beyond the traditional neo-Piagetian/Vygotskian dichotomy or other situated/socio cultural accounts of learning? Can cultural-historical activity theory provide an adequate theoretical framework to clearly document and explain how these moments of complex learning are reflected in the actions of lower and higher order agents, and in the continually evolving artifacts produced by learners?

While I agree that "establishing collaborative, knowledge-building communities in the mathematics classroom is a significant and challenging goal for the advancement of students' mathematical learning (Scardamalia, 2002)", I remain pragmatic about the adequacy of existing theoretical frameworks to capture the learning that takes place in such environments. The question of how this burgeoning area of research can have a systemic impact on school, curricula, textbooks and teacher education is even more complex (pun intended).

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PRACTICAL RATIONALITY AND THE JUSTIFICATION FOR ACTIONS IN MATHEMATICS TEACHING

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The action happens in a high school geometry course in late November. The class has spent some time learning to use triangle congruence to prove statements and has begun the study of quadrilaterals. Mr. Jones starts the day congratulating students for their progress learning to do proofs. He then draws Figure 1 on the board and asks the class to prove a statement about the relationship between the sides of the rectangle *ABCD*. There is some hesitation. Somebody is heard to ask whether they could prove that $\overline{AB}$ is longer than $\overline{BC}$ while another kid asks what they have to go on; the teacher ignores them. A student asks whether triangles *ADE* and *BCE* are congruent. Mr. Jones writes down this question on the board and draws two arrows from it. One arrow points toward a question he writes, "how would it help to know that those triangles are congruent?;" the other arrow points toward another question he also writes, "what would you need to assume to be able to say that those triangles are congruent?" You can hear somebody say that it's obvious they are congruent while another says that they could then say the triangles are isosceles. Another student says "you'd need to know that *AEB* is a right angle;" Mr. Jones writes this on the boardand asks the class what they have to say about that. Some students claim to not really know what the teacher means with that question but others raise their hands. One of these students says that she thinks it would be useful if the angle were right because then the angles at the top would be congruent with the small angles at E. Some kids perk up and one kid says, "and you could then say that AB is twice BC." The teacher asks them to take a few minutes and see if they can prove that the ratio between the sides is 2 assuming as little as possible. You see a kid write, "Prove: The ratio is 2 " while others have written "Given:" and are pensive.

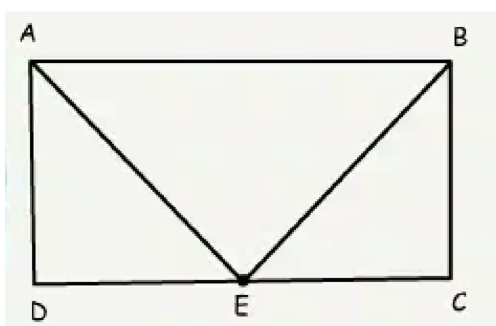
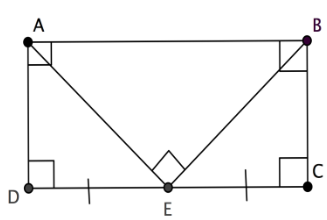


Figure 1. Mr. Jones diagram

I want to use that episode to raise a few questions around mathematics instruction in school classrooms. Some of these questions concern the substance of the episode—in particular, what is the nature of students' engagement in proving and of the teacher's work managing that activity? Other questions are about theory: What kind of considerations about classroom instruction could help us describe and explain how teacher and students ordinarily transact mathematical ideas, in such a way that we could also account for possible avenues for change and foresee their consequences? Finally, other questions are about methodology: What kind of data can help us ground those theoretical considerations? How to obtain it? These questions, though large, serve to introduce a program of research that falls under the name of practical rationality of mathematics teaching (Herbst & Chazan, 2003; Herbst, Nachlieli, & Chazan, in review).

What mathematical work are students doing in the episode above? I'd describe it as listing plausible statements about a figure and considering whether they could be connected through logical necessity. The source of some of those statements is perceptual—e.g. the observation that angle *AEB* is right. But regardless of their origin, statements are being connected through deduction in two directions—what statements would enable one to infer the plausible statement made and what inference could be made taking that plausible statement for granted. The assertion about the relative length of the sides of the rectangle eventually derives from the plausible truth of those earlier statements. Students are thus reducing a question of truth (what could be true about an object) to a question of deducibility from possible statements about an object. They are using proof as a method to find things out.

That mathematical work is valuable from a mathematical perspective as well as from a more general epistemological perspective. Such use of proof as method in knowledge inquiry is essential to the discipline of mathematics (Lakatos, 1976). It is also behind the drive to model mathematically other fields of experience: The expectation that in those fields it will also be possible to reduce the problem of truth to the quest for deducibility, which can then warrant new, still unknown, possible truths is important in pure and applied science (Jahnke, 2007). Insofar as that kind of work could empower students with such way of knowing, I'd argue that participating in problem solving of the kind presented in the scenario above is not only authentic mathematical work to do but also a skill that, if learned, would enable students to contribute to society. It is likely the case, however, that few students encounter such opportunities to learn about proof in school mathematics. The work they do during those years rarely includes chances to acquire the skill or the appreciation of the methodological, model-making function of proof or even experiences doing work that could have had that exchange value.



Given: *ABCD* rectangle,

E midpoint of $\overline{CD}$,

$\angle AEB$ right angle

Prove: $\frac{AB}{BC} = 2$

Figure 2. A more likely proof exercise

It is more likely that the problem above would be presented to high school geometry students as shown in Figure 2. In particular, while students are responsible to prove propositions in high school geometry, it is normative that the teacher (or the book) will state the givens and conclusion of the propositions they prove. I call that statement a *norm* of "doing proofs" in the sense that an observer describes teachers and students acting *as if* they expected this would be the case. The sense to which 'normative' means 'usual' could be corroborated empirically by observing, over large number of high school geometry classrooms, the recurrence of this feature in proof activity. Yet other techniques would be needed to corroborate that the statement is a norm in the sense that participants act as if they expected such behaviors as appropriate or correct. I return to this below, after more considerations about how the study of instruction and its rationality can help improve opportunity to learn.

Traditionally the question of what opportunities to learn are created in instruction has been thought of as a question of resources. For example, one might think that the limited nature of students' encounters with proof result from instruction having insufficient or inadequate resources: unsophisticated curricula, teachers that don't know enough, etc. Traditional theorizing in education has had instruction playing an instrumental role, feeding from resources and producing student learning; improvement has been conceptualized as improving resource quality. But more recently researchers have realized that resource *use* in instruction is what makes a difference (Cohen, Raudenbush, and Ball, 2003; Stein, Grover, and Henningsen, 1996). A more fundamental inquiry into the nature and function of instruction itself is therefore warranted. Within that inquiry I am interested in what in how instruction regulates itself.

I started my work from pondering whether the kind of mathematical work described above—the use of proof as a tool to know with—could feasibly be deployed in classrooms. Of course that question includes questions of resource development and my instructional experiments have included developing resources (e.g., Herbst, 2003, 2006). But behind that feasibility question is the fundamental hypothesis that classrooms are complex systems where actions are not merely a projection of the will or capacity of the actors or the richness of their resources. Rather, actions of individual actors contribute to the deployment of a joint activity system whose performance also feeds back, and thus gives shape, to the actions that the participants can take in that system. The question then is not simply how to design materials that enable desirable mathematical work or how to create in teachers the desire to promote that work. The questions also are what is the structure and function of the activity system where that work might be deployed and how this system might accommodate or resist attempts to deploy that work. In particular this requires thinking of mathematics instruction in school classrooms as a system of relationships that are deployed under various conditions and constraints. A conceptualization of this system could enable us to think in a more sophisticated and potentially accurate way about what teacher and students do and thus be able to foresee if given improvement efforts have a prospect of success.

An analogy with how mathematics educators have evolved in their thinking about students' errors can illuminate this conceptualization of instruction as a system. There used to be a time when student errors were seen as indications of misfit, mishaps, or forgetfulness. Things changed when research on students' mathematical work started to be treated within a cognitive paradigm. For example Resnick and colleagues' (1989) study on decimal fractions showed that students' errors had conceptual basis: Their errors could be explained on the existence of tacit controls such as the "fraction rule" or the "natural number rule". Students that made errors did so not out of the lack knowledge but out of the possession of some knowledge. Our stance toward students' errors thus changed from a judgment stance early on to an inquiry stance later on: Rather than judging students as irrational when they make errors, we strive to understand what rationality leads them to make those errors.

I want to propose that we think of the actions of teachers (and students) in the classroom in analogy with how we have come to think about error in students' mathematical work. The analogy I propose is that we could think of "error" in teaching—really teaching that deviates from what we might deem desirable—not as an indication of misfit, ill will, or lack of knowledge, of the practitioner. Rather, we can think of it as an indication of the possible presence of some knowledge, knowledge of what to do, which is subject to a practical rationality that justifies it. This is a rationality that we should try to understand better before judging teachers or legislating their practice. Teachers and students act in classrooms in ways that attest to the existence of specialized knowledge of what to do; knowledge that outsiders of those classrooms are less likely to have even if they know the knowledge domain being taught and learnt. For example teachers and students of geometry would likely see it as strange for Mr. Jones to ask the students for the givens of the problem. I want to focus here on the rationality associated with the role of the teacher and how this might warrant or indict actions like that one.

The teacher of a specific course of mathematics studies, such as high school geometry, is an institutional role to play, not just a name to describe an aspect of an individual's identity (Buchmann, 1986). There is a person who plays the role, for sure, and that person comes to play the role with personal assets that are likely to matter in what he or she chooses to do. These assets are likely to include mathematical knowledge for teaching and skill at doing some tasks of teaching (Ball, Thames, & Phelps, 2008). These assets make a difference; teachers who have these assets may be able to figure out and do things that others may not be able to do. But while teachers' causes and motives to do things may have personal grounds, it is unlikely that their actions could always be justified on personal grounds. One could imagine that Mr. Jones in the scenario above was bored with the prospect of giving his students another routine proof exercise or wanted to have a fun day teaching geometry. But we could not really expect him to use any of that as the warrant for doing what he did—his job is not to find activities that amuse him, but rather to teach geometry to his students. How could he justify having done that? The notion of practical rationality points to a container of dispositions that could have currency in a collective, for example within the set of colleagues who teach geometry in similar settings. By dispositions we mean what Bourdieu (1998) describes as the categories of perception and appreciation that would compel agents in a practice to act in specific ways. Dispositions tend to be tacit but they can be articulated to others when justifying to one's peers (or to other stakeholders) why one might or might not do something like what Mr. Jones did with that proof problem. The high school geometry course and the work of doing proofs, in particular, have been particularly fertile grounds for me to develop theory about instruction and the practical rationality of mathematics teaching. I want to use this context to present some of those theoretical ideas.

A basic notion to describe the role the teacher (and the student) play in classrooms is that of didactical contract (Brousseau, 1997): The hypothesis that student and teacher have some basic roles and responsibilities vis-à-vis a body of knowledge at stake. These responsibilities include the expectation for the teacher to give students work to do which is supposed to create opportunities to learn elements of that body of knowledge, and the expectation for the student to engage in the work assigned, producing work that can be assessed as evidence of having acquired that knowledge. I use the word *norm* to designate each of those statements that an observer makes in an effort to articulate what regulates practice: Actors act *as if* they held such statement as a norm, though they may be quite unaware of it. From the perspective of the teacher, the didactical contract authorizes a basic exchange economy of knowledge that he or she has to manage: An exchange between work designed for, assigned to, and completed by students and elements of knowledge, prescribed by the contract, at stake in that work, and hopefully embodied in students' productions. A fundamental role of the teacher is to manage those exchanges. This management includes, first, enabling and supporting mathematical work; and second, interpreting the proceeds of this work, exchanging it for the knowledge at stake, acting on behalf of the discipline as well as of other stakeholders. Evidently, the hypothesis of a didactical contract only says that a contract exists that has those characteristics; the hypothesis means to describe any mathematics teaching inside an educational institution. But it is also obvious that the teacher and student roles and responsibilities are under-described by that hypothesis: There are many ways in which the didactical contract could be enacted to have at least those characteristics; contracts could be quite different from each other not the least because the mathematics at stake could be very different from course to course and thus require very different forms of work to be learned. Even for the same course of studies, say high school geometry, different contracts could further stipulate the roles and responsibilities of teacher and student differently. In classes that work under a contract for teaching through problems (Lampert, 2001), where students' work on problems enabled them to come across and use the geometric propositions in the curriculum, students might have a responsibility for recognizing new knowledge. It is not expected that they do so in the usual geometry course, where students only use in problems those propositions that have been previously installed in class.

While some research has endeavored to conceptualize, enact, and study the characteristics of alternative contracts (e.g., Chazan, 2000; Yackel & Cobb, 1996), in my work I have been interested in using a variety of approaches to study the usual high school geometry contract and the practical rationality behind the teachers' work managing the exchanges enabled by that contract. The reason for that has been the thought that durable change in instruction will need not only to provide new and better resources but also to be able to deal with the inertia and possible reactions from established practice. Knowing how instruction usually works and what rationality underpins its usual operations is key for designing reforms that are viable and sustainable. Furthermore, knowledge of how usual instruction works can encourage piecemeal, incremental changes that don't throw the proverbial baby with the bathwater.

"Doing proofs" has been a useful starting point in that research agenda. With my research group we have studied "doing proofs" in high school geometry using several approaches: looking at intact geometry classrooms (Herbst, 2002a; Herbst et al., 2009), historical textbooks and documents (Herbst, 2002b; González & Herbst, 2006), geometry lessons that accommodate alternative proving work (Herbst, 2003, 2006), students' responses to different tasks that might involve proving (Herbst & Brach, 2006) and geometry teachers' responses to problematic scenarios of geometry instruction (Nachlieli & Herbst, 2009; Weiss, Herbst, & Chen, 2009). The historical analysis has showed how the general skill "how to do proofs" became an object of study in and of itself, leaving behind the important role that proofs played in the construction of specific concepts, theorems, and theories that result from mathematizing a field of experience (Boero, 2007). The work that students do has also evolved. When students "do proofs" what matters is not (anymore) *what* they can prove given what they avail themselves of but just *whether* and *how well* they prove whatever they prove. In exchange for a claim on that knowledge students are to show that they can connect a "given" with a "prove" by making a sequence of statements justified on prior knowledge. I argue that this exchange is facilitated by a specialized set of norms that elaborate how the didactical contract applies.

From observing work in geometry classrooms we have noted that implicit expectations of who is to do what and when vary depending on the specifics of the object of study. In relation to diagrams, for example, the extent to which students can draw objects into a diagram or draw observations from a diagram varies according to whether the work is framed as a construction, an exploration, or a proof. While a contract for a course may have some general norms that differentiate it from a different contract, there is also differentiation in the more specific norms within the course, depending again on what is at stake. Much of those rules are cued in classroom interaction through the use of selected words such as *prove*, *construct*, or *conjecture*. These words frame classroom interaction by summoning special, mutual expectations, or *norms*, of who can do what and when. I have used the expression *instructional situation* to refer to each of those frames. Instructional situations are specialized, local versions of the didactical contract that frame particular exchanges of work for knowledge, obviating the need to negotiate how the contract applies for a specific chunk of work. “Doing proofs” is an example of an instructional situation in high school geometry; “solving equations” is an example of an instructional situation in algebra I (Chazan & Lueke, 2009). We contend that these frames for classroom interaction, these instructional situations, are defaults for classroom interaction, tacit knowledge of what to do that the classroom as an organization has (Cook & Brown, 1999), perpetuated through socialization (and with the aid of textbooks and colleagues) that, in particular, provide cues for the teacher on what to do and what to expect the student to do. Instructional situations are social units of analysis; they organize joint action with content. Of course explaining causally their empirical realization (and the chances for deliberate alteration through teacher development) requires some use of psychological constructs to understand precisely how individuals come to recognize that they are indeed in an instructional situation or how they come to perceive alternatives for action in an instructional situation. Our work has not progressed that far. Thus far we have created models of those situations that consist of arrays of norms that describe each situation in terms of who has to do what and when. Those models facilitate research on the content of practical rationality.

Practical rationality is a container whose content includes the categories of perception and appreciation that are viable within the profession of mathematics teaching to warrant (or indict) courses of action. The notion of an instructional situation is the point of departure to study this rationality empirically. We build on the ethnomethodological notions of *breaching experiment* and *repair strategy* (Mehan & Wood, 1975) to propose, as a methodological hypothesis, that if participants of an instructional situation are immersed in an instance of a situation where one of its norms has been breached, they will engage in repair strategies that not only confirm the existence of the norm but also elaborate on the role that the norm plays in the situation. Our technique of data collection relies on representations of breached instances of instructional situations—representations in video or comic strips, using real teachers and students or using cartoon characters. We confront usual participants of such instructional situation with one such breached representation. For example the classroom scenario narrated above is quite close in content to an animated classroom story, “A proof about rectangles,” that we produced in order to study with it the rationality behind the tacit norm that the teacher is in charge of spelling out the givens and the prove. To find out about that rationality we attend to participants’ reactions to the representation: Do they perceive the breach of the norm? Do they accept the situation in spite of the breach? What do they identify as being at risk because of the breach? What opportunities, if any, do they see being created or lost because of the breach? Our aim is not so much to understand the participants themselves as it is to use the participants’ experience with the situation to understand the situation better. In particular we want to understand what elements of the practical rationality of mathematics teaching teachers can see as viable justifications of breaches of situations that would arguably be desirable, say because they might create a more authentic kind of mathematical work. In the case of the story narrated above our question concretely would be on what account could a teacher justify (or indict) an action like the one Mr. Jones took. Clearly, researchers might have some good reasons why what Mr. Jones did is justifiable and I have tried to articulate that from a mathematical perspective above; but in spite of the fact that some of us have had experience teaching we don’t know teaching now in the way practitioners do. By virtue of the role that they play they have to respond to specific obligations that shape their decisions. This leads me to introduce the last element of the theory.

When teachers respond to a breach in an instructional situation, they may reject the situation. For example teachers might say that Mr. Jones is leading students in an *exploration* rather than a proof. But participants might also accept the situation as “doing proofs” and engage in repairs that rehearse grounds for justifying the breach Mr. Jones made. One set of justifications of it might address the nature of proof in mathematical practice. Participants might also repair by rehearsing grounds for indicting the breach. For example they might note how the time taken in having students state the givens precluded them from doing other proofs that day. In general, we propose that four professional obligations can organize the justifications (or indictments) that participants might give to actions that depart from a situational (or contractual) norm. We call these obligations disciplinary, individual, interpersonal, and institutional (Herbst & Balacheff, 2009; see also Ball, 1993). The *disciplinary obligation* says that the mathematics teacher is obligated to steward a valid representations of the discipline of mathematics. The *individual obligation* says that a teacher is obligated to attend to the well being of the individual student. The *interpersonal obligation* says to all members of the class that they are obligated to share and steward their (physical, discursive) medium of interaction. And the *institutional (schooling) obligation* says that the teacher is obligated to observe various aspects of the schooling regime including policies, schedules, and such. We contend that those obligations can be present in participants’ justifications or indictments of breaches of norms and that combined with the norms of contracts and situations they span the practical rationality of mathematics teaching. Within that rationality one can see specific contracts (high school geometry, algebra I) and their instructional situations (doing proofs, solving equations) as sociohistorical constructions dependent on elements of practical rationality; more importantly, one can see possibilities for improved practices as subject to similar grounds for justification. Practices that are close to existing instructional situations (as gauged by how many norms of a situation a practice breaches) may be easier to justify than others. The theory also provides the means for the researcher to anticipate how instruction may respond to new practices: A novel task such as “what is something interesting that could be proved about the object in Figure 1” conjures up by resemblance one or more instructional situations (e.g., “doing proofs” and “exploration”) as possible frames for the work to be done. Models of those situations provide the researcher with a baseline of norms that could be breached as the work proceeds. Researchers can then use the obligations to anticipate what kinds of reactions the teacher may perceive in and from practice that feedback and thus shape how they manage the work. This can be useful in examining the potential derailments in the implementation of new practices in classrooms as well as the examination of teachers’ responses to assessment or development interventions. Thus the theory provides not only the basis for the design of probes for the rationality of teaching (Herbst & Miyakawa, 2008) but also a framework for an analysis of the reactions of participants. Combined with finer tools from discourse analysis (e.g., Halliday & Matthiessen, 2004) teachers’ responses to representations of breaching (but arguably valuable) instances of an instructional situation can help us understand not only what justifies teaching as it exists today but also how new practices could be justified in ways that practitioners find compelling. The psychology of mathematics teachers may still be useful to inform what enables and motivates individual teachers to do things, but the logic of action in mathematics teaching addressed by practical rationality may help us understand why some of those actions can be viable and sustainable.

Endnotes

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A Critique and Reaction to

PRACTICAL RATIONALITY AS A FRAMEWORK FOR MATHEMATICS TEACHING

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Grossman and McDonald (2008) recently argued that the research community needs to move its "attention beyond the cognitive demands of teaching … to an expanded view of teaching that focuses on teaching as a practice (p. 185)." Building on the work of Bourdieu (Bourdieu & Wacquent, 1992; Bourdieu, 1985, 1998), Pat Herbst and colleagues (Herbst & Chazan, 2003, 2006) have written about mathematics teaching as a practice, just as law and medicine are considered practices, in an attempt to better understand the rationality that produces, regulates, and sustains mathematics instruction. This *practical rationality* is the commonly held system of dispositions or the "feel for the game" (Bourdieu, 1998, p. 25) that influences practitioners as to those actions that are appropriate in the classroom. It is practical rationality that:

> *…not only enables practices to reproduce themselves over time as the people who are the practitioners change, but also regulates how instances of the practice are produced and what makes them count as instances* (Herbst & Chazan, 2003, p. 2).

To better understand the practice of mathematics teaching, whether to communicate or improve it, one must understand the practical rationality that guides it. However, practical rationality often "erases its own tracks" (Herbst & Chazan, 2003, p. 2) so that its practitioners come to view these practices as being natural. Because this rationality provides the regulatory framework that socializes its current and future practitioners into ways of thinking and acting that conform to expectations, it is important to bring to the forefront a deliberate, conscious understanding of the rationality that drives the practice of mathematics teaching.

While practical rationality allows for a certain amount of diversity in its similarity, it is driven by norms. These norms provide the persistent continuity of the practice. Before future teachers ever enroll in education courses, they have ideas about schools in general and mathematics instruction in particular (Ball, 1988). Through an apprenticeship of observation, they develop deep-seated ideas about mathematics and its teaching and learning (Lortie, 1975). These ideas often form the foundation on which they will eventually build their own practice of mathematics teaching (Millsaps, 2000; Skott, 2001).

Herbst has suggested four obligations of teachers that frame instruction and that have the potential to organize a departure from normative practice: disciplinary, individual, interpersonal, and institutional obligations. Of the four, I would like to focus on the disciplinary obligation. What is taken as:

> *Mathematically normative in a classroom is constrained by the current goals, beliefs, suppositions, and assumptions of the classroom participants. At the same time these goals and largely implicit understandings are themselves influenced by what is legitimized as acceptable mathematical activity* (Yackel & Cobb, 1996, p. 460).

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This raised an initial series of questions in my mind. First, what is “acceptable mathematical activity” in the words of Yackel and Cobb? Is it the same as “authentic mathematical work” using the words of Herbst? Next, what role does authentic mathematics work[1] play in the practice of mathematics teaching as well as in the practice of mathematics teacher education?

Herbst’s plenary paper encouraged me to ponder if the rationality that often drives the teaching of mathematics (or more specifically the teaching of geometry) overlaps with the rationality associated with doing mathematics. While I confess to having studied neither systematically, the following assumptions and questions are based on my experiences[2] and seem to be supported by the work of Michael Weiss (2009), one of Herbst’s students[3]. Mathematicians, those whose goals are to generate new and refine existing mathematical ideas and methods (Weiss, 2009), are more than just proficient at mathematics. While they demonstrate exactly those qualities and competencies that have been identified by the National Research Council (2001) as goals of mathematics learning (namely conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition), mathematicians also demonstrate a certain mathematical wonder and an appreciation of mathematics that extends past their professional careers into their personal lives. They often tweak problems; at times this is done out of curiosity, other times to make the problem more accessible. To what extent do the activities commonly seen in classrooms nurture authentic mathematical work? Do current norms in mathematics instruction promote either mathematical proficiency or curiosity? Does the rationality that drives mathematics teaching help encourage an appreciation of mathematics?

Herbst has drawn our attention to the practice of geometry instruction. He has provided a scenario and suggestions that should provoke thought as to the norms surrounding the teaching of proof, but what about other key components of geometry courses? For example, definitions play a critical role in geometry. What norms exist for the teaching of definitions in geometry? Are students presented with finalized definitions or are they given opportunities to create, reflect on, and compare definitions (de Villiers, 1998)?

What rationality underpins other aspects of geometry instruction? What is normative in regards to the introduction and use of the diagrammatic register often seen only in geometry classes? What rationality guides teachers’ and students’ expectations in regard to the role of perception in the reading of geometric diagrams? What norms influence the teaching of subtle, yet key, concepts of geometry like existence and uniqueness? Are students given impossible problems[4] as a means to discover existence? Are students allowed to explore situations that demonstrate uniqueness?[5]

While the above questions are particular to geometry, others apply to the many branches of mathematics. Is it normative to encourage students to tweak existing problems or to introduce their own assumptions when solving problems? How often are students encouraged to pose their own problems? Are they taught strategies like Brown and Walters’ (2004) “what-if-not” strategy as a relatively simple means to generate new problems?[6]

Unfortunately, a large number of teachers view mathematics “as a discipline with a priori rules and procedures that … students have to learn by rote” (Handal, 2003, p. 54). For many teachers in the U.S. “knowing” mathematics is taken to mean being efficient and skillful in performing rule-bound procedures and manipulating symbols (Thompson, 1992). As a consequence, mathematics students are “not expected to develop mathematical meanings and they are not expected to use meanings in their thinking” (Thompson, 2008, p. 45).

Herbst has suggested that it is crucial to recognize how instruction typically works, understanding the practical rationality that underpins teaching, if we are to design reforms that are viable and sustainable. He has claimed that through incremental changes that recognize current practice permanent transformation is most likely to occur, but how might incremental changes be introduced? What form might such changes take?

One method that has been shown to have a profound, transformative effect on future teachers' beliefs about the nature of mathematics and its teaching and learning is through experiences with mathematical discovery (Liljedahl, 2005). Through engagement in authentic mathematical activities, teachers might come to view mathematics differently. If they come to view mathematics differently, the disciplinary obligation that frames their instruction could lead to changes in what they deem valid representations of mathematics.

Undergraduate mathematics courses should not be the only opportunities for future teachers to experience mathematics. Mathematics teacher educators need to realize that they "have the dual responsibility of preparing teachers, both *mathematically* and *pedagogically* (Liljedahl, Chernoff, & Zazkis, 2007, p. 239)." Besides providing future and current teachers opportunities to engage in authentic mathematical activities during their mathematics education courses, teacher educators should also provide opportunities for teachers to witness authentic mathematical work in secondary classrooms through episodes of instruction such as written and video cases (much like the scenario of Mr. Jones' class that Herbst presented). Such experiences would provide teacher educators a first-hand experience of teachers' reactions to breaches in normative practice. Moreover, such experiences might even transform teachers' views of the nature of mathematics and its teaching and learning, possibly influencing the rationality that underpins their instruction.

While Herbst made a strong case for the use of practical rationality as a lens for research, I conclude by challenging those of us in mathematics and mathematics education to use this same lens as a means to look introspectively at our practice. What is the rationality that undergirds the way we represent doing and teaching mathematics in our own courses? Do our normative practices include opportunities for authentic mathematical work?

Endnotes

1. Here I assume that "authentic mathematical work" would correspond to the work of mathematicians.
2. This experience includes sixteen years of full-time university teaching including nine years in a Department of Mathematics and seven years in a Graduate School of Education.
3. After reading Herbst's plenary paper, I used my familiarity with his work to pen a draft of my response. To be sure that Herbst's ideas were accurately represented, I reviewed his work including that of some of his students. I was struck when I first read Weiss' dissertation at how much my response overlapped with his study. While the final version of this paper has not changed much from the draft, I would encourage anyone who is interested in the issues I have raised to read Weiss' dissertation, which develops the idea of mathematical sensibility and attempts to answer some of the questions that I have presented.
4. Existence is involved in relatively simple, yet impossible, activities like: Form a triangle with sides of lengths 2 cm, 3 cm and 10 cm. It is also involved in more complicated problems like: Find a circle tangent to two non-equidistant points from the vertex of an angle, such that one point lies on one ray of the angle and the second point lies on the other ray of the angle. This

second problem is shown as a part of an instructional episode modeled in the ThEMaT (Thought Experiments in Mathematics Teaching) animations found at http:grip.umich.edu/themat.

5. An obvious example of allowing students to consider uniqueness would involve the SSA case of triangles.

6. For example of a what-if-not application, consider how a compass and straightedge are used to construct a perpendicular bisector for a given line segment. Applying the "what-if-not" strategy could lead to the following questions. What if you wanted to construct a bisector that was not perpendicular to the line segment? How could you construct a perpendicular that did not bisect the segment?

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REPRESENTATIONS OF LEARNING FOR TEACHING: LEARNING PROGRESSIONS, LEARNING TRAJECTORIES, AND LEVELS OF SOPHISTICATION

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Learning progressions (LP) are playing an increasingly important role in mathematics and science education (NRC, 2001, 2007; Smith, Wiser, Anderson, & Krajcik, 2006). They are strongly suggested for use in assessment, standards, and teaching. In this article, I discuss the nature of learning progressions and related concepts, and I illustrate issues in their construction and use. I also highlight the different ways that LP represent learning for teaching.

Definitions and Constructs

According to the National Research Council, "Learning progressions are descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic" (2007, p. 214). A similar description of LP is given by Smith et al. who define a LP "as a sequence of successively more complex ways of thinking about an idea that might reasonably follow one another in a student's learning" (2006, pp. 5-6). Unlike Piaget's stages, but similar to van Hiele's levels, it is assumed that progress in LP is not "developmentally inevitable" but depends on instruction (Smith et al., 2006, pp. 5-6).

Common Characteristics of the LP Construct

In the research literature, LP possess several commonalities and differences. The characteristics most common to descriptions of LP are as follows:

- LP "are based on research syntheses and conceptual analyses" (Smith et al., 2006, p. 1); "Learning progressions should make systematic use of current research on children's learning " (NRC, 2007, p. 219).
- LP "are anchored on one end by what is known about the concepts and reasoning of students [entering the period covered by the LP]. … At the other end, learning progressions are anchored by societal expectations. … [LP also] propose the ***intermediate*** understandings between these anchor points that " (NRC, 2007, p. 220).
- LP focus on how core ideas, conceptual knowledge, and connected procedural knowledge (not just skills) develop. LP organize "conceptual knowledge around core ideas" (NRC, 2007, p. 220).
- LP recognize that not all students will follow one general sequence (NRC, 2007).

Differences in LP Construct

There are several differences in how the LP construct is used in the literature.

- LP differ in the time spans they describe. Some progressions describe the development of students' thinking over a span of years; others describe the progression of thinking through a particular topic or instructional unit.

- LP differ in the grain size of their descriptions. Some are appropriate for describing minute-to-minute changes in students' development of thought, while others better describe more global progressions through school curricula.
- LP differ in the audience for which they are written. Some LP are written for researchers, some for standards writers, some for assessment developers (formative and summative), and some for teachers.
- LP differ in the research foundation on which they are built. Some LP are syntheses of extant research; some synthesize extant research then perform additional research that elaborates the syntheses (the additional research may be cross-sectional or longitudinal).
- LP differ in how they describe student learning. Some numerically "measure" student progress, while others describe the nature or categories of students' cognitive structures.

Learning Trajectories (LT)

A construct that is similar to, different from, and importantly related to, LP is that of a "learning trajectory." I define a LT as a detailed description of the sequence of thoughts, ways of reasoning, and strategies that a student employs while involved in learning the topic, including specification of how the student deals with all instructional tasks and social interactions during this sequence. There are two types of LT, hypothetical and actual. Simon (1995) proposed that a "*hypothetical learning trajectory* is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process—a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (p. 136). In contrast, descriptions of *actual learning trajectories* can be specified only during and after a student has progressed through such a learning path. Steffe described an actual LT as "a model of [children's] initial concepts and operations, an account of the observable changes in those concepts and operations as a result of the children's interactive mathematical activity in the situations of learning, and an account of the mathematical interactions that were involved in the changes. Such a learning trajectory of children is constructed during and after the experience in intensively interacting with children" (2004, p. 131). Clements and Sarama's (2004) view of LT emphasizes the relationship between levels in the LP and the sequence of tasks in which this progression occurs.

One critical difference between my definitions of LP and LT is that trajectories include descriptions of instruction but that progressions do not. One of the most difficult issues facing researchers who are constructing hypothetical LT for curriculum development is determining how instructional variation affects trajectories. That is, how specific is the trajectory to the instructional sequence that accompanies it? How do trajectories vary with curricula? How similar are trajectories for the same concept within similar and different curricula? If one constructs a prototypical hypothetical LT for a particular topic, how do the actual LT for individual students vary about this prototypical path? One might think of a prototypical trajectory as a "mean" of the actual student pathways, so the "standard deviation" of the distribution of actual trajectories is also relevant.

Purpose: Fixedness versus Reactivity

A major difference between LP and LT arises from the purposes for which they are developed. If one is developing curricula, one is more likely to develop a LT, with a fixed sequence of learning tasks. If, in contrast, one is focusing on formative assessment, one is more likely to develop LP (with associated assessment and instructional tasks), arranged in ways that allow the flexibility and reactivity needed in day-to-day and moment-to-moment teaching. Indeed, if one is taking a constructivist approach to teaching, flexibility and reactivity are key. For instance, Simon argues that hypothetical LT should help teachers with (a) "advanced planning and *spontaneous decision making*," and (b) making instructional decisions based on their "best guess of how learning might proceed" (Simon, 1995, p. 135). I believe, however, that LP can accomplish the same thing for expert users who use their understanding of LP and underlying learning mechanisms to generate local hypothetical LT "on-the-fly." Thus, from the constructivist perspective, hypothetical LT and LP must help teachers understand, plan, and react instructionally, on a moment-to-moment basis, to students' developing reasoning.

Theoretical Frameworks for Learning Progressions

LP can also be differentiated by examining their theoretical frameworks. For instance, van Hiele related progress through his levels to his phases of instruction. In contrast, Battista used constructivist constructs such as levels of abstraction to describe students' progress through the van Hiele levels (see also Pegg & Davey, 1998).

The Nature of Levels

A critical component of LP is the notion of "levels" of sophistication in student reasoning. Because the concept of level is not straightforward, and because how one defines levels determines how one views (and measures) level attainment, I examine this concept in more detail, using the van Hiele levels as an example. The issues discussed in the van Hiele context are critical because any attempt to develop, assess, and use levels in LP must address these same issues.

Levels, Stages, and Hierarchies

Clements and Battista (1992) differentiated researchers' use of the terms *stage* and *level*. A *stage* is a substantive period of time in which a particular type of cognition occurs across a variety of domains (as with Piagetian stages). A *level* is a period of time in which a distinct type of cognition occurs for a specific domain (the size of the domain is an issue).

Types of Hierarchies

Hierarchies of levels in LP come in two types. A "weak" hierarchy is a set of levels that are ranked in order of sophistication, one above another, with no class inclusion relationship between the levels. A "strong" hierarchy is a set of levels ranked in order of sophistication, one above another, with class inclusion relationships between the levels; that is, students who are at level n are assumed to have progressed through levels 1, 2, … (n-1). The van Hiele levels were originally hypothesized to form a strong hierarchy (which is generally supported by the research), while Battista's length levels (discussed below) form a weak hierarchy.

Being "At" a Level

What, precisely, does it mean to be "at" a level? Battista (2007) argued that students are ***at*** a van Hiele level when their overall cognitive structures and processing causes them to be disposed to and capable of thinking about a topic in a particular way. So students are "at" van Hiele Level 1 when their overall cognitive organization and processing disposes them to think about geometric shapes in terms of visual wholes; they are at Level 2 when their overall cognitive organization disposes and enables them to think about shapes in terms of their properties. Also in this view, when students move from familiar content to unfamiliar content, their level of thinking might decrease temporarily; but because students are disposed to operate at the higher level, they look to use that level on the new material, and quickly become capable of using that level (Battista, 2007). However, even if we develop an adequate definition for what it means to be "at" a level, the periods of time when students meet the strict requirement for being at levels may be short, with students spending much time "in transition."

A Different Approach: Vectors and Overlapping Waves.

Some studies indicate that people exhibit behaviors indicative of different van Hiele levels on different subtopics of geometry, or even on different kinds of tasks (Clements & Battista, 2001). So an alternate view of the development of geometric reasoning is that students develop several van Hiele levels simultaneously. Consistent with this view, Gutiérrez et al. (1991) used a vector to indicate the degrees of acquisition of each van Hiele level. Similar to the vector approach, several researchers have posited that different types of reasoning characteristic of the van Hiele levels develop simultaneously at different rates with overlapping waves of acquisition, and that at different periods of development, different types of reasoning are dominant (Clements & Battista, 2001; Lehrer et al., 1998).

Although these alternate models of the van Hiele theory have merit, they both face an important issue—intermingling of *type of reasoning* from *level of reasoning.* That is, sometimes the term *visual-holistic* is used to refer to a type of reasoning that is strictly visual in nature, and sometimes it is used to refer to a period of development of geometric thinking when an individual's thinking is dominated and characterized by visual-holistic thinking.

Level Determination

Empirical determination of individual students' levels of thinking is a major issue in LP. For instance, consider some of the different ways that researchers have determined van Hiele levels. Gutiérrez and Jaime (1998) defined four mental processes that were used in each van Hiele level then used these processes as indicators of a student's level of reasoning. In a collaborative effort, Battista, Clements, and Lehrer developed a triad sorting task, that, with variations, was used in separate research efforts (Battista, 2007). Students were presented with three polygons, such as those in Figure 1, and were asked, "Which two are most alike? Why?"

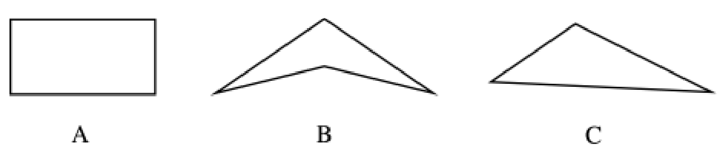

Figure 1. Triad polygon sorting task.

Lehrer et al. (1998) construed each triad as an indicator of type of reasoning. So students' use of different types of reasoning on different triads was taken as evidence of jumps in levels. In contrast, Clements and Battista (2001) used a *set* of 9 triad items as an indicator of student levels. To be classified at a given level, a student had to give at least 5 responses at that level. If a student gave 5 responses at one level and at least 3 at a higher level, the student was considered to be in transition to the next higher level.

Another difference between these researchers' approaches is that Lehrer et al. (1998) classified student responses solely on the basis of type of reasoning, while Clements and Battista (2001) also accounted for the "quality" of reasoning—each reason for choosing a pair in a triad was assessed to see if it correctly discriminated the pair that was chosen from the third item in the triad. The van Hiele levels for students were determined using a complicated algorithm that accounted for both type of reasoning and discrimination score[1].

Cognition Based Assessment (CBA): Levels, Progressions, Trajectories, and Profiles

I now describe my work on CBA to illustrate the relationship between LP, LT, and levels of sophistication as *representations of learning for teaching*[2].

The CBA View of Learning and Instruction

According to the "psychological constructivist" view of learning *with understanding*, the way students construct, interpret, think about, and make sense of mathematical ideas is determined by the elements and organization of the relevant mental structures that the students are currently using to process their mathematical worlds (e.g., Battista, 2004). A major component of psychological constructivist research is its attention to students' construction of meaning for specific mathematical topics. For numerous topics, researchers have found that students' development of conceptualizations and reasoning can be characterized in terms of "levels of sophistication" (Battista, 2001). These levels lie at the heart of the CBA conceptual framework for understanding and building upon students' learning progress. Selecting/creating instructional tasks, adapting instruction to students' needs, and assessing students' learning progress require detailed, cognition-based knowledge of how students construct meanings for the specific mathematical topics targeted by instruction.

CBA Assessment and Instruction

To implement mathematics instruction that genuinely and effectively supports students' construction of mathematical meaning and competence, teachers must not only understand cognition-based research on students' learning of particular topics, they must be able to use that knowledge to determine and monitor the development of their own students' reasoning. CBA supports these activities with four critical components.

1. Descriptions of core mathematical ideas and reasoning processes that form the foundation for students' sense making and understanding of elementary school mathematics.
2. For each core idea, research-based descriptions of levels of sophistication (LP) in the development of students' understanding of and reasoning about the idea.
3. For each core idea, coherent sets of assessment tasks that enable teachers to investigate their students' mathematical thinking and precisely locate students' positions in the cognitive terrain for learning that idea.
4. For each core idea, descriptions of instructional activities specifically targeted for students at various levels to help them move to the next higher level.

Learning Progressions and Trajectories for Length

The CBA levels of sophistication, or LP, for a topic (a) start with the informal, pre-instructional reasoning typically possessed by students; (b) end with the formal mathematical concepts targeted by instruction; and (c) indicate cognitive plateaus reached by students in moving from (a) to (b). As an example, the table below outlines the CBA LP for length.

Non-Measurement Reasoning	Measurement Reasoning
N0: Student Compares Objects' Lengths in Vague Visual Ways	M0: Student Uses Numbers in Ways Unconnected to Iteration of Unit-Lengths M1: Student Iterates Units Incorrectly
N1: Student Correctly Compares Whole Objects' Lengths Directly or Indirectly	*M1.1: Iterates Non-Length Units (e.g., Squares, Cubes, Dots) and Gets Incorrect Count of Unit-Lengths* *M1.2: Iterates Unit-Lengths but Gets Incorrect Count* M2. Student Correctly Iterates ALL Unit-Lengths One-by-One
N2: Student Compares Objects' Lengths by Systematically Manipulating or Matching Their Parts	*M2.1: Iterates Non-Length Units (e.g., Squares, Cubes) and Gets Correct Count of Unit-Lengths for Straight Paths*
N2.1. Rearranging Parts to Directly Compare Whole Shapes	*M2.2: Iterates Non-Length Units (e.g., Squares, Cubes) To Correctly Count Unit-Lengths for Non-Straight Paths* *M2.3: Explicitly Iterates Unit-Lengths and Gets Correct Counts for Straight and Non-Straight Paths*
N2.2. One-to-One Matching of Parts	M3: Student Correctly Operates on Composites of Visible Unit-Lengths
N3: Student Compares Objects' Lengths Using Geometric Properties	M4: Student Correctly and Meaningfully Determines Length Using only Numbers—No Visible Units or Iteration M5: Student Understands and Uses Procedures/Formulas for Perimeter Formulas for Non-Rectangular Shapes

The CBA LP for length is graphically depicted in Figure 2a. Also shown, are an ideal hypothetical LT (solid path) and a typical actual LT for students (dotted path). The CBA LP represents the "cognitive terrain" that students must ascend during an actual LT.

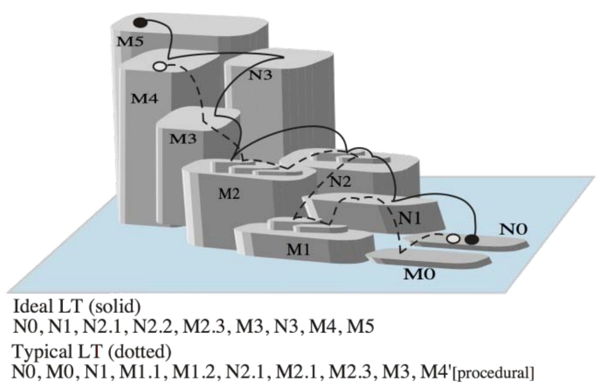

Figure 2a.
CBA Levels of Sophistication Plateaus and LT

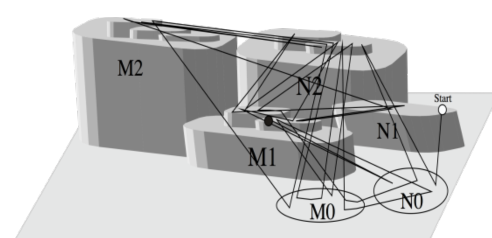

Figure 2b.
A portion of RC's actual LT

A CBA LP for a topic describes not only cognitive plateaus, but what students can and cannot do, students' conceptualizations and reasoning, cognitive obstacles that obstruct learning progress, and mental processes needed both for functioning at a level and for progressing to higher levels. The levels are derived from analysis of both the mathematics to be learned and empirical research on students' learning of the topic. The jumps in the ascending plateau structure of a CBA levels-model represent cognitive restructurings evidenced by observable increases in sophistication in students' reasoning about a topic. A CBA LP indicates jumps in sophistication that are small enough to fall within students "zones of construction." That is, a student should be able to accomplish the jump from conceptualizing and reasoning at Level N to conceptualizing and reasoning at Level N+1 by making a significant abstraction, in a particular context, while working to solve an appropriate problem or set of problems.

Because the levels are compilations of empirical observations of the thinking of many students, and because students' learning backgrounds and mental processing differ, a particular student might not pass through every level for a topic; he or she might skip some levels or pass through them so quickly that the passage is difficult to detect. Even with this variability, however, the levels still describe the plateaus that students achieve in their development of reasoning about a topic. They indicate major landmarks that research has shown students often pass through in "constructive itineraries" or LT for these topics.

Delving Deeper into LP/LT Representations

The LT depicted in Figure 2a are simplifications of actual LT traversed by individual students. To illustrate, I describe one portion of the actual LT of fifth grader, RC, who was having great difficulty with the concept of length (the trajectories of other students were usually much simpler). Figure 2b shows RC's LT for 34 consecutive length items (starting with the white circle, end with the black circle). This actual LT is extremely complex because it contains so much back-and-forth movement between levels.

Figure 3 provides a better representation of this complicated portion of RC's LT. The period shown starts with RC's levels on initial assessment items, moves to his responses during an instructional intervention, and ends with his reasoning on reassessment items.

But even Figure 3 does not adequately portray RC's actual LT in enough detail to be maximally useful for instruction. We need to return to the data to develop a summary characterization of RC's reasoning in terms of the CBA conceptual framework for length. Although space does not permit showing the data, I carefully re-examined the critical period of instructional intervention in which RC made progress (see the three starred items in Figure 3).

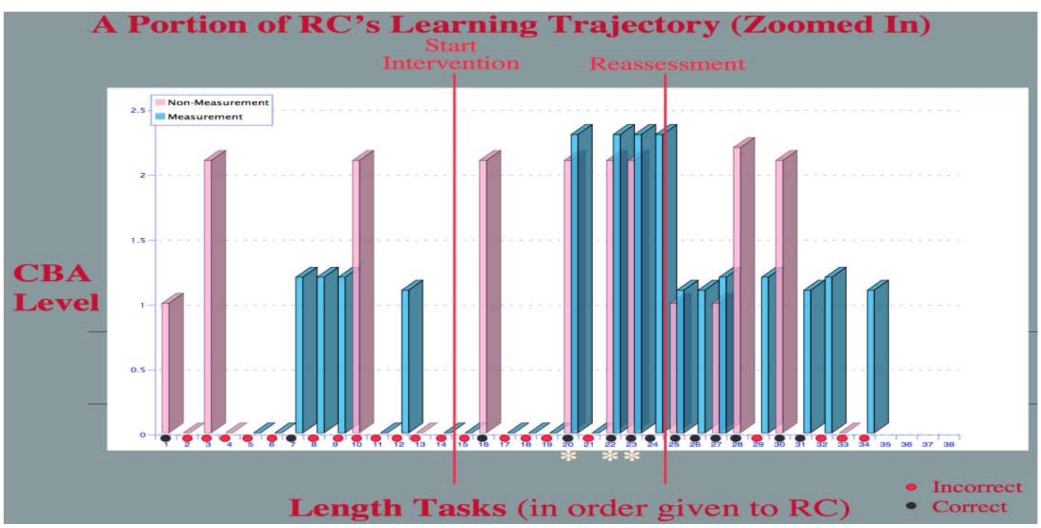

Figure 3

Given the data on RC's reasoning, how should we represent his current knowledge structure with respect to length in a way that is most helpful for instruction? Rather than using an actual LT, the CBA approach is to construct a "profile" of RC's reasoning. To see what this profile looks like, the data indicate that on problems like Item 23, in which the "wires" could be stretched using actual inch rods, RC saw *empirically* that counting unit lengths could predict which was longer. So, for these problems, he adopted the scheme of comparing wires by counting unit lengths in them. At first, he checked his answers by physically straightening a set of inch rods for each wire; but he curtailed this physical check on the last problem. We can conclude that in this context, RC had abstracted a particular reasoning scheme. However, in the different contexts used in the reassessment, where dots and squares were salient, RC did not apply his new scheme (but he also did not apply his original M0 scheme).

So, in future instruction, we need to help RC reconnect to the scheme he abstracted for Item 23. To broaden his scheme to these new contexts, RC needs to iterate inch rods (M2.3) and connect this iteration to straightening paths (N2.1). For instance, in problems in which the lengths of paths appearing on square-inch grids had to be compared, we would encourage RC to use inch rods to check his answers. [Using this type of intervention, many students constructed more generally applicable schemes, overcoming the fixation on the visually salient squares.]

Importantly, the best instruction for RC is determined not by knowing the predominant level number of RC's reasoning, but by using the constructs of the CBA LP to analyze and characterize RC's reasoning. It is this conceptual profile that enables us to appropriately characterize and diagnose RC's reasoning.

Qualitative versus Quantitative Approaches to Developing LP

I believe that both qualitative and quantitative methods are equally rigorous and "scientific" for developing LP. Generally, both approaches involve (a) synthesizing, integrating, and extending previous research to develop conceptual models of the development of student reasoning about a topic (hypothesized LP); (b) developing and iteratively testing assessment tasks; (c) conducting several rounds of student interviews in support of steps (a) and (b); and (d) iteratively refining LP levels. In qualitative approaches, the cycle of iteration, testing, and revising eventually "stabilizes" into final levels, as determined by current levels being used to reliably code all data. Quantitative methods compare observed data to statistical model predictions (often with mathematical iteration) to adjust assessment item sets and levels.

Rash Rush to Rasch? Issues with Quantitative Methods

There have been numerous recommendations to use quantitative techniques to develop LP (e.g., NRC, 2001), with a hint that using non-quantitative techniques is less "scientific." For example, Stacey and Steinle state that there have been "repeated suggestions made by colleagues over the years, which implied that we had been remiss in not using this Rasch analysis with our data" (2006, p. 89). However, using Rasch and other IRT approaches raises serious issues that are often ignored.

First, Rasch/IRT models are "measurement" models. For instance, Masters and Mislevy state that "The probabilistic partial credit model … enables *measures* of achievement to be constructed" [italics added] (1991, p. 16). Wilson describes Saltus as an example of "psychometric models suitable for the analysis of data from assessments of cognitive development" (1989, p. 276). However, the whole enterprise of "measuring" in psychological research has been criticized, with less than compelling rebuttals (Michell, 2008).

Second, many of the assumptions of numerical models do not seem to fit our understanding of the process of learning and reasoning in mathematics. For instance, the Saltus model "assumes that each member of group *h* applies the strategies typical of that level consistently across all items" (Wilson, 1989, p. 278). Or, "The Saltus model assumes that all persons in class *c* answer all items in a manner consistent with membership in that class…. This means that a child in, say, the concrete operational stage is always in that stage, and answers all items accordingly. The child does not show formal operational development for some items and concrete operational development for others" (Draney & Wilson, 2007, p. 121). But LP levels do not necessarily form a strong hierarchy, making quantitative models problematic:

> *When [situations in which students appear to reason systematically] arise, evidence about student understanding can be summarized by [numerical] learning progression level diagnoses, and educators can draw valid inferences about students' current states of understanding. Unfortunately, inconsistent responding across problem contexts poses challenges to locating students at a single learning progression level and makes it unclear how to interpret students' diagnostic scores. For example, how should one interpret a score of 2.6? A student with this score could be reasoning with a mixture of ideas from levels 2 and 3, but the student could also be reasoning with a mixture of ideas from levels 1, 2, 3, and 4* (Steedle & Shavelson, 2009, p. 704).

Thus, use of Rasch-like models to examine cognitive development, such as Wilson's Saltus model or latent class analysis, assumes that students are "at a level," which returns us to the problem discussed earlier about a student being at a level. Research on learning suggests that quite often, the state of student learning is not neatly characterized as "being at a specified level," which causes problems for interpretation of model results: "Students cannot always be located at a single level … Consequently, learning progression level diagnoses resulting from item response patterns cannot always be interpreted validly" (Steedle & Shavelson, 2009, p. 713).

Third, Rasch/IRT models are based on measures of "item difficulty," which might not capture critical aspects of the nature of student reasoning, as Stacey and Steinle argue:

> *Being correct on an item for the wrong reason characterises DCT2 [their decimal knowledge assessment]. It is one of the reasons why the DCT2 data do not fit the Rasch model, because these items break with the normal assumption that correctness on an item indicates an advance in knowledge (or ability) that will not be 'lost' as the student further advances. … A student's total score on this test might increase or decrease depending on the particular misconception and the mix of items in the test. This does not fit the property of Rasch scaling … that 'the number right score contains all the information regarding an examinee's proficiency level, that is, two examinees who have the same number correct score have the same proficiency level' … Neither the total score … nor Rasch measurement estimates provides a felicitous summary of student performance* (2006, pp. 87-88).

Indeed, Stacey and Steinle further state that, "Conceptual learning may not always be able to be measured on a scale, which is an essential feature of the Rasch approach. Instead, students move between categories of interpretations, which do not necessarily provide more correct answers even when they are based on an improved understanding of fundamental principles" (2006, p. 77). They concluded that there is nothing to gain in applying the Rasch approach to their study and many others. "Learning as revealed by answers to test items is not always of the type that is best regarded as 'measurable', but instead learning may be better mapped across a landscape of conceptions and misconceptions" (2006, p. 89)[3]. Even more, how to place rote performance on items becomes extremely problematic in such models. For instance, in Noelting's hierarchy for proportional reasoning, in the formal operational stage the "child learns to deal formally with fractions, ratios, and percentages" (Draney & Wilson, 123). But using a formal procedure rotely is not a valid indication of formal operational reasoning.

Methods for Collecting Data on Students' Levels in Learning Progressions

The most accurate way to determine students' levels in LP (once the framework has been developed) is administering individual interviews, which are then coded by experts, using the LP levels framework. Many teachers can learn to make such determinations, both with individual interviews and during class discussions. However, the difficulty with this approach is that it is time consuming. Another way to gather such data is using open-ended questions. Again, students' written responses must be coded, and many students do not write enough—far less than they say in interviews—for proper coding.

An alternate, less time-consuming, way to gather data is through multiple choice items that have distracters that are generated from interviews and that correspond to specific levels (Briggs et al., 2006, have labeled this format "Ordered Multiple-Choice [OMC]"). CBA has also experimented with having students orally describe their reasoning to a teacher or a classroom volunteer, who then chooses an option on a multiple-choice-like coding sheet. However, beyond convenience, there are several issues that one must consider when using these alternate formats. For instance, how accurate are Open-Ended and OMC tasks in revealing students' thinking (e.g., Alonzo & Steedle, 2008; Briggs & Alonzo, 2009; Steedle & Shavelson, 2009)[4]? For example, students may not recognize which OMC choice matches the strategy they used to solve a problem even when a researcher-generated description of their strategy is one of the choices.

When assessments are used summatively, taking a numerical approach can be both practical and useful. However, to use numerical data for instructional guidance, teachers must consult the theoretical model on learning that underlies the levels framework.

Summary

When using quantitative methods to develop levels in LP, the validity and usefulness of interpretations of results depends on (a) the adequacy of the underlying conceptual model of learning, (b) the fit between the statistical model (including its assumptions) and the conceptual model of learning, and (c) the fit between the data and the statistical model's predictions. Unfortunately, use of quantitative methods often ignores (b). For example, adopting the Saltus model might cause one to neglect explicit consideration of the critical issue of what it means to be at a level. Also, although many users of quantitative approaches argue that implementing such approaches enables them to test their models, too often, these tests are restricted to factor (c). Researchers in mathematics education need to resist external pressures to apply quantitative techniques without deeply questioning their validity, because such adoptions result in the techniques being applied in ways that we would call in other contexts instrumental or rote procedural. Instead, researchers must investigate much more carefully the conceptual foundations of these techniques, reconciling them with our research on learning (a daunting task, given the statistical/mathematical complexity underlying the procedures)[5].

Endnotes

1. Quantitative methods for determining levels face many of the same issues as qualitative methods. For instance, with Saltus there can be many students who cannot be clearly placed at a level (e.g., Draney & Wilson, 2007).

2. Much of the work described in this paper was supported by the National Science Foundation under Grant Nos. ESI 0099047and 0352898, DRL 0554470 and 0838137. Opinions, findings, conclusions, or recommendations, however, are those of the author and do not necessarily reflect the views of the National Science Foundation.

3. Proponents of Rasch-like methods might argue that the PCM model is more appropriate in the Stacey and Steinle study. Debate on this issue should carefully examine the match between the model's assumptions and the learning theory it proposes to model.

4. Also at issue is whether Rasch techniques appropriately model OMC tasks (Briggs & Alonzo, 2009).

5. Also at issue is whether Rasch techniques appropriately model OMC tasks (Briggs & Alonzo, 2009). One way to investigate the conceptual foundations of the approaches is to apply both to the same sets of data.

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A Critique and Reaction to

REPRESENTATIONS OF LEARNING FOR TEACHING: LEARNING PROGRESSIONS, LEARNING TRAJECTORIES, AND LEVELS OF SOPHISTICATION

OPTIMIZING RESEARCH ON LEARNING TRAJECTORIES

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As I was thinking about my response to Michael Battista's informative overview of learning trajectories (this volume), I wondered what it was about *learning trajectories* that was new and helpful. After all, have we not been involved in studying learning since the inception of our field? I thought about Piaget and his colleagues and their careful analyses of the development of children's domain-specific thinking (Piaget et al., 1960, for example). I thought about chains of inquiry that had stretched over decades. There was Glenadine Gibbs's (1956) study of students' thinking about subtraction word problems, which helped to pave the way for later researchers such as Carpenter and Moser (1984) to create frameworks portraying the development of children's thinking about addition and subtraction; and Les Steffe's and John Olive's recent (2010) book on *Children's Fractional Knowledge,* which detailed the evolution of children's conceptual schemes for operating on fractions. I even thought about Robert Gagné (1962), whose task analyses represented a topic as a hierarchy of discrete tasks learnable through stimulus-response associations. It seemed that the idea of learning trajectories was not new at all.

Yet the term has only recently become popular in research on thinking and learning and is regarded as a new construct. It seems to be making an impact. There have been special journal issues (Clements & Sarama, 2004; Duncan & Hmelo-Silver, 2009) devoted to learning trajectories in mathematics and science and it, or its cousin *learning progressions,* has shown up in prominent policy documents, such as the recently released *Common Core Standards in Mathematics* (CCSM) (2010), whose authors asserted that the "Development of these Standards began with research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time" (p. 4). Researchers, curriculum designers, and standards writers are turning their attention to learning trajectories as a way to bring coherence to how we think about learning and the curriculum.

As I continued to think about Battista's paper, I was drawn to reread Marty Simon's influential 1995 paper on Hypothetical Learning Trajectories. It was the first instance I could recall of the use of the term *learning trajectory* in mathematics education. The most important things I noticed were that a) a learning trajectory did not exist for Simon in the absence of an agent and a purpose and b) the term was introduced in the context of a theory about teaching. A hypothetical learning trajectory is a teaching construct – something a teacher conjectures as a way to make sense of where students are and where the teacher might take them. In Simon's framework, teachers are agents who hypothesize learning trajectories for the purposes of planning tasks that connect students' current thinking activity with possible future thinking

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

activity. A teacher might ask, “What does this student understand? What could this student learn next and how could they learn it?” and create a hypothetical learning trajectory as a way to prospectively grapple with these questions.

This musing led me to think more critically about what learning trajectories were and what their potential for us as researchers and educators could be. They are complex constructions that include, according to Clements and Sarama (2004), “the simultaneous consideration of mathematics goals, models of children’s thinking, teachers’ and researchers’ models of children’s thinking, sequences of instructional tasks, and the interaction of these at a detailed level of analysis of processes” (p. 87), Yet, because the term is almost faddish in its popularity, the idea is frequently simplified as “a sequence of successively more complex ways of thinking about an idea” (Smith et al, 2006, cited in Battista).

Spurred by reading Battista’s paper, then, I pose three talking points for us to consider as we continue to conduct research on learning trajectories:

1) Learning trajectories do not exist independently of tasks and learners’ interactions around tasks. And especially, they do not exist independently of teaching.
2) Learning trajectories tend to focus on content and leave out disciplinary practices.
3) Learning trajectories need to be useful for teachers and teaching to survive and flourish in the ecology of the classroom.

Learning Trajectories Do Not Exist Independently of Tasks and Teaching

I’ve noticed that when we talk about learning trajectories, it’s easy to forget that they exist only in relationship to other education constructs. For example, Battista reminded us, the National Research Council (2007) described learning progressions as “successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic” (p. 214). This characterization ignores the significance of tasks and teaching to the development of students’ thinking and implies that students’ thinking unfolds following a natural, singular path.

Many researchers point to the fact that how students think about a topic depends upon the types of tasks in which they have been asked to engage. Take fractions, a topic that teachers often find difficult to teach. Many typical misconceptions could be avoided if teachers used tasks that supported reasoning about fraction relationships, instead of using all-too-common tasks that reduce children’s fraction reasoning to counting parts. For example, Equal Sharing problems create the need for students to partition units that they have previously conceptualized as “one”(Empson & Levi, in press; Streefland, 1991). Measurement problems create the need for students to iterate subunits of measure (of their own creation) to account for measures that are between whole-number measures (Brousseau, Brousseau, & Warfield, 2004; Davydov &Tsvetkovich, 1991). Simon and Tzur (2004) underscored the necessary relationship between tasks and student thinking in learning trajectories when they said, “the selection of learning tasks and the hypotheses about the process of student learning are interdependent” (p. 93).

Beyond tasks, students’ mathematical thinking and its development also interacts with teaching. This fact may be more difficult to take into account in learning trajectories. Design experiments are one solution to this problem. They allow researchers to see the possible interactions between teaching and learning under conditions that are conducive to optimizing student learning. Often the teachers are researchers themselves. But then there is the question of how student learning and its interaction with teaching are represented. The easiest and most common solution is to represent the development of students’ thinking as snapshots at different levels, developmentally arranged. The teaching is essentially not represented; it is treated as a condition to produce student learning but not represented as integral to student development. More complex solutions would include representing teaching in terms of mechanisms of learning or its relationship to significant changes in students' thinking.

Battista noted that "One of the most difficult issues facing researchers … is determining how instructional variation affects trajectories." This variation is a product of tasks, as many researchers recognize, and to a profound extent, teaching. What would it mean for researchers to not only be cognizant of the possible interaction of teaching with trajectories of learning but to incorporate it in a meaningful way?

Learning Trajectories Tend to Focus on Content Over Disciplinary Practices

Usually, when we talk about what we want students to learn in mathematics it involves a complex and integrated set of content understanding and disciplinary practices (e.g., Kilpatrick, Swafford, and Findell, 2001; National Council of Teachers of Mathematics, 2000). Not only do we want students to understand the major concepts of a given domain, we also want them to be able to solve problems, construct models, and make arguments. One of the ways that researchers have made the study of mathematics learning more tractable is by focusing narrowly, such as on conceptual development in a specific content domain. Steffe and Olive's (2010) research on the development of fraction concepts and Battista's research on children's understanding of measurement are examples of such an approach. This work, like a great deal of the research in mathematics education including my own, is informed by a Piagetian-like view of learning, if not in its emphasis on levels, then certainly in its emphasis on a conceptual trajectory, in which less sophisticated concepts give way to more sophisticated concepts. We have learned a lot about children's learning in this way.

However, only part of what we value as a field about mathematics is represented in this kind of trajectory. Research on students' mathematical argumentation and modeling (Lehrer & Schauble, 2007; Lesh & Doerr, 2003), for example, highlights some of the practices that are just as critical to a well developed capacity to think mathematically but are less amenable to analysis in terms of sequences of development. One difficulty for researchers may be that students engaged in these practices draw on multiple content domains. Another may be that these practices, ideally, involve several students engaged in a complex task that has many possible resolutions (e.g., Stroup, Ares, & Hurford, 2005).

Our challenge as a field is to integrate research on learning the content and practices of mathematics, in order to better understand learning and to better inform teaching and the design of curriculum. When the CCSM first appeared, I was pleased to see a focus on mathematical practices, which included standards such as "make sense of problems and persevere in solving them," "construct viable arguments and critique the reasoning of others," and "model with mathematics" (pp. 6-7). But in the body of the document, these practices were just listed in a box at each grade level, isolated from the content standards, which were elaborated in detail (e.g., see pp. 10 & 53). There are no guidelines in the CCSM for integrating the learning of mathematics content and practices and the risk is that, in practice, educators will focus on content standards with a quick nod to the practice standards.

Some researchers have cautioned that representations of learning as progressive sequences of content understanding could lead teachers to funnel students through the sequences at the expense of allowing students to "express, test, and revise their own ways of thinking" (Lesh & Yoon, 2004, p. 206; see also Sikorski & Hammer, 2010). A focus on integrating mathematical content and practices in our representations of learning may lead us away from the notion of a trajectory, in the sense of a sequenced development or pathway.

Learning Trajectories Need to Be Useful for Teachers and Teaching

I agree whole heartedly with Battista when he writes that one purpose of learning trajectories is to help teachers “understand, plan, and react instructionally, on a moment-to-moment basis, to students' developing reasoning.” I believe that a critical part of our mission as researchers is to produce something that is of use to the field and serves as a resource for teachers and curriculum designers to optimize student learning. But what makes a learning trajectory useful?

We may find some guidance for answering this question in emerging research on *teacher noticing* (Sherin, Jacobs, & Philipp, in press). Useful representations of learning trajectories would draw teachers’ attention to specific aspects of students’ mathematical thinking activity, such as solving problems or making arguments, and help teachers to interpret them and respond to them instructionally. What is a reasonable unit of students’ mathematical activity for teachers to notice? If a unit is too small or requires a great deal of inference (e.g., a mental operation), then teachers in their moment-to-moment decision-making may not be able to detect it and respond to it; likewise if a unit is too broad, or stretches over too long a period of time (e.g., “abstract thinking”). The most productive kinds of units of mathematical activity would allow teachers to see, or notice, clearly defined instances of student’s thinking during instruction and to gather information about students’ progress relative to instructional goals. For example, in research in elementary arithmetic, *strategies* and *types of reasoning* are productive units because we know that teachers can learn to differentiate students’ strategies and use what they learn about students’ thinking to successfully guide instruction (e.g., Fennema, et al., 1996). Battista’s work on creating assessment for teachers to use that are informed by research on learning trajectories appears to be another promising route.

Conclusion

The idea that students progress in some way as a result of instruction is at the very heart of the enterprise of mathematics education in which we are engaged. Research on learning trajectories has the potential to move the field forward. Yet the notion of learning trajectories is not a panacea to the many difficulties that we face in educational research and practice. In this paper I have raised some issues and posed some challenges.

Learning trajectories are essentially provisional. We can think of them as the provisional creation of teachers who are deliberating about how to support students’ learning and we can think of them as the provisional creation of researchers attempting to understand students’ learning and to represent it in a way that is useful for teachers.

I urge researchers to continue to create, test, and refine empirically based representations of students’ learning for teachers to use in professional decision-making and, further, to investigate ways to support teachers’ decision-making without stripping teachers of the agency needed to hypothesize learning trajectories for individual children as they teach. This focus would add new layers of complexity to our research on learning. Instead of researching children’s learning alone, we would be researching how teachers incorporate knowledge of children’s learning into their purposeful decision-making to support children’s learning in classrooms. I do not know if research on learning trajectories can or should be optimized. But I do know that it is worth thinking hard about what learning trajectories are and how they can be used by teachers and in teaching to optimize the development of students’ capacities to think mathematically.

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Chapter 1: Advanced Mathematical Thinking

Research Reports

Brief Research Reports

Posters

COLLEGE STUDENTS' REFLECTIVE ACTIVITY IN ADVANCED MATHEMATICS

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This study explores how an instructional intervention, called the ε-strip activity, would help students understand the meaning of the convergence of sequences in advanced mathematics. The subjects of this study were undergraduate mathematics students in a real analysis class. Dewey's theory of reflective thinking was used as a theoretical framework to characterize students' activity in the class. Results show that the students initially felt perplexed and confused due to uncertainty about the meaning of convergence. However, the students developed their conception of convergence compatible with the ε-N definition as they engaged to the ε-strip activity. The students continued reflecting on the ε-strip activity when they worked on subsequent problems related to the convergence of sequences.

Introduction

The ε-N definition of the convergence of sequences is fundamental in studying advanced mathematics, such as advanced calculus and real analysis; however, many students experience difficulty understanding it. Research related to students' conceptions of limit and convergence has been well developed by mathematics education researchers (e.g., Cornu, 1991; Williams, 1991). Students' difficulties with the ε-N definition are, on one hand, influenced by their images of limit, which is not compatible with the ε-N definition (Davis & Vinner, 1986; Roh, 2008); on the other hand, these difficulties are related to their confusion over multiple quantifiers in the ε-N definition (Duran-Guerrier & Arsac, 2005; Roh, 2010).

An aim of this paper is to explore how an instructional intervention would help students understand the limit of a sequence. This study suggests the ε-strip activity as the instructional intervention and examines how students induce an image of limit that is compatible with the ε-N definition of convergence. This study also explores how the ε-strip activity promotes the students' reflective thinking in the case of the limit of a sequence.

Theoretical Framework

In this study, Dewey's theory of reflective thinking is used to analyze students' activities in understanding the limit of a sequence. According to Dewey (1933), reflective thinking consists of (1) the *pre-reflective* situation, "a perplexed, troubled, or confused situation" (p. 106); (2) the *reflective* situation, "an act of searching, hunting, inquiring" (p. 12) as an effort to resolve the perplexity; and (3) the *post-reflective* situation, "a cleared-up, unified, resolved situation" (p. 106). This study relates these three situations of reflective thinking to the development of student understanding by (1) experiences of perplexed feeling or confusion, (2) reasoning and (mental) actions to figure out the problem, and (3) experiences of mastery and enjoyment, respectively. A powerful instructional intervention for students' reflective activity should create a situation that gives rise to intellectual problems, fosters mathematical reasoning to figure out the problem, and represents fundamental underlying ideas in the context. This paper aims to illuminate how the ε-strip activity plays a role, as an instructional intervention, in promoting students' reflective thinking in the case of the limit of a sequence.

Research Methodology

This study was conducted as part of a larger study from a semester-long design experiment (Cobb et al., 2003) at a southwestern public university in the USA. The author of this paper, as the instructor of the course, conducted a series of teaching sessions with 22 students in a real analysis course. The students were mathematics students or preservice/inservice secondary mathematics teachers, and they had already completed calculus and a transition-to-proof course. During the entire semester, the class worked in small groups, each of which consisted of 4 to 5 students.

In the days of this study, the students in groups first determined the convergence of a sequence by exploring the existence of limits of sequences via their own conceptions of limit. The sequences that the instructor suggested to the class included monotone, constant, and oscillating sequences. The students then examined whether or not their conception of convergence could be regarded as proper to determine the limit of a sequence.

Second, the ε-strips were then introduced to the class as follows: "The ε-strips are made of translucent paper of a rectangular shape. You may imagine each ε-strip as a strip with indefinite length and constant width. The center of each ε-strip is marked with a red line, and half of the width of an ε-strip is called 'ε'." The students were then asked to put ε-strips of different widths on the same graph of a sequence to cover a possible limit value of the sequence. They were also asked to determine how many points on the graph of the sequence are outside/inside each ε-strip. After providing enough opportunities to work with the graphs of the sequences and the ε-strips, the instructor introduced two statements (called ε-strip definitions A and B) to students as follows: "(ε-strip definition A) A certain value L is a limit of a sequence when for any ε-strip centered at L, infinitely many points on the graph of the sequence are inside the ε-strip; (ε-strip definition B) A certain value L is a limit of a sequence when for any ε-strip centered at L, only finitely many points on the graph of the sequence are outside the ε-strip." Afterwards, the students applied ε-strip definitions to particular sequences and evaluated whether or not the ε-strip definitions gave the correct answers to the sequences.

Third, the instructor induced the ε-N definition from ε-strip definition B. Also, the instructor asked the class to evaluate ε-strip definitions A and B, to compare ε-strip definition B with the ε-N definition; and to construct proofs of properties about convergence, for instance, "the sequence $\{1/n\}_{n=1}^{\infty}$ does not converge to 1/10," and "every convergent sequence is a Cauchy sequence."

Results

Pre-Reflective Situation: Students Feel Frustrated, Perplexed, and Confused

Although the students took calculus and learned the limit of a sequence in their previous calculus classes, most of them did not seem to feel confident on the topic.

Steve: Oh, Jesus! I haven't seen these [sequences] since I was a senior [in high school].
Sean: I hated sequences.
Stan: I like them…Aah, heck yea. Tell us about thing, geometric [series].
Susie: I don't even really remember being that far. [Laughs]

As shown above, the students also felt perplexed and frustrated in talking about the notion of sequences, which is a characteristic of the *pre-reflective situation*.

Reflective Situation: Suggestion and Intellectualization

The students then *suggested some ideas* about the meaning of convergence to each other, seeking a proper justification to their claim. Some of their initial suggestions for the convergence of sequences were not accepted by other students. For instance, Sean asked to his group if a sequence should decrease or increase toward a value in order for it to be convergent. This idea was declined by Steve, who suggested a counterexample of a sequence that does not decrease but oscillates and converges to 0. However, Steve's idea was not accepted by Susie because, according to her conception of convergence, an oscillating sequence does not have a limit and Steve's example was an oscillating sequence. At this moment, none of the students in the group agreed upon the meaning of oscillating sequences. Steve seemed to agree with Susie; however, he believed his sequence is not oscillating because his sequence is not a sine or cosine function. Sean was then against Steve's conception of oscillating sequences and believed that Steve's sequence was in fact "a version of sine or cosine" function.

Sean: Doesn't it, for it to converge, wouldn't it have to— every succeeding—every succeeding point have to be either decreasing or increasing towards a value?
Steve: No, you can have something like this. Or like, if you have a grid, if you have a graph that goes like this [drawing a graph of an oscillating convergent sequence] and it goes back and forth across as it [*n*] goes infinity. And that [sequence] converges to 0.
Susie: I thought that [the limit of Steve's sequence] didn't exist because it is now oscillating.
Stan: No, that's alternating series.
Steve: No, an oscillating would be like sine or cosine.
Sean: But that's a version of a sine or cosine. That's oscillating.

On the other hand, when Stan examined the convergence of the sequence $\{(-1)^n / n\}_{n=1}^{\infty}$, he in fact thought of a *p*-series $\sum_{n=1}^{\infty}(1/n)^p$ with $p=1$, and hence he believed it to be divergent. However, at the same time, he also believed that the sequence would have a limit. Then he got frustrated by the fact that the sequence is divergent but has a limit.

Stan: So that's, it would be like something like $a_n = (-1)^n / n$. Well, I think that converges. You could, you could… No, it doesn't though, does it? Because it's not a *p*-serie...
Sean: I don't remember what the definition of it [*p*-series] is.
Stan: I can't remember. Well, *p*-series is where your variable and numerator, umm, denominator, when the exponent [of the denominator] is greater than, or greater than 1, strictly greater than 1, then it converges. But this is 1/*n* , so I don't know if that converges. Umm .... I don't think it comes fast enough, but it still has a limit though. So it doesn't converge. Convergence and diverging is different than a limit, huh?

As shown above, the students were unable to properly account for the meaning of the convergence of sequences. From this moment, the students' own conceptions of convergent sequences became a *true intellectual problem* to them, which is another characteristic of the reflective situation.

Reflective Situation: Students Continue Conjecturing, Reasoning, and Testing the Ideas

After the students realized that their conceptions of the convergence of sequences were inaccurate to account for the convergence and the limit of a sequence, the instructor suggested ε-strip definitions A and B. The students then discussed whether ε-strip definitions would properly describe the notion of convergence. Their first conjecture was that only ε-strip definition A would be proper for a description of the limit of a sequence. It should be noted that the students instantly accepted ε-strip definition A without seeking for a justification. On the other hand, it took a while for the students to understand the meaning of ε-strip definition B. In fact, the students seemed not to recognize that the convergence of a sequence is equivalent to the finiteness of the number of terms outside any given error bound. Instead, they seemed to think that an infinite number of points clustered around a point would be essential for the convergence. When Sean pointed out that ε-strip definition B does not describe the points inside ε-strips, Steve and Susie rejected ε-strip definition B for a proper description of the limit of a sequence.

Sean: I think A, I like A so far. I'm gonna get to B.
Group: [silence]
Stan: So they [B]'re saying that the when…
Sean: So the points that aren't covered are finite.
Susie: Yeah.
Stan: The points that aren't covered are finite.
Sean: But it [B] doesn't say anything about the points covered being infinite.
Susie: Yeah, that's true.
Steve: It [B] doesn't say anything about the points covered being infinite. So that would incorrectly describe a limit.
Susie: Right.

Now the students *tested their conjecture*, which is another characteristic of the *reflective situation.* In particular, Sean suggested that his group should look for some examples of sequences and check the number of points inside and outside the ε-strips. The sequence he chose first was a sequence whose odd terms were defined as 1 and even terms were defined in the form of $1/n$. The students then observed that for some ε-strip, when it was centered at y=1, infinitely many points of the sequence are inside the ε-strip. Therefore, accepting ε-strip definition A would force them to determine 1 as a limit of the sequence, which contradicted their belief of the divergence of the sequence.

Sean: […] Let's look for examples. We have the graphs, we have two…Where's those— where's those graphs at? [Choosing the sequence $a_n = 1$ when n is odd; and $a_n = 1/n$ when n is even; then placing an ε-strip on y=1 so that all points on the x-axis are outside the ε-strip] So if we could cover finite or infinite ….can we?
Susie: But that one [the sequence] doesn't have a limit to begin with.
Sean: Well, I know, that's what I'm trying to say. I'm trying—, I'm saying we're setting an ε-strip here [y=1], and we're covering an infinite amount.
Susie: Right.

Sean: [pointing to the dots on y=0] And we're not covering in it, an infinite amount too. So, but it doesn't have a limit— I don't know.
Stan: Well…
Steve: Hmm….
Group: [silence]

After realizing that ε-strip definition A would not work for the sequence, the students seemed to deliberate making a change in their conjecture. Sean proposed ε-strip definition B to be a better description for the convergence, whereas Susie suggested combining ε-strip definition A with ε-strip definition B. Since accepting the combination of ε-strip definitions A and B would force them to describe both points inside and outside the ε-strip, the students seemed to consider Susie's idea more informative than Sean's idea. Consequently, they followed Susie's idea at this moment.

Sean: B is better.
Group: [silence]
Susie: Maybe it's a combination of both.
Sean: Yeah.
Susie: The number of points *in* the strip has to be infinite, and the number of points *not* covered by the strips has to be finite.
Steve: Isn't that what A was saying?
Sean: No.
Susie: No.
Sean: They only talk about only one or the other, so we can combine them. Let's go with that. That's right. That's a right thing for now.
Susie: Okay.

Although the group seemed to be satisfied with combining ε-strip definitions A and B for a description of the convergence, Sean wondered why ε-strip definition B itself would not be proper for a description of the convergence.

Group: [silence]
Sean: So, I haven't thought of any argument against why B can't just be.

The instructor moved to the whole class discussion by collecting students' conjectures on the ε-strip definitions. This group insisted on combining ε-strip definitions A and B, whereas another group said ε-strip definition B itself would be enough for a proper description of the convergence. The instructor then asked if ε-strip definition A would imply ε-strip definition B, or vice versa. The class tested the implications by putting particular ε-strips on various sequences as well as by imagining the compliment of an infinite set. The students then realized that ε-strip definition B would imply ε-strip definition A, but its converse would not always be true. From that moment, the class agreed that ε-strip definition B itself would be proper for a description of the convergence of sequences.

Post-Reflective Situation: Students Experience Their Confusion Cleared Up and Resolved

Once the students accepted ε-strip definition B as a proper description for the convergence of sequences, they developed their conception of convergence compatible with the ε-N definition. Moreover, the students continued reflecting on the ε-strip activity whenever they worked on subsequent problems related to convergence of sequences. For instance, when the instructor asked them to discuss how they could tell the sequence $\{1/n\}_{n=1}^{\infty}$ does not converge to 1/10, the students attempted to show it via ε-strips. To be more precise, Susie aligned an ε-strip on y=1/10 and pointed out that only a finite number of points were inside the ε-strip. Since the rest of the points would then be outside the ε-strip, there should be infinitely many points outside the ε-strip. Sean agreed with Susie, but pointed out the necessity of specifying "what your ε is" in order to show that infinitely many points would be outside. Steve then suggested 0.05 as an appropriate value for such an ε.

Susie: …Well, okay. So this is saying that L shouldn't be 1/10. If we say let L =1/10, it has a finite number of points [inside the ε-strip].
Sean: But then we have to discuss what your ε is gonna be first, because you can have ε equals this [using two fingers to illustrate an ε-strip, a wider one than Susie's ε-strip, that would cover the x-axis], that would cover it [0].
Susie: Oh, right!
Sean: That's why I was saying we need to find ε that's less, so that we can [Susie: I see.] include the finite portion and exclude the infinite portion.
Sean (or Steve?): So we'll let, first let L equal .1 and ε has to be <.1. We could just declare it first, ... Should we talk about it for all values of ε is [less than] .1 or just ε being .05? 'Cause we just have to find it for any ε, so we just have to disprove it for one ε.
Stan: Yeah.
Steve: So we should just use .05.
Sean: So ε equals .05, sort of.

Next, the group explored how to show there would be only finitely many points inside a 0.05-strip. Sean suggested that by checking "the entry point and exit" point to the ε-strip, they would show there would be only finitely many points inside the ε-strip, which would then imply infinitely many points outside the ε-strip. The problem now became to solve an equation $1/n = 0.1 + 0.05$ for the entry point and an equation $1/n = 0.1 - 0.05$ for the exit point, and they were able to solve these equations successfully.

Sean: … So, would you have to say, would you have to find the, the two spots where you have your finite, like your first your entry point and your exit? And then that would show that you have finite numbers [inside the ε-strip] and then…
Susie: Yeah, I like that.
Stan: That's cool. That would be the intersection of the ε-strip across uh... And we were just talking about sets, too, if these are sets, the intersection of sets.
Sean: So we'd have to solve for 1/n equals .15 and 1/n equals .05 [Susie agrees] and that would be these two points. So [the entering] n = 1/.15 and [the exit] n = 1/.05.

Post-Reflective Situation: Sean's Attempt to Prove "Every Convergent Sequence of Real Numbers is a Cauchy Sequence" via ε-Strip

The following episode illustrates another example of the *post-reflective situation*, in which Sean reflected on the *ε*-strip activity to prove that every convergent sequence is a Cauchy sequence. First, he and his group members tried to make sense of the definition of Cauchy sequences by comparing differences in symbols between the *ε*-*N* definition of convergence and the definition of Cauchy sequences. They then set up their proof by assuming that a sequence is convergent. Accordingly, they imagined that for any *ε*-strip, every point but a finite number of points, *N*, of the sequence is contained in the *ε*-strip. In particular, Sean drew a graph of a sequence that was oscillating but converged to 0 (He seemed to draw the graph of $\{(-1)^n/n\}_{n=1}^{\infty}$). He then marked *N* in the *x*-axis and drew two lines, each of which was parallel to and had the same distance from the *x*-axis; thereafter, every term whose index was greater than *N* was contained in the *ε*-strip. In fact, the region bounded by these two lines is the interior of an *ε*-strip, and hence Sean called such a region an *ε*-strip. He also chose two values on the *x*-axis, each of which was greater than *N*, and marked them as *m* and *n*.

Sean: So you have like, here's the big *N* that satisfies it. Let's say this is going like this, and so your *ε*-strip is such that everything in here is contained past that *N*. And you choose two values *n* and *m* regardless of where, like… Let's say this one [a_n] is here and this one [a_m] is here ….This distance [$|a_n - 0|$] here has to be less than this distance [*ε*] here, right?

Since *m* and *n* were greater than *N*, students in his group could see that both a_m and a_n were within the *ε*-strip, and hence each of $|a_m - 0|$ and $|a_n - 0|$ was less than *ε*. Sean then realized that based on his reasoning, he could argue that the distance between a_m and a_n is less than 2ε, i.e., $|a_m - a_n| < 2\varepsilon$. However, Sean was unable to conclude $|a_m - a_n| < \varepsilon$ because he knew that *ε* was positive, and hence $2\varepsilon > \varepsilon$. Although Sean also checked $|a_m - a_n| < \varepsilon$ in the case of a monotone convergent sequence by drawing a graph of $\{1/n\}_{n=1}^{\infty}$, he believed that finding an example of a sequence satisfying the inequality $|a_m - a_n| < \varepsilon$ does not mean that such an inequality would always be true for any sequence.

Sean: … it would be for sure that this distance [$|a_m - a_n|$] here would be less than 2*ε*, because 2*ε* is this whole thing. But how can you be sure that this distance [$|a_m - a_n|$] is going to be less than a single *ε*? I mean that's just in the special case, I mean other cases like, where it's just that and there's your *ε*-strip and your two values; obviously this distance here... [is less than *ε*].

The instructor moved to the whole class discussion by collecting students' ideas of proving the given theorem: Every convergent sequence is a Cauchy sequence. Sean, on the other hand, was off task and continued his thinking to figure out how to fill the gap between what his group had so far and what to prove at the end. The aha-moment came to him when he realized that the value of *N* in the definition of Cauchy sequences does not have to be the same as the value of *N* in the ε -*N* definition that he assumed in the beginning of his proof, although the same symbol *N* was used in each definition. He explained to his group that by choosing an index *N* in the definition of Cauchy sequences "farther down" than the index *N* in the ε -*N* definition of convergence, the difference between the two terms a_m and a_n would be less than ε.

Sean: [whispering] Let me check this out. …
[Instructor talking to class]
Sean: [whispering] Oh wait!
[Instructor talking to class]
Sean: Yeah, this *N* [in the definition of Cauchy sequences] isn't doing the same thing as it was doing before [in the ε-*N* definition of convergence].
Steve: Yeah, exactly.
Sean: So this could be farther down here, and this could be an *N* where it's right here.
Steve: As long as…
Sean: It's farther down.
Steve: … small *m* and small *n* are greater than [*N*]… yeah.
Sean: Yeah.

It is worth noting that Sean was no longer in the status of just comparing symbols in the ε -*N* definition of convergent sequences and the definition of Cauchy sequences. Rather, he realized that although the symbol *N* is used in both definitions, it does not mean their values are the same. In fact, he came to consider the symbol *N* in the definitions as a bound variable. Such a shift in understanding the role of the variable in the definitions seems to resolve the group's obstacle in showing the inequality $|a_m - a_n| < \varepsilon$. Sean and his group members also found that the instructor used the same oscillating convergent sequence with ε-strips to make sense of the theorem, and therefore they convinced themselves that their ideas would lead a proof to the theorem.

Discussion

This paper illustrates how the ε-strip activity plays a role in promoting students' reflective thinking in the case of the limit of a sequence. Brousseau (1997), Cornu (1991), and Fischbein (1987) have proposed the necessity of creating a learning environment in which students are made aware of difficulties and given the opportunity to reflect on their own ideas. Concerning instructional treatment to help students reflect on their conception of convergence, it is worth noting that the ε -strip activity was effectively used in this class. The students were initially situated in such a way that they felt perplexed, frustrated, and confused due to the impreciseness of their conceptions of convergent sequences. Students' preconceived notion of convergence became a true intellectual problem to them. Students then suggested one idea after another, seeking a possible way to properly describe the convergence sequences, and tested their hypotheses by (mental) actions with ε-strips. It is also worth noting that after the ε-strip activity, the students continued reflecting on their new conception of convergence and utilized it to make sense of other properties of convergent sequences by using the ε-strips.

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FACTORS OF CALCULUS STUDENTS' MATHEMATICAL PERFORMANCES AND PREFERENCES FOR VISUAL THINKING

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From the perspective of developmental research, this study completed a research cycle of a new instrument, the Mathematical Processing Instrument for Calculus (MPIC), to examine calculus students' mathematical performances and preferences for visual or analytic thinking regarding derivative and antiderivative tasks presented graphically. It extends previous studies by investigating factors meditating calculus students' mathematical performances and their preferred modes of thinking. Data were collected from 183 Advanced Placement calculus students. Students' performances on the MPIC were not influenced by gender or visual preference. There was no significant difference between the two sexes, but AP high- and low-performing students differed in terms of visuality Thus, the results suggest stronger preference for visual thinking was associated with higher mathematical performances.

Introduction

Past studies analyzing factors of differences in mathematical performance have generated inconclusive findings. We hypothesized that preferred mode of thinking might underlie differences in mathematics learning and designed a new instrument, the Mathematical Processing Instrument for Calculus ([MPIC], Haciomeroglu, Aspinwall, Presmeg, Chicken, & Bu, 2009), to determine students' mathematical performances and preferences for visual or analytic thinking regarding derivative and antiderivative tasks presented graphically. This study sought to investigate factors meditating calculus students' mathematical performances and their preferred modes of thinking.

Interest in the relationships between gender, spatial ability, and mathematical performance has existed for decades, and spatial ability and its various definitions constitute a long-standing topic of discussion within the mathematics education community (e.g., Clements, 1979; Bishop, 1980; Lohman, 1988). Many researchers have used diverse definitions to examine components of spatial ability (e.g., Guay & McDaniel, 1977; Lohman, 1979; McGee, 1979; Bishop, 1983; Linn & Petersen, 1985; Tartre, 1990).

The problems associated with working with multiple, and at times divergent, definitions are exacerbated by several issues that have been reported in literature. Spatial tests may be complicated by analytic strategies (Bodner & Guay, 1997), or students with spatial ability may prefer not to use visual methods (Krutetskii, 1976; Clements, 1984; Presmeg, 2006). Moreover, spatial tests with inconsistent correlations measure various components of spatial ability, and these tests often measure different abilities as they relate to individual differences in solution strategies (Guay, McDaniel, & Angelo, 1978; Lohman, 1979). Generally spatial tests use visual or pictorial tasks, but visual strategies can be used for visual or non-visual tasks in any content area of mathematics (Bishop, 1983). Bishop (1989) contends that the psychometric approach might not be appropriate for studying the visualization process due to students' idiosyncratic approaches to solving spatial tasks. Since research conclusions are varied, this study omits spatial tests and instead investigates the relationship between students' solution strategies determined by the MPIC and their mathematical performances based on the number of correct answers on the MPIC.

There is extensive research relating differences in mathematical performance to solution strategies. Halpern and Collaer's (2005) extensive literature in this field suggests that gender differences in mathematical performance may be related to strategies males and females use to solve visual or spatial tasks. Fennema and Tartre (1985) reported a study in which students in grades 6, 7, and 8 were classified as either high or low in spatial visualization and verbal skills. They found that boys with low spatial visualization and high verbal skills had the highest mathematics achievement scores compared to boys with high spatial and high verbal, girls with low spatial and high verbal, or girls with high spatial and high verbal. Moreover, high spatial/low verbal and low spatial/high verbal students differed in their strategies, but not in their ability to solve problems. This is consonant with Battista's (1990) observation that logical reasoning was an important factor in geometry achievement and geometric problem solving and that low achieving geometry students used more visual solutions than analytic. Linn and Petersen (1985, p. 1492) attributed gender differences on spatial tasks to the selection of efficient strategies: "The pattern of sex differences could result from a propensity of females to select and consistently use less efficient or less accurate strategies for these tasks." In a longitudinal study with elementary students, Fennema, Carpenter, Jacobs, Franke, and Levi (1998) concluded that there were no gender differences in solving problems but strategies used to solve problems were different throughout the study. Gallagher and De Lisi (1994) examined solution strategies of high-ability students with high mathematics scores on the Scholastic Aptitude Test (SAT– M) on routine and non-routine SAT–M problems and found differences in performances on routine problems favoring females and on non-routine problems favoring males. Gallagher and De Lisi attributed the results to gender differences in solution strategies, suggesting that the use of conventional and unconventional strategies were significantly high for female and male students respectively.

Krutetskii (1969, 1976) went so far as to exclude visual thinking from essential components of mathematical abilities. In his studies, Krutetskii identified types of mathematical giftedness based on students' preferences for analytic (or verbal-logical) or visual (visual-pictorial) thinking. According to Krutetskii, the ability to visualize abstract mathematical relationships and the ability for spatial geometric concepts are not essential components of mathematical abilities; that is, the level of mathematical abilities is determined by the strength of the analytic component of thinking, and the visual component, which is not considered to be of the components of mathematical giftedness, determined the type (though not the extent) of mathematical giftedness. Lean and Clements (1981) reported a similar finding: spatial ability and knowledge of spatial conventions were not factors significantly affecting mathematical performance of engineering students, and that the students who preferred analytic thinking outperformed those who preferred visual thinking on both spatial and mathematical tests. Moses (1977) and Suwarsono (1982) analyzed students' solutions strategies to determine their visuality and conlcuded that visuality and problem solving performance did not correlate significantly. Presmeg's (1985, 1986) study confirmed Krutetskii's findings and reinforced the roles of differences in preferred mode of thinking in learning mathematics. Other studies (e.g., Clements, 1984; Habre, 2001; Presmeg, 2006; Haciomeroglu, Aspinwall, & Presmeg, 2010) have also shown that students who have the ability to visualize may prefer not to do so. Ben-Chaim, Lappan, and Houang (1989) investigated the effect of instruction on middle school students' preferences for visual or analytic solutions and performance in solving spatial tasks. Before the instruction, boys and girls did not differ in their preferences and performances. After the instruction, there were significant differences between the two sexes in the preference of solutions but not in solving the tasks. Girls preferred mixed and visual solutions, whereas boys demonstrated strong preference for visual solutions.

The work of Lean and Clements (1981), Moses (1977), Suwarsono (1982), Presmeg (1985), Ben-Chaim et al. (1989), and Lowrie and Kay (2001) stands apart from other research on students' mental processes because they identified students' thinking processes based on their preferences for visual or analytic solutions and compared their mathematical performance in their preferred mode of thinking. We have seen a relationship between preferences and mathematics learning, and preferred mode of thinking might underlie differences in mathematics learning. In our review of existing literature, we have found a dearth of research studies examining mathematical performances of students with different preferences. Our students have developed a preferences to think visually or analytically, and their preferences can be different from their abilities to think visually and analytically (Haciomeroglu et al., 2009). Differences in students' mathematical performances may arise from their preferences and not necessarily from their abilities. We believe this suggests research studies focusing on students' preferences, and that a distinction has to be made between studies focusing on mathematical ability testing and studies focusing on students' preferences.

The primary goal our study was to examine possible factors contributing to calculus students' mathematical performances and preferences. In the present study, the following research questions were investigated in high schools:

1. Differences in Performance. Do gender, visual preference, and AP performance affect mathematical performance on the derivative and antiderivative tasks of the MPIC?

2. Differences in visual preference. Do males and females and high- and low-performing students differ in preferences for visual thinking?

Methods and Data Collection Procedures

Sample

We visited ten classrooms in five high schools in two school districts in North Florida in the United States and collected data from *188* Advanced Placement (AP) calculus students with seven teachers. The students were enrolled in five *AB* and five *BC* calculus courses. All *188* students in the ten classes agreed to participate in the study. Five students who failed to take both the derivative and the antiderivative test were excluded from the data. There were *103* (*56*%) students in *AB and 80* (*44*%) students in *BC* calculus courses. Eighty-two (*45%*) of the participants were female, and *101* (*55%*) of the participants were male. Of the *183* students who completed the MPIC, we were able to collect *174* students' scores on AP calculus test at the end of the year. Therefore, results are based on the most available data in this study.

The Mathematical Processing Instrument for Calculus (MPIC)

In the study, we have administered the MPIC whose reliability and validity are well established (see Haciomeroglu et al., 2009). The MPIC is designed to determine students' mathematical performances and preferences for visual or analytic thinking as they attempt to sketch derivative and antiderivative graphs. The MPIC consists of graphical tasks because research (e.g., Eisenberg & Dreyfus, 1991) has shown that most calculus students tend to use analytic strategies to compute derivatives and integrals, and this tendency makes it more difficult to infer students' visual strategies when they are solving procedural tasks.

The MPIC consists of two main sections — Derivative and Antiderivative — and each section consists of five graphical tasks requiring students to draw derivative (see Figure 1) orantiderivative graphs (see Figure 2) of basic functions. Every graphical task has at least two possible ways of obtaining a solution because we think that visual students prefer to work directly from graphical information, and analytic students prefer to translate to an algebraic representation when this option is available. We use the terms visual and analytic to mean graphical and algebraic solutions respectively. We will use the terms interchangeably and think this is an advantage for the analysis of students' methods of solutions. Graphical solutions such as estimating slopes are considered as visual solutions. Algebraic solutions such as estimating equations are considered to be analytic solutions. One task from each section of the MPIC is illustrated in Figures 1 and 2. We considered a solution to be analytic if the student estimated and integrated the equation of the graph to draw its derivative or antiderivative graph. We considered a solution to be visual if the student estimated the slopes of tangent lines at various points on the graph of the function and used this to draw the graph of the derivative or antiderivative. Upon completion of the Derivative or the Antiderivative section of the instrument, students were given the Method part, a questionnaire, consisting of an analytic and a visual solution method for each task, and asked to choose for each task a method of solution that most closely described how they sketched their graphs.

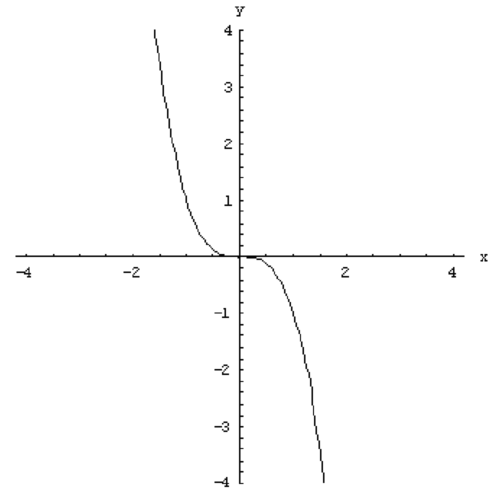

Figure 1. Derivative task

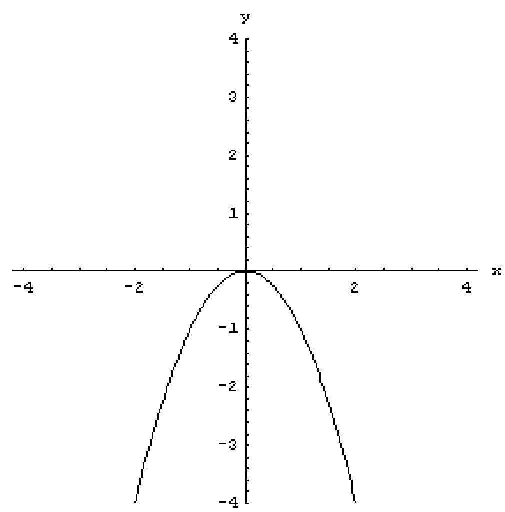

Figure 2. Antiderivative task

Variables

To further explore the students' preferences for derivative and antiderivative tasks presented graphically, we administered the MPIC, which yielded two scores for each student: 1) a score of mathematical performance determined by the number of correct responses, and 2) a score of visual preference. There were two mathematical performance variables: AP calculus test scores (*AP*) and Antiderivative/Derivative scores on the MPIC (*AD*).

On the MPIC, to determine the students' visual preference scores, *Vd (Derivative), Va (Antiderivative),* and *VdVa (Derivative/Antiderivative),*for the derivative and antiderivative tasks presented graphically, they were given a score of *0* for each analytic solution and *1* point for each visual solution regardless of whether their answers were correct or incorrect. When determining preference on the MPIC, the primary goal was to identify the students' methods as visual or analytic; whether their answers were correct or incorrect mattered less than their method(s) in determining preference. In assessing students' understanding of the calculus derivative or antiderivative graphs, the students were given a score of *0* for each incorrect answer and *1* point for each correct answer. Thus, for each of the two sections, the total possible score is five points.

For instance, in the derivative section consisting of five tasks, for a visual preference score of *0* (*Vd*) and a derivative score of *5* (*D*), we can say that the student solves all the tasks correctly and has a strong preference for analytic thinking. In the antiderivative section consisting of five tasks, a visual preference score of *5* (*Va*) and an antiderivative score of *0* (*A*) indicate a strong preference for visual thinking and an incomplete understanding of the antiderivative graphs.

In our analyses, we made use of *V.alpha* based on the best combination of the visual preference scores, *Va* (Derivative) and *Vd* (Antiderivative), from the MPIC (*V.alpha = 0.23 * Va + 0.77 * Vd*). Moreover, logistic regression (see Figure 3) was used to model the probability of being visual (or analytic), given a student's score on the MPIC. Figure 3 illustrates the likelihood that a student's solution will be *0* (analytic preference) or *1* (visual preference) given a visual preference score of the student on the MPIC. Since it is the lowest on the left, and the highest on the right, and the steepest in the middle, *V.alpha*, the visual preference score of the derivative and antiderivative tasks, is the best model (see Figure 3).

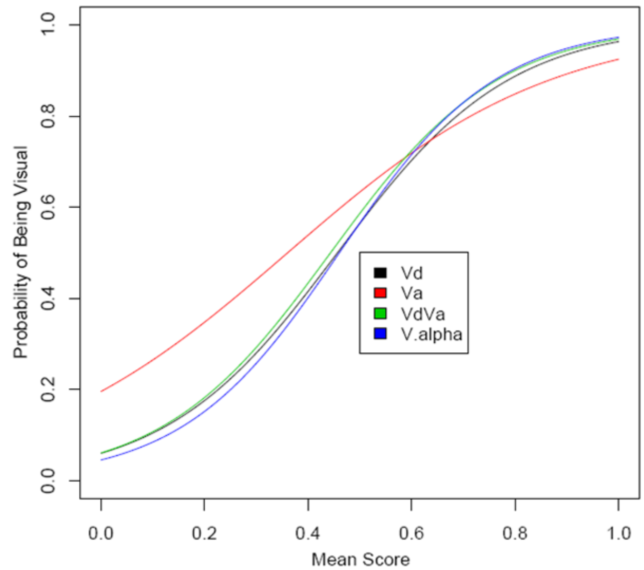

Figure 3. Logistic regression

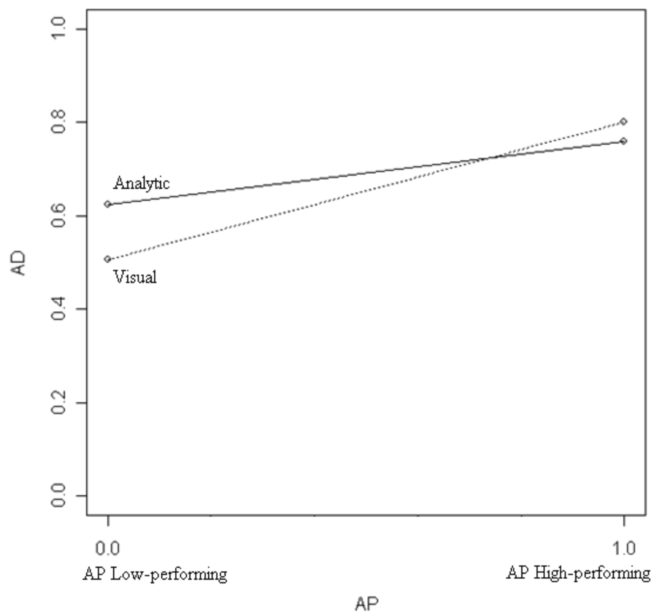

Figure 4. Modeling AD scores on AP performance and Visual preference

To compare visual and analytic students' performances on the MPIC, we used their visual preference scores, *V.alpha*, to distinguish between visual and analytic preferences. If a student has a *V.alpha* score greater than *0.5*, then the student has a visual preference. If the student has a *V.alpha* score less than *0.5*, then the student has an analytic preference. We also divided the students into subgroups based on their scores on the AP tests. The students with AP scores of *4* or *5* were considered as high-performing and those with AP scores of *1* or *2* as low-performing. The students with a *V.alpha* score of *0.5* or an AP score of *3* were excluded from these analyses. All students could be classified as visual or analytic in preference since no students had a *V.alpha* score of *0.5*.

Results

Differences in Performance

We used *Antiderivative/Derivative* (*AD*) as the measure of mathematical performance on the MPIC, and the students were divided into subgroups according to their gender, visual preference, and AP performance. A three-way analysis of variance (ANOVA) was conducted to explore the effects of visual preference, gender, and AP performance on *AD*.

The overall F-test showed that there was a difference in the *AD* means due to the factors ($p < 0.001$). As expected, the ANOVA showed there was a significant effect due to AP performance ($p < 0.001$). The mean *AD* score for the high-performing AP students was *0.77*, while the mean *AD* score for the low–performing AP students was *0.56*. Additionally, there was a significant effect due to the interaction of AP performance with visual preference ($p = 0.02$). There were no significant effects for gender ($p = 0.80$) or any of the other interaction terms. Figure 4 shows the mean *AD* scores on the vertical axis and AP scores on the horizontal axis. Visual and analytic students are denoted by the dashed and solid lines respectively. If these students are divided into subgroups according to their preferences for visual or analytic thinking (see Figure 4), the increase for the analytic students is not as large as the increase for the visual students.

Differences in Preferences for Visual Thinking

We used *V.alpha*, the continuous measure of student visual preference, to determine whether the subgroups (i.e., males and females; high- and low-AP-performing students) differed in terms of visual preference scores. The two-way ANOVA was borderline significant ($p = 0.06$). However, from the two-way ANOVA analysis, there was significant effect on *V.alpha* due to the AP score ($p = 0.035$), but no significant effects on *V.alpha* due to gender or the interaction of gender and AP score. The high-performing students had a mean *V.alpha* score of *0.59*, while the mean *V.alpha* score for low-performing students was *0.46*.

Conclusions

In the present study, we investigated contributing factors to calculus students’ mathematical performances and preferences. Our results suggest that gender and visual preference were not significant factors influencing the students’ performances on the derivative and antiderivative tasks presented graphically on the MPIC. There was a significant difference in scores for high- and low-performing students. Moreover, the interaction between visual preference and AP performance in their effect on MPIC scores was significant, and thus there was a difference in scores over AP performance for the visual and analytic students. Students’ gender did not have a significant influence on their preferences for visual or analytic thinking, which is in agreement with the findings reported by Ben-Chaim, Lappan, and Houang (1989). Statistically significant differences in visual preference scores were found among high- and low-performing students. The high performing students had significantly higher visual preference scores than the low-performing students. Thus, the results suggest that stronger preference for visual thinking (or solutions) was associated with higher mathematical performances. Our results are not consistent with the results of the study by Lean and Clement (1981) and do not support Battista’s (1990) contention that low achieving students used more visual methods.

Our work with AP calculus students and the Mathematical Processing Instrument for Calculus has generated new information about calculus students’ mathematical performances and preferences. Considering students’ differing and idiosyncratic methods, we suggest that differences in mathematics learning can be explained by students’ preferences to think visually or analytically, not necessarily by their ability to think visually or analytically. Our field tests confirm the potential for use by researchers and teachers to identify students' preferences and to understand students' comprehension of derivative and antiderivative graphs and difficulties associated with their preferences. We think these are topics worthy of continued study and additional research cycles with the MPIC in calculus classrooms.

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PART / WHOLE METAPHOR FOR THE CONCEPT OF CONVERGENCE OF TAYLOR SERIES

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During a detailed analysis of interviews conducted with university calculus, real analysis, and numerical analysis students concerning the convergence of Taylor series, we discovered that several students consistently relied on a single metaphor throughout several tasks. In this paper we characterize one such metaphor based on Part / Whole relationships. We provide a detailed illustration of one student's commitment to this metaphor (emphasis) and the degree to which it influenced his reasoning (resonance).

Introduction

Taylor series are frequently used in physics and engineering to simplify complicated mathematical models and play a foundational role in the theory of complex analysis. In introductory-level calculus sequences, Taylor series are typically developed in four or fewer sections of the textbook emphasizing computation and algebraic manipulations (e.g., Hass, Weir, & Thomas, 2007; Larson, Edwards, & Hostetler, 2005; Stewart, 2008). Students are likely to revisit Taylor series in more theoretical or applied depth in courses such as differential equations, introductory analysis, numerical analysis, complex analysis, modern physics or physical chemistry, or a variety of engineering courses. We ask the question, what images guide students' reasoning about the convergence of Taylor series as they establish their initial conceptual foundation?

Background

Portions of this paper are part of an initial study conducted in partial fulfillment for a degree of Doctor of Philosophy in Mathematics (Martin, 2009). The initial study sought to analyze and describe expert and novice conceptualizations of the convergence of Taylor series by identifying and categorizing particular reasoning patterns while accounting for different levels of exposure to series. To address this goal, data were collected from 131 undergraduate students, 10 graduate students, and 6 faculty from a mid-size four-year university and from a regional community college. To help better account for the effect of the amount of exposure undergraduate participants had with Taylor series and with series in general, undergraduate student participants were selected from calculus, real analysis, and numerical analysis classes after having prior exposure to Taylor series. All 131 students completed an in-depth questionnaire about their understanding of Taylor series, and eight of these students subsequently participated in no more than two face-to-face, task-based, individual interviews. Martin (2009) categorized some of the different ways in which these experts and novices conceptualized Taylor series using the construct of concept images developed by Tall and Vinner (1981), as well as other influential work on Taylor series and limits in general (e.g., Alcock & Simpson, 2004, 2005; Kidron, 2004; Kidron and Zehavi, 2002; Oehrtman, 2002; Williams, 1991).

For the purposes of the study represented in this paper we focused our attention primarily on undergraduate student participants, and attempted to identify patterns of metaphorical reasoning employed by individual students.

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Theoretical Framework

We analyzed interview data for student reasoning employing a theoretical perspective of conceptual metaphor based on Max Black's (1962, 1977) interaction theory and as used by Oehrtman (2002, 2009). In general, metaphorical reasoning involves conceiving of unfamiliar aspects of a target domain in terms of similar aspects of a more immediately understood metaphorical domain. Black distinguished emphasis and resonance as necessary characteristics of strong metaphors, those that have the potential to be ontologically creative for the user. Emphasis is the degree to which the user is committed to applying the chosen metaphorical domain and resonance is the degree to which the metaphor can support "elaborative implication," the development of additional inferences not contained within the original metaphor. According to Black (1977) strong metaphors require an active response in the conceptualization of the metaphorical domain as well as projection of aspects to the target domain. The resulting dialectic allows for conceptual innovation that far exceeds what is possible by reasoning entirely within either domain.

Results

Some of the metaphors that emerged from our analysis of calculus and analysis students reasoning about convergence of Taylor series include the approximation, collapse, and proximity metaphors as previously described in Oehrtman (2002, 2009). An additional metaphor based on part / whole relationships emerged as distinct from those already observed by Oehrtman. In this metaphor partial sums are seen as "part" of a bigger whole (the infinite series) and convergence may be viewed as being accomplished through a "massing" of points from the whole that contributes to a preponderance of evidence for convergence.

In this section we will highlight one student, Brian's, use of such a "part / whole" metaphor when discussing tasks related to Taylor series convergence and the influence of this metaphor on his understanding of Taylor series. In the following excerpt, one can see Brian hinting toward an idea of a large number of points yielding a resulting limit.

> Interview Task 3: What is meant by the "=" in "$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$ when x is any real number?"
>
> Brian: I'm thinking it has something to do with like a Riemann sum. That's just what comes to mind. Uh, if I add up, if I were to add up all of these it would give me a definite point.

In this excerpt Brian appeared to cue off of his prior notion of Riemann sums. Even though it is unclear what "these" was for Brian (he did not explicitly say that "these" meant the terms of the Taylor series), the effect of adding up "all" of them yielded a "definite point." Brian will eventually refer to this effect as a "massing" of points.

Following his comments in the first excerpt, Brian was asked to elaborate about the meaning behind his reference to "Riemann sum." In response, Brian initially took note of the order of the terms in the series and then appeared to notice that the denominators of each term in the Taylor series were going up by multiples of two. In the next excerpt, Brian detailed some of the meaning behind his reference to "Riemann sum" and continued to allude to the effect of all points "massing".

> I'm thinking that because these are all fractions [pointing to the terms of the series] of I guess cosine curve or function, it's gonna give me one single point. Um, kinda of a sum-, it's gonna, it's gonna give me a summary more or less is what I'm thinking since it's all these little points adding up to one point. I'm thinking it's gonna converge into something.

It appears very possible that the "fractions" that Brian was referring to were the coefficients of *x* in the expanded Taylor series. Alternatively, he may have imagined plugging in a number for *x* in each term of the Taylor series obtaining "fractions" to be added up. In either case, Brian conceived of "all fractions" as giving "one single point" that may be a "summary" of the larger whole. Again in this same excerpt, Brian alluded to "all these little points" that add up to "one point." Subsequently in the interview, Brian was asked what he meant by his reference to "converge into something" found at the end of the above excerpt, and Brian reiterated that he meant that "it's going to come to a point" and that this point was a specific number.

Later in the same interview, when asked "How would you go about estimating $\sin(103)$?" Brian responded by saying the following:

> Well, I'm thinking that if we did go, just for arguments sake, to 103, I mean that would be any-every number from 1 to 103. So just, I'm thinking the summation of all those numbers would probably get me that one number that we're looking for.

Although Brian clearly exhibited confusion in this excerpt, it is important to note that in the first three excerpts Brian persisted in his references to "all's", whether the all be composed of terms of a Riemann sum, fractions of a Taylor series, a collection of little points, or numbers, that through some limiting process, like an "add up" process, yield a single result, usually in the form of a point or a number that is sought after. In these cases, the "all's" may be seen as the "whole" which has a characteristic of "massing" around a single point. The following excerpts will reveal further entailments of Brian's "part / whole" metaphor and the emerging role of "massing". Consider his response to Interview Task 4:

> Task 4: What is meant by the "$(-1,1)$" in, "1/(1 - *x*) = 1 + *x*+ x^2 + x^3 + L when *x* is in the interval $(-1,1)$?"
>
> Brian: Well it uh. I didn't really look at it there. Um, I'm guessing those are the parameters. I mean, it's gonna go, if I were to set up an integral, you know, it would be from one to negative one, any infinite amount of numbers between negative one and one. Um, let's see, I know it can't be, well I'm thinking it can't be one for this particular function [pointing to $1/(1-x)$] just because it doesn't exist.
>
> I: Okay.
>
> Brian: Negative one would be two, so I'm thinking it might be equal to a negative one and just up to one. So, just any infinite amount of numbers in between there.

As Brian was attempting to make sense of the interval of convergence of this Taylor series, he first appeared to focus on the interval $(-1,1)$ and referred to setting up a definite integral using –1 and 1 as "parameters" for the limits of integration. Based on his response to the previous task found in the first excerpt in which he referred to the Taylor series of cosine as having "something to do with like a Riemann sum", it appears possible that the definite integral may have been triggered in Brian's mind because of its relationship to Riemann sums. The generating function $1/(1-x)$ gained Brian's attention as he pointed to the function and commented that it does not exist at 1 and the value is 2 when $x = -1$ (seemingly referring to the value of the denominator). Twice Brian talked about the infinite amount of numbers between –1 and 1, prompting the interviewer to ask his next question.

> I: Okay, and what can we do with this infinite amount of numbers between there?
> Brian: Well, like in number 3, I'm thinking in these infinite amount of numbers you're going to find some type of mass. I mean it's gonna be… If I were to add up all of them, it would somehow equal one finite number [holds both hand up as if holding something between] as opposed to all these infinite numbers [moves both hands away from each other]. I mean, it's gonna be just around this number consist-consistently.

This excerpt illustrates Brian's use of "mass" in which an infinite amount of numbers are "around" some "number consistently." He then appears to have nearly duplicated his response to Task 3 when he said "If I were to add up all of them, it would somehow equal one finite number [holds both hand up as if holding something between]". Based on his previous responses it appears that "all of them" may have referred to the infinite amount of numbers in the interval $[-1,1)$. Although mathematically imprecise in his language, his usage of "all" and how it can be "added up" to yield one "finite number" is consistent with his prior usage of "all" found in his preceding excerpts.

> When discussing how to estimate sine using its Taylor series, Brian said,
> Well, it's called an estimate because it's not exactly that specific number that it's, uh, revolving around. It's just gonna be somewhere in the ball park of that specific number.

Following these comments, Brian alluded to plugging in numbers whose numerical representation contained several "9's" following the decimal, and then concluded

> If I were to think of a [holds both hands up with palms facing each other], just a number line, you know, I'm coming from the left hand side [moves left hand inward], I'm coming from the right hand side [moves right hand inward], and this is the number it's gonna stop at [moves both hands very close to each other]. It's an estimate, it's not exactly reaching it, but it's the best we can do.

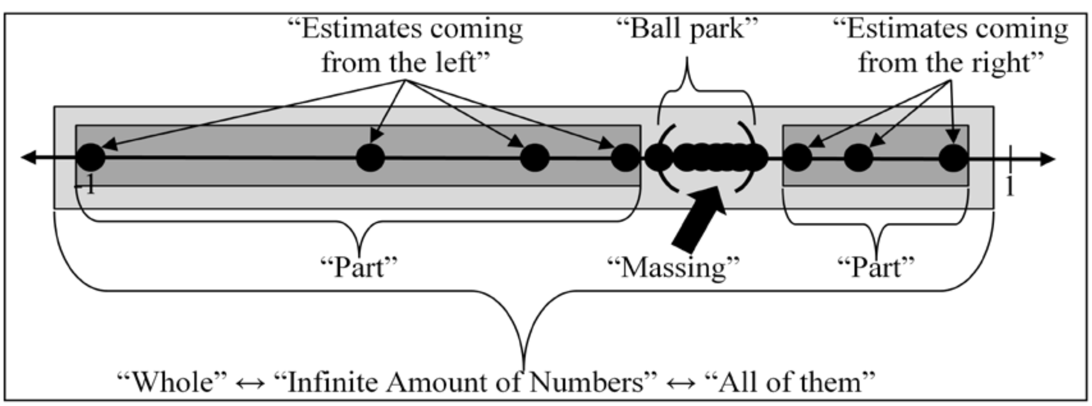

Figure 1. Brian's "Part / Whole" Metaphor on a Number Line

In the previous two excerpts, Brian revealed a variation of his “part / whole” metaphor. Previously, he spoke of a specific number or point that may be sought after using some limiting process applied to the “all”, but in these excerpts, he addressed the role of estimation and concluded that an estimation is “not exactly” but in the “ball park” of the sought after specific number. The estimation is seen as dynamic in that it may be “revolving around” the “specific number”, and when being viewed as a “number line”, estimates “come from the left”, “come from the right”, and “stop”. Taken together, all excerpts indicate an iconic representation of convergence on a number line in which points “mass” around a sought after specific number (see Figure 1).

The next excerpts will illustrate the roles of the “part” and “whole” in Brian’s “part / whole” metaphor. Brian begins with a spontaneous analogy using Google Maps.

> Interview Task 8: What is meant by the “approximation” symbol in “$\sin(x) = x - x^3/3!$ = a Taylor polynomial for sine when x is near 0?”
>
> Brian: Just because it says approximation and nothing else, I’m, I'm guessing it's gonna equal only a portion of what the whole Taylor series would equal. It's not gonna equal the whole answer, it's just gonna get me one little section of it [holds up left hand with thumb and index finger extended close together]. So-like it may be close to divergence, but it's not gonna be-it's almost like Google Maps. I'm gonna show, you know, [holds up left hand with thumb and index finger extended close together] this one little section, but if I, if I pan out or whatever, gonna show me [circular motion with left hand] exactly everything. I think the Taylor series is like the whole view [holds up both hands extended across body with palms facing each other]. And any time I show an approximation [pointing to the Taylor polynomial in Interview Task 8], it's just gonna give me that [holds up right hand with thumb and index finger extended close together] little piece.

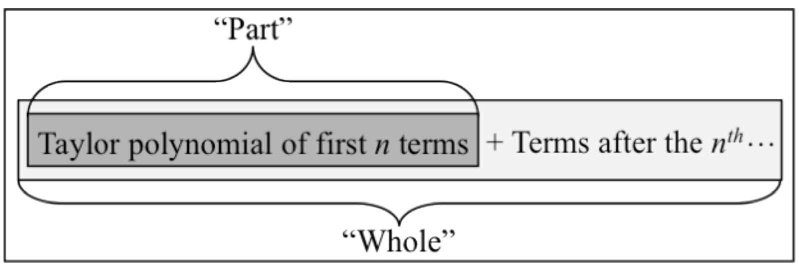

Figure 2. Brian's "Part / Whole" Metaphor for Taylor Series

In the above excerpt, Brian viewed the third degree Taylor polynomial $x - x^3/3!$ as equaling a “portion of the whole Taylor series” that does not equal the “whole answer” but merely a “little section” of the “whole answer”. Furthermore, the “little section” of the Taylor series “may be close to divergence.” He then proceeded to reason by analogy in which a “little section” of Google Maps was equated with the Taylor series “approximation” by Taylor polynomials and the “panned out” view of Google Maps corresponded to the “whole view” of Taylor series. It is also worth noting that when talking about the “little section” of Google Maps and the “approximation” using Taylor series, he used the exact same gesture of extending an index finger and thumb as if grasping the “little section” or “approximation” between these two small appendages. In contrast to this small gesture, when "panning out" on Google Maps or discussing the whole view of the Taylor series his gestures embodied a larger scale. These utterances and gestures reveal a structure of the "part / whole" metaphor for Taylor series illustrated in *Figure 2*.

In the next excerpt, Brian describes a portion of the whole Taylor series, a Taylor polynomial, as an approximation that can be made more accurate by using more terms from the Taylor series.

> How can we get a better approximation for sine than using $x - x^3/3!$?
> Brian: I'm thinking that if we were to keep with the Taylor series we'd go, we'd keep going on, you know, with more of the actual equation. So *x*, you know, equals five over five factorial, and *x* equals seven. I mean-
> I: Uh-huh.
> Brian: -sorry, *x* to the fifth over five factorial, and then *x* to the seven over seven factorial. I think that would give a better approximation.

Even though, Brian alluded to increasing accuracy by increasing the degree of the Taylor polynomial, the extent to which Brian understood this increasing accuracy relative to the degrees of the Taylor polynomials is unclear in the above excerpt. For example, Brian made no reference to the graphical effect of adding more terms to the Taylor polynomial. Therefore, the interviewer asked Brian what adding more terms to the Taylor polynomial did "graphically"?

> Graphically I think it just, it gives us more, it gives us more numbers to consider when trying to find the absolute number, like the number that we're looking for as far as converging or diverging. If it's to converge, I think the more numbers that you have, you know, the better feel you get for what type of convergent it's going to go to. It's like, it's like a grading scale, you know, you can have two tests in one semester and if you, you know, if both are, if one's an F and one's an A, you're not going to know which one the student was. If you got a couple more, you know, well maybe this student's more of an A student and that one F was just a fluke. I'm thinking if this one little piece of the Taylor series [pointing to the Taylor polynomial] shows, just this one little piece [holds right hand up with thumb and index fingers extended close together], it's not going to give as much as opposed to maybe, you know, these other numbers are around that but it's going to zero in [points hands at each other with palms facing toward body] or home in on [points hands at each other with palms facing toward body] something more definite.

Brian initially equated adding more terms with giving "more numbers to consider" for finding the "number that we're looking for." He then stated that "more numbers" yield a "better feel" for the sought after cluster point. He then extended his analogy to a student's grades in which knowing only a couple of grades were not enough to correctly determine the student's overall grade. Only more grades will better reveal the student's actual grade and allow the teacher to determine if the F was just a "fluke". Here, the idea of looking at only a couple of grades which only give a small, potentially misleading piece of the student's actual grade was equated with the Taylor polynomial which only gives "one little piece" of the Taylor series which is not "much" compared to when more terms are added. Just as more grades can better reveal a student's overall grade, "other numbers" can "zero in or home in on something more definite." When later asked what the Taylor polynomial approximation gave, Brian elaborated on his reference to it not giving "much" found above. At first he simply restated that the Taylor polynomial gave an "approximation", he then added that it gives what the Taylor series "might be" and "could be" in the form of an "estimate" and a "guess" that is "just one piece of the puzzle."

Discussion

Throughout the entire questionnaire and subsequent interviews Brian never revealed a formal graphical understanding of Taylor series convergence. He was unable to draw Taylor polynomials when given a graph of a generating function and was unsuccessful in stating any relationship to Taylor series when given graphs of Taylor polynomials. As the previous excerpt illustrates, even when presented directly with task of describing graphically the effect of adding more terms to a Taylor polynomial, Brian responded in non-graphical language. Therefore, his reference to "little pieces" of Taylor series and accompanying gestures with thumb and index finger extended, should not be interpreted as suggesting intervals but "pieces" of formulas of expanded Taylor series. On other occasions, his gestures suggest a graphical understanding of convergence restricted to a number line. For example, in one of the previous excerpts Brian gestured by moving his hands inward when approaching a number from the left and right and later he pointed his hands at each other for "zeroing in or homing in" on a number. Thus, one should not view Brian as having no graphical understanding of Taylor series convergence, but instead, like many students encountering Taylor series for the first time, he has an emerging graphical notion of Taylor series convergence that has not yet clearly distinguished itself from prior graphical notions of convergence, such as convergence on a number line.

Even though Brian's understanding of Taylor series may be less than desirable, all of these excerpts point to a similar structure used to reason about convergence across various contexts. We call this structure the "part / whole" metaphor. Evidence of this structure is not only seen in his consistency of language across the different contexts, but in his consistency of gestures. Depending on the context, the "part" may be composed of some points on a number line, the first few terms of a Taylor series, a "little section" zoomed in on a map, or a couple of grades. It was embodied as minimalist gestures, such as "thumb and index finger extended close together" or "moving both hands very close to each other with palms facing each other." The "whole" is all the points on the number line, all the terms of the Taylor series, a "panned out" view of a map, or all the grades. It was embodied as global gestures, such as "moving both hands away from each other" or "circular motion with left hand." The "part" was consistently viewed as an insufficient portion of the "whole" that could potentially lead to a misleading estimation of the "whole". The "part" can give one a "better feel" for the "whole" as more "pieces of the puzzle" are added to the "part." Once all the pieces are in place to yield the "whole," a sought after convergent can be clearly determined. In the context of the number line, this determination is achieved by noticing where points "mass." The implications of the "part / whole" metaphor elaborated above gave Brian a facility to discuss convergence in multiple contexts, and thus, demonstrated the high degree of the metaphor's resonance within Brian. His commitment to this metaphor throughout the interview tasks and his omission of other metaphors indicate the high degree of emphasis that Brian placed on the "part / whole" metaphor.

What separates this metaphor from other metaphors previously identified (see Oehrtman, 2002, 2009) lies in the entailments of the metaphor. For example, the approximation metaphor, which is commonly used by students engaging in Taylor series activities, contains entailments referencing the remainder. However, the "part / whole" metaphor contains no direct entailments to the remainder because in this metaphor, there is only the "part" and the "whole" and the difference between the two is ignored. The insights gained by understanding this metaphor can help instructors recognize students utilizing this metaphor and engage them in a productive discourse that reveals the metaphor's potential pitfalls and develops scientific reasoning.

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THE IMPLIED READER IN CALCULUS TEXTBOOKS

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Textbooks have the potential to be powerful tools to help students learn mathematics. However, many students struggle to read in a meaningful way. This paper presents a framework to analyze the implied reader of a mathematics textbook; this idea is adapted from the field of reader-oriented theory and enables us to identify the skills and understandings that are required of a student to learn mathematics by reading a textbook.

Introduction

Textbooks are an integral part of most undergraduate calculus courses. They serve many purposes, acting as a compendium of homework problems and examples, a reference for definitions and theorems, and a "road map" through the course content. In addition, most calculus textbooks are designed to be read by students, as evidenced by their detailed exposition of calculus concepts. Educators have suggested many ways to encourage their students to read their calculus textbook (e.g. Boelkins & Ratliff, 2001). However, students often have difficulty using their textbook as a tool for learning mathematics. For example, consider the excerpt from a widely used calculus textbook in Figure 1:

2.7 DERIVATIVES AND RATES OF CHANGE

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit, as we saw in Section 2.1. This special type of limit is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the sciences or engineering.

TANGENTS

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Figure 1. Excerpt from Stewart (2007, p. 143)

Many instructors would have little difficulty interpreting this excerpt in a meaningful way: they could tell that the authors are formally defining the concept of "tangent" and connecting this to the concepts of slope, secant, velocity, and function. However, students often complain that explanations such as this one are difficult to understand: they describe the textbook as simultaneously "too chatty" and "too technical." For example, the following is an excerpt from an interview with an undergraduate calculus student:

Interviewer: Are there some [parts of the textbook] that you look at more than others?

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

Student: I think with the textbook I'm using now, I kind of skip to the definitions and the highlighted parts, especially if I'm in a hurry…. A lot of times I just feel like they're just kind of rambling or reiterating things.

Interviewer: Do you think your books do a good job presenting… the "big ideas" in the chapter? Or… you [were] talking about how sometimes they seem to ramble on a little bit and maybe aren't particularly clear at expressing those big ideas?

Student: I feel like the big ideas are there, but when I start to break it down, sometimes the terminology or the wording they use, you can tell it makes perfect sense to them, I'm trying to decipher it and I have to read a few sentences over a couple of times in order to get it straight in my head

In order to help students view reading their textbook as a useful way to learn mathematics, instructors need to understand what skills and characteristics a student needs to successfully read their textbook. This paper presents a framework that enables us to describe these characteristics and explain why students struggle to read meaningfully. Although the framework will be specific to calculus textbooks[1], it can be easily adapter to describe other mathematics textbooks. After describing the framework, we will present an example of how it may be used and discuss implications for pedagogy.

Implied Reader Framework

In order to explain why students express these difficulties with reading their textbooks, we will use the idea of the *implied reader*, which is a concept that originated in the field of reader-oriented theory. Much like the way modern theories of learning posit that students actively create their own understanding of mathematics, reader-oriented theory conceives of readers as actively constructing the meaning of a text as they read (Rosenblatt, 1938). When reading a textbook, each person uses their own ideas and experiences to construct an interpretation of the textbook that they find personally meaningful: as Morgan (1996) notes, "the meanings constructed from a text by its readers will vary with the resources of individual readers and with the discourse(s) within which the text is read" (p. 3).

Although the idea of the implied reader was created to describe aspects of literary texts (e.g. Wilson, 1981), Weinberg and Wiesner (In-press) extended the definition to apply to technical texts such as textbooks: they define the *implied reader* of a mathematics textbook to be "the embodiment of the behaviors, *codes*, and *competencies* that are required for an empirical reader to respond to the text in a way that is both meaningful and accurate." These aspects are determined by the text itself, as opposed to being determined by the author. (In contrast to the implied reader, the author's image of the reader is called the *intended reader*; the person who actually reads the textbook is called the *empirical reader*).

Codes

Based on ideas from semiotics (e.g. Eco, 1976), a *code* is a way of ascribing meaning to symbols, words, and other elements in a text. A reader's codes enable him or her to interpret the textbook. While there are multiple ways to interpret an element of the text, there are only a handful of interpretations that will be valid and meaningful. Consequently, the implied reader of the textbook possesses a specific collection of codes. In addition to the codes that are required to interpret "regular" literary works, the implied reader of a calculus textbook has codes for interpreting the *formatting*, *language*, and *symbols* that are specific to these texts.

Formatting codes

Love and Pimm (1996) note that the formatting in the text—such as font size or color—imbues the words with particular significance. In Figure 1, the implied reader interprets the italicized words *derivative* and *tangent* as terms that are being defined. In contrast, interpreting these words as simply being emphasized would not allow the reader to understand their true significance. Similarly, the formatting of the phrases "Derivatives and Rates of Change" and "Tangents" indicate that they are describing the main idea of the section; this allows the reader to connect the ideas within the section back to this main idea. If an empirical reader does not possess these codes, they may interpret the formatting as simply a cue to begin and end reading.

Formatting codes can be categorized in the following ways:

- Typeface (e.g. font, boldface, italics, color)
- Page layout (e.g. indentation, positioning of figures)
- Delimiters (e.g. text boxes or background colors)

Language codes

Words and phrases that are used in mathematical writing often have particular meanings that go beyond their everyday usage. These can be classified into four categories: mathematical words, definitions, logical statements, personal pronouns, and imperative verbs.

There are many words that have specific meanings when used in mathematical contexts. The implied reader interprets *mathematical words* such as "approach," "point," and "curve" as having specific meanings in a mathematical context. If the empirical reader lacks these codes, he or she will not be able to think about the mathematical objects and ideas in appropriate ways. These mathematical words can be divided into four categories: words that have been *formally defined* for the reader in a mathematical context (e.g. "limit" or "equation"); words that are *not formally defined* (e.g. "let" or "exist"); *metaphorical* use of words (e.g. "nearby" and "approach" rely on a metaphor of position); and *terms from other domains* (e.g. "velocity" and "displacement"). The implied reader has codes to recognize and interpret *definitions of terms*, *definitions of notation*, and basic *logical statements*. For example, phrases such as "a curve *C*" and "the point *P(a, f(a))*" indicate both that a mathematical object is being created and that it is being connected with the notation and that a mathematical object is being created. Statements such as "If a curve *C*… then we consider…" create a logical relationship between the two clauses. The implied reader to mentally creates a "curve" and then responds to the imperative in the second clause. If the empirical reader lacks this code, he or she will not see the important conceptual relationship.

The use of the *personal pronoun* "we" is pervasive in mathematics textbooks. Morgan (1996) notes that there are two ways it is used: to invoke the authority of the community of mathematicians or to include the reader as an active participant in the mathematical activity (p. 5). The implied reader of a calculus textbook distinguishes between these two uses and thinks of their relationship with the textbook in the corresponding way.

Textbooks contain numerous *imperative verbs*, some of which appear as part of "we…" statements (such as "we consider" and "we… compute"). Rotman (2006) describes two basic functions of these verbs: *inclusive* verbs invite the reader to "establish a shared domain, … introduce a standard, mutually agreed upon ensemble of signs," and "share some specific argued conviction about an item in such a world" (p. 104); examples of inclusive verbs are "consider," "define," and "prove." In contrast, *exclusive* verbs "dictate that certain operations meaningful in an already shared world be executed" (p. 104); examples of exclusive verbs include "integrate", "count", and "compute". The implied reader distinguishes between inclusive and exclusive forms to determine how they must interact with the ideas in the textbook.

Codes for mathematical symbols and pictures

In addition to words, calculus textbooks typically contain many printed symbols and pictures. For example, in the symbol string "*P(a, f(a))*," the implied reader of the textbook interprets this string of symbols in a mathematical context as opposed to a literary context. Then the implied reader recognizes the symbol *P* as the name of a point and the outer parentheses as delimiting the point's coordinates; the *a* is interpreted as a specific (but undetermined) value of the independent variable and the $f(a)$ is interpreted as the corresponding value of the dependent variable (using the previously-defined equation $y=f(x)$).

The symbols used in calculus textbooks are primarily algebraic; these are most frequently used to represent numbers, variables, functions, infinitesimals (e.g. *dx*), arithmetic operations, limits, and operators (such as derivatives and integrals). In addition to these symbols, calculus textbooks frequently include Cartesian *graphs, pictures* (such as a depiction of a falling object) and other *diagrams*. The implied reader has codes to interpret the common features of graphs (e.g. axes, units, points, lines, curves, intervals/distances). The codes that are required to interpret other pictures and diagrams vary widely. For example, consider the graph in Figure 2 (which appears in the textbook after the excerpt in Figure 1). In addition to the "standard" codes for a Cartesian graph, the implied reader also has codes to interpret the three points labeled *Q* as depicting multiple steps in a limiting process in which secant lines approach the tangent line at *P*.

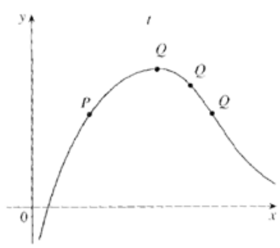


Figure 2. A graph/picture from Stewart (2007, p. 144)

Competencies

While codes enable readers to interpret symbols, words, and phrases, *competencies* enable readers to work within the established context. These competencies are comprised of the reader's mathematical knowledge, skills, and understandings, their knowledge and understanding of real-world concepts, and their ability to "objectify" mathematical ideas.

For example, in the excerpt in Figure 1, the implied reader knows that a function is a relationship between inputs (x) and outputs (y and $f(x)$), has a graphical representation (a "curve"), and is comprised of coordinate pairs; the implied reader also knows how to compute the slope between any two points. In terms of real-world concepts, the implied reader knows what "velocity of an object" and "rate of change" are, how position can be thought of as a function, and how velocity can be computed by finding the slope between two points. If the empirical reader lacks these competencies, they will be unable to understand functions in a meaningful way or connect and use the abstract ideas in other contexts.

There are many concepts in algebra and calculus that must be thought of as having both process- and object-like properties. For example, a reader might think of the symbols $P(a, f(a))$ or $Q(x, f(x))$ as representing the process of picking an x-coordinate and computing the value of

$f(a)$. However, to correctly work with the concept that is described by the phrase "then we let Q approach P," the implied reader thinks of P and Q as "objects" that can be manipulated and can themselves act by "approaching" one another. Morgan (1996) described this transformation of processes into objects as "nominalization" and other researchers have described similar transformations (e.g. Sfard, 2000).

Behaviors

Behaviors are sequences of actions (either physical or mental) that the implied reader performs in order to understand the textbook. Calculus textbooks contain both *structurally embedded behaviors* and *directives*.

Structural embeddings

Calculus textbooks are "linear" in that each chapter, section, sub-section, paragraph, and sentence in the textbook "presumes that the functions and tasks assigned explicitly or implicitly to the reader have been carried out satisfactorily" (Love and Pimm, 1996, p. 381). Consequently, the implied reader develops ideas, vocabulary, and symbolism in the same order as the book presents them.

Directives

In Figure 1, phrases such as "we consider" and "compute" require that the implied reader imagines the appropriate point or thinks about computing the slope; these imperative statements are examples of explicit directives. After interpreting the directive, the implied reader undertakes various actions such as following a procedure or engaging in more complicated mental behaviors

In addition to these explicit directives, the implied reader also responds to requests that are not made explicit. For example, the opening clause—"The problem of finding the tangent line to a curve"—prompts the implied reader to recall the "problem" that the text previously described. Similarly, when the implied reader sees the expression $x \neq a$, they imbue this with meaning by referring back to the definitions of these symbols. These are examples of *implicit directives* The behaviors undertaken by the implied reader in response to these imperatives and directives can be classified in the following way:

- Make a connection with the ideas, symbols, or definitions in another text section
- Make a connection to an external idea (from real life)
- Generalize important mathematical ideas from examples
- Follow a procedure (e.g. follow and replicate the steps of an example)
- Engage in other mental behaviors

Applying the framework

The method for identifying the implied reader of a calculus textbook is for an expert reader to read and recognize what he or she must know, understand, or do in order to generate a correct interpretation. The excerpt in Figure 1 is analyzed here; all of the numbers referenced in the analysis refer to the corresponding numbered boxes in the figures.

Analysis of Codes

First we can investigate the language codes in Figure 3. The personal pronouns (L6, L8, L12 and L17) serve to include the reader as an active participant by inviting them to share the goals and mathematical activity of the author. For the same reason, the imperatives in L12 and L17 are inclusive: they ask the reader to seek motivation for defining the “tangent line,” examine the “nearby point,” interact with the points “*Q*” and “*P*,” and name and construct the concept of “tangent line.”

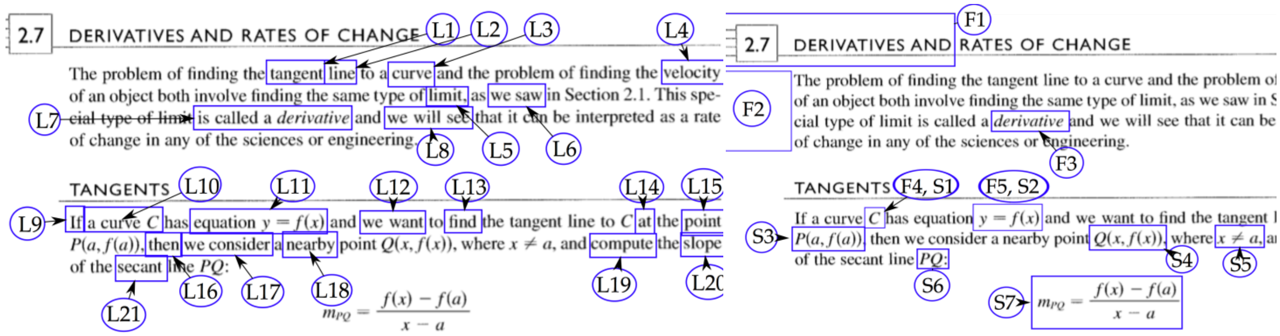
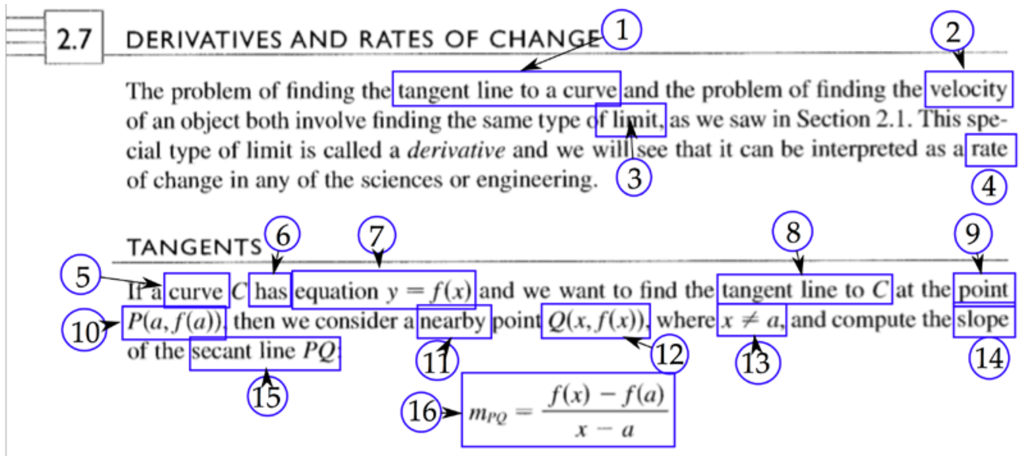


Figure 3. Analysis of codes

In contrast, L19 requires the reader to produce a new mathematical object by operating in an already-established semiotic space. There are numerous mathematical words that the implied reader interprets, some of which have been defined (L1, L5, L11, L20, L21) and others that may not have been defined (L2, L10, L15). There is a reference to a physical concept (L4) and several places in which the implied reader applies a metaphor of physical position (L13, L14, L18). The implied reader recognizes that L7 is connecting a term with its definition, while L10, L11, L15, and L21 are defining notation. In addition, the reader must interpret the logical connection in the “if… then” pair (L9 and L16) as described previously.
Next, we can investigate the formatting and symbol codes in Figure 3. The italics indicate that “derivative” is being defined (F3) and also that the letters represent mathematical entities (e.g. F4, F5). The position and typeface of words in F1 mark the beginning of the section and also describe its main idea; the indentation (F2) indicates that the subsequent text is an elaboration on this idea. The symbol groups S1, S3, S4, and S6 indicate that a mathematical object is being named, with S1 representing an object with one component and the others representing objects with two coordinates or endpoints. In S2, S3, S4, S5, and S7, the *x* and *a* are interpreted as a variable and an undetermined value of the variable, which are connected to the concept of a coordinate pair by equating *y* and *f(x)*. The equals signs in S2 represents an identity between two different types of objects; in S5 it shows the (non-)identity of the same type of object, while in S7 it serves to define notation.

Analysis of Competencies

Figure 4 shows the competencies of the implied reader. The implied reader knows that a curve is a geometric representation of an algebraic relationship (1, 5, and 7) and thinks of it as an object that possesses properties (6). The implied reader knows that the curve is comprised of points (9), each of which has coordinates (10, 12) that are linked by the underlying function, which can be expressed algebraically (7). The implied reader understands the concept of slope (14), how it is computed (16), that two points (11, 13) can be connected to form a secant line (15) which approximates a tangent line at a point (1, 8) through a limiting process (3) which generalizes the idea of slope. The implied reader understands and is fluent with the algebraic notation (e.g. 13, 16), interprets the notation as an algebraic generalization, thinks of the notation as representing an object (16), and connects all of these ideas to their understanding of real-world concepts of rate and velocity (2, 4).


2.7 DERIVATIVES AND RATES OF CHANGE

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TANGENTS

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Figure 4. Analysis of competencies

Analysis of Behaviors

Figure 5 shows the behaviors of the implied reader. The structurally embedded behavior means that the implied reader will have read previous sections of the textbook prior to the section in the excerpt; the implied reader also reads all of the numbered boxes in order.

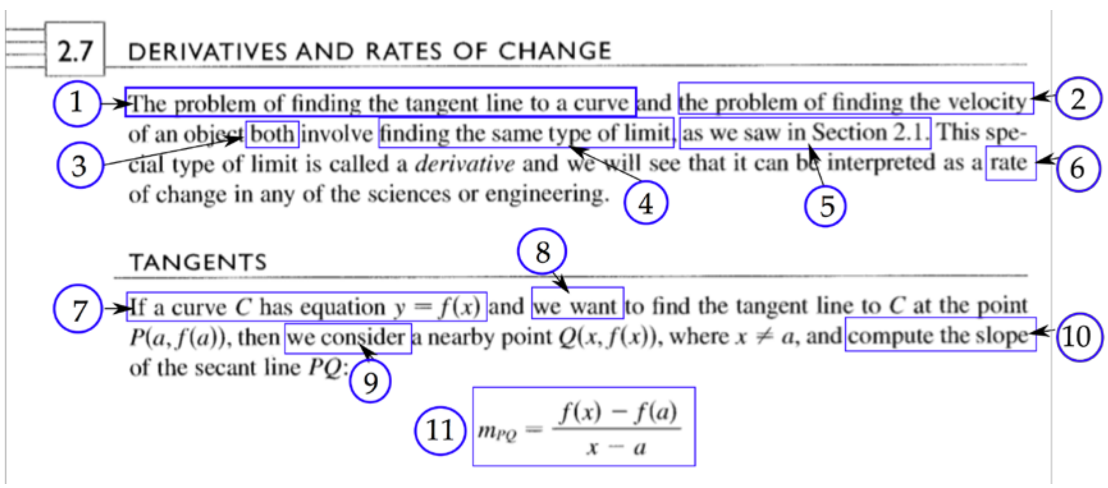
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Figure 5. Analysis of behaviors

There are three explicit directives (8, 9, and 10) that the reader follows. There are also numerous implicit directives. In 1 and 2, the implied reader makes a connection to the previous text sections in which the indicated "problems" were first described. In 3, the implied reader connects these two problems and then connects the problems to the concept of "type of limit" in 4, referencing the previous section in the text as necessary in 5. In 6, the implied reader connects these ideas to the real-world concept of "rate of change". In 7, the implied reader assigns meaning to the symbol "*C*" and associates this definition with the concept and symbolism of the "equation" (this behavior is replicated when the symbols *P*, *Q*, *PQ*, and m_{PQ} are defined). After the implied reader assigns meaning to the individual symbols in 11, they mentally perform the indicated arithmetic operations and think of the result as a new mathematical object.

Analysis

The concept of the implied reader enables us to describe what an empirical reader must understand and do in order to read a mathematics textbook meaningfully. In the analysis of the implied reader, we see that the reader must perform a wide range of mental activities, that each text element may have multiple associated codes, competencies, and behaviors, and that there can be many of these "components" in a very short space.

The analysis of the textbook excerpt (Figure 1) sheds light on why instructors may view a textbook as a useful source of explanations but students may find it difficult to read. Instructors are more likely than students to possess the codes, competencies, and behaviors of the implied reader, which enables them to construct correct mathematical meaning while reading.

The students' difficulties can be explained by the characteristics of the codes, competencies, and behaviors of the implied reader as well as the density of these components. If the components are similar to those that students already possess, then students should be successful at constructing knowledge from reading their textbook; if the implied reader's codes, competencies, and behaviors are closer to those of a mathematician, or if they are too densely packed, students will struggle to interpret the text correctly. As a result, students might view a textbook as "rambling" when they do not possess the correct codes to understand the underlying meaning, and they may see it as "too technical" when many components occur close together.

Analyzing the implied reader of a textbook can be useful for instructors, who play an important role by mediating the way students interact with the textbook (Luke, de Castell & Luke, 1989); that is, instructors choose the textbook, decide which parts of the book to use, and encourage particular uses of the textbook. In order to help students use a textbook as an effective tool for learning mathematics, instructors need to identify the codes, competencies, and behaviors of the implied reader and then help their students develop these characteristics. However, instructors must take care to not solely focus on "translating" specialized mathematical vocabulary, syntax, and symbolism. The theoretical foundation of the idea of the implied reader rests on the notion that the reader actively constructs the meaning of the text. Consequently, instructors must help their students build the codes, competencies, and behaviors of the implied reader in a way that enhances their ability to generate meaning as they read. This use of the textbook is in contrast to a typical perspective of the use of mathematics textbooks, in which reading is seen as "only… a necessary kind of ability in order to become active in situations where learning can take place (i.e. solving given tasks)" (Osterholm, 2006, p. 326).

The idea of the implied reader also has consequences for authors of textbooks. Specifically, calculus textbook authors should analyze their own writing to identify the codes, competencies, and behaviors that will be required of their reader. Even though some textbooks claim to be written in "plain English", the implied reader may still possess sophisticated aspects that will make the reading process difficult for undergraduate mathematics students.

Endnotes

1. Each type of mathematics textbook requires specific types of knowledge and interpretations. For example, an undergraduate abstract algebra textbook typically follows a "theorem-proof-example" format and requires the student to understand particular types of logical arguments; this is very different from a college algebra or a calculus textbook.

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CALCULUS STUDENTS' CONCEPT IMAGES OF ASYMPTOTES, LIMITS, AND CONTINUITY OF RATIONAL FUNCTIONS

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This study was designed to investigate college students' concept images of rational functions, asymptotes, limits and continuity beyond the algorithmic knowledge. A two-hour long problem solving interview was conducted by the use of CORI, the problem solving instrument. Nineteen students taking a university calculus course participated in the video taped one-to-one interview.

Introduction

During the past few years, I was confronted with high school and college students' incomplete conceptions regarding asymptotes of rational functions. Inconsistent conceptions often interfere with students' learning of mathematical concepts. The identification of asymptotes plays a central role in the study of rational functions and its limit properties. The study of asymptotes and the exploration of rational functions are based on algebraic manipulations of expressions. Students don't always fully understand the concept. They still move on to calculus courses. There, they are introduced to the concepts of limits and continuity. They do not fully understand them either. During the instruction of limits, asymptotes are touched upon again. The formal definition of continuity is presented again in terms of limits and students struggle with this definition too. May be the concepts of asymptotes, limits, and continuity together could be emphasized so that students' understanding of all of these concepts could be enriched. Before doing that, it is important to find out student notions of these concepts beyond the algorithmic level.

Theoretical Framework

According to Piaget (1970), students do not come to our classes as *blank slates*. They might have encountered certain aspects of a concept before that concept was formally introduced to them. Therefore, at times, their prior experiences with a certain concept could interfere with the newly introduced aspects of the concept. In this case, learners will need to modify their pre-existing internal schemes of the concept in accordance with the new conceptual input. According to Piaget, schema stands for a person's internal conceptual configurations. Similar to Piaget's view Tall and Vinner (1981) stated that mathematical concepts that students learn informally are often incomplete. They referred to students' interpretations of the *concept definition*, which were influenced by their pre-existing notions of the concept, as *concept images*. Misconceptions or incomplete conceptions occurred as a result of students viewing the concept definition through the lens of pre-exiting concept images that are at odds with the formal concept definition.
For example, before formally learning the *concept of limits*, students are familiar with the *terminology limit* in daily life. The meaning of this word is interpreted differently by different people. For example, when referred to an *off limits* situation, the meaning of *limit cannot be attained* is implied while in another situation such as *age limit* 5, the implication is that the *limit cannot be surpassed*. Due to this confusion, students acquire different meanings and interpretations while dealing with the concept of limits. One such dilemma causes uncertainty in students whether limit can be obtained, or can be surpassed. Students' incomplete conceptions

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could be analyzed from the perspectives of the innate nature of mathematical truth (Moru, 2006; Cornu, 1991; Grey & Tall, 1994), the limitations of instruction and instructional resources (Clement, 2001; Sajka, 2003; Zaslavsky, 1997), and students' beliefs and attitudes regarding the nature of mathematics knowledge and the goals of mathematics instruction (Szydlik, 2000).

Research Questions

1. What are student notions of rational functions?
2. What conceptions do students possess regarding asymptotes, in particular the horizontal and vertical asymptotes of rational functions?
3. What connections do students make between the concepts of asymptotes, continuity, and limits of rational functions?

Methodology

Qualitative methodology was used to conduct this research. Nineteen Calculus 2 students from a large midwestern university participated in the study. Participants were freshmen students among which, 15 of them were engineering majors, one a mathematics major; one a psychology major, and two geology majors. Student conceptions regarding rational functions, asymptotes, limits, and continuity were assessed using the Concept Organizer Response Instrument (CORI). Interview questions were semi-structured and were designed to gather information on students' thinking. The goal of the interview was to gain an in-depth understanding of students' thinking processes while solving problems. Students were asked to solve problems on the test booklet while talking aloud explaining their thought processes. In addition, the researcher asked probing questions of students to help further clarify the reasoning behind the work they performed.

The interviews were video taped. The video tape analysis was conducted and common traits in student responses were compiled. Students' written work was examined to further clarify researcher's interpretations of students' explanations of the problems solved.

Results

Students' incomplete concept images of rational functions fell mainly into three categories: *the rational number image, the fraction image,* and *the discontinuity image.* Students who possessed the rational number image described that the graphs of rational functions are "nice," "whole," "even," and "symmetric." They further clarified that the graphs of rational functions looked like that of linear and quadratic functions and that they were "one-piece," "continuous," and "without any complications." Some students specified that like rational numbers, such as $\sqrt{4}$ and $\sqrt{9}$, these functions were "whole." These students' concept image of rational numbers were restricted to that of numbers like $\sqrt{4}$ and $\sqrt{9}$. The rational number conception was the most prevalent conception that students held of rational functions.

Some students believed that rational functions assumed fraction forms with no variable in the denominator. According to this conception, like fractions, rational functions could only have constants in the denominator. Those students who held the discontinuity image believed that all rational functions had variables in the denominator, and therefore were discontinuous- they came in several pieces, and had vertical asymptotes.

Student notions about asymptotes in general fell in to the categories of the *three-piece graph image*, the *invisible line image,* and *the no-concurrency image*. The three-piece graph image reflects graphs with two vertical asymptotes and one horizontal asymptote that were symmetric

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with respect to the Y-axis. Three-piece graphs were also comprised of graphs with two vertical asymptotes and no horizontal asymptote and were symmetric about the origin. Students also believed that asymptotes were dotted lines the graph approached but never reached. The no-concurrency images held by students lead them to believe that a graph must never be concurrent with any of its asymptotes.

Regarding particularly on vertical and horizontal asymptotes, students held a variety of notions. All students associated vertical asymptotes in connection with multiple views of undefinedness. Most students stated that *vertical asymptotes occurred at points were the function was undefined*. However, several students did not know when a rational function, such as $f(x)=\dfrac{2x-1}{3x+5}$ became undefined. Some believed that all functions including functions of the form $g(x)=\dfrac{17}{x^2+1}$ had a vertical asymptote since there was a variable in the denominator. While 4 students knew about the possibility of having either a hole or a vertical asymptotes at the points were the rational function was undefined, two students believed that both a hole and a vertical asymptote occurred simultaneously at points were a rational function was undefined. Such cases were described as point asymptotes by one student. In addition, the belief that vertical asymptotes could occur at jump discontinuities was also noted.

Among the students who knew about the possibility of having either a hole or a vertical asymptote at the points were the rational function was undefined, the majority of the students were unable to distinguish between the conditions under which a hole occurred for a rational function. Even though students had seen graphs with holes and have realized that the function would be undefined there, many of them were unable to differentiate between the function behavior around holes and around vertical asymptotes. The confusion between holes and vertical asymptotes seems to have stemmed from noticing no distinction between the $\dfrac{0}{0}$ and $\dfrac{b}{0}, b \neq 0$ forms.

In many ways, student notions of horizontal asymptotes were similar to that of vertical asymptotes. Some dilemmas were centered on the inability to find the equation of the horizontal asymptote, and the belief that a graph cannot be concurrent with its horizontal asymptote. Other problems seem to have stemmed from not knowing the function behavior around its horizontal asymptote, and the inability to identify horizontal asymptote from the limit form. The inability to use appropriate terminology while describing horizontal asymptotes and the failure to write the function term corresponding to a specified horizontal asymptote was also a problem. In addition students believed that horizontal asymptotes occurred at cusps.

In regards to the concept of limits, students experienced problems with the correct usage of terminology. While referring to limits, in regards to terminology, instead of stating "as x approached a, f(x) approached L," the roles of x and y and x and f(x) were often interchanged, or instead of stating x, or f(x), or y, the word *it* was used. Other difficulties included problems with computing limits, and problems with connecting limits with asymptotic behaviors of functions. While computing limits students used direct substitution when there was no graphing calculator available. In some instances, students substituted infinity directly in place of x and wrote, $\infty/\infty = \infty$, $\infty/\infty = 0$, $\infty/\infty =$ undefined. They were unable to compute infinite limits and limit at infinity. While dealing with forms such as b/0, b $\neq$ 0, $\dfrac{\infty^2}{\infty}$, $\dfrac{\infty}{\infty}$, $\dfrac{\infty}{\infty^2}$ students gave up and stated that limit (not necessarily in the finite sense) cannot be found, or did not exist.

In addition, they were unable to recognize function limits from their graphs. Identification of limit from function graphs posed challenges for many students. Some students believed that no limit existed at a point if the function was undefined at that point. In some instances, students stated that if f(x) had a hole at the point (2, -7), as x approached 2, the limit of the function would be -7. However, if the function f(x) had a removable discontinuity at (2, -7) such that f(2) = 3, then as x approached 2, f(x) would approach 3. Some students, while computing limits stated that in cases such as $\lim_{x\to 7}\frac{11}{x-7}$, which produced the $\frac{11}{0}$ form, the function limit would be 11. In this student's view, since $\frac{11}{0}$ was *undefined* and therefore cannot be attained by the function, the limit of the function should be 11 the *attainable* part of $\frac{11}{0}$.

Regarding continuity, some students believed that a function was discontinuous at sharp corners and a function was continuous everywhere in its domain if the domain was all real numbers. They also believed that both jump and removable, discontinuities produced vertical asymptotes at the points of discontinuity. Some others believed that whenever the left hand limit was equal to the right hand limit, the function was continuous at that point.

It must be noted that the majority of students demonstrated correct understanding of the concept of continuity by specifying that in order for the function f(x) to be continuous at a point, say, x = a, $\lim_{x\to a} f(x)$ must exist and must also be equal to $f(a)$. While answering the question whether a function with all real number domains should always be continuous, several students immediately answered yes, but, as soon as they started sketching a function, they recanted their answer by creating a hole on the curve and placing a point that indicated a removable discontinuity. It should be noted that these students did not display a function with jump discontinuity as a counter example. Understandably, it would be a lot easier to alter a continuous function to one that had a removable discontinuity rather that altering it into a function with a jump discontinuity.

Conclusion

Nineteen Calculus II students' concept images of asymptotes, limits, and continuity were investigated by the use of a problem solving interview. The instrument for this interview, CORI, was developed to solicit student intuitions by the use of atypical problems that required thinking *out of the box*. Each interview lasted about two hours. I was particularly interested in students' incomplete concept images so that I could try to employ instructional methods that would help students reconfigure their inconsistent concept images.

Students generally possessed a process-oriented view towards mathematics problem solving. While holding numerous incomplete conceptions regarding rational functions, their asymptotes, limits and continuity, students failed to establish connections between these highly related concepts. Students were unable to explain the behavior of functions around its asymptotes and they were unable to relate the asymptotic properties of functions in terms of limits. Without being able to relate to the ways in which function behavior was affected by its asymptotes, students were unable to find limit at infinity and infinite limit without a graphing calculator.

Endnotes

1. This paper is a part of my dissertation, *College Students' Concept Images of Asymptotes, Limits, and Continuity of Rational Functions*, completed under the direction of Dr. Douglas T. Owens; Professor, College of Teaching and Learning, The Ohio State University, Columbus, Ohio.

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CALCULUS STUDENTS, FUNCTION COMPOSITION, AND THE CHAIN RULE

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This study focused on calculus students' routines of function composition while working on chain rule tasks. The functions used in these tasks are functions with which they are familiar, somewhat familiar and not familiar. Previous research on the chain rule has indicated that function composition is an important concept for understanding the chain rule. The significance of function composition to the chain rule, provided a useful context in which to study function composition. Sfard's (2008) commognitive framework was used to analyze transcript data and identify these routines. The results identify nine different routines across the different types of functions.

Introduction

Although the teaching and learning of the function concept has received much attention in the research literature (Ferrini-Mundy & Graham, 1991; Harel & Dubinsky, 1992; Monk, 1994; Oehrtman, Carlson, & Thompson, 2008; Vinner & Dreyfus, 1989) there have been few studies that have focused on student learning and/or understanding of function composition (Engelke, Oehrtman, & Carlson, 2005). That limited research has noted that students experience difficulties with the notation used for composite functions as well as the representation (algebraic, tabular, graphical) that they use. Meel (1999) noted that students interpreted both $(f \circ g)(x)$ and $f(g(x))$ notations as function multiplication resulting with either $(f(x) \cdot g(x)) \cdot x$ or $f(x) \cdot g(x)$. Hassani (1998) and Engelke, et al. both reported that students were more successful in tasks using algebraic representation than in either graphical or tabular representation. Engelke, et al. studied precalculus students at the end of their course and documented that these students answered function composition tasks correctly 94%, 43%, and 41% of the time algebraically, graphically, and tabularly, respectively. Similarly, Hassani performed a teaching experiment on calculus students. Students' performance on function composition tasks on pre-review and post-review tests were all lower than those reported by Engelke, et al. for each representation. Hassani's study, however, did not end there. Following the post-review test, the students were instructed on the chain rule. After this instruction, students' performance improved remarkably and correctly answered function composition tasks 84% of the time graphically and 63% tabularly. She concluded that studying the chain rule improved students understanding of function composition. Additionally, she reported that this prior lack of function composition knowledge did not have a significant role in students' understanding of the chain rule rather that the knowledge needed to be "gained by the time they apply the chain rule" (p. 193). She concluded that there seemed to be a relationship between the chain rule and function composition that improves the understanding of each.

Research on the learning of the chain rule has highlighted the importance of function composition to the chain rule concept (Clark, et al., 1997; Cotrill, 1999; Hassani, 1998; Webster, 1978). Using a teaching experiment Webster (1978) concluded that there was no significant difference on the chain rule post-test between the groups receiving extra instruction on function composition and those that did not. In contrast, Clark et al. (1997) using the APOS framework (see Asiala, et al., 1996) "came to the conclusion that the [students'] difficulties with the chain rule…could be attributed to student difficulties in dealing with the composition and decomposition of functions" (p. 360). Cottrill (1999) completed a follow-up study to the Clark's et al. study. Cottrill's findings were inconclusive whether "understanding of composition of functions is fundamental to understanding the chain rule" (p. 58). Cottrill further suggested that a new study that collected data from interviews instead of a written questionnaire was needed to address this issue.

These mixed findings and conclusions about the relationship between the chain rule and function composition illustrate the need for both a clear research focus and good task design. Each of these studies were either broad or had purposes other than or in addition to studying function composition. These studies acknowledged that function composition played a role in understanding the chain rule, but its role was elusive to studies which had different foci. One major change in my study is that I am reversing the emphasis. Instead of studying the chain rule as the object of study and using function composition as a lens, function composition is the object of interest and the chain rule is being used as the lens.

Using chain rule tasks as a window to study function composition enables this concept to be observed in a rich context. As noted above function composition is an integral part of the chain rule. Furthermore, this extends the existing research beyond simply asking students to find $f(g(4))$ or $(f \circ g)(4)$ algebraically, graphically, and/or tabularly. In the chain rule context they must create a composite function from two given functions, take the derivative of the composite function which may involve function composition, and then evaluate the derivative function which again may involve function composition. Thus, the chain rule has the potential to create multi-step function composition tasks and students' use of function composition concepts can be tracked across each step as well as the whole problem. This can give a wider and richer view of students' use of function composition.

Theoretical Framework

This study utilized Sfard's (2008) commognitive framework. As the term commognition suggests, this perspective combines communication with cognition. In this perspective thinking is a form of communicating. Thus thinking and communicating are merely "different (intrapersonal and interpersonal) manifestations of the same phenomenon" (Sfard, 2008, p. 296). Thus, observable words (and nonverbal forms of communication) are still just words and are studied as such. This is in contrast to a strictly cognitive perspective which considers observable words as pointers that are used to study and make claims about unobservable phenomena that exists inside the head.

The focus of this study will be on students' routines. From the commognitive perspective a routine is a set of rules "that repeats itself in certain types of situations" (Sfard, 2008, p. 301). This means that it must happen a minimum of two times otherwise it does not satisfy the repetition requirement of a routine. A routine has three parts to it. First is the *applicability condition*. This is the set of rules that evokes or conjures up the need to start an action. The action itself is called the *course of action* or procedure of the routine (e.g., an algorithm). At some point the procedure comes to an end. These ending rules are called the *closure* or the indication to stop the course of action. The applicability condition and closure are part of the *when* of a routine, while the course of action is the *how* of a routine.

The when of a routine is difficult to identify. Two routines may differ only by the when of the routine. For example a mathematics student may learn the steps of an algorithm and does them the same way that his or her instructor does it. On a subsequent homework assignment or test the student encounters a problem that he or she is unable to solve. While the teacher considers this new problem as needing the algorithm discussed in class, the student did not have the same conditions that would evoke the need to apply the algorithm. Even though the course of action was the same for both the student and the teacher, the applicability conditions were different. Not only can the applicability conditions be different with the same course of action, but the closure can also be different between individuals. For example two individuals solving a system of linear equations may stop at different points. One person may stop when he or she comes to the statement x = 5 (or some number). Another may not stop until the equations are evaluated at x = 5 to check for consistency. All three parts are essential components of a routine.

Research Methods

While the focus of this study was on students' routines, given the difficulty of identifying applicability conditions and closure, most of the findings are about students' course of action(s). The specific question that guided this research was: What are calculus students' routines of function composition in chain rule problems? Specifically, what do students do with functions with which they are familiar, somewhat familiar, and not familiar? The familiarity level of a function was determined by when students typically encounter each type of function (calculus, precalculus, never). For example, polynomial and trigonometric functions were classified as Familiar because students experience them in many calculus contexts including the chain rule. Transcendental functions were classified as Somewhat Familiar because students encountered them in precalculus, but at the time of this study they had not encountered them in any calculus context. A third task called the Flowers-Colors Task was classified as Not Familiar. It was expected that students had not come across functions like the ones used in this task. It involved functions with the names of flowers and colors. Figure 1 contains the Flowers-Colors task.

Flowers – Colors Task:

Suppose $roses(x)$, $violets(x)$, $red(x)$, and $blue(x)$ are all differentiable functions and have the following derivatives:

$$\frac{d}{dx}(roses(x)) = red(x) \qquad \frac{d}{dx}(violets(x)) = blue(x)$$

$$\frac{d}{dx}(red(x)) = red(x) \qquad \frac{d}{dx}(blue(x)) = \frac{3x+1}{x^2}$$

1. $f(x) = roses(violets(x))$
   a. Find $f'(x)$
2. Find the derivative of $h(x) = f(g(x))$ where $f(x) = blue(x)$ and $g(x) = violets(x)$
   a. Find $f'(x)$ and $g'(x)$
   b. Find $h'(x)$
3. Find the derivative of $h(x) = (f \circ g)(x)$ where $f(x) = roses(x)$ and $g(x) = red(x)$
   a. Find $f'(x)$ and $g'(x)$
   b. Find $h'(x)$

Figure 1. Flowers-Colors Task

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The Flowers-Colors Task needed to explain that these were names of functions and that these functions were differentiable. This task was modified from a task created by Webster (1978). Modifications included adding the derivatives of the functions $red(x)$ and $blue(x)$. These were added for two reasons.

First, so that it could be determined where students would stop multiplying by the derivative. Instructors of calculus informally consulted prior to data collection mentioned that students may answer part a by claiming that $f'(x) = red(violets(x)) \cdot blue(x) \cdot blue'(x)$ with the extra $\cdot blue'(x)$ in the answer. By including $\frac{d}{dx}blue(x)$ in the problem statement, students that apply the chain rule in this manner would not be affected by this information being absent.

Second, the derivatives of the colors were made to be similar to transcendental functions. Thus $red(x)$ is similar to exponential functions and $blue(x)$ is similar to the natural logarithmic function in that the derivative is a rational expression. In a pilot study it was difficult to determine if and what kind of effect the rational expression in the derivative statement of $\ln x$ had on students' answers. This second rational expression was included to learn more about its effect.

Participants & Data Collection

Ten first-year calculus students enrolled at a large Midwestern university during the summer semester volunteered to participate in this study. The textbook being used for the course was *Thomas' Calculus 11e*. Each participant was interviewed individually for approximately one hour. The task-based interviews took place after the students had been tested over the chapter that covered the chain rule (Chapter 3) and before any in-class instruction on differentiation techniques for transcendental functions (Chapter 7). Each interview was both audio- and video-taped. Additionally all written work of the participants was collected for analysis.

Data Analysis

The data was analyzed in multiple ways. First, transcripts were also made following the interviews detailing what was said and what was done. The relationship between what was said and what was done was also included on the transcript. Participants' pointing helped to make clear the meaning of words like "this," "that," "here," and "there." Pointing also helped to identify what participants were considering when they were silently thinking.

The written work and transcript data was first evaluated for the correctness. Categories were then made that described the students' solution methods. These categories came out of the data and were not decided upon prior to data analysis. Due to the complexity of the tasks, a single category could not appropriately or adequately describe students' solution methods or course of action. For example given the functions $f(x) = 5x + 7$, $g(x) = x^2 + 3x$, and $h(x) = (f \circ g)(x)$, one participant stated that $h(x) = f(x) \cdot g(x)$ or $h(x) = (5x+7) \cdot (x^2 + 3x)$ and that $h'(x) = f'(x) + g'(x)$ or $h'(x) = 5 + (2x+3)$. Since deriving a formula for *h(x)* involved multiplication and deriving a formula for $h'(x)$ involved something that appeared similar to the addition rule, a more complex categorical system was needed.

This was achieved by dividing the tasks into two parts and each part receiving its own categorization. The first part was the participants' solution methods to derive a formula for the composite function *h(x)*. The three categories from this part of the task were Composition, Multiplication, and Addition. The Composition category involved the participant using

$h(x) = f(g(x))$ while the Multiplication and Addition categories involved $h(x) = f(x) \cdot g(x)$ and $h(x) = f(x) + g(x)$, respectively. The remaining part of a task was the solution methods participants used to find a formula for $h'(x)$. Eight categories emerged from this part of the data. Half of these were actual rules of differentiation which included the power rule, the product rule, the addition rule, and the chain rule. The remaining four categories included alterations to these rules. These categories were Composition & No Chain Rule, Multiply Derivatives, Chain Adding, and Chain Times-ing which involved $h'(x) = f'(g(x))$, $h'(x) = f'(x) \cdot g'(x)$, $h'(x) = f'(x) + g(x) + g'(x)$, $h'(x) = f'(x) \cdot g(x) \cdot g'(x)$, respectively. For example if $f(x) = \sin x$ and $g(x) = x^2$, then Composition & No Chain Rule would be $h'(x) = \cos x^2$, Multiply Derivatives would be $h'(x) = (\cos x)(2x)$, Chain Adding would be $h'(x) = (\cos x) + (x^2) + (2x)$, and Chain Times-ing would be $h'(x) = (\cos x) \cdot (x^2) \cdot (2x)$. Further illustrations are included in the next section for each of these categories.

Since a routine is something that is repeated, a solution method was not categorized as a course of action until it appeared a minimum of two times. This minimal requirement could be satisfied in two ways: (a) a single participant doing the same solution method on two different tasks or (b) at least two different participants doing the same method on any of the tasks. If two different participants did the same method on different tasks, then the two times requirement was satisfied.

Results

This section is organized according the familiarity level of the functions in the tasks. The findings of familiarity tasks will be presented starting with familiar, then somewhat familiar, and ending with not familiar.

Familiar Tasks

The familiar tasks included the polynomial functions $f(x) = 5x + 7$ and $g(x) = x^2 + 3x$. All ten participants found $f'(x)$ and $g'(x)$ by using the regular Power Rule and Constant Rule. These were described as multiplying the coefficient by the exponent and subtract one from the exponent to form a new exponent and zero, respectively. The second part of this task defined $h(x) = (f \circ g)(x)$, the composition of $f(x)$ and $g(x)$. The specific task was to find the derivative of $h(x)$ at $x = 3$. All ten participants first derived a formula for $h(x)$ before proceeding to the derivative. To derive $h(x)$, six were categorized as Composition, three as Multiplication, and one as Addition. After deriving a formula for $h(x)$ the participants proceeded to find the formula for $h'(x)$. Six participants used the Power Rule, one used the Product Rule, two used the Addition Rule, and the remaining one used Chain Adding. After each participant derived a formula for $h'(x)$, all ten evaluated that formula at three in the same manner, which was to "plug in three for x."

Somewhat Familiar Tasks

The Somewhat Familiar Tasks involved the natural logarithmic function. The first task defined the composite function $g(x) = \ln(-x^3)$ with the instructions to find $g'(2)$. The participants first derived a formula for $g'(x)$ and then evaluated $g'(2)$. Nine of the ten participants completed this task. The remaining participant focused on properties of logarithmic functions (e.g., $\ln x^2 = 2\ln x$) and decided that she could not do this problem because she could not remember all of the properties. All nine participants that completed this task derived a formula for $g'(x)$ and then, as occurred in the Familiar task, plugged in 2 to evaluate his or her function. Five participants' solutions were categorized as Composition & Chain Rule, three as Composition & No Chain Rule, and one as Composition & Chain Adding. For this task these categories took the form of $\frac{1}{-x^3}\cdot -3x^2$, $\frac{1}{-x^3}$, and $\frac{1}{-x^3} + -3x^2$, respectively.

Each of these methods was similar in that they entailed composition. The $-x^3$ term of $g(x)$ was "plugged into" the x in the statement of the derivative of $\ln x$. The only difference between these categories was the way in which the chain rule was applied. The participants classified in the Composition & Chain Rule category multiplied the rational term by the derivative of $-x^3$, those in the Composition & No Chain Rule category left off the derivative of $-x^3$ completely, while Mark, of the Composition & Chain Adding category added the derivative of $-x^3$.

The second and third Somewhat Familiar Tasks involved $f(x) = \ln(x)$ and $g(x) = x^2 + 1$. The second *task* defined $h(x) = (f \circ g)(x)$, while the third task defined $s(x) = (g \circ f)(x)$,. The categorizations of these tasks were similar. The second task included one as Product Rule, one as Addition Rule, five as Composition & Chain Rule, one as Composition & No Chain Rule, one as Composition & Chain Adding, and one as Multiplying Derivatives. The only difference for the third task was two as Product Rule and none for Composition & No Chain Rule.

In summary, the Somewhat Familiar tasks that were not pre-composed had more variability in the methods to derive a formula for the derivative than those that were already composed. Both pre-composed and non-pre-composed tasks used the methods Composition & Chain Rule, Composition & No Chain Rule, and Composition & Chain Adding. Additionally participants used the methods Product Rule, Multiply Derivatives, and Addition Rule for non-pre-composed tasks. Similar to the Familiar tasks, evaluating a derivative function at a specified value was always done by plugging in that value for the x's in the formula.

Not Familiar Tasks

The Flowers-Colors Task was the final task related to the function familiarity research question. In it participants answered one question with a function already composed. The first Not Familiar Task used a function that was already in composed form. Seven of the participants used the Chain Rule to determine $f'(x)$ and all of the participants used methods that have been discussed in previous sections. Table 1 shows these methods and how they compare to the other Not Familiar tasks and the previous tasks. Two additional techniques occurred on this task. Three students decomposed the function into two parts (e.g., $f(u)$ and $u = violets(x)$) and another student attempted to solve a similar task with a function more familiar to him (e.g., $\ln(3x)$) and that had similar features to the function in this task. After seeing how the more familiar function behaved, he then applied the same principle, the chain rule, to the unfamiliar function. He explained the common features of his problem and the task in the following episode.

In the second Not Familiar Task the function was not pre-composed. All ten participants derived a formula for $h(x)$ first and nine of them did so using the Composition method. More variability occurred when deriving a formula for $h'(x)$ (see Table 1). This is the only task where the Chain Times-ing category appeared. This category is similar to Chain Adding in that it contains all of the components of the chain rule only with different operations. In this task Chain Times-ing appeared as $h'(x) = \frac{3x+1}{x^2}(violets(x)) \cdot blue(x)$

The third Not Familiar task defined $h(x)$ as $(f \circ g)(x)$,. Similar to other tasks, nine of the ten participants derived a formula for $h(x)$ before finding $h'(x)$. Again Table 1 lists the methods used to derive $h'(x)$. Additionally, similar to the first Not Familiar Task one participant performed decomposed the $h(x)$ that was just derived and another student solved a similar task with a more familiar function.

Table 1. Distribution of the methods used to solve different types of Familiarity tasks

Category	Familiar	Somewhat Familiar			Not Familiar		
		task 1	task 2	task 3	task 1	task 2	task 3
Power Rule	6						
Product Rule	1		1	2	1	1	1
Addition Rule	2		1	1			
Chain Adding	1				1	1	1
Composition & Chain Rule		5	5	5	7	6	6
Composition & No Chain Rule		3	1				
Composition & Chain Adding		1	1	1	1		
Multiplying Derivatives			1	1			2
Chain Times-ing						2	

Discussion

One of the driving questions of this study was: What are students' routines with functions with which they are familiar, somewhat familiar, and not familiar? Based on the findings, there was less variety in these students' methods to solve tasks which contained familiar functions. The somewhat familiar functions created the largest number of methods. This large number of methods may have been due to previous (potentially negative) experiences with logarithmic functions whereas the not familiar functions may not have come with the same kind of anxiety.

Also shown was that students used different operations (composition, addition, or multiplication) when finding the derivative $h'(x)$ than they did for finding the original function $h(x) = (f \circ g)(x)$. A majority of the students used the "plug g into f" method to find *h(x),* but up to half of the participants used other operations for the remaining portion of the task.

These resfults have implications for the teaching and learning of the chain rule. The participants that were the most successful in attaining the correct answer were those that were flexible in their use of function composition. They were able to see a composite function and create another one similar to it. They could compose and decompose functions in multiple notations (e.g., function and Leibniz or circle and parentheses). Thus, mnemonics or other teaching tricks to remember the chain rule may not be as effective as encouraging deeper understanding of composing and decomposing. For example the Outside-Inside rule currently found in many the chain rule section of calculus textbooks could be related to composing and decomposing functions as opposed to simply a way to perform the chain rule.

Additionally this study has shown the usefulness of commognitive research. By paying close attention to what the participant says and does, one is able to see what is correct as well as what is wrong. For example the Chain Adding method would typically be quickly dismissed as completely incorrect once the plus sign appeared. Whereas it has been shown here that the three pieces of the chain rule *f'*, *g*, and *g'* are all there and that it was only the operation that was incorrect. Thus, students might not be closer to the correct answer than one might suspect.

Future studies could include an exponential function or another somewhat familiar function. It would be interesting to see if the relationships between familiar, somewhat familiar, and not familiar seen in this study hold with other functions. Additionally, determining a function with which students are not familiar but is more mathematical than the Flowers-Colors task could be useful. Tasks involving multiple representations could be designed to study the consistency of routines in different representations.

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CHARACTERISTICS OF FIVE INTERNATIONAL MATHEMATICAL OLYMPIAD WINNERS BASED ON KRUTETSKII'S FRAMEWORK

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A lack in fundamental understanding of the gifted is one of the causes that hinder blooming their potential. To attain better understanding of them this study investigated attributes of five former Olympians. A series of interviews with these individuals is used as the method of the research. The five elements of "readiness for an activity" presented by Krutetskii (1976) were used to indentify characteristics of the five Olympians including four general psychological conditions in addition to natural ability. All five Olympians displayed combinations of five elements of the readiness for an activity. Findings of the study suggested that the mathematically gifted without numerical evidences or of early ages may be identified be careful observations by their caretakers.

Introduction

There is no single strategy or formula to develop all students' abilities to their fullest, especially those who are highly gifted (Gallagher, 2006; Muratori, Stanley, Ng, Ng, Gross, Tao, & Tao, 2006). To serve these students successfully, first, proper identification should be conducted so that appropriate educational practices and environments can be created to develop students' potential to their fullest.

Although the use of multiple identification methods is strongly suggested (Coleman & Cross, 2003), common numerical methods such as standardized test scores, which may not reveal students' full potential, are still considered highly because of their convenience. Approaches to identify characteristics, not solely relying on test scores, of the gifted are imminently needed so that both students with manifested giftedness and those with latent potentials can be recognized. Krutetskii (1976) hypothesized and later confirmed that a child's successful mathematical development derives from a combination of qualities, including *a positive attitude toward mathematics*; *characteristic traits* such as diligence, persistence, and self-discipline; *a favorable mental condition to its implementation*; *a definite fund of knowledge, skill, and habit*; and *ability* as demonstrated in Figure 1.

While schools and gifted programs vary in how they identify and admit students, common admissions criteria include standardized test scores, teacher recommendations, school grades, and interviews. Another criterion is performance from competitions. In fact, performances in Olympiads are widely used to indicate a young person's potential, with the International Mathematical Olympiad (IMO) considered the most prestigious competition for mathematically gifted secondary students (Evered & Nayer, 2000; Karp & Vogeli, 2003). It is true that Olympians are very selective group of the mathematically gifted, however, their examples stand "as a benchmark for all gifted students" (Subotnik, Miserandino, & Olszewski-Kubilius, 1996).

Historically, Asian countries are among the top performers in international competitions. For instance, Korean teams have consistently demonstrated highly successful accomplishments in the IMOs, achieving 4th, 3rd, 3rd, and 5th places between 2005 and 2008, respectively. Other Asian countries such as China, Thailand, North Korea, and Japan have been among the top 10 countries in recent years. Therefore, to better understand the attributes of and influences on highly gifted students, this study investigates factors that helped in shaping these students' success so that more adequately developed systems of identification that take characteristics of mathematically gifted students into consideration can be created.

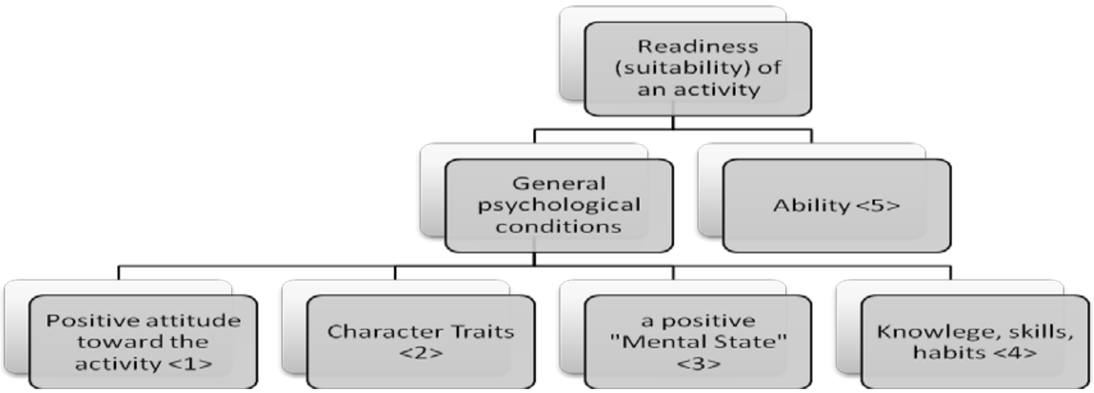

Figure 1. Suitability or Readiness for an Activity

Historically, Asian countries are among the top performers in international competitions. For instance, Korean teams have consistently demonstrated highly successful accomplishments in the IMOs, achieving 4^{th}, 3^{rd}, 3^{rd}, and 5^{th} places between 2005 and 2008, respectively. Other Asian countries such as China, Thailand, North Korea, and Japan have been among the top 10 countries in recent years. Therefore, to better understand the attributes of and influences on highly gifted students, this study investigates factors that helped in shaping these students' success so that more adequately developed systems of identification that take characteristics of mathematically gifted students into consideration can be created.

Several practical implications are expected from this study. First, an analysis of former IMO winners' characteristics based on Krutetskii's (1976) study will reinforce understanding of the mathematically gifted and provide resources to educators, school personnel, and parents. Second, educators in the fields of mathematics education and gifted education will be able to explore unique characteristics of former IMO winners and ascertain whether characteristics that Krutetskii outlined are also present in the Korean context in spite of cultural and educational differences. Third, the selection processes that specialized schools and programs are using (e.g., teacher recommendation, the use of achievement scores, and previous achievements) to identify and admit students will be diversified. Lastly, parents and teachers who are interested in their child's mathematical development will benefit in understanding such characteristics even if their child's achievement score does not reflect a high level.

Purpose of the Study

The purpose of this study is to examine characteristics of highly gifted individuals, in particular, Korean students who have participated in IMOs in the past. School administrators, mathematics educators, and parents can employ these findings to assist in the development of mathematical potential of their students and children and for early identification of their mathematical giftedness. This study will answer the research question: What are characteristics of Korean mathematically gifted IMO winners based on Krutetskii's (1976) readiness for an activity?

Theoretical Framework

Quite a few researchers have uncovered various characteristics of gifted individuals. Among them, this study is following models of three studies. First, Krutetskii's characterization of psychological conditions was the model to categorize what former IMO winners possess and exhibited. Krutetskii studied individual children through qualitative methods such as interviews and observation, and, in this study, the interview is the method of research. Secondly, a study of two mathematicians, 'former greatest child prodigies' as the article calls, to revealed that various distinctive characteristics and influential factors were closely associated with their talent developments (Muratori et al., 2006). In-depth interviews with four individuals – two mathematicians and their fathers – enriched the understanding of their different personal qualities and environmental factors. The last model that this study follows is Karp's (2003) study tracking former Olympians especially on the side to understand their perception in Olympiad experiences.

Krutetskii's report (1976) on 'readiness for an activity' organizes characteristics of the mathematically gifted into four general psychological conditions in addition to *ability*, which covers most other classifications of characteristics of this special population. He examined the development of children's inclination and interests; attitudes toward school subjects, especially in mathematics; and their character traits. After 12 years of experimental and non-experimental studies with 201 gifted children, he concluded that a student's success was derived from the combination of five characteristics (p. 74): (1) *an active, positive attitude toward the activity* and an interest in and an inclination to study it, which becomes passionate enthusiasm at a high level of development; (2) *character traits* that primarily include diligence, self-discipline, independence, clearness of purpose, persistence, as well as stable intellectual feelings (a feeling of satisfaction from intense mental work, joy in creation and discovery); (3) a positive *mental condition* toward its implementation such as winning for competitions or high achievement in assessments; (4) a definite collection of *knowledge, skill, and habits* in the appropriate field; and (5) *ability*, that is, specific individual psychological characteristics (Figure 1). The combination of these attributes determines the suitability or readiness for an activity.

Krutetskii (1976) separated ability from the other four general psychological conditions. According to Krutetskii, mathematical ability consists of three components: obtaining, processing, and retaining mathematical information. Remarking on the importance of these three components in building giftedness, Krutetskii stated, "These components are closely interrelated, influencing one another and forming in their aggregate a single integral system, a distinctive syndrome of mathematical giftedness, the mathematical cast of mind" (p. 351).

Research Method

To answer the research question, a series of interviews was conducted with five IMO participants and their parents. This study used a qualitative case study approach. Participants were selected through Snowball sampling (Gay & Airasian, 2003) started from existing information provided on the official IMO website (http://www.imo-official.org).

According to the IMO official website, there were120 Korean participants (some participants are counted more than once) since 1988, the first year of Korean team's appearance to IMO. Past participants' names taken from the IMO website were searched on web engines with a purpose to obtain their contact information. Initially, the email addresses of three Olympians were obtained. When the first contact was made with the three individuals explaining the purposes and description of the study, all three agreed to participate in the study. Through correspondences with these former Olympians by email, two other fellow Olympians were contacted and agreed to be involved in the study. Eventually, the sample consisted of these five former Olympians (Table 1). Moreover, three of the five participants agreed to have their parents contacted by the researcher and provided phone numbers. Contact was immediately made with all three parents, who were willing to spare time to be interviewed.

Table 1. Demographic Information of Participants

	A	B	C	D	E
Age	From 25 to 27 at the time of interviews				
Gender	Three females and two males				
Graduate School/Major	All in well-known graduate schools/ Four in mathematics and one in non-mathematics				
Undergraduate School/Major	All entered Seoul National University (SNU) and four graduated from SNU and one transferred to a college in the U.S.				
High School Type	Four attended science high schools and one public high school				
IMO participation and awards	Years from 1998 to 2001. Six Golds, one Silver, and one Bronze*				
Other	One competed at Putnam Mathematical competition				

Before the interview, the participants were asked to provide basic information about themselves and their educational background to prepare individualized interview questions. At the beginning of the initial meeting for the interview, each interviewee was informed about the intent of the study and signed the consent form. Interview questions were partially structured with open-ended questions and their order of presentation was determined, however, follow-up questions were asked during the interview, if necessary. Interviewees' responses were audio taped and immediately transcribed verbatim.

Three one-hour interviews were planned and conducted in Korea with each Olympian, with additional interviews scheduled if needed. During the first session, each interviewee was asked questions about their general background, such as educational and family experiences. After the initial meeting when the researcher and the interviewee became familiar with each other, the researcher posed more personal and in-depth questions in the following sessions. Additional interviews after the initial three hours were made with four of the five interviewees to gather all of the desired information. Further questions were asked depending on the circumstance to explore more detailed experiences of the interviewees. Several post-interview email corresponding were made to confirm and refine information gathered during interview sessions.

Parents were interviewed twice – the first time, on the phone, and the second, in-person – to verify or append the Olympians' statements and provide anecdotal evidences. A consent form was signed by each at the beginning of the interview as well. For further questions that might arise during analysis of their responses, all parents agreed to be contacted by phone or email to elaborate on their responses.

Interviews with a total of eight people (five IMO winners and three parents) took place in various locations of their convenience. Each interviewee consented for the interview to be audio recorded. Interviews were immediately transcribed in Korean. The transcripts were translated into English except for technical remarks such as names of schools they attended, time and places that interviews took place. Then, two independent coders analyzed the translated transcripts sentence by sentence with keywords noted. Next, statements related to themes based on keywords were selected across the interviews. Notes about facial and emotional expressions during the interviews were also reviewed so the feelings and emotions expressed by the interviewees would not be lost.

In reviewing the transcripts, each interviewee's responses were categorized based on Krutetskii's characteristics. Next, their responses were categorized into themes. In this way, the constant comparison approach was carried out, where interpretation focuses on patterns and perspectives of the participants. Each interviewee's response was coded based on each component/key word of Krutetskii's characteristics until there is no more new ideas to what were already found about a category, its properties, and its relationship to the core category. Next, their responses were categorized into themes, where interpretation focuses on the patterns and perspectives of the participants.

Credibility of findings and interpretation depends on "careful attention to establishing trustworthiness" (Glesne, 1999, p.151). To ensure trustworthiness of a study, one must spend sufficient time in the field and make detailed observations (Lincoln & Guba, 1995). In this study, both factors were successfully executed. First, three one-hour interviews with each IMO winner were conducted. By meeting multiple times with the researcher, the interviewees became comfortable and provided personal thoughts and experiences. Also, by corresponding via email for additional information enabled the researcher to gather and explore more the data in more detail. Moreover, parent participation authenticated information provided by the Olympians, making the findings more trustworthy. Second, the transcripts were reviewed multiple times on separate occasions with two independent coders that increased the reliability of the findings. Transcript data were carefully and continuously reviewed when they were re-organized by keywords and themes.

Results

Indications of Natural Intellectual Abilities

It was apparent that all five former Olympians displayed the presence of ability not limited to mathematics from early ages. Some of them showed the presence of abilities through accomplishments from mathematics contests, while others were through episodes that were provided by their parents.

Indications of Krutetskii's Psychological Conditions (Characteristics)

General psychological conditions or readiness for an activity that contribute to a student's success include characteristic traits (i.e., persistence, motivated and self-disciplined behaviors), a positive attitude toward the activity, a positive mental condition, knowledge, skills, study habits in the field, and the presence of ability (Krutetskii, 1976). Interviews with the five participants revealed the presence of combinations of these psychological conditions, which are described below.

Characteristics traits that Krutetskii presented were revealed through self-asserted, disciplined, goal-oriented, competitive, confidence and acquisitive behaviors of the five former Olympians. All five Olympians clearly displayed some of these traits through their lives. Despite of parental advices and observing what professions in pure science would be like, remaining firm in pursuing mathematics showed independence, clearness of purpose, and persistence in one Olympian. Being self-motivated and competitive was also evident in two Olympians when they were clear about their goals. Studying and preparing mathematics Olympiads was not easy when resources are scarce for many former Olympians, however, their self-disciplined nature, a feeling of satisfaction from intense mental work, and joy in creation and discovery were found to encourage and sustain them to reach the goal to win atIMO.

Knowledge, skills and good study habits were found among all five Olympians as well. In various occasions, these Olympians' knowledge shined through their school works, accomplishments and while helping others. Study habits and skills contributed to individuals' learning experiences and developing mathematical and academic advancement. Many Olympians not only enjoyed learning advanced mathematics but also looked for additional resources to learn more and loved reading them on their own.

Positive attitude toward mathematics and an interest in and an inclination to study mathematics, which becomes passionate enthusiasm at a higher level of development, was displayed by all five Olympians. Some of instances that uncovered positive attitudes toward mathematics were overcoming obstacles such as parental objection to study further in pure mathematics, reading and completing mathematics books for older pupils, returning back to mathematical competition after participating in science competitions, and completing assignments in an unexpectedly shorter period of time than older peers, and a presence of natural fondness in solving challenging problems. Fondness in mathematics of one Olympian was nourished through exploration of various academic fields such as computer science, philosophy, and others.

The fourth psychological condition, the positive mental condition favorable to its implementation was imbedded in four Olympians. Achievements from competitions served as a main implementation in these people. Also, tuition-exemption policies motivated some individuals to work harder. In general, goals or incentives were types of implementations that allowed these students to drive themselves to achieve highly.

Discussion and Conclusions

To create and provide an environment that gifted students are able to develop their potential to the maximum, it is vital to understand characteristics that they possess. Characteristics not only tell us how to identify such individuals, but also guide us to ways in how to educate and encourage them in a way that they respond to. Results from this study confirm that the five Korean IMO winners possess combinations of characteristics that Krutetskii (1976) found in mathematically gifted children in Russia. Thus, it will be a good guidance for educators to recognize hidden giftedness of a child without numerical indicators even though they were in different cultural contexts.

All of the former IMO winners hold characteristic attributes that Krutetskii (1976) found in his own study participants. Although each individual revealed different traits, all five former Olympians displayed persistence, self-discipline, self-assertiveness, competitiveness, confidence, and diligence throughout their life course. These characteristics are indicators of mathematical giftedness that teachers and parents can detect from careful observation and without holding expertise. Only when these traits are accurately recognized as a sign of mathematical giftedness, will underachieving mathematically gifted students be correctly identified.

First, all five participants displayed knowledge, skills, and good study habits, that are another strong gauge of mathematical giftedness. Krutetskii (1976) asserted that without having minimum knowledge, skills, and habits, a person might not be suited for mathematical activities no matter how great the person's mathematical ability may be. Undoubtedly, these former Olympians have shown their mathematical knowledge in their accomplishments in competitions and as reflected in their GPAs. Two participants referred to themselves as lucky to have possessed certain knowledge, which is extremely scarce to happen, in encountering familiar problems in competitions. Taking into consideration the small number of problems in these competitive exams (e.g., there are only six problems on the IMO and Final Korean Mathematical Olympiad), their so-called “luck” actually is reflective of their tremendous preparation and depth of knowledge.

Second, habits of learning and studying are another factor that distinguishes some of the Olympians from their peers. One interviewee with extreme anxiety in taking high-stake situations coped with it by being perfectly prepared, which helped build her confidence, and acquire a definite fund of knowledge so that she would not experience “bad luck” in forgetting what she had previously studied. Another Olympian also exhibited good study habits through hard work and effort. He showed high achievement in all subjects and significant mathematical knowledge by spending a large portion of his time after school studying mathematics. Therefore, teachers’ and parents’ observations of a child’s study habits and its consequences (e.g., knowledge acquisition) should not be disregarded simply as a personality trait but as a possible indicator of giftedness.

Third, a positive attitude towards mathematics or its achievement also points to mathematical giftedness. Some were fascinated by the “beauty” of mathematics and indulged in it by spending hours in solving challenging problems. For others, the goal was to do well in competitions and participate in summer camps where they could spend time with other similarly gifted peers, to enter prestigious colleges, to pursue free tuition programs, or to beat their own records they set previously. They invested time and energy, which they could have spent elsewhere, studying mathematics and relishing in it. Consequently, parents need to learn to recognize whether their child enjoys or dislikes studying mathematics because a successful test score does not necessarily indicate mathematical giftedness if the individual presents negative attributes in doing it.

Prevailing among all five Olympians was a display of mathematical ability from early ages. Some grasped mathematics information more quickly than others, including older peers and siblings. Others processed mathematical knowledge easily, reading mathematics books, solving higher level mathematics problems, and displaying quick calculation skills. Retaining mathematical information well was another aspect of their ability. While learning advanced mathematics at an early age, these students used what they previously learned as a stepping stone to learn at the next level. Without the retention of mathematical facts or concepts, it would be impossible for them to move on to the next advanced level of mathematics.

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DESIGN PRINCIPLES FOR THE WIDESPREAD USE OF DYNAMIC MATHEMATICS

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Although research has demonstrated the effectiveness of representational technologies in mathematics education, there have been barriers to broad use. Across multiple projects, we have refined an approach to curriculum development that lowers the barriers to the use of representational technologies—specifically through a dynamic math approach to mathematics learning—while ensuring that students can benefit from their unique affordances. In this paper we describe a set of principles for the design of units using a dynamic-math approach and discuss how they were applied to two different middle school units.

Introduction

Research has shown the effectiveness of using representational technologies in mathematics to scaffold and support student learning (Marzano, 1998; Mayer, 2005). However, there have been barriers to broad use, such as the perception that technology is too difficult to implement in diverse classrooms (Becker, 2000), and inconsistent findings on the benefits of educational technology in mathematics (Dynarski et al., 2007; National Mathematics Advisory Panel, 2008).

In this paper we describe our approach to curriculum development to help overcome these barriers and best ensure benefits to students who use the materials. We derive design principles based on lessons learned across multiple projects that relied extensively on representational technologies. We focus on two units using two different kinds of dynamic mathematics software and curricular approaches, both of which support student learning through use of technology-based representations: SimCalc Mathworlds™ and Geometer's Sketchpad™ (GSP). In designing the units we incorporated the perspectives of different stakeholders—students, teachers, and school districts— to minimize barriers to implementation and increase the chance of having the intervention be successful with a variety of teachers. We addressed teacher and district concerns about current policy demands (e.g., NCLB and accountability testing) and the need to meet local standards. We considered multiple teaching styles and designed materials so teachers with a wide variety of mathematical and technological backgrounds could use them. Through representational technologies and scaffolded curriculum, we met the cognitive, linguistic, and social needs of a diverse student population. At the heart of this approach is a refinement in our conceptualization of the use of innovative technology in the classroom.

Perspective

For more than 15 years, research in dynamic math in general and the SimCalc project in particular has had the goal of ensuring that all learners have the opportunity to learn complex and important mathematics. For SimCalc, this is expressed in the mission statement "democratizing access to the mathematics of change and variation" (Kaput, 1994). A foundational belief of the dynamic math movement is that reconceptualizing middle and high school mathematics can yield

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

a more coherent and fruitful mathematical experience for all learners, including those who have not traditionally been successful in mathematics (Kaput & Roschelle, 1997). Our work continues this tradition and attempts to remain faithful to the core principles of the tradition while incorporating new principles to allow for broader impacts in a wider variety of contexts.

Results from the Use of Our Dynamic Math Materials

A series of studies found our SimCalc unit to be successful in meeting the needs of a diverse set of students and teachers. Ninety-five seventh-grade teachers and their students across varying regions in Texas participated in a randomized controlled experiment in which they implemented a SimCalc-based 3-week replacement unit. An analysis of the results showed a large and significant main effect with an effect size of .8 (Roschelle et al., 2007; Roschelle et al. in review). This effect was robust across a diverse set of student demographics. Students who used the SimCalc materials outperformed students in the control condition regardless of gender, ethnicity[1], teacher-rated prior achievement, and poverty level[2] (Figure 1 below). In addition, an implementation study in Florida, called the SunBay Digital Math project, used the same materials but with no control group. The SunBay project replicated the gains found in the Texas study (Figure 2 below), again with learning gains across a wide variety of teachers and learners.

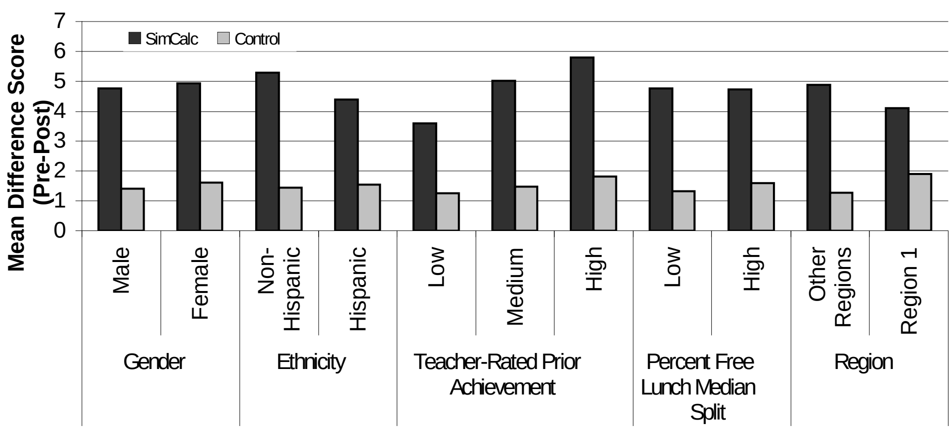

Figure 1. Mean student learning gains by subpopulation group

SimCalc Unit Content

The SimCalc-based curriculum unit addressed core state standards and also included topics that were more challenging than those in the standards. Beginning with simple analyses of motion at a constant speed, the unit followed a learning progression that culminated in the more complex topics. The unit addressed unit rate and proportional functions—topics from the seventh-grade Texas standards that are also core to Florida standards—and ended with multirate functions and the meaning of positive, negative, and zero slope, expressed informally. It underwent minor revisions for use in Florida, but the core principles underlying the design remained the same.

By combining paper materials with guiding questions and SimCalc MathWorlds software files, the units provided a structured exploration of algebraic representations through their connections to real-world topics. The students had opportunities to use various motions and other "accumulation" contexts (distance is accumulated as a runner moves along; money is accumulated when added at a given rate). The unit presented soccer players running races and team buses traveling from one town to another, and students were to find speeds and write stories to explain patterns of motion. Non-motion contexts included saving money when buying uniforms and predicting how much fuel vehicles would use, in miles per gallon. Knudsen (2010) contains a more complete description of the content of this SimCalc unit.

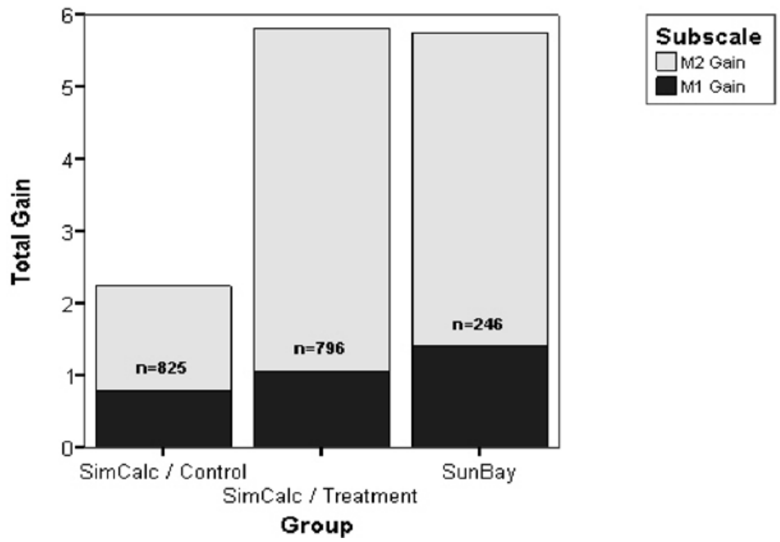

Figure 2. Florida SunBay results

Results

Many education research studies depend on curriculum as a vehicle for representing the researchers' ideas in the classroom. We note that the curriculum teachers use and how they use it are influenced by factors that extend beyond the classroom, yet rarely is attention paid to design principles that account for this wide range of factors. The design principles should allow a faithful instantiation of the theories, ideas, and innovations of researchers while also resulting in materials that are accessible to and easily usable by the studies' participants. The developers of the curriculum, then, have to pay attention to a number of contextual factors when writing curriculum and designing professional development.

Curriculum materials are used in settings with particular sets of *people*, *conventions*, *resources*, and *political considerations*. The minimal set of *people* to include is teachers and students. Teachers have a wide range of subject matter knowledge, comfort with technology, and teaching experiences. Students come to class with different languages and cultural traditions, as well as with levels of past success in mathematics. *Conventions* include the teaching practices expected in local classrooms and the relationship between teachers and parents. Relevant *resources* include available technology (or lack thereof), available support for the use of technology, release time for professional development, and even how many electrical outlets are in classrooms. The school, the district, the state, and the nation also provide *political considerations* that developers must take into account, including district policies and accountability expectations.

People: Teachers and Students

For our materials to be usable by a wide variety of teachers, they had to support a range of teacher math knowledge and a range of pedagogical styles and preferences. In the original Texas study, we addressed this variety in several ways, starting with the professional development (PD). Simple pedagogical routines were introduced and reinforced in the materials and PD, so that beginning teachers could simply establish the routine, and veterans could use it as a platform from which to improvise. The PD addressed basic and advanced content knowledge by having teachers study the math of the unit, the math that was prerequisite to the unit, and the math beyond the unit.

To support teachers as they were teaching the unit, we created the student workbook so that it could be used as a substitute lesson plan. This was based on our experience that most current teachers no longer create written lesson plans and typically teach from the student materials. In our materials, teachers could rely on the questions written in the student workbook as a template of key question and activities. The workbook, however, was not a prescriptive script. Experienced and knowledgeable teachers could easily use their own questions or use the teachers' notes to create their own lesson plan. For additional support, the teacher notes provided an outline of the activity flow for each lesson, extra questions to ask students, and sample student responses. To meet teachers' accountability needs, a list of standards addressed in each lesson was included. A suggested pacing chart was provided, and teachers also received forms they could use to write their own lesson plans.

These choices instantiate two key design principles. The first is that student materials meet teacher as well as student needs. The second is to design teacher notes to meet teacher needs without providing an overwhelming amount of information. For the SimCalc study, this meant including only what was necessary in the teacher notes. Today's teachers rarely have the luxury of significant planning time and so, by definition, do not spend much time reading teacher notes. Bulleted, highly relevant information is most likely to be read and used.

Teachers may have been the first audience for the SimCalc materials, but students were the primary audience. Features of the units were designed to address the needs of a wide variety of students. Each unit's theme provided continuity across the real-world contexts represented in the software simulations. Numbers used in these contexts were for the most part realistic, so that students could use their knowledge of speed and prices to gauge the correctness of their answers. The text used simple sentence structure and consistent vocabulary, never going beyond a fifth-grade reading level, in order to accommodate those with low-level reading skills and those learning English. The workbook used graphical conventions to indicate various kinds of activities and content; for example, definitions and other critical information appeared inside boxes on the page. The amount of white space provided with a question indicated the type and length of an expected answer. Even the fact that the workbook contained all the student activities physically bound together provided another organizational aid to students. Last, the workbooks were printed with as much color as the budget would allow to appeal to media-savvy students.

These choices instantiate the design principle that materials need to account for more than just mathematics content. This means adopting methods from special educators, from experts in second language acquisition, and others—as well as considering popular culture. Our units have been (and will continue to be) successively refined as we learn more about student needs and analyze student work with our materials.

Conventions

The SimCalc unit included a lesson on slope, connecting the lessons on steepness of lines quantified as rate to this highly related topic. As it turned out, teaching slope in the seventh grade was not conventional in Texas. The developers had mistakenly understood the opposite: that it was important to introduce slope connected to rate. Fewer teachers used this activity than any of the other activities in the unit. We inferred a design principle to be used in the future: Pay attention to local conventions while at the same time retaining all the topics important to the research project.

Access to Resources

Access to resources is always a concern when creating a dynamic math curriculum. Almost by definition, dynamic math activities are technology-based activities—where the technology could be computers, calculators, or even electronic whiteboards. Where these resources are scarce, the curriculum can compensate by providing plenty of activities not dependent on technology. It is essential, however, to ensure that non-technology activities contribute to the progression of content and are not just filler. We addressed this problem in two ways in the SimCalc unit: We created activities that leveraged technology where necessary, and we created practice materials that did not require technology. Whereas our learning activities supported student use of computers (and we strongly advocated student access to computers), we wrote our curriculum to be flexible enough to support learning in a wide variety of technical contexts, including one-to-one computer classrooms, classrooms in which members of a small group share a computer, and even where only the teacher had the opportunity to drive the technology using one computer at the front of the class.

The Political Context

Curriculum is used in a political context—at local, state, and national levels. At the national level, No Child Left Behind and other federal policies set a tone that encourages a less open-ended curriculum than had been popular in the previous decade. Current national policy requires that school-wide test scores increase at a rapid rate, and these scores are based on tests designed at the state level, so state-level standards become a critical influence on the curriculum. At the building level, political “with a small p” considerations can include a teacher’s need to comply with school policies by both producing good test scores and by keeping a fairly quiet classroom atmosphere.

The SimCalc unit was shaped by these political influences in several ways but most importantly by state standards. Not only did standards and standardized tests determine what was taught in the classroom, but also many teachers followed the pacing guides telling them what to teach each day, to make sure that they covered all the standards. These teachers needed special assistance to match their guides with the suggested timeline in the unit, and they were accommodated during training. The first part of the unit addressed content directly from the standards, focusing on rate as a proportional linear function. The second part of the unit went beyond state standards. A demarcation was clear in the materials and in the training. The beyond-standards content was vetted with experts in state and local policy to ensure that teachers would still be willing to use the materials, given pressures at the school level.

The general design principle to be derived here is to be aware of and respond to political considerations at all levels. Design decisions made in response to federal policy might support teachers at the school level—or not. The consequences need to be thoroughly considered. For example, some teachers may be monitored by district officials to make sure that their teaching methods are in line with local expectations. Replacing an activity that would require students walking about the classroom with one that accomplishes the same goal with students in their seats could help teachers stay in line with local policies and classroom behavior norms.

Applying Principles to and Deriving Principles from a Dynamic Geometry Unit

On the basis of the success of the SimCalc unit, we have designed and are in the process of implementing a geometry unit that uses Geometer's Sketchpad. This unit is of a shorter duration (1 week) than the SimCalc unit, is designed to be used in Florida, and covers different mathematical content. Despite these differences, however, it is based on the same design principles discussed above.

Content. As in the SimCalc unit, the content of the geometry unit was designed to address core standards while also including topics more challenging than those in the standards. We determined that the most appropriate focus of the geometry unit would be geometric similarity, an aspect of proportionality that is in the Florida standards but was not addressed in the SimCalc unit. We again began with simple analyses and moved to more complex topics. In this case, we started with informal notions of similarity and followed a progression that ended in formal definitions of similarity that applied to all figures.

We again combined paper materials with guiding questions and different contexts. Early in the unit, students analyze images of the Statue of Liberty to investigate how nonsimilar images look warped or distorted. Later in the unit, they analyze images of currency (all are rectangular bills) and again determine which images are distorted and which are scaled proportionally. As a result of this activity, they are introduced to the notion that similar rectangles have the same height to width ratio. Students are presented with quadrilaterals with corresponding sides with equivalent ratios but without congruent corresponding angles. Students then are guided to revise their working definition of similarity to address polygons' angles. Finally, students are presented with an activity to consider non-polygonal figures and introduced to a definition of similarity using dilation.

Design principles: people. The new unit is designed to meet particular needs of teachers and students**.** Teachers can use the student materials as a substitute lesson plan. As in our previous materials, the student guide provides a template for student activities and teacher questioning. We have also included teacher notes that contain additional questions for students and expected student answers. To meet the needs of students, we again use realistic contexts and numbers whenever possible and use the same formatting rules as in the SimCalc-based unit.

Design principles: conventions Dynamic geometry software and curriculum afford students with opportunities to make geometric constructions, creating and modifying software files. During the implementations of the SimCalc unit in Florida, however, we found that many teachers were uncomfortable with students altering software files. The local convention seemed to be more of a plug-and-play model of using the software. When our analysis of existing GSP lessons on similarity revealed that the lessons assumed a high degree of student fluency with the software, we knew that we would need to design new activities where students could explore the mathematical representations without requiring the fluency required to build such representations on their own.

Design principles: access to resources. As in our SimCalc unit, we again use technology judiciously to exploit the unique affordances of representational technology. As a result, about half our activities require technology, and half do not. In addition, we have found a significant difference in those resources available in the Texas SimCalc study and those now available in the SunBay study: the introduction of small-screen NetBooks. To ensure that our materials work with the resources available to our teachers, we are now creating new file sets that work on the smaller screens on NetBooks.

Design principles: politics and standards. We found that our SimCalc unit, which was originally created for Texas, only required a small number of changes to meet key Florida standards. In particular, our SimCalc unit was aligned with Florida's Next Generation Sunshine State Standards' *Big Idea 1: Develop an understanding of and apply proportionality, including similarity*. Although several possible Florida standards would have been appropriate to address using GSP, we determined that the most effective use of GSP was to continue to address Big Idea 1 and specifically the topic of geometric similarity. In so doing we created a set of materials with a consistent goal and that focus on one core aspect of Florida's standards. On the other hand, the expectations within the standards focus on the application of similarity to find missing numbers, while our unit helps students devise increasingly sophisticated versions of the definition of similarity. This led us to pose the unit as a supplementary unit, to be used after the state tests are administered and to be used in addition to textbook chapters on similarity, rather than to replace them. This is an example of fulfilling local political requirements while still staying true to dynamic math ideas.

Conclusion

This paper described several aspects of the curriculum used in two experiments using a dynamic math approach with a wide variety of teachers and students. Paper materials and software served to guide students in an exploration of real-world contexts and associated mathematics representations, focusing in one unit on rate, proportionality, and linear function and in another on definitions of similarity. Developers of the unit took into account not only the mathematics that could be learned using a SimCalc approach, but also a set of other constraints on the curriculum. The developers addressed then-current political and social considerations, mostly about the dominant influence of state standards and assessments on instruction. Teachers were supported in their classroom use of the materials through a set of teacher notes and professional development that focused on teachers' mathematics learning and effective implementation of the unit. Special considerations were made for the needs of teachers and students, including meeting local conventions.

Endnotes

1. This paper is based on work supported by the National Science Foundation (Grant number 0455868), by the Helios Education Foundation, Pinellas Country Schools, and by the Pinellas Education Foundation. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation, the Helios Education Foundation, Pinellas Country Schools, or the Pinellas Education Foundation.

2. We focus on Hispanic students because they consisted of a majority of our student sample, numbers of other minority groups in the study were negligible, and Hispanic students have traditionally underperformed in measures of mathematics achievement (Education Trust, 2003).

3. We take as our measure of poverty the percentage of the campus eligibility for the free and reduced-price lunch program.

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THE KILLER PROBLEM

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In the spirit of a series of studies by Selden and other collaborators (Selden, Mason & Selden, 1989, Selden, Selden & Mason, 1994, and Selden, Selden, Hauk & Mason, 2000), an analysis of student performance on a simple trigonometry problem shows it to be void of predictability and trends. We extend the analysis given by Selden et al to explain the phenomenon.

Introduction

In traditional pre-calculus courses like trigonometry and algebra, teachers must assess their students and, in doing so, it may be in their standard practice to develop problems that incorporate a variety of conceptual tasks. The number and the type of conceptual tasks involved in solving a problem will generally contribute to its perceived difficulty, and the most difficult problems will involve a number of these tasks which would be considered to be abstract. More abstract problems might come later in a course (and course of study), whereas those with fewer, less abstract concepts would come on assessments earlier in these courses. In Math 128: Trigonometry at West Virginia University, four regular midterm examinations are given throughout the term. The problems on the first examination largely review algebraic concepts and test the basic, simple concepts of trigonometry. By the second examination, students have been exposed to the unit circle – as we refer to it, the sines and cosines of the special angles between 0 and 2π radians – and have been using this information in problems for several weeks.

By the time of the second examination of the semester, the unit circle information is used regularly in lecture, appears regularly on the daily homework assignments, is referred to regularly in laboratory experiences, and, as expected, is embedded into many of the examination problems. Because of this level of exposure to unit circle information, performance on a problem that involves nothing more than recalling the basic tenets of unit circle trigonometry, combined with simple algebraic simplification, would be expected to be quite good. Contrary to this expectation, data collected from several terms of math 128 that includes more than 4500 attempts at one such problem shows uniformly poor performance. At WVU, we refer to this problem as the "killer problem". This report begins an attempt to explain the killer problem phenomenon.

History

By 2006, math 128 at WVU had become deeply entrenched within the framework of a subset of the mathematics department known internally as the Institute of Math Learning (IML). The mission of the IML incorporates many components with regard to teaching, research and service, but on the surface, students of an IML course can expect to see a myriad of technological advances like applet-based interactive labs, online assessments, and the electronic personal response system, all of which are intended to supplement the teaching of courses in large lectures. In the case of math 128, every one of these technologies had already been in place, and so the maximum enrollment is generally equal to the capacity of the auditorium in which the course is held. In recent semesters that capacity has been on the order of 225 students.

With online assessments comes the vast archival of assessment data, and trends typically noted and perhaps discussed in classroom building hallways can be explored and analyzed. Such is the case with the killer problem, an online problem embedded in the bank of examination problems used over several years. The killer problem involves little actual trigonometry and some basic algebraic simplification and so it is often expected that students will do generally well on it. Yet, the problem has produced uniformly poor results from students, hence its moniker and reputation. In every case between the fall semester of 2007 and the fall semester of 2009, the classes of students exposed to their first attempt at the killer problem earned less than 68% of the total points available to them, and this statistic is routinely on the order of 30% to 50%. In a tally of all 4583 online attempts at the killer problem over this period, only 53.2% of the points available to students were awarded.

Solving the killer problem involves an understanding of basic unit circle information and algebraic manipulation but the skills and knowledge required to solve it are essential. Because of this and the generally dismal results, the problem deserves to be examined. Is the killer problem phenomenon an issue that students have with algebra? Is the issue trigonometry? Or, is there a deeper explanation?

The Killer Problem

Because of its poor results, the killer problem is regularly assigned to several course examinations in any given term. It made twenty-three appearances on exams between the fall 2007 and fall 2009 terms. The counter-intuitive results have stood the test of time and variability. Four different instructors have taught the course at various times in this period in 6– and 15–week terms with a number of different textbooks using several formats of online and traditional homework platforms. Aside from one change in format in the killer problem itself which we discuss later, the correct solution to the killer problem has consistently proved to be elusive to students.

The actual title of the killer problem is *Linear Combination of Trigs*, and at least five versions of the problem exist in the bank of online assessment problems. For the sake of efficient communication, we focus on one of those versions, which looks like this:

> *Simplify* $4\sin\frac{2\pi}{3} - 8\sin\frac{\pi}{6}$ *so that it is expressed in the form* $A + B\sqrt{C}$, *where A, B, and C are all integers (NO DECIMALS).*
> *What is A?*___
> *What is B?* ___
> *What is C?* ___

The format of the killer problem is what makes it unique. If the original author (who is unknown) of the problem was interested in knowing only whether a student can use a calculator to compute $4\sin\frac{2\pi}{3} - 8\sin\frac{\pi}{6}$, then the problem would simply ask, "*What is* $4\sin\frac{2\pi}{3} - 8\sin\frac{\pi}{6}$?" In that case, a single solution field would have been sufficient and students would type in a numerical solution. This is because a student equipped with a calculator in radian measure mode can type in the expression *4*sin(2*pi/3)-8*sin(pi/6)* and get the decimal approximation, –0.5359. Instead, the killer problem's unorthodox method of reporting its solution eliminates the usefulness of all but a few advanced calculators without prohibiting their use for the assessment at large. Because of the format of the problem, a student must know that $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, that $\sin\frac{\pi}{6} = \frac{1}{2}$ and then be able to algebraically simplify the expression, arriving ultimately at $2\sqrt{3} - 4$.

If the problem had asked for a single numerical solution, then students with very little understanding at all can arrive at the correct solution. The killer problem intends to assess knowledge beyond this.

Tucker (1996) reports that calculators had been widely used in introductory undergraduate mathematics courses by the mid 1990’s. Regarding precalculus courses such as trigonometry, Quesada & Maxwell (1994) confirmed their merit by showing that students who were taught precalculus with the use of a graphing calculator scored higher on a comprehensive final exam than their counterparts who did not. Thus, the IML’s mission promotes the role of technology and, in particular, the use of graphing calculators. Still, many are concerned that technology can interfere with conceptual understanding and so this tactic of eliminating the usefulness of the calculator is implemented often in math 128 to span the dichotomy.

The killer problem asks for three numbers and so partial credit is given for any correct value reported. For example, in the version given above, the solutions are $A = -4$, $B = 2$, and $C = 3$, and so each of these values is worth one-third of the number of points allotted for the killer problem on the examination on which it occurs. For example, if the problem was given a weight of 12 points for an examination, then a student giving the solution $A = 4$, $B = 2$, and $C = 3$ would receive 8 points for getting B and C correct. Partial credit for each blank in the problem is binary; students are either given the points for a value or they are not.

Data and Evolution

The data below show the success rates for students on the killer problem in the terms for which data is available. For any particular term, the number in the column on the right gives the portion of the total points awarded for all attempts made at the problem (the number of points awarded for the killer problem varies by exam and term, and so portions must be used for comparisons). The statistic in the last column corresponds to two different point distributions. As an example, if all students got one of the three numbers in the killer problem correct and two incorrect, the percentage would be 33.3%. Alternatively, if one-third of the students got full credit and two-thirds of the students got no credit at all, the percentage in this case would also be 33.3%. Obviously both distributions are at work in these data.

In Table 1, a major change in the killer problem goes unnoticed. Because of the poor student performance, a hand-holding measure was introduced in the spring 2009 term. In the original version of the problem, many students failed to enter integers and instead entered non-integer numbers like 3.5 or irrational or nonsensical responses such as the syntactical “sqrt(3)” and so it appeared that failing to accurately follow instructions may have been responsible for a portion of the incorrect responses. This happened despite clear language in the problem specifying that all values should be integers. Those responses are clearly wrong and indicate meaningful information to the instructor, but in order to assist our students, the problem was reformatted to eliminate this type of incorrect response. The actual problem did not change, but the response format did. Instead of three blanks in which students could type anything, three drop-down menus appeared so that the student was presented with a limited number of options, from which they were to choose A, B, and C. Now, the problem looked like this:

$4\sin\frac{2\pi}{3} - 8\sin\frac{\pi}{6} = A + B\sqrt{C}$. *Select A, B and C from the menus below.*

A

B

C

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

Table 1. Descriptive Statistics on the Killer Problem, 2007 – 2009

Term	Examination	N	Average of total points awarded on the Killer Problem
Fall 2007	Exam 2	318	30.1%
Fall 2007	Final Exam	282	60.5%
Spring 2008	Exam 2	294	29.4%
Spring 2008	Final Exam	270	54.1%
Summer 2008 – I	Exam 2	41	30.1%
Summer 2008 – I	Final Exam	37	71.2%
Summer 2008 – II	Exam 2	45	54.8%
Summer 2008 – II	Final Exam	44	53.0%
Fall 2008	Exam 2	318	38.9%
Fall 2008	Final Exam	292	44.1%
Spring 2009	Exam 1	282*	66.8%
Spring 2009	Exam 2	420	36.0%
Spring 2009	Final Exam	403	74.7%
Summer 2009 – I	Exam 2	108**	67.6%
Summer 2009 – I	Exam 3	54	59.3%
Summer 2009 – I	Final Exam	54	92.0%
Summer 2009 – II	Exam 2	80**	51.25%
Summer 2009 – II	Exam 3	40	55.00%
Summer 2009 – II	Final Exam	40	84.17%
Fall 2009	Exam 3	402	49.8%
Fall 2009	Exam 4	381	6.12%
Fall 2009	Final Exam	378	78.8%

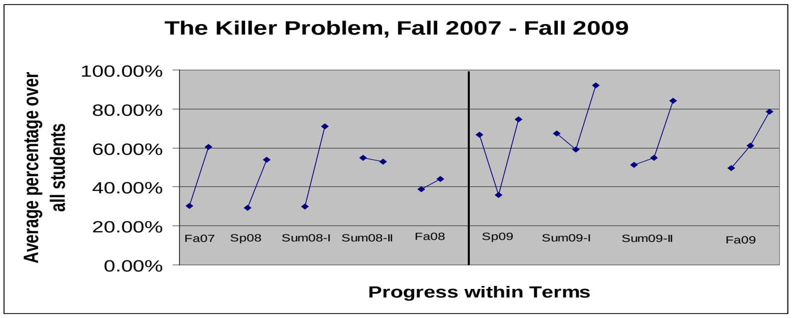
F.I.T.B. = "Fill in the blank"

* *For Exam 1 in the Spring 2009 term, an ambiguity was discovered in a new version of the problem and so data for that version has been removed for analysis.*

** *In both summer 2009 terms, two versions appeared on each student's Exam 2 assessment.*

The new format of the killer problem gave the same drop-down menu options for each of *A*, *B*, and *C*: the integers from -10 to 10. By doing this, students were now at least forced to follow the instructions of the problem. Integers were the only options, and this new format was scribed in time for spring 2009 examinations.

Pigeonholed by the drop-down menus, students did the best ever on an initial attempt (67.6%) at the newly formatted killer problem only to do dramatically worse (36.0%) on the next attempt in the same semester. From there, the killer problem continued to live up to its name with poor results. The chart below shows the success rates reported in table 1 with a vertical line separating the fill-in-the-blank format and the new drop-down format. It is clear that the new format had little, if any, effect.

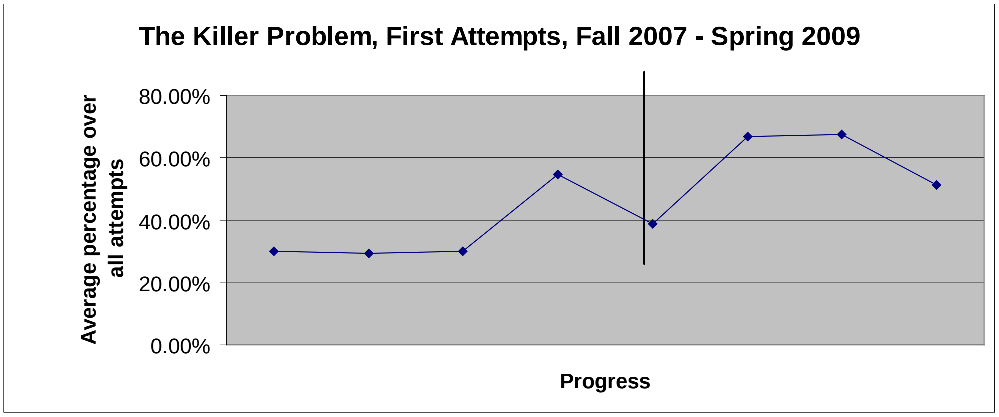


Motivation

In their famous series of studies, Selden et al studied the phenomenon of non-routine problems in calculus courses and came to the conclusion that students who pass calculus (those who earned the grades A, B, or C) are generally unable to solve these problems (Selden, Mason & Selden, 1989, Selden, Selden & Mason, 1994, and Selden, Selden, Hauk & Mason, 2000). A *non-routine* problem, as defined by the researchers, is defined to be one "for which they had not been taught a method of solution" (1994, p. 19).

A student's first attempt at the killer problem fits the definition the Selden studies use for non-routine problem because the problem does not appear on any prior assessments, is not taught in class, and had not been attempted by the student until that time. There are several reasons that it is not seen prior to the examination on which it first appears. First, the unorthodox format is not necessary outside an electronic testing environment. Secondly, there are too many different types of problems to cover them all as in-class examples. After the killer problem's first appearance, there are typically questions about it by students which then may lead to an in-class algorithmic explanation, disqualifying latter attempts from fitting Selden's definition of non-routine. For this reason, intuition borne out of the Selden studies would indicate that the results seen on the first attempt of the killer problem should in fact be expected. And, as it turns out, this intuition is accurate.

To see this more convincingly, it is helpful to separate the first attempts at the killer problem from latter attempts. Success rates, even at their best, are lower than one might expect given the simple nature of the problem. This view does, however, illuminate that some increase in student performance occurred after the change in format, but the overall success rate is still under 70% in all cases.



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A Deeper Look

A few basic questions are essential to understanding students' difficulties when confronted with the killer problem or any non-routine problem. Primarily, it is important to know whether students are fluent with the arithmetic and algebraic operations underlying the simplification of

(1) $4\sin\frac{2\pi}{3} - 8\sin\frac{\pi}{6}$.

Would students have faired similarly well if the problem examined here had asked for *A*, *B*, and *C* in

(2) $4\left(\frac{\sqrt{3}}{2}\right) - 8\left(\frac{1}{2}\right) = A + B\sqrt{C}$?

One would imagine that a majority of students in trigonometry can correctly solve (2) because they either passed or were placed above algebra. Secondarily, is it that students' difficulties with the killer problem pertain to knowledge of trigonometry? Mathematically, the only difference between the expressions in (1) and (2) is the conversion of $\sin\frac{2\pi}{3}$ and $\sin\frac{\pi}{6}$ to their exact values, and so a natural step is to assess a specific sample group of students, asking them to simplify $4\left(\frac{\sqrt{3}}{2}\right) - 8\left(\frac{1}{2}\right)$ and to transform $\sin\frac{2\pi}{3}$ and $\sin\frac{\pi}{6}$ in separate problems. Student performance upon these tasks may expose deficiencies in Quantitative Knowledge (Gq) (Horn & Noll, 1997; McGrew, 2005, 2009) related to trigonometry or in algebra. The subset of students of specific interest to this inquiry are those who possess the Gq to simplify the algebraic expression and the ability to transform the transcendental expressions but who fail to solve the killer problem in its initial form, regardless of those abilities.

To be clear, solving (1) demands that students first recognize the necessity of performing two simple transformations using unit circle information and then to make those transformations, which are $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$. (Note that for $\sin\frac{2\pi}{3}$, students are likely to implement a two-step process by first computing the sine of the reference angle $\frac{\pi}{3}$ and then assessing its sign – positive or negative.) Thus, the difference between solving $4\left(\frac{\sqrt{3}}{2}\right) - 8\left(\frac{1}{2}\right)$ and solving $4\sin\frac{2\pi}{3} - 8\sin\frac{\pi}{6}$ (from a mathematical perspective) is solely confined to the concepts of the unit circle and leads to an important conclusion about the solver's trigonometric knowledge.

If resolving (1) is not a matter of Gq pertaining to the trigonometric or algebraic functions, then is the issue not one of mathematics at all? The deficit these students are manifesting may not be related to their core understanding of the basic principles of unit circle trigonometry. Rather, a more plausible explanation in the face of any accumulated evidence is that the observed functional deficiency is related to students Quantitative Reasoning (Gr) ability rather than Gq. Gr is a sub-domain of Fluid Reasoning (Gf) (Horn & Noll, 1997; McGrew, 2005, 2009). Fluid Reasoning is a higher order cognitive function referring to the ability to execute deductive and inductive thinking. Reasoning inductively means employing strategies that work from the part to the whole, from the specific to the general, or from the individual instance to the universal principle. Reasoning deductively is the opposite; conclusions are derived from the general or universal to the specific.

Based upon the understandings of the cognitive processes at play, a reasonable explanation of students' failure to resolve (1) is that the Gr process breakdown occurs in relation to the inductive rather than deductive aspects of reasoning. When students are confronted with parts of this multi-step problem they are unable to reason from the parts to grasp the universal principle.

This inductive failure requires additional explanation. As discussed above, the targeted defect is unlikely to be one of Gq. In other words, if cued we imagine these students will be capable of correctly recalling from their memory stores (Glr) (Horn & Noll, 1997; McGrew, 2005, 2009) the appropriate algorithm to execute a transformation of a trigonometric expression (in Piagetian (1928) terms the Gq could be thought of as a *scheme* or in Selden's terminology a *problem situation image*). In general this seems to be the case based upon examination problems that specifically ask problems like $\sin\frac{2\pi}{3} = \pm\frac{\sqrt{X}}{Y}$ and $\sin\frac{\pi}{6} = \pm\frac{X}{Y}$. The components that comprise the ability to make an inductive leap include

1. the recognition of the need to apply a specific element of Gq (the correct scheme),
2. fluent recall of the Gq, and
3. the Gq itself.

So the question remains, what is involved in recognizing the need to apply a specific Gq strategy? Or to restate the question, how is it that from the parts of (1) a student could perceive the expression in a context whereby the process of solving the killer problem is evident? To apprehend that a particular solution avenue is correct necessitates a glimpse of the big picture (an inductive process). The successful student executes this inductive leap perhaps because they can *read* the meaning in visual-spatial terms; they can quite literally see what is being asked.

The implied requisite cognitive abilities then include inductive Quantitative Reasoning (Gr), visual-spatial thinking (Gv), and long term memory (Glr). Though it may seem counter intuitive we hypothesize that working memory (Gsm) contributes little to no variance predicting the difference between those students who can and those who cannot solve the killer problem.

In a way, the killer problem in its simplicity presents a scenario that in the future can be more easily targeted by the assessment of cognitive functions than those used by Selden et al. Despite subtle differences, the fact remains that the killer problem presents to students a basic problem that they cannot reliably solve.

We conjecture that the analysis taken by Selden is robust enough to be applicable to a basic problem such as the killer problem but that the cognitive explanation offered by Selden et al should be extended. The *problem situation image* is defined in the third Selden report (2000) with the following statement. "*When a problem situation is recognized, most of the features in its image do not immediately come to mind. i.e. into consciousness. Rather they seem to be partly activated"* (p. 145). Our reading of the idea that "*the features in its image do not immediately come to mind… they seem to be partly activated"* is consistent with our suggestion that students' failure to solve (1) is a failure of inductive Gr. The similarities between our account of the cognitive process involved and Selden's account pertain to the inability of students to appreciate the problem's gestalt though the differences are related to our various understandings of the cognitive origin of that failure. Selden seems to understand the students' deficiencies in recall of long term memory stores by suggesting that students' difficulties are attributable to an inability to sufficiently activate retrieval of Gq. Rather than a problem with long term memory, we suggest that neither the language of Baddeley's (1996) central executive nor of Norman and Shallice's (1986) Supervisory Attentional System accurately portrays the cognitive failure at work for these students. The issue in our view is not simply a problem of retrieval of a Gq scheme from long term memory, but rather a problem of interpreting and understand the semantic meaning of the expression (1) in a context whereby the process of solving the killer problem is conducted in a visual-spatial framework so as to trigger or cue (activate) retrieval of the appropriate Gq scheme.

As in non-routine problems, the killer problem strays from a student's knowledge base due to its format and the fact that a linear combination of basic trigonometric expressions may not have been introduced before. Continuing with Selden's argument, it is reasonable to suggest that students reach into this knowledge base for *tentative solution starts*, or "*tentative general ideas for beginning the process of finding a solution*" (2000, p. 145). Only an analysis of the work done in student attempts of the killer problem can confirm this suggestion. Unfortunately, because the killer problem is completed online, no archived data exists aside from the response given by each student. Future research might incorporate hard copy work done in attempting the killer problem, perhaps along with interview protocols to more specifically retrieve information.

To articulate a summary, we offer a simplifying analogy. Selden's analysis suggests that students are for some reason or another unable to access the "folder" that contains the knowledge necessary to solve the killer problem, going to a filing cabinet of familiar mathematical facts and algorithms and rifling through them. We suggest that the students do not initially go to the filing cabinet; they look at the problem for something to tell them which folder to go to. The students are missing the appropriate cues that tell them, "it's in this folder."

One important difference between the killer problem and Selden's non-routine problems lies in the subsequent attempts that were made by the students. If a problem is no longer non-routine after it has been seen by a student, then the consistently poor performance on later attempts of the killer problem remains unexplained. Future analysis may discern these attempts, find trends, and uncover why, after repeated attempts, students cannot solve the killer problem.

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AN EXPLORATION OF FACTORS THAT INFLUENCE STUDENT ACHIEVEMENT IN DIFFERENTIAL EQUATIONS

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This study is part of a larger curriculum comparison between two versions of a service course for engineering and physical science majors on differential equations (Baker, n.d. Boyce & DiPrima, 2009). The research questions stem from attempts to revise differential equations courses, and service courses in general, to be more relevant to non-math majors and to be in better alignment with the expectations of client departments. Thus, this investigation was oriented toward discovering what nonmajor students and faculty consider relevant. Of particular interest is whether the students, and their engineering professors perceive the content of their mathematics classes as related to the content in their chosen majors (Schoenfeld, 1989): (1) How do students in these courses perceive the role of mathematics, and the utility of differential equations in particular, in their majors? (2) How do engineering faculty characterize their own goals for a course on differential equations?

The study involved two lectures of engineering and physical science majors. The most common majors enrolled in the courses were Mechanical Engineering and Electrical/Computer Engineering and thus chosen as the target majors. Engineering faculty were invited to participate in semi-structured interviews to elicit their views the role of differential equations in their classrooms. The following data were collected: (1) a Likert-type questionnaire administered to ascertain student attitudes toward the utility of differential equations, and (2) interviews conducted with engineering faculty to assess the needs of the client departments and their expectations for the mathematics instruction of their students (Bingolbali & Monaghan, 2008). Questionnaires were administered to students at the beginning of the course and interviews with engineering faculty were conducted throughout the quarter. The questionnaires were analyzed using quantitative methods while the faculty interview transcripts were analyzed through the method of constant comparison.

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CREATING MEANING FOR VECTOR EQUATIONS: SYMBOLIZING IN A CLASSROOM COMMUNITY OF PRACTICE

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Symbolizing in mathematics is an important part of learning to do mathematics and is an important focus of analysis for mathematics education researchers (Arcavi, 1994; Duval, 2008). In the context of linear algebra, researchers have analyzed how students think about various symbolic representations involved in understanding systems of linear equations and vector equations (Sierpinska, 2001; Stewart & Thomas, 2007). These studies have focused on how individual students interpret the formal symbolic systems of linear algebra. Extending this work on how individuals construe meaning for symbolic systems, I focus on how a classroom community negotiates meaning for the symbolic forms of vectors and vector equations. As emphasized by Wenger (1998) the production of knowledge is a process of developing a collective experience of an object of scrutiny. In this particular case, the objects of scrutiny are vectors and vector equations. The presentation will discuss data collected from a recently completed classroom teaching experiment (Cobb, 2000) in linear algebra. Specifically, I focus on how the classroom community interpreted vectors and vector equations in relation to the central ideas of span, linear dependence /independence, and eigen-theory. In addition, I examine how the classroom negotiated between the geometric representations, systems of equations, and equations in vector form. The analysis considers verbal argumentation, gesture, and symbolizing that are prevalent in the classroom community. The significance of this work is that it illuminates the diversity of meanings that can be produced by a classroom community.

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GUIDING STUDENTS TO AN UNDERSTANDING OF THE FORMAL DEFINITION OF LIMIT

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While limits are foundational to the central concepts of calculus, our experiences with students as well as educational research abound with examples of students' alternative conceptions about limits and infinity. Furthermore, most students struggle to make sense of the formal ε-*N* definition of the limit of a sequence. For the purposes of this study, we designed a series of lessons and a collection of dynamic sketches using *The Geometer's Sketchpad* which we hoped would enable Calculus I students to value the formal limit concept and to construct the formal syYmbolic definition on their own. Rather than presenting students with the definition of limit at the outset, we used Heid's (1988) approach of allowing concepts to develop first then attaching the symbolic definitions to the experiences of the students' investigations. The lessons began by having students brainstorm about the colloquial meanings of the word "limit." Students then investigated dynamic sketches of eight carefully-chosen sequences, using their intuitive ideas to decide if each sequence appeared to have a limit or not. After an in-depth class discussion, students were asked to evaluate the adequacy of possible informal definitions for the limit of a sequence. As students played the ε-N game using dynamic sketches of a variety of sequences, students attempted expressing the formal limit concept using both their own words and using formal symbolism.

We examined 17 college-level Calculus I students' initial intuitive conceptions of limit and then investigated how their conceptions changed over the course of the lessons. Data were collected through a pre-test evaluating students' initial understanding of limits, students' written work, recordings of class discussion and individual interviews with six specific students, and a post-test. Analysis was conducted within the framework of conceptual change theory (Smith et al., 1993). Students initially held alternative conceptions of limit cited in the literature, such as the *limit as unreachable* conception, the *limit as boundary* conception, the notion that a constant sequence does not have a limit, or the idea that a sequence has a last term. Interestingly, intuitive ideas about cluster points and subsequences also arose. As the study progressed, students' mental repertoire of sequences enlarged, thus allowing students to recognize the unproductiveness of certain conceptions in certain contexts. By the end of the study, many students could successfully explain the ε-N game in their own words, could express the formal concept symbolically, and could apply the formal limit concept graphically to conjecture if a sequence had a limit.

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INDIVIDUAL AND COLLECTIVE ANALYSES OF THE GENESIS OF STUDENT REASONING REGARDING THE INVERTIBLE MATRIX THEOREM

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This poster presentation will be a summary of my dissertation, which has two aspects: (a) research into the learning and teaching of linear algebra, and (b) research into analyzing the development of mathematical meaning for both students and the classroom over time. Specifically, I analyzed how students—both individually and collectively—reasoned about and with the Invertible Matrix Theorem (IMT) over time. The IMT consists of over seventeen equivalent statements for n x n matrices, and these statements encapsulate the fundamental ideas of linear algebra that are developed over the duration of the course. Of methodological interest, I developed a way to coordinate the analytical tools of adjacency matrices and Toulmin's (1969) model of argumentation at given instances during the semester as well as over time. Synthesis and elaboration of these analyses was facilitated by the notion of microgenetic and ontogenetic analysis (Saxe, 2002) and an approach for documenting classroom mathematics practices (Rasmussen & Stephan, 2008). Finally, a coordination of both the microgenetic and ontogenetic progressions documented by adjacency matrices and Toulmin was carried out in order to illuminate the strengths and limitations of utilizing both of these analytical tools in parallel on the given data set.

The data for this study came from a semester-long classroom teaching experiment conducted in an inquiry-oriented linear algebra course at a large university in the southwestern United States. The main data sources were video and transcript of whole class and small group discussion, as well as individual interviews with the five focus students for the individual component. My analysis revealed rich depictions of the ways in which students reasoned about and with the IMT that was not apparent through use of only one of the analytical tools. Adjacency matrices proved an effective analytical tool on arguments consisting of multiple connections that were for explanation, whereas Toulmin models proved illuminating for arguments with complex structure for the purposes of conviction. These and other results, as well as my methodological approach, will be discussed in the poster presentation.

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REAL NUMBERS AND MATHEMATICAL THINKING

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Although it is universally agreed upon that a solid understanding of rational numbers is necessary for quantitative literacy (QL), little attention has been paid to the continuum both in school mathematics and in the research. Although, irrational numbers may not be necessary for the advancement of QL directly, a more complete understanding of the real numbers is important to the extent that incomplete knowledge of same may have some impact on the learners understanding of the rational numbers. The goal of this study is therefore, to explore pre-service teacher understanding and or misunderstanding of the real numbers. The motivation for this study may be attributed to work by Fischbein, Jehiam and Cohen (1995), involving the exploration of learner formal knowledge of the real numbers, real number hierarchy, definitions, and the location of various real numbers relative to the real line. With these concepts in mind a questionnaire was developed and administered to 26 undergraduate pre-service math teachers at a large university situated in the eastern United States. Drawing from this questionnaire, the focus was on four items:

1. *Circle all the irrational numbers in the given set of numbers:*
   $\{0,\ \sqrt{5},\quad (3\frac{1}{2})/7\,,\quad 0.123152687943,\quad \sqrt{\frac{1}{16}},\quad 3i+5,\ \ 13,\quad \frac{1}{2}\Big/\frac{2}{3}\}$
2. *Irrational numbers are a subset of the rational numbers.*
3. *Rational numbers are a subset of the irrational numbers.*
4. *Can the exact location of √7 be found on the real number line?*

The findings of this study demonstrate that participants had either incomplete knowledge, uncertainty, or both, relative to the real numbers. For example, with respect to item 1, 88% of the sample correctly identified $\sqrt{5}$ as irrational but only 46% circled only $\sqrt{5}$. 54% of the sample identified rationals as irrational or mistook $3i + 5$ as being irrational. With respect to item 4, 46% of the sample believed that it was not possible to find the exact location of √7 on the real number line.

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THE EFFECTS OF A NON-TRADITIONAL TEACHING METHOD OF THE CHAIN RULE ON MATHEMATICAL ACHIEVEMENT

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In the wake of calculus reform, a number of interactive approaches have been proposed to better calculus teaching generally, and the chain rule in particular. Some use computer algebra systems and have found that students scored higher on tests of conceptual knowledge as a result (Palmiter, 1991). Students' difficulties with the chain rule are symptomatic of a broader difficulty with functions, evidenced by Clark et al.'s (1997) finding that poor understanding of function compositions inhibits understanding of the chain rule, and Vinner and Tall's (2004) description of how conceptual difficulties with topics such as the chain rule arise when a student's concept image and/or concept definition are not the same as the formal mathematical definition.

The present study used a quasi-experimental approach with two freshman-level college calculus classrooms to explore the question: does a self-guided activity approach to teaching the chain rule increase student knowledge when compared to a traditional lecture approach?

The 120 participants were primarily engineering majors who were taught by the same instructor. The control group received a traditional teaching method grounded in a composition-of-functions approach while the treatment group engaged in a self-guided activity during which the instructor monitored progress, answered questions, and used the terms "inside" and "outside functions" in place of "composition" vocabulary. Students in the treatment group graphed functions and their derivatives on a handheld calculator while answering questions designed to probe the relationship between the two. The operating theory guiding the activity was that technology could promote conceptual reflection beyond procedural mimicry.

Pre- and post-test item comparisons showed no statistically significant gains in student knowledge resulting from the treatment. However, the treatment produced modest improvements in student understanding of the chain rule. An unanticipated outcome was the instructor's satisfaction with the treatment activity and proposal to use the same activity in future classes.

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TRIGONOMETRY AND DYNAMIC SITUATIONS: GESTURES, GRAPHS AND CO-VARIANT RELATIONSHIPS

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A small group teaching experiment was conducted using an applet of a double Ferris-wheel. Three groups, with two students in each group, were videotaped solving a series of problems intended to help them symbolize the vertical movement versus time of a rider on the wheel. These students were part of a class for prospective teachers that examined topics in high school mathematics. Each set of problem-solving sessions lasted approximately 2.5 hours and was spread over two sessions. In this study, three episodes from two students', Andrew and Oscar's, interview demonstrate how these students made sense of the problem situation and the symbolization, and in turn used the symbolization to clarify and elucidate the dynamic system.

The theoretical perspective that this research takes extends from Freudenthal's assertion that mathematics is a human activity. Freudenthal calls this activity mathematizing, which involves transforming realistic problems into mathematical symbols and then utilizing those symbols in order to derive greater understanding of the problem and the mathematics involved. Specifically, horizontal mathematizing is the development of informal ways of speaking, symbolizing and reasoning that students use in order to make sense of problem situations (Gravemeijer, 1999).

The analysis demonstrates how the two students made sense of the dynamic system through a process of horizontal mathematizing. Consistent with Rasmussen, et al. (2005) the students used their previous knowledge of trigonometry and algebra to symbolize the height versus time of the rider. In their creation of a graph of the height versus time of the rider on the double Ferris-wheel, they coordinated their gesturing of the movement of the wheel with their drawing of the graph. This led to a discontinuity between how they perceived the movement of the wheels (as each having constant velocity) with the inflection points on the graph. This argument came to a head in their symbolization of the angular velocity of each of the separate wheels. As they began to interpret their symbolization and make sense of the various co-varying quantities that had been symbolized, the students recruited the use of their gestures and body movement in conjunction with their argumentation in order to clarify how their symbolic function and graph fit their perception of the movement of the rider on the wheel.

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Chapter 2: Algebraic Thinking and Reasoning

Research Reports

Brief Research Reports

Posters

RE-THINKING ALGEBRA IN MIDDLE GRADES: MATHEMATICAL THINKING VS. MATHEMATICAL TOOLS

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Using a mathematical problem solving test, we documented approaches used by thirty 8th grade students who had either completed or were enrolled in an Algebra I course at the time of data collection. In solving the problems, students could use and show their knowledge of linear equations and graphing. These topics were included since their mastery is considered the core goal of Algebra I curriculum. Results indicated that only 3 of the participants used algebraic techniques to solve problems. Prominent problem solving heuristics used included guess and check and setting tables. When children attempted to use algebra to solve problems they were not successful. Justifying and explaining results was problematic for all participants.

Introduction

The movement towards requiring an algebra course in the middle grades is widespread. More and more school districts across the country, particularly in urban communities, are now offering Algebra to 7th and 8th graders in hopes to increase enrollment in more advanced mathematics courses in high school. As a result of this push, in the past decade the number of students taking algebra and more advanced courses in 8th grade has increased by 30% (Loveless, 2009). An increase in the size of population taking Algebra however, can't be interpreted as success in preparation (Smith, 1996). Opponents of the "pushing algebra down" movement have argued that while honorable in intention this push may not lead to desired results. Some empirical data indicate that these concerns are legitimate (Loveless, 2009). Political push for requiring Algebra in middle grades persists regardless, appealing it to be a civil right (Moses, 1996). Debates surrounding what middle school children may or may not gain, conceptually, from a formal course in Algebra remain prominent in the field (Silver, 2002). Many maintain that middle grades should be treated as a bridge between the informal and formal worlds of mathematics (Steen, 1996); hence study of symbolic Algebra should be delayed. Instead, students are better served if middle grades curriculum is focused on Algebra as multiple representations (Kaput, 1989) or ways in which it may be used to model real life contexts (NCTM, 2000). While a formal course focused on symbolic Algebra might be beneficial to a handful of students, time is better invested if middle grades children are immersed in activities that build their mathematical thinking and problem solving skills (Silver, 1997).

The purpose of the study we report here was twofold. First, we aimed to document how 8th grade children who had either completed or enrolled in a formal course in Algebra I would go about solving problems that utilized algebraic skills and techniques. Second, we wished to identify the problem solving heuristics that the participants commonly used. This study is a part of a longitudinal research project in which we trace development of mathematical thinking of approximately 80 students from grades 7 through 9 as the result of exposure to an after school enrichment program. The data used for analysis in this paper was collected at the beginning of year I of the project and prior to program implementation.

Context

Algebra for all children was initiated in 1994 and motivated by a number of educational, political and social forces (Steen, 1999). The worthwhile goal attached to the agenda is allowing access to advanced mathematics courses in high school and ultimately increasing participation in STEM areas. National data indicate that the push for early algebra has in fact been successful. Not only more children are taking Algebra in Middle Grades, but also more students are taking courses in calculus and geometry in high schools. While the reported statistics speak positively to the enrollment of children in mathematics courses, a careful examination of students' performance on achievement exams paints a grim picture of the conceptual gains resulting from this movement (Loveless, 2009). For instance, results of NAEP indicated that a large percentage of these students who scored in the lowest 10% of the eighth grade exam had either completed a course in Algebra or were enrolled in one at the time of testing. Further, many of the students who took Algebra or higher remain intellectually and conceptually at a low achievement level. Indeed, Loveless (2009) found that an increasing number of lowest-performing students were those that were pushed into algebra in 8th grade The results of the 2005 NAEP scores of different student groups highlighted that the low performing eighth graders in advanced classes scored even below the average fourth grade students. Additionally, among the lowest-scoring 10% of children in the sample, nearly 29% were taking advanced math, showed skills typical of second-graders. According to Loveless (2009) *Misplaced Students* need to be helped to first develop knowledge needed for entry into advanced and abstract mathematical work. He concluded that delaying a course in Algebra until basis arithmetic skills are mastered might be necessary. Supporting or challenging Loveless' conclusion is not of particular concern to our inquiry; neither is it central to the discourse that concerns what mathematics might be of value to children in middle grades or what students might gain from a formal course in Algebra. Several questions persist: Would courses in Algebra then be beneficial if students have indeed mastered the basic computational skills? What do middle grade children gain from a course in Algebra I after completing the course with honorable grades and suitable ranking on the standardized achievement exams? Would these students be able to access their knowledge of algebra when solving problems? The need to address these questions motivated our study.

In this work, we hoped to document how a group of successful 8th graders performed on problem solving tasks that drew on central concepts from Algebra. In particular, we sought to seek evidence of whether efficiency of algebraic techniques was internalized by the participants as demonstrated in their work. Our goal was not to record gaps in computational skills but to identify ways in which they solved problems.

Setting

The study reported here is a part of a longitudinal research project in which we trace development of mathematical thinking and problem solving of approximately 80 middle grades children as the result of exposure to an after school mathematics enrichment program. Participants are of minority heritage and come from various urban schools and communities. The enrichment program provides opportunities for students to work on authentic tasks and explorations using technology and inquiry methods. The current study was conducted as an effort to document algebraic and problem solving skills of 8th graders at the time of their entry into the enrichment program with the intent to develop appropriate curricular materials that could enhance their existing knowledge.

Participants and Methods

The sample for the study consisted of 30 8th graders from urban communities. Three of the participants were of Hispanic, two of Caucasian and all others of African American heritage. All children had earned or maintaining either a letter Grade of A or A- in their Algebra I class.

At the time of data collection eighteen of the children were enrolled in either a regular or an Honors Geometry course, having completed an Algebra I course in 7th grade. Among this group 12 reported having been studying trigonometry in school at the time of data collection. The remaining six were studying either similarity and congruence criteria (proving triangles being similar using two column proofs) or constructions using compass and straight edge. The twelve students enrolled in Algebra I were studying either FOIL method or solving system of linear equations.

Data collection and analysis

All children initially completed a Personal Inventory Survey (PIS) consisting of 12 questions. The survey consisted of three parts. On the first part of the survey the children reported their feeling towards mathematics; and ranked their ability to make sense of mathematical ideas as well as their confidence in their ability to think mathematically. On the second part of the survey the children identified mathematical areas in which they felt most and least successful. The last portion of the survey elicited information from children on two issues: (1) what they considered to be features of a person with a mathematical mind and; (2) skills they felt were needed to be successful in mathematics. The last portion of survey asked students to state what they believed the study of Algebra was about and its major ideas and concepts.

Table 1. Student performance

Task	Geometry (n=18)	Algebra I (n=12)	t-tests
Problem 1 a	M=2 SD=0	M=2 SD=0	
1 b	M=0.88 SD=0.78	M=0.86 SD=0.69	0.93
Problem 2	M=0.44 SD=1	M=0	0.13*
Problem 3	M=1.44 SD=0.88	M=1.14 SD=0.69	0.46
Problem 4	M=1.6 SD=0.41	M=1.51 SD=0.5	

Upon completion of PIS, all children were administered a test of mathematical problem solving consisting of four questions (See table 1). The questions could be solved using a variety of different approaches and techniques and concerned topics traditionally covered in Algebra I curriculum. Selection of problems was guided by both theoretical and practical considerations. On a theoretical level, in line with Schoenfeld's (1994) characterization of mathematical thinking, we opted to use tasks that provided opportunities for children to show their competence in solving novel problems. Additionally, we aimed to use contexts that allowed children to use the tools of the trade (Schoenfeld, 1994) with which they felt most comfortable (including algebraic tools) and using them to solve problems. Further, in line with Cobb et al.'s perspective (1991) we agreed that the purpose for engaging in problem solving is not just to solve specific problems using pre-defined tools, but to encourage and elicit the interiorization and reorganization of the schemes as a result of the activity. The test items were piloted among children in grades 5 through 7 in the previous to assure various approaches could be used to solve them. The items did not restrict students to the use of specific Algebraic tools. Therefore, they had the potential to elicit "natural" mathematical behaviors on the part of learners. On a practical level, in choosing specific content pieces to be addressed in the test, we focused on linear relationships since mastery of this topic is the central goal of Algebra I curriculum. Students were asked to solve each of the problems using any method they wished. They were also asked to explain why they felt their answer was correct in each case.

1. Consider the two pay options: \$300 a week or \$7.50 an hour. A) What factors will affect you're your choice of option to take. (B) Draw a graph that compares the two pay options, allowing the reader to determine which option might be best for them to take.
2. Water lilies are growing on a lake. The water lilies grow rapidly, so that the amount of water surface covered by lilies doubles every 24 hours. On the first day of summer, there was just one water lily. On the 90th day of summer, the lake was entirely covered. On what day was the lake half covered?
3. Karim lends \$104 to 3 friends. He gives the second one twice as much as the first one and the third friend 5 times as much as the first one, how much did he give to each friend?
4. In a crowd of horses and people we see 18 heads and 52 legs. How many humans and how many horses were in the crowd?

Data analysis followed two stages. At the first level, students' responses were analyzed along four criteria: (a) Appropriate representation of the problem, (b) appropriate implementation of procedure for solving the problem, (c) completion of response and, (d) adequacy of justification. Each question was scored on a scale of 0-2 (0=No/wrong response; 1= some error; 2=complete response). Descriptive data was compiled for each item for the group. T-tests were conducted to see whether significant differences existed among the group according to the course in which they were enrolled at the time of data collection.

Second, reviewing students' responses to each question, we identified and tallied the various problem solving heuristics children had used on each problem. Common difficulties children had seemingly encountered on each task were also recorded. In particular, we noted where students had used Algebraic techniques for solving problems and whether they were successful in their effort. Inferences concerning ways in which Algebra could have enhanced or impeded students problem solving were then made.

Results

The participants' responses were analyzed using both qualitative and quantitative means. Each response was scored on a scale of 0 to 2. A score of 2 was assigned to a response where correct procedure was used and correct answer was obtained. A score of 1 was assigned to each response when either an incorrect answer was obtained using a correct procedure, or a correct answer was reported without showing work. A score of zero was assigned with either no response was provided or the answer was incorrect or irrelevant to the question. A second review of each item was completed with attention focused on the justifications provided on each response by the participants.

Overall results according to different groups

Table 1 summarizes the performance of children on each task. Note that while some differences existed among the overall scores of each group on items 1b, 2, and 3, the only statistically significant difference appeared on the second problem.

Common Approaches to tasks

Graphing. Drawing and interpreting graphs, representing data graphically, and communicating information and relationships using graphs are among the most important skills children are expected to develop in Algebra I curriculum (NCTM 2000). In response to part a of problem 1 all students successfully identified that the number of hours worked per week should influence the choice of job option, indicating an understanding of dependent and independent variables (Note that three students also identified location of the job and health benefits as critical considerations when deciding on a job). Part b of the problem asked students to illustrate the relationship between two job options graphically, capturing all necessary data that could assist an individual make a decision regarding each option. Among 30 students only one student successfully produced a “Pay vs. Hours Worked” graph for the two options, highlighting “40” as the point of intersection of the two lines. Other responses produced fell under four different categories: (1) documentation of instances of pay vs. hours without realization of a need for continuity in representation; (2) Separate graphs representing a cumulative count of pay for different weeks; (3) Two distinct bar graphs representing each of the options using different scales for amount of pay in each case, and (4) Cumulative “Pay vs Week” graphs of options with (0,0) as the common point (point of intersection of the two lines).

Wate Lily problem. The water Lilly problem was designed and used so to determine children’s facility with the use of working backward heuristic. The problem highlighted the negative influence of algebra on students’ problem solving process. All but one of the participants approached solving the task by setting up a table of values, assuming 1 water lily on day one and doubling the value each time (representing growth in days). Among the 29 who used this approach, 12 students seemingly stopped computations after approximately 10 levels and wrote 45 (90/2) as the answer. Five of the students crossed out the table they had set initially and reported 89 as the answer. Neither one of these groups students explained how they had reached the final solution or whether they were certain of the accuracy of their answer. Five students noted 30 as the answer, stating that 30 was the square root of 90. Only one student had tried to represent the problem pictorially (See Figure WL1). This approach did not lead to an answer.

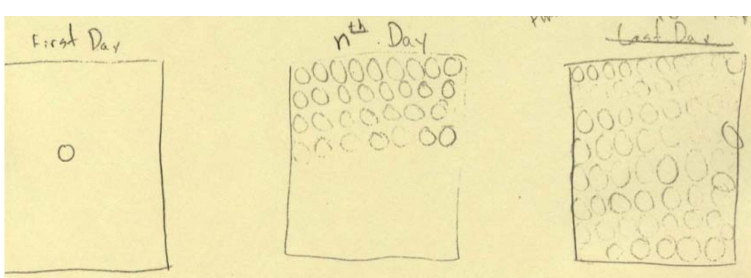

Figure WL1. Drawing a picture

Lending Money problem. The lending money problem is arguably the most typical of problems used in Algebra I curriculum. As such, we had hypothesized that a majority of the students would use an algebraic approach for solving the problem. However, only three students attempted to represent the relationships algebraically and only one of them was successful in establishing a formula to solve the problem. Figure LM1 is illustrative of the unsuccessful efforts used by children to solve the problem algebraically. Among the remaining 27 students, 15 first attempted to equalize the amount among the three individuals and used the established initial value as a way to determine the amount of money borrowed by each of the three friends. These individuals reached incorrect answers by rounding up and rounding down answers (see Figure LM2) without attention to the context. No student in this group attempted to check the results for accuracy. Twelve students used a guess and check strategy with an equal number of students using either systemic (educated first guesses and refining the guess to reach the final answer) (See Figure LM3) or un-systemic approach (beginning randomly with one extremely low or extremely high initial guess) (See Figure LM4). Five of the 6 students who used a systemic approach for solving the problem answered the question correctly. Among those who had used an un-systemic guess and check approach, four abandoned the problem after 10 numerical trials. Only one student in this group correctly answered the question.

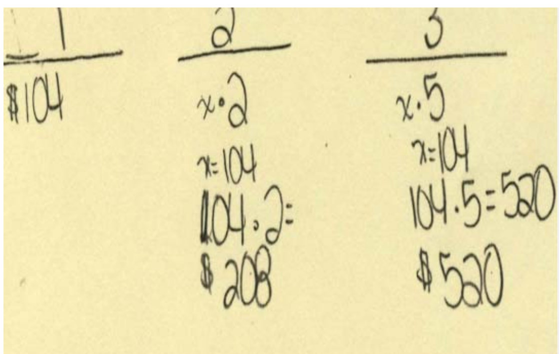

Figure LM1. Use of Algebra

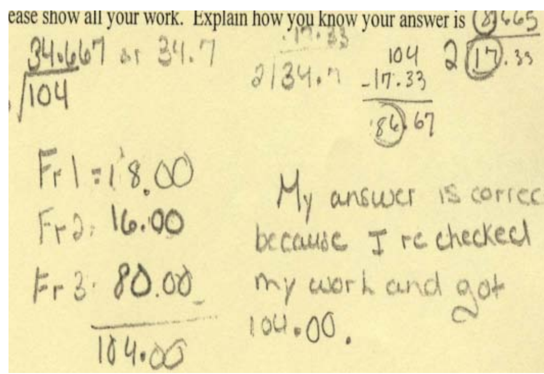

Figure LM2. Equal partitioning and check

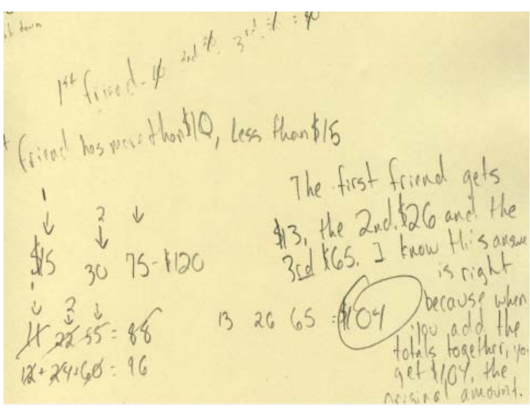

Figure LM3. Systemic guess and check

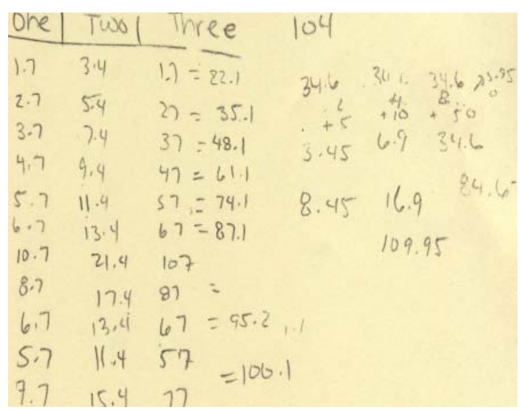

Figure LM4. Un-systemic guess & check

Horses and People problem. Figures HP1-3 illustrate the most commonly used problem solving strategies by children when solving Horses and People problem. Among the thirty participants only two of them used algebraic techniques (solving the problem using a system of linear equations). In case of one of these students, due to an un-noticed arithmetic error, an incorrect answer was obtained. Other approaches used by children included: drawing a picture (n=12)(See Figure HP1), guess and check (n=10) (See Figure HP2), and repeated subtraction (n=5)(See Figure HP3).

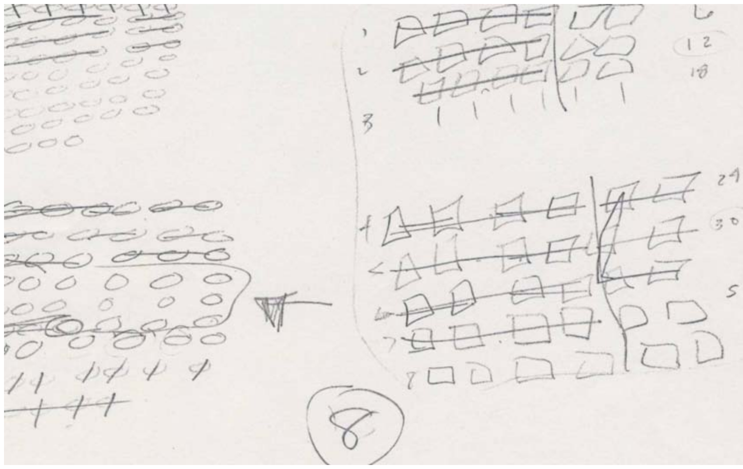

Figure HP1. Drawing a picture/Counting

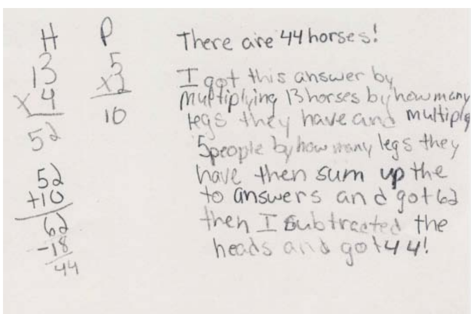

Figure HP2. Guess and Check

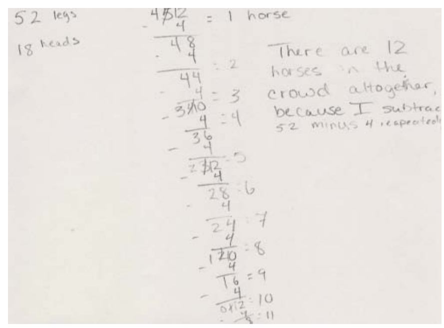

Figure HP3. Repeated subtraction

Discussion

Algebra for All is certainly an honorable educational goal. Requiring Algebra in middle grades, however, may not be wise if a serious consideration is not given to how and what children are taught in previous years. Increasing participation in advanced mathematical coursework is a major concern to the discipline. However, success in such courses rests on students' ability to reason, think mathematically, use mathematical tools to solve problems, and communicate ideas in writing. Our data indicate that an early course in Algebra did not appear to have led to development of either efficient problem solving skills or algebraic approaches among the participants. Indeed, among the thirty participants only 3 tried to solve problems using algebra. More importantly, only one of these individuals was successful in this effort. Problem solving heuristics used by children included the use of guess and check and setting up table of values without evidence of thoughtful consideration of efficiency of the methods they used. Considering that the thirty participants in this study were from 24 different classrooms in 10 different schools, the results are of particular importance.

Success in advanced mathematical coursework and continued participation in pursuing such courses beyond high school depend largely on the ability to think mathematically. Thinking mathematically has more to do with using the tools of trade to solve problems (Schoenfeld,

1994) and less with the ability to recall or accumulate these tools. Algebra courses in their current form and content share several major shortcomings. First, they focus on mastery of a narrow range of skills for solving equations and inequalities, graphing, and simplifying expressions. As such, they remove context from learning and emphasize manipulation of symbols to the point of hindering the development of algebraic reasoning and mathematical thinking skills among children. Second, they tend to be prescriptive in terms of what students are expected to do as they complete exercises and tasks. An early emphasis on Algebra courses that focuses on mechanical mastery neglect the importance of helping children develop an understanding of Algebra as a language for communication and prediction. Children need to learn to answer questions about quantitative patterns and relationship and should be provided a chance to realize how algebra may empower them to engage in such activities. There is consensus that mathematical power consists of the ability to work with mathematics flexibly and in different contexts. There is also consensus that a major educational goal for all children is to assist the development of their mathematical power. We argue, based on the data we presented in this paper, that requiring a formal course in Algebra may not be an appropriate pathway towards reaching the goal of making mathematics children.

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SUPPORTING THE UNDERSTANDING OF ALL STUDENTS: TEACHER MOVES THAT FACILITATE SUCCESS WITH RELATIONAL THINKING

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In this paper, we examine classroom norms and teacher moves that support the equitable participation and growth of all students in a third-grade inclusion classroom during a routine focusing on relational thinking. Analysis of classroom video supports findings that a highly conceptual approach like relational thinking can be used successfully with an inclusion classroom.

Introduction

In this study, we focus on one teacher's implementation of an algebraic routine in a third-grade inclusion classroom. Student participation is an issue of equity and achievement; students who participate more generally learn more from the lesson, and low rates of participation can predict low achievement in the early grades (Cohen, 1984; Finn & Cox, 1992). The paper speaks directly to the conference theme of Optimizing Student Understanding in Mathematics, as we examine how establishing classroom norms and assigning competence to low-status students promoted the participation of *all* students, including those students with Individual Education Plans (IEPs).

Theoretical Perspectives

The work of Carpenter, Franke, and Levi (2003) around relational thinking examines the use of number sentences (equations) designed to develop concepts of equality and relational thinking. These include two types of number sentences. One type is complete number sentences about which the question is asked as to whether they are true or false. Examples of these are: 2=2; 3+0=3; 5=1+4. These number sentences are used to challenge children's notions of the meaning of the equal sign. As Carpenter and colleagues (2003) note, children often reject the previous three examples (and others) as being equal because respectively (a) there is no operation; (b) adding zero is not really adding anything so it isn't allowed; and (c) the order is wrong. A second type of number sentence is one where an unknown is present such as 3+10=‡+7. Children who believe the equal sign indicates that the answer comes next will predict that 13 is the correct response for the unknown (Falkner, Levi, & Carpenter, 1999). Challenging children's naïve or emergent understanding of the equal sign is one aspect of supporting them in developing relational (or algebraic) reasoning.

A second aspect of supporting children is developing the capacity to use relational strategies instead of computational strategies when solving for an unknown (Carpenter et al., 2003). Through working on series of number sentences containing unknowns and carefully selected values for the given numbers, children begin to see that it can be easier NOT to compute to find an unknown value. For example, in the case of judging the truth or falseness of the number sentence 27+37= 25+39, instead of computing to find that the value for each side of the equation is 64, children will begin to use relational strategies to determine whether the expression is true. The will, for example decompose 27 into 25+2 and 39 into 37+2 resulting in an expression that now is clearly equal: 25+2+37=25+2+37.

In order to understand the teacher moves that supported the achievement of *all* the students, we situate the discussion of mathematical content and strategies within a larger sociocultural framework, in which the co-constructed norms of a classroom community shape the learning of the students (Yackel & Cobb, 1996). Establishing norms such as the necessity of (a) explaining one's thinking around a solution to a problem, and (b) carefully attending to the presentation of thinking by classmates, helps to create an atmosphere in the classroom where everyone is expected to engage in and explain substantive mathematical thinking and everyone is respected for his or her contribution. We analyze these norms developed over time in this classroom, through particular teacher moves in interaction with students. In other words, this teacher invested in creating norms that would support learning, including the learning of students with IEPS.

Next, we draw on the work of Cohen and her colleagues (Cohen, 1997; Cohen & Lotan, 1995; Cohen, Lotan, Scarloss, & Arellano, 1999) around complex instruction in heterogenous classrooms. Among other things such as using rich and worthwhile mathematical tasks, complex instruction involves teachers explicitly *assigning competence* to low-status students by making comments that are public, specific to mathematics, and valid. It is this aspect of complex instruction that we focus on in this study. Children with low-status (social and academic) are often reluctant to participate in mathematical discussions and thus defer to their higher status classmates. Assigning competence by explicitly noticing and calling public attention to work of lower status students challenges the notion that students have that there are those who are good and competent with mathematics and others who are not.

As we are particularly concerned with the inclusion of low-achieving students in conceptual mathematical practices such as the relational thinking routine, we also draw on Empson (2003), who specifically analyzes the mathematical participation of low-achieving students in a discussion based classroom. Research has documented the unequal participation of low achieving students in discussion-based classrooms, including lack of participation and less mathematical contributions than their high performing peers (Baxter, Woodward & Olsen, 2001). Empson (2003) found many different strategies used by low-achieving students, such as reading teacher cues and attempting to avoid work. After days, months and years of limited participation, low achieving students are less likely to take up opportunities to learn, such as making presentations in class. Our study analyzes how one teacher sought to interrupt this cycle. Empson (2003) uses socio-interactional linguistics to determine participant frameworks, or a unit of activity in which relationships, roles, and domain-specific content affect classroom interactions (Goffman, 1981). These frameworks are particularly useful to understand the relationship between the learners, teacher, and the mathematical content as it is constructed within classroom discussion. In our study, we focus on the whole group discussion of relational thinking problems, as individual student present their solutions. We look, then, to see how the participants, especially low-achieving students, are positioned into particular roles.

In order to understand the representations that students used to solve problems like the ones described above, we draw on earlier work of Carpenter and colleagues (Carpenter, Fennema, Franke, Levi, & Empson, 1999). We extend a framework (direct modeling, counting, and numeric strategies) previously used to analyze solution paths to contextualized problems to work done around relational thinking. Direct modeling involves modeling each value in a problem with concrete materials. In the original framework for example, a child might use direct modeling with cubes or other manipulatives to solve the following problem: I have 3 pencils. I pick up 4 more pencils from the classroom floor. How many pencils do I now have? Following the action of the problem, the child might lay out three cubes, representing the initial number of pencils, add four more cubes to represent the pencils found on the floor, and then count the entire number of cubes to find the solution of seven pencils. A child using a counting strategy might solve by counting on from four, saying 4, 5, 6, 7 to arrive at the solution. A child using a numeric strategy might know the math fact 4+3=7 or might derive the fact thinking: I know 3+3=6 and so one more will be 7. This framework supports our analysis that children with poor numeric strategies (often in our case, children with IEPs) were able to display relational thinking through the use of concrete materials.

For a final note, we are taking a social constructivist view of competence and disability in this paper. Within this study we are considering the label of students with IEPs, not as an inherent and static determinant of individual ability, but as a school-based designation which reflects and recreates differential ability within the classroom (Dudley-Marling, 2004; McDermott, Goldman & Varenne, 2006). Because of the importance of this designation in the culture of schools, we choose to use this classification to focus attention on how the teacher successfully managed a discussion-based classroom that included all students.

Research Question

The goals of the research reported in this paper were to examine the teacher moves and classroom norms that supported the development of relational thinking (Carpenter et al., 2003) with particular attention to the participation of students with IEPs. Our data collection and analysis processes were guided by one question: What are the norms and teacher moves that support the equitable participation and growth of all students in an inclusion classroom during a routine focusing on relational thinking?

Methodology

During the course of one school year, a teacher in a third grade inclusion classroom employed a weekly routine focused on developing children's competency with relational thinking (Carpenter et al., 2003). Once a week for approximately 30-45 minutes, the teacher presented the class with number sentences to solve. These were either (a) complete number sentences to be judged true or false, or (b) number sentences that had to be solved for an unknown. Number sentences had already been written on the board when the children entered the classroom. They were encouraged to get to work quickly on solving the problems. After 20-30 minutes of independent work, students were asked to present their thinking to the class. It was these presentations that were video-taped. A total of 25 of the presentation portions of these weekly sessions were video-taped and comprise the data set.

There were 12 participants, seven boys and five girls. Seven students were African American, five were White. Four students had IEPs; all of these were African American. These 12 participants are the students who were enrolled in this class for the majority of the school year and who participated in the routine; there was one student with Downs Syndrome who was in the class for the entire year but did not participate in the routine.

For analysis, the video was segmented by student presentation so that one student presenting his/her solution path to a given problem on a given day constituted one unit. For each unit, a detailed narrative description of the student solution path, including interactions with the teacher, was constructed. These solution paths were then coded for relational or computational thinking (Carpenter et al., 2003). Following this initial coding, a second pass was taken through the data. At this time the solution paths were coded as to what type of representation was used: direct modeling, counting, or numeric (Carpenter et al., 1999). For each unit, teacher moves were then identified. In addition we catalogued which participants presented their thinking at each session so as to examine to what extent all students (particularly those with IEPs) were supported in participating in this portion of the activity.

Results

The teacher was able to create a classroom in which students with IEPs were not isolated, but were contributing, developing members of the classroom mathematics community. She did this in two ways: (a) by developing and consistently employing classroom norms that applied to all students and (b) by using several strategies for assigning competence to students with IEPs (Cohen & Lotan, 1997). After discussing the classroom norms that the teacher established, we turn to an extended example of the work of one student with an IEP, although this student is not an isolated example. By the end of the year, there was significantly greater participation in the routine by students with IEPs.

Classroom Norms

From our analysis of the data, it seems clear that the teacher created a supportive environment for all students to learn complex concepts. While at the beginning of the year, higher-achieving students were the ones that presented their thinking most often, by the middle of the year, students with and without IEPs made presentations of their thinking and contributed to the ensuing discussions a similar percentage of time. Some of the norms that supported student success included:

- Every student was expected to explain his/her thinking out loud at the board.
- Students were allowed to use notebooks to scaffold their presentation without loss of status.
- Every student was expected to attend carefully to the presentations, and encouraged to compare the strategies that they used and the strategies of the presenter.
- Emerging strategies were named after children who used them; this was as likely to be a student with as a student without an IEP.
- When a atudent struggled to present his/her ideas, the teacher asked questions and re-voiced statements, but only after allowing considerable wait time, often at several points throughout the student's presentation.
- Students were encouraged to use manipulatives to solve problems without loss of status.
- Teacher focused attention on thinking, not on correctness of answer.
- Teacher had high expectations that all students would move from computational to relational thinking strategies.

Most of these norms were enacted in a strikingly consistent way for all students. This consistency, along with the continually high expectations that the teacher held for all students, were key factors in creating a supportive environment for all students including those with IEPs and others who struggled to think relationally. In what follows, we present an example that demonstrates the development of one student (with an IEP) over the course of the school year with respect to the enactment of several of these norms. In the example, this includes (a) the student explaining his thinking, (b) the use of teacher scaffolding, (c) the use of manipulatives for direct modeling, and (d) movement from computational to relational thinking. The example of this one student is meant to be indicative more generally of the ways in which the teacher managed the classroom to support the mathematical performance and participation of all students.

In one exchange, early in the fall, the teacher asked Caleb, a student with an IEP (all names are pseudonyms), to present his solution for the problem: 8+7=15+1, True or False? The teacher began the exchange by attempting to relate his presentation to the strategies of other students, "Do you want to do it the way Jacob did it or Serena did it or Karen or Jessica? Is there something about what they did that you would like to try?" Caleb did not answer, instead he turned to the board and wrote 15 under 8+7, then wrote 16 under 15+1,.and circled false. Although correct, the teacher did not accept this workas sufficient. She asked:

Teacher: Tell us in words what you were thinking please?
Caleb: (No response, looks down)
Teacher: How did you know that this was a false equation, a false number sentence?
Caleb: (Looks down)
Teacher: Look at the board please.
Caleb: (He turns and looks at the board.)
Teacher: What is 8+7 equal to?
Caleb: 15
Teacher: What is 15 + 1 equal to?
Caleb: 16
Teacher: Is 15 equal to 16?
Caleb: (Under his breath, he repeats the question to himself.) No.
Teacher: No, so false is the correct answer.

The next time that Caleb presented a solution, for the problem 19+3=0+9+3, both teacher and student had a different strategy. Caleb used a direct modeling strategy to solve the problem and his teacher validated this strategy, simultaneously scaffolding his presentation. Caleb brought connecting cubes to the board and began his presentation by meticulously arranging his cubes into stacks of 10, 9, 3 and then 9 and 3.

Teacher: What number of cubes do you have, Caleb?
Caleb: (He doesn't answer, but continues arranging his cubes.)
Teacher: Look. Here you've got 10 and 9 and 3. And here I see 9 and 3. (Teacher looks into his notebook). Well what did you do on this side, Caleb? (Indicating the 19 and 3.)
Caleb: I made 19 and 3.
Teacher: You started with the 10 . . .
Caleb: And then I put the 9.
Teacher: And that made how much?
Caleb: 19
Teacher: You've got the 19+3 on this side (showing his cube representation). What about this side of the equal sign?
Caleb: On this side I had 9 and 3.
Teacher: So what was missing here?
Caleb: The ten (showing his separated stack of ten cubes).

In a final example, in the spring of the year, Caleb was called on to quickly present a solution from his seat. Here he did not need additional teacher scaffolding, and was able to present a purely verbal solution. The teacher, focused on interrogating student understanding of the nature of the "answer" in these problems, asked Caleb what the answer was in the problem 5+1+1=8. When Caleb answered "False", she responded with "Because why?"

Caleb: False, because this (referring to the 5) would have to be a 6.
Teacher: What is the answer?
Caleb: False.

Discussion

In this classroom, the teacher positions all students as problem solvers and solution reporters (Empson, 2003). Students with IEPs in mathematics are expected to present, just as high-achieving students are. These roles, problem solver and solution reporter, were not immediately taken on by Caleb or by other students with IEPs in the classroom. In fact, the teacher invested considerable resources of time and scaffolding in students with IEPs so that these students became full participants in the classroom routine. The first exchange in the above example lasted seven minutes, the second 11 minutes. Clearly, the amount of time invested in this student during these class sessions is significant.

The consistent application of a set of classroom norms resulted in a classroom that supported the participation of all students. The remarkably consistent expectations of this teacher, resulted in a classroom that allowed all students to participate equitably. Mathematics classrooms are too often focused on a single ability: executing procedures correctly (Boaler, 2006). One important strategy used by this teacher in creating an equitable classroom was de-emphasizing procedures and correct answers. Instead, as our example shows, the teacher consistently focused on student thinking. In the first exchange, we can see the teacher attempted to position Caleb as a competent solution reporter, even though he seemed to need considerable verbal scaffolding to present his solution. Many teachers may have accepted Caleb's initial non-verbal presentation. This teacher consistently expected all students to provide both a visual and a verbal explanation of their solution. She did not let Caleb off the hook- and chose rather through direct questioning to introduce language for a simple proof. We would argue that this approach allowed students to feel safe taking risks, whether that attempt was at presenting an idea to the class or in trying the strategy of another student

As the results indicate, the teacher used several strategies for assigning competence to this student with an IEP: valuing his problem solving methods; holding high expectations of the justification in his presentations; and making positive comments that were specific and valid about his thinking. Another critical aspect of teacher practice in promoting equity was allowing solutions using direct modeling, or manipulatives, to have equal status with numerical solutions. Most of the students who were struggling (those with IEPs) often used direct modeling with linking cubes to arrive at their solutions. As we saw in the extended example, the teacher encouraged the students to share this strategy during presentations, even asking them to demonstrate how they used cubes to find the answer. In the second exchange, the teacher continued to expect that Caleb would present his work both visually and verbally. Caleb needed fewer prompts to explain his thinking than he had in the first exchange. The concrete manipulatives supported both his thinking, and the presentation of his strategy. The cubes allowed him to demonstrate the decomposition of 19 into 10 and 9, which was central to his thinking relationally about the problem. The teacher attached status to the connecting cubes as a strategy, here, by choosing Caleb to demonstrate this strategy. As the results demonstrate, students who struggled significantly with computation were able to think relationally and demonstrate this thinking using direct modeling.

More generally, the level of scaffolding provided by the teacher was based on whether or not a student was stalled in his or her presentation, not on his or her status within the classroom. Thus, although in this case, we see the teacher scaffolding a student with an IEP, both high and low-status students gave presentations where the teacher provided significant scaffolding. In this way, variations in scaffolding were evenly applied across all students regardless of classroom status.

Our results show a student who began the year using computational strategies to solve problems and who was a reluctant participant. By the end of the year, this student was more confident in his participation as the last student-teacher exchange demonstrates. By the end of the year, not only he, but most students, including those with IEPs, were solving problems relationally a great majority of the time.

Conclusion

Through the analysis of Caleb's participation in the classroom routine, we can see a development of his algebraic thinking from computational to relational. This thinking was scaffolded by his use of direct modeling to solve the problems. In addition, we can see the development of his class participation in the community, from a student who was reluctant to speak (first exchange), to a student who confidently engaged in the discussion from his seat (final exchange). This work, including and engaging the students with an IEP in mathematical thinking, is hard work. It would be easier to let a shy, reluctant student like Caleb get away with limited participation. Instead, this teacher made participation a requirement, and supported Caleb through encouraging alternative strategies for solving problems that had equal status in the classroom, in this case direct modeling with connecting cubes.

In our work with teachers and students, we are frequently asked how to include all students in high-level mathematical thinking. Teachers, faced with low-achieving students who may not often participate in whole group discussion, assume that this kind of instruction is not for "them." As Empson (2003) suggests, teachers want to help students save face, and not to embarrass those who are struggling. This classroom presents an equitable resolution to this conflict. Through high expectations of participation, and a supportive community, all students were able to present relational thinking to their classmates. We believe that the results of this study are significant, as they demonstrate that a teacher can successfully use a highly conceptual approach like relational thinking with an inclusion classroom.

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THE NATURE OF TRANSFORMATIONAL ALGEBRAIC ACTIVITIES ADDRESSED IN DIFFERENT CLASSES OF THE SAME TEACHER

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This paper compares the nature of the transformational algebraic activities addressed in four beginning algebra classes that used the same innovative algebra curriculum materials. Two classes were taught by one teacher; the other two by another teacher. We developed categories of transformation-related ideas and used them to analyze the transformational activities enacted in each class. The analysis reveals that some ideas were addressed in all classes, whereas others were addressed in only some of the classes. Moreover, in the case of one teacher, transformation-related ideas were addressed similarly in her two classes, whereas major differences were found between the two classes of the other teacher. The findings suggest that the differences found between teachers and between classes of the same teacher are related to complex interactions among the teachers, the curriculum, and the classes.

Introduction

By and large, meaning-making work in algebra has been mostly associated with generational algebraic activities, i.e., with developing meaning for the objects of school algebra: expressions and equations (Kieran, 2004). However, the construction of meaning in algebra need not be related solely to generational activities, since constructing meaning for the concept of equivalence and for using properties and axioms in the manipulative processes occurs within transformational algebraic activities (Ibid). Kieran (2007) pointed out a recent change in research in the area of transformational algebraic activity, from emphasis on the manipulative processes used in simplifying expressions and in solving equations, to attention to the theoretical foundations of students' manipulative work. Thus, in recent years attention has been given to studying the processes of learning the concept of equivalence and using meaningfully properties and axioms in the manipulative processes. Similarly, in recent years a major change has occurred in the way transformational activity is treated in algebra curriculum materials. Algebra textbooks have traditionally centered on the transformational aspects of algebraic activity, emphasizing rule following and rote symbol manipulation, often without attention to conceptual understanding and meaning (Kieran, 2004). However, innovative algebra curricula developed in recent years (e.g., *Everybody Learns Mathematics* in Israel; *Connected Mathematics Project* (CMP) in the USA) often emphasize conceptual understanding of algebraic processes, and developing meaning for equivalence, and for the use of properties and axioms in the manipulative processes, thus, giving meaning to algebraic manipulative processes.

Obviously, students that use different curriculum materials may be exposed in class to different natures of transformational activity. For example, studying from a textbook that focuses primarily on rule following and symbol manipulation would probably result in more emphasis on rote transformational activities than in the case of using a textbook that focuses on conceptual understanding and construction of meaning. But do students that use the same curriculum materials experience the same nature of transformational activity? Accumulating research suggests that the enacted curriculum is not identical to the written curriculum (e.g., Remillard et al., 2009), and that different teachers use the same curriculum materials differently

(Manouchehri & Goodman, 2000; Tirosh, Even & Robinson, 1998). Studying the same teacher teaching in different classrooms has also recently begun to be the focus of research (e.g., Herbel-Eisenmann, Lubienski, & Id-Deen, 2006). Still, seldom did the teacher in such studies use the same curriculum materials in different classrooms. Furthermore, these studies focused mostly on pedagogy – rarely did they examine the enacted mathematics.

Recently, as part of the *Same Teacher – Different Classes research program* (Even, 2008), Eisenmann and Even (2009, in press) examined the enactment of the same algebra curriculum materials in four classes that used the same innovative algebra curriculum materials. Two classes were taught by one teacher; the other two by another teacher. Eisenmann and Even's study, which employed quantitative analysis that compared the distributions of three types of algebraic activity – generational, transformational, and global/meta-level – revealed statistically significant differences between the written and the enacted curricula, between the two participating teachers, and between the two classes of each teacher. Our study builds on these quantitative findings. Its aim is to compare, this time using qualitative methods, the *nature* of the transformational algebraic activities addressed in Eisenmann and Even's four classes, between the two participating teachers, and between the two classes of each teacher.

Methodology

Two teachers and four 7th grade classes – all using the same innovative algebra curriculum materials – participated in this study. Each teacher taught in two classes, each from a different school. Sarah taught classes S1 and S2; Rebecca taught classes R1 and R2. Interviews with the teachers revealed that each perceived one class (S2 and R2) as encountering more difficulties in engaging in mathematics. The curriculum materials that Sarah and Rebecca used in their classes were part of the *Everybody Learns Mathematics* program, one of the innovative 7th grade curriculum programs developed in Israel in the 1990s. The curriculum materials include suggestions on enactment, including detailed lesson plans. Sarah followed rather closely the written lesson plans whereas Rebecca did not. Rebecca also was more attentive and responsive to students' mathematical behavior and performance (Eisenmann & Even, 2009, in press).

The main data source included video- and audio-tapes of the teaching of a whole learning unit that introduced the topic *equivalent algebraic expressions* (intended for 15 lessons in the curriculum materials) in each of the four classes (a total of 67 lessons). Our analysis centered only on whole-class work. First, we analyzed the nature of each transformational activity in the written curriculum materials, using open coding. We then used the categories generated to analyze the enacted transformational activities in each of the four classrooms. Activities that the teachers used that were not from the curriculum materials were also categorized. We constantly compared the categories with new data from the videotapes and refined them. The occurrence, explicitness, and inter-connections of various central ideas embedded in the transformational activities were also analyzed, as well as the contribution of both the teacher and the students to the progression and implementation of the activities.

Transformation-related Ideas Addressed in Class

Analysis of the whole class work in the four classes revealed the following, more or less chronological, 12 categories; each linked to a different transformation-related idea. The first six categories, briefly listed below, represent ideas addressed in the written materials:

(1) Substitution of numerical values into expressions to show that an expression does *not* represent a generalization of a given pattern.
(2) Substitution of numerical values into expressions to develop a sense of the behavior of expressions.
(3) Substitution of numerical values into expressions to show non-equivalence.
(4) Substitution of numerical values into expressions is not an appropriate means to show equivalence.
(5) Expanding and simplifying expressions to maintain/show equivalence.
(6) Expanding and simplifying expressions as a means of engineering expressions for a desired purpose.

The upper row in Figure 1 illustrates the teaching sequence of the above six ideas as addressed during the suggested whole class work of the 15 units in the written materials.

The next five ideas (#7-#11) were not offered in the written materials, but were addressed in at least one of the classes:

(7) Substitution of numerical values into expressions as a means to show that an expression represents a generalization of a given pattern (mathematically invalid).
(8) Substitution of numerical values into expressions to examine the potential of equivalence.
(9) Substitution of *all* numerical values into expressions as a means to show equivalence.
(10) Substitution of numerical values into expressions as a means to show equivalence (mathematically invalid).
(11) Expanding and simplifying expressions to show non-equivalence.

Another transformational activity addressed in class was:

(12) The technical practice of simplifying expressions and substituting numbers in expressions.

Note that technical practice was not recommended in the written materials as part of whole class work; therefore, it is not included in the list of categories generated by analyzing the written curriculum materials. Yet, it was suggested in the written curriculum materials as individual work or homework.

The above 12 categories can be divided into three conceptually different types: (a) ideas related to *developing an understanding of generational activity* (#1, 2, 6, 7), (b) ideas related to *developing an understanding of equivalence of expressions* (#3, 4, 5, 8, 9, 10, 11), and (c) technical practice with no attention to meaning and understanding (#12).

Similarities and Differences between Teachers and between Classes of the Same Teacher

Sarah, S1 and S2

Sarah taught the topic *equivalent algebraic expressions* in both S1 and S2 using the same curriculum materials, covering by and large the same written units. She followed rather closely the suggested lesson plans outlined in the curriculum materials. As can be seen in the second and third rows of Figure 1, all six transformation-related ideas suggested in the written materials were enacted in S1 and S2 – and they were enacted similarly by means of proportion, order, and distribution, as well as with respect to the written materials. Moreover, transformation-related ideas that were not offered in the curriculum were rarely addressed in her classes (only ideas #10 and 11).

In both classes Sarah tended to make clear presentations of the above-mentioned ideas and to lead students to applying them in the different assignments. For example, when working on unit #6, students in both classes were presented five pairs of expressions and were asked to substitute several numeral values in the given pairs of expressions, compare the results, and cross out the non-equivalent pairs. Eventually, two pairs of expressions were left not crossed-out on the board. In both classes, Sarah stated that substitution could not be used to prove that two given expressions are equivalent (#4). She explicitly incorporated in her presentation of this idea its underlying justification: that there might exist a number that was not yet substituted, but its substitution in the two given expressions would result in different values. Sarah presented this idea as a motivation for finding a method to show equivalence, and immediately proceeded to work on using properties in the manipulative processes as a means to prove equivalence (#5):

> We substituted two numbers, and it was the same… And then the third number gave something else. Who says that these expressions (points to one pair), if we substitute more numbers, there wouldn't suddenly be a number that will give a different result? Therefore, it isn't enough; substitution isn't enough in order to know that expressions are equivalent. What do we need to do in order to know that expressions are equivalent? We need to use properties.

Rebecca, R1 and R2

Rebecca used in both R1 and R2 the same teaching sequence that Sarah used. However, unlike in Sarah's case, major differences were found in the transformation-related ideas between the written materials and Rebecca's classes. As can be seen in the fourth and fifth rows of Figure 1, in both R1 and R2 a central idea suggested in the written materials was not addressed at all (#6). Also, compared with Sarah's classes, in R1 and R2 considerably more ideas that were not suggested in the written materials were addressed in class (#7-#12). Moreover, ideas that were not suggested in the written materials took a great deal more class time in Rebecca's classes than in Sarah's classes. In addition, in both R1 and R2 Rebecca assigned technical practice, with no attention to conceptual understanding and meaning (#12); this activity was not recommended in the written materials for whole class work, nor was it addressed in Sarah's classes.

Unlike Sarah's case, major differences were found in the transformation-related ideas addressed between R1 and R2, both regarding ideas suggested in the written materials and ideas not suggested in the written materials. For example, in R2 significantly less time was devoted to working on the central idea of expanding and simplifying expressions as a means to maintain/show equivalence (#5) than the time devoted to it in R1. Moreover, the idea that substitution is not an appropriate means to show equivalence (#4) was addressed only in R1. In addition, only in R2 were ideas suggested in class (by students) (#7, 10), which are mathematically invalid. Finally, the last units enacted in R2 centered on technical practice (#12), devoting significantly more time to it than in R1.

Unlike Sarah, who tended to make explicit presentations of main ideas, Rebecca attempted to draw ideas from her students and to follow students' suggestions. For example, when working on unit #6, Rebecca also presented students with the problem of deciding which of five given pairs of expressions are equivalent. Similarly to the activity in Sarah's classes, after substituting several numeral values in the given pairs of expressions and crossing out non-equivalent pairs, three pairs of expressions were left not crossed-out on the board. In contrast with Sarah, who suggested herself the use of properties as a method to show equivalence, in both R1 and R2, Rebecca pressed her students to find a method that works, and different scenarios developed in the classes, as is illustrated in Figure 1, unit #6. In R1, the class concluded that the remaining pairs appeared to be equivalent but that it was impossible to know for certain. Following students' proposal, the discussion centered on the possibility of proving equivalence by substituting *all* numbers or by substituting representative numbers:

Rebecca: When will I be sure that these three [she points to the pairs not crossed-out on the board] are indeed equivalent? That each pair is equivalent? When will I be sure?
Student1: When you check all the numbers.
…
Student2: There is an infinite number of numbers so you will never finish.
Rebecca: So I am not going to substitute infinite numbers. I need to find some other trick.
Student3: Substitute different kinds of numbers… Negative, positive…

After the class discussed the ideas suggested by students and eventually rejected them, Rebecca changed the focus of the activity to looking for a connection between pairs of expressions. The class quickly embraced the discovery that by using properties, it was possible to move from one expression to another and to show equivalence.

In contrast with R1, R2 embraced the idea that many successful substitutions are a valid means for determining equivalence of expressions. For example,

Rebecca: Why are they equivalent? Why do I say that these are equivalent?
Student1: Because we checked at least thirty [numbers].
Rebecca: We didn't check thirty, but I am asking: Why are these equivalent, in your opinion?
…
Student2: Because we checked.
Rebecca: Because you checked, but we said that maybe there is one number that you did not check.
Student3: But we checked almost all the [inaudible].

Rebecca then asked the class to find new expressions that would be equivalent to the given ones. Eventually, R2 embraced the idea that equivalence can be determined by manipulating the form of expressions, using properties of real numbers.

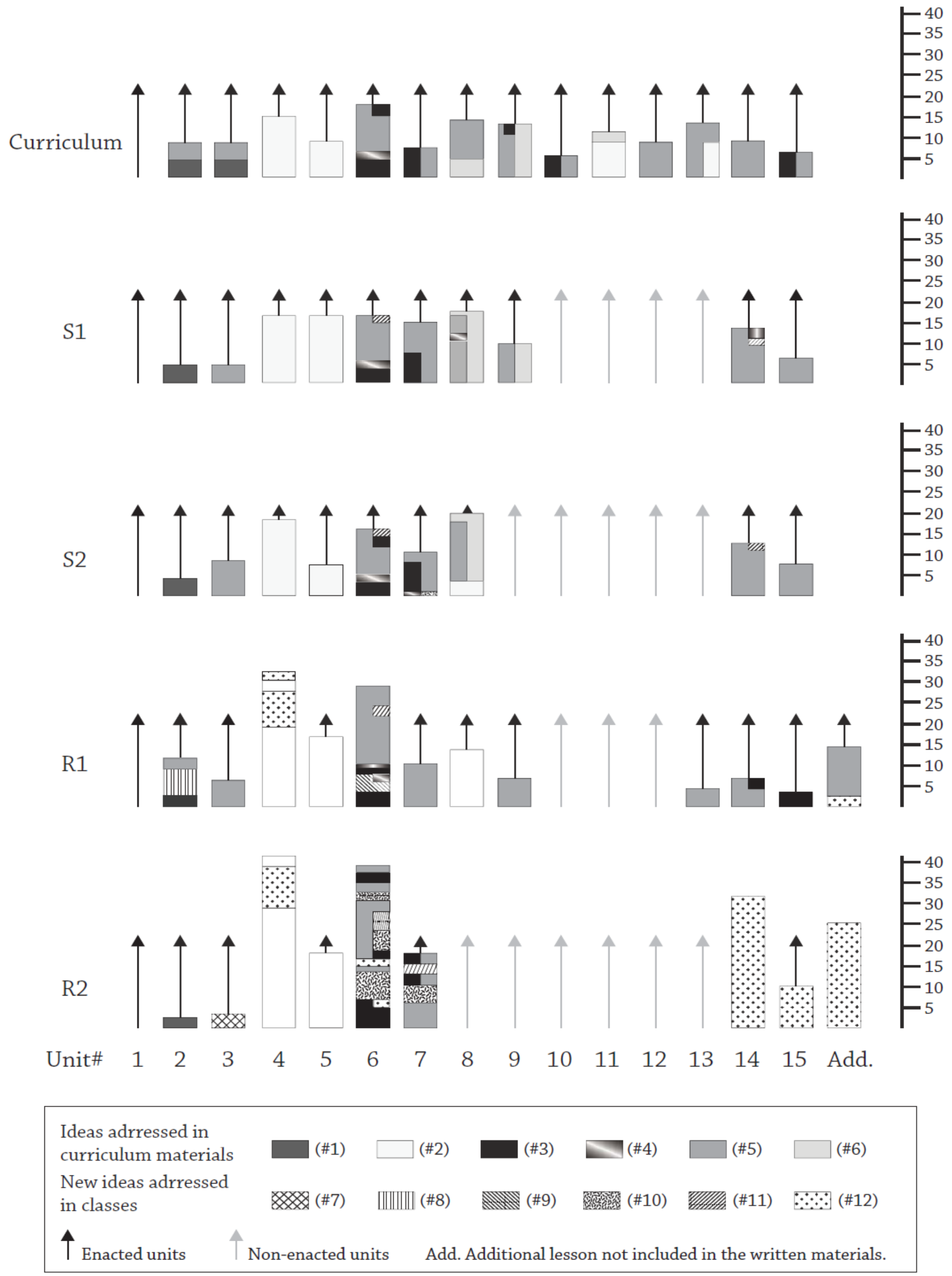

Figure 1: Teaching sequences of transformation-related ideas, addressed during whole class work, in the written materials and in the classes.

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Discussion

Sarah and Rebecca taught the topic *equivalent algebraic expressions* using the same curriculum materials and a similar teaching sequence. However, differences were found between the two teachers in the ways transformation-related ideas were addressed in class. All transformation-related ideas suggested in the written materials were addressed in Sarah's classes, whereas not all were addressed in Rebecca's classes. Moreover, compared with Sarah's classes, a relatively large number of ideas that were not suggested in the written materials were addressed in Rebecca's classes – some of which were not mathematically valid. Thus, eventually, Sarah's students and Rebecca's students were introduced to some extent to different mathematical ideas when the 15 units were taught. These differences between the two teachers regarding the mathematics that was offered to students in class appeared to be linked to their teaching approaches. Sarah chose to follow rather closely the suggested written lesson plans with only minute deviations, and stated explicitly the main ideas. In contrast, Rebecca used the suggested written lesson plans in a flexible way, was attentive and responsive to students' mathematical behavior and performance, and spent time probing students, expecting them to suggest and explicate important ideas. Naturally, Rebecca enabled various scenarios to develop in her classes, and innovative ideas to emerge – some mathematically invalid – that were not addressed in the written materials. Yet, at times, depending on students' contributions, important ideas were not made explicit.

Teaching approaches also appeared to be linked to the similarity or differences found between the two classes of the same teacher. Whereas there was a similarity between Sarah's classes, major differences were found between Rebecca's classes. In the case of Sarah, who followed rather closely the lesson plans in the curriculum program, and who made explicit the main ideas, the same transformation-related ideas suggested throughout the 15 units in the written materials were addressed in both classes, and they were addressed similarly. Likewise, transformation-related ideas not offered in the curriculum materials were rarely addressed in her classes. In contrast, in the case of Rebecca, who did not follow the lesson plans so strictly, and who was attentive and responsive to students' suggestions, and who seldom stated in her own words the main ideas, major differences were found in the transformation-related ideas offered to students in her two classes. For example, ideas related to proving equivalence, such as that substitution may not be an adequate means to show equivalence of given expressions (#4), and that the use of properties of real numbers is an adequate means to show equivalence (#5) – known to be problematic for students (e.g., Mariotti & Cerulli, 2001) – were addressed in R1, but were never made explicit, or were relatively briefly addressed in R2. In addition, because of the difficulties that R2 students encountered, Rebecca chose to assign in this class, but not in R1, extensive technical practice, skipping assignments related to developing understanding.

The significance of this study is twofold. First, is its contribution to the development of conceptual frameworks for analyzing the nature of school-level algebraic activities. Refining Kieran's (2004) model, focusing on the algebraic meanings and ideas associated with transformational activities, we identified several ideas addressed in beginning algebra teaching, related to three conceptual types: (a) developing understanding of generational activity, (b) developing understanding of equivalence of expressions, and (c) technical practice. Another significant contribution of this study lies not in evaluating the mathematics learning opportunities offered to students in one class or another. It is rather in showing how transformation-related ideas may be addressed differently, even in classes taught by the same teacher using the same written materials, and in pointing out links between the differences found and teaching approaches. There is a broad agreement today that teachers are key to students' opportunities to learn mathematics. We agree with that. Yet this study points to complex dynamic interactions among teachers, curriculum, and classrooms, the nature of which remains to be further explored.

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ALGEBRA AND NO CHILD LEFT BEHIND: STANDARDIZED TESTS AND ALGEBRAIC COMPLEXITY

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As No Child Left Behind (NCLB) places increasing emphasis upon high-stakes standardized tests, the phenomena of teaching to the test becomes of paramount concern to the education field. Given the role of algebra as a gatekeeper to advanced mathematics and STEM fields, I examine the algebra items that appear on NCLB-mandated tests with two cognitive complexity frameworks. With an understanding of the importance of algebra and the impact of NCLB on states, teachers, and what is being taught in the classroom, this research project explores the mathematical complexity of items found in the tests from 40 states, and discusses the far-ranging impact of the findings.

Introduction

Algebra holds a place of great importance in mathematics learning, due both to its common usage in the workplace and its status as a gatekeeper in the mathematical education community (e.g., Moses & Cobb, 2001; National Research Council [NRC], 1989; National Council of Teachers of Mathematics [NCTM], 2000). Consequently, a considerable amount of attention has been paid to improving the currently dismal situation surrounding algebra (Steen, 1992) by exploring ways to consistently allow students access to the necessary algebraic concepts (e.g., Knuth, Stephens, McNeil, & Alibali, 2006; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Moses & Cobb, 2001; Stephens, 2005; Nathan & Koedinger, 2000). As the call to focus more on the importance of algebra in the K-12 curriculum came from mathematicians and educators alike, another widely reaching shift occurred in the US educational system: the No Child Left Behind Act (NCLB) passed in 2001.[1] NCLB's emphasis on school accountability for student performance requires all states to individually develop standardized tests that satisfies NCLB requirements, as a method for assessing which schools provide adequate educations. Although the language of NCLB emphasizes challenging all students, there is concern that—given the pressure to perform well—schools are "teaching to the test" (e.g., Center on Education Policy [CEP], 2007; Apple, 2007; the Commission on No Child Left Behind, 2007), in which case standardized tests may guide the mathematics undertaken in the classroom, instead of instruction focusing on conceptual and coherent understanding or mathematical skills applicable beyond these test items.

A wide variety of research geared towards unpacking the impact of standardized tests on teaching has revealed powerful and complex effects that include "teaching to the test." Wilson (2007) notes that "high-stakes testing in mathematics…can affect teaching practices, curricular decisions, and the decisions made about schools or individual students" or individual teachers (p. 1107). The National Mathematics Advisory Panel (NMAP; 2008) states flatly that tests "can drive instruction" (p. 57), and Taylor, Shepard, Kinner and Rosenthal found that teachers reported a considerable impact upon their classroom time and instruction due to high-stakes testing (2003). "Teaching to the test" is not a new NCLB-linked phenomenon, but rather a situation present pre-NCLB (see Resnick & Resnick, 1992; NRC, 1989) that is increasing as all states increase the number and importance of standardized tests as mandated by NCLB.

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

The scenario of "teaching to the test" becomes particularly problematic when considering the results of Hyde, Lindberg, Linn, Ellis, and Williams (2008): they analyzed a number of mathematics items from NCLB-mandated standardized assessments in order to compare the performance of males and females. While examining various grade levels and NCTM strands of mathematics, they found very few high cognitive demand items according to Webb's Depth of Knowledge (DOK) framework (one of the frameworks used in this paper). Given evidence that standardized assessments may at times guide mathematics in the nation's classrooms, Hyde et al.'s findings stimulate the research question: *What difficulty level of algebra content are NCLB-mandated standardized tests measuring?*

The exploration undertaken here is devoted specifically to determining which algebra items are present on standardized tests, and what level of cognitive complexity they represent according to two different and well-respected frameworks: Webb's DOK for mathematics (2002) and the National Assessment of Educational Progress' (NAEP) Levels of Mathematical Complexity (NAGB, 2004). With an understanding of the importance of algebra and the acknowledgement of the impact of NCLB on states, teachers, and what is being taught in the classroom, this analysis explores the mathematical complexity of items found in the mandated tests from 40 states, and discusses the far-ranging impact of the findings.

Motivation For Analysis

No Child Left Behind

The 1983 release of *A Nation at Risk* (National Commission on Excellence in Education) directed the public's attention to the state of public education (Bullough, Clark, & Patterson, 2003), and in the following decades the public's demand for accountability increased. A reauthorization and drastic overhaul of the Elementary and Secondary Education Act from 1965, NCLB was a comprehensive response to the call for accountability, requiring various demonstrations of achievement from schools in order to avoid sanctions or receive rewards. Signed into law in 2002, schools were given until 2014 to fully achieve NCLB's requirements of 100% proficient students. Now, eight years after being signed into law, as NCLB is being re-examined and revised by a new administration, it is a topic of conversation that inevitably engenders controversy. In this research project, the 2001 bill is the one under discussion due to both the incomplete status of the Act II version and the fact that the original bill has dramatically shaped what standardized assessments written for fulfillment of NCLB look like now and in the future.

NCLB specifically indicates mathematics and language arts (reading) as the two topics of the greatest importance, emphasizing the need for "challenging" curriculum and content in these areas and others. NCLB requires tests that are developed and designed by each individual state which are used to provide yearly "report cards" that are used to make decisions at a variety of policy levels. Failure or success is measured by what student percentage achieves "proficiency" on the tests, and each year's percentile benchmark is defined by each state under the guidance of NCLB's Adequate Yearly Progress (AYP) requirements.

AYP is defined individually by state, with a number of guiding points designed for flexibility so that each state can implement the policy as it sees fit. For example, NCLB mandates that the calculation of AYP be based primarily upon the standardized assessments, but that graduation and attendance rates must also be a factor of importance. Exactly how graduation and attendance rates are weighted in respect to the test scores is up to individual states, so long as their formula meets those NCLB requirements. To add another layer of flexibility, each state was permitted to determine which scores count as "proficient." This was, naturally, taken up differently by different states (Fuller, Wright, Gesicki, & Kang, 2007).

As the assessment scores carry a considerable amount of weight in determining whether a school is identified as failing to meet AYP or not, the standardized tests are high-stakes for individual schools, districts, and states. And as Wilson (2007) notes, although these tests were designed to provide aggregate results, in some cases the test scores are used to make decisions at the individual student or classroom level. The design and annual administration is thus of considerable importance to all educational stakeholders.

Algebra

As NCTM is an authority that is not bound up tightly in the content standards of a specific state, using NCTM's algebra standards allows for a more comprehensive picture than what would be available by choosing a single state's standards. NCTM's (2000) summary of algebra is that it "emphasizes relationships among quantities, including functions, ways of representing mathematical relationships, and the analysis of change" (p. 37). *The Principles and Standards* then reminds us that "algebra is more than moving symbols around": "Students need to understand the concepts of algebra, the structures and principles that govern the manipulation of the symbols, and how the symbols themselves can be used for recording ideas and gaining insights into situations" (p. 37). NCTM continues beyond this definition to provide examples of algebraic concepts that are appropriate and important for every grade from pre-K to 12th, noting that "a strong foundation in algebra should be in place by the end of eighth grade" (n.p.).

Their call for a "strong foundation" has been echoed recently NMAP, who go even further than NCTM by recommending an increase in the number of students taking algebra by the end of middle school: "All school districts should ensure that all prepared students have access to an authentic algebra course—and should prepare more students than at present to enroll in such a course by Grade 8" (2008, p. xviii). Finally, the increasing trend across the country is to provide algebra as an 8th grade subject, and one state, California, has even gone so far as to require algebra for all 8th graders (see California Department of Education, 2009, for more information about this edict). This decision has spawned a plethora of discussions by scholars and the popular press (Cavanaugh, 2008), and the Association of California School Administrators (ACSA) has filed a temporary restraining order to prevent the implementation of the new law (2008).

NCTM, NMAP, and others are rightly concerned about access to algebra, as the subject is widely considered to be a gatekeeper to success in future mathematics classes, in college, and Science, Technology, Engineering, and Mathematics (STEM) fields (Moses & Cobb, 2001; Stinson, 2004). Algebra provides the needed foundation for abstract and advanced mathematics, and lack of access to algebra concepts limits access to mathematics classes, college options, and career choices. Furthermore, NMAP notes that "a strong grounding in high school mathematics through Algebra II or higher correlates powerfully with access to college, graduation from college, and earning in the top quartile of income from employment" (2008, p. xii). The powerful impact of algebraic understanding on future opportunities for students mandates that we make algebra a priority for all, whether that includes offering algebra at lower grade levels or weaving key algebra elements throughout elementary and middle-school classes. The quality of teaching, curriculum, and equitable student access are of considerable importance in this goal, as are the high-stakes standardized tests that concretize the requirements for algebra learning.

Teaching to the Test

Everybody Counts (1989) declares, "What is tested is what gets taught. Tests must measure what is most important" (p. 69). This concern over the connection between standardized tests and the shape and content of classroom teaching has continued to be an urgent topic of discussion in classrooms across the country, in the educational research field (CEP, 2007), and even in presidential speeches (Obama, 2009). Apple (2007) notes that NCLB could be considered "a situation in which 'the tail of the test wags the dog of the teacher'" (p. 109). Taylor et al. (2003) found that teachers reported an impact on classroom instruction due to high-stakes tests, and noted that teachers viewed some of these changes positively and others negatively. For example, some teachers added mathematical content to their classes in order to prepare their students more fully for the test, while other teachers bemoaned the considerable amount of classroom time they spent on sample test items in order to prepare the students for the test format.

"Teaching to the test" is not inherently bad or inappropriate; rather, "teaching to the test" makes it even more important that standardized tests are designed carefully and thoughtfully so that they can serve as curricular guidelines. The controversy over "teaching to the test" indicates that standardized tests are designed to be assessments of student achievement, and are not universally accepted as impacting classroom level instruction. Given considerable evidence of the effect of high-stakes tests on mathematics instruction, a careful examination of the content and quality of the tests themselves becomes of paramount importance. Asking *What mathematical content do these tests contain?* becomes the same as asking *What mathematical content are teachers being guided to emphasize in their classrooms?*, and we must begin looking at tests and test items with such a question in mind.

Research Methods

First, the data collection process is described, and the examination that each data point went through. More specifically, each data point was examined individually for algebra content, then all algebra problems were coded for problem format, DOK level, and LMC level (the problem format coding is not discussed here for lack of space). The author's coding was compared with each state's coding wherever possible in order to provide an ad hoc inter-rater reliability.

Data

As discussed previously, algebra begins to take a more prominent role in middle and high school, with many students taking Algebra I in 8th or 9th grade. Given this fact, the standardized tests at these two grades were deemed most appropriate and most likely to provide a sufficient number of algebra-coded items. Each of the states with NCLB-mandated test items available contributed items from 8th grade tests, and 7 of those states also contributed items from voluntary (non-NCLB mandated) 9th-grade tests. I chose not to include 10th grade tests because there are few tests designed for 10th grade, and those that are administered in 10th grade are very likely to be end-of-class tests or preparation tests for graduation exams. In total, the initial data set contained 1729 items from 40 states.

Released and sample items from states were found on state education websites or received through personal communiqué.[2] When multiple years were available from a single state, the most recent two years of items were used; and when a state had both released and sample items available, released items were given preference. A number of states released items that were considered representative of their mathematical strand in the assessments, but did not release the complete exams. Thus, data collection was conducted with an eye towards acquiring as many items as possible, and not towards accurately reproducing actual assessments.

Coding Algebra

The framework for determining what qualifies as "algebra" comes from NCTM's outline of thorough content standards for grades K to 12 (2000). As discussed in the Algebra section, NCTM's status as one of the foremost leaders in the field of mathematics education makes this an appropriate choice, especially given the widely disparate state of mathematics education in the US. The nature of decentralized educational practices and state-specific policies results in very different definitions of "algebra" across the country, but NCTM provides a common ground for analysis that does not privilege any particular state.

The first round of analysis used NCTM's standards to the purpose of examining the complete data set in order to determine which of the 1729 problems were algebraic in nature. In total, 421 of those items were classified as primarily Algebra. Of the 40 states that contributed items to this analysis, 24 provided standards information of their own, with a total of 847 items coded independently by states. Of the items coded Algebra by these states or the author, there were 58 irreconcilable coding disagreements, which is a 93.15% agreement of codes. The majority of these disagreements were occasioned by states coding items "Algebra" instead of "Measurement" or "Number and Operations," apparently because of the presence of a variable.

Coding Webb's DOK

When Webb's DOK levels were provided for individual items by the states themselves, they were used as an ad hoc inter-rater reliability test. When differing codes were encountered, the author's codes trumped the state's unless the author's original code was on the cusp between levels, in which case it was changed to the state's code (items can fall between two of the four DOK levels, see the methodology used by Webb in a similar situation, 2008). The state-provided DOK levels were taken with a grain of salt, as Webb was not explicitly cited in all of the resources. A total of 56 problems from 4 states (Mississippi, Nevada, Oklahoma, and the New England Common Assessment Program[3]) provided DOK levels, there were 36 agreements between the author and the state, and 20 disagreements.

Coding NAEP's LMC

When NAEP's levels were provided for individual items by the states themselves, they were used as an ad hoc inter-rater reliability test that gave feedback regarding accuracy to the author. However, only 19 of the 421 items provided used a Low/Moderate/High framework, and of those 10 were agreements between the author and the state, and 9 were disagreements.

Results

According to Webb's DOK, a total of 11 of the 421 problems were rated level 3 (2.61%) by Webb's framework, and 0 problems were ranked 4. The rating of 1 was applied to 177 (42.04%), and the remaining 233 problems were ranked 2 (55.34%) (see Table 1). Note the asymmetric distribution, with the majority of the problems Level 2, and the remaining problems distributed primarily towards Level 1, with fewer Level 3.

According to Webb's DOK framework, the majority of the items (55.34%) fall in the Level 2 range, which is not problematic. What *is* problematic, however, is the skewed distribution of the remaining items, the majority (42.04%) of which are Level 1, leaving a small number (2.61%)

ranked Level 3 and absolutely zero problems ranked Level 4. This finding duplicates the unexpected results published by Hyde et al. (2008), and brings to the forefront the fact that our nation's assessments are failing to assess truly complex algebraic and mathematical thinking. The amount of time required by Level 4 problems may explain in part their lack of presence in standardized assessments, but the combination of few Level 3 and absolutely no Level 4 items clearly indicates that our algebra items are measuring only mathematical thinking that is not particularly complex, according to this framework.

According to NAEP's LMC, a total of 23 items were ranked High (5.46% of the total), 102 were ranked Moderate (24.23%), and the remaining 296 were ranked Low (70.31%) (see Table 2). Note that the distribution is positively skewed, with the majority of items ranked Low, then a smaller number of Moderate and a still smaller number of High rankings. Note that the vast majority (70.31%) of these items were of Low complexity, considerably more than the DOK percentage at the least complex level. Fuller et al. (2007) have noted that NAEP assessments tend to be more difficult than the individual state assessments, which may explain this distribution of rankings. However, these results serve to underscore the DOK framework results: our NCLB-mandated tests tend towards the less complex in their algebraic items, and include very few cognitively complex items.

Table 1. DOK Results

DOK level	number	percentage
1	177	42.04%
2	233	55.34%
3	11	2.61%
4	0	0%
Totals	421	99.99%

Table 2. LMC Results

LMC rank	number	percentage
Low	296	70.31%
Moderate	102	24.23%
High	23	5.46%
Totals	421	100%

Conclusion

The NCLB-mandated state tests are undoubtedly impacting teachers and their classroom practices, as they strive to tailor their instruction and content to the test. Given the status of algebra as a gatekeeper, we as a field want students to develop deep conceptual understandings of the subject. Sadly, the cognitive complexity of the nation's algebra items tend strongly to the simplistic and the rote, instead of providing truly challenging problems that measure the level of abstraction and sophistication of our students. By designing such assessments, states are supporting a situation in which teachers may be teaching to a mediocre (in terms of algebraic content) test.

Despite the bleak outlook, there is a remedy for this situation. Stakeholders must require that the designers of each state's standardized tests begin to examine the impact of their standardized tests, and strive to design the tests in such a way that they support complex algebraic thinking in the classroom. In order to inform and assist those involved in creating these assessments, additional research must be conducted on the relationship between tests and classrooms. A nuanced understanding of the long-term and far-reaching effects of NCLB-mandated tests on curricula, teachers, and students must be achieved before we can design assessments that steer us to the more complex algebraic understanding that our students need to achieve.

Endnotes

1. All information regarding NCLB was retrieved from http://www.ed.gov/nclb.
2. Special thanks to Janet S. Hyde, Sara M. Lindberg, Marcia C. Linn, & Amy B. Ellis for sharing standardized assessment items previously collected.
3. The New England Common Assessment Program (NECAP) is composed of three states: Rhode Island, New Hampshire, and Vermont. Given that the items provided by each state were identical, NECAP is being counted only once for the purpose of analysis of items.
4. This paper reports the study and analysis from Williams (2010), an unpublished master's thesis.

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AN EXAMINATION OF ALGEBRAIC REASONING: ELEMENTARY AND MIDDLE SCHOOL STUDENTS' ANALYSES OF PICTORIAL GROWTH PATTERNS

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One approach to help children develop algebraic reasoning is the examination of pictorial growth patterns. The purpose of this study was to explore and compare how elementary and middle school students analyzed pictorial patterns, with a focus on figural and numerical reasoning. Task-based interviews were conducted with a second, fifth, and eighth grader in which they were asked to describe, extend, and generalize a pattern. A cross-case analysis showed figural reasoning was more evident with the younger students, but all students did not exclusively use figural or numerical reasoning. The students' generalizations included informal notation, descriptive words, and formal notation.

Introduction

Over the past twenty years, the focus on algebra in school curricula and state standards has shifted from less to more emphasis as a primary content strand across K-12 mathematics. Traditionally, algebra has been considered a formal course students take in eighth, ninth, or maybe tenth grade. As recent as the late 1980s, algebra was a course deemed abstract in nature and only appropriate for college-bound students who were developmentally ready. Proportionally, African-American, Hispanic, and White working-class students were underrepresented in formal courses in algebra (Gamoran, 1987). In today's global economy, algebra is considered the "gatekeeper" course to higher level mathematics and in turn, a college education (Moses and Cobb, 2002; RAND Mathematics Study Panel, 2003). Increasingly, algebra is becoming the focus of policymakers, researchers, and practitioners with a movement towards all students taking a formal course in algebra in eighth grade.

Regardless of when students take algebra, students need opportunities throughout their elementary and middle school years to develop their algebraic thinking. When students analyze pictorial or geometric patterns, they use algebraic reasoning. Further, the patterns serve as a rich context for exploring generalization, a major part of algebraic thinking (Kaput, 1999). The purpose of this study was to explore and compare how elementary and middle school students analyze pictorial patterns.

Theoretical Perspective

The release of the *Curriculum and Evaluation Standards for School Mathematics* (1989) and the *Principles and Standards for School Mathematics* (2000) by the National Council of Teachers of Mathematics (NCTM) called for new thoughts about algebraic thinking being integrated into work in elementary school mathematics. Furthermore, the Standards propose integrating algebraic thinking early and throughout elementary school to promote students' abilities to make generalizations. The idea of early algebra "encompasses algebraic reasoning and algebra-related instruction among young learners – from approximately 6 to 12 years of age" (Carraher & Schliemann, 2007, p. 670).

Since there is a push to integrate algebraic reasoning across the grade levels in K-12 mathematics, describing algebra and defining algebraic reasoning are necessary. First, algebra

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includes generalized arithmetic, a problem-solving tool, the study of functions, and modeling (Benarz, Kieran, & Lee, 1996). Carraher & Schliemann (2007) define algebraic reasoning as "psychological processes involved in solving problems that mathematicians can easily express using algebraic notation" (p. 670). Algebra is typically thought of as a study of symbol systems while algebraic thinking is often used to "indicate the kinds of generalizing that precede or accompany the use of algebra" (Smith, 2003, p. 138). NCTM's *Principles and Standards* (2000) explain that algebraic thinking focuses on analyzing change, generalizing relationships among quantities, and representing these mathematical relationships in various ways.

One of many approaches to help young children generalize about patterns is through the examination of geometric or pictorial growth patterns (Ferrini-Mundy, Lappan, and Phillips, 1997; Orton, Orton, & Roper, 1999). A pictorial growth pattern is a "sequence of figures in which the objects in the figure change from one term to the next, usually in a predictable way" (Billings, 2008, p. 280). When using pictorial growth patterns, the goal is for students to analyze, describe, and extend the pattern and ultimately, generalize about relationships in the patterns. Pictorial growth patterns provide a context for students to make generalizations about what they notice. Further, the nature of the pattern tasks allows for a variety of approaches to making generalizations.

Two modes of reasoning are often used when analyzing pictorial growth patterns, figural and numerical. "A numerical mode of inductive reasoning uses algebraic concepts and operations (such as finite differences), whereas a figural mode relies on relationships that could be drawn visually from a given set of particular instances" (Rivera & Becker, 2005, p. 199). When analyzing pictorial growth patterns, children often use a "covariation analysis of the pattern" (Smith, 2003) in which they focus on the changes from one figure in the pattern to the next. They use the previous figure to build up to a new figure and identify what stays the same and what changes in the pattern. This can also be called recursive induction. When asked to generalize to figures later in the sequence, children may switch to a "correspondence analysis" (Smith, 2003) in which they are able to describe the relationship between the figure number and the changing aspect of the dependent variable. Further, they generalize about what any figure in the pattern will look like. When generalizing about any figure in a sequence, students are sometimes able to provide a closed or explicit formula.

Becker and Rivera (2005) conducted a five-year study in a school district assessing ninth graders' abilities to analyze patterns and functions. Over the period of five years, they assessed close to 60,000 students. They used pictorial growth patterns and found that 70 percent of the participants could extend patterns one at a time, but only 15 percent could develop a closed, explicit formula through generalizing algebraically. To follow up this study, Rivera (2007) conducted interviews of 22 ninth graders in which he asked them to analyze pictorial growth patterns. He analyzed the strategies of the students to find that most students who used only numerical reasoning and developed a closed formula were not able to explain their formulas.

In their work with second and fourth graders, Moss et al. (2006) worked with students on integrating figural and numerical patterns. They facilitated a series of lessons to promote students translating between the two representations of patterns. After the treatment, they found students who participated in the lessons were stronger on number knowledge and on making generalizations with explicit reasoning than students who did not receive the lesson instruction. Students who used both numerical and figural reasoning were better able to make generalizations, develop closed formulas, and make connections to what the terms in the formula mean.

Research Questions

The purpose of this study was to explore how students of various ages and grade levels engage in algebraic reasoning when presented with pictorial growth patterns, through a qualitative research design. Specifically, there were three research questions. First, what are the processes and strategies a second grader, fifth grader, and eighth grader utilize to *describe* and *extend* a pictorial growth pattern? Second, what are the processes and strategies the three students utilize to *generalize* about a pictorial growth pattern? Finally, what are the similarities and differences in the employed strategies of the three students in their analyses of pictorial growth patterns?

Methodology

Participants

The participants in this study were a second-grader, a fifth-grader, and an eighth-grader. I refer to the participants as Dara, Jenny, and David, respectively.

Dara, a second grader at an elementary school in a suburban area of a southeastern state, said her teacher “makes stuff fun” in math. Dara’s parents expressed that Dara does not exhibit the same high level of confidence in mathematics as she does in reading and writing. Despite this lower level of confidence, she has been successful in school mathematics.

Jenny, a fifth grader at an elementary school located in a rural area of a mid-Atlantic state, was enrolled in an advanced mathematics class in which she learned sixth grade concepts as outlined by state standards. She described her sixth grade math as mostly review with only some “new things”. Her teacher usually says, “what we’re going to learn about and he explains it. And, then we take notes, and he helps us on worksheets and stuff”.

David attended eighth grade at the only middle school in the same rural district as the elementary school that Jenny attended. David was enrolled in Geometry, the high school course, as an eighth-grader. He completed Algebra I when he was in the seventh grade. David expressed that he likes math. He shared, “a typical day is taking notes with examples that look a lot like what we’re doing on our homework and then we do our homework”.

Data Collection: Task-Based Interviews

The participants engaged in a task-based interview for approximately thirty minutes. The interviews occurred at a site convenient for the participants. During this time, the participants were asked to analyze two pictorial growth patterns (Smith, Silver, & Stein, 2005). The first of these patterns is examined in this paper and shown in Figure 1.

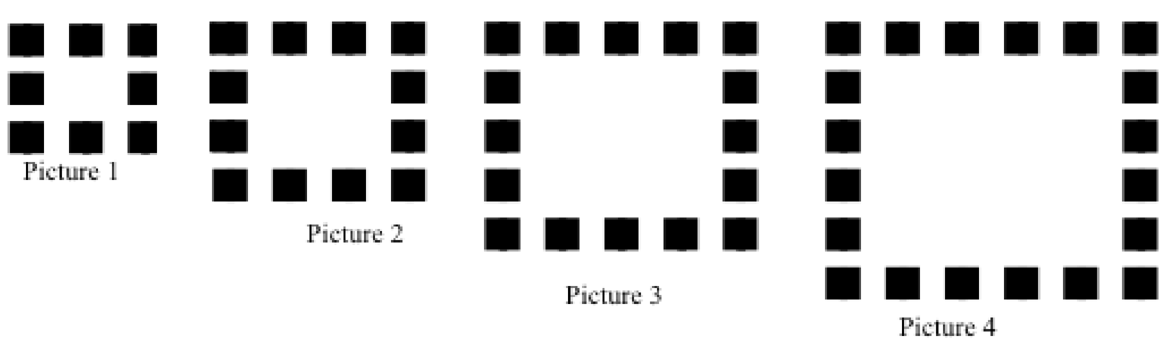

Figure 1. Pictorial Growth Pattern

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The task-based interviews were semi-structured. Participants were presented with one picture in the pattern at a time and were first asked to describe the picture. Next, they had Figure 1 to explain the next picture in the pattern, reason about later entries in the pattern (e.g., picture #20, picture #100), and generalize about any entry in the pattern. During the interviews, participants had access to plain white paper, pencils, the papers with the pictorial growth pattern, and square tiles. The interviews were videotaped with the camera focused on their workspace as they described their thinking.

Data Analysis and Reporting

Miles and Huberman's (1994) systematic data analysis was used to obtain descriptions from and relationships among the data. The three components to this type of data analysis are: data reduction, data display, and conclusion drawing and verifying. The data is reported through individual case studies followed by a comparative, cross-case analysis. Comparisons are made between the three students as to the type(s) of reasoning each used, figural or numerical. Validity for systematic data analysis is attained through checking for representativeness, checking for researcher effects, triangulating across data sources, and weighing the evidence

Results

Dara

When Dara, a second grader, was presented with picture #1 and asked to describe what she sees, she immediately said, "3, 6, 9, 12." When asked to elaborate, she circled four groups of three (see Figure 2), double circling the corner squares. Subsequently, when she was asked to compare picture #1 to #2, she said, "I can say they can still be in the same way." Dara proceeded to circle four groups of four and said, "4, 8, whatever".

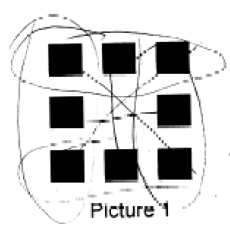

Figure 2. Dara circled four groups of three to explain her "3, 6, 9, 12"

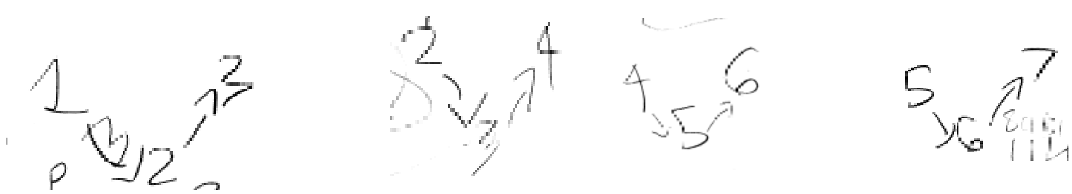

Figure 3. Dara's description of the relationship between the picture # and # of tiles in each "row"

When presented with picture #3, Dara says she noticed how this could be a tool for helping students with counting and says, "5, 10, 15, 20." Again, she circled the four groups of five like she did on picture #1. When asked what picture #4 would look like, Dara said "probably sixes" and draws the correct picture. Next, I asked Dara to predict what picture #10 would look like.

She drew a large square made up of twelve smaller squares on each side. Dara proceeded to explain the relationship between picture #10 having 12 squares “in each row”. She says 12 represents the “number in each row” and 10 represents the “picture”. She explained her numerical notation (see figure 3) by saying “you start with picture 1, you skip a number, and you come up to 3.” Similarly, she used the same notation to describe pictures #2, #4, and #5.

Next, I asked Dara to predict picture #25. Dara applied the same strategy as illustrated in Figure 3. She started at 25, skipped 26, and said there would be 27 squares in each row of the square. In reference to the squares in the corner, she recognizes they overlap. She says, “even these that overlap, it doesn’t matter. It’s still a row.” She knows she is double counting the corner squares, but she recognizes this does not change the description of her pattern. When asked to consider picture #100, Dara immediately applies the same strategy. She says there will be 102 squares in each row.

I asked Dara to describe her strategy for finding picture #100, and she started to generalize on her own using invented symbolic notation. She said, “start, to skip, to answer”. Her symbol for “start” was a circle with a vertical line. For “skip”, she used a backwards “C”. For “answer”, she sketched a circle with a horizontal line (see Figure 4).

Figure 4. Dara’s notation for her strategy of “start, skip, answer”

She explained her rationale for having notation by saying, “if you have like one hundred, um, one thousand, four hundred, ninety-three, I don’t think you could just pop that out of your mind, the next number and the next number after that, if you’re in second grade. So, it’s easier to do that *(referring to her notation and strategy)*. All you have to do is skipcount. Once you do it, poof, you know it.”

Jenny

Jenny, a fifth grader, described picture #1 as squares making up a square. When asked to compare picture #2 to picture #1, she noticed there are three squares in the base and height in picture #1 versus four squares in the base and height for picture #2. When asked to predict what picture #4 would look like, Jenny said, “it’s going to be a six height and a six base because it gets bigger by one.” When asked to determine picture #10, Jenny says there will be twelve squares on each side “because there are six numbers in between ten and four. And, on picture #4, there are six and there’s six in between so you do six plus six and you get twelve.”

Next, Jenny described picture #25. After doing some work (see Figure 5), she explained, “I just subtracted twenty-five minus four to find out how many numbers are in between four and twenty-five. It would be twenty-seven squares because I found out there were twenty-one numbers in between. So, it would be six plus twenty-one, and I get twenty-seven. The six comes from six squares in that *(points to the bottom row of picture #4).*”

Figure 5. Jenny's work to determine how many numbers are between 4 and 25

Jenny's strategy for generalizing was not dependent on the picture number given. Instead, she used information about the largest picture she had (#4) to determine any other picture in the pattern. For picture #100, Jenny thought quietly, then wrote 96 + 6 on her paper to determine 102. She said, "102 squares on the bottom, on the height, on the top, and on the other side because I subtracted 100 minus 4 and got 96 so there are 96 numbers in between. Then, you already have 6, so you plus 6 plus 96 and you get 102." When asked about any picture in the pattern, she said, "draw it out?" When asked again for any number in the pattern such as 144, she included 144 in her description. She said, "you take the number and subtract it from the highest picture you have, like 144 minus 4. Then, you add that to six in this case, so 146."

David

When David was asked to describe picture #1 in the first task, he said, "it's made up of eight tiles, it's a square, it forms a perimeter instead of an area, it's not like a filled-in figure." His prediction of picture #4 was: "A square that only forms a border made up of 6, 6, 4, and 4, made up of 20 tiles." He drew picture #4 and explained how he divided the tiles into groups of six, six, four, and four (six on the vertical sides of the square and four on the horizontal sides).

As we moved to picture #5, David said, "I'm noticing a pattern that it's four more tiles each time." To move beyond consecutive pictures, I asked David about picture #10. David said, "It would be the tenth picture in the sequence. The first one was eight. So, it would be eight plus four for every picture after that. So, you could come up with a formula." David proceeded to write the formula (see Figure 6). His description of his formula follows: "the number of tiles, t, equals eight plus four times the number there is minus one."

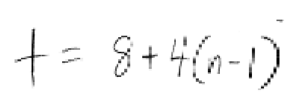

Figure 6. David's "formula" for the picture pattern in the first task

Cross-Case Analysis

The analysis of these three cases resulted in two interrelating themes. The two themes are: an intersection between figural and numerical reasoning and making generalizations using symbols and/or words.

The intersection between figural and numerical reasoning. The three students in the study did not use one type of reasoning exclusively. By combining figural and numerical reasons, the students were able to make generalizations about the pictorial growth patterns. The participants would not have made generalizations without examining the spatial features of the pictures and applying their knowledge of number relationships. Dara's reasoning was primarily based on figural reasoning. She focused on how many squares are "in each row." To relate the picture number to the number of squares in each row, she used her ability to reason numerically to describe her skip counting technique. Jenny's strategy for describing the first pattern was based on the figure as she notes the number of squares in the base and the height of each square. When asked to extend and generalize about the pattern, Jenny drew more on her numerical reasoning than figural reasoning. David's strategy for describing the first pattern was focused on the figural characteristics of the pattern. When he moves to extending, he primarily used numerical reasoning as his explanations centered on the increase of four tiles from one picture to the next.

Making generalizations using notation and/or descriptive words. Making generalizations using notation and/or descriptive words was common among the three case studies. The types of generalizations the students used were descriptive words or notation. The notation was invented or formal in nature. Dara used descriptive words to describe how she determined how many square tiles are in each row. She explained her "start, skip, answer" strategy as starting with the picture number, skipping the next number, and the answer is the next number. As she described her generalization, she used invented notation for the start number, skip number, and answer. Jenny used descriptive words to describe her generalization of the pattern focusing on the distance between picture numbers, but she did not use notation for her generalization. David used formal, algebraic notation to write a closed, explicit formula. David was able to explain in words why his formula worked. David was able to generalize with a formula, explain his formula, and describe how any picture in the pattern would look.

Discussion

There are three primary findings from this study. First, students did not use figural or numerical reasoning exclusively. The pictures appeared to promote both types of reasoning as the students described, extended, and generalized the patterns. This supports the work of former researchers who found students use a combination of numerical and figural reasoning, although one type may be the dominant choice for a student (Becker and Rivera, 2005; Moss et al., 2006).

Second, the younger students are more apt to rely more heavily on figural reasoning when analyzing pictorial growth patterns. Both Dara and Jenny used some level of figural reasoning for every stage (describe, extend, generalize). This is developmentally appropriate in a learning trajectory of algebraic reasoning. While David used figural reasoning, he was very focused on the number of tiles in both tasks. This raises the need to incorporate pictorial growth patterns of various levels of challenge for students. For these students, their choice of reasoning (figural, numerical, or both) is likely affected by their experiences in their mathematics classrooms.

Third, students will generalize using various ways to communicate – informal notation, language, and formal notation. Dara created her own notation system to describe how to determine the number of squares on each side of the first pictorial pattern. Jenny used language as she generalized about the pattern. David used formal algebraic notation. Developmentally, the progression of generalizations among students is expected, although oral language may precede informal notation for younger students.

This study suggests pictorial patterns would serve as valuable tools in elementary and middle school classrooms to promote algebraic reasoning. When students make sense of their generalizations in terms of a pictorial context, students will likely be able to describe mathematical relationships, the heart of algebraic thinking. Presumably, then, they would be able to make meaning of formulas rather than just manipulating numbers. Generalizing the patterns can potentially connect to their understanding of the constant and slope in linear equations. As evident in the findings, providing picture patterns also seems to elicit several different ideas about the same pattern.

In summary, incorporating the analyses of pictorial growth patterns into elementary and middle school mathematics promotes the use of figural and numerical reasoning together. Most importantly, students use algebraic thinking to generalize about the relationships, a critical factor to prepare students for success in their formal algebra courses.

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ASSESSING ELEMENTARY STUDENTS' FUNCTIONAL THINKING SKILLS: THE CASE OF FUNCTION TABLES

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Functional thinking is an appropriate way to introduce algebraic concepts in elementary school. We have developed a framework for assessing and interpreting students' level of understanding of functional thinking using a construct modeling approach. An assessment was administered to 231 second- through sixth-grade students. We then developed a progression of functional thinking knowledge. This investigation elucidates the sequence of acquisition of functional thinking skills. This study also highlights the utility of a construct modeling approach, which was used to create criterion-referenced and ability-leveled assessment. This measure is particularly suited to measuring knowledge change and to evaluating instructional interventions.

Introduction

Research into the effectiveness of teaching and learning interventions has become increasingly more mainstream, particularly as the call for evidence-based research grows louder. Unfortunately, instruments that measure learning outcomes from focused interventions are often researcher-created and are not tested for reliability and validity. This makes it difficult to compare research results and generalize findings into instructional recommendations.

The goal of this project is twofold. The first is to develop an assessment of elementary-school student's functional thinking abilities, with a specific focus on students' ability to find a rule of correspondence within a function table. The second is to develop a model of knowledge progression of elementary-level functional thinking skills. We identified a set of skills important for elementary-level functional thinking, designed an assessment which tapped these skills, administered the assessment to students in Grades 2 to 6, and used a construct modeling approach (Wilson, 2005) to developed a construct map, or model of knowledge progression, based on the student performance data. The findings provide insight into the typical sequence in which learners acquire functional thinking skills. This proposed sequence of functional thinking skills, and the assessment, are useful tools for measuring student knowledge. Eventually, the assessment can be used to measure learning gains after instructional interventions.

Theoretical Perspective

The transition to algebra is a notoriously difficult one for many students. Our students must be better prepared in order to take on the challenges of an increasingly technical and complex world. Traditional mathematics instruction, which focuses on teaching arithmetic procedures, followed by a similarly procedure-based algebra instruction in middle and upper grades, has not been successful in supporting student learning (Blanton, 2005).

One way to overcome this difficulty is to inculcate appropriate forms of algebraic reasoning into early math instruction. Algebraic reasoning is defined as a type of reasoning in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways (Kaput, 1999).

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Functional Thinking

Functional thinking is a particular kind of generalized thinking that lends directly to the development of algebraic thinking. It is a type of representational thinking that focuses on the relation between two varying quantities (Smith, 2008). Functional thinking is one of the core strands of Kaput's (2008) framework of algebraic reasoning and a core expectation for mathematics curriculum. For grades 3 through 5, students are expected to "describe, extend, and make generalizations about geometric and numeric patterns; represent and analyze patterns and functions, using words, tables, and graphs" (NCTM, 2008). At the heart of functional thinking is a relationship between two particular quantities; this can be referred to as a *rule of correspondence* (Blanton, 2005). This relationship can be used to find other sets of particular quantities that adhere to the same rule. The functional relationship binds together the set of numbers to which it applies.

Elementary-school students often focus on particular numbers as outcomes. In order to think in an algebraic way, and in a way which allows for generalization, one must understand that there are many possible outcome values. Finding a functional relationship between two sets of numbers is a way to jump from thinking of particulars to sets (Carrahar et al., 2008). This is an accessible way to get students thinking about numbers in a general way, and in a way which eases their transition to algebra. Thinking about functional relationships makes explicit the fact that many sets of values are possible for a given constraint (i.e., the rule) (Carrahar et al. 2008). This functional relationship can be described in terms of algebraic syntax, such a $y = 3x – 3$, and a prolonged consideration of functional relationships can scaffold a learner up to the logic behind such syntax (Carrahar et al., 2008).

Children's Functional Thinking Abilities

Exploring ways to support early functional thinking has been the focus of several recent teaching experiments, and commonalities across these studies indicate particular abilities that can be cultivated in elementary school.

Evidence from teaching experiments have shown that elementary-school students have the ability to understand the functional relation between X and Y values (Carraher, Schliemann, & Brizuela, 2003; Carraher, Martinez, & Schliemann, 2008), identify the rule or pattern (Carraher et al., 2003), use it to predict new values (Carraher et al., 2003, Carraher et al., 2008), and articulate the general rule verbally (Carraher et al., 2008; Cooper & Warren, 2008; Warren & Cooper, 2005; Warren & Cooper, 2006) and symbolically (Carraher et al., 2003; Carraher et al., 2008; Carraher & Earnest, 2003; Cooper & Warren, 2008; Warren & Cooper, 2006). In the teaching experiments, researchers typically use geometric or numeric patterns and introduce function tables as a representation for capturing functional relations. However, researchers have used their own assessments of student thinking and have not provided evidence for the validity of their assessments. In addition, little is known about the emergence of these skills in students in typical classrooms. In the current study, we focused on assessing students' knowledge of function tables, working with students in classrooms not receiving special interventions.

Assessment Development

We chose to focus on numeric patterns presented in function tables because they are a common and foundational component of functional thinking (Schliemann, Carraher & Brizuela, 2001). In an informal review of textbooks and national and state tests, we found function tables to be the most common problem format used at the elementary level. In particular, we focused on students' ability to identify rules of correspondence in function tables and use the rule to predict new instances.

A review of existing test items as well as a task analysis suggested at least 6 skills related to understanding function tables. (1) A precursor skill is to apply a given rule. When presented with a table with X and Y values, and a verbal or symbolic rule which describes how to compute the Y value from an X value, students should be able to use that rule to compute a Y value given a particular X value. (2) Students should also be able to recognize a function rule out of a selection of possible rules for a given table. In this case, students can test the rule against values in the table, rather than needing to generate the rule. (3) Students should be able to determine the next instance in a function table. However, students may be able to predict the next instance without thinking about the relation between the X and Y values (Schliemann et al., 2001). (4) Students should be able to extrapolate the function rule and articulate the function rule for a table in words. (5) Students should be able to predict a variety of instances in a function table, particularly instances that require identifying the function rule to make the prediction (e.g., predicting the 100th instance). (6) Students eventually learn to articulate the rule in the function table symbolically (Schliemann et al., 2001). Although there is a logical ordering of the relative difficulty of some of these skills (e.g., skill 1 is easier than skill 6), the relative difficulty of some skills could not be determined from previous research, and exploring the relative difficulty was one goal of the current research.

With these skills in mind, we created an assessment meant to tap each skill for working with functions presented in function tables. The functions were additive, multiplicative and two-operator functions. After data collection, we evaluated the progression of mastery of these skills, and used the data to inform the creation of a construct map (Wilson, 2005) of the development of functional thinking.

Methodology

The assessment was administered to a wide range of grade levels, as we expected differential performance and an increase in functional thinking skill through the grade levels. Participants were 231 2nd – 6th grade students attending two suburban schools. There were 52 second graders (24 girls), 50 third graders (30 girls), 25 fourth graders (15 girls), 60 fifth graders (28 girls), and 44 sixth graders (16 girls). Approximately 3% of the students were from minority groups. About 27% of students at the schools were eligible for free or reduced lunch.

The assessment contained 11 items, and was divided into three parts. Some of the items were broken into sub-items. The first section (one item) asked students to identify the number of eyes that a certain number of dogs would have, and to articulate the rule. It was designed to see to what extent students could engage in functional thinking given a supportive and grounded context without scaffolding from the problem formats that follow. The second section (six items) asked students to apply (two items) and recognize (four items) function rules. Of these four items, the function rules were presented as verbal statements (two items), and as algebraic equations (two items). The third section (four items) contained two problem types. Three of the items were function table problems, which required students to determine missing entries in the table, and then formulate both a verbal and symbolic rule. There was also a multiple-choice item asking students to identify the next value in a sequential numeric pattern.

The assessment was administered during a single class period. A member of the research team read the directions for each section, answered any questions, and enforced a time limit for each section. For 2nd and 3rd graders, each item was read aloud to reduce the reading demands. If a student had a question, the researcher would provide a helping prompt from a script.

All items were coded as correct or incorrect. A few items had open ended responses, and these were also coded as correct or incorrect according to a strict coding rubric. Each item or sub-item in the assessment only assessed one isolated skill.

Results

We used student accuracy on each item, in conjunction with other findings in the functional thinking literature, to place the skills into hierarchy and develop a construct map. A construct map is a representation of the continuum of knowledge that people are thought to progress through for the target construct (Wilson, 2005). Our preliminary construct map is presented in Table 1, with lower level skills at the bottom of the table. Student accuracy at each level of the construct map is presented in Table 2.

Table 1. Elementary Function Skill Construct Map

Level Description		**Abilities**	**Examples**
Level 4: Generate Symbolic Rule	• Generate an explicit symbolic rule	Can write rule of correspondence using algebraic symbols.	Y = X +4
Level 3: Generate & Use Verbal Rule	• Generate an explicit verbal rule • Complete a function table with missing values	Can write rule of correspondence verbally. Can fill in missing values of a function table, including a far instance.	-"You add 4 to the number in the X column to get the number in the Y column" -Completed function table
Level 2: Recognize Rule	Recognize Explicit Rule • Select a correct verbal rule out of several choices • Determine the next Y value in a function sequence	Can recognize a rule of correspondence for a table. Can also determine the next Y value in function tables and in patterns.	X Y 1 5 2 6 3 **7** Is the rule: -multiply by 5 -multiply by 3 **-add 4**
Level 1: Apply Rule	Application of an Explicit Rule • Use a given rule to determine new Y values	When given a rule for a table, can use it to determine new values of the table.	Complete the Table: *X+4=Y* X Y 1 5 2 6 3 **7** 4 **8**

Level One: Apply Rule

The easiest items were the application of a given function rule items (85% correct). This ability level included items that asked students to apply a rule to determine missing values in a

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table. This application of an explicit rule is level one of our construct map. These items required that students understood enough about a relationship between X and Y values in a table that they could determine new Y values when the rule is provided. To complete items like these, computational skill is required, but not yet any deep understanding of a functional relationship.

Table 2. Performance on Items by Level and Grade

	Knowledge Level			
Grade	1	2	3	4
2 (n=52)	71%	35%	11%	1%
3 (n=50)	82%	45%	23%	5%
4 (n=25)	94%	74%	50%	32%
5 (n=60)	91%	71%	53%	38%
6 (n=44)	91%	87%	69%	68%
Average	85%	61%	40%	28%

Level Two: Recognize Rule

Level two of our construct map is Recognize Rule. Within this level are two of our original skills: recognizing a rule, and determining the next Y value. Performance on items at this level was at 61%. Recognizing a rule for a table from a number of other options is presumed to be a skill children acquire before they can generate a rule on their own, and our data bear this out. This level also includes determining the next sequential Y value in a function table. This is different than determining other missing values in a function table, as the next value can be determined by extending the pattern in the Y values, if the table is ordered sequentially.

Level Three: Generate & Use Verbal Rule

Level three in our construct map incorporates the main set of skills students must have in order to have a grasp of function table problems. The skills included within this level are determining further missing values in a table and generating a verbal rule. The ability to generate a correct rule for a table coincided with the ability to determine the missing values. This should not be so surprising, as one would have to determine the rule to find the missing values. Performance on items that utilized these skills was at 40%.

Level Four: Generate Symbolic Rule

The fourth and final level in our construct map is Generate a Symbolic Rule. These items entailed writing the rule using algebraic notation, such a Y = X + 4. This was more difficult for students, presumably not only because of the use of variables, but because of the generality that variables imply. Performance on Symbolic Rule items was at 28%.

Assessment and Construct Map Evaluation

To further evaluate the assessment and refine the construct map, classical test and item response methodologies were used. The alignment of the relative difficulty of the items and the leveled construct map was considered. An item-respondent map (i.e., a Wright map) generated by a Rasch model (a type of item response model) was used in this evaluation. The left column of a Wright map is the respondent, or participant, column. Respondents with the most estimated ability are placed near the top of the column, and those with the least estimated ability are on the lower end. In the right column, the most difficult items are listed at the top, and the easier items are at the bottom of the column. This Wright map allows for a visual evaluation of construct map.

The purposed levels roughly clump together in the Wright map (Figure 1). There are some issues with Level 3 items overlapping with Level 2 and 4 items. The fact that there are different arithmetic operations in the underlying functional relationships in the items was the suspected cause for this overlap. Specifically, some items have an underlying functional relationship that involve addition (i.e., y=x+2), some multiplication (i.e., y=2x), and others, a combination of both (i.e., y=2x+2). Students in grades two through six have different proficiencies with these arithmetic skills, and so it is reasonable that this would affect item difficulty. When the items were separated by operation of underlying function, the Wright maps have good grouping of items by level, and good separation between levels (Figures 2-4). There is some compression of levels 3 and 4 in the combination-only Wright map. This is likely due to the difficulty of the items; if a student is of high enough ability to do the level 3 combination items, they are also likely to be of high enough ability for the level 4 combination items. Viewing the Wright maps separately by underlying function type, or operation, seems to be quite useful for delineating and clarifying the underlying functional thinking ability progression. Now the levels have much more separation, and the item difficulties are no longer confounded by arithmetic difficulty. This multidimensional model, with operational difficulty considered as a separate factor, is a much clearer way of tracking the progression of early functional thinking knowledge. This model will be developed further in future work.

Several analyses were performed to evaluate the validity and reliability of the assessment. To evaluate validity from measures of internal structure, the expected rank order of difficulty from the construct map was compared to the empirical rank order difficulty from the Wright map analysis (r_s=0.916). An item-mean location analysis was performed to determine if getting an item correct implicated a greater ability level on the part of the respondent than getting it incorrect would. All items behaved appropriately according to this metric. Internal consistency of the assessment was high. The classical test index of internal consistency for binary data (the Kuder-Richardson 20) was quite high, indicating that 93% of the variance was accounted for by the model. The analogous measure from item response methodology, the separation reliability index, was at 0.99. These measures can be thought of as similar to Cronbach's alpha. Additionally, several other analyses from item response methodology were employed, and all metrics of item analyses and person and item fit indicated that the assessment functions well.

Discussion

Functional thinking has been argued to be a useful way to introduce young students to fundamental algebraic concepts. Several teaching experiments have suggested instructional techniques for bringing functional thinking to elementary classrooms. As this topic becomes a focus of more research studies, it becomes increasingly important to have a reliable and valid measure that can be used to capture knowledge change and to have a well defined knowledge construct, so that general claims can be made across studies. In this project, we have identified key skills that are important for elementary-level functional thinking, with a focus on function table problems. These skills were then incorporated into an assessment, which was given to 231 2nd through 6th grade students. Student performance data was used to develop a construct map, or proposed knowledge progression, of elementary-level functional thinking skills. The resulting construct map provided insight into the acquisition of functional thinking knowledge in elementary-school students, and can be used to guide future research.

Benefits of a Construct Modeling Approach

This approach to measurement development, based on Wilson's construct modeling approach (2005), was useful for several reasons. First, it elucidated the relative difficulty of functional thinking skills, and at times this was not in line with our predictions. Second, the resulting assessment is a criterion referenced measure which is particularly appropriate for assessing the affects of an intervention on individuals (Wilson, 2005).

In regard to relative skill difficulty, skills fell together in ways which we did not predict. For instance, Level 2 includes the ability to recognize a rule, as well as determine the next sequential Y value. Based on our literature review and intuitions, we predicted that determining the next Y value would be the easier skill. Also, Level 3 includes the skills of completing a function table and generating a verbal rule. We had predicted that there would be a gap in students' ability to master these two skills. This construct modeling approach allowed us to see that these skills were of the same difficulty, as they could be completed by students of the same ability level. Another aspect we did not predict to play such a large role was the arithmetic operation of the underlying function. We initially predicted that functional thinking skill would function somewhat independently of operation type, but the data indicated that operation profoundly effects students' ability to successfully complete a function table problem. As such, operation type must be given attention in future functional thinking research. We are working to more formally incorporate arithmetic operation of the underlying function into our model.

Future Directions and Conclusions

Our elementary-level functional thinking assessment and construct map are important first steps, but they could each be further refined. Based on the construct map, we can now edit the assessment to more evenly include items at the different skill levels. Additionally, since it is now clear that the operation in the underlying function is a highly important factor, we will include more items so that there are items of each difficulty using each operation type. This revised assessment will then be used in a new iteration of this project. We hope to develop a multidimensional item response model that incorporates both functional thinking and operational skill level. This will allow us to more accurately measure change in students' knowledge.

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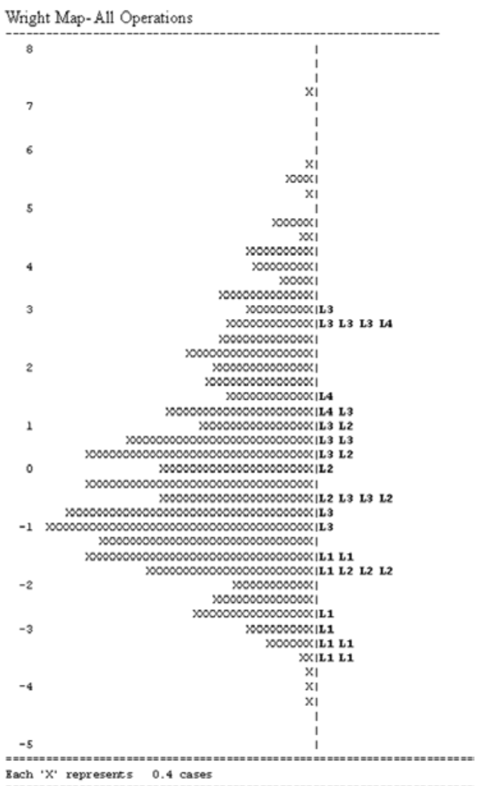
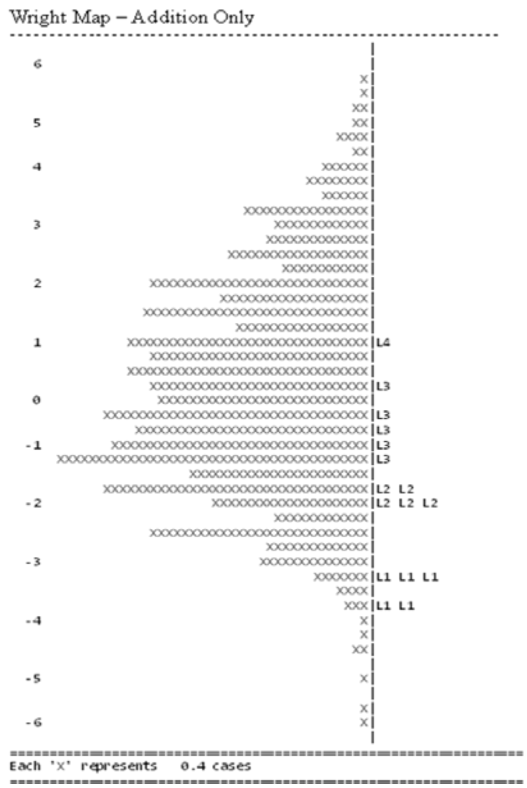
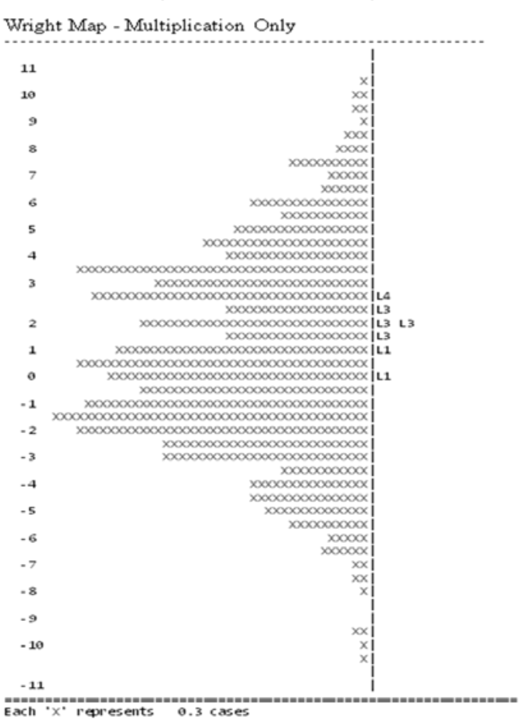
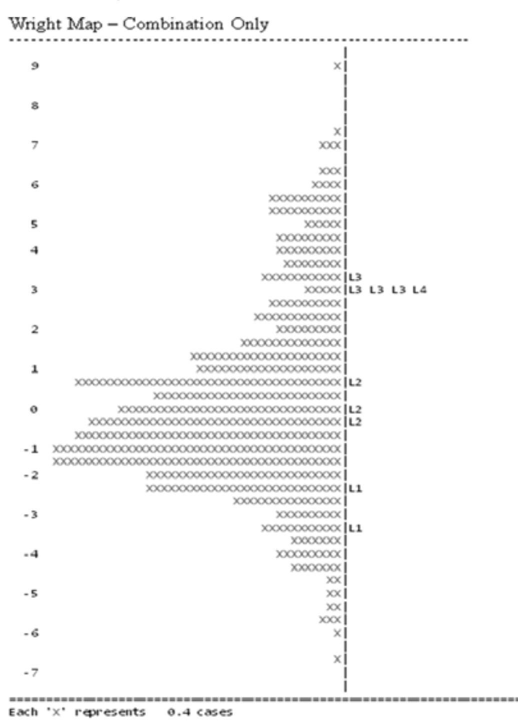
Figures 1 & 2. Wright Map & Wright Map, Addition





Figures 3 & 4. Wright Map, Multiplication, and Wright Map, Combination





ASSESSING KNOWLEDGE OF MATHEMATICAL EQUIVALENCE: A CONSTRUCT MODELING APPROACH

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Knowledge of mathematical equivalence is a foundational concept in algebra. We developed an assessment of equivalence knowledge using a construct modeling approach. Second through sixth graders (N = 174) completed a written assessment of equivalence knowledge on two occasions, two weeks apart. The assessment was both reliable and valid along a number of dimensions. The relative difficulty of items was consistent with the predictions from our construct map and accuracy increased with grade level. This study provides insights into the order in which students typically learn different aspects of equivalence knowledge and illustrates a powerful, but under-utilized, approach to measurement development.

Introduction

To increase students' success in algebra, there is an emerging consensus that educators must re-conceptualize the nature of algebra as a continuous strand of reasoning throughout school rather than a course saved for middle or high school (National Council of Teachers of Mathematics, 2000). Part of this effort entails assessing children's early algebraic thinking. In the current paper, we describe development of an assessment of one component of early algebraic thinking – knowledge of mathematical equivalence. Mathematical equivalence is the principle that two sides of an equation represent the same value. We employed a construct modeling approach (Wilson, 2003, 2005) and developed a construct map (i.e., a proposed continuum of knowledge progression) for students' knowledge of mathematical equivalence. We used the construct map to develop a comprehensive assessment, administered the assessment to students in Grades 2 to 6, and then used the data to evaluate and revise the construct map and the assessment. The findings provide insights into the typical sequence in which learners acquire equivalence knowledge. The study also illustrates an approach to measurement development that is particularly useful for detecting changes in knowledge over time or after intervention.

Theoretical Perspective

Too often, researchers in education and psychology use measures that have not gone through a rigorous measurement development process, a process that is needed to provide evidence for the validity of the measures (AERA/APA/NCME, 1999). For example, Hill and Shih (2009) found that less than 20% of studies published in the Journal for Research in Mathematics Education over the past 10 years had reported on the validity of the measures. Without this information, we cannot know if measures assess what they intended or if conclusions based on them are warranted.

A construct modeling approach to measurement development is a particularly powerful approach for researchers interested in understanding knowledge progression. The core idea is to develop and test a *construct map*, which is a representation of the continuum of knowledge that people are thought to progress through for the target construct. Although Mark Wilson has written an authoritative text on the topic (Wilson, 2005), there are only a handful of examples of using a construct modeling approach in the empirical research literature (e.g., Claesgens, Scalise, Wilson, & Stacy, 2009; Wilson & Sloane, 2000). We illustrate how this approach was used to develop a reliable and valid measure of mathematical equivalence knowledge.

Although few previous studies have paid careful attention to measurement issues, a large number of studies have assessed children's knowledge of mathematical equivalence (sometimes called mathematical equality). Understanding mathematical equivalence is a critical prerequisite for understanding higher-level algebra (e.g., Kieran, 1992; MacGregor & Stacey, 1997). Given the importance of mathematical equivalence, it is concerning that students often fail to understand it. Many view the equal sign *operationally*, as a command to carry out arithmetic operations, rather than *relationally*, as an indicator of equivalence (e.g., Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Kieran, 1981; McNeil & Alibali, 2005). Evidence for this has primarily come from three types of tasks: (1) solving open equations, such as $8 + 4 = \square + 5$, (2) evaluating the structure of equations, such as deciding if $3 + 5 = 5 + 3$ is true or false, and (3) defining the equal sign (Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1980; e.g., Carpenter, Franke, & Levi, 2003; Li, Ding, Capraro, & Capraro, 2008; Seo & Ginsburg, 2003). The primary source of U.S. children's difficulty understanding mathematical equivalence is thought to be their prior experiences with the equal sign (e.g., Baroody & Ginsburg, 1983; Carpenter et al., 2003). What is less clear is how a correct understanding of mathematical equivalence develops without specialized interventions.

Table 1. Construct Map for Mathematical Equivalence Knowledge

Level 4: Comparative	Compare the expressions on the two sides of the equal sign, including recognizing that doing the same thing to both sides maintains equivalence. Relational definition considered the best definition.
Level 3: Relational	Accept and solve equations with operations on both sides and recognize and generate a relational definition of the equal sign (although it co-exists with an operational definition).
Level 2: Flexible Operational	Accept and solve equations not in operations-equals-answer format that do not directly contradict an operational view of the equal sign, such as equations with operations on the right or with no operations. Continue to think of equal sign operationally, or in other non-relational ways.
Level 1: Rigid Operational	Only accept and solve equations in operations-equals-answer format correctly; define the equal sign operationally (e.g., it means "get the answer").

The primary goal of the current study was to develop an assessment that could detect systematic changes in children's knowledge of equivalence across elementary-school grades (2nd through 6th). To accomplish this, we utilized Mark Wilson's Construct Modeling approach to measurement development (Wilson, 2003, 2005). The core idea is to develop and test a *construct map*, which is a representation of the continuum of knowledge that people are thought to progress through for the target construct. Our construct map for mathematical equivalence is presented in Table 1. We hypothesized that there would be a transition phase between a rigid operational view and a relational view of equivalence, which we labeled *Level 2: Flexible* *operational view*. We also included a fourth level of understanding to capture a more flexible and sophisticated relational understanding of equivalence – comparing the expressions on the two sides of the equal sign.

Method

We used our construct map to guide creation of an assessment of mathematical equivalence knowledge, with items chosen to tap knowledge at each level of the construct map using a variety of problem formats. We developed a pool of possible assessment items from the past research and selected and modified items so that there were at least two per construct map level for each of the three common item types identified in the literature review - solving equations, evaluating the structure of equations, and defining the equal sign.

We administered an initial long version of the assessment to 174 children in grades 2 through 6. Analyses of these results and input of a domain expert informed the creation of two shorter, comparable forms of a revised assessment, which were administered to the same students two weeks later.

Results

In presenting our results, we focus on the revised forms of the assessment, with data from the initial assessment used as supporting evidence when appropriate.

Evidence for Reliability

Internal consistency, as assessed by Cronbach's α, was high for both of the revised assessments (Form 1 = .94; Form 2 = .95). Performance on the assessment was also very stable between testing times. There was a high test-retest correlation overall for both Form 1, $r(26) = .94$, and for Form 2, $r(26) = .95$.

Evidence for Validity

First, experts' ratings of items provided evidence in support of the face validity of the test content. Four experts in mathematics education rated most of the test items to be *important* (rating of 3) to *essential* (rating of 5) items for tapping knowledge of equivalence, with a mean rating of 4.1.

Next, we confirmed that we could equate scores across the two forms. Our forms were administered to equivalent groups, demonstrated similar statistical properties, and received similar difficulty estimates for most paired items. Having met these criteria, it is reasonable to use a random groups design in IRT to calibrate the scores from the two forms, placing all item difficulties and student abilities on the same scale.

To evaluate the internal structure of the assessment, we evaluated whether the *a priori* predictions of our construct map about the relative difficulty of items were correct (Wilson, 2005). An item-respondent map (i.e., a Wright map) generated by the Rasch model was used to evaluate our construct map. In brief, a Wright map consists of two columns, one for respondents and one for items. On the left column, *respondents (i.e., participants*) with the highest estimated ability on the construct dimension are located near the top of the map, while those with the least ability are located near the bottom. On the right column, the *items* of greatest difficulty are located near the top of the map and those of least difficulty are located near the bottom of the map. The Wright map seen in Figure 1 allows for quick visual inspection of whether our construct map correctly predicted relative item difficulties.

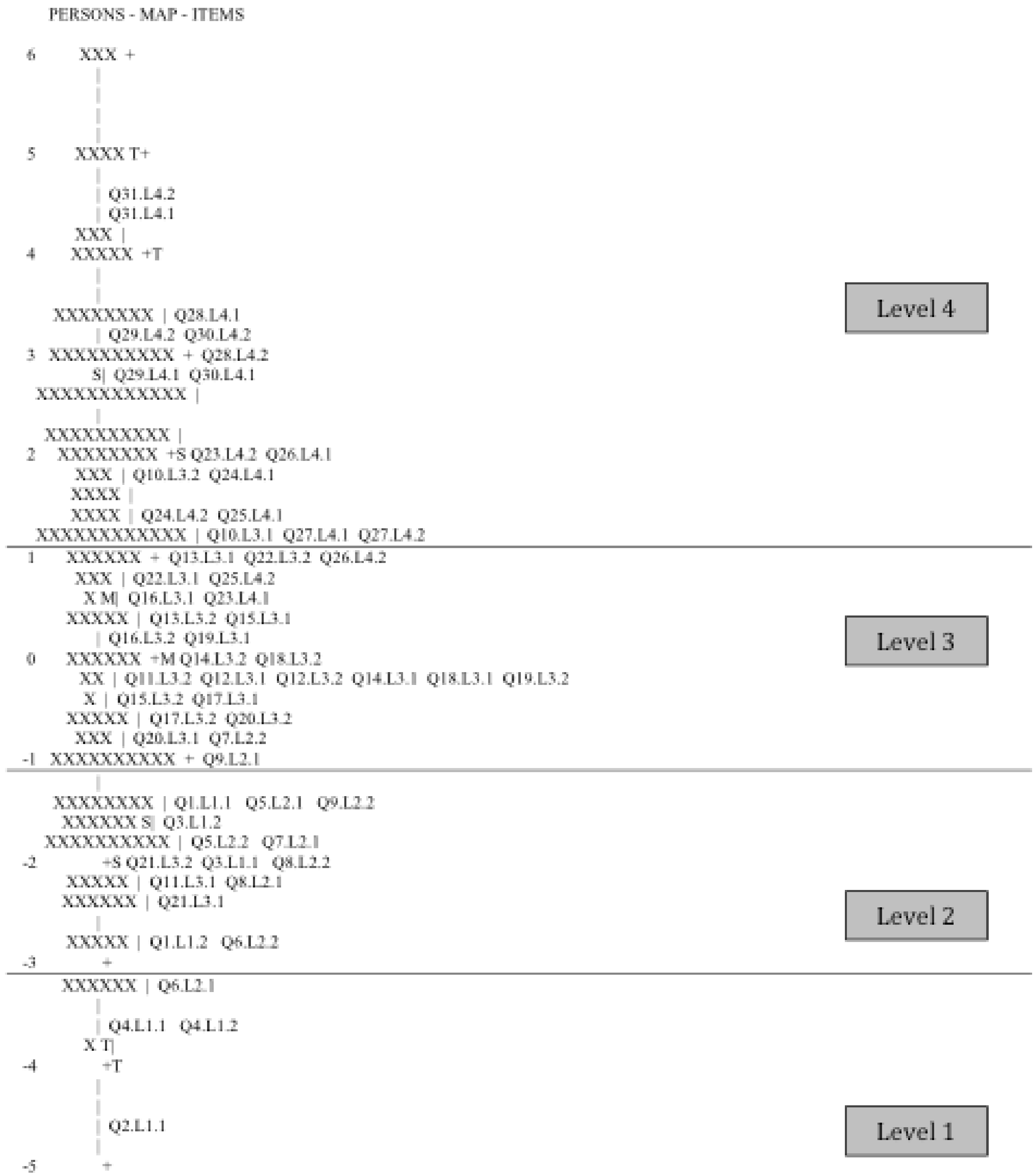

Figure 1. Wright Map for the Mathematical Equivalence Assessment. Easiest items are at the bottom of the map (L1, L2, L3 & L4 indicate which Level the items was expected to be at). Each X represents one person

As seen in Figure 1, the items we had categorized as Level 4 items were indeed the most difficult items (clustered near the top with difficulty scores greater than 1), the items we had categorized as Levels 1 and 2 items were indeed fairly easy items (clustered near the bottom with difficulty scores less than -1), and Level 3 items fell in between as predicted. Overall, the Wright map supports our hypothesized levels of knowledge, progressing in difficulty from a rigid operational view at Level 1 to a comparative view at Level 4. This was confirmed by Spearman's rank order correlation between hypothesized difficulty level and empirically derived item difficulty, $\rho(62) = .84$ $p < .01$. Additional evidence for the validity of the assessment is that ability estimates should increase with grade level. Ability estimates can be thought of as students' predicted success on the equivalence assessment. As expected, mean ability estimates progressively increased as grade level increased, $\rho(173) = .76$, $p < .01$.

To gather evidence based on relation to other variables, we examined the correlation between students' standardized math scores on the Iowa Test of Basic Skills (ITBS) and their estimated ability on our equivalence assessment. As expected, there was a significant positive correlation between students' scores on the equivalence assessment and their grade equivalent scores on the ITBS for mathematics (r (86) = .85 and r (83) = .87, p's < .01, for Forms 1 and 2 respectively), even after controlling for their reading score on the ITBS (r (86) = .79 and r (83) = .80, p's < .01, for Forms 1 and 2 respectively). This was true within each grade level as well. This positive correlation between our assessment and a general standardized math assessment provides some evidence of convergent validity.

Characterizing Students' Knowledge Levels

Much of the power of IRT results from the fact that it models participants' responses at the item level, as opposed to classical test models which are modeled on responses at the level of test scores. Thus, with IRT, we can use participant ability and item difficulty estimates to glean specific information about individual student knowledge and about the relative skill levels of cross-sections of students. For summary purposes, we can partition students into groups according to knowledge levels of our construct map. Students can be classified as working on developing knowledge at a particular level based on their ability estimate. The distribution of ability levels by grade is listed in Table 2.

Table 2. Percentage of Students Performing at Each Knowledge Level

	Knowledge level			
Grade	1	2	3	4
2 (n = 37)	14	70	14	3
3 (n = 42)	2	35	40	23
4 (n = 33)	3	24	21	52
5 (n = 34)	0	3	18	79
6 (n = 28)	0	0	0	100

In second grade, almost all students were at Level 2, or a flexible operational view, meaning they were in the process of gaining a more flexible, operational view of equivalence. In third grade, a majority of the students had gained a flexible, operational view and were working on a relational view of equivalence (Level 3) or had advanced to beginning to compare the two sides of an equation (Level 4). From fourth to sixth grade, an increasing number of students were advancing to Level 4, with all of the sixth graders reaching Level 4. Note that classification at a particular level indicates that students are working on learning this knowledge, not that they have mastered it.

Discussion

Numerous past studies have pointed to the difficulties elementary-school children have understanding equivalence (e.g., Behr et al., 1980; Carpenter et al., 2003), underscoring the need for systematic study of elementary students' developing knowledge of the concept. We used a construct modeling approach to develop and validate an assessment of mathematical equivalence knowledge. We proposed a construct map that specified a continuum of knowledge progression from a rigid operational view to a comparative view. We created an assessment targeted at measuring the latent construct laid out by our map and used performance data from an initial round of data collection to screen-out weak items and to create two alternate forms of the assessment. The two forms of the revised assessment were shown to be reliable and valid along a number of dimensions, including good internal consistency, test-retest reliability, test content, and internal structure. In addition, our construct map was largely supported. Below, we discuss possible sources of increasing equivalence knowledge, benefits of a construct modeling approach to measurement development, and future directions.

Developing Knowledge of Equivalence

Our construct map specified a continuum of knowledge progression from a viewing the equal sign operationally to comparing the two sides of an equation. Inspection of the Wright map (see Figure 1) indicated that items of a given level clustered together. There did appear to be a transition level between a rigid operational view and a relational view. In particular, items with operations on the right or without operations were easier than items with operations on both sides. In addition, some students in elementary school were advancing beyond a basic relational view (Level 3) and were comparing the relations between the two sides of an equation (Level 4). By fifth and sixth grade, a majority of students were successful on Level 3 and some Level 4 items. A comprehensive assessment of children's knowledge of mathematical equivalence requires items detecting transitional knowledge as well as items detecting more advanced relational thinking.

We have also analyzed the use of the equal sign in the textbooks used at the participating school and gathered teacher reports of student exposure to equations in different formats. Both analyses suggest that mere exposure to the equal sign is not the primary driver of knowledge change. Rather than simple exposure, explicit attention to ideas of equivalence in classroom discussion, with attention to the equal sign as a relational symbol, may promote knowledge growth, even in typical classrooms. This is in line with teaching experiments conducted by Carpenter and colleagues on the effectiveness of classroom discussions of non-standard equations and what the equal sign means (Jacobs et al., 2007).

Benefits of a Construct Modeling Approach to Measurement Development

A construct modeling approach to measurement development is a particularly powerful one for researchers interested in understanding knowledge progression, as opposed to ranking students according to performance. We found construct modeling to be very insightful and hope this article will inspire other educational and developmental psychologists to use the approach. This measurement development process incorporates four phases that occur iteratively: 1) propose a construct map based on the existing literature and a task analysis, 2) generate potential test items that correspond to the construct map and systematically create an assessment designed to tap each knowledge level in the construct map, 3) create a scoring guide that links responses to items to the construct map, and 4) after administering the assessment, use the measurement model, in particular Rasch analysis and Wright maps, to evaluate and revise the construct map and assessment (Wilson, 2005). The assessment is then progressively refined by iteratively looping through these phases. Overall, repeatedly evaluating our theory against performance on individual items helped us to evaluate our commonsense assumptions that otherwise would have gone unquestioned.

Another benefit of a construct modeling approach is that it produces a criterion-referenced measure that is particularly appropriate for assessing the effects of an intervention on individuals (Wilson, 2005). We developed two versions of our equivalence assessment so that different versions could be used at different assessment points in future intervention or longitudinal research. Our equivalence assessment could also help educators modify and differentiate their instruction to meet individual student needs.

Conclusions

Although we have made an important first step in validating a measure of equivalence knowledge, much still needs to be done. A critical next step is to provide evidence for the validity of the measure with a larger and more diverse sample; a task we are currently undertaking. Further, we need to know the predictive validity of the measure – for example, does the measure help predict which students need additional math resources or who are ready for algebra in middle school?

In conclusion, our assessment and accompanying construct map for developing knowledge of mathematical equivalence are quite promising. The construct map provides a means for tracking students' developing knowledge of equivalence across elementary school and a construct modeling approach provides a criterion-referenced analysis of performance. Multiple measures of reliability and validity support the promise of our measure. In the future, this measure should be a valuable tool for researchers who are evaluating the effectiveness of different educational interventions and for teachers who want to differentiate their instruction.

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BEHIND AND BELOW ZERO: SIXTH GRADE STUDENTS USE LINEAR GRAPHS TO EXPLORE NEGATIVE NUMBERS

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Ten sixth grade students utilized their understanding of creating graphs for linear rules to explore negative numbers, and mathematical operations with negative numbers. Students extended the axes of their graphs (the first quadrant) to include numbers "below" zero on the vertical axis, and "behind" zero on the horizontal axis. They then created graphs for linear rules with either a negative constant (y=mx-b) or a negative multiplier (y=(=m)x+b). Results indicate that constructing linear graphs supported students in developing both a unary and binary understanding of negativity. In addition, working within the four quadrants of the graph, and the process of plotting trend lines for linear rules may also have supported a multiplicative understanding of negativity.

Introduction

The purpose of this study was to determine whether Grade 6 students' previous work with linear growing patterns, and graphical representations of linear growing patterns, could support their understanding of negative numbers and operations with negative numbers. Three lessons were developed as a way of extending students' understanding of graphical representations of linear rules that contained only positive numbers to incorporate negative numbers. Activities included extending the boundaries of the graphing space from the first quadrant to include all four quadrants, and working with linear rules with a negative constant (y=mx-b) or a negative multiplier (y=(-m)x+b).

Theoretical Framework

When coming to understand negative numbers, students must develop an integrated understanding that the minus sign performs several roles, which then leads to an overall understanding of 'negativity' (Vlassis, 2004). Two roles are particularly pertinent when beginning to think of 'negativity' (Gallardo & Rojano, 1993) – the first is the unary role of the minus sign that acts as a structural signifier to indicate that an integer is negative. The second is a binary role of the minus sign that is an operational signifier, that is, the sign is an indication of the operation of subtraction. Studies have shown that students do not consider that the minus sign could have a double status, that is, have either a unary or binary function and instead tend to have a rigid idea of a minus sign as indicating subtraction (Carraher, 1990). Other studies have outlined the deep-rooted and widely held misconceptions students have about signed numbers and the kinds of operations that can be performed on them (Vlassis, 2002, 2004; Gallardo, 2002; Gallardo & Romero, 1999).

Past studies have looked at two general types of models for teaching negative numbers. One model is based on the embodiment of negative numbers in practical situations – for instance, a witch's pot where hot cubes are positive and cold cubes are negative, with the goal of achieving equilibrium through the addition or subtraction of negative or positive cubes (Kemme, 1990, as cited in Streefland, 1996). Research suggests that these kinds of models are not beneficial (Streefland, 1996) primarily because students have difficulty understanding the connection between the magnitude of number (in terms of its proximity to zero) and the temperature of an object.

Researchers have demonstrated that when teaching negative numbers, a more successful model is a number line, which has been shown to be a more intuitive representation for students (Bruno & Martinon, 1999; Streefland, 1996; Hativa & Cohen, 1995). Hativa and Cohen (1995) conducted a study with Grade Four students, and demonstrated that younger students can develop an understanding of negative numbers by extending their understanding of the positive number system. By situating students' problem solving on a number line, students learned how to locate numbers on a number line (either side of 0), compare the magnitude of two numbers, and estimate the distance between two numbers. This in turn enabled them to perform some simple addition and subtraction (a+b, a-b, -a+b, -a-b). The researchers found that students did have an intuitive sense of negative numbers even at this young age, but that operations with signed numbers were still problematic. Similar results have been reported in studies with older children, for whom the operations with two negative numbers were found to be difficult (Bruno & Martinon, 1999). Other researchers (Streefland, 1996) have explored children's conceptions of negative numbers as being the mirror opposites of positive numbers on the number line with 0 playing a prominent role in the approach to distinguishing between positive and negative integers.

There have been no studies to date that have specifically looked at how children's intuitions about negative numbers on the number line can be incorporated into their understanding of linear graphs and linear rules, and how this in turn can support their conceptions of negative numbers. Graphs offer an opportunity to visually represent both the location of negative numbers in 2-dimensional space, and the outcome of operations with negative numbers, for instance, the fact that multiplication does not always result in a greater number if the number being multiplied is less than 0 (Greer, 1992). When plotting points for linear rules students can explore how a negative multiplier (y=(-m)x+b) or a negative constant (y=mx-b) or a negative x-value (y=m(-x)+b) affect the value of y.

Methods

This study is part of a larger three-year study using design research methodology to develop and assess new learning situations that support students' understanding of linear relationships. In year 1 of the study, students developed an understanding of the covariation of two sets of data, position cards and numbers of tiles in growing patterns, and expressed this as a pattern rule, for example "total tiles = position number x 2 + 3." In the second year of the study, the instruction was extended to include graphical representations of linear relationships. Results indicated students developed an initial understanding of the connections among growing patterns, pattern rules, and graphs (for example, author, 20xx). These results formed the basis for the instructional design of the present study.

As outlined in the research literature, the number line can be a powerful support for learning negative numbers with 0 playing a prominent role in distinguishing between negative and positive integers. In this study the learning of negative numbers was integrated into students' developing understanding of linear graphs, thus allowing them to work in a Cartesian graphing space bounded by both a vertical and a horizontal number line that intersect at 0 (the origin). Three lessons were developed for this study. The instruction centered the inclusion of negative numbers in linear rules y=mx+b and the effect on graphical representations, both in terms of the trend lines of the pattern rules themselves, and how negative numbers are represented in the graphing space.

Lesson 1. The goal of this lesson was to have students think about extending the graphing space they had been working in (the first quadrant) in order to think about where on the graph negative numbers could be represented. Initially students were asked to brainstorm about where they had seen negative numbers in order to see what metaphors the students used when thinking about negative numbers, with the goal of establishing two models of negative values – a horizontal number line and a vertical number line. This was so that students could think about how both the vertical and horizontal dimensions of negative numbers could be used to extend the two axes of the first quadrant graph that they were familiar with.

Lesson 2. In this lesson, students were shown the rule "y = 4x-2." Students were asked how they would graph the rule. I was interested to see whether they would utilize a previous heuristic, "the constant part of the rule is responsible for where the line "starts" on a graph at the y-intercept." The goal was to determine how the students would carry out calculations with negative numbers, and whether the graph supported this.

Lesson 3. The students were asked to create graphical representations for rules with negative multipliers, such as y= (-4)x + 20.

Teaching Intervention

The three research lessons were delivered during 6 class lessons over the course of three weeks, and were alternated with the classroom teacher's math lessons that covered other strands of the curriculum. The lessons were taught during the students' regular math classes. The lessons were taught by the classroom teaching intern, who was in her second year of a two year Master of Arts degree in teaching, child development and education.

Participants

This study included one grade six class of 10 students (5 boys and 5 girls). They came from a class of 22 students, 11 of whom were in grade 5. One grade six male student did not wish to participate in the study. This was an opportunistic sample of students who had participated in the previous 2 years of the larger 3-year study. The students were from a University Laboratory School, and had been taught mathematics using reform-based teaching approaches with an emphasis on inquiry-based tasks, use of manipulatives, and mathematical discussion. The Laboratory School was chosen (rather than one of the classrooms in the public school system that had participated in the previous two years' research) primarily because of their mandate to "provide an environment that fosters research and professional inquiry and involvement in supporting new ideas related to improving education." This allowed for the content of the lessons to include material that went beyond the expectations set out in the provincial curriculum documents for grade 6.

Data Collected

The primary forms of data collected were comprised of videotaped observations of classroom lessons and transcripts of those videotapes. Videotape included observations of classroom whole group discussions, with a particular focus on capturing student-student interactions.

During classroom sessions, students were also interviewed as they engaged in individual or small group work. Other supporting data included all student artifacts and field notes.

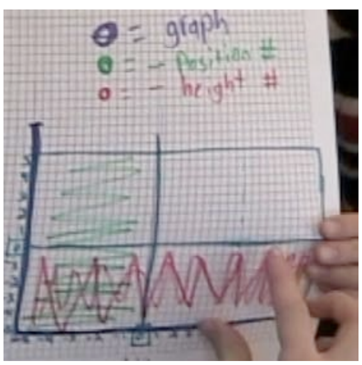
Data Analyses

Video data and transcripts were viewed and coded, and codes were used to merge categories together to establish trajectories of understanding. These trajectories underwent both a descriptive analysis in order to identify the elements of the trajectory, and a theoretical analysis as a means to identify the theoretical constructs of the understanding of negativity. Cross-referencing of trajectories from the codes identified in transcripts to other sources of data (particularly field notes and student work) was undertaken for the purpose of complementarity (Greene et. al., 1993).

Results

Extending the Graphing Space

When discussing their experiences with negative numbers, the students introduced several metaphors including typical examples such as debt and temperature, and a less typical example of a timeline. By using these commonly understood metaphors, the values along the number lines were imbued with meaning. Temperature became the basis for a vertical model of negative numbers, representing negative (cold) and positive (warm) values. The concept of the timeline underpinned the horizontal model, with years BCE representing negative values and years CE representing positive. In both cases the students were familiar with how to represent negative values, "below zero" starting with negative 1 and numbering down on the vertical model, and "behind zero" starting with negative 1 and extending left on the horizontal model. Zero was used as the division between positive and negative numbers, with the two (positive and negative) number lines mirroring each other on either side. The students knew that the farther away from zero, the larger the numeral (signed or unsigned) became.



Students used three distinct ways to extend their first quadrant graphs to include the three other quadrants. One approach was to start with the vertical number line in the middle of the grid, and the horizontal line along the bottom. Both lines had zero in the middle. The vertical axis was labeled with positive and negative values – the negative values from -1 to -10 but after -10 came the 0 of the horizontal axis. One student explained the problem with this initial representation:

> *I realized that on the thermometer line it didn't make sense to count down from -1 to -10 and then have 0 again. So I decided to lift the bottom number line up to the middle and kind of overlap the zeros [gesture lifting the horizontal line from the bottom of the page and placing it in the middle of the page]. That way there was room for negative numbers to go behind the zero and to go below the zero.*

Another way was to extend the vertical axis down "below zero" and the horizontal axis left "behind zero."

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

> *We put the 0 in the centre of the page. We went down like this (extending down from the origin to extend the vertical axis) and across like this (extending horizontally to the left of the origin).*

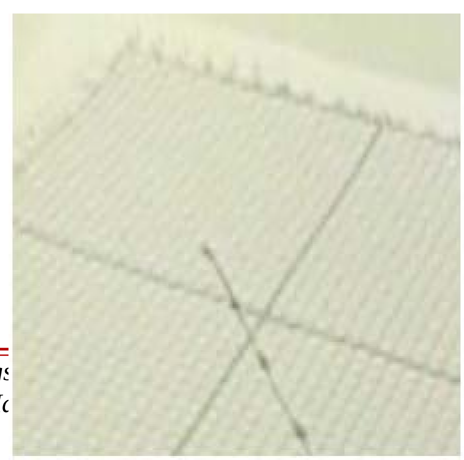
A third approach was to reconfigure the numbering of the existing horizontal and vertical axes along the bottom and left-hand side of the page. The fact that the two number lines "ran into each other" in the lower left corner did not seem to bother the students.

> *I did my numbers along the side and bottom, like we usually do. Then I colour-coded it, to make it easier for me visually. So, the bottom half of the graph is red and that's all the negative numbers, because it's under the line…the zero line (the horizontal axis) and then these are all the positive numbers above the zero line. And here, this half (the left half) is all green, and the green is all the negative numbers that are behind zero (the vertical axis). And these (to the right) are all the positive numbers in front of zero. When you look at where the zero lines intersect, that's the ultimate zero.*

Plotting a Rule with a Negative Constant

These students had previous experience creating graphs for rules with a positive constant. In order to plot points, the students knew that the y-intercept represents the value of the constant in a pattern rule. They plotted points by carrying out the operation of the rule with each x-value, which meant multiplying the x-value by the multiplier and then adding the constant.

The students were introduced to a negative constant both as a signed number and as representing the operation of subtraction. The negative constant meant that the y-intercept was a point at a negative value on the y-axis. The negative constant also meant that a constant amount had to be subtracted as each point was plotted. For example, Jack created a graph of a rule with a negative constant by calculating the rule at successive x-values and subtracting the constant amount instead of adding. The physical plotting of points, and the contrast of counting down for a negative constant (subtracting a constant amount from each point) reinforced the notion of directionality of negative and positive numbers, in this case oriented to the vertical axis as the number line. Jack demonstrated how to plot the points for the first few x-values.

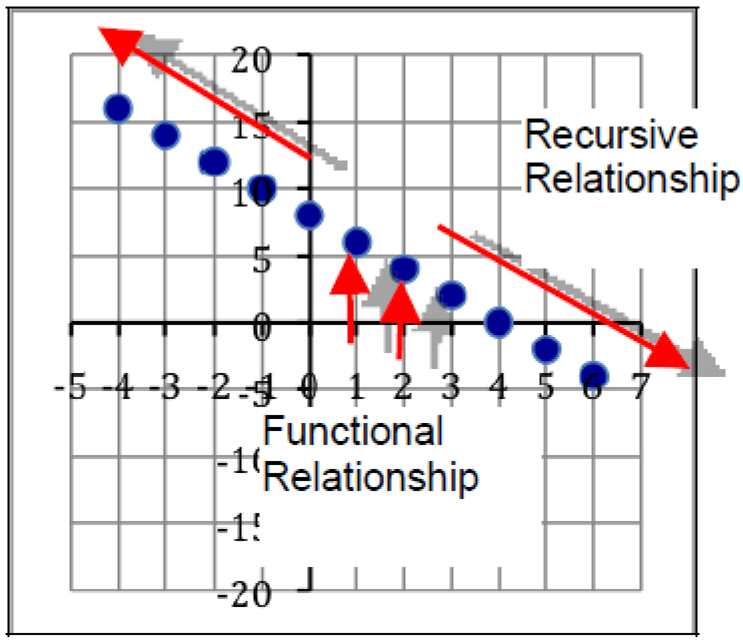


> *The first x-value is times 4, so 1 times 4 equals 4 and then you do minus 2, which equals 2 [plots point on the graph at (1,2)]. Two times 4 equals 8 and then you subtract 2, which is 6 [plots point at (2,6)]. Three times 4 is 12 minus 2 is 10.*

Elsie started by plotting the y-intercept. "Now, the rule is 0 times 6 minus 3 so 0 times 6 is 0 minus 3 is negative 3.” This is a blending of the understanding of both meanings of negative (subtraction and negative integer). She continued calculating the rule for each positive x-value. She then calculated the rule for an x-value of negative 1. “Negative 1 times 6 is negative 6, minus 3 is negative 9.” She plotted a point at (-1,-9). I asked her if she thought that was the correct answer. She confidently replied it was, “Because it makes a straight line. It’s a nice way to visually check and see if I’m doing it right.” Elsie used the trajectory of the trend line to check the correctness of her calculations.

Plotting a Rule with a Negative Multiplier

In the final activity, students were asked to construct graphs for rules with a negative multiplier and a positive constant. The students used two strategies to plot the trend line for the linear rule with a negative multiplier. Students used a strategy based on the functional relationship between the x-value and y-value, carrying out the calculations with the x-value to find the value of y for the first two or three points. They then relied on a recursive strategy, which involved subtracting the amount of the multiplier from the previous y-number (from left to right in the upper right quadrant) so that the negative multiplier represented the successive subtraction of 2. When plotting points for negative x-values all students relied on a recursive approach of adding the value of the negative multiplier to previous y-numbers, or “going up 2 each time” from right to left. The trend line of the graph in the upper left quadrant was a tool for checking their calculations, and the correct placement of the points. The students developed the understanding that multiplying positive x-values with negative multipliers resulted in smaller values, and that as the value of x increased, the resulting value of y decreased. They then related this to the slope of the trend line on the graph.



> *The numbers (y-axis numbers) if you start at the zero position (y-intercept) and keep going, the numbers got smaller. So instead of getting bigger and the line going like this [shows angle positive slope with forearm] they just went like that [angled arm downward]. The slope was going down instead of up.*

However the students noticed that the opposite was true for negative x-values.

> *Ya, what’s weird is if you have a negative multiplier and a negative x-number it gets higher (the trend line in the upper left quadrant), cause usually you’d think a negative multiplier and a negative x-number must be getting lower, because negatives are lower, but it keeps getting higher in the positives (y-values) when you go into negative x-values.*

> *It's really cool. This can help you think about why a negative times a negative is a positive.*

Discussion

The purpose of this study was to examine how building on students' work with graphical representations of linear relationships could support their understanding of negative numbers. During the course of the study, we identified that students were able to use their intuitions about vertical (thermometer) and horizontal (timeline) number lines to develop an understanding of the numeric values represented in the four quadrants of the Cartesian plane. Although the students utilized three different strategies to create their four quadrant graphs, all of the students were successfully able to extend their graphing space by integrating a vertical and horizontal model of negative numbers to include numbers "below zero" on the vertical axis, and "behind zero" on the horizontal axis.

Once they had meaningfully created spaces "below" and "behind" zero the students were able to then apply their understanding of working with linear rules to incorporate negative numbers. They included negative numbers as the constant or the multiplier of the pattern rules, and calculated points for negative x-values. Students were able to make connections between the pattern rules and the resulting trend lines on the graph – including developing an understanding that pattern rules with a negative constant would have a y-intercept below 0, and pattern rules with a negative multiplier would have a trend line that sloped downward. All students were able to carry out addition and subtraction with negative numbers, including subtracting a number from a negative number, or -a-b, which has been shown to be problematic (Streefland, 1996). In this case, the students were able to utilize the trend line as a way of checking the "correctness" of their answers, since they understood that linear trend lines are straight, and so calculating linear rules with x-values (positive or negative) had to result in a value somewhere along the trend line.

As previously outlined, there are two kinds of negativity that are important to develop when considering negative numbers. The first is unary understanding, an understanding of negativity as a point on the number line. The second is binary understanding, negativity as indicating movement along a number line that corresponds with the operation of subtraction. When considered on a horizontal number line, positive and negative integers allow two directions to be used – to the right of 0 (positive) and to the left of 0 (negative). In this study the integration of two number lines introduced two more directions, above 0 or up (positive) and below 0 or down (negative). Just as number lines are defined by two directions, positive or negative, the four areas of the graph allowed for 4 different combinations of values represented by points in each of the four quadrants: positive/positive (PP), negative positive (NP), positive negative (PN) and negative, negative (NN). The four quadrants as area models of positive and negative values supported students in their ability to identify the position of negative numbers in the Cartesian graphing space. Creating graphs for linear rules that had either negative constants, or negative multipliers allowed students to develop an understanding of carrying out arithmetic operations with negative numbers. Plotting points allowed students to reason about adding and subtracting with negative values, with the added support of the trend line to determine if their results were correct by viewing whether the resulting point on the graph was in line with other points.

Students in this study seemed to also develop a multiplicative understanding of negativity, in addition to a unary and binary understanding. Multiplicative negativity is the understanding that the product of multiplying negative or positive x-values with negative or positive multipliers results in a product that is represented by a point in one of the four quadrants of the graph. The four quadrants acted as area models of positive and negative values, which supported students in their ability to think multiplicatively about negative values. When multiplying positive or negative multipliers with positive or negative x-values, the resulting point on the graph expressed the sign of the x-value, and the sign of the y-value based on the influence of the sign of the multiplier. The resulting points could be above or below the x-axis, and in front of or behind the y-axis in different combinations of positive and negative space. All of the students began to identify points within the four quadrants of the graph as representing the relationship between positive and negative multipliers and positive and negative x-values. However, this was a limited study (3 lessons with 10 students), and further research is required.

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FOURTH GRADE STUDENT CONCEPTIONS OF PIECEWISE LINEAR POSITION FUNCTIONS

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This study examined fourth grade student concepts of graph shapes in piecewise linear position functions and their relationship to the motion of an animated runner by focusing on the student predictions of the runner's motion. Descriptions from six students reveal interpretations of the graph and strategies used to answer questions related to rate of speed. Two conceptions are identified as being common to all the students. When students considered graph shape they usually exhibited an iconic interpretation. Alternately, students read values from the axes and apply those in coming to a prediction story.

Introduction

What students can understand about piecewise-defined linear position functions necessarily begins with what they do understand – what their conceptions are about graphs. Discovering students' interpretations of piecewise-defined linear position functions graphically represented is a step in understanding how to communicate expert conceptions of linear position graphs. DiSessa (1991) speaks of "conceptual expertise" (p. 152) being present in students. These conceptions are how students make sense of graphs, and the point of view through which they will interpret graphs. Specifically, DiSessa (1991) discusses speed as a "principal and direct resource in students' thinking about motion" (p. 153). Would this accessibility be present in early grade students, and would it be available to them as they thought about the shape of graphs?

Theoretical Framework

Algebra is a vital component of mathematics education, included as its own thread in the standards of the United States' National Council of Teachers of Mathematics (2004). One skill targeted in these standards is the ability to relate change in one variable to changes in other variables, even in the early grades. However, this is not associated with graph representations until much later in the strand.

Students can grasp algebraic concepts at much earlier ages than is suggested by the current curriculum, even in the early grades (Carraher, Schliemann, & Brizuela, 2000). Kaput (1999, p. 136) has described a number of forms of algebraic thinking including algebra as the ability to generalize and formalize patterns and constraints. These patterns may be shapes that students recognize in graphs, for example. Students can engage in these sorts of activities and this sort of mathematical thinking in the early grades, "making qualitative interpretations of symbolic but situated representations" (Tierney & Monk, 2008, p. 199).

Among graphical representations which carry mathematical meaning, the notion of the slope of a linear graph is an important subdomain, giving students access to algebraic notions of joint variation of quantities (Kaput & Blanton, 2001; Kaput, 2000). Students have an ability to talk about the rate of change of a variable over time by looking at slope, even at the 4th grade level (Tierney & Monk, 2008).

Smith, diSessa, and Roschelle (1993) have described certain student conceptions as being based on prior knowledge, but having been extended in a faulty way. Student conceptions of slope which are often labeled “misconceptions” can tell us something about student prior knowledge upon which algebra concepts can be built. While work is being done to examine some of these conceptions (Zahner, Moschkovich, & Ball, 2008), much of it is focused on the late middle grades. My approach to learning about conceptions follows from an assumption that knowledge construction is “mediated by symbol systems including language” (Moschkovich, 1988, p. 212). Student stories and descriptions, predicting how a character is going to move based on their examination of a graph, tell us more than their conclusions; they reveal some of the processes students use to make sense of what they see.

Research Questions

This study aimed to answer three questions:

1. How will students describe the motion defined by a plot of a simple (single piece) linear function on a position graph?
2. How will students describe a slightly more complicated 2-piece function plot involving two positive rate pieces?
3. Given a race which includes positive, zero and negative slopes, how will students describe that motion?

Method

The teacher of a Massachusetts fourth-grade mathematics classroom allowed me to work with students in the back of a classroom while class was in session. This presented some technical limitations, but it provided a comfortable, familiar setting for the students.

Following some initial start-up experiences, I collected data from six students for approximately 20-minutes per each session, returning over a number of days to work within the time constraints of the hour-long class.

For the purpose of being able to present students with a number of piecewise-defined position functions, I used SimCalc MathWorlds software (Hegedus & Dalton, 2006). The software has the ability to display editable piecewise position graphs, and to animate a runner character according to the motion defined by the graph (see Figure 1).

This software was developed at the University of Massachusetts Dartmouth as part of the SimCalc project in the Kaput Center for Research and Innovation in STEM Education. The software provides a number of representations related to joint variation of quantities using a motion context. The graphs are dynamically, graphically editable. An executable representation is provided in the form of animated runners on a soccer field. Once the graphs are edited into a desired configuration, running the animation activates the executable representation. The runners move according to the position graphs and a number of other visual cues represent the passage of time, and highlight the current time on the graph.

The software is designed to be used with a curriculum targeted at middle grade and high school students; it is used here without its curriculum, merely as a delivery platform for the graphs. However, students did observe graph editing and participate in graph editing during the sessions.

Activity

Sessions with the students would roughly follow the following pattern:

1. I would briefly describe the software interface.
2. I would explain the association between the graphs and the runners ("the blue graph is describing the motion of the blue runner.")
3. I would ask the student to make a prediction about what the red runner was going to do when the animation is allowed to run. ("Does the red graph tell you anything about what the red runner is going to do when the race is run?")
4. I would allow student to describe prediction/story.
5. I would run the animation.

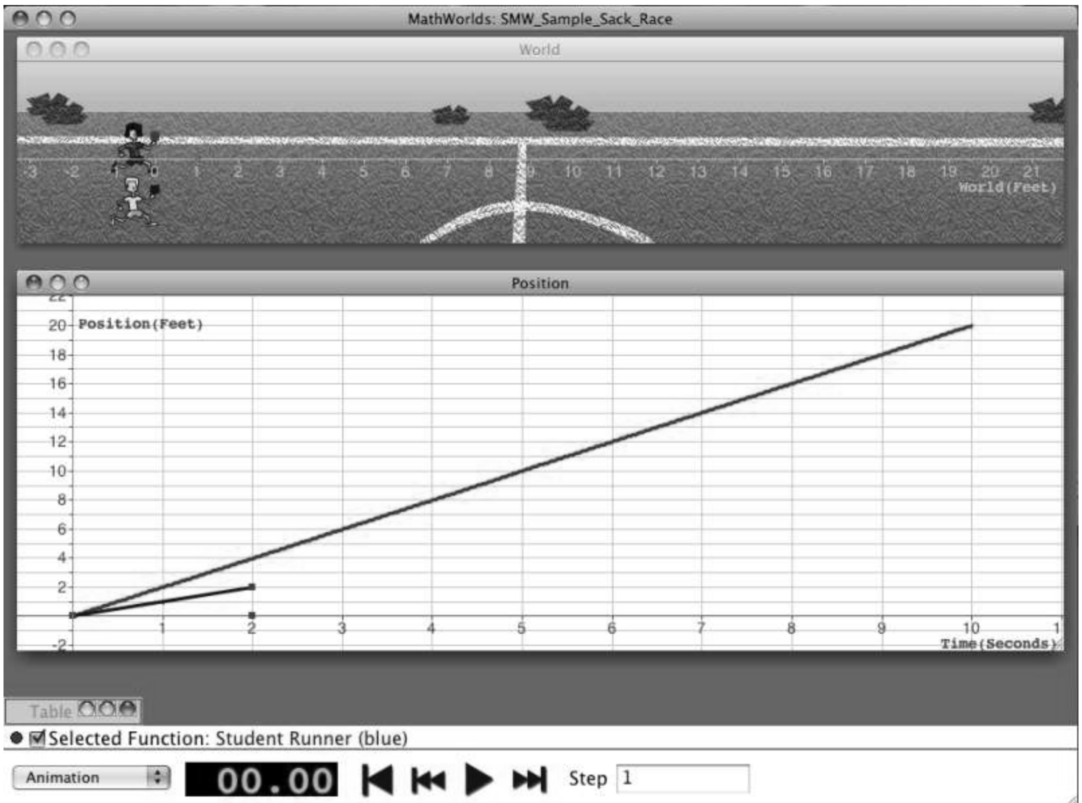

Figure 1. SimCalc MathWorlds Software

Figure 2. Example of student-created piecewise position graph and the constant plot that served as an example

This pattern would then repeat with a slightly more complex graph containing two positive-slope pieces for the blue runner, reaching the same position at the same time as the red runner at the end of the race.

For the third iteration I allowed the student to build a graph and then tell me a story about what sort of a race that graph would cause the blue runner to run (see Figure 2). (“Tell me a story about the race you’ve created.”)

Finally, I would sometimes explore further by creating a graph with features that had been neglected in the earlier iterations. For instance, I would introduce zero or negative slope pieces if they had not been introduced by the student.

I asked questions throughout the session comparing the runners to get the students to elaborate about their conceptions (“Which runner is running faster at the beginning of the race?”).

The data gathered was on discussions that each student participant had with the interviewer through each approximately 20 minute session; these discussions took place before and after the executable representation was allowed to run. However, the bulk of the discussion took place in the form of predictive stories that students told before they witnessed the executable representation of the piecewise-defined position function.

Analysis

After transcribing the student sessions, I decided to focus on the language that students used before they saw any animation of the graph under discussion. Seeing the animation often changed a student’s description of the “race.” Post-animation discussions were characterized by students discussing what they had observed the runner doing, or attempts at reconciling their conception of the graph with the runner’s apparent motion. By focusing on the pre-animation discussion, the data pertained to student’s understanding of the race based solely on the static aspects of the software – primarily the graphs.

I coded the language for a number of identifiable patterns in the descriptions that appeared to be references to graph shape, or relating graph shape to motion, or relating some other aspect of the graph to the motion. Two conceptions dominated the students’ language, appearing in every one of the students stories. I called these conceptions “direct reading” and “graph as path.”

Direct reading

Direct reading occurred when students read the quantities which were directly represented by the graph. In this case, they were reading position and time from the axes. Every student was comfortable doing this when there was a question of one runner’s position in the animation world, or the relative positions of the runners.

For example, for the single-piece graph representing a motion going from 0 to 20 in 10 seconds, this typical exchange occurred:

Interviewer:	Can you tell me by looking at the graph what you think the red line is telling us about how the red runner is going to run?
John:	Uh he’s gonna run 20 feet in 10 seconds?
Interviewer:	OK and how do you know that?

John: Because the line ends. When you go across it ends on the 20 and go straight down it ends on 10 seconds.

This sort of direct reading also occurred in more complicated graphs. Direct reading is usually characterized by using numbers to talk about the graph, or relationships between two numbers. Usually this discussion took place without any mention of graph shape.

Graph as path

Graph as path has also been called "graph as picture" (Mokros & Tinker, 1987) and iconic interpretation (Leinhardt et al., 1990); it happens when students take the graph shape to directly represent some other type of object the student is familiar with. The student may take the graph to be like the path on a map. The consequence of this is that students describe the motion as if it is in two dimensions rather than the one-dimensional motion that the runners are limited to in their animation world. For example, if a student knows that the two runners will end the race in the same place at the same time, but one runner has a single, straight piece from beginning to end (20 feet over 10 seconds) and the other runner has two pieces of different slope, the latter graph appears to be a longer "path." To meet someone at the same place and time yet take a longer path means you must have gone faster, since you covered a greater distance in the same time. This is the sort of conclusion that students come to using graph-as-path. Here is a specific instance:

Interviewer: Ok, so can you tell me anything about who's running faster in this race?
Issac: Um, I mean he's going to make it to 20 in that time. Running faster?
Interviewer: Yup.
Issac: Blue, 'cause he's going kinda fa—'cause he's and take—it's like taking 2 routes and they make it to the same place

There is only one route in the animation world; the runners are sharing a road. The student has concluded here that the two graphs represent different actual paths, each with their own independent total distance. These iconic representations dominate the shape descriptions that the students used. None of the students escaped some form of this conception, in which the graph stood in for a picture of something else.

Collapsing representations

My other codes attempted to describe other conceptions the students had. "Flat is stopped" was identified when a student described a zero slope as a lack of motion. "Down slope is slower" was identified when a student described a negative slope as slowing down. "Bend as stop" appeared when a student described a bend in the graph as a moment when the runner stopped.

Many of these other codes for language used in predictions were only related to graph shape because of my own efforts to identify them; students' actual spontaneous references to graph shape were more rare. Often these codes could be collapsed to either direct reading or graph as path. For example, here is a description of backwards motion which appears to be related to graph shape at first, but as the student elaborates, it appears that she has reached her conclusion by reading position off the graph. The student's description is of a negative slope section:

Interviewer: And what's he doing there?

Gloria: He's running back because the line... the line goes down.
Interviewer: OK.
Gloria: Um, which means, um, that like he's not going as many yards as he was. This would be… 16 yards? and this would be… can I just say less than 16 yards?

Similarly, one student, Fred, repeatedly described steeper slopes as being slower motions. Steep downward slopes appeared to him to be going faster. In this case, the graph is not a path for him, but I suspect his iconic conception of the graph is more like the runner having to negotiate a hilly terrain. So, 'steeper is slower" collapses to my iconic representation code.

At times, an iconic interpretation of the graph conflicted with what a student was able to read off the graph. This was apparent in some halting descriptions. For example, here a student is trying to describe a zero slope segment of the graph:

Interviewer: Ok, and how does he move after that?
Helen: Straight—well, uh. [pause] He stays there.

Helen's initial answer is an interpretation of a straight, flat line as representing motion. However, she corrects herself when she realizes that the runner is at the same place at the beginning of the segment and at the end. She doesn't say the runner has stopped, she says "he stays there" choosing to describe the motion in terms of position.

Another Common Conception

Another conception was present in half of the six students. Students resisted describing changes in speed as instantaneous, even though they were represented that way on the graphs. Issac stated that the runner would "start getting faster" after a slope change. While this does not directly relate to slope, it does represent a description or prediction of the motion that is not based on what the student is seeing in the graph and may be an assumption carried in from experience in the world that change in speed is not a naturally instantaneous phenomenon.

Conclusion

I have identified two very strong student conceptions that were apparent in the work of students who participated in the study. *Direct reading* off the graph was a familiar conception and students relied on it comfortably. The *iconic conception* of the graph was also common among all students, and resulted in shapes being interpreted as paths, hills, or other familiar objects. This conception resulted in inaccurate student predictions.

Direct reading of position was a skill that students utilized often when they were asked about velocity—a quantity that they did not appear to see represented directly on the graph. When they turned to direct reading, students were often forced to describe motion in terms of position.

Graph as picture is different in that it includes awareness that the shape of the graph has some meaning that can be understood by shape alone. Students working on an unfamiliar question about a secondary characteristic of the graph extend some conception from their prior knowledge to this situation. Students using the direct reading method sometimes came to conclusions similar to what you might get from an understanding of slope; they read quantities off the graph and used a procedure to combine them in an effort to build on their prior conception in order to come up with an answer about rate of speed. However, the students did not associate this with the shape of the graph; they had not generalized a pattern of slope as it relates to rate.

The common ability to read position quickly off the graph implies that if they were given some instruction on slope, they could verify it for themselves with their skills of reading position off the graph. By this verification process, perhaps they could convince themselves that the direction of slope or relative degree of slope between two graphs has meaning.

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OF THE ARITHMETIC-ALGEBRAIC NATURE OF WORD PROBLEMS AND OF SOLUTION STRATEGIES

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During the initial stages of reading/transforming the statement of a word problem, one must necessarily undertake what we call the logico-semiotic outline, which includes processes of analysis and synthesis, and not just the recognition of a previously learned solution scheme. Outcomes from analyzing solution protocols of students in a pre-service mathematics teachers program, suggest that as of the stage of the logico-semiotic outline, it is feasible to anticipate whether the solvers will use the algebra sign system or not, and this wields an influence on the algebraic –or arithmetic- nature of the remainder stages of the entire problem solution process.

Introduction

In the Cartesian method symbolic algebra operates as the linguistic vehicle that makes it possible to detach oneself from the meanings referred to in the text of the problem, and thus reach the solution by way of transformations -at the syntactic level of algebra- of the equation or equations that represent(s) the situation described in natural language. This is what gives the Cartesian method its algebraic nature. One of the goals most sought after in secondary school is for students to master that method. Yet sundry studies have documented a marked preference for non-algebraic methods, both informal (such as trial and error) and school-based methods (such as arithmetic solutions) within this population of students (e.g. Bednarz & Janvier, 1996; Stacey & MacGregor, 1999; Lins, 1992). The foregoing has lead several authors to attempt to decipher the arithmetic-algebraic nature of the problems themselves (e.g. Puig & Cerdán, 1990), trying to see what response could be given to the question raised by Wagner & Kieran (1989, p. 226) "Are there word problems that are intrinsically algebraic rather than arithmetic?"

Other studies that have focused on analyzing the solution processes used by students at varying school levels reveal that in many cases, from the problem reading, interpretation and analysis stage, it is possible to identify whether the predominant nature of the strategy adopted is arithmetic or algebraic (Rubio, 1990; Filloy, Rojano & Rubio, 2001). In this article we broach the topic of the nature of the "structure of the problem" (a priori theoretical analysis) and of the resolution processes utilized (empirical data) jointly. From a theoretical standpoint that emphasizes interaction of the sign system of algebra with natural language and with other sign systems, such as that of arithmetic, we analyze episodes of the solution processes involving a problem "with an algebraic structure", carried out by two pairs of students in a pre-service mathematics teachers program.

Solving methods and the logico-semiotic outline

In addition to the already mentioned preference of secondary school students for non-algebraic methods of solving word problems, documentation exists to specifically support that together with strictly arithmetic strategies junior secondary school and middle school students show that they are inclined to use number substitutions in representations of the relations present in the problem (Filloy & Rubio, 1991; Kutscher & Linchevski, 1997; Malara, 1999; Filloy, Rojano & Rubio, 2001). Based on the previous studies, one can say that the non-algebraic

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methods most used can be grouped into two general methods, as follows: the arithmetic method and the exploration by number substitution method. The Cartesian method, on the other hand, consists of translating the problem statement into algebraic code and of finding the solution by solving an equation or system of equations in the field of algebra. It is important to note here that the arithmetic method, a method that shall be known here as the *successive analytical inferences method*, differs from the Cartesian method in that in it one does not anticipate the formulation of an equation, rather that the statement of the problem is conceived of as the description of "possible states of the world" and that text is transformed by analytical phrases, using "facts" that are valid in "any possible world". These analytical sentences constitute logical inferences that act as logical descriptions of transformations of "possible situations", until such time as the solver arrives at one that he/she recognizes as the solution to the problem (Filloy, Rojano, Puig, 2008, p. 217).

Analyzing the statement of a problem enables one to reduce that wording down to a set of relations among quantities; during this process some contextual elements become no longer relevant for the solver. Said relations may be represented by means of a graph in which the vertices correspond to quantities and the edges to the relations among quantities. In these graphs, which we have adapted from those formulated by Fridman in his work "Trinomial graphs as meta-language of problems" (Fridman, 1990), one can show the network of relations that exist among quantities that have been established in the analysis of the statement and the known quantities or data of the problem appear as black vertices, while the unknown quantities of the problem (including auxiliary unknowns) appear as unfilled boxes. The criterion for establishing the arithmetic-algebraic nature of the solution consists of whether the relations among quantities, that is to say the operations that must be carried out in order to reach the solution, involve operating with the unknown, with what one is searching for, or not. Figure 1 portrays the *hay problem* (taken from Kalmykova, 1975, p. 90, and slightly modified) and the corresponding graph from an analysis of the statement. The graph shows that any solution that uses that particular network of relations necessarily means having to operate with the unknown because there is no edge in which there is only an unknown quantity. In the cases to be discussed here, we have used the same type of graph to represent and analyze the reading and interpretation of the text of the *hay problem* made by participant pre-service teachers who had received instruction on general practices applied to solve problems in mathematics. However prior to going on to discuss the cases chosen, we will introduce several theoretical elements that will enable us to refer to such cases in terms of the sign systems involved in that reading process.

The hay problem

Some farmers stored hay for 57 days, but since the quality of the hay was better than they initially thought, they saved 13 kilos per day, as a result of which they had hay for 73 days. How many kilos of hay did they store?

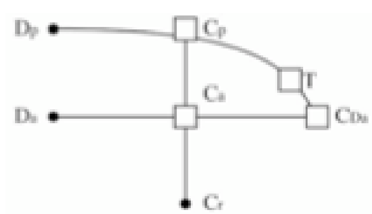

Figure 1

The studies "Heuristic tools, plausible patterns and management in problem solving"

This article reports some of the findings of the series of studies entitled "Heuristic tools, plausible patterns and management in the process of solving problems". The main objective of the portion of the work reported here is to study the influence on student actions of explicitly teaching problem solving management and process control. In more particular terms, we report two of the study's cases that correspond to solution of arithmetic-algebraic word problems. The students -around 18 years of age- were studying in a teacher training school and had sufficient knowledge of the mathematics contents involved in the problems they were given. They had particularly received prior instruction on algebraic solution of problems, but it was the first time they had received instruction on general issues related to problem solving. The instruction was given to a natural group made up of 34 students, to whom a test was applied both before and after receiving the instruction. The objective, *inter alia*, of said tests was to select the students who would be observed in the clinical study. The data reported here are derived from the clinical study which was undertaken by video-taping the students as they solved problems in groups of two with no intervention by the researcher.

In our work, we have used the idea of Mathematical Sign System (MSS) in order to describe the phenomena of teaching, learning and using symbolic algebra (Filloy, Rojano & Puig, 2008). In addition, given that the mathematical texts include signs of natural language as well as mathematical signs, and that in no case do the signs appear in isolation in the texts, we speak of a mathematical system of signs and not of a system of mathematical signs. In other words, the adjective "mathematical" is qualifying the system in its entirety and not just the signs (Puig, 2003). By the same token we conceive of MSSs from the standpoint of semiotics (basing ourselves on the semiotics of Pierce, 1982-) in which it is the processes rather than the very signs that are emphasized (Hoopes, 1991). And the processes we are interested in studying are those of the reading/transformation of a text through which sense is produced (Filloy, Rojano, Solares, 2010) (in our case, the initial text is a word problem). Those processes involve the mathematical sign systems of algebra and of arithmetic, which is why we have abbreviated them as AlSS and ArSS , respectively.

During the initial stages of reading/transforming the statement of a problem, one must necessarily undertake what we have denominated here as the logico-semiotic outline, which includes processes of analysis and synthesis, and not just the recognition of a previously learned solution scheme (Filloy, Rojano & Puig, 2008, p. 35). That is to say that the outline includes, among other things, a mental representation of the problem that contains the basic information concerning the problematic situation and that identifies the relations that are central to the possibility of implementing any sort of solution strategy –or method- (Filloy, Rojano & Puig, 2008, p. 250). In the analysis –carried out using *graphs*- of the *hay problem* solution protocols undertaken by two pairs of students in the afore-mentioned clinical study, one can observe that as of the stage of the logico-semiotic outline in these cases it is feasible to anticipate whether the solvers will use the AlSS or not, and this determines to a great extent the algebraic –or arithmetic- nature of the remainder of the stages and of the entire solution process. The foregoing is regardless of the result of an *a priori* analysis using *graphs*, of the nature of the problem or of implementation by the solvers of practices learned that were incorporated into the solution process (such as, for instance, systematic review of the expressions that are used and of the actions carried out at each step). The section below consists of a presentation and discussion of the cases referred to above.

The protocol of student pair A & J

Pair A & J ends up solving the problem by translating the statement into the equation $x=\left(\frac{x}{57}-113\right)73$ and solving it. Yet they had previously written other erroneous algebraic expressions, which they corrected as a result of the fact that their process management is effective enough that it enables them to detect mistakes. In successive analytical readings they extracted, either explicitly or implicitly, the following quantities and relations from the statement:

Quantities: days planned (D_p), actual days (D_a), additional days (D_m), planned consumption 3of hay (C_p), actual consumption of hay (C_a), daily reduction in consumption of hay (C_r), consumption on the additional days (C_{Dm}), consumption on the actual days (C_{Da}) consumption on the planned days (C_{Dp}), total saving (St), hay stockpiled (T).

Relations: $D_p + D_m = D_a$, $C_a + C_r = C_p$, $C_{Dp} + C_{Dm} = C_{Da}$, $D_p \times C_p = T$, $D_p \times C_a = C_{Dp}$, $D_p \times C_r = S_t$, $D_a \times C_a = T$, $D_m \times C_a = C_{Dm}$, $C_{Dm} = S_t$, $C_{Da} = T$.

This network of quantities and relations can be represented in the graph depicted in Figure 2, which shows the space of quantities and relations within which the solvers have moved during the overall solution process.

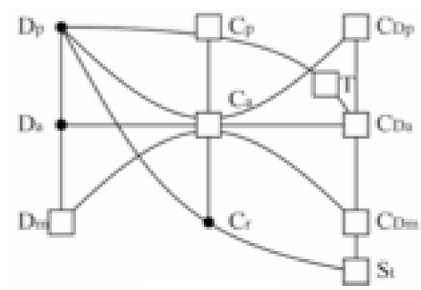

Figure 2

In the final correct solution they have only used the quantities and relations indicated in bold in the graph. These are the quantities and relations that serve to build the equation $x = \left(\frac{x}{57} - 113\right)73$, as can be seen in the graph in Figure 3.

The previous erroneous attempts are always guided by the logico-semiotic outline that pertains to the Cartesian method. From the very first, they anticipated operating with the unknown in the same way as with known quantities. Their plan of action can be described as a version of the Cartesian method, made up of three steps as follows: a) call the unknown *x*; b) examine quantities that appear in the story narrated in the statement of the problem and calculate them numerically or write them down by way of algebraic expressions that solely involve the letter that refers to the unknown, until an equation is obtained; and c) solve the equation.

The errors originate with the lack of analysis or bad analysis of the relations that exist among quantities, which leads the students to make calculations or write algebraic expressions that fail to respond to the correct relations among quantities, responding rather to incorrect quantity relations.

The first attempt, after calling the unknown in the problem *x*, consists of calculating the total hay saved by using the mistaken relation $S_t = C_r \times D_a$ (they write "hay saving = 113 × 73"). The calculation is then cast aside because of J's doubt "…it is per day, for each of the days for which there was a planned consumption of hay", which implies the correct relation $S_t = C_r \times D_p$. The graph in Figure 4 shows the erroneous relation used with a dotted line.

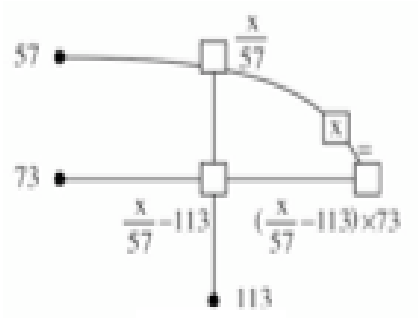

Figure 3

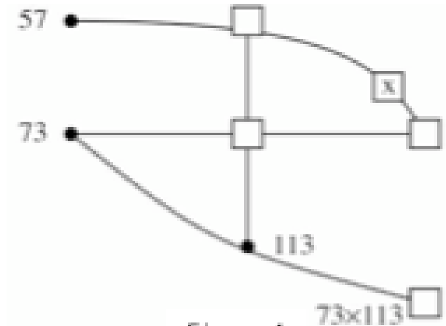

Figure 4

In the second attempt, they begin by correctly calculating the planned consumption of hay. A says “So every day they planned to consume …”, writes “x/57 every day initially” and adds “this is what they had initially planned”.

Once the quantity *Ca* had been correctly analyzed, and the corresponding algebraic expressions had been written, A proposes to continue the analysis using quantity *Cp*: “So then, what they actually consume is …” However to do this he does not appear to analyze the quantity, rather he combines the unknown and the data in some way: “every day they consume *x* minus 113 divided by… And so, what do we divide it by?”

Not having analyzed the relations leads to writing an algebraic expression that is void of meaning, $\frac{x-113}{73}$, and in which the erroneous relations used are those represented in the graph depicted in Figure 5.

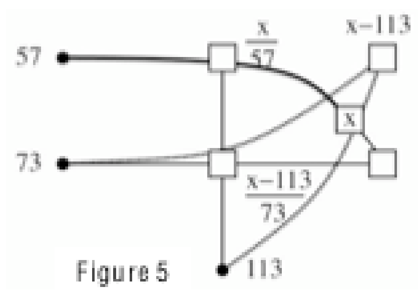

Figure 5

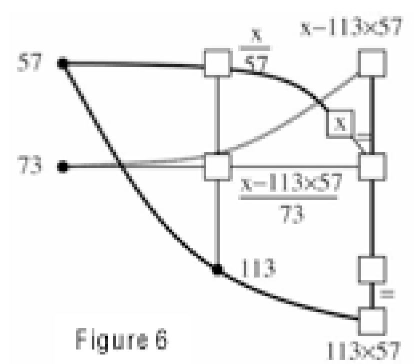

Figure 6

The discussion between A & J leads them to examine the meaning of the expression they have written because J maintains that they must divide by 57. Their collaborative management work means that A must reanalyze the quantities and relations, and discover that the algebraic expression is mistaken: “not 113 either, 113 daily … Well this is wrong.”

During the third attempt the expression they write for *Ca* is $\frac{x-113\times 57}{73}$ adding the multiplication times 57 in the numerator of the foregoing expression.

The discussion between A and J is once again a source of process management since it forces them to discuss the meaning of the expressions written. J ascribes an erroneous meaning to the numerator (“this would be the amount of hay that you would have left”), while A does not see it the same way. Since they are unable to find a meaning for the expression that they can both agree to, J decides that “This is wrong”, and deletes the numerator. In Figure 6, we have shown in a graph just how the expression on the numerator could have been imbued with sense, but with the division taken from an erroneous relation.

The definitive and correct attempt is unleashed thanks to a discussion that enable them to make sense of the correct algebraic expression for $Ca\left(\frac{x}{57}-113\right)$, by calling the relation “consume every day saving”. That denomination in daily language contains what they had previously called “consume every day initially” ($x/57$) and the subtraction in the verbal form “saving”, which represents the action of saving every day 113 applied to $x/57$. As a result they are able to conclude their analysis by building the following equation: “Anyway it’s the initial hay. They consume this every day, they do so over 73 days, and then during that entire time what they consume is the initial quantity, which is the x.”

In short the solution process used by A & J is therefore algebraic from the very beginning, and they do not depart from that process to fall into arithmetic or trial and error strategies in spite of discovering their errors. Finding and correcting the errors was possible because of their process management. The students have been given instruction on review and control of the expressions they write. Control is undertaken by resorting to an examination of the very meaning of the expression and to a description that uses a vernacular name to express that meaning. By working together and in collaboration, the students are compelled to arrive at a shared meaning, thus preventing them from working in a syntactically senseless manner.

The protocol of student pair M & H

During the entire solution process M & H move within the field of arithmetic solution: calculating solely with the data. And this is the case as of the initial analysis (M: “And what other data do we have?”) and the first actions that deal with the data and are aimed at obtaining further data (J: “Look, every one of the 57 days, every day they saved 113 kilos, let’s see how many kilos they saved.”). Once they had correctly calculated the total saving (57×113=6441), they continue analyzing the statement in search of relations among the data (J: “And the kilos they saved are the ones that allowed them to have 16 more days of hay”), and by transforming that assertion into a hypothetical relation (J: “if they were able to make it through 16 days with the kilos that they planned…, stored to make it through the 57 days … I don’t know if it’s 57 or 73”). The proportion is set erroneously because the option of 57 or 73, that is opting between the days planned and the actual days, is discussed without examining the arithmetic meaning of the relation, in other words the ratio involved in that proportion. Instead of examining the arithmetic meaning, they –mistakenly- examine the meaning in the story that is narrated in the wording “it is 57 days because those are the days they had calculated that they had to store for, not the time they were able to withstand; that came later because of the fact that they had saved.”

Not having taken into consideration that ratio $C_{Dm}:Dm$ is Ca and that ratio $C_{Da}:Dp$ is Cp led them to establishing the erroneous proportion of $C_{Dm}:Dm::C_{Da}:Dp$, rather than the correct proportion of $C_{Dm}:Dm::C_{Da}:Dr$. M & H express the foregoing mistaken proportion by way of the scheme of the rule of three, which also provides them with an algorithm for obtaining the result.

The quantities and relations used in the solution have been represented in Figure 7. The figure shows that, with the reading given the problem has an arithmetic solution because it can be run through without the need to work with unknowns. (In the graph the erroneous proportionality relation is represented by way of a dotted edge that has four vertices.)

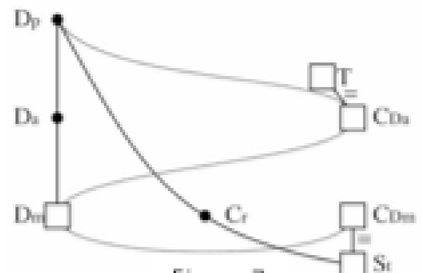

Figure 7

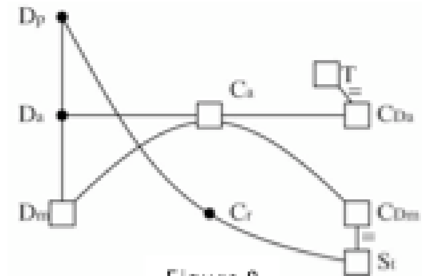

Figure 8

After obtaining the result (erroneous) M & H go through a very long series of verifications that we shall not go into in detail here. The number of times they verify the mistaken result is indeed high: seven. On each of those occasions, their verification is based on a correct analysis of the relations among quantities and, consequently, ends up contradicting the initial result. Yet M & H have greater faith in their initial result than in their verifications. Hence upon completion of each verification, they do not decide that their initial result is incorrect but rather search for a different verification. It is only their persistence to continue verifying that leads them, on their seventh attempt, to a reflection that is absolutely crucial. M says: “What is truly clear is that if we have formulated a verification strategy, that is the way to solve the problem; it’s another path, sure let’s see if it’s the right path, since we know…”

The solution that they then derive from that “verification strategy” is also arithmetic and is based on calculation of the ratio that they had alluded to in their incorrect initial solution when they used the rule of three algorithm. Consideration that the ratio of the proportionality relation is *Ca* corrects the initial error and leads to the solution depicted in the graph shown in Figure 8, which is once again arithmetic.

On the nature of solution strategies: Final remarks

The objective of incorporating into school training the teaching of general review and monitoring strategies or practices that enable students to look at their own actions as they carry out the steps needed to solve problems is that of allowing the students to become competent problem-solvers, who are even able to change their solution strategy or method when their monitoring efforts throughout the process show them that their efforts are not being met with success. In the cases of student pairs A & J and M & H, one can see that despite having systematically undertaken reviews and corrections of their steps, both pairs of students remain on the solution path that each pair had chosen from the very start.

A & J decide to use the AlSS and as of the graphs that correspond to the episodes analyzed, one can see the algebraic nature (Cartesian) of their approach to the problem. That is to say, the algebraic nature of the solution is not solely derived from their choice of the AlSS to represent the relations found in the logico-semiotic outline, but also from the nature of the network of those relations, in which operations on unknown quantities also appear. In this case the corrections are made to the equation that A & J were first able to produce, but this does not only happen in the AlSS. The back and forth between this sign system and the elements of the problem in its verbal form also involves, at the very least, natural language. In these processes of reading/transformation of the algebraic text initially produced and in those that involve elements of the statement, sense is produced and that sense guides the correction actions. One could say that in this case the algebraic nature of the logico-semiotic outline permeates the entire solution path.

Student pair M & H resort to data calculations and to proportional reasoning as of the logical outline stage, and remain in the field of arithmetic throughout all of the verifications that they carry out on the result they obtained initially, despite the fact that those verifications contradicted the initial result time and again. The text of the problem statement is subject time after time to readings/transformation, generating quantities and relations. Yet at no time do they calculate with an unknown quantity. Only reflection on the verification process enables them to carry out the final correction, without ever leaving arithmetical reasoning behind.

The foregoing cases demonstrate just how the algebraic or arithmetic nature of solution strategies for a word problem are intimately related both to the sign system chosen to express the network of relations among quantities –determined as of an interpretation of the statement- and to the fact of whether or not that network includes operating with the unknown quantity (ies) of the problem. Analysis of the succession of reading/transformation actions of the text of a problem by the problem solvers suggests that the logico-semiotic outline stage significantly influences the choice of mathematical sign system and, consequently, also wields an influence on the nature of the solution strategy.

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RATIO AND PROPORTION: HOW PROSPECTIVE TEACHERS RESPOND TO STUDENT ERRORS IN SIMILAR RECTANGLES

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The idea of interpreting and responding to student thinking is one of the central tasks of reform-minded mathematics teaching. This study examined elementary and secondary prospective teachers' interpretations of and responses to a student's errors involving finding a missing length in similar rectangles. Analysis results revealed that although the student's errors came from conceptual aspects of similarity, a majority of prospective teachers identified the student's errors from procedural aspects of similarity. They tried to cope with the student's errors by invoking procedural knowledge. This study also revealed the two different forms of address by prospective teachers to student errors.

Introduction

Ratio and proportion are a core topics in elementary as well as secondary mathematics education (NCTM, 1989, 2000; Kilpatrick, Swafford & Findell, 2001). The NCTM document (1989) describes proportionality to be "of such great importance that it merits what ever time and effort that must be expended to assure its careful development" (p.82). Very often multiplication and division tasks in the lower grades are presented in unit-rate form, which is a special form of ratio and proportion. In the middle grades, word problems involving equivalent fractions and fraction comparisons can also be thought of as ratio and proportion situations (NCTM, 2000). The ability to recognize structural similarity and multiplicative comparisons illustrated in such proportional reasoning processes are the cornerstone of algebra and more advanced mathematics (Kilpatrick, Swafford & Findell, 2001). Nevertheless, research has consistently shown that many students have difficulty with developing proportional reasoning. Hart (1984), for example, reported that less than 42% of students in grade 7 succeeded in solving simple problems of enlargement, a most common error is additive reasoning (Lamon, 2007). According to these studies, students tend to focus on the difference between the given quantities rather than proportionality illustrated in given contexts. What instructional support do teachers need to provide? How do teachers need to use student errors in instruction?

In this study I set out to investigate prospective elementary and secondary teachers' reasoning, their responses to student errors on the topic of ratio and proportion, and the relationship between their knowledge and approaches. I used aforementioned student error in exploring prospective teachers' reasoning and approaches. I was curious about how prospective teachers would interpret and respond to student errors in finding a missing length in similar rectangles, and how their approaches related to their mathematical knowledge. Although a growing body of research has focused on teachers' treatment of student errors (e.g., Schleppenbach, Flevares, Sims, & Perry, 2007; Stevenson & Stigler, 1992), prospective teachers' responses and their strategies have received limited attention in the research literature. If teachers are called to use student errors as springboards for inquiry into mathematical concepts, it is important to explore prospective teachers' responses and strategies to student errors, and prepare them to make better use of student errors through teacher education programs. The purpose of this study is not to add to the collection of studies documenting prospective teacher weakness, but rather to inform the design of teacher education in this area. Exploration of prospective teachers' interpretations of and responses to student ideas and, in particular, student errors will help enrich a dialogue among reformers, teacher educators, and professional developers in ways they could help prospective teachers learn to teach math to promote student understanding. More specifically, the research questions guiding the study included:

1. How do prospective teachers understand ratio and proportion problems?
2. How do prospective teachers interpret and respond to student errors on ratio and proportion in similar rectangles?

Conceptual Framework

Teacher Knowledge, Approaches and its Relationship

Although many researchers have devoted considerable attention over the last two decades to what teachers should know and be able to do, there are still large gaps in this research, in particular, research studies on teachers' knowledge of ratio and proportion. Fisher (1998), for example, investigated secondary mathematics teachers' understanding and strategy usage in ratio and proportion problems. He used word problems involving direct proportional reasoning and inverse proportional reasoning. He reported that, overall, teachers were more successful with direct proportional reasoning problems (i.e., $y=kx$) than with inverse proportional problems (i.e., $xy=k$). However, the strategy chosen by teachers varied with the type of problem. In terms of teaching approaches, he reported that most teachers said they would use the same strategies they used while teaching the problems tested. Lim (2009) investigated twenty-eight preservice teachers with four types of invariance in miss-value problems: ratio, sum, product, and difference. He found that preservice teachers had different levels of understanding depending on the type of problems. While prospective teachers had less difficulty with the first and second problems (ratio and sum), they showed greater difficulty with a missing value task involving product and difference. In particular, he reported that preservice teachers generally do not pay close attention to the meaning of ratios when they set up a proportion to solve a missing value problem, indicating that prospective teachers tend to use the same approach for a missing value problem regardless of different contexts. Although the findings from these studies help us understand prospective teachers' understanding and strategies, there remains a need for research to unfold how the teachers' own understandings impact their interpretations of the students' misconceptions. An investigation of this important topic is described here.

Strategies in Ratio and Proportion problems

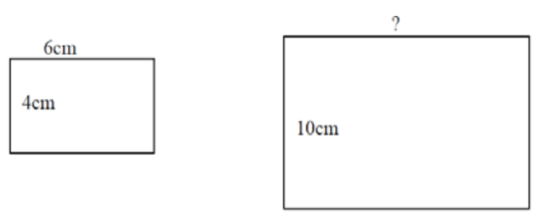
In analyzing prospective teachers' knowledge, interpretation, and approaches to student error(s), I was guided by the studies of Rittle-Johnson and Alibali (1999), who distinguished between the use of procedural and conceptual knowledge. I also referred to the studies of Fisher (1999) and Lamon (2007), who articulated different levels of understanding of ratio and proportion. Rittle-Johnson and Alibalie defined *conceptual knowledge* as explicit or implicit understanding of the principles that govern a domain and of the interrelations between pieces of knowledge in a domain. They defined *procedural knowledge* as action sequences for solving problems. These two types of knowledge lie on a continuum and cannot always be separated; however, the two ends of the continuum represent two different types of knowledge. More detailed examples will follow of these two forms of knowledge regarding ratio and proportion in similar rectangles as I discuss the findings of the study.

Methodology

This study investigated pre-service teachers' interpretation of and responses to student error(s) through a classroom scenario in which an imaginary student incorrectly solved a similar rectangles problem. The study also examined how prospective teachers' pedagogical strategies used to address the student errors are related to their content knowledge of ratio and proportion.

Two tasks (content knowledge and pedagogical content knowledge) were developed based on the literature review (e.g., Hart, 1984) and textbook analysis (e.g., Connected Mathematics). The content knowledge task was aimed at assessing prospective teachers' understanding of ratio and proportion. The first question, called *similar rectangles problem*, required prospective teachers to find the missing side of a rectangle given the condition the two rectangles are similar, and to explain their solution. The pedagogical content knowledge followed the content knowledge task. After completing the similar rectangles problem, they were asked to interpret and respond to a student's incorrect solution. According to Hart (1984), the most common incorrect strategy in finding the length of the missing side in similar rectangle is to use additive reasoning (A-B=C-B) focusing on the difference between the given length in similar rectangles, rather than focusing on proportional relationship between two figures (A/B=C/D). I based my exploration of prospective teachers' pedagogical content knowledge (PCK) on this common incorrect strategy. The pre-service teachers were asked to identify one fictitious student's (Sally) error(s) and then to provide a written description of how they would respond to her (see Figure 1).

You are teaching 6th graders. You asked the students to find the length of the missing side in similar rectangles shown below. After a few minutes, you asked Sally, one of your students, to explain how to solve the problem. Sally explained that the side would be 12 cm long because 4+2=6.



1. Evaluate Sally's reasoning and explain whether it is mathematically correct or incorrect. If it is not correct, identify the error(s) in Sally's reasoning.
2. How would you respond to Sally? Explain what type of guidance you would give Sally as much detail as you can.

Figure 1. Pedagogical Task: "What is the length of the missing side?"

Fifty-seven prospective teachers participated in this study. Thirty-one were in their senior year of the elementary teacher preparation program and twenty six were math majors seeking middle and secondary school certification at a large Mid-Western University. Prospective teachers in elementary and secondary programs were included in order to obtain a broad range of responses to the study's task.

The tasks went through multiple phases of revisions and were pilot tested with two volunteers who were then interviewed to check for possible misunderstandings. The final version of the task was then administered as an in-class survey in three mathematics methods course sections, two elementary and the other secondary, towards the end of the semester. This study only includes data from prospective teachers who signed the study's consent form. For data analysis, responses to the content knowledge task were first identified based on correctness and then solution strategies were identified. In analyzing prospective teachers' interpretations of and approaches to student error(s), I was guided by the research of Rittle-Johnson and Alibali (1999), who distinguish between the use of procedural and conceptual knowledge. The final phase of analysis consisted of the identification of emergent patterns in teaching strategies.

Summary of Results

How Do Prospective Teachers Understand Ratio and Proportion?

The findings from the content knowledge task were helpful in providing an initial framework for analyzing the PCK task. In the first problem, prospective teachers were asked to find a missing length in similar rectangles where a rectangle that began as a 4 cm by 6 cm was enlarged to a rectangle with a short side of 10 cm. All prospective secondary teachers answered correctly and 73% of prospective elementary teachers (22 out of 31) answered correctly. Among the prospective teachers who provided incorrect answers, 6 prospective teachers used a common strategy, additive reasoning, by focusing on the differences between the given quantities. One typical response was as follows:

> I think that the missing side is 12 cm because the difference between 4 cm and 6 cm is 2 therefore the difference between the 2 sides of the larger rectangle would also be 2. 10 cm + 2 =12cm

Among prospective teachers who provided correct answers, three different solution approaches were used—within ratios, between ratios, and unit-rate (or scale factors) as addressed in Figure 2.

Between Ratios	Within Ratios	Scale factor method
$\frac{4}{10}=\frac{6}{x}$. If the rectangles are similar, their sides are at a constant ratio. Thus you can compare the ratio between the width and length and use this proportion to find missing length from the width of same rectangle.	$\frac{6}{4}=\frac{x}{10}$. Assuming the 4cm side is similar to the 10cm side I must find a relationship between 6 cm and x. I create a fraction with 6cm and 4cm and set it equal to x and 10 cm. Cross multiply and solve for x	Ratio of smaller to bigger similar rectangle is 4: 10. $6\times\frac{10}{4}$, $6\times 2\frac{1}{2}$=12+3=15

Figure 2. Example of different methods used by prospective teachers

Table 1 shows the frequency of each type of solution strategies by prospective teachers. While the most frequently used strategy is the within ratio approach among prospective elementary teachers, the between ratio approach is used most often in prospective secondary teachers.

I was also curious to know if prospective teachers pointed out the lying idea of similarity of rectangles, the concept of similarity, when explaining their solution approaches because finding a missing length in similar rectangles involves not only understanding the concept of similarity but also procedural knowledge of setting up a proportion and performing calculations. Among prospective elementary teachers who provided correct solution strategies, 48% of prospective elementary teachers (11 out of 23) referred to the concept of similarity while 81% of prospective secondary teachers (21 out of 26) pointed out the property of similarity of figure. This indicates that a smaller percentage of prospective secondary teachers carried out three methods as a rote procedure that requires little proportional reasoning.

Table 1. Solution Strategy used in finding a missing length by prospective teachers

	Category	Elem. (N=31)	Second. (N=26)	Total (N=57)
Incorrect	Additive	7 (22%)	0 (0%)	7 (12%)
Correct	Within ratio	14 (45%)	7 (27%)	21 (36%)
	Between ratio	8 (25%)	15 (58%)	23 (40%)
	Scale factor methods	3 (1%)	4 (15%)	7 (12%)

How Do Prospective Teachers Identify Sally's Learning Difficulties?

In a classroom setting, our fictitious student, Sally was asked to solve the same problem prospective teachers completed--find a missing length in similar rectangles. Sally concluded that the long side is 12 cm since 4 cm + 2 cm =6 cm. As I addressed earlier, finding a missing length in similar rectangles involves at least four big ideas: (1) understanding the concept of similarity, (2) determining a *between* ratio, *within* ratios, or a scale factor, (3) setting up a proportion, and (4) carrying out calculations correctly. In Sally's case, she did not understand the concept of similarity--the lengths of the corresponding sides in similar rectangles increase (or decrease) by a constant ratio. As such, she focused on the difference, in particular, *within* difference, by comparing the difference between the length and the width within a rectangle. Although she carried out the calculation correctly based on additive reasoning, she was not able to find the correct missing length in similar rectangles. The fundamental error in Sally's case results from not understanding the concept of similarity.

In analyzing the responses of the prospective teacher participants, three categories of interpretation were identified and are illustrated below. The first interpretation was to classify different ways of identifying Sally's learning difficulties using a conceptual approach—in this case, focusing on the meaning of similarity of rectangles, in which the following hold: two figures are similar if (1) the measures of their corresponding angles are equal, (2) the lengths of their corresponding sides increase (or decrease) by the same factor, called the scale factor, and (3) the perimeter from one rectangle to another rectangle also increases by the same scale factor. The conceptual approaches also included the idea of enlargement or reduction in similar rectangles. I call this type of identification of Sally's error as *similarity-based*. One typical response was as follows:

> Sally does not understand that similar means proportion or she may not understand what proportional means. A proportion is a ratio of two numbers, where Sally looked at the sum (or difference, depending on how you think about it) of sides.

The procedural approach involves finding the missing value in a proportion, which relates to big ideas 2 through 4 outlined above (i.e., determining ratios, setting up a proportion, and carrying out the calculation). In this approach, prospective teachers also indicated the need for a ratio, proportion, or a scale factor for calculation. This type of identification of Sally's error is called *procedure-based*.

> Sally did not calculate the ratio of corresponding sides, i.e., 4cm/10cm =ratio of sides. What Sally did was 6-4=2cm difference then added 10cm +2cm=12cm.

In addition to these two categories of interpretation, a third category involved responses indicating misdiagnosis of Sally's error based either on additive reasoning or incorrect focus. In most cases of additive reasoning, prospective teachers indicated Sally's errors stemmed from not comparing the difference between rectangles. The following are examples:

> Sally is explaining the relationship between the sides 4 and 6 rather than first comparing sides 4 and 10 then 6 and x. So she should be looking at how side 4 is related to side 10, then use that same relation with side 6 to get side x.

Table 2 shows subcategories of each approach and the distribution of response in terms of similarity-oriented vs. procedural-oriented.

Table 2. Pre-service Teachers' Identification of Sally's error

Type of Identification	Subcategory	# of response	Total
Similarity-based	Meaning of similarity	2	19 (34%)
	Recognizing the similarity by a constant ratio	12*	
	Comparing lengths between rectangles	3	
	Visualization of enlargement in similar figures	2	
Procedure-based	Use of addition or difference in calculation	11	
	Use of a ratio or proportion in calculation	13*	30 (53%)
	Scaling up by a constant scale factor in calculation	3	
	Comparing lengths between figures in calculation	3	
Misdiagnosed	Additive reasoning or incorrect identification	8	8 (13 %)
Total		57	57

*Note: * represent the most frequently referred category*

Four subcategories were devised with respect to the similarity-oriented approach. While prospective teachers in category one did not provide a definition of similarity, prospective teachers in category two stated specifically that Sally did not see the relationship between two similar rectangles as a constant ratio. Although teachers in category three and four stated Sally's limited understanding of the concept of similarity, teachers in category three pointed out specifically that Sally's errors came from not comparing the lengths between the two rectangles. In the case of category four, prospective teachers indicated Sally's difficulty as not visualizing enlargement of the second rectangle. In terms of the procedural approaches, four subcategories were devised as well. The major difference between the conceptual vs. procedural approach lies in the focus of indication of Sally's errors. Interestingly, when prospective teachers were asked to identify Sally's errors, they tended to rephrase Sally's method by pointing out the use of difference in calculation, which is coded into the first category in the procedural approach. Table 2 shows that, although Sally's errors came from her limited understanding of similarity rather than from procedural knowledge of setting up an equation, prospective teachers in this study tended to identify her errors more from a procedural perspective.

How Do Prospective Teachers Respond to Sally's Work?

I performed the same analysis on the responses the prospective teachers provided for the question asking them to describe how they would respond to Sally. Table 3 shows the distribution of response in terms of concept-oriented vs. procedure-oriented.

Table 3 shows that more than half of the prospective teachers provided guidance from procedural aspects of similarity. I was intrigued to find out how the prospective teachers who recognized Sally's error in terms of conceptual aspects of similarity would respond to Sally. A comparison of Tables 2 and 3 shows that although the frequency of focusing on the meaning of similarity slightly increased from identifying the learning difficulties and responding to it, prospective teachers tended to provide their intervention based on the procedural aspects of similarity.

In addition to this approach, I explored two forms of approaches addressed by prospective teachers to student errors—teacher-focused vs. student-focused. While 81% of the prospective secondary teachers provided teacher-directed approaches of telling and explaining, less than half of the prospective elementary teachers used a student-focused approach by engaging Sally in activities or questions. Furthermore, I also observed prospective teachers' using the three different approaches to student errors identified by Son (2010) in their examination of prospective teachers' pedagogical strategies to student errors in reflective symmetry: (1) *Generalization*, (2) *Return to the basics, and (3) A Plato-and-the-slave-boy approach.*

Table 3. Prospective Teachers' Pedagogical Strategies

Type of Strategy	Subcategory	# of response	Total
Similarity-based	Recognizing the similarity by a constant ratio	9	21 (37%)
Procedure-based	Comparing lengths between figures	2	33 (58%)
	Idea of enlargement in similar figures	10*	
	Finding a ratio or proportion in calculation	22*	
	Scaling up by a scale factor in calculation	3	
	Comparing lengths within a figure in finding a proportion	8	
Misdiagnosed	Additive reasoning or incorrect guidance	3	3 (5 %)
Total		57	57

*Note: * represents the most frequently referred category*

Discussion

This study investigated prospective elementary and secondary teachers' understanding and pedagogical strategies applied to students making errors in finding a missing length in similar rectangles. It was revealed that prospective secondary teachers had better understanding of ratio and proportion in similar rectangles than prospective elementary teachers. While all prospective secondary teachers solved the similar rectangles problems correctly, a large portion of prospective elementary teacher struggled with the problem. In explaining their solution strategies, and even though similar strategies appeared both from prospective elementary teachers and prospective secondary teachers, a majority of prospective secondary teachers pointed out the underlying idea of similarity, whereas less than half of the prospective elementary teachers explained their reasoning for using ratios and proportion.

This study also showed that, although a student's error stemmed from lack of understanding of the concept of similarity, a majority of prospective elementary and secondary teachers identified the student's errors from a procedural perspective of similarity. When they responded to the student's errors, they guided the student by invoking procedural knowledge.

Another interesting finding was that, although prospective secondary teachers showed better understanding of the mathematical concepts presented in this study than prospective elementary teachers when responding to student errors, prospective secondary teachers tended to rely on a teacher-focused approach of telling or explaining students' errors whereas prospective elementary teachers tended to use a student-focused approach by asking questions or providing related activities to help students overcome conceptual misunderstanding. These results are consistent with the findings reported by Son and Crespo (2009).

This study has implications for teacher educators, teacher education programs, and researchers. For example, our study joins others (e.g., Ball & Bass, 2003) in highlighting the importance for teacher educators to emphasize the importance of effective interpretation and response to student errors when teaching mathematics. In particular, by identifying different types of responses and their resulting effects on students, teacher educators are in a position to raise prospective teachers' awareness of the implications of their interactions with students. Teacher educators can also develop prompts and facilitate discussions of similar tasks to challenge and expand prospective teachers' initial understandings and imagined practice.

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SECOND GRADE CHILDREN'S UNDERSTANDINGS AND DIFFICULTIES WITH PATTERNS

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The goal of our research project was to understand how children extend patterns, what they know about patterns, and what difficulties they have with patterns and patterning. Sixty-four second grade children took a test and eighteen of them were individually interviewed. Sixty-one children correctly extended the simple repeating pattern (○Δ○Δ○Δ○Δ) and nineteen children correctly extended the growing pattern (ABABBABBBABBB). Only five children correctly extended all the four patterns. We explored five categories of children's misunderstanding of growing pattern extensions. The findings add to the research about children's pattern understandings and provide a stronger basis for instruction, professional development and curriculum development.

Introduction

Several researchers have focused on the effects of teaching patterns (Mckillip, 1970; Hendricks, Trueblood, & Pasnak, 2006). For example, Burton (1982) emphasized the importance of teaching and learning patterns. Alsom Clements and Sarama (2007) noted that children have to be able to identify patterns and generate patterns throughout kindergarten and the primary grades. The processes of patterning develop early algebraic thinking and make young children's skills in mathematics meaningful (Waters, 2004) because children can experience comparing, counting, symbolizing, classifying, measuring, representing with pattern activities (see, English, & Warren, 1998). Preschool children have been found to engage heavily in pattern analysis in their play (Ginsburg, Inoue, & Seo, 1999). Patterns are not only important in mathematics; they are a big part of children's lives. The purpose of the study was to investigate the pattern understandings of second grade children, who have experienced patterns in their life and in school; specifically, we explored how children extend patterns, what they knew about patterns, and what difficulties they have with patterns and patterning.

Theoretical Background

Patterns are important to children's mathematical understanding. Through patterning children can build a foundation for temporal and spatial sequences; expand the ability to recognize similarities and differences which are basic to their success in mathematics (Burton, 1982); and develop the variable concept (English, & Warren, 1998) and number concept (Mckillip, 1970).

Patterns in algebra are divided into two categories: repeating patterns and growing patterns. Repeating patterns are patterns with a recognizable repeating cycle of elements, referred to as 'unit of repeat' (Zazkis, & Liljedahl, 2002). For example, ABABAB is a repeating pattern and AB as the unit of pattern, A and B serve as elements. Growing patterns have discernible units commonly called terms and each term in the pattern depends on the previous term (Queensland Studies Authority, 2005). For instance, "ABABBABBBABBBB is a growing pattern and AB,ABB,ABBB, or ABBBB are terms.

Recognizing of repeating patterns requires knowledge of similarities, differences, and ordinal relations (Hendricks, Trueblood, & Pasnak, 2006). Warren (2005a) investigated young children's thinking about repeating patterns and the instructional process. She found even though children can generalize patterns with symbols, they need to articulate patterns with “variables”.

Although early adolescent classroom uses growing patterns to bridge the gap between arithmetic and algebra, students have difficulties with the transition (Warren, 2005b). Growing patterns are more difficult cognitively than repeating patterns for children to extend. Many children struggle with growing patterns due to their lack of prior knowledge and experience. Extending growing patterns does not mean copying the given elements, but requires children to perceive the relationship between the previous term and the next term. Extending growing patterns also involves an understanding of some number concepts, such as the concept of “one more than” (Mckillip, 1970). For example, to extend a growing pattern, ABABBABBBABBBB, children need to realize that the number of B’s is growing one more than the number of B’s in the previous term. To support children’s understanding of growing patterns, it is necessary to know what kinds of difficulties children will encounter with growing patterns.

The central research question explored in this study, is how second grade children extend repeating and growing patterns, and what are their difficulties and errors to extend patterns?

Methods

Sixty-four children took a test and eighteen of them were individually interviewed in May of second grade on tasks involving patterns. Second graders were chosen since by then children had already have experienced repeating patterns and growing patterns in kindergarten and first grade.

Subjects

The subjects for the study consisted of sixty-four children in three second grade classes in Phoenix, Arizona. The school chosen was a public school and was located in an urban area. At this school, classes were organized according to children’s English level since almost all the children were of Hispanic origin [with more than 50% English language learner] and their teachers taught them in English. One of the three classes was high English proficient class, another was middle level and the third was designated a low level English proficient class. Ninety-six percent of children in the school were Hispanic in origin; two percent of the children were African-American and two percent of children were Caucasian. Ninety-three percent of the children in the school received free lunch and the other seven percent got reduced lunch.

Tasks

The test had two repeating patterns and two growing patterns listed in Table 1. It asked children to extend each pattern for as long as possible. Each child had an individual test sheet. Children were interviewed with the same tasks.

Table 1. Test Problems

Category		Pattern
Repeating pattern	Simple	○Δ○Δ○Δ○Δ
	Complex	GGRRRGGRRRGGRRR
Growing pattern	Simple	ABABBABBBABBBB
	Complex	●●○○●●○○○○●●○○○○○○○●●

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

We assumed that the simple pattern was easier than the complex pattern since in the simple repeating pattern an element comes once in the "unit of repeat" (○ come once in every unit of repeat, ○Δ) but in the complex repeating pattern an element comes twice or three times in the "unit of repeat" (two Gs come in every unit of repeat, GGRRR). This is the similar for the growing patterns. In the simple growing pattern, the fixed element comes once in a term (*A* comes once in every term), and the number of increasing is one (*B* is increasing by one each time). However, in the complex growing pattern, the fixed element comes twice (two ●s come in every term) and the number of increasing is two (○ is increasing by two each time).

Data Collection

Children took the test and were interviewed in May, as they were completing second grade. When a child could not read the problem, we read the problem for the child. After they took the test, we interviewed eighteen children to better understand their pattern extensions. We chose children, who extended patterns correctly or who extended patterns incorrectly with particular misunderstanding, to interview. Initial questions were "why did you make it longer like this?" and the follow up questions were developed according to a child's initial responses. When we could not understand what a child had explained, we asked the child for further explanation until we understood his or her explanation. The interviews were conducted in the classroom. During the interview we took notes. The notes and children's test sheets were gathered to be analyzed.

Data Analysis

Answers were coded both in terms of whether the extensions of the patterns were correct or not and the strategy the students employed in extending the pattern. The correct extension was coded as 1 and incorrect extension as 0 for each question. There was not special strategy coded for the correct answers but errors were coded for the incorrect answers. The categories of students' errors will be explained in results section which follows immediately.

Results

Five children (7.8% of the total) extended all the four patterns correctly, although two of them made a minor error on one problem, using a lower case 'r' instead of a capital 'R'. Results for each of the four pattern extension are summarized in Table 2. In the next section we focus in children's incorrect extensions of patterns as they provide critical insights about children's pattern understanding. Consequently, the analysis focuses on both correct and incorrect renderings.

Table 2. Number of Children Correctly Extending Each Pattern (N=64)

Pattern	Number correct
○Δ○Δ○Δ○Δ	61
GGRRRGGRRRGGRRR	51
ABABBABBBABBBB	19
●●○○●●○○○○●●○○○○○○●●	8

Difficulties with the Extension of Repeating Patterns

Three children extended simple repeating patterns incorrectly and thirteen children extended complex repeating patterns incorrectly. It was hard to find something in common from their errors not only because the number of children who got wrong answers was small, but also because their answers were very different with each other except for the three children who shared the same error, which was GGRRGGRRGGRR for the complex repeating pattern, GGRRRGGRRR. The following example was John's incorrect extension of repeating patterns (Figure 1). His extension had a repeated unit without anything changing but the unit was incorrect.

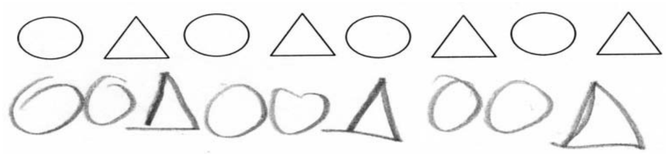

Figure 1. John's Incorrect Extension of a Repeating Pattern

Mathew's way of extension showed his recognition of elements. He identified the elements in the unit of repeat but he did not consider the number of elements in a unit. Therefore, he extended correctly the pattern in which an element comes only once in each unit, ○Δ○Δ○Δ○Δ, however he did not correctly extend the pattern in which an element comes twice or three times in a term (Figure 3).

Figure 2. Mathew's Incorrect Repeating Pattern Extension

Difficulties with the Extension of Growing Patterns

Nineteen children extended the simple growing pattern correctly and eight children extended the complex growing pattern correctly. It is not quite true that the eight children who correctly extended the complex growing pattern correctly extended the simple growing pattern as well, since the three of the eight children made a counting error on the extension of the simple growing pattern. Children's incorrect extensions of growing patterns displayed commonalities. There are listed in Table 3. No answers, unfinished extensions or extensions which did not have a unit of repeat were categorized as "None", which we did not focus on in this study.

Table 3. Number of Children for the Growing Pattern Extension Incorrectly

The type of incorrect extension	Number of children	
	Simple growing pattern (*N=45*)	Complex growing pattern *(N=56)*
A long repeating pattern	16	21
Repeating of the simplest term	6	8
Partly growing then repeating of the whole	2	5
Repeating of one term	7	0
Increasing incorrectly	0	5
None	14	17

The most common misunderstanding of growing pattern extensions was that children considered growing patterns as long repeating patterns. Since children copied the elements from the starting of the pattern without a full comparison of each term, they did not recognize the increasing number of elements. They perceived the whole elements on the task as one unit of a repeating pattern.

The following examples were from Mike's growing pattern extension and showed a long repeating pattern (Figure 3, and Figure 4). Mike copied from the starting term. Although the end of given pattern was two blacks, he started with two blacks again since the original given pattern started with two blacks

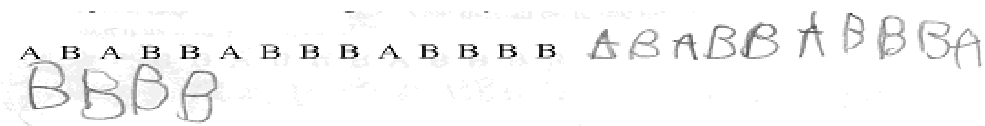

Figure 3. Mike's Simple Growing Pattern Extension as a Long Repeating Pattern

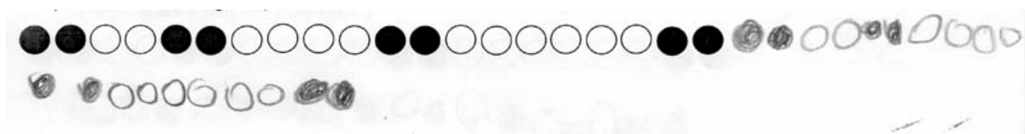

Figure 4. Mike's Complex Growing Pattern Extension as a Long Repeating Pattern

One of the most common misunderstandings of the growing pattern extension was to repeat the simplest term to extend it. The simplest term for a pattern, ABABBABBBABBBB was AB so Janet used only one A and one B to extend the pattern (Figure 5). A child extending with repetition of the simplest term did not consider the number of elements in each term important. Janet circled two whites and two blacks and figured out several groups of two white in each term but she did not focus on the number of two whites (Figure 6).

Figure 5. Janet's Simple Growing pattern Extension as a Repeating of the Simplest Term

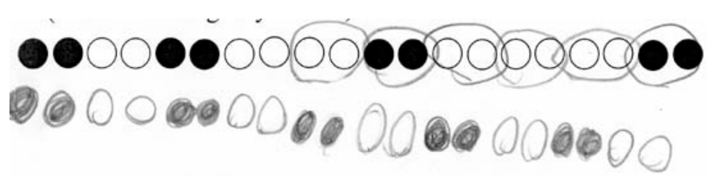

Figure 6. Janet's Complex Growing Pattern Extension as a Repeating of the Simplest Term

Another common misunderstanding of the growing pattern extension was to extend some terms correctly then to copy the whole as a long repeating pattern. At first, Jose knew that the white circles increased by two. However, when the number of white circles was over ten, he started to draw two white circles (Figure 7).

I: How did you make this longer?
Jose: I think we have to follow this (pointing the previous black and white circles).
I: Can you explain how you followed this?
Jose: by 2's. Six, eight (he counted white circles by two).
I: Then, you put here?
Jose: 10. It is by 2's.
I: Then, you put two (whites).
Jose: I think this is 10 (the number of previous white circles), starts all over again.

Figure 7. Jose's Growing Pattern Extension as Partly Growing then Repeating of the Whole

The other misunderstanding was found only from simple growing patterns but not from complex growing patterns. Children repeated one of the terms (e.g., ABBB) to extend the growing patterns. Some children repeated ABBB and others repeated ABBBB. Children who had this misunderstanding did not explain why they extended it with only ABBBB or ABBB. Marisa only said "because you need to do the order" without further explanation (Figure 8).

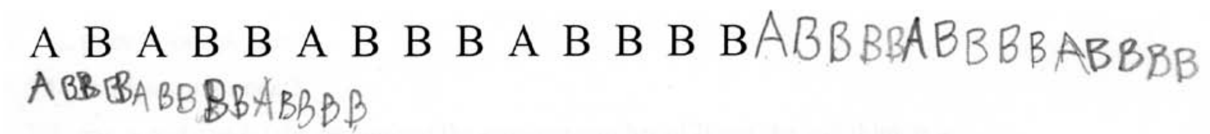

Figure 8. Marisa's Repeating Pattern Extension as Repeating of One Term

The last shared misunderstanding was found only from complex growing patterns and not from the simple growing patterns. Alex said "keep two blacks and one more white" (Figure 9). Even though five children made the same error, it could have been considered as their counting error. Since four of the five children correctly extended the simple growing pattern in which B was increased by only one, more pattern problems are needed to give them to for the researcher to fully disentangle this issue.

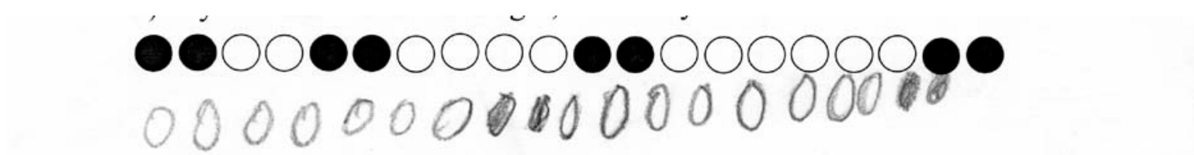

Figure 9. Alex's Growing Pattern Extension as Increasing Incorrectly

Extending of Repeating Patterns Correctly but a Growing Pattern Incorrectly

Thirty-five children extended repeating patterns correctly and a simple growing pattern incorrectly. That is, they recognized similarities, differences, and ordinal relations in the repeating patterns (Hendricks, Trueblood, & Pasnak, 2006) but did not recognize the relationship between the previous term and the next term, and the concept "one more than" in the growing patterns (Mckillip, 1970). Carlos extended repeating patterns correctly and growing patterns incorrectly. The following example shows Carlos' comparison of repeating patterns with growing patterns. Here, what Carlos found as a difference was between shapes and letters, but not between patterns. He did not focus on the number of elements between terms.

I: Do you think ○Δ○Δ○Δ○Δ is the same with ABABBABBBABBBB or not?
Carlos: Different.
I: Why do you think so?
Carlos: This one (○Δ○Δ○Δ○Δ) is circle and triangle and this one (ABABBABBBABBBB) is A and B.

Children who extended growing patterns correctly recognized the difference between growing patterns and repeating patterns not by the shape or letter but by the number of elements. They explained the difference by "gets bigger", "add one more", "goes one more" or "have more". The following example showed Susie's explanation about the difference between repeating patterns and growing patterns.

Susie: This (○Δ○Δ○Δ○Δ) is different to this (ABABBABBBABBBB). This is circle, triangle, circle, triangle, circle, triangle. This one is ABABBABBBABBBB. B keeps bigger.

Extending a Simple Growing Pattern Correctly but a Complex Growing Pattern Incorrectly

Five children extended a simple growing pattern and a complex growing pattern correctly. In contrast, fourteen children extended a simple growing pattern correctly and a complex growing pattern incorrectly. Tom identified the growing number of elements for a simple growing pattern but his explanation for a complex growing pattern did not show that he understood the growing number of elements when it was not one. He said "the two patterns (ABABBABBBABBBB and ●●○○●●○○○○●●○○○○○○●●) are different because this one (●●○○●●○○○○●●○○○○○○●●) is circle and this one is (ABABBABBBABBBB) letters." Tom used a simplest term (●●○○) to extend the pattern so his pattern extension was ●●○○●●○○●●○○. Tom's explanation about the difference was similar with the explanation of the children who did not recognize the difference between repeating patterns and growing patterns, which was to focus only on the type of the element (e.g., shape or letter). However, Jane, who extended a complex growing pattern correctly, focused on the difference of the growing number of elements as following.

I: Is this (ABABBABBBABBBB and ●●○○●●○○○○●●○○○○○○●●) similar or different?
Jane: Similar.
I: How is it similar?
Jane: Because it is counting by 2's. Two blacks, two whites, two blacks, four whites, two blacks, six whites, two blacks, eight whites.
I: So it is similar?
Jane: Because this one gets one bigger, ABBBBB (which was coming next to ABBBB). It is not by 2's.

Jane recognized that the element of both patterns was increasing, and that the number of elements that were increasing was different. One pattern was increased by two whites and the other was increased by one B. Jane identified the similarity and the difference between both patterns.

Conclusion

Overall, the second grade children in this study demonstrated success in extending of repeating patterns. On the other hand, over 70% children misunderstood growing patterns and did not extend them correctly. There were five categories of children's misunderstandings of growing patterns: they extended a growing pattern as a long repeating pattern; they repeated the simplest term to extend growing pattern; they extended one or two terms correctly but children repeated the whole as a repeating pattern; they repeated one of terms; or they increased incorrect number of elements.

Children's reasoning presents for why repeating patterns and growing pattern are different lay in their focus on the shape and the letters but not on the patterns themselves. That is, children focus mostly on the type of element. Children need to have more experiences working with patterns where the elements are varied across the dimensions of color, shape, letter, or number. With these experiences, they need to be guided to focus more on the similarities, differences, ordinal relations, and the relationship between terms rather than element type.

To connect pattern understandings with the idea of a function, children need to generalize patterns such as "it goes one A and one B" or "white grows by 2" or "B gets one bigger". In this study, children who used copy from the start could extend patterns without recognizing of the relationship among terms. They need to have chance not only to extend patterns but also to generalize them. This will help broaden children's patterning understanding with critical linkages to the central mathematical concepts of variable and function.

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SECOND GRADE STUDENTS' PREINSTRUCTIONAL COMPETENCE IN PATTERNING ACTIVITY

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The study assessed 21 2nd grade students' preinstructional competence in patterning activity. Over three rounds of clinical interviews, they solved tasks that addressed various aspects of patterning work. The theoretical framework uses the phenomenological notion of structural awareness (SA). Some results: (1) SA and functional thinking (FT) were not evident in numerical patterning tasks and a semi-free figural patterning task. Empirical counting was used frequently in dealing with near and far generalization tasks. (2) SA and FT were evident in well-defined figural tasks. (3) Students employed narrow specializing on the last known stage (versus specializing with the use of analogy) in dealing with far generalization.

Introduction

In the California state standards in mathematics, elementary students begin patterning work in first grade by "describing, extending, and explaining ways to get to a next element in simple repeating patterns" using "numbers, shapes, sizes, rhythms, or colors" (California Department of Education, 1997, p. 6). In second grade, they "demonstrate an understanding of patterns and how patterns grow and describe them in general ways," which includes skills of "recognizing, describing, and extending patterns and determining a next term in linear patterns" and "solving problems involving simple number patterns" (ibid, p. 10). By the end of third grade, they should be able to "extend and recognize a linear pattern by its rules" (ibid, p. 12). In fourth grade and beyond, students pursue patterns in a variety of arithmetical, algebraic, and geometric situations. Patterns (as objects) and patterning (as a process) in third grade are also meant to introduce students to functional relationships, while patterns and patterning in the upper elementary grades involve finding and reasoning with structure ("observe patterns," "draw patterns," "construct patterns," and "use patterns" to calculate and predict outcomes).

This study addresses preinstructional cognitive issues relevant to second grade students' (ages 7 and 8 years) competence on growth patterns, numerical and figural. Currently, there is very little information about second-grade students' conceptions of patterns prior to formal instruction. In this report, I address three research questions, as follows. *Prior to a formal study of patterns*:

1. What strategies do second-grade students employ in dealing with a near generalization task (stages 9 and below in a pattern)? A far generalization task (stages 10 and above)? An inverse situation task (given an outcome, find the stage number)?
2. To what extent are they capable of discerning a structure relevant to a well-defined pattern?
3. To what extent are they capable of constructing a structure relevant to a semi-free pattern?

Literature Review

In this review, I exclude findings from patterning studies done with prekindergarten to elementary students that have been drawn from design-driven teaching experiments. Briefly,

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

these studies (e.g.: Papic, Mulligan, & Mitchelmore, 2009; Radford, 2009; Carraher, Martinez, & Schliemann, 2008; Blanton & Kaput, 2004) provide sufficient evidence of the significance of effective mediation from different sources in helping young children develop function-based generalizations involving figural patterns. Here I focus on a few studies that assessed children's pre-instructional knowledge of patterns.

Lee and Freiman (2004) were recently involved in a long-term study on pattern generalization involving K children (ages 5-6), Grade 3 students (ages 8-9), and Grade 6 students (ages 11-12) in a French Montreal private school. When patterning tasks were presented to each cohort, the 35 K children were simply asked to extend the patterns, while the 31 Grade 3 participants had to obtain the answer for the 10th stage and the 23 Grade 6 students were required to obtain a generalization. Findings with the K children show success in extending repeating patterns but not so in the case of growing patterns. They had difficulty transitioning from repeating to growing patterns or at least failed to recognize one from the other unlike the Grade 3 cohort. Also, a few other K children interpreted patterns as repeating with the three given stages as the repeating unit, while two other children tried to "correct" patterns when the numerical dependent values are, say, nonconsecutive. The authors note that "(i)t is difficult to compare K and Grade 3 students on the criteria of continuing their growing patterns. Although a couple of patterns seem to be of comparable complexity, the K children had the added obstacle of having to make the abrupt leap from repeating to growing patterns. If we had accepted the solutions of children who continued to treat all patterns as repeating ones, they would have performed as well or better than the Grade 3 group" (p. 248).

Mulligan and Mitchelmore (2009) tested 103 Australian Grade 1 students (ages 5.5 to 6.7 years) on 39 items that focused on patterns and structures. Results indicate the existence of at least four stages of structural development, namely: prestructural; emergent; partial structural, and; stage of structural development. The authors also saw that the level of a student's structural development seems to be determined by the complexity of a patterning task.

When 45 Australian Year 2 students (mean age of 8.5 years) in Warren and Cooper's (2008) study were (pre-)tested on three pattern tasks, many students had difficulty continuing and creating figural growth patterns. When they were asked to describe a general rule in relation to a pattern task that had a rate of change of 2 units, the most frequent response (49%) is that the pattern "grows in twos" or "goes up by two, one more on each end," followed by a no response (27%), and then the general response, "it grows" (13%).

Billings, Tiedt, and Slater (2008) collected data from 8 US students in Grades 2 and 3 through two separate clinical interviews with an intervening teaching experiment. Prior to formal instruction, the children manifested covariational generalization process that had three levels, as follows: 1-analyzing change between consecutive figures; 2-using previous figure to build a new, and; 3-identifying what stays the same and what changes. Level 3 was difficult among students especially in cases when visual patterns were complex.

Stacey's (1989) study probed 371 Australian students (Years 4 to 6, ages 9 to 11) on linear pattern generalization tasks that were given to them near the end of the school year. With respect to the 82 Year 4 students, her data shows a decrease in their ability to obtain generalizations for a particular well-defined pattern from the 12th stage (59% correct) to the 20th stage (33%) and then to the 1000th stage (17%). Counting was used quite frequently as a strategy in obtaining a value, but in most cases it produced incorrect answers. The other strategies use multiplication, which is not relevant in this study and, thus, omitted. Stacey notes that "(t)he overall impression from the responses of the primary school students was that the children were not reluctant to generalize, but rather that they constructed the generalization too readily with an eye to simplicity rather than accuracy" (p. 153) and that there was a "willingness to grab at relationships without subjecting them to scrutiny" (p. 163).

The three research questions stated in the preceding section address crucial issues in the second grade mathematics curriculum in the US and in other countries, for that matter, when young children's understanding of structures is being extended from repeating to (linear) growth patterns. Also, considering the well-established research knowledge base on pattern generalization at the middle school level (see Rivera (2010) for a synthesis account), it would be interesting to assess elementary students' competence in various aspects of pattern generalization that researchers have found valuable to investigate in the upper grades with older children.

Theoretical framework

In this study, I use the phenomenon of *awareness of, or attention to, structure* in making sense of young children's capacity for structural thinking. Mason, Stephens, and Watson (2009) define a *mathematical structure* of, say, a set (of objects, of relations, etc.) as consisting of general properties that are instantiated through relationships between and among particular elements in the set. Structural thinking does *not* involve merely recognizing a relationship that exists as it is primarily about having a "disposition to use, explicate, and connect these [general] properties in one's mathematical thinking" (p. 11). There is, thus, a difference between recognizing a relationship between elements in a set, which is a tendency to dwell in the specific and particular, and perceiving general properties as being instantiated in particular situations relative to the set being investigated. Mason et al's *awareness of, or attention to, structure* captures this particular sense of structural thinking in which the focus "lies" in one's "experience of generality, not a reinforcement of particularities" (p. 17). In more practical terms, learners manifest awareness of or attention to structure when they begin to focus on, say, what stays and what changes – that is, "becom[e] accustomed to considering invariance in the midst of change" (p. 13). Mason et al also use their notion of structural thinking in making a distinction between *empirical counting* and *structural generalization*, which have implications in how we can interpret young children's actions relative to some patterning task, that is, it is interesting to know whether they perform such actions to basically obtain and generate a sequence or to become aware of an underlying structure.

Methodology

Participants, Access, and Relevant Background Contexts

Twenty-one 2nd grade students (7 girls, 14 boys; 20 Hispanic-Americans, 1 African-American; mean age of 7.5 years) participated in the first two of three clinical interviews. One student (a girl) moved to a different school, leaving twenty children who participated in the third clinical interview. On the basis of periodic district assessments, the previous year's data on 18 of the 21 students indicate that 10 of the 18 were above and at first-grade standards. The remaining three students came from different school districts. On the basis of the first benchmark conducted this year, only one was above standard.

The three clinical interviews were conducted over five months with about a month separating each interview. A graduate student conducted the interviews in my presence. Because there were issues that came up as a result of a preliminary data analysis on the first interview, we (graduate student and myself) were permitted to conduct a second interview with the same graduate student conducting the interviews. A third interview then took place, which addressed various gaps in our preliminary data analysis on two earlier interviews. All interviews took place outside of math time. During the one-hour math time, which was daily, we were in class doing collaborative work with the two Grade 2 teachers. Also, I needed to make sure that the children were familiar with us so they would feel comfortable during the interviews.

Interview Protocol and Data Collection and Analysis

Each individual interview lasted between 15 and 20 minutes. The students were asked to think aloud and to write their answers on a construction pad. A number line was provided for them to use during the first round of interviews, which was not used in the second round since they could exhibit facility with the addition algorithm. Concrete objects were always provided. Videotaped interviews and their written work were collected. I kept a log that contained my ongoing data analysis. Data analysis followed a grounded theory framework, which I have used in all my published work (see, for e.g., Rivera (2010)) as follows: an analysis template was set up for each task; individual student responses were categorized accordingly; emergent themes were then established and empirically verified with the construction of case studies and relevant transcripts drawn from the interviews.

Results

First interview

Tasks. Three tasks were presented to the students. In the first two tasks, three puppet dogs and plastic model zebras were shown one at a time and they were asked to determine the total number of eyes in the case of dogs and legs in the case of zebras. Then they were asked to figure out the number of eyes and legs in the case of 4, 5, and 6 dogs and zebras. Next they determined the number of eyes and legs for 10 and 20 dogs and zebras, respectively. Finally, they solved inversion problems that involved determining the number of dogs and zebras when 17 eyes and 21 legs were known instead. With the two inversion tasks, I was interested in how they interpreted the “odd” values. In the third task, they were presented with the semi-free pattern task in Figure 1 with the intent of identifying the kinds of structures they were capable of developing at their level.

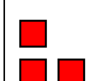
Let us begin with a square and call it step 1. Now suppose step 2 looks like as shown. How many squares do you see? (A) How might step 3 appear to you? Show me with the blocks. (B) Show me steps 4 and 5. How many squares do you see? (C) Pretend we don’t have any more blocks and suppose we skip steps. If someone asks you how step 8 looks like, how might you respond? Can you describe or draw for me? (D) If someone asks you how step 10 looks like, how might you respond? Can you describe or draw for me?	 Step 1  Step 2

Figure 1. A Semi-Free Task

Results. Table 1 shows the frequencies of responses relative to the first two tasks and the strategies they used in dealing with the near and far generalization and inversion tasks. Results indicate their disposition toward using a count-all strategy from stage to stage, their difficulty in employing a count-on strategy from the preceding to the succeeding stage, and the absence of any other sophisticated arithmetical counting strategies. A combination strategy involves using two or more counting strategies. For example, in dealing with the total number of eyes in the case of 20 dogs, Sammy first counted on “2, then 4 plus 2 is 6, plus 2 is 8, …, plus 18 is 20” and then used the number line to count to 40 from 20. In dealing with 6 dogs, Carl began with a count-on strategy in the case of 2 to 5 dogs but shifted to a count-all strategy in the case of 6 dogs (“1, 2, 3, 4, …, 12”). In dealing with the total number of legs in the case of 4 zebras, Manuel initially skip counted by 4s (“4, 8”) and then counted by 1s (“9, 10, 11, …, 16”). Overall, they had greater success in dealing with the far generalization task relative to the dogs pattern compared with the zebras pattern. The inversion task was difficult for most of them. Those who drew circles and sticks (representing eyes and legs, respectively) but did not group by 2s and 4s were unable to determine the correct number of dogs and zebras.

Table 1. Students’ Responses Relative to the “Dogs / Zebras” Tasks (*n* = 21)

Task	Count All	Count On	Skip Count	Number Line	Combination	Drew and counted	Drew/grpd by 2s/4s	Could Not Do
Near	16 / 13	2 / 5	2 / 0	0 / 1	1 / 2			
Far	6 / 2	3 / 0	3 / 0	4 / 3	1 / 1	2 / 0		2 / 15
Inverse						3 / 1	5 / 5	13 / 15

Table 2 shows the frequencies of responses relative to the third task and the extensions on the basis of the first two given stages. None of them offered a structure for the semi-free pattern. All of them ignored step 1. The ten students who decided 4, 5, and 6 circles to be plausible extensions relied on the successor notion of whole numbers (“after 3 comes 4, then 5, then 6”). None paid attention to the shapes of the two given stages. When they were asked them to determine what stayed the same and changed from one step to the next, none saw any kind of change. None could deal with the far generalization task.

Table 2. Students’ Extensions Relative to the Semi-Free Task

(n = 21; CND means “Could Not Determine,” F means “Frequency,” S means “Stage”)

“Patterns”	F	“Patterns”	F	“Patterns”	F	“Patterns”	F	“Patterns”	F
1, 3, 5, 7, 9, CND S10	2	1, 3, 3, 4, 5, S10: 10	2	1, 3, 5, 6, 7, CND S10	1	1, 3, 1, 1, CND S10	1	1, 3, 6, 9, 19, CND S10	1
1, 3, 3, 4, 5, CND S10	2	1, 3, 4, 5, 6, CND S10	0	1, 3, 6, 10, 12, CND S10	1	1, 3, 1, 5, CND S10	1		

Discussion. In all three tasks, none of them manifested structural awareness. Skip counting was not employed efficiently beyond single-digit cardinalities. While this observation might not be novel, what is interesting is the level of empirical counting strategies they employed in dealing with near and far generalization tasks. Their predisposition toward counting all (from stage to stage) versus counting on (from a current stage to the next) actually distracted them from attending to, or developing an awareness of, some structure that might have allowed them to successfully deal with both the far generalization and inversion tasks. Counting all prevented them from structurally thinking in terms of (equal) groups. It is also interesting to note how their ability to perform far generalization appears to depend on group size, that is, the students failed to deal with the far generalization task with zebra legs than with the far task involving dog eyes. Finally, relative to task 3, none of the students: (1) offered a repeating sequence of two objects, a first-grade skill, and: (2) manifested an “illusion of linearity,” in which growth is perceived to be linear and, thus, underdetermining the given values. Most of them *specialized* on the second stage without connecting their extensions to the first stage of the semi-free pattern.

Second interview

Tasks. Five tasks were presented to the students, but only four are discussed in this report due to space constraint. After more than six weeks of daily instruction, I wanted to assess possible changes in their counting strategies relevant to the *first two tasks*, that is, the dogs/zebras pattern. Also, drawing on the thinking of the five students in the first interview who were successful with the two inversion problems, I wanted to assess if providing all of them with drawn pictures of dog eyes and zebra legs might encourage them to focus on counting by groups. The *third task* was a slightly modified version of the semi-free task shown in Figure 1. Instead of two initial stages, they were given three initial stages in a growing L-shaped pattern. On the basis of the Table 2 results, I wanted to assess whether they needed a minimum of three initial stages to be able to recognize and construct a structure for some pattern of their making. Two additional tasks were added. The *fourth task* involves assessing their ability to "guess a rule." In this task, they were presented with a magic paper so that: when 1 circle has been placed under the paper, 4 circles came out; 2 circles under the magic paper produced 5 circles; 3 circles yielded 6 circles, and 4 circles produced 7 circles. They were then asked to predict what would happen when 5 and 6 circles were placed under the magic paper and to describe the effect of the magic paper in a general way.

Results. Table 3 shows the results in relation to the dogs and zebras tasks. More than half employed count on, skip counting, and combination strategies in dealing with both near and far generalization tasks. Examples of a combination strategy are as follows. Dexter, in obtaining the total number of eyes for five dogs, saw that there were initially three puppet dogs on the table. He then lined them up vertically and saw "3+3." Then he counted on 2 eyes, giving him 8 and finally "10 eyes" for five dogs. In Zena's case, she counted "3, 6, 7, 8, 9, 10," which combines skip counting by 3s and count on. In dealing with ten dogs, Juan used a doubles strategy ("10 + 10 equals 20 eyes").

Table 3. Students' Responses Relative to the "Dogs / Zebras" Tasks (n = 21)

Task	Count All	Count On	Skip Count	Combination /Other	Guessed	Drew and grouped	Grouped by 2s/4s	Could Not Do
Near	7 / 7	3 / 7	9 / 4	2 / 3				
Far	4 / 4	1 / 10	9 / 1	3 / 1	1 / 0	1 / 0	2 / 5	
Inverse							6 / 3	15 / 18

Table 4. Students' Extensions Relative to the Semi-Free Task (n = 21, S = "Stage")

"Patterns"	F	"Patterns"	F	"Patterns"	F	"Patterns"	F	"Patterns"	F
1, 3, 5, 8, 5, S10: 10	1	1, 3, 5, 6, 7, S8: 8	6	1, 3, 5, 7, 9	1	1, 3, 5, 7, 8, 10, S10: 11	1	1, 3, 5, 7, 7, S8: 8	2
1, 3, 5, 6, 8, S8:14	1	1, 3, 5, 8,10, S10:10	1	1, 3, 5, 7, 23, 22	1	1, 3, 5, 7, 8, 10, S8:15	1	1, 3, 5, 6, 6, S8: 8	2
1, 3, 5, 4, CNDS4, S8	1	1, 3, 5, 7, 9, S8: 15	1	1, 3, 5, 6, 6, CND S10	1	1, 3, 5, 7, 8, 10, S10: 10	1		

Table 4 shows the sequences generated by the students. Again, most of them ignored the first two stages and specialized on the third stage to generate the succeeding stages. Only two students gave some indication of structure. Dexter produced a growing L-shaped pattern by adding two squares (row and column). When asked to determine stage 10, he reasoned by saying "there are 10 across and 10 up." Gerry's pattern of a few stages is as follows: "1, 2+1, 3+2, 4+1, 5+2, 6+1, 7+2, 8+1."

In Table 5, none of the students articulated "adding 3." Eight of them generated the correct dependent values 8, 9, and 10 circles in cases 5, 6, and 7, respectively. But when asked to explain the dependent values, they all relied on the fact that since the dependent terms 4, 5, 6, and 7 were consecutive numbers, the next number in the sequence would be 8, followed by 9, and then 10. Mig claims, "we are plusing it by the ones."

Table 5. Students' Guesses Relative to the Guess My Rule Task (n = 21)

Guess	F	Guess	F	Guess	F	Guess	F	Guess	F	Guess	F
5 -> 8 6 -> 9 7 -> 10	8	5 -> 7 6 -> 8 7 -> 12	2	5 -> 8 6 -> 7 7 -> 8	1	5 -> 5 6 -> 10 7 -> 7	1	5 -> 10 6 -> 9	1	5 -> 10 CND	1
5 -> 6 6 -> 7	1	5 ->4 6 ->7	1	5 -> 6 CND	2	5 -> 5 6 -> 6	2	CND	1		

Discussion. Comparing Tables 1 and 3 on the same two numerical tasks, while there was a significant change in students' empirical counting strategies, there still was no evidence of structural awareness. Also, even when they were provided with drawings of eyes and legs, must still could not solve the two inversion problems even when the outcomes were even. This failure might be attributed to the fact that they still do not have a facility for grouping objects. Comparing Tables 2 and 4, providing three initial steps that suggested a structure in a semi-free task still did not help. Except for two students, most of them specialized on the third stage in order to generate the dependent terms after the third. Results of Table 5 indicate their inability for functional thinking *in the context of a numerical task*. Instead of seeing a relationship between an input and an output, they focused on the dependent terms despite repeated probing from the interviewer about the effect of the magic paper relative to each input.

Third interview

Tasks. Three tasks were presented to assess if they could discern a structure for patterns with "more familiar" features than the semi-free task used in the first two interviews. Figure 2 shows the three tasks and in each case they were asked to use pattern blocks and circle chips to determine stages 4 and 5 and then to describe stage 10 in ways other than using the blocks and chips.

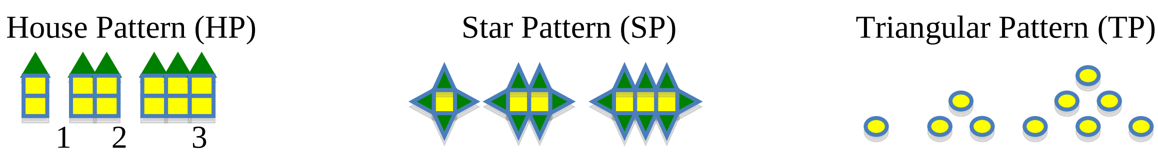


Figure 2. Well-defined Pattern Tasks (HP and SP taken from Greenes et al, 2001, p. 79)

Results. In Table 6, their responses were initially categorized according to degree of structural awareness (SA). Relative to the three tasks, when they exhibited SA, it meant they attended to both shape and count on the basis of their success in both near and far generalization tasks. Having a partial SA meant they attended to either shape or count, and no SA meant a relative inability to discern a structure in both shape and count. Table 7 reports frequencies of those students who had SA.

Table 6. Responses Categorized by SA (n = 20)

Tasks	No SA	Partial SA	SA
HP	2	6	12
SP	1	7	12
TP	12	6	1

Table 7. Frequencies of Responses of Those with SA (n = 12)

Tasks	Near (S4)	Near (S5)	Far (S10)
HP	12	12	11
SP	12	12	10
TP	1	1	1

In Table 6, among those who exhibited partial SA relative to the HP and SP, they were primarily influenced by how they saw the "general" shapes of the patterns. Some ignored the first two stages and began to specialize on the third case, which led to a new pattern. For example, Nikki initially interpreted stage 3 of the TP to be consisting of two groups of 3 circles. Hence, her stage 4 had two groups of 4 circles, stage 5 had two groups of 5 circles, and stage 10 had two groups of 10 circles. Damien interpreted stage 3 of the TP as 3 + 2 + 1, so his stage 4 had 4 + 3 + 1 circles, stage 5 had 5 + 4 + 1 circles, and stage 10 had 10 + 9 + 1 circles. Soraya interpreted stage 3 of the TP as a triangle with a "base" of 3 circles and two growing side lengths. Thus, her stages 4 and 5 resembled triangles having a stable base of 3 circles and the two adjacent sides growing according to stage number. In the case of the SP, Jomer saw a monster with teeth and fangs. So his stages 4, 5, and 10 had two pairs of fangs, two triangles at both ends of squares ("teeth") that grew by the stage number. In the case of the HP, six students saw houses but paid attention only to the number of rooftops (triangles) that grew by the stage number.

Those who exhibited no SA disregarded what they saw and resorted to guessing. For example, relative to the TP, some constructed circles, a fish, and rectangles from stage 3 and did not assess the corresponding cardinalities for pattern consistency. In Table 7, of the 12 who manifested SA relative to the HP and SP, all of them drew stage 10 on paper. In the TP, Juan was the only one to successfully discern a triangle structure that followed the sequence of triangular numbers (1, 3, 6, 10, 15, 21, …).

Discussion. Consistent with the findings on the semi-free task (Tables 2 and 4), many of them employed *narrow specializing on the last known stage* when confronted with what they perceive to be a complex pattern. Those with SA who obtained correct far generalizations in the case of the HP and SP *coupled specializing with the use of analogy*. From the videotapes, this coupling was present in eye-tracking gesture when their eyes began to move from stage to stage, indicating inspection of cases, followed by a pause before finally constructing the tenth stage. In the case of complex patterns, those who engaged in narrow specializing oftentimes became fixated with the third case at the expense of the prior stages. Considering all the tasks given to the students in all three interviews and the data results, it seems that they had more difficulties with numerical than figural patterning tasks. This could be attributed to their lack of more sophisticated arithmetical strategies for counting objects beyond count-all. In the third interview, those students who exhibited partial SA discerned shape structure globally without paying attention to shape details, which affected how they constructed the succeeding stages. Finally, among those students who exhibited SA, their primary mode of expressing a far generalization is more visually (drawing) than verbally oriented. Furthermore, while no explicit symbolic responses (i.e., use of variables) were offered, their incipient generalizations exhibited early manifestations of functional thinking *in the context of figural tasks*.

Endnotes

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USING MULTIPLE REPRESENTATIONS TO MAKE AND VERIFY CONJECTURES

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This article reports on the results of research whose objective was to document and analyze the manner in which students relate different representations when solving problems. A total of 20 students took part in the study, students attending their first year of university studies. In order to design the problem, the underlying information in each representation was deemed to be the starting point of different inferences and of different cognitive processes. The findings obtained make it possible to assert that the underlying information in each representation is not visible to all students and that a problem can foster handling of different representations, the making and verifying of various conjectures and the transfer of knowledge acquired in previous courses.

Introduction

A consensus seems to exist among researchers in mathematics education with respect to the importance of relating different representations of a mathematical concept in order to attain the solution to a problem. One example of this is that when students solve a problem that involves the concept of function, they draw a graph or build a table of values to represent it. From the graph or the table it is possible to obtain the information relevant to the function, which can then be employed so as to arrive at the solution of the problem.

Parnafes and Disessa (2004) have expressed themselves along such lines. On the one hand they indicate that student reflections are tied to the representation and context that they are using. They further state that each representation either highlights or hides aspects of a concept, and that when the students use several representations they develop a more flexible understanding of the concept (p. 251). While on the other hand, the same authors state that the relationship among different representations also provides information regarding the cognitive processes of the students in the problem solving process. The underlying information in each representation is the point of departure of different inferences and, consequently, of different cognitive processes (p. 252). The latter idea is the starting point of the research reported on here, which was undertaken with students attending first year of university. The objective of the research is to document and analyze the manner in which the students relate different representations while solving a problem.

Theoretical Framework

Since the relationship that exists among representations is the core of the research reported on here, the authors of this article believe it is advisable to clarify that in this document the term “to relate representations” is used in the sense of Goldin and Kaput (1996). The foregoing authors point out that “a person relates representations when he is able to integrate his cognitive structures in such a way that given an external representation, the individual is able to predict or identify its counterpart” (p. 416). Several authors (Duval 1988; Parnafes & Disessa, 2004; and Goldin & Kaput 1996) stress the relationship among representations as an important element in stimulating reflection in students. In particular terms, Parnafes and Disessa (2004) refer to use of varying representations in two different ways, to wit: a) use of representations in order to achieve greater understanding of a concept; and, b) use of a representation in order to promote cognitive processes, such as abstraction and generalization.

Use of Representations in Learning a Mathematical Concept

As regards use of representations in comprehension of a concept, Duval (1988) mentions that each representation underlies information concerning the concept. As an example, he alludes to the case of a function represented by a line, in which he indicates that the relevant elements are the direction of the slope of the line, the angles that the line forms with the axis and the position of the line with respect to the origin of the vertical axis. Duval (1988), associates each visual variable with a meaningful unit in algebraic writing. The direction of the line's slope is related to the sign of coefficient X in the expression y=ax+b; the angle formed by the line's intersection with the axis, to the absolute value of parameter a; and the position of the line with respect to the origin of the vertical axis, to the value of b. According to Duval (1993) adequate knowledge of a concept is considered the invariant of multiple semiotic representations of the concept and it is accomplished when a student skillfully handles changes between representations.

Use of Representations to Make the Cognitive Processes of Students Visible

Parnafes and Disessa (2004) explain that each representation provides specific information and promotes certain cognitive processes. For instance they document the work of students within a computer-aided learning environment in which two representations are used. The first is a representation that simulates the movement of two turtles, while the second is a number representation made up of two lists of values, one of positions and the other of speeds. The two representations characterize the same structure –movement of two objects- yet each one is the starting point of different cognitive processes (p. 252). Based on their analysis of the data, the authors identify two types of reasoning. In one the students arrive at the solution by assigning a value to each variable in the problem, ensuring that all of the restrictions are complied with. In the other, the students create a mental model of the movement of the turtles and analyze those images in order to infer qualitative descriptions from those images.

Digital Technologies and Representations

One can also identify studies that refer to the role of digital technologies and their contribution to having students relate representations. In this respect Goldin and Kaput (1996) state that digital tools, the likes of computers, provide resources for an individual to relate representations. The foregoing is achieved when based on an action in an external representation, the subject is able to predict and identify the results of that action in another representation.

As mentioned by Parnafes and Dissesa (2004), each representation is associated with specific information. The problem reported on in the latter document was redesigned so that in addition to using verbal and iconic representations, the students also build a table of values. Usage of an electronic spreadsheet (Excel) can be very useful in this type of activity. Wilson, Ainley and Bills (2004) identify three characteristics of these spreadsheets that may help students to make generalizations and to gain a better understanding of the variables, as follows: a) analysis of the calculations, b) use of notation, and c) the possibility of feedback.

Since the students had used an electronic spreadsheet in three previous activities, there were minimal technical problems related to use of the software and such problems are not cause for thought in this document. Having said this, we would not like to detract from the importance of the role of digital technologies in student reflections.

Subjects, Methods and Procedures

The research was carried out with 10 pairs of first year university students. The activity reported on here was undertaken during two sessions of one hour and one half each. The sessions were videotaped and the students submitted their work both in writing and in electronic files.

The Problem: The Triangular Stack of Boxes

The problem is proposed by Bassein (1993). It consists of determining how the quantities in a sequence of steps change and of identifying that from the very first step a relationship among the quantities is maintained.

The problem was presented to the students in the following manner:

The idea is to build a triangular stack of boxes, as shown in Figure 1.

a) Determine the number of boxes needed to form a stack of a given height.

In this problem the boxes are rectangular and are all the same size.

The height of the stack will be measured in terms of the number of levels in the stack.

For example, a stack made up of three levels will have a height of 3.

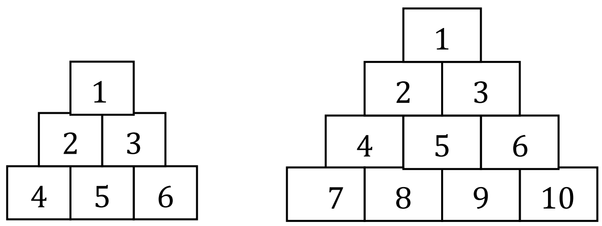

Figure 1. Source: Bassein, (1993 p. 7)

b) You can begin your explorations by drawing stacks made up of different levels.

c) Write down in a table the number of levels and of boxes needed in each drawing.

Two representations are referred to in the problem wording: verbal and iconic. The instruction provided under clause c) suggests to the students that they should make drawings and write down the values represented by different triangular stacks. The clause was included in order to guide the work of the students toward using a numerical representation, and having them prepare a table of values that included the variables of levels and boxes. This was done re-broaching the ideas of Parnafes and Disessa (2004) who indicate that tables of values contribute to facilitating identification by students of patterns and regularities among the series that make up the table, and that they are the basis of particular cognitive processes.

The table of values suggested in clause c) was also included with the intention of orienting the work of the students toward using an algebraic representation. It is important to mention at this point that to obtain a mathematical model that makes it possible to determine the number of boxes needed to build a stack of a given height, it is indeed advisable to work with an algebraic representation.

The students worked on the problem during two sessions. A detailed analysis of their work was subsequently undertaken and this analysis provided information on the manner in which they related the variables present in the problem and the manner in which they developed a mathematical model to represent the relation between them. The actions carried out by the students were followed very closely, paying special attention to their handling of representations.

The questions outlined below arose from the analysis of the data, questions that are the focal point of this document:

- What information do the students obtain from each representation that they use while solving the problem?
- In what manner do the students relate different representations?

Upon analyzing the data, evidence was identified that enabled us to respond to the questions. The manners of relating representations arose and are exemplified with the work of three students: Carlos, Luis Miguel and Leonardo. The work of the three foregoing students is representative of the work carried out by the remainder of the group. The data used below come from different sources. Brief episodes of the videotaped sessions were taken up again, hence the actions and gestures are explained in parenthesis and pauses are indicated by use of ellipsis.

First Manner. Carlos Related Verbal, Iconic, Numerical and Algebraic Representations.

Carlos read the wording of the problem (verbal representation), drew stacks containing different levels (iconic representation) and carried out a numerical exploration (numerical representation). The instruction to draw stacks with different levels was a contributing factor in his being able to identify a relationship between the height of the stack and the number of boxes in the base level, as is shown in the following episode:

> Carlos: …so, analyzing the problem you gave us, the base is equal to the height sought; so we have … if the base is two we add two plus one, then if we have a height of seven, the base is seven.

The drawings done together with the numerical exploration also contributed to Carlos' ability to identify that the total number of boxes needed to create a stack of a given height was equal to the sum of the number of boxes in the base, plus the number of boxes in a stack with one level less, as he explained to the group:

> Carlos: … if its height is seven, its base is seven. So finding the total number of boxes is seven, plus six, plus five, plus four, plus three, plus two, plus one *(He places his hand in a horizontal position and raises it to indicate a different level in the stack each time he adds; seven, plus six, plus five, etc.)*

The explanation provided by Carlos attests to his having associated the total number of boxes with an expression that enabled him to calculate the sum of the natural numbers from 1 to n. Carlos was already familiar with this expression.

> Carlos: I don't know if you remember, but this formula was given to us by our Physics professor when he asked us what the sum of one though one hundred was. And that was when he gave us the formula, *(writes the formula on the whiteboard)* Figure 2.

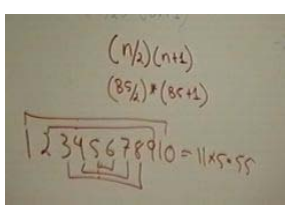

Figure 2. Carlos writes a formula on the whiteboard

Carlos: So to find the total number of boxes, the base is seven, plus six, plus five, plus four, plus three, plus two, plus one, in other words, all of its previous numbers, and thus we have the formula *(writes the formula on the whiteboard)*, n is the number of levels.

With a few examples, Carlos checked that the formula worked to calculate the number of boxes needed to create a stack of a given height. It is important to point out here that it was not simply a matter of his having memorized a formula; the explanation he gave his classmates and what he wrote down on the whiteboard confirm that he had established a relationship between the numerical representation of the sum of the natural numbers from 1 to n, and the algebraic representation that corresponds to that sum, as can be seen in Figure 2. What is more, Carlos was able to transfer the information he obtained from working with the verbal, iconic and numerical representations to a new context.

Second Manner. Luis Miguel Related Verbal, Iconic and Numerical Representations

Luis Miguel began his exploration on a sheet of paper, in which he represented particular cases. It is noteworthy to mention that unlike Carlos, Luis Miguel counted the number of boxes needed to obtain a stack of a given height. Apparently he did not consider the number of boxes needed to create a stack with one level less. Figure 3 depicts Luis Miguel's drawings, submitted in his written report, where he has one stack made up of six levels with twenty-one boxes, one stack made up of seven levels with twenty-eight boxes.

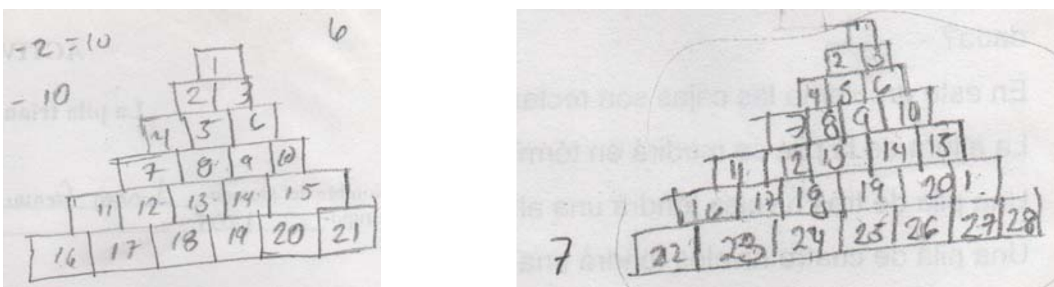

Figure 3. Luis Miguel draws particular cases.

In the episode included below, Luis Miguel explains to his classmates how he went from the wording of the problem (verbal representation) to the drawings of stacks containing different levels (iconic representation), from which he obtains the information needed to build a table of values (numerical representation).

Luis Miguel: I created a table in Excel *(inaudible)* and by levels I started with zero …
Researcher: You worked directly in Excel?
Luis Miguel: No *(inaudible)* first on a sheet of paper *(points with his hand to the shape of a triangular stack).*

In figure 4, Luis Miguel explains the manner in which he developed a table of values on the electronic spreadsheet. As can be seen during his explanation the student used Excel syntax.

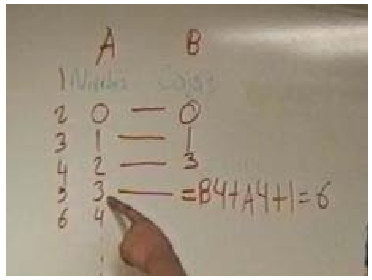

Figure 4. Luis Miguel explains his work with an electronic spreadsheet

The episode outlined below includes a transcription of Luis Miguel's participation, and represents evidence that the numerical representation contributed to his identification of a relationship among the values in the table. The relation enabled him to determine the number of boxes needed to build a stack of a given height. Nonetheless the operation that he proposes, B4+ A4 + 1, which corresponds to adding the number of boxes needed to create a two-level stack, plus the number of boxes in the base level of that stack, plus one, makes it possible to infer that Luis Miguel did not identify the same relation as Carlos did. Carlos pointed out: if its height is seven, its base is seven.

Luis Miguel: For one level it's one box: for two levels, 3 boxes... And what I did was: add the two previous ones, plus 1; for example, for 3…, *(He writes the formula =B4+A4+1, and using it he calculates the number of boxes needed to create a three-level stack).*

Luis Miguel: So I would be left with three plus two is five, plus one is six. And that's the way I would fill it in up to 50.

In order to encourage Luis Miguel to reflect upon the fact the expression he had come up with only worked in particular cases, the researcher intervened as follows:

Researcher: And what about if you have a stack made up of 75 levels?
Luis Miguel: I would have to make the list all the way down to 75.
Luis Miguel: The formula depends on the whole list *(Moves his hand upward to indicate the previous numbers in the table).*

Luis Miguel found a relation based on his analysis of particular cases, but he was unable to find a general expression to determine the number of boxes needed to build a stack of a given height.

Number of levels	Number of boxes
1	1
2	3
3	6

Figure 5. Excel table built by Leonardo

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

Third Manner. Leonardo Related Verbal, Iconic, Numerical and Algebraic Representations.

Leonardo initially worked with the verbal representation, then changed to the iconic representation and drew stacks with different heights. From his analysis of the figures, he obtained information that he used to built a table in Excel. Figure 5 shows the table he built. Column 1 indicates the number of levels, while column 2 indicates the number of boxes.

The numerical exploration undertaken by Leonardo in the table facilitated his ability to identify relations between number of levels and number of boxes variables. The extract provided below portrays the analysis he carried out and the explanation he gave to his classmates.

> Leonardo: The way I solved it is you have levels and numbers of boxes: for one level, one box; for two levels, three boxes; then for three levels, six boxes. And here what I found was that this quantity of boxes *(Points to the value three in the number of boxes column)* is equal to the sum of the previous [level] *(Points to the value one in the number of boxes column)*, plus the number of levels that you want to have *(Points to value two in the number of levels column).*

The work done subsequently by Leonardo was divided into two sections. In each section one can identify a relation between the number of boxes and number of levels variables.

First Relation Established by Leonardo

In the extract provided below, Leonardo proposes an expression to obtain the number of boxes needed to create a stack containing n number of levels.

> Leonardo: I found that the number of boxes is equal to the number of levels that you want to have, plus the number of boxes that you had before, and that gave the result. So I applied the formula to calculate them, equal to … *(In Figure 6, he has drawn a table on the whiteboard and uses Excel syntax)* cell B2 + A3 and it gives me a result of 6. So then I just copy and paste *(Uses Excel-related language, copy and paste)*

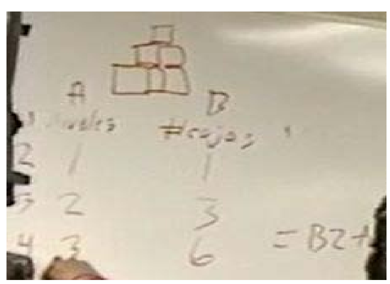

Figure 6. Table built by Leonardo

Leonardo identified that the number of levels in each stack coincided with the number of boxes in the base, he then used that relation to determine the total number of boxes in each stack.

Second Relation Established by Leonardo

Leonardo obtained the total number of boxes by adding the number of boxes needed to create a stack with one level less, plus the number of levels from the next stack. In the episode transcribed below Leonardo explains that in order to obtain the total number of boxes, it is necessary to build a table with a consecutive number of levels.

Leonardo: The consecutive continued to be maintained *(Points to the column with the number of levels)*, but when there was a later one, from 2 to 7 for example, then it didn't give the result, it didn't work. I agreed that this result was correct for consecutive stacks that increased one by one.

Leonardo: So I began to investigate what relation existed between those two columns *(Point to the number of levels and number of boxes columns)*, and thought that this result was a function of this number, with something that could be a sum with something or the product of something, and so I ended up dividing this column *(Points to the number of boxes column)* by this one *(Points to the number of levels column)* and found that this was the result *(Added an additional column, and writes down the result of the divisions).*

Leonardo: I divided B1 by A1 *(Divides the number of boxes by the number of levels)* which gives me one, then one point five, then two, then two point five, and so on and so forth, point five more each time.

Leonardo wanted to determine a formula that related the number of levels (column A) with the results of quotients (column C), so as to obtain the number of boxes without having to depend on previous results.

Leonardo: And then I began to investigate what relationship existed between column A and column C. I knew that the values in column C times the values in column A give the values in column B.

Leonardo: I cannot depend on these results *(Points to the values in column B)* because that is the result I want to arrive at. So what I did was to see what the relationship between column A and column C was …And I found that it was 3; number of levels divided by 2 plus 0.5; then he does calculations for the remaining values in the table.

Leonardo: So to be able to arrive at that result, I inputted in the cell the same number as the number of levels, say A1 over 2 plus .5, all of this times A1, and that way it no longer depends on the result of B. So… *(He calculates a result for a 10-level stack).*

$$\left(\frac{10}{2}+.5\right)(10)=55$$

In Figure 7 one can identify the operations that Leonardo carried out in Excel.

C8 = =((A8/2)+0.5)*A8

A	B	C	D
NIVELES	No. DE CAJA	((A6/2)+0,5)*A6	(A6/2)*(A6+1)
1	1	1	1
2	3	3	3
3	6	6	6
4	10	10	10
5	15	15	15
6	21	21	21
7	28	28	28
8	36	36	36

Figure 7. Table built by Leonardo in Excel

Conclusions

From our analysis of the data, we were able to find evidence that makes it possible to respond to the questions raised in this document. Each representation provides specific information concerning the problem. The underlying information in each representation is not visible to all of the students, as can be seen in the work of Luis Miguel and Leonardo. The problem put to the students triggered usage of different representations, the making and verification of conjectures, as well as the transfer of knowledge acquired in previous courses, as can be seen in the work done by Carlos.

Use of the numerical representation, which was suggested in the wording of the problem, was a contributing factor in enabling the students to identify regularities and relations among the variables –number of boxes and number of levels- in the manner pointed out by Parnafes and Disessa, (2004). The language used by Carlos and Leonardo provides proof that they kept the context of the problem in mind at all times. The written reports submitted by Luis Miguel, as well as his participation, enable us to infer that he did not relate the numerical representation with the wording of the problem and the figures of the stacks, even when he proposed an expression to calculate the number of boxes needed to build a stack.

Endnotes

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CLASSROOM PRACTICES THAT LEAD TO STUDENT PROFICIENCY WITH ALGEBRAIC WORD PROBLEMS

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This study focuses on mathematics classroom practices that result in students' proficiency with word problems in algebra. Our goals are to produce a deeper understanding of "teaching for mathematical proficiency," research tools for the study of teaching for robust mathematics learning, and practical tools that can be used on a large scale for benchmarking and improving teaching practice. Word problems are central to mathematical proficiency because fluency in solving them demands a wide range of sense-making, modeling, representational, and procedural skills. Specifically, we explore the following two questions: (1) What instructional practices are frequently used by teachers judged to be doing an exceptional job of helping students to develop proficiency in solving word problems? (2) What analytic procedures can be used to characterize these promising teaching practices, with low enough cost so that connections between teaching and learning can be examined for a large number of classrooms?

Project activities the first year have included: (1) analysis of existing classroom observation schemes (including, e.g., IQA (Junker, et al., 2005) and CLASS (Pianta et al. 2006); (2) identifying tasks to assess students' robust mathematical understanding related to solving algebraic word problems; (3) interviewing students as they work on these tasks; (4) developing and refining a new analytic system for observing mathematics classroom instruction. In developing our analytic system we are considering multiple analyses, at varied levels of grain size – from the establishment of classroom norms (Horn, 2008; Yackel & Cobb, 1996) to the detailed study of turn-by-turn classroom exchanges (Schoenfeld, 2002) – to identify mathematically productive classroom practices and related student understandings.

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ELEMENTARY STUDENTS' USE OF REPRESENTATION IN DEVELOPING GENERALIZATIONS

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Representation is a critical component in the process of generalization. Unfortunately, the interpretations and development of representations in the minds of students may differ, as each mathematics learner "must progress from idiosyncratic and ad hoc representations of particular problems to conventional, abstract, and general representations that function for a class of problems" (Smith, 2003, p. 264). Of particular interest are internal and external representations. Goldin and Shteingold (2001) state, "the *interaction* between internal and external representation is fundamental to effective teaching and learning" (p. 2). However, the interaction between the internal and external can lead students to internalize instruction differently, leading to a variety of misunderstandings. The aim of this study is to describe the interaction between external and internal representations and how this interaction influences students' ability to generalize.

The study focused on 16 elementary students. Each student was videotaped as they worked a generalization task; a researcher prompted them for their thinking. Data relating to the students' use of representation were coded into the following categories: External Contextual, External/Internal Contextual, Internal/External Contextual, Internal Contextual, and Procedural. Schematics were developed to chronologically track the use of these representation categories during the generalization process. These charts were retrospectively analyzed for patterns of representation use.

Several patterns were found: (1) when students recognized an error, or become uncertain of their results, they often returned to a more external level, (2) students who progressed quickly to an internal representation were more likely to develop misunderstandings and discrepancies among their representations, and (3) errors, recognized or unrecognized, were commonly linked to a mismatch between internal and external representations. One of the major obstacles faced by the students in our study during the generalization process was the transition from external to internal representations. This ability to successfully transition between internal and external representations, and the use of context for this purpose, are paramount to the development of efficient generalizations.

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FROM THE LABORATORY TO THE CLASSROOM: DESIGNING A RESEARCH-BASED CURRICULUM AROUND THE USE OF COMPARISON

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Substantial research on cognition and its development support the benefits of comparison for learning (e.g., Loewenstein & Gentner, 2001; Namy & Gentner, 2002). Grounded in this body of work, recently, building on existing laboratory studies, we have been engaged in small-scale experimental classroom studies to explore the benefits of comparison for students' learning of mathematics, focusing on equation solving (Rittle-Johnson & Star, 2007, 2009). Our goal has been to transform a body of research emerging from the psychological literature into usable, palatable, and effective materials for use in the classroom, bridging the gap between experimental lab-based research and classroom practice.

The central focus of our curriculum is a "worked example pair," a one page, side-by-side presentation of two problems that differ either by problem type or solution method. The worked example pairs serve as a medium to facilitate students' comparison of and reflection on multiple strategies.

This poster shares the development process and decisions we made concerning several aspects of our curriculum, including the scope of our materials within the Algebra I curriculum; the "types" of worked example pairs included; adaptation of our laboratory research materials for classroom use; flexibility in instructional formats; student engagement with the materials; and differentiation, homework, and additional discussion support.

The creation of our comparison curriculum represents the first step in an iterative development process. Both initial refinement and pilot testing of our curriculum are in progress; the curriculum is being tested for efficacy as well. As we look ahead to further iterations, we will continue to consider questions regarding the nature of research-based curricula and curriculum development more generally.

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ILLUMINATING INEQUITABLE LEARNING OPPORTUNITIES UNDERLYING EFFECTIVE COLLABORATIVE PROBLEM SOLVING IN ALGEBRA

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A thematic objective of mathematics-education researchers focusing on algebra content is to develop theoretical frameworks that account for students' difficulty with problem solving. Yet while these frameworks are being developed, the national achievement gap is increasing, even amid calls for equity in mathematics education and, in particular, to improve the accessibility of algebra content for students from minority groups and economically disadvantaged backgrounds.

Radford's (2003) *semiotic–cultural* approach theorizes mathematics learning as the personal construction of meaning for canonical semiotic artifacts (e.g., algebraic symbols such as the variable "*x*") through authentic discursive utterance. By repeatedly expressing their presymbolic notions in available semiotic means of objectification, students build personal meaning for the mathematical notation and develop procedural fluency. Specifically in pattern-finding problems, students solve by progressing through factual, contextual, and symbolical generalizations.

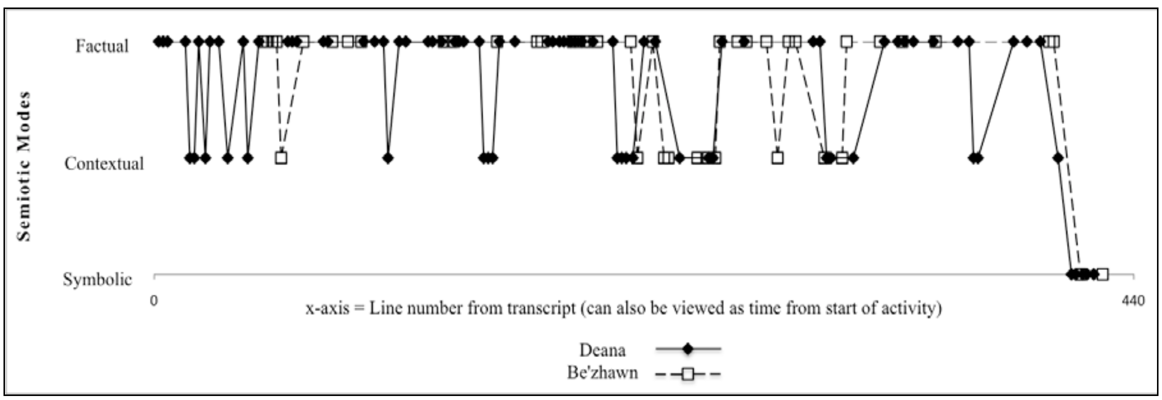

Figure 1. Two collaborating students' utterances over 28 minutes, by semiotic mode.

I propose an elaboration on the semiotic–cultural approach that enables researchers to delve into cognitive planes below external discursive manifestations. By applying this elaboration to empirical data from an intervention conducted at a school for academically at-risk students, I expose tension between one student's overt, positive contributions to shared problem solving and his covert ungrounded reasoning as illustrated in Fig. 1.

Students' effective contributions may blind teachers to underlying discontinuities in meaning construction. I thus propose: (a) a theoretical refinement to the semiotic–cultural approach; and (b) a methodological approach for distinguishing between process and product in collaborative (algebraic) problem solving.

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REDUCING ABSTRACTION: THE CASE OF LOGARITHMS

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This study investigates learners' mental process while coping with abstraction level of the concept of logarithms through the lens of Reducing Abstraction (Hazzan, 1999). Reducing abstraction "refers to the situations in which learners are unable to manipulate concepts presented in a given problem; therefore, they unconsciously reduce the level of abstraction of the concepts to make these concepts mentally accessible" (Hazzan & Zazkis, 2005, p.101). Hazzan (1999) categorizes three abstraction levels: 1) Abstraction level as the quality of the relationships between the object of thought and the thinking person, 2) abstraction level as reflection of the Process-Object duality and, 3) degree of complexity of mathematical concepts. However, because of the space limit, this paper discusses only the first two levels. The research was guided by two questions:

1) What errors are found in students' work when working with logarithmic expressions?
2) Can the Reducing abstraction framework suggest a plausible explanation for the source of these errors?

In order to answer these questions, I used students' written work and informal interviews with them. Three students (Ali, Beth and Cayce- pseudonyms) who were enrolled in a college level foundation course on mathematics served as the participants for the study.

Analysis of written work and interviews showed that, Ali, Beth and Cacey, tended to relate unfamiliar logarithmic expressions to more familiar algebraic expressions and treated them with properties such as commutativity and distributivity. For Ali, $\log_3 4$ and $4 \log_3$ were equivalent; for Cacey, $\log_5(x+1) + \log_5 (x-3) = 1$ was equivalent to $\log_5 (x+1 +x-3)= 1$ and for Beth $\log_3 4 + \log_3 5 = \log_3 (4+5) =\log_3 9$. This can be interpreted as an act of reducing abstraction level (1). Beth's work on evaluating the numerical value of $\log_3 9$ is another example of how learners tend to reduce abstraction level (2). Beth seemed to correctly evaluate $\log_3 9$ as evidenced in her written work, but her arguments (during interview) shows that she knew how to do it (process), but did not understand what it means (object). This is in line with Sfard's (1991) theory of process-object duality according to which the process is less abstract than an object conception. The results emphasize the importance of paying attention to the nature of students' understandings and possible misconceptions in designing instruction.

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Chapter 3: Affective Issues and Learning

Research Reports

Brief Research Reports

Posters

PRIVATE AND PUBLIC DIMENSIONS IN TECHNOLOGY-RICH CLASSROOMS

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We present a model of how moving from private to public is a deeply affective affair that can lead to students feeling alienated especially when their contributions are carefully controlled by teachers and deemed right or wrong. We use a post-modern feminist perspective to draw analogues between the oppression of voice and identity historically with women in society and children in high school classrooms today and how this can be transformed with new classroom connectivity technology.

Introduction

This work is situated within a larger program of research that has studied the profound potential of combining the *representational* innovations of the computational medium with the new *connectivity* affordances of increasingly robust and inexpensive hand-held devices in wireless networks (Roschelle & Pea, 2002) linked to larger computers (Kaput & Hegedus, 2002).

Classroom connectivity has roots in more than a decade of classroom response systems, which enabled instructors to collect, aggregate and display (often as histograms) student responses to questions, and, in so doing, create new levels of interaction in large classes in various domains (Crouch & Mazur, 2001).

Our focus has been to develop software and curriculum that aims to generate student engagement and participation in non-traditional forms by structuring the participation of students in mathematically meaningful ways. Integrated curriculum and software allow for the exchange and collection of student work that can be projected for classroom discussion and debate. For example, students can be assigned a count-off number to create a position function so that the motion of a character in a microworld (see http://tinyurl.com/64wls6 for more details of the software) moves for a duration of 6 seconds at a speed equal to their group number. So Groups 1, 2, and 3 create y=x, y=2x, and y=3x respectively for a domain of [0,6]. The important concept is slope as rate—an underlying concept of the mathematics of change and variation—and a family of functions is created by the whole class via varying the parameter k, in $y=kx$, where k is group number. Such an activity is generative where students can individually or in small groups construct a wide variety of functions with certain constraints focused on either the coefficients of polynomials of the rate of change of elementary functions. The focus of this paper is not on the design of these classrooms or the results of the intervention—this can be found elsewhere (Hegedus et al., 2007)—but more so on the affective dimensions of moving between various zones of privacy.

We take a historical perspective to present an analytical framework that accommodates various zones of private to a theoretical concept of purely public. We propose that moving between these zones is affective and deals with many deep emotional structures that modify our identity and how we interact socially. Layering a post-modern feminist perspective on this model

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of private to public will illustrate how this framework is rooted in demonstrative dimensions such as issues of power, intimidation and identity conflict. This chronological perspective will present the consequences of expressive writing and its influence on social identity for women in the United States, from the first European settlers in 1600s to pre-Civil War 1900s. We use this approach to illustrate how such zones of privacy (see Figure 1) can be analyzed with a historical perspective, drawing analogues with the oppression of woman and their role in society pre-21st century with present day mathematics classrooms. To quote Fried & Amit (2003, p.107): “In the history of general education, a tension between public and private domains in education is not an uncommon theme”. Our conclusions position new classroom connectivity technologies as an advancement in the development of expressivity for students in mathematics classrooms, offering the potential to address some of the deeply debilitating motivation problems existing in the United States.

Theoretical Framework

Prior work (Hegedus & Moreno, 2009) discovered three important products when representational infrastructures (e.g., dynamic mathematics software) and communication infrastructures (e.g., wireless networks) are combined. These include: (1) new forms of mathematical expression, (2) identity formation and identification of self, and (3) new activity structures that focus on intentionality and attentionality. Our focus on a private-to-public space is most related to Item 2.

There we found at least two kinds of identity-formation processes occurring simultaneously within these classrooms. On the one hand, fairly stable social identities (Wortham, 2004), often negotiated and maintained within classrooms, are enacted. These types of social identities are often stable and well-defined both inside and especially outside the classroom and this plays a role in how a student participates in a classroom discussion, whether calls for attention are upheld and how their contributions and participatory role is accepted, e.g., the “smart kid”.

We also observe a more local identity within networked classrooms that is temporal, less stable or well defined, and constructed through the mathematical activity made possible by a networked environment. Here identity can be virtual as work is projected away from a local self to a representation space managed by a teacher. The enhanced mathematical meanings produced by that intersection is to offer an opportunity for students to reinforce their identity by gaining the floor in classroom debates. We have witnessed the outburst of gestural and more general symbolic activities as students gain a kind of emergent identity. These identities are less stable but more flexible as the technology makes it feasible for more students to participate and be engaged within the class.

Our analytical framework (See Figure 1) illustrates how a truly private to a truly public affair moves are theoretical and instead we operate through expanding zones of privacy. The space of action or interaction moves from something closed to something more open and the process becomes more and more social until it is at full disclosure (E) where nothing is private and everyone knows everything about each other (one can say theoretical and never achievable). At the other extreme is Ego, regarding the self—the inner being—and can be unknown to the human (Nietzsche, 1969). What we wish to express here is the affective dimensions involved in moving across zones. These are where we see fundamentally historically charged identity related issues that determine whether someone wishes to participate, learn, express or even be in a situation. Before we present such affective dimensions, we describe the components of the model.

Private affairs are either external forms of actions with oneself (e.g., writing journals, diaries, or self-pleasurable acts) or egocentric speech. We uphold the positions of certain authors (Furrow, 1992; Wertsch, 1979) that the origins of ego-centric inner speech are products of social speech and interaction. In fact, as we begin to apply this model to our particular types of technology-enhanced classrooms, we posit that it is through certain types of metacognitive activity such as *other-regulation* (observing how others makes sense and regulate their thinking) and *self-regulation* that we come to understand more about our self and our own thinking hitherto unknown. It is this process that one's identity can be modified or changed through the enhanced communication infrastructure (as described earlier). Shared private affairs are with a few others. They are intimate affairs as often the members of this semi-closed zone share ideas or thoughts that they have individually constructed as a private affair. It might be deemed risky for the person to share these thoughts with one or more other people and so such a zone assumes norms of trust. A *semi-open space* is less intimate yet involves social norms and practices often situated within how ideas can be shared, vocalized and discussed. Aspects of agency become relevant and support meaning-making as actions advance collective understanding, especially when it engages the perceptions of the participants. This is often the case for a school classroom. Finally, a *purely public space* is where all ideas are public and shared. Theoretically aspects of intimacy or privacy are no longer relevant. All members of the public space are free and unemotionally attached to their place or contribution to such a space.

Extension of Private Zone

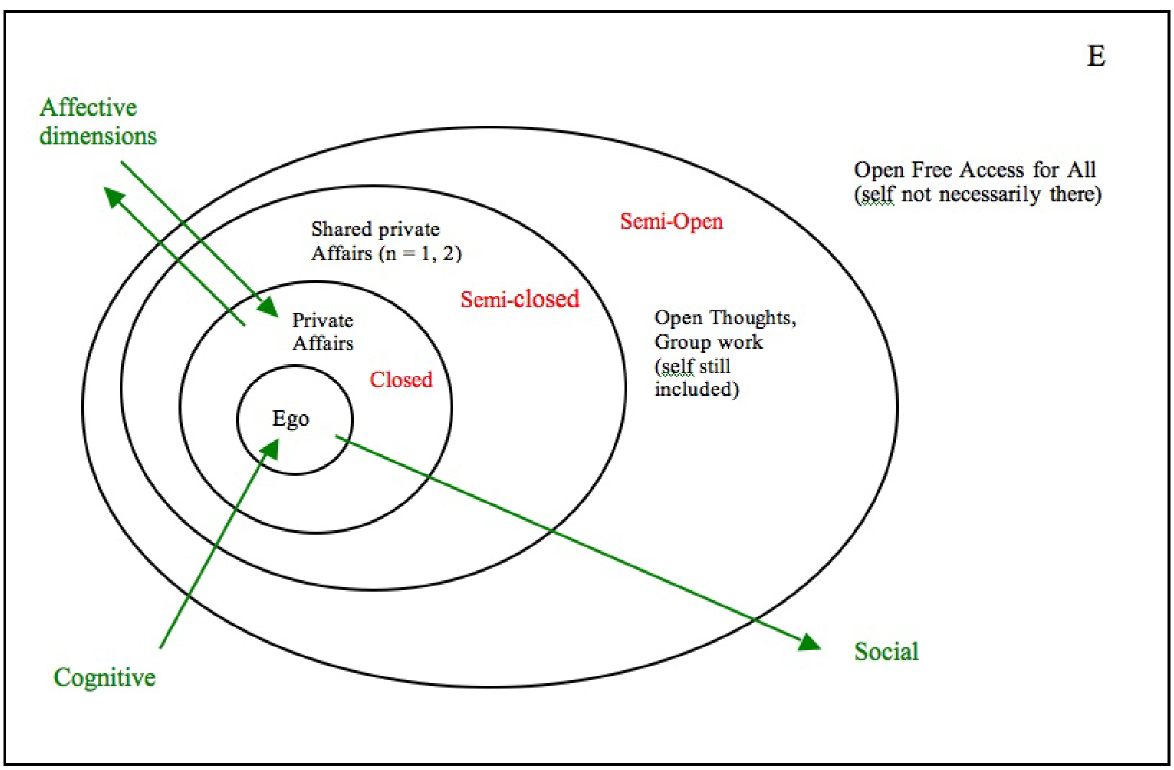

Figure 1. Modeling zones of privacy

It is our position that expressivity occurs as a cognitive response as we are affected by our own feelings and others. These affective responses can be a progression through expanding zones of privacy and interplay between them. Sometimes we are confident to move from a personal space to a semi-open space but this is still an affective response. Our paper aims to provide an analytical framework of affective dimensions.

Expressivity occurs because you want to show others what you can do, who you are and how you have internalized the world and because you want to incorporate others into your world. Traditionally, public and private are perceived as two distinct zones whereas we have tried to separate them out further and will elaborate with historical and classroom examples later.

Private can be thought of as *personal work*, and public as *shared work* but we posit that these are not realistic definitions. We are not focused on other forms of shared-ness such as small group work or constructivist teaching but acknowledge their relevance. It is the intent of journal or diary writing, which establishes some of what is understood about privacy as a conscious and cognitive act, to serve a purpose for a relationship with someone else. This can also include oneself to better understand how one is thinking. Such is not the case in some settings, especially where power and identity conflict amongst participants. In some societies and in some schools, individuals agitate their positions in relation to power in an oppressed system denied of closed and semi-closed spaces to engage in expressive actions. We draw on a postmodern feminist perspective to illustrate how the social identity of women of the past can format our understanding of present classroom interactions. Such identities evolved via the interplay of zones enhancing private expressivity of thought and reformation of individual and collective public actions.

In the early 17th century American women were objects and had no *identity*. Common law "ensured the submergence of the married women's identity under her husband as a 'feme covert'…silent in church, subservient at home occupying a reserved space" (Woloch, 1984, p. 18). Sharing thoughts was viewed as meddling and was not condoned by religious or legal doctrine. In letter books and in public, self-expressions were tied to biblical reference. Public "communicative actions" (Coulter, 2001) were socially invisible which conditioned an absence of themselves in private thoughts. Diaries allowed women to cope with oppression and to establish modes of social interaction. In the next century, women conversed with brief home visits, after church or in exchange of letters with other women.

In the late 18th century, primary schooling expanded, women graduated from academies and entered into teaching and ministries. Legitimacy of women's education enhanced feminine self-esteem and formulated a social identity in a civic function. "The influence women had on children, especially sons, gave them ultimate responsibility for the future of the nation"(Kerber, 1980, p. 229). In *semi-open spaces,* women joined literacy circles and women-libraries to reflect and exercise freedom to challenge ideas collectively. Reformation of private thoughts led to ideologically-based social changes. Themes of this ideological shift included the symbolic power of the classes, the subtleties of class stratification and the idea of home life for each class. (Delamont & Duffin, 1978).

The late 19th century, appearance of ladies magazines publicly revealed feminine thought. Ladies submitted private letters to be critically viewed and judged by women and men. Women offered creative views and an innovative way of thinking about society. Group formulated topics were now projected into a less-private sphere. An unexpected consequence was that the "woman's sphere" gradually enlarged from the home to the larger society as women pursued the women's role beyond the limits of the parlor and church. Education became a special talent of women and the education of the youth moved from the home to the school. With support and integrated action of others, women were able to use journals for closed and semi-closed expressivity. Collective groups formed to share emotions, empower creativity and foster innovation, such actions led to a modification of the social identity of women in society.

More recently, Fried and Amit (2003) describe the interplay of student ideas and public thought of the classroom in math journals. The journals were to be in a semi-closed zone whereby “reflection, internalization, visualization, and the creation of math meaning” (p.92) could be represented in the student’s writing. They found an objection to this notion as the students reconstructed the design of the teacher’s notes and lacked expressive writing. Classroom instruction and pedagogy enhanced a well-defined identity structure mitigating the teacher’s judgment of students’ reasoning and thought processes.

Modes of Inquiry

We draw on a large dataset collected during a 4-year project in two high schools in Massachusetts. The experimental design was a controlled study design. We used two districts of similar type in terms of Socio Economic Status (SES) and achievement (as measured by the Massachusetts Department of Education), and assigned a content test as a pre-test to all incoming freshmen (9th grade/14-year-olds) and assigned a 6-week algebra replacement unit using a quasi-experimental methodology. There were 160 students who received the treatment and we observed 236 other high school students as a comparison. We also visited every classroom implementing the replacement unit and recorded observation notes and took video of the classroom from various angles. We also collected video data on some of the comparison classrooms. It is from this rich dataset that we draw examples to support our framework, which we believe will inform future analysis of similar technology-enhanced classrooms. To inform our framework in terms of its broader context which is germane to aspects of identity, privacy and oppression of voice we draw upon historical accounts grounding the significance of journals for 17th/18th Century women as a means of expressivity.

Analysis

We present three main affective dimensions that provide the dynamics of moving between zones of privacy. For each category, we provide a feminist historical perspective and compare and contrast this with data from our present classroom connectivity project. This will be in the form of classroom enactments or personal student data (e.g., attitude surveys, content tests, student interviews) to illustrate forms of learning and identity formation.

Affective Dimension 1: Space and Time

Like that of 17th century women, certain classrooms silence the thoughts of students. Self-reflection was almost absent then and expressivity was protectively melded with those in public authority. Here we begin to see public as a theoretical notion of oppression not necessarily a collection of people.

Today, in our technology-enhanced classrooms, we observed students working independently or in small groups with computers or similar technologies developing forms of closed spaces within the whole classroom. This can lead to private discussion or work that may or may not be shared. The closed, private space might be a place to explore creative tendencies, which involves risk. Students can be protective of their work in a semi-closed or independently private zone, possessive of their work and wary of sharing their creations.

With classroom connectivity, students can be involved in a *dynamic semiotic enterprise* where the construction they make, or the ideas they are personally invested in can be updated or reconstructed because of the affordances of the software and be something to be shared or used by others to make larger mathematical objects (e.g., families of functions). It is this shared purpose that changes the nature of closeness and how space is structured within the classroom. The invisible yet guiding nature of the wireless network allows teachers to collect and display individual and whole class sets of constructions that are personal and identifiable to each student. It is time that shapes meaning as the teacher gradually and iteratively displays a student's work. We develop a shared intentional space. And if we project this outside of the classroom, ontogenetically, this is the world we meet through our own development. In a sense, the world is too open and we need to find ways to close it partially, because we need others. Inside a connected classroom, the nature of how work can be shared and discussed with varying degrees of openness—even with the software itself (i.e., testing a conjecture by running a simulation)—can modify our needs of others.

In an interview with one of the students (*Student A*) in our study who had gone from scoring below the class mean on a pre-test to above the class mean on a post-test, we discover how the interaction between traditional worksheets and public discussion modifies her comfort with her learning environment. The shift between privacy spaces affects her in a positive way.

> For *Student A* the most enjoyable part was working as a class and having the class discussions; she felt she really understood it afterwards. When asked about the prediction questions on their worksheets, *Student A* said at first they were confusing but when everyone's work was aggregated on the board she understood it. And as they continued with these activities, it got easier.

On a survey we developed using a 5-point Likert scale to measure changes in their attitude towards learning mathematics, we noted an increase in *Student A*'s nervousness in talking out loud in front of her classmates (even though she was often quite vocal) from pre to post intervention, yet a significant increase in her confidence regarding her mathematical ability. She also felt more comfortable talking with her own girlfriends in class.

Affective Dimension 2: Intimacy

With advanced literacy skills, women in the 18th century agitated power by alliances with one or two other women via letter writing. Fears and challenging thoughts were shared in a *semi-closed* intimate fashion by exchanging letters with a close friend. Again, for protection of prosecution caused by a wrongful response, women voiced thoughts in a collective action within a *semi-closed space* of one or two others. In our own study, many students in the comparison classrooms exercised limited power by not sharing their ideas with others and this was often structured by the teacher. Fried & Amit (2003) refer to such actions as "transactional" (p.104) whereby the intent of the expressions are for others and not for self-reflection.

Schorr & Goldin (2008) discuss how the software environments used in our study provide emotionally safe environments but even within such environments negative feelings can occur (e.g., frustration, impasse, or anxiety). At the same time, the dynamics of the environment "evoke the affect of curiosity excitement and challenge" (p.145). In such a framework, affective structures such as mathematical self-identity and mathematical intimacy allow for integrity and a willingness for someone to increase or open up their understanding for public scrutiny.

Intimacy is based on trust, comfort with the environmental setup and reciprocity of emotion. We have observed this to be evident in a certain kind of discourse mediated by classroom connectivity. In the connected classroom, we see frequent efforts on the part of students to convince their peers to attend to a particular object and then "see" it in a particular way that supports their position in an argument. A common occurrence is for speakers to bid for their peers' and teachers' attention to the shared display in the front of the room. In some respects, this display is a rhetorical resource for students in making mathematical arguments. Language is personal, representative of the mathematical objects and ideas that they are projecting into a shared space and are often powerful markers to signal solidarity or adherence to others (see Hegedus & Penuel, 2008 for many examples and further discussion). We claim it is the technology that allows such mathematical intimacy an individual has with their personal constructions to a group level mode of debate and argumentation. So as women in the 18th century began to voice their opinions in a collective action, so we see how such technological environments can develop voice and identity among many. The communication infrastructure integrated with a safe and well-articulated participatory framework (in terms of activity structure and pedagogical intentions) can allow many students of differing types of personality the ability to contribute and observe others ways of thinking.

Affective Dimension 3: Intentionality

With 19th century social changes and equal public privileges such as access to secondary and higher education and legal rights to own property, women gained access of others and executed expressivity in *semi-open spaces* such as publications. Ladies magazines were semi-open journals and served as an accessible forum for group emphatic formal statements as well as private individual letters. Women challenged topics such as education for children, the quality of home making vs. mercantile goods that evolved to joining with men to address public causes such as abolishment of slavery. *Semi-open* classrooms can empower the individual ideas of students; provide them with forum for democratic dialogue and exercise freedom to challenge the thoughts of others (Coulter, 2001). But how does this affective structure realize itself in terms of how students communicate with each other?

Bakhtin (1981) describes how we borrow words so we can mean to others, and we populate them with our own meanings and intentions so we can signify our relationship, attitude, and identity with others. Burke (1969) explains how people using representations direct others' attention to some things, and deflects them away from others. This is called external intentionality. In directing attention, intentionality can be externalized through various forms of expression and action.

Intentionality can be structured by an intimate pedagogy that appreciates the participatory nature of the classroom. In our connected classrooms, the teacher can scaffold the classroom discourse in a highly structured way, and in a manner that allows the structure of the underlying mathematics to emerge. For example, questions in our curriculum follow a pattern of ''where are you in the world?'' i.e., which colored dot or animated character represents your work? And then ''what do you expect to see for Group 1?'' by selecting to see just Group 1 in the world. These are all representative features of our technological environment that can show and hide student work, and representations of such work (e.g., graphs, tables, motions of dots/objects/actors in a simulated world, etc.). Intentionality can also be structured by the expressive needs of an individual.

It is important to ascertain for whom a certain private act is written or composed. Is its intended purpose prescribed or an organic process by which an individual finds a safe place within a conversation or public place? At these decision-making points, loaded with affect, the self is on the dividing line between containment and projection. And once projected, it becomes increasingly difficult for the individual to return to that point as it will be modified by others (however intimate or close they are in terms of privacy). So this stability-instability *Rubicon* is a place where identity is formed or modified. Belonging to a space loaded with intentionality indicates a movement away from semi-closed privacy as it has a pre-determined structure that is transformed by affective responses.

Conclusion

We have presented how private and public are dimensions that are not exact, occur in different forms, and are primarily cognitive affairs that as individuals in society, we move between affective responses to engage with others. The cognitive interplay and expressive actions between zones of privacy can involve positive and negative affect in a safe environment. A historical perspective using postmodern feminism provides an analytical backdrop to understand the complexity of how certain mathematics classrooms manage expressivity and identity. We have proposed that a certain kind of technology-enhanced classroom can establish new forms of communication that can support identity transformation. Such technologies can format the discursive and pedagogical practices of classrooms that can enhance participation in meaningful ways and inform curriculum development in the future.

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PURPOSES OF SMALL GROUP WORK IN SIXTH-GRADE MATHEMATICS CLASSROOMS: WHAT DO STUDENTS PERCEIVE AND VALUE?

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To construct insights about whether and how students develop productive values for engaging in small group work, I analyzed how sixth-grade students from two different classrooms talked about working in small groups. Students' talk was analyzed to assess students' perceived purposes and values for small group work. I interpreted these findings in relation to students' histories with mathematical performance and teachers' efforts to facilitate small group work. I considered whether teachers' implied purposes for group work in each classroom were perceived and valued by students, and developed conjectures for facilitating small group work to promote productive values and dispositions.

Introduction

Small group work in mathematics classrooms can be conducted for different purposes, such as promoting learning and communication through collaborative peer interaction or to enhance direct instruction (Noddings, 1989). Not all purposes for group work or efforts to facilitate group work are similarly productive, and students may or may not interpret opportunities to participate in group work in ways that teachers intend. Following Flores and McCaslin (2008), students' voices about their experiences in small group work can serve as an additional measure of purposes of group work (in complement to observations) and afford an understanding of affective outcomes of teachers' efforts to facilitate group work.

In this study, I sought to understand relationships between the ways in which two middle school teachers facilitated small group work, what sixth-grade students perceived to be the purposes of small group work, and what students valued about small group work. This analysis had three purposes: (1) To determine whether students' perceptions of the purposes of small group work aligned with the purposes promoted by the teacher, I contrasted observations how two sixth-grade teachers facilitated small group work with student interview data in which their students talked about purposes for small group work. (2) To assess affective outcomes of engaging in group work, I analyzed what students with different histories with mathematics performance valued about participating in group work. (3) To construct conjectures about how teachers can promote values for engaging in collaborative work in mathematics classrooms, I contrasted students' values from each classroom.

Theoretical Perspective

Much of the scholarship on small group work focuses on the role of the teacher in facilitating classroom discourse (e.g. Cohen, 1994; Fuchs, Fuchs, Hamlett, Phillips, Karns, & Dutka, 1997). Fewer studies focus on students' perceptions of purposes of small group work or values for group work. In studies when students' roles in group work are studied, the focus is often on observable behaviors (e.g., Gresalfi, 2009; Esmonde, 2009). Outcomes of group work are usually assessed through achievement data (e.g., Fuchs, Fuchs, Hamlett, Phillips, Karns, & Dutka, 1997).

This study complements and extends those studies by listening to students' voices rather than focusing on observations of their behavior, incorporating an alternative cognitive outcome to achievement (students' perceptions), and an affective outcome (values) for examining teachers' facilitation of classroom discourse. Additionally, contrasting students' voices from two different classrooms supported the development of conjectures about teachers' roles in facilitating discourse and group work that can help students develop productive values about group work in mathematics class. Students' engagement can be understood through examining their opportunities to participate in groups, whether students are aware of (or perceive) these purposes (Levenson, Tirosh, & Tsamir, 2009), and whether students merely *cooperate* with these opportunities or come to *value* them (Cobb, Gresalfi, & Hodge, 2009).

Prior research provides insights on how teachers can effectively facilitate collaborative work. For instance, explicitly teaching students to provide more elaborated conceptual explanations to each other correlated with higher student achievement (e.g., Fuchs, Fuchs, Hamlett, Phillips, Karns, & Dutka, 1997). Effective groups work together as knowledge building communities that are truly collaborative and interdependent; knowledge is generated jointly (Scardamalia, 2002). Elizabeth Cohen's (1994) work on Complex Instruction indicates that productive small groups are equitable - students all have equal opportunities to participate, all students' contributions are valuable, and group members have equal status. When students work together on challenging tasks, there are a number of ways to enter into and solve the problems, so there are more opportunities for more students to be competent. However, teachers may vary in the degree to which they promote engaging in group work for the purposes of promoting elaborated, conceptual explanations and collaborative, interdependent, and equitable interactions among students on challenging tasks.

In complement with understanding teachers' efforts to productively facilitate students' engagement in small group work, listening to students' voices can provide insight about why they do or do not engage productively during small group work. Students' talk about their experiences in group work affords an understanding of the degree to which students' perceptions of the purposes of group work are similar to or different from the purposes promoted by their teachers. Students' perceptions are worth investigating because even when students enact behaviors promoted by teachers, students' perceptions of the purposes of participating may or may not align with what the teacher promotes (Levenson, Tirosh, & Tsamir, 2009). Students' perceptions may serve to mediate relationships between teachers' facilitation of students' participation and how students participate (Peterson & Swing, 1985). In this way, students' perceptions provide a lens to understand the nature of students' engagement during group work.

Students' voices can provide evidence of affective outcomes. Students with positive dispositions toward mathematics (National Research Council, 2001) believe that they can do mathematics, value the utility of mathematics, and value engaging in mathematics. Opportunities to develop these positive dispositions are situated in opportunities to enact particular roles in classrooms. As students participate in school mathematics, they receive messages about their competencies and develop their sense of efficacy in mathematics. Higher self-efficacy correlates with higher effort and interest (Zimmerman, 2000), so students with a higher sense of competence in mathematics are likely to persist in the face of challenge and be more interested in mathematics. Similarly, students who value collaborative activity may be more likely to put forth effort into sharing their thinking with peers and trying to understand the thinking of others to develop new understandings as a result of collaboration.

In this study, I described how two teachers facilitated small group work, assessed students' perceptions of purposes of group work and values for engaging in group work through listening to students' voices, and explored which instructional practices appeared to promote productive dispositions and values.

Methods

Data were collected in two sixth-grade classrooms in the same school in the Mid-Atlantic region of the United States. The demographics of the school's student population were the following: 50.6% African-American, 27.5% White, 21.5% Latino/a, and 0.5% Asian-American. According to school district data, 69% of the students were from low-income families. Teachers at this school used the *Mathematics in Context* (2006) textbook series, which is a set of curriculum materials developed with funding from the National Science Foundation. The materials include mathematical problem solving tasks that have the potential to foster dialogue among students.

The two teachers in this study were Ms. Summers and Mr. Winters. Mr. Winters was a second year mathematics teacher at the time of this study. Ms. Summers was in her sixth year of teaching, but this was her first year teaching at this school. Both teachers had experience using these curriculum materials for at least one school year prior to this study. As participants in a statewide professional development activity, the two teachers studied students' engagement with mathematics through video recordings of focus students and developed interventions for their mathematics classroom based upon what they learned about their students. These teachers asked me to interview a range of students for them as a part of their professional development work.

Data Sources

There were two data sources for this study: video recorded classroom observations and one-on-one interviews with students. Video recorded observations from each classroom were analyzed to describe the ways in which the two teachers facilitated classroom discourse and small group work. The set of observations was comprised of five class periods from each teacher across the school year (November to March). Data about students' perceptions and values were gathered through one-on-one interviews that lasted approximately 30 minutes. I conducted interviews with 24 sixth-grade students (12 from each teacher's classroom) in February and March of that school year. Interviewees were purposefully selected; I asked their teachers to identify students who they considered to be generally successful problem solvers, generally struggling problem solvers, and students who were sometimes successful and sometimes struggling. Students responded to a range of interview questions, such as, Why do you think your teacher asks you to work in groups during mathematics class? Do you like (or dislike) working in groups during math class? Why?

Data Analysis

When analyzing the video recorded observations, I sought to describe how each teacher monitored and debriefed group work to infer what teachers could be communicating to students about the purposes of group work. This analysis was conducted through an emergent process (Corbin & Strauss, 2008), with multiple passes through the data. Initial conjectures of significant themes were developed based upon the first two passes through the data. During additional passes through the data, I revisited the conjectures and sought to revise them by seeking confirming or contradictory evidence.

When analyzing the interviews with students, during the first phase of analysis, I sought to identify what students perceived to be the purposes for small group work in their classrooms.

This process involved multiple passes through the data, as described above. For the second phase of analysis through the interviews, I sought to distinguish between students' *own* purposes for group work (values for group work) and what students perceived to be their *teacher's* purposes for group work. Following Cobb, Gresalfi, & Hodge (2009), I conducted discourse analysis of student interview responses to distinguish between obligations-for-others (identifying perceptions of their teacher's purposes) and obligations-for-self (what students valued). Obligations-for-others, or perceptions of teacher's purposes, were revealed when the students said that they engaged in a behavior because the teacher required them to do so. Their perceptions of teachers' perceptions were analytically determined through students' use of verbs with high modality (e.g., "have to" or "need to"). Obligations-for-self, or students' values, were revealed when students reported that they wanted to engage in a behavior or that they preferred or liked engaging in a behavior.

Results

Two Teachers' Efforts to Facilitate Small Group Work

When monitoring small group work in progress, Mr. Winters did more cognitive work for the students than Ms. Summers. For instance, when he visited a group of students that was trying to estimate the equivalent percentage for a fraction (e.g., 156/216), after the student divided using a calculator and held it up to the teacher, Mr. Winters looked at the calculator and said, "Point 72, 72 hundredths, what percentage is that? 72% What is that, almost? 75% what do we know about that? It's close to ¾, you could have estimated…" If a student asked Ms. Summers for help when she monitored the small groups, she would say, "That's why you're in groups – to be open to multiple strategies and teach each other." Then, she would step away from the group.

During whole-class discussions after group work, teachers posed different types of questions and appeared to hold different expectations for students' participation. Mr. Winters' questions were "how" and "what" questions, such as asking students what answer they found, how they found it, and about which step came next in a procedure. When a student responded to his question, he would ask students to raise their hands to see how many of them agreed their classmate. The discussion continued if there was disagreement, and the class went on to the next problem if there was agreement. In contrast, during whole class discussion after small group work, Ms. Summers's questions included many "why" questions, such as "Why does that work?" These questions arose when students went up to the overhead as a group and spent time writing out solutions at the overhead and explaining the solution strategy in detail. Students would ask their peers questions or request elaboration from one another (e.g., "I don't understand why you…") No students posed questions to their peers in Mr. Winters's class.

Looking across the ways in which the teachers facilitated small group work, these teachers appeared to promote different purposes for group work. Given that Mr. Winters asked the students for single answers or individual steps during whole-class discussion, the purpose of group work appeared to be to prepare for whole-class discussion by finding a single correct solution. In contrast, the purpose of group work in Ms. Summers's classroom appeared to be to prepare for whole-class discussion by trying to find more than one solution strategy.

Sixth-graders'Voices and Purposes of Small Group Work: Perceptions and Values

To explore whether the teachers' purposes for group work (implied by how the teachers facilitated small group work) were perceived or valued by students, I will present a subset of the sixth-grade students' voices. These four students were selected as cases based upon two criteria:

data from these students represented the most prevalent themes from the students interviewed from each classroom and these students represented a range of performance in mathematics. Below, I will share how these students talked about obligations-for-others (perceptions of their teacher's purposes) and obligations-for-themselves (what they valued) regarding small group work in their mathematics classrooms.

Enrique: Successful problem solver, Mr. Winters's classroom. Enrique was one of four (out of 12 sixth-grade students) that Mr. Winters identified as a successful problem solver. Enrique said that math had been his favorite subject ever since the 4th grade. Enrique reported earning near perfect grades that year, even a grade above 100% the most recent marking period. He was aware that he was considered by peers and his teacher to be mathematically competent, as he reported that his peers relied on him for help and that his teacher said that he was the highest performing student in his class.

Enrique perceived that the purpose of group work was to complete his work efficiently. When I asked Enrique about working in small groups, he said, "Sometimes I would rather work by myself." When I asked why, he said, "Because I'm good at math. I'm done faster by myself." When given a choice of talking to a teacher or a peer during mathematics class, he said, "More to a teacher. He knows more." When he talked about group work, he emphasized efficiency. At times, he thought it would be more efficient to talk with his teacher rather than his peers. Other times, he sought efficiency through dividing the work among his peers.

> We can get more done, like, faster… sometimes we just like, tell each other so we all get the one part. Like, there's four, um, four kids in each group, and there are four problems, we just say, 'you do this problem, you do this problem, you do this, you do this.' Then we just share.

Although he said that he would rather work alone or get help from a teacher, Enrique was willing to work in a group of peers if it helped him complete his work efficiently. His desire to efficiently completing his work aligned with the focus of whole-class discussion in Mr. Winter's class, which was usually to find a single strategy that reached a correct answer.

Enrique did not actively resist group work, but he did not appear to value working with peers. When I asked him why he worked with his classmates in a group, he said, "The teacher tells us to work with a group." This suggests that he *cooperated* with his teacher's request to work in groups (for the purpose of efficiently obtaining an answer), but he didn't necessarily *value* collaborating with peers (as he would rather talk with teacher).

Kiara: Struggling problem solver, Mr. Winters's classroom. Mr. Winters identified Kiara as one of four struggling problem solvers (out of 12 students interviewed from his class). When asked about what I should know about her as a mathematics learner, she self-identified as "one of the middle-low students, but not the lowest." Kiara said that she had been having trouble with math lately, but she said that she liked math, particularly when she had a nice teacher, as she said that did that year. Kiara said, "I like doing math, I just have problems… I've been struggling through math, but Mr. Winters and my group helps [sic] me out." She also said that she attended help sessions after school with her teacher.

When Kiara talked about group work, she perceived that the purpose of small group work was to develop social skills.

> …when we grow up we are going to have to work with people that maybe we might not like, he [Mr. Winters] tells us this, with people we might not like, but we have to work with them, anyway. So he kind of puts us with people maybe that we don't talk much with, and then we start getting used to working with people...

She talked about being aware of and cooperating with her teacher's expectations when she mentioned what they "have to" do and what her teacher told the class ("he tells us this").

However, even though Kiara appeared to be willing to work in a group, her preference was to get help from her teacher; she did not appear to consistently value working in a group. I asked her with whom she would rather work during math class. Kiara said, "Sometimes the group, but mostly, in my opinion, would be the teacher. … Well, Mr. Winters says he puts geniuses in our groups, so I listen to [classmate], and he gets it right." Her preference appeared to be working with whoever would be most likely to help her obtain a most direct route to the correct response, such as the teacher or a "genius."

Carolina: Successful problem solver, Ms. Summers's classroom. Ms. Summers identified Carolina as one of the two strongest problem solvers in her class. Carolina self-identified to me as one of the first students to participate and get involved with her group during small group work.

Carolina perceived that the purpose of group work was to obtain multiple solution strategies.

I: Would you rather have groups, or would you rather have individual time to work?
Carolina: Groups! [I: Why?] Because, like I said, we get like multiple strategies.

When I asked Carolina for advice to give her teacher, she said, "Like to her, like not to always, like, talk. Like, she can talk and whatever, but… we can, like, brainstorm it." I asked her whether her teacher let them brainstorm it, and she said, yes, her teacher allowed them to brainstorm, and she wanted her teacher to continue to let them do so. This suggests that Carolina believed that students were able to develop and contribute their own mathematical insights.

Carolina not only appeared to *cooperate* with Ms. Summers's expectation that students find multiple solution strategies, but she appeared to *value* finding multiple solution strategies.

I: And what do you like about working in groups, if anything?
Carolina: Like how we, you know, like she says, find multiple strategies? [I: Yeah.] So, we, like, we found, let's say we take…like we are working by ourselves? Then after we are done we share with each other and we have two strategies, or something like that…
I: And you like that? (Carolina nodded.) Why?
Carolina: Because we can learn from each other.

Carolina said that she liked doing what her teacher asked them to do. Also, Carolina expressed that she wanted to exercise her own autonomy when working with peers.

I: … what helps you learn math the most [in groups]?
Carolina: Like, they [peers] can, like, explain it better…like, let's say if Ms. Summers is talking about something that I don't understand, I ask them first, and then I ask her. So I ask them first, and they explain it better… like, he [a peer] like, tries to take care of everybody's work. Like we got different work--
I: And you don't like that?
Carolina: It's taking over our work. It's like, he's taking over the whole group.
I: And what do you think he should do instead?

Carolina: Like, work, do his part, and everybody else does their part.

Carolina wanted to learn from her classmates, and she did not like it when peers tried impose their way of thinking onto her or others. She valued interactions that were more collaborative.

Julius: Struggling problem solver, Ms. Summers's classroom. Ms. Summers expressed concerns about Julius as a problem solver. Julius self-identified to me as "not good at fractions," which was the unit they were doing when I interviewed the students, but he self-identified as being generally "good with numbers." He self-reported that he was earning a D in math at the time, and he said that he had gotten behind in his homework.

I: I don't know you yet, so how would you say you are in math?
Julius: Well, I don't really think nobody's the best. I think we're all the same, but, it's to the point where someone, I think if they try hard enough, then they'll all be, like, we'll all be the same.
I: So, anybody can.
Julius: Yeah. They're just not trying. They're not trying to become better. They just decide, some people just decide not to do it, and that's how they get bad grades.
I: Oh, it's not that they can't.
Julius: Everybody can do it!

Notice that he attributed lack of success in mathematics to lack of effort. He did not speak of mathematical ability as fixed, as if some students were geniuses and others were not.

Regarding obligations-for-others, Julius appeared to be aware of his teacher's expectations. "When the teacher's stopped talking, she'll say go, and she'll say 'math talk' and all that other stuff, that's when I know it's time to start talking." He did not appear to resist.

In terms of obligations-for-himself, Julius appeared to value multiple strategies, as he internalized the process of finding more than one solution. I asked Julius what he would do if he was stuck when working on a mathematics problem.

Julius: I would ask myself, like.
I: You would ask yourself a question. What do you ask yourself?
Julius: Like, how would you explain it? I would explain it another way, just in case, like, somebody asked, I still don't get it, then I would explain it another way to make it easier.

Julius described a metacognitive process of asking himself for another strategy, which suggests that he internalized his teacher's expectations and used them as a strategy for learning.

Discussion

Students from both classrooms appeared to perceive purposes for group work that aligned with what their teachers implicitly promoted as they facilitated small group work, but Ms. Summers's students were more likely to value their teachers' purposes. Mr. Winters's students did not actively resist working in a group, but they reported that they would rather circumnavigate group work and go directly to a teacher or a "genius" to efficiently complete their work. In contrast, Ms. Summers's students appeared to value their teacher's request to work with peers and learn multiple strategies for solving a problem, as exemplified by Carolina's self-reported preferences and Julius's internalization of the process of seeking multiple strategies. A contribution of this study is the use of students' self-reported values to assess the effectiveness of group work.

Another contribution of this study is that the results serve as an existence proof that relationships between performance and affect (productive dispositions and values) can be complex. Enrique was a case of a student who had a history of success with problem solving, but he may not necessarily be developing a productive disposition toward mathematics. Even though he was performing well in mathematics, he spoke about mathematical authority as lying in teachers or more knowledgeable others. He happened to be a peer authority in his classroom at this point in time. However, if a student like Enrique encounters challenge in mathematics repeatedly over time, will he give up or will he persist? Compared to Enrique, Carolina was an example of a successful problem solver who was developing a more productive disposition toward mathematics, as she talked about students being able to make sense of mathematics. Julius was a case of a student who appeared to be struggling with problem solving, but he appeared to be developing a productive disposition toward mathematics. Although he was not performing well in a unit about fractions, he expressed that everyone and anyone could do math, and he attributed lack of success to lack of effort. This suggests that he could be likely to persist in mathematics when he encounters difficulty because he said that it was possible for him to be mathematically successful, in contrast to Kiara (who talked about others being geniuses).

I conjecture that efforts to facilitate group work similar to Ms. Summers's efforts would be likely to promote values for engaging in collaborative work. Consider that a purpose of small group work may be to prepare for whole-class discussion. Mr. Winters's classroom discussed one solution strategy for each problem, with a focus on the correct answer. Ms. Summers's classroom discussed multiple solution strategies, even if the strategy was not necessarily correct, with a focus on understanding the process. A focus on answers in Mr. Winters's classroom could lead to a decreased need to engage with mathematics collaboratively; successful students would be those who correctly answered his questions first, and then the conversation would end. Alternatively, if there is a focus on multiple solution strategies, as in Ms. Summers's class, there can be more opportunities to be successful (Cohen, 1994) and more reasons to interact about mathematics. Values for collaborative work may be promoted through an emphasis on understanding relationships between multiple solution strategies, because the conversation does not end once a correct answer is obtained.

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EXAMINING MOTIVATIONAL CONSTRUCTS IN MATHEMATICS FOR STUDENTS ACROSS ELEMENTARY GRADES

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This study examined how cognitive and social cognitive constructs of motivation might relate to elementary students' math achievement. A motivation survey was constructed and administered to 1,018 students in grades one through five. Factor analysis of the survey yielded three underlying scales, which were labeled math anxiety, math self-efficacy, and value of math. First grade students reported significantly lower math anxiety than third, fourth, and fifth grade students. First grade students also reported significantly higher self-efficacy in mathematics than third, fourth, and fifth graders. Fifth grade students reported significantly lower value for mathematics than second and third graders. These results are discussed in relation to student motivation and achievement in mathematics as well as implications for mathematics education.

Introduction

The purpose of the present study is to investigate students' motivational constructs in mathematics across grades one through five in schools that are currently implementing an inquiry-based mathematics curriculum. Results of this study can provide data about motivational patterns in elementary mathematics in early to late grades and may have implications for curriculum implementation and instructional strategies of teachers. This study also sought to establish the reliability of a survey instrument designed to measure motivational constructs related to mathematics in elementary students. Though the survey was previously piloted with a smaller sample (N=79 students), the current study focuses on examining the reliability of the instrument with a larger student sample (N=1,018).

Perspectives

Motivation is a psychological characteristic integral for students to be successful (Schunk, 1991). Studies have shown that highly motivated students are more likely to engage in behaviors that enhance academic performance (DiPerna, Volpe, & Elliott, 2005; DiPerna & Elliott, 1999; Whang & Hancock, 1994), including effective goal-setting, focusing effort, and persisting in academic challenges (Ormrod, 2006). Highly motivated students also are more likely to view academic tasks as valuable and important (Pintrich & Schunk, 2002; Eccles & Wigfield, 1994).

Motivation has also been connected to students' level of cognitive engagement and use of metacognitive strategies (Pintrich & DeGroot, 1990). Students who exhibit high motivation for a task are more likely to utilize effective cognitive strategies for encoding new information (Ryan, Ryan, Arbuthnot, & Samuels, 2007). These students display a tendency to employ critical thinking skills in problem-solving situations and integrate prior knowledge with new information. Highly motivated students may also employ more effective metacognitive strategies such as planning how to approach a new learning task, evaluating progress, and monitoring comprehension of new material (Pintrich & Schunk, 2002; Pintrich, 2000). Motivated students, in other words, are better equipped to learn than unmotivated students.

Research indicates that student motivation is domain-specific and can vary across curriculum areas (Ormrod, 2006; Stipek, 1988). Students as young as 8 years have demonstrated the ability

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to differentiate between subject areas in relation to motivational constructs (Alderman, 2004). In the area of mathematics, student levels of math motivation have been associated with math achievement. For example, DiPerna, Volpe, & Elliott (2005) found a significant positive correlation between motivation and math academic achievement and engagement in math tasks for primary students. This boost in achievement might be explained by other research that suggests students who have high intrinsic math motivation may also have a greater conceptual knowledge of math topics (Stipek, Salmon, Givvn, Kazemi, Saxe, & MacGyvers, 1998; Nichols, 1996) and may persist at challenging math tasks (Eccles & Wigfield, 1994). In short, if a teacher wants to ensure students are successful in math, a critical first step is to ensure an appropriate level of student motivation.

Measuring psychological constructs in young children has proven problematic and inconsistent (Thorndike, 2005). Motivation research typically has focused on the academic drive and related learning outcomes of older students. Thus, the needs of early elementary school teachers who wish to measure and then attempt to enhance their students' motivation have not been fully addressed. This is unfortunate, because early elementary years (Kindergarten through third grade) are critical to a child's development of attitudes about schooling, as well as a child's academic motivation (Perry & Weinstein, 1998). Such formative years can often set a precedent for either academic success or failure in later grades. The current study aims to address the topic of math motivational constructs in elementary children and examine possible interactions between grade levels.

Methods

Participants

Participants were 1,018 students from a suburban school district in the northwest area of South Carolina. Students ranged from grade one to grade five, with the following numbers represented from each grade level: grade one (137 students), grade two (209 students), grade three (241 students), grade four (200 students), and grade five (231 students). Students were recruited from five school sites, all of which were participating in a three year grant from the Commission on Higher Education in South Carolina dedicated to improving teacher quality through the implementation of an inquiry-based mathematics curriculum and professional development program. This program is described briefly in the section below. The only criterion for student participation was parental consent.

Program implementation

Participants at these five schools were in the process of implementing a K-5 mathematics curriculum development at Clemson University through the college of Engineering and Science in an effort to increase student achievement and improve teacher quality in mathematics. This curriculum, called Math Out of The Box, is designed with four interrelated strands: Developing algebraic thinking, developing Geometric logic, and developing number concepts. Teachers at these schools taught three out of the four strands in the 2007-2008 school year when this study occurred. The fourth and final strand, developing number concepts was implemented in the school year following data collection for this study. Teachers were trained on the Math Out of the Box materials through a professional development model where they were immersed in inquiry-based tasks while the facilitator supported a collaborative learning environment.

Instrumentation

A 17-item survey was designed to measure cognitive and social cognitive motivational constructs related to math for elementary students. Differing items contained positively or negatively phrased statements of disposition toward math, prompting students to respond with a four-point Likert-type scale (4 = *Just like me*, 3 = *Sort of like me*, 2 = *Not really like me*, and 1 = *Definitely not like me*). Items were evaluated and reviewed for clarity and appropriateness by a measurement expert. Following this evaluation, the measurement expert and researchers refined the instrument together. Items in the instrument were written to prompt a judgment related to (1) anxiety (Shipman & Shipman, 1985), (2) interest (Hidi & Anderson, 1992), (3) task value (Wigfield & Eccles, 2002), (4) self-efficacy (Bandura, 1986), and (5) goal orientation (Elliott & McGregor, 2000).

Procedure

A total of 117 classrooms participated in this study in the spring of 2008. Each student's involvement was contingent on informed consent. All surveys were administered by the teacher or a school math coach in the students' regular classroom setting and no surveys were given to students individually. In order to control for reading ability, each item was read aloud to the students in grades one through three, who then made their rating selections before the next item was read aloud. Participants were given a wait time of 10-15 seconds to complete each item. All surveys were completed during the same two-week time period during the last month of the school year.

Results

Reliability

Reliability for the math motivation scale was estimated by computing the Cronbach's Alpha (0.833). This coefficient demonstrated high reliability for the scale. Further, an examination of "Cronbach's Alpha if Item Deleted" suggested that the Cronbach's Alpha increases to 0.846 upon deletion of Items 4 and Items 9. However, this increase was not significant enough to warrant the removal of these items from the scale.

Factor Analysis

The dimensionality of the 17 items from the motivation scale was analyzed using principle components factor analysis. A varimax rotation yielded three factors with eigenvaules greater than one, and each factor yielded an interpretable factor solution. Three items loaded on more than one factor. Factor 1 (which accounted for 29 % of item variance) was defined by six of the scale items. Because these items were related to varying constructs of math anxiety, Factor 1 was labeled *Math Anxiety*. Factor 2 (which accounted for an additional 11 % of item variance) was defined by seven of the scale items and was labeled *Math Self-Efficacy*. The third Factor was defined by five of the scale items, accounted for 8 % of item variance, and was labeled *Value of Math*.

Analysis of Variance

A one-way analysis of variance was conducted to evaluate the relationship between grade level and each of the following factors: (1) math anxiety (2) math self-efficacy and (3) value of math. The independent variable, grade level, included five levels, represented by grades one through five. The dependent variable was the student score for each factor on the motivational scale, identified through factor analysis.

The ANOVA for grade level and math anxiety was significant, $F(4, 1013)=13.554$, $p<0.001$. Follow-up tests were conducted to evaluate pairwise differences among the means. Levene's Statistic revealed equal variances, resulting in the use of Bonferroni's post hoc comparison. Students in grade one reported significantly lower math anxiety than grades three, four, and five. Grade two also reported significantly lower math anxiety than grades four and five.

The ANOVA for grade level and math self-efficacy was also significant, $F(4,1015)=5.928$, $p<0.001$. Bonferroni's post hoc was selected to evaluate pairwise differences among the means, since variances were equal. Students in grade one reported higher math self-efficacy than students in grades three, four, and five. There were no additional significant differences between self-efficacy and grade level.

The ANOVA for grade level and value of math was significant, $F(4,1013)=3.215$, $p<0.001$. Finding equal variances, Bonferroni's post hoc was conducted to examine differences among the means. Students in grade five reported significantly lower value for math than students in grades two and three.

Results from this study indicate that students in the lower grades at these five schools report less math anxiety and higher self-efficacy in mathematics. These results raise questions about possible reasons for this discrepancy, including the need to investigate issues of increased math anxiety as a result of testing pressure or rising levels of difficulty in content. The correlation existing between math anxiety and achievement in mathematics has been well documented in the research literature (Ashcraft, 2002; Beasley, Long, & Natali, 2001). The results from this study introduce the possible correlation between math anxiety and reduced self-efficacy in mathematics.

These results also raise questions about factors at the schools that may account for the reduced levels of motivation at the higher grades such as classroom environment, teacher quality, or fidelity of implementation. Fifth grade students at these schools report a significantly lower value of mathematics than students in other grades. Value is essential for students to be motivated in mathematics (DiPerna, Volpe, & Elliott, 2005). It is necessary to explore how the role of the teacher and the curriculum account for such a change in attitudes. It would also be beneficial to compare the motivation of upper elementary students whose teachers are using an inquiry-based curriculum to students whose teachers are using a more traditional mathematics curriculum where procedural knowledge is emphasized over the development of conceptual understanding. While students at the upper elementary grades report less motivation for mathematics than students in lower grades, it is unclear if they have more motivation than students in other schools where a traditional mathematics curriculum is implemented.

Conclusions

This study provides insight into students' motivation in mathematics in relation to grade level. This data may be helpful in informing curriculum decisions and instructional practices that may sustain students' motivation as they progress from early to late elementary grades. Special attention should also be given to increasing levels of math anxiety as students advance through the grade levels. Identifying and reducing this anxiety in students in the mid to late elementary years may result in more positive experiences with mathematics in the middle school years. In addition, a drop in reported value for mathematics during fifth grade may suggest a need for authentic experiences that help make mathematics meaningful for these students.

This research also provides information for stakeholders at these schools acting in a supportive role as teachers implement this new mathematics curriculum. These stakeholders include the math coaches at each of the schools who provide daily support to teachers in addition to facilitators from Clemson University who provide weekly support through content-focused professional-development for grade-level teams of teachers. The need to increase student motivation at the upper elementary levels in these schools will become a primary goal of these professional development sessions. Next steps for this line of research include investigating the impact of instructional practices on student motivation in mathematics. A mixed methods sequential explanatory design will be used to examine this correlation. Researchers will use the survey that was developed and validated in this study as a method of participant selection to identify a sample of teachers whose students display either a high level or a low level of motivation. A case study design will then be utilized to determine how instruction can influence student motivation in mathematics.

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HIGH SCHOOL STUDENTS' PERSPECTIVES ON COLLABORATION IN THEIR MATHEMATICS CLASS

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Collaborative learning formats are being used increasingly in mathematics classrooms. However, students working in groups often remain dependent on their teacher as the distributor of knowledge, or turn to peers to fill this role. In this project, I used viewing session interviews to investigate high school students' perspectives regarding collaboration in their mathematics class. I highlight two students who characterized their responsibilities as class members differently, but expressed similar beliefs about the purpose of group work and the nature of mathematical understanding. I argue that viewing collaboration as "helping" may limit the potential benefits of having students work together.

Introduction

Students' experiences in mathematics class, including their experiences with collaborative learning formats, can affect their identities as learners and doers of mathematics (Boaler, 2002). Cobb, Gresalfi, and Hodge (2009) used interviews to show how students in two classes with very different pedagogical approaches constructed identities differently in terms of their mathematical obligations to each other, their opportunities to exhibit agency, and their view of the teacher as a mathematical authority. But even in a class where student interaction is a central component of instruction and where the teacher seeks to establish a community approach to learning, students may construct mathematical identities that are individualistic and passive in nature. These identities are influenced not only by the nature of instruction, but also by their beliefs about the purpose of collaboration in their mathematics class, as well as their beliefs about mathematical understanding more generally. These beliefs may be resistant to change, and may constrain the ways that students are able to engage in collaborative work.

In this paper, I compare and contrast the perspectives of two high school students in the same mathematics class where collaborative work was common. Sevanye (a pseudonym) described her responsibilities primarily in individualistic terms, rarely expressing concern for contributing to the understanding of her peers. In contrast, Jordan's description of learning was more community-focused, acknowledging a responsibility not only to improve her own knowledge, but to help improve others' knowledge as well. At the same time, both students seemed to hold similar beliefs about the purpose of collaboration and also about the nature of mathematical understanding. Both Sevanye and Jordan described working together primarily as passing knowledge from capable students to struggling peers, and seemed to see mathematical understanding as knowing procedures. I argue that these beliefs may be related to each other and that they have implications for the strategies that teachers use to implement collaborative learning in their classrooms.

Framework and Research Questions

Particularly when they are working on problems for which solutions are not already known, students will inevitably struggle. Clearly, teachers in classrooms with 20 or 30 students will not be able to attend to each student and help them resolve their confusion. Putting students into

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groups can be seen as one way of resolving this dilemma, as the stronger students can help those who are struggling. Some research has suggested that students working in groups tend to position themselves asymmetrically, where select students take on an "expert" role, telling other students what to do, while their peers defer to their authority (Esmonde, 2009; Kotsopoulos, 2007).

Some strategies for implementing collaboration seem to assume (and may even promote) asymmetric positioning by, for example, training students to give better explanations when helping their peers (Fuchs et al., 1997; Webb & Farivar, 1994). Even in a study which claimed that students created a "collaborative zone of proximal development" in their classroom, descriptions of student interaction suggested asymmetric positioning: "Some students additionally adopted teacher-like scaffolding strategies to assist less capable peers, for example, by asking questions that led their partner to locate an error or reconsider a plan" (Goos, 2004, p. 282). While these interactions deemphasized the *teacher* as the distributor of knowledge, some students still seemed to be positioned as knowledgeable while others were positioned as less able to contribute.

Another view of collaboration suggests that students could be positioned more symmetrically while generating knowledge. According to this view, knowledge need not be possessed by any individual in order for it to emerge among a group of learners (Davis & Simmt, 2003). That is, students may *co-construct* new ideas through activities such as reiterating, redefining, or expanding on the ideas of others (Mueller, Maher, & Powell, 2007). Scardamalia and Bereiter (2006) argue that this is precisely what takes place in research institutions—groups of people come together, construct new ideas through discourse, and together advance the knowledge of society. This process of *knowledge building* should not be limited to the research laboratory, these researchers argue; it should be integral to schooling.

One of the goals of this paper is to argue that the former view (collaboration as helping) may undermine the benefits that might be gained from having students work together, particularly if this view is adopted by students and teachers. I set up this argument by using multiple lenses to examine two students' perspectives on working together. First, I investigate the extent to which each student describes personal obligations to help her peers, as well as the extent to which each describes her classmates as resources for helping her. Then I examine each student's beliefs about the purpose of collaboration; in particular, does she describe working together as a process of building new knowledge through the contributions of multiple individuals, or as a process of passing ideas from knowledge-giver to knowledge-receiver? Finally, I explore each student's beliefs about the nature of mathematical understanding in order to explore connections between these beliefs and beliefs about the purpose of group work.

Methodology

Sevanye and Jordan were both in Mr. Neal's "Integrated Math 3" class, the third in a sequence of courses taught with the Core Plus mathematics curriculum materials (Hirsch et al., 2008). They were both in 10^{th} grade and both reported typically receiving B's in mathematics.

I spent 23 days in Mr. Neal's class, recording video of whole class discussions and small group work. I interviewed eight students from the class twice, including Sevanye and Jordan. In the first interview, I asked them questions about working with their peers, such as whether they liked working together and why, what they saw as the goal of group work, what they saw as their role or "job" when working in a group, and under what circumstances they felt comfortable sharing their ideas publicly in whole-class discussions. In the second interview, I showed each student video clips of their own participation in various collaborative activities and asked them questions about how they participated. Using these viewing sessions helped students be more specific about their roles, and allowed them to address any inconsistencies between stated beliefs and observed behavior. The data was analyzed by categorizing portions of interviews according to recurring themes that were developed and refined throughout the analysis process (Glaser & Strauss, 1967). In this paper, I focus on three of these themes: how Sevanye and Jordan characterized their personal responsibilities as members of the class, how they described the purpose of collaboration, and how they talked about mathematical understanding.

Results

Responsibilities: Individual Versus Community

When Sevanye discussed her role during group work, she emphasized a responsibility for making sure *she* understood rather than a responsibility to the members of her group for improving collective understanding. During the initial interview, I asked her about situations where her group mates might get off-task, and she shrugged, saying,

> If you don't want to do your work, I'm not going to say anything to you. We're all in high school, graduating in two years. If you can't be smart enough to know I need to get this work done 'cause there's a test next week then…that's not my problem.

In another section of the interview, she said, "It's up to everyone…. I can't expect anybody else to carry me 'cause I don't know what I'm doing. That's not fair to anybody. Especially if they know what they're doing 'cause they paid attention." These comments suggest a rejection of obligations to contribute to her peers' understanding.

In one video clip, Sevanye was seen allowing a group mate to copy her answers. She explained the behavior in the interview: "It doesn't bother me, as long as in the end she understood what we were doing. If you wanna copy my answers, I mean…I don't always get the right answers, so if I mess up I'm gonna have to erase." Here Sevanye does mention her classmate's understanding, but since she made no effort to help her peer understand, this can be interpreted as a responsibility she put on her classmate, not herself. In general, for Sevanye, the responsibility for understanding seemed to lie with the individual rather than the community.

Unlike Sevanye, Jordan did express more of a concern for helping others understand. She said, "…we're supposed to help each other or whatever, and I don't know, I just don't feel as though we should go on, leaving one person sitting there stuck, because…I don't want to be left there stuck." Jordan seemed to see helping others in terms of reciprocation—helping others was important because she might need help from them at a later time. This came up when discussing the nature of help as well: "Like, I don't like giving people the answers, like I like explaining it to them, because I don't want nobody giving me the answer. Like I like if somebody explains it to me so I can know how to do it." Here Jordan seems to be expressing a responsibility not only for sharing her answers, but for explaining how to solve problems.

Sevanye and Jordan also differed in the extent to which they saw their peers as able to help them understand. In many cases Sevanye described her classmates as incapable of helping her. In one episode where she became confused, rather than ask her group members about their ideas, she raised her hand to call the teacher over for help. When I asked her about this, she replied,

> They didn't know what they were doing…. I asked Lindsay a little bit about like what she was doing, and she didn't really seem to know. If I would have asked her some more, she probably would have confused me more than what I was already confused about…so I just went right to Mr. Neal.

In this case, Sevanye did not seem to see her peers as resources for helping her understand.

Sevanye also said, “When I work in group I like to work with people who already know what they’re doing,” but other comments suggest that this was a rare occurrence for her. In general, Sevanye seemed to prefer explanations from Mr. Neal rather than from her classmates. In one episode, Mr. Neal conducted a whole class discussion in which he gave an explanation for a solution that a student had written on the board. I asked Sevanye if she would have preferred that the student explain his solution rather than Mr. Neal. She replied, “No. I like it when Mr. Neal explains it better. Like, he just breaks everything down…. I’d rather Mr. Neal do it because he knows exactly what he’s doing.” This quote again suggests that Sevanye did not see her peers as able to help her understand, and seemed to see Mr. Neal as the primary mathematical authority in the classroom.

In contrast, Jordan explicitly described her peers as capable of helping her learn. She stressed the importance of sharing ideas with others:

> Yeah I like group work because you get to share your ideas, and everybody has different thoughts or whatever. I think it’s better to see what everybody else thinks about the problems or whatever, because….like Mr. Neal says, more approaches to a problem, whatever…so we don’t have to just do it the way he shows us.

Jordan acknowledged that it was possible to learn from others without relying on Mr. Neal. In the case of a disagreement between students, she said that “if you both explain how you got your answer, then maybe you all can come up with something. Maybe if you go through it again, you see somebody make a mistake, something like that.” Jordan acknowledged that “some students come up with their own ways, like ways that the teacher doesn’t even explain yet,” although she claimed that this was rarely the case for her.

Jordan indicated that she valued other students’ explanations during whole class discussions as well as during group work. She said that rather than Mr. Neal providing all of the explanations, he should allow students to explain because “maybe the student had a different approach than the teacher had. Or maybe some students understand it better from a student.” Jordan seemed to see students as capable of contributing to each others’ understanding, and wanted Mr. Neal to provide opportunities for students to share their ideas with their peers.

The preceding sections have outlined some ways that Sevanye and Jordan described their roles in Mr. Neal’s mathematics class. While Sevanye primarily described her responsibilities in individualistic terms, Jordan expressed a sense of responsibility for helping others and also saw her peers as resources for improving her own understanding.

Beliefs About Collaboration

While Sevanye and Jordan’s expressed obligations as members of their mathematics class seemed different, they described similar beliefs about the purpose of collaboration. In particular, both Sevanye and Jordan expressed the view that working together was primarily about those with knowledge “giving” that knowledge to those without. In other words, they described collaborative work in asymmetric terms, with some students positioned as knowledge-givers and others as knowledge-receivers. Sevanye was quite explicit about this. When asked about the purpose of group work, she said, “You put the stronger people with some people who may not know the subject as well and it helps them.” She also indicated that “being knowledgeable ” was a prerequisite for being able to help others. “I wouldn’t want to tell somebody how to get the answer if I wasn’t one hundred percent sure how to do it,” she said. As I described in the previous section, she also was hesitant to listen to her classmates if they did not have a solid understanding. This suggests that for Sevanye, working together was primarily about giving knowledge to others who were confused. This activity seemed to hold little value for her, which predictably resulted in a generally negative view of group work.

Jordan also seemed to believe that working together that was primarily about students "helping" each other. When I showed her a video clip where there was very little discourse between her and her group members, she explained: "When somebody needs help, we help each other.… Even though we work at a different pace, we still stop if you don't understand something, we'll explain it." Here Jordan is again expressing a view that learning is a collective effort, but she is also describing this collective activity in terms of *stopping* and *helping*, particularly when someone *does not understand*. This implies that some members of the group already do understand and could go on if they were not obligated to help their classmates. It also suggests that for Jordan, a primary reason for discourse is to express confusion and get or give help. Later she talked specifically about a partner who helped her understand. "I know my partner, it makes sense to him, 'cause he always knows. I'm like, 'do you understand this?' And he's like, 'yeah.'" She described a specific case where she was confused: "…so I asked my partner. He understood it. He explained it to me." In this sequence she describes her partner as knowledgeable, herself as lacking knowledge, and working together as the process of him explaining to her. These quotes suggest that for Jordan, working together was often a process where students who had good understanding explained to students who did not.

So while Jordan described her responsibilities in the classroom as extending beyond improving her own understanding, both she and Sevanye described working together in terms of passing knowledge between students. This finding suggests that promoting collaboration in the classroom may mean more than getting students to feel obligated to help each other. It may also involve addressing students' beliefs about the purpose of group work, beliefs that may be tied to students' views about the nature of mathematical understanding.

Beliefs About Mathematical Understanding

During the interviews, Jordan and Sevanye articulated the belief that mathematical understanding consists primarily of knowing procedures. This was apparent in their descriptions of what they valued when giving and receiving help, which often included concerns with *what to do* rather than providing and evaluating arguments for *why* ideas were true.

When I asked Sevanye about how working in a group might help one understand, she said, "Because whenever you do something wrong you know how not to do it…and how you should do it." Her emphasis on *how to do it* suggests a procedural focus, and this was repeated throughout both interviews. Here she explains an episode where she had the wrong solution:

> I was in a group…and we had both gotten the answer but my answer was wrong. And I asked [my partner] how she got it. She had, her answer was right. I didn't put a negative sign where it was supposed to go and she was like, 'oh, well this is how you're supposed to do it. But you just have to remember to put your negatives in the right place or you're gonna…get the wrong answer again.'

Again, the emphasis appears to be on *what to do*, rather than on providing justifications for why the solution was correct. Later, she said that she asked questions in whole-class discussions because "I wanna know how each step goes or how to answer the question." She did not talk about giving help in terms of providing mathematical justifications, and rarely expressed a desire for the teacher or her peers to explain *why* their solution was correct.

Jordan also expressed an emphasis on procedures. An example can be seen in an interview regarding a video clip in which she was helping a group mate with a problem from her textbook (Hirsch et al., 2008, p. 336), reproduced in Figure 1. When I asked Jordan about the explanation she gave her classmate for this problem, it became clear that she was merely mapping the regions on the diagram to the terms in the equation $2x^2 + 7x + 6$: "I just told him that the two x squared was the two blue ones…plus the three green down there, and the four green up here, that's the seven x, that's the b. And then c was the, uh, the six yellow blocks." When I pressed her to explain how the picture showed the relationship between $(x + 2)(2x + 3)$ and $2x^2 + 7x + 6$, the only explanation she provided was the standard "FOIL" procedure (multiply the First, Outside, Inside, and Last terms, and add them together). This explanation, and that which she offered to her group mate, suggest that Jordan missed the point of the diagram, which was to use an area interpretation of multiplication to show how and why the FOIL procedure works ($2x + 3$ can be seen as $x + x + 1 + 1 + 1$, which is the length of the rectangle, $x + 2$ as $x + 1 + 1$, which is the height. Multiplying them gives regions that can be seen as having areas of x^2, x, and 1, and these regions can be separated into four parts to show the four products in the FOIL procedure).

Not only did Jordan fail to describe to her group mate how the areas were generated by the multiplication of the binomials, she also seemed to be satisfied with her explanation and did not ask the other members of her group about their solutions. This can be seen as an indication that knowing why $(x + 2)(2x + 3)$ is equivalent to $2x^2 + 7x + 6$ is either not important, or that the FOIL procedure is a sufficient justification for Jordan, although there was no evidence to suggest that her understanding of FOIL included anything more than the ability to complete the four steps it entails. This suggests that Jordan was satisfied with knowing the procedure for multiplying binomials, failing to press her group mates or herself for justification. This example is consistent with other comments in Jordan's interviews.

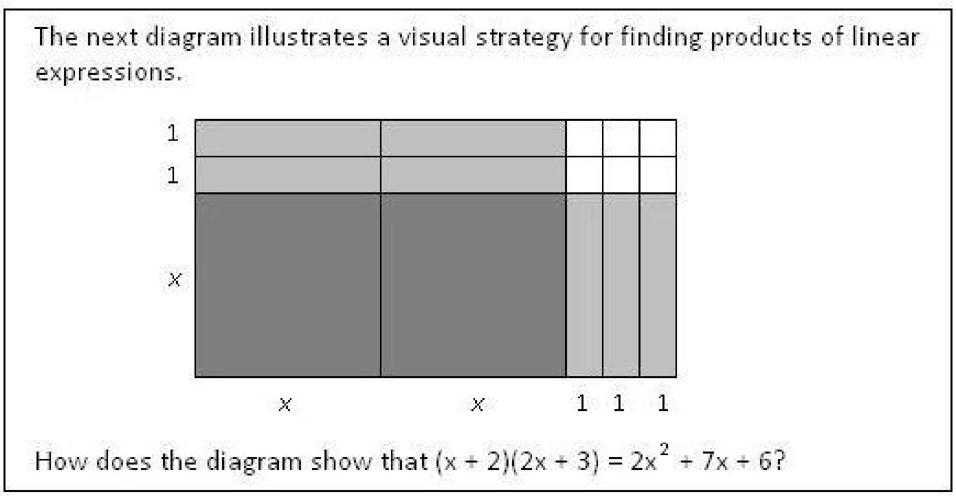

Figure 1. A task designed to reveal the conceptual meaning of binomial multiplication
Note: In the textbook, the large squares were colored blue, the rectangles green, and the small squares yellow.

Discussion and Implications

The fact that both Sevanye and Jordan tended to describe working together in terms of passing knowledge between individuals could be related to the fact that they seemed particularly concerned with knowing procedures. That is, if mathematical understanding is primarily about being able to complete a procedure, then mathematical understanding is dichotomous—one can either do the procedure or one cannot. This dichotomy extends into the social fabric of the classroom. If you know the procedure, then you are likely to be positioned as a knowledge-giver. If you do not know the procedure, an efficient way to learn it is finding someone who knows it and getting them to explain it to you.

Thinking about mathematical understanding as grasping concepts and the relationships between them makes it possible to view collaboration in a way that does not position students asymmetrically. This perspective acknowledges that it is possible for one to understand quite a bit and still not be able to complete a procedure, and also for one to be able to complete a procedure with very little conceptual understanding. Mathematical understanding is much more complex than simply knowing what to do. Students are likely to have a variety of ideas, which, rather than being seen as correct or incorrect, could be seen as revealing different aspects of a mathematical concept. With this view, working together might be easier to see as a process of improving what is known collectively rather than a process of passing knowledge between individuals. Students might position themselves more symmetrically when working together. They might more readily accept that one not need be "knowledgeable" in order to contribute, and be more inclined to persist even in the absence of a knowledge-giving peer (or teacher).

On the other hand, if students see working together only as passing knowledge between individuals, this may affect the extent to which they are able to engage in truly collaborative work. In particular, students who take this perspective and consider themselves to be knowledge-givers have little incentive to work with their classmates. For Jordan, the possibility of reciprocation was an incentive for her to help her peers, but this incentive was not enough for Sevanye—perhaps because Sevanye more often was positioned in a knowledge giving role and saw fewer opportunities for reciprocation from her classmates. It should also be noted that, in light of the beliefs that both students seemed to share, Sevanye's lack of a sense of responsibility for her classmates' understanding is perfectly reasonable. If solid understanding (i.e., knowing a procedure) is a prerequisite for making contributions, and her peers rarely demonstrate this kind of understanding, then there is not much for her to gain by working with others.

One implication for teaching is that if Mr. Neal attempts to improve collaboration in his classroom by trying to help Sevanye be more like Jordan—to develop in her an obligation for helping others—then he may be missing the core problem. Perhaps the more fundamental issue is that neither one of the students may see mathematics as a complex set of relationships that can be understood in many ways and from many perspectives. Thus, they may reject the possibility that others can contribute if specific guides for action are not present. Simply placing students in groups and relying on goodwill (or the belief that one might at some point need help from others) may not be a reliable strategy for encouraging consistent collaboration.

Teachers might address this problem by paying attention to the way they portray mathematical knowledge and what it means to work together. Webb, Nemer, and Ing (2006) suggest that students are likely to adopt the helping behavior that is modeled by their teacher. In particular, if the teacher models knowledge-giving behaviors and emphasizes procedures, then students may not be challenged to see working with each other in less asymmetric ways. In this class, Mr. Neal did not often directly help students during group work; however, during whole-class discussions, he did resolve issues of correctness and provided explanations for problems that students had worked on. A more detailed analysis of Mr. Neal's teaching practices might reveal more about how they could have been adapted to challenge students' beliefs about collaboration and mathematical understanding.

Conclusion

In this paper, I used the perspectives of two high school students in the same mathematics class to argue that the beliefs students have about the nature of mathematical understanding and the purpose of group work can affect the extent to which their participation in “collaboration” is truly collaborative. Promoting collaboration means more than making sure that students adopt responsibility for helping other understand; it means addressing and challenging these underlying beliefs.

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JUXTAPOSING MATHEMATICAL IDENTITIES: SAME STUDENTS WITH DIFFERENT CONTEXTS, PERSPECTIVES AND LANGUAGES

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In mathematics the intricate relationship between learning and identity is predicated upon students' quality of social engagement with mathematical practices. In this paper, I explore five Latino/a students' actions in and narratives about a mathematics afterschool program over a three-year period. Students juxtaposed their experiences within the program and those in the regular mathematics classroom, claiming a difference in the nature of mathematics and in the type of interactions with the participants. Their mathematical experiences, however, were mediated by both their language preference and their reified positions in mathematics, which, in turn, informed their perspectives on what counts as mathematics.

Introduction

In this paper, I explore Latino/a students' interactions in a mathematics afterschool program and their narratives about their experiences in both the program and their regular mathematics classroom; and at the same time, how these experiences mediate their mathematical identities. Thus, I explore the formation of subjectivities or actors' thoughts, sense of self, and self-world relations (Holland et al., 2003). My exploration is on the socialization processes, on the quality of the students' interactions and experiences in specific environments, especially in terms of how they situate themselves and on how they are situated by others with respect to mathematics and language preference. This focus bridges contents and contexts, a situated perspective of students' experiences (Martin, 2006), here bilingual Latino/a students. Students' ideas about what mathematics is and its use seem impacted by the different ways students engage with others around mathematical practices (Lave & Wenger, 1991; Nasir, 2002), this simultaneously shapes participants' *dispositions*, i.e.,"ideas about, values of, and ways of participating with a discipline" (Gresalfi & Cobb, 2006, p. 50). Learning and identity evolve dialectically (Martin, 2006, in press; Lave & Wenger, 1991).

Previous research on after-school settings have capitalized on children's interests and choices and promoted interactions in flexible environments where participants enacted multiple identities (Cole & the Distributed Literacy Consortium, 2006; Vásquez, 2003), and allowed the blurring of categories, such as: student/teacher, play/learning, and school/home. In *La Clase Mágica*, this system nurtured students' social identities as they needed e.g., *amigo* (friend), jokester, wizard's assistant, etc., though the *Mexicano* identity was the prevalent across cases. Vásquez explains that despite instances refusing to speak either language, students continuously constructed a bilingual identity for immediate purposes.

Latina/o students represent the largest (15.4%) minority in the US (US census, 2007). Llagas & Snyder (2003) report Hispanic students having higher high school dropout rates than White or Black students (p. 40), over one-half of Hispanic students speak mostly English at home (p. 72), and that most Hispanic students attend schools where minorities are the majority of the student body (p. 26). More recently, Flores (2007) demonstrated unequally distributed opportunities to learn mathematics for all students. Specifically for African American, Latino, and low-income students who are less likely to have access to experienced and qualified teachers, more likely to

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face low expectations, and less likely to receive equitable funding per student. He reframed the achievement gap problem to opportunity gaps for minorities. Latinos' social situation further aggravates under deficit model approaches—on language, race, or ability (Cashman, 2008). In fact, arguments against languages and cultural diversity are based on the belief that diversity brings divisiveness. Students need to understand that loyalty to the country does not have to go along with denial of their ethnic and linguistic heritage (Brisk, Burgos, & Harmela, 2004, p. 167). As a result, negative attitudes awareness toward their heritage language results in refusing it. Language then is a tool for socialization, and the use of language is socialized (Jackson, 2009, p.177).Thus, given with ideas in mind, I deem pertinent an exploration of the processes conforming their mathematical and bilingual identities.

Setting and Cases

The afterschool program *"Los Rayos de CEMELA"* was developed as an adaptation of the *Fifth Dimension* (Cole, 2006) and *La Clase Mágica* (Vásquez, 2003). *Los Rayos*—a non-remedial mathematics program—is a hybrid space promoting both languages; with playful approach, but not play; with a school subject, but not school (Khisty & Willey, in press). Its curriculum included probability, proportions, geometry, and pre-algebraic thinking. Students met twice a week for ninety minutes and solved problem-solving tasks in small groups that they selected (2-3 students, 1-2 facilitators—pre-service teachers called "UGs," and at times mothers).

Participants, mostly (US-born) from a Mexican background, were all bilingual (Spanish and English). Thirty-one students participated over a three-and-a-half period having an average of seventeen per semester, though twelve of them consistently attended the program. It started with their 3rd and ended with their 6th grade. Students self-selected to be part of *Los Rayos.* The hosting public school in Chicago—dual language school—had 98% Latino, 91% low income students, 9 % special education students, and 62% English language learners—ELLs (Office of Research, Evaluation, and Accountability, 2008). For this project, I selected five student cases—Ramón, Rolando, Candy, Graciela, and Letty (general contexts with multiple embedded cases, Yin, 2009). For comparative purposes (Miles & Huberman, 1994), I selected students that differed from each other in their mathematics performance in regular school. They respectively had the following average scores in mathematics: A, C, B, A, and C.

Procedure

The process of exploring students' bilingual and mathematical identity had various stages. Students were interviewed at different times during the program. For the purpose of this paper, I include data obtained from: the debriefing session (about 20 minutes) at the end of school year 1; five student interviews (40 minutes) held respectively during the end of school year 2 and 3; and finally, three focus-group sessions (30 minutes) developed during last semester. Questions explored purposes of the program: students' experiences in the afterschool activities, their experience with using and learning mathematics in and out of school, their language use and preference (Spanish, English, or both). They ranged from semi-structured to open-ended formats; when students brought up ideas, the interviewer followed up that lead until finished and then moved onto another item. Facilitators conducted interviews either in Spanish or English depending on students' preference. In addition, I analyzed teachers' interviews, UGs' field notes, and videotaped interactions at *Los Rayos,* but not from classrooms. Here I explored students' *discursive identities* by selecting episodes from the beginning, middle, and end of the program.

Analysis and Framework

I based my analysis on theoretical prepositions, case descriptions and rival explanations (Yin, 2009). First, I started observing the videotaped interview sessions identifying general themes. My notes also included any aspect related to following questions:

- *What meaning do these students make from their own participation in mathematical practices?*
- *What does it mean to these students to do mathematics as a bilingual person?*

From my pool of notes, I separated episodes referring to mathematical or bilingual identity and developed an open coding on them. I noticed students were differentiating between the quality of mathematics they experienced in regular school and in the afterschool. With respect bilingual identity, they seemed to address language preference differently. Second, I revisited the selected episodes using transcriptions, videotaped interaction, teachers' interviews, and field notes developing conceptual themes and concomitance with already identified concepts.

My conceptual frame has an eclectic approach rooted in a sociocultural perspective. I draw from the *multilevel framework* for mathematics socialization and identity (Martin, 2000). Correspondingly, I delineate *mathematical identity* as socially negotiated the dispositions, beliefs about one's ability to participate and perform effectively in mathematical contexts (Martin, 2006); and I identify mathematics as a type of *practice* (Moschkovich, 2004; Nasir, 2002). Thus, mathematics learning, development, and identity (Boaler, 2002; Martin, 2000; Nasir, 2002) are determined by way students *engage* in these practices —or not—(Lave & Wenger, 1991). I observe students' *narrative* (i.e., relevant, reifying stories, collectively co-constructed practices about being a kind of person, Gee, 2001) and *discursive identities* (i.e., actions that students engage in during practices). Along this process, I examine how students may align (completely fulfilling), merely (hardly), or not with the obligations and sociomathematical norms (Yackel & Cobb, 1996) co-constructed in the mathematical spaces they share (*normative identities).* These obligations imply a *control or distribution of authority* (*to whom,* the persons in a space; and *about what*, actions or concepts in that space) in their mathematical learning and agency (Cobb, Gresalfi, & Hodge, 2009). I understand *agency* as a reactive and proactive self-efficacy process, "beliefs in one's capabilities to organize and execute the course of action required to produce given attainments" (Martin, 2000); agency being either *conceptual*—advancing human action—or *disciplinary* —linked to discipline structures, formal procedures—(Boaler, 2002).

Finally, as mathematics learning and identity develop by doing and thinking, and these are linked to speaking (Vygotsky 1,978), and so the use of language is connected to identity (Brisk et al. 2004; Cashman, 2008). The connection of language and mathematics becomes relevant as one enacts and makes meaning of life situations inseparably from who one is. Thus, I envision *bilingual identity* as a set of recollections on experiences and beliefs from intimate and socialized events about language skills, and a sense of value on one's linguistic preference.

Key Themes and Results

I present results on students' mathematical identities in tree sections: 1) in different contexts, 2) in different perspectives, and 3) different languages. I describe and contrast students' identities as they are depicted through their *narrative and discursive identities*.

Development of mathematical identities in two contexts:

Students' descriptions seemed to juxtapose their experiences in each context. For '*Context*' I refer to the relationship between a setting and how participants interpret it (Moschkovich &

Brenner, 2000). Students distinguished context in three main dimensions: (a) the quality of interactions, (b) the access to resources, and (c) the quality of mathematics.

(a) Students asserted that the nature of interactions they had in each context mediated the meaning and quality they saw for each context. Partly it relates to the number and the kind of adults they worked with in each setting. Graciela (G) asserted having enjoyed working with older people—their UGs (undergraduate facilitators,) because they are not only “cool and popular,” but also they understand them and being able to speak about “outside school stuff:”

F (facilitator): So you just think that with the UGs you can talk about things at another level?
G: Yeah, it is like having an older sister that I never had. Se siente bonito porque nos tienen confianza.

It seems that the level of trust with the UGs is connected to a greater level of intersubjectivity that they share with students, maybe due to a similar age. While with teachers there was a gap connected to the school structure. Similarly, Letty (Ly) and a friend (A) argued a qualitatively different attention from classroom teachers:

A: Because teachers do not really . . . , okay when they give us a problem, they just say to do it. They do not really ask us, they ask us if we need help when we are doing the work. We usually need more time, I do not know more information about what is it going to be about. You are usually there and if you understand fine, and maybe it is going to take a different way.
Ly: Yeah, they say you can ask them, but then when you ask they say: “Why weren’t you paying attention!!!” yeah…But here the UGs see if I really get it.

Students reported a more formal mathematics environment in class, where they had less of a central role and less support during the meaning making process in mathematics; aspects that seem connected to the type of negotiation that students had access to as well as to the level of authority exercised by the teacher. These circumstances transferred to unproductive dispositions about the way they did mathematics in their classroom. Contrarily, they seemed to identify with the normative identity at Los Rayos, and noticed different mathematical support, generating different dispositions about the context and themselves. Rolando and Ramón reported themselves interacting differently where mathematics is part of their social interaction and vice-versa. Letty said “at school we really get to speak to each other only during lunch time.” Conversely to classroom practices, . . these interactions involved specific distribution of authority, collaboration, problem solving, discourse, meaning making, and fun—their sociomathematical norms. Students engaged in various games that nurtured relationships among themselves and their interest in mathematical concepts. Students enhanced their understanding on the underlying mathematical concepts by manipulating and playing with them. Thus interaction patterns in afterschool nurture networks across and within generations around mathematics.

(b) Students described differently accessing resources, materials (e.g., manipulatives) in each setting. Ramón and Rolando contrasted their experiences based on their opportunity for choices:

Ramon: In class the teacher picks the groups, but in CEMELA we pick our own.
Rolando: Yeah, in CEMELA we can choose our friends and whatever we want to use.

Ramon: Yeah, in class we just have paper and stuff.
C (facilitator): Are you saying that you cannot use manipulatives like blocks, calculators and stuff in your class?
Ramon: Yeah, but the teacher decides when, then we just got pens and paper.

Students deemed relevant not being able to access materials and friends as resources, but also choosing when to access them while working in mathematics. Los Rayos capitalized on students' choices by not only having them select their resources, but also the tasks they wanted to work on. Thus, students asserted different distribution of authority and negotiation in each setting: facing passive-active roles on decision making, and open-limited access to resources. Schoenfeld (1985) declares selection of heuristics in problem solving nurtures metacognitive development.

(c) Students also expressed they engaged in a different kind of mathematics in the afterschool. One way was about the role of control in each setting especially to whom students were accountable. Ramón and Rolando described having a critical role by being accountable to each other in the afterschool, arguing for a collaborative process and exercise of conceptual and disciplinary student agency in mathematics; while in the classroom, the authority and control resides on the teacher. Similarly, Rolando characterized his mathematics experiences in class as "boring", thus limiting the development of productive dispositions towards engaging and becoming a problem solver. In a way, he negotiated these practices by resisting them. During the second interview, Rolando role played what he meant by boring, he stood in front of the blackboard asking me to copy what he wrote (i.e., 5 x 1 = 5, 5 x…, etc.). And when he role played Los Rayos' math, he constructed a small clay dinosaur and compared it with a taller toy dinosaur and asked me: "How many times it [small one] needs to grow to become as big as this one [tall one]?" He solved it by developing a multiplicative structure. He affirmed it was fun math: "you do something not only numbers and reading; you talk, make, and learn;" a 'dance of agency' (Boaler, 2002).

The rest of cases coincided with Rolando's perspective. Furthermore Graciela, who was consistently a high-performing math student in their class, argued that math started becoming different to her when she joined Los Rayos; this transformation was connected to discovering and experiencing mathematics with comfort and enjoyment. Finally, Candy developed an activity creating various perimeter/area designs—which she argued having learned in class:

Candy created "a flying snake." She did not want to find out the perimeter because it was too difficult, so she only found the area. She used a strategy of grouping and counting cubes by colors and even though she was finding the area, she added. So, there was no presence of the previous rule/formula she mentioned. Alma suggested getting the area by multiplying chains of blocks that were the same length (CL field note).

Though she brought this math content from class, she not only 'rehearsed' and 'played' with it, but she deconstructed and recreated it widening and exploring its applications and concepts.

Summarizing, students seemed to have developed "a particular relationship with the discipline of mathematics" (Boaler, 2002, p. 10) and with each other at Los Rayos, which differed from mathematical practices associated with the classroom: ways of reasoning, arguing, and symbolizing mathematical ideas (Moschkovich, 2004). Thus, students indentified two ways of doing and relating to mathematics, two parallel rather than intersecting mathematical identities. A regular school teacher during an interview commented: "Los Rayos helped kids to strengthen their self-esteem…these students take leadership roles in the classroom. I think the UGs provided lots of strategies, it's like what Beth Warren says about the hidden curriculum."

Development of mathematical identities in two perspectives:

Although all student cases described having productive dispositions about the mathematics in the afterschool and taking different leadership roles; still some students reported having, belonging to a marginal participation in mathematics:

C (facilitator): How do you share your strategies and solutions in class?

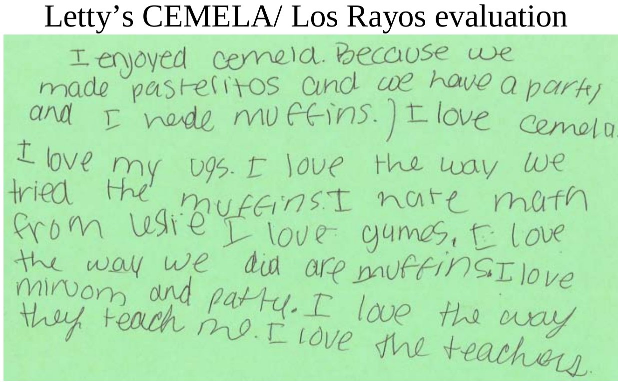
Ro: I don't know, then the teacher just asks… I just ask Graciela, she always knows, and then she tells us.

C: But you can have your answer, right?

Ro: The thing is that we ask them [pointing to Ramón] because they are so smart. They have like a big, big, big brain. They are like the fastest. And us [pointing to another male peer], ours are like dead!

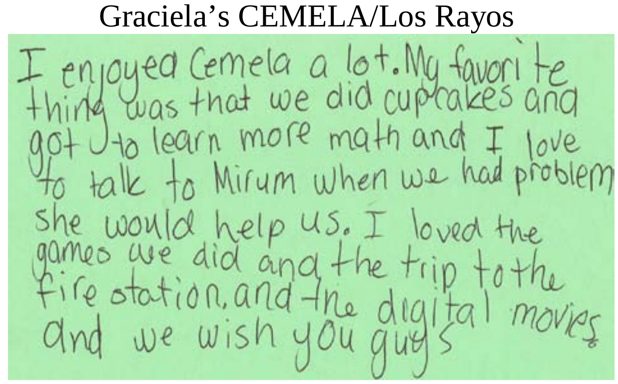
As Rolando (Ro) detached himself from the practices promoted by his teacher, he also declared having certain difficulty understanding the task, consequently he willingly relied on others' reified (Graciela and Ramón or the teacher) positions—having a more central role in the mathematical practices; at the same time, it seemed to inform his beliefs about himself—belonging to more a marginal role and with less disciplinary agency. It seems that the socialization and reification of certain roles in his regular class may have been internalized and used as a way to consistently identify one another. Nevertheless, Rolando claimed that that social construction and structure of performance trespassed settings, privileging some and excluding others—like him—in mathematics (Cohen, 2000): Ramón "sometimes gets to do all the work" in the afterschool. Even though Rolando identified a pattern across contexts, he seems bothered by that privilege in the afterschool and not in the classroom. In fact, Rolando took up leadership roles at Los Rayos. At the end of the program, when Rolando 6th "tutored" 3rd graders, he made various tasks; these showed not only what Rolando knew, but the math he thought as "fun."

Letty's CEMELA/ Los Rayos evaluation



I enjoyed cemela. Because we made pastelitos and we have a party and I nede muffins.) I love cemela. I love my ugs. I love the way we tried the muffins. I hate math from lesie I love gumes. I love the way we did are muffins. I love mirvom and patty. I love the way they teach me. I love the teachers.

Graciela's CEMELA/Los Rayos



I enjoyed Cemela a lot. My favorite thing was that we did cupcakes and got to learn more math and I love to talk to Mirum when we had problem she would help us. I loved the games we did and the trip to the fire station. and the digital movies And we wish you guys

Figure 1. Students' write-ups

These conflicting perspectives in Rolando's case are similar to Letty's, who also took active roles doing mathematics in the afterschool. For example, during a recipe project she changed the mathematical course of a project by making claims based on realistic considerations about the measurement proportions they were using. Despite some episodes of ownership, there was a disconnection to their self-perspective. In a way, they engaged in these tasks, but they did not exactly see themselves engaging with mathematics, but just with an activity with math as a functional tool. Thus, they associated the meaning of belonging to the activity, but not to the mathematical practice. To illustrate these perspectives I use two students' write-ups, as illustrated in Figure 1.

Even though these students worked continuously together, notice how Letty contrasted every practice as either 'loving' or 'hating' it; and locating math in the latter category. Graciela's write-up partly resembles Letty's; however, she pointed: "got to learn more math." So, as Graciela loved the projects, she learned/loved the math. Participants—though in the same task and context—developed subjectivities and connections to mathematics at different levels. Cohen (2000) informs status carries over settings, here it seems about connecting meaning and status to belonging, a transferability of a fixed experiential identity. This perspective could be related to students' mathematics definition. Perhaps Letty's is what she hates: a decontextualized, formal subject demanding disciplinary agency. While Graciela's includes a broader definition: "math with a fun side." In any case, situations call for reconsidering students' experiences, not only on what and how they are imagining their participation (Nasir, 2002) but with what perspective.

Development of mathematical identities in two languages:

Although students spoke either language in the afterschool, English dominated, but in some projects students used more Spanish (e.g. recipes); a plausible explanation is that those projects drew more from home-related experiences. Regardless of student's discursive language preference, overt negotiation of language was tensional. Students' language choices appeared to give students different identity perspectives aligning—or not—to powerful (English) or less powerful (Spanish) paradigms. For example (italicized text has been translated from Spanish):

Ramon: [reading] *that are common to the circle and the pentagon, but not in the triangle or the rectangle*
A: Do it [read] in English!
J (facilitator): He can do it however he wants. He can do it in English or Spanish.
A: I can't understand him!
J: [to Ramon] Can you tell him what you mean?
J: [to Ramon] Ok, let's take a look at it. *That are common to the circle and to the…, what is it?*
A: Read it in English!
J: [to A] I will.
[to Ramon] *But not in the triangle or the rectangle. Does it mean that it is …*

Students' demand to use English often resulted with facilitators switching to English sometimes during the rest of the activity. Conversely, instances demanding Spanish never emerged. The argument here is not whether students choose one language or the other, but that the process represents a much contested space, in which the dynamics are likely to privilege the dominant paradigm, English. In this regards, all students reported "having to speak Spanish at home because their parents do not speak English." Their statements deemed a necessity to justify using other language than English at home. Students used verbs implying obligations rather than an option or an asset. Twelve out of fifteen students from the afterschool self-reported higher fluency in English. Consequently, students privileged a disciplinary agency by identifying with the dominant language, though they were in the position to pick both languages.

Conclusions

Students' narrations and actions asserted experiencing differently the interactions in each space: *Los Rayos* and regular classroom. These dimensions included reconsidering the perspectives of students in learning, interacting, and being; thus becoming bilingual mathematical learners. It is argued that an education that honors the diversity of Latino students' experiences is constructed on a genuine knowledge of their real identities—not those projected onto them (Gutierrez, R., 2002). This tensional space may illuminate new realizations about what it means to promote and support students' bilingual resources. Perhaps it goes beyond simple assumptions of providing bilingual translations and having bilingual facilitators and students in educational contexts. I argue for a re-consideration of the connection between language, learning, social spaces, mathematical practices, and students' identities and perspectives, so we may promote students' more genuine motivation and engagement in mathematics, where goals, identity, and learning intersect (Esmonde, 2009; Nasir, 2002).

Endnotes

1. The data used in this article were originally collected in an after-school research project conducted by Dr. Lena Licón Khisty, Principal Investigator, at the University of Illinois Chicago (UIC) as part of the Center for the Mathematics Education of Latinos (CEMELA), University of Arizona. CEMELA is supported by the National Science Foundation under grant ESI-0424983. The views expressed here are those of the author(s) and do not necessarily reflect the views of the funding agency.

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MANIFESTATIONS OF MATHEMATICAL INVENTIVENESS IN AT-RISK (IMMIGRANT) HIGH SCHOOL STUDENTS PERFORMING ARITHMETICAL CALCULATIONS

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The paper reports a research inquiry with 40 mathematically disadvantaged high school students in Israel, including 30 students who indicated that their first language was Russian. The data constituted the students' responses to the Algorithm Collection Project Questionnaire and videotaped task-based interviews with seven students. We looked at the students' computational strategies through the lens of the concept of relational creativity or inventiveness and inquired which conditions might stipulate its manifestations. We argue that though mathematical knowledge of the participants was weak, their thinking bore signs of such creativity-related constructs as flexibility and relative novelty.

Introduction

The invention of a *new* problem solving method or algorithm is one of the essential manifestations of both absolute and relative mathematical creativity (Silver, 1997; Shye & Yuhas, 2004). The distinction between *absolute creativity* and *relative creativity* (Leikin, 2009; Liljedahl & Sriraman, 2006) is rooted in the observation that an adjective "new" can be utilized for characterizing a problem solving method or an algorithm in two different meanings. In the absolute meaning, *new* means that one's mathematical product is unknown and can be essential for mathematical community at large. In the relative meaning, a mathematical product is *new* mostly for a person or a group who created them (Leikin, 2009). The development of the *relative creativity* in all students is being nowadays considered one of the most important objectives of school mathematics education (e.g., Leikin, Berman & Koichu, 2009; Sriraman, 2008).

Creativity may be manifested only in response to a challenging task that cannot be otherwise solved (e.g., Shye & Yuhas, 2004). Shye and Yuhas (2004) wrote that a person, when faced with a challenge, tackles it in the simplest and most straightforward way, in a sense he or she utilizes mental functions. Similarly, Koichu and Berman (2005) and Koichu (2008) suggested that human beings solve challenging problems in accordance with the principle of intellectual parsimony, i.e., the tendency to minimize intellectual effort. According to Shye and Yuhas (2004), a mere recall method of solution is a low mental function associated with problem solving. If an appropriate method cannot be straightforwardly retrieved from the memory, the would-be solver tries to infer the solution from some known examples that can be considered analogous to the given problem. Finally, if no recalled or inferred method can help to solve the problem, the would-be solver may turn to inventing a new method or a new kind of analogy by raising very openly a number of ideas. This process constitutes the essence of *creative* or *inventive thinking*.

Goldin (2009) asserted that inventiveness can be manifested in an emotionally safe environment where students' mistakes, problem solving impasses, and criticisms of one another's ideas are not avoided or downplayed, but come to be regarded as the productive outcomes of bold and praiseworthy intellectual efforts.

Performing arithmetical tasks as an arena for manifestations of inventiveness

Performing arithmetical tasks by high school students is rarely seen as an arena for manifestations of inventiveness. In part, this is because all students are normally expected, when in high school, to fluently perform basic arithmetic calculations and understand their meaning (OECD/PISA, 2006). However, extensive research teaches us that this is not the case. For example, Clarke, Clarke and Horne (2006) found that by the end of Grade 6 less than 60% of students were able to perform arithmetic exercises with multi-digit numbers by any mental or pencil and paper method; Thomas (2002) reported that 47% of 14-year old students thought 6÷7 and 6/7 were not equivalent.

The availability of calculators in high school can partially camouflage the deficiencies in students' arithmetical knowledge. Consequently, the request to perform a multi-digit arithmetic computation without the use of a calculator may constitute a challenge for many high school students. Further, one can argue that computational tasks are comprehensible for the majority of high school students. Indeed, arithmetic tasks are brief and contain few words. The considerations of briefness and clarity of the tasks are especially important when the first languages of the students differ from the language of schooling, as in our study. Often the need or ability to compute mentally in their native language aids in accuracy and fluency.

For all these reasons, we considered computational tasks both feasible and challenging for mathematically disadvantaged or at-risk (immigrant) students and decided that such tasks are appropriate for exploring the students' inventiveness. Specifically, this report[1] focuses on two interrelated research questions:

- How do the mathematically disadvantaged (immigrant) high school students utilize standard computational algorithms when working on which for them may be challenging arithmetical tasks?
- Which calculation methods do the students invent in cases that they cannot (or prefer not to) utilize standard computational algorithms?

Method

Participants

Forty 12th grade students, 22 boys and 18 girls, took part in the inquiry. On average, the students were 16.7 (SD=0.57) year old. The students studied mathematics at the lowest (one credit point) level in a school situated in underprivileged urban area in Northern Israel. The socio-economical status of the students' families is described as low to average. Thirty students indicated that their first language was Russian, and 10 students – Hebrew. Hebrew was indicated as an additional language by 26 students. Thirteen students were born in Israel, and the rest – in the countries comprising the former USSR. Many of the students were actually bi-lingual, however, for the sake of clarity the students who indicated that their first language is Russian or Hebrew are referred to as Russian-speaking students and Hebrew-speaking students, respectively. The seven students who took part in the interviews represent well the entire group in terms of gender, language and educational history.

Instruments

During 30 minutes all forty students individually filled-in the Hebrew version of the Algorithm Collection Project (ACP) questionnaire (Orey, 1999). The students were asked not to use calculators and show their work on the questionnaire sheets; those who had difficulties in understanding Hebrew instruction were provided explanations in Russian. It was explained to the students that their questionnaires were not be graded. The questionnaire consisted of 16 tasks, as shown in Table 1. Note that all of the tasks allow the use of standard algorithms of long addition, long subtraction etc. Simultaneously, the tasks are chosen so that computational shortcuts can be implemented. In addition, the order in which the tasks are given is designed to see whether the students would use the result of the previous calculations in the next ones.

Table 1 also presents the percentages of the correct answers to the ACP questionnaire; 40 is taken for 100%. Only seven students correctly responded to all 16 APC exercises, 6 students – 13-15 exercises, 17 students – 10-12 exercises and 10 students – between 5 to 10 exercises. These findings are in good agreement with past research (Clarke, Clarke, & Horne, 2006). They show that the questionnaire was not too simple for the majority of the participants and that most of the difficulties were related to multiplication and division, in general, and to multiplication and division involving fractions, in particular. We did not find essential differences between Russian-speaking and Hebrew-speaking students' rates of success.

Table 1. APC tasks and success rates

Task	Correct answers	Task	Correct answers	Task	Correct answers	Task	Correct answers
$37+23=$	95%	$37-23=$	90%	$23\cdot 6=$	85%	$37\div 3=$	45%
$4001+199=$	97.5%	$4000-199=$	77.5%	$230\cdot 60=$	70%	$307\div 3=$	32.5%
$295+86=$	95%	$295-86=$	67.5%	$203\cdot 6=$	82.5 %	$0.37\div 3=$	27.5%
$255+93=$	80%	$255-93=$	77.5%	$23.16\cdot 2.5=$	45%	$370\div 100=$	47.5%

The interviews were conducted immediately after the students completed the questionnaire. Each interview took from 35 to 40 min and was done using the first language to ease the comfort level of the interviewees. The interviews were videotaped, and consisted of the following stages. First, the interviewees were asked to reflect on their math experiences in Israel and, in the case of immigrants, in their country of birth. We then asked them to talk us through several ACP questionnaire problems. Specifically, the students were asked to explain their computational strategies and to deliberate when and how they had learned them. The chosen conversational interviewing methodology (Patton, 1990) enabled us to probe for answers, while taking care to be flexible and sensitive to the students' preferences[2] as well as quite supportive when the students found themselves in an unusual position of inventors, and not just persons trying to recall the appropriate methods.

Results and Discussion

How did the students utilize standard computational algorithms?

Table 2 presents the numbers of students who showed their work as performing long addition/subtraction/multiplication/division algorithms. The numbers of students who wrote anything in response to an item is given in parenthesis. For every item, the difference between the second and the first number represents the number of students who only wrote the final answer or made brief notes suggesting that they had used some form of mental calculation or computational shortcuts.

The first observation from Table 2 is that the numbers of responses were decreasing from 40 for addition tasks to less than 25 for the last two division tasks; the most abrupt change in responses is from 36 (for $203\cdot 6$) to 28 (for $23.16\cdot 2.5$). We suggest that for some of the students who did not have a clue how to calculate the latter exercise, it created expectations of failure also from the rest of the questionnaire. Thus, it is possible that at that stage several students decided to skip the rest of the questionnaire or even to cheat in order "to maintain face" (cf. Schorr et al., 2008). This hypothesis is strongly supported by the interviews. For example, Luda, one of the Russian-speaking interviewees, confessed that she fairly approached most of the exercises, and cheated when solving the task $0.37 \div 3$. Note that such a confession became possible only when an atmosphere of trust was developed in the interview.

Table 2. The use of long addition/subtraction/multiplication/division

Long addition		Long subtraction		Long multiplication		Long division	
37+23	22 (40)	37-23	23 (40)	23·6	23(38)	37 ÷3	8(29)
4000+199	21(40)	4000-199	20 (40)	230·60	17(37)	307÷3	5(28)
295+86	25(40)	295-86	25 (39)	203·6	21(36)	0.37÷3	5(22)
255+93	25(40)	255-93	25(38)	23.16·2.5	10 (28)	370÷100	2(24)

The second observation is that the number of students who consistently used long-calculation algorithms varied throughout the questionnaire and appeared to be the lowest for the division exercises. The first two addition and subtraction exercises were calculated mentally by about 45% of the participants, and the third and forth addition and subtraction exercises – by about 38%. For example, Becky, a Hebrew-speaking student, calculated the first and the second addition and subtraction exercises mentally (4001+199=4000+200=4200) and used the long algorithms in order to calculate the third and the second ones. She explained in the interview that she could calculate all the exercises mentally, but for situations like 295-86 it was easier for her to use the long-calculation algorithms. This finding means that Becky had two available strategies for the addition and subtraction exercises and chose one of them, apparently, in accordance with the principle of intellectual parsimony (Koichu, 2008).

Which calculation methods did the students invent?

It seems that many students did not have much choice when it came to multi-digit multiplication and division exercises. Some of them merely did not remember the algorithms and tried to invent computational shortcuts or approximate the results, and the others implemented only the algorithms.

Michel, a Hebrew-speaking student, calculated all multiplication exercises using the repeated addition scheme. For example, she calculated 203 times 6 in two steps: 203+203+203=609; 609+609=1218. The intermediate additions were done by the long addition algorithm, so some measure of flexibility and inventiveness, though rooted in her lack of knowledge, can be seen in her calculations. She also wrote that $37 \div 3 = 12.1$ and explained her method as follows:

> Michel: Let's say that you have 37 sticks [she drew 37 sticks]. I now divide them into the groups of three sticks [she crossed out the sticks by three]. How many groups do we have? Twelve [she counted the groups], and one stick is left, so the answer is 12.1.

When asked how she had got familiar with this method, Michel answered that she invented it herself when in elementary school. Four more students obtained the answer 12.1 in that exercise, and, based on the described interview episode, we deem that their mistake (12.1 instead of 12.333… or $12\frac{1}{3}$) was of the same origin as the Michel's one.

Valentine, who had attended elementary school in the former USSR, wrote in the questionnaire only the final answers (11 correct responses). He explained to us that he forgot all the algorithms and invented various effective calculation methods by himself. For example, he mentally calculated 23.16 times 2.5 as follows: (i) 20 times 2.5, which is 20 times 2 and half of 20; overall 50; (ii) 3 times 2.5, which is 2.5+2.5+2.5; overall 7.5; (iii) 0.16 times 2.5, which is 0.16 times 2 and half of 0.16; overall 2.40; (iv) the answer is 57.90.

We were astonished by his ability to keep in mind all the intermediate results. When asked about this, Valentine told that it is easy for him, but then confessed that some of the intermediate calculations he wrote not in the given sheets but "on the table". Three more interviewees, including two Hebrew-speaking students, indicated that, despite the request to show their work, they preferred to write in the questionnaire sheets only the final answers or "standard" computations. In other words, they declined to show us their private methods of calculations in the format of a written test and agreed to do so during the interviews, when some trust was developed. Looking back, we suggest that some of the 16 students who computed $230 \cdot 60$ by long multiplication as though they had not calculated $23 \cdot 6$ a minute ago, in fact, could know the shortcut but preferred to show us the more "standard", in their opinion, method.

These findings shed new light on the phenomenon described in work by Sfard and Prusak (2005). Sfard and Prusak observed that most of the Russian-speaking students wrote little in spite of the teacher's request to show their work in writing. Their actual learning has been later proven more substantial than that of many Hebrew-speaking students who wrote more readily. The researchers interpreted this phenomenon using the distinction between *substantial learning* (i.e., learning for inner understanding which is not necessarily shared with the teacher) vs. *ritualized learning* (i.e., learning by mere conforming the teachers recommendations and requests), and further, in terms of different *identities* (i.e., narratives of the participants about themselves as learners) of Russian-speaking and Hebrew-speaking students. In our study, the desire for mathematical intimacy and trust seems to be directly responsible for both Hebrew-speaking and Russian-speaking students' sharing behaviors. Note however that the comparison between the two studies is constrained by the fact that Sfard and Prusak (2005) studied a class following an advanced mathematics program, whereas our study's participants were mathematically disadvantaged students.

Concluding Remarks

Our inquiry shows that creativity-related constructs, such as flexibility and relative novelty, can be found even when the students demonstrated the lack of knowledge bordering with mathematical ignorance. In line with Goldin's (2009) suggestions, we observed that even mathematically disadvantaged students can be inventive if they succeed to overcome anxiety and anticipation of failure when facing a challenge.

Two potentially powerful, from the creativity development perspective, situations, were observed in the study. The first one occurs when students are in a position to choose the best method from *several* available ones; this situation potentially evokes the manifestations of flexibility. The second one is when students do not have any readily available method of solving the problem; this situation can potentially evoke the manifestations of relative novelty. It seems that the realization of this potential is stipulated by positive answers to subtle yet crucially important dilemmas of an affective and social nature that students resolve for themselves. Examples of these dilemmas include: "Is it praiseworthy to engage in solving the given task?", "If a solution method is not readily available, is it praiseworthy to keep trying?" "Is it safe enough to show the invented method of solution?" Consequently, our task, as mathematics educators, is to assure that our students have a fair chance to positively answer these and such questions in our classrooms.

Endnotes

1. We *wish* to thank Yaniv Biton for his help in making arrangements that made data collection in Israel possible. The second-named author thanks the office of the President at California State University, Sacramento and the Center of Learning Sciences of the Technion who generously supported his travel to Israel.
2. An extended paper based on the results of our study is to appear in *Mediterranean Journal for Research in Mathematics Education* in 2010.
3. For example, some of the students preferred to be interviewed individually, and others – with a friend; some of them wanted to think in silence before voicing their responses and others were ready to think out load from the beginning.

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CARING AND COMMUNITY IN A COMPLEX INSTRUCTION FRESHMAN ALGEBRA CLASS

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This study grew from a larger research project supporting teachers' implementation of Complex Instruction (CI) in secondary mathematics of three urban high schools in a large Pacific Northwest school district. The pedagogy of Complex Instruction (CI) interweaves teacher moves and student participation aiming to form a learning environment of mutual engagement, that is, placing both students and teachers in dual roles as intellectual resources and learners (Jilk, 2007). CI pedagogy lies at the intersection of its three facets: multiple ability curricula, instructional strategies, and status and accountability (Cohen & Lotan, 1997). This study examined secondary students' mathematics experiences to understand the four affective requirements of caring in community as delineated by Osterman (2000): sense of belonging, membership matters, shared faith, and commitment to group.

Semi-structured interviews were conducted with four students in a mixed-ability algebra class whose teacher demonstrated strong implementation of CI. Interviews were transcribed and coded according to each of the four criteria above. Two examples are provided below.

Sense of Belonging. Absolon, a 14 year-old African-American male demonstrated a sense of belonging when he described everyone in his class as a potential friend through learning math. Through interactions with his peers, Absolon expressed that he learned about math, himself, and others, indicating friendships built through the learning process because of a strong sense of belonging to the class.

Membership Matters. Tom, a 14 year-old European-American male demonstrated how important his membership was to the class in his desire to help his peers. He insisted that learning was not a competition but rather an opportunity for everyone saying. Membership in the classroom matters to Tom because he sees his role as a learner and teacher, something he both receives and gives.

Further analysis of all participants' interviews suggested that a classroom with strong CI pedagogy promotes mathematical learning concern for peers' success—mathematical gains while fulfilling social interaction. This implies mathematical discourse and reform teaching practices can support student relationships with a potential bend towards social justice.

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RE-CONCEPTUALIZING TEXTBOOKS AS MATHEMATICAL STORYLINES: LESSONS FROM DEWEY

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This poster introduces a re-conceptualization of mathematics curriculum as a mathematical storyline and argues that this conceptualization can enable teachers and curriculum developers to represent the momentum of curriculum, rather than particular moments. This analysis builds from Dewey's (1902/1998) conception of mathematics curriculum as both logical and psychological, as well as his metaphor of curriculum as a journey, where the psychological layer represents the continuous experience of a journey and the logical layer represents the mapped points along the way.

To understand how the logical and psychological layers of mathematical development may appear in mathematics curriculum, a geometric investigation described by Hofstadter (1992) is presented and analyzed. It is shown that when taken separately, neither the logical nor the psychological layers of his story encapsulate his intellectual and emotional experience. However, when taken together, his mathematical story emerges, which include his mathematical insights, motivations, and emotions. Hofstadter's journey and all of his "logical" conclusions were influenced by his aesthetic desire for balance, surprise, and completeness.

I argue that when teachers craft and enact daily experiences, students also produce mathematical stories that, while different than Hofstadter's, contain both logical (the content) and psychological (their personal connection to the experience) layers. These stories contain a certain level of drama – sometimes very little (which is possibly why many students describe mathematics as boring), and at other times ample, when there is much interest and students feel compelled to move forward. Thus, the story is more than the scope and sequence of the content, which is solely the logical. It is also the way that content is developed and experienced. These stories develop over time – within a task, within a lesson, and over multiple lessons. The curriculum is the progression through these moments: the momentum.

This conceptualization of a mathematical story can inform curriculum developers, who can aim to develop "better stories." As textbook writers develop and sequence content (the logical), additional attention needs to be paid to the aesthetic experiences intended for the student (the psychological), as well as the continuity that strings the curricular events together. To illustrate, an example of a intended mathematical storyline from a textbook is introduced.

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Chapter 4: Assessment and Instrumentation

Research Reports

Brief Research Reports

Posters

THE ETHICS OF INTERVENTION: RESEARCH, PEDAGOGY AND HIGH-NEEDS SCHOOLS

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This presentation explores, both empirically and conceptually, the obligations researchers may have to use educative tasks when performing assessments for research purposes, particularly when working in high-poverty, rural schools. The presentation draws on video data of student assessments gathered during the pilot of a three-year longitudinal study. Early analysis suggests that informal activities may provide rich opportunities for both assessment by researchers and learning by students.

Introduction

This presentation emerged out of the pilot year of an NSF-funded, longitudinal study that proposes to follow children's formal and informal mathematical learning as they move from prekindergarten to first grade in a rural school. Nearly all of the students at the school are African-American and all receive free lunch. In addition to ethnographic work inside and outside of classrooms, the study proposes to formally assess children at the beginning and end of each of the three years of data collection. This assessment process is the focus of this presentation.

Initially, the research team selected items for the assessment protocol (Appendix A) because they had been used successfully by previous researchers to elicit young children's thinking around number concepts and geometry. As part of the literature review in selecting these items, we found a number of recommendations for teachers about choosing assessments that not only measure children's learning, but that also create pedagogical opportunities for the children who engage in them (e.g., NCTM, 1991; Wilcox & Lanier, 2000). Although many of the tasks presented by researchers as part of studies about young children's mathematical thinking had pedagogical features, little attention was paid to this quality and there seemed to be no expectation that questions used by researchers did more than uncover children's current understandings. That is, children were not expected to know more mathematics after participating in the research. In fact, in many ways, the notion that the assessments themselves teach children is problematic and complicating for researchers who are seeking to describe children's mathematical learning in relation to some other variable or intervention (a new curriculum, a parent workshop, a teaching strategy, etc.).

In our own study, our initial instinct was to minimize the impact of our assessment sessions so that we could more accurately describe the mathematical learning that goes on in our focal school and surrounding community without the intervention of university researchers. However, as we talked more about our particular context – an isolated, high-poverty school with few opportunities for teacher professional development and a back-to-basics curriculum, we began to wonder if we had an ethical obligation to foster, rather than minimize, opportunities to intervene in students' mathematical learning through our assessment process.

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Purpose

In this article we will address two questions:

- How can researchers balance ethical obligations they have as educators while preserving the integrity and validity of their research?
- What kinds of assessment questions and tasks are most likely to be productive for children's mathematical learning?

The first of these questions we explore conceptually, examining previous thinking on researcher roles. This occurs in the perspectives section of this paper. The second question we explore empirically in the second half of the paper, examining video data from the pilot study for evidence of opportunities for mathematical learning. The goal of the presentation will be to solicit feedback on the research team's decisions in relation to both of these questions in order to inform the official launch of the three-year data collection period.

Perspectives

Ethics of Involvement

For some time now, qualitative researchers have concerned themselves with the ethics of involvement with their participants, concerning themselves with the impact of involvement on the integrity and validity of the research (Bogdan & Biklen, 2003; Douglas, 1976) as well as the moral obligation that researchers may have to make a positive contribution to the lives of their participants (Duneier, 1999; Taylor, 1987; Weis & Fine, 2000; Whyte, 1992). As researchers, we found ourselves struggling over this dilemma. The expressed goal of the project is to document the mathematical learning of young children in their school and home lives. To do this, we planned to observe children in their school and homes and to conduct formal assessments periodically. In crafting the assessments, we debated whether we ought to design tasks where the goal was only to reveal the mathematical learning taking place in the home and school settings or to design tasks that would create opportunities for children to learn math, potentially in different ways that those they were exposed to in the home and school. The argument for teaching seemed particularly compelling in this setting where the adopted curriculum of the school system focused on memorization of numbers and shapes and, later, on test-taking skills, rather than on mathematical reasoning and problem solving. We reasoned that interactions with the research team might provide one of few opportunities for children in the study to engage in more sophisticated mathematical thinking and could potentially provide an opportunity to teach mathematical play that students could adopt productively in their homes and their classroom.

However, we did not make the move toward intervention lightly. A number of researchers have noted that continued observation of troubling practices, even including abuse, allows researchers to document conditions in schools and institutions in ways that may lead to broader and more systematic changes than intervening only in the site of research (e.g., Bogdan & Biklen, 2003; Taylor, 1987). Furthermore, the conditions at the proposed research site in our study in no way approached the level of abuse. Children were generally happy and treated kindly by their teachers. However, the back-to-basics mathematics curriculum significantly limited children's opportunities to learn and attempts to alter this may unwittingly paint too positive a picture of the learning at this rural school serving poor children, which in turn, might make broader curricular interventions as a result of the research findings less likely. From this perspective, the argument would be to use assessments that impacted children's learning as little as possible so as to most accurately report the learning opportunities available in the natural settings of the school and then to use these research findings to argue for broader change.

Ultimately, we decided against trying to claim validity by limiting our interventions as much as possible, in part because of the impossibility of non-intervention. Bogdan and Biklen (2003, p. 35) argue that because researchers *cannot* study participants' lives in a natural setting, they instead much use their rich knowledge of the social context to interpret "what they actually study – 'a setting with a researcher present'." Following this thinking, our research team decided that even questions designed to be unobtrusive would, in fact, intervene in the lives and the mathematical thinking of the children we intended to study. Given this, we decided that the risk to the integrity and validity of our data – our representation of what the children at this site knew and could do mathematically – was not as significant as the ethical obligation that we had as people with financial, educational, and mathematical resources to offer opportunities for learning to the children we encountered. Drawing on Bakhtin (1993), we decided that we were answerable to the children and their parents in the moments of our interactions with them as well as to the broader researcher community and to children in other contexts. At the same time, we acknowledge a need for us to carefully record and analyze ways that our own engagements with the children may shape their mathematical performances in other contexts (in the classroom, with parents, etc.). We anticipate that our plan to collect data in many sites will help to place these assessments in an appropriate context.

Research and Assessment Tasks

Once we decided that we would attempt to maximize, rather than minimize, our child-participants' opportunities to learn mathematics through our interactions with them, we needed to find assessment tasks that would both appropriately uncover children's thinking around number and geometry (the focus of the larger study) and create opportunities for further thinking. Appendix A lists our original list of tasks. We included a number of tasks that asked children to count in various ways (orally and in writing, with and without objects, forwards and backwards) and tasks that asked children to identify and describe shapes because these tasks are common in the research literature (e.g., NRC, 2005; Saxe, Guberman & Gearhardt, 1987; Thompson & Van de Walle, 1980). However, once we began to consider the educative properties of tasks, we found some evidence that these counting tasks may not help children develop skills and understandings that transfer to other contexts (Aunio, Hautamaki, & Van Luit, 2005). At the same time, we found no literature on the use of games as assessments, but found a few articles that described children's productive interactions with mathematical games in other settings (e.g., Anderson & Gold, 2006; Ramani & Siegler, 2008). We decided as a result of this work to include games in our assessments and also decided to add puzzles. Again, the use of puzzles was uncommon – although not unheard of -- in the assessment literature of young children. We hoped that they might present the same sorts of productive opportunities as game playing (e.g., Clements & Battista, 1992; Hannibal, 1999). Overall, based on the literature on assessment aimed at teachers, we decided to evaluate the proposed tasks in the pilot based on their ability to produce interactions that other researchers and teachers had found to be pedagogically productive, including those that promoted dialogue, encouraged students to ask questions, offered students opportunities to manipulate materials (Schwerdtfeger & Chan, 2007; Zack and Graves, 2001).

Modes of Inquiry

This study draws on theories that see learning as participation in social contexts (e.g., Greeno, 1997; Lave, 1988; Lerman, 2000). From this perspective, as Lerman (2000, p. 26) noted, "knowledge has to be understood relationally, between people and settings: it is about competence in life settings." Thus, mathematical knowledge and learning cannot be seen as fixed or as easily summarized. Instead, mathematical learning is seen as evidenced in enacted practices related to the discipline of mathematics in a variety of social settings. For this presentation, we are concerned with a small part of the larger study: children's mathematical performances during formal assessments.

The site of the study, Oliver County Public School (a pseudonym) is a PK-12 school in central Georgia with fewer than 300 students. The county does not have a single stop light, fast food restaurant, or pizza parlor. Most of the students are African American, and, because so many students qualify for free lunch, the school decided not to charge any of its students for meals. These characteristics make it an ideal setting to study the mathematical learning of underrepresented students in a *rural* (as opposed to the more-commonly studied urban) context.

At the same time that we were piloting observation instruments in the classrooms, we also used the tasks described in Appendix A with a small group of children in preschool, kindergarten and first grades, the three grade levels that we intend to follow children through in the larger study. We planned to have five children in each grade level respond to the tasks (about one-third of each class) over the course of two months. We video taped these interactions for analysis and presentation later. Our discussion of the results focuses on what children did and said during these formal assessments, actively acknowledging that they in no way summarize all of the mathematical performances of which the children were capable.

Assessment interviews took approximately 20 to 30 minutes, depending on the age of the child. Not all the tasks in the assessment protocol were used in each interview, although all tasks will be used at least three times. Interviews took place in the school library with the first author. The video recorder was placed on a tripod.

Technology designed for examining video segments was used during analysis. The Video Analysis Tool (Recesso et al, 2008) allows researchers to identify and code small segments of video tape. For this presentation, the research team developed codes related to mathematical learning around number, geometry, problem solving, and representation. In addition, codes were developed to identify qualities of engagement with the task based on previous research. The research team identified instances of conversation and silence, active use of materials and reluctance, and volunteering of questions and thoughts. Segments of tape were then sorted by the codes for analysis.

Because data collection and analysis will be on-going throughout the spring, the results presented in the following section are preliminary. A major goal of the presentation will be to share video data with other researchers and to solicit feedback.

Results

We began our pilot assessments with the current preschool students so the results reported here come from just this classroom. Many of the school-like tasks in interviews seemed to produce little engagement and conversation from students. In most cases, students completed the tasks and we were able to determine something about their knowledge of number or geometry in this setting. However, in most cases they seemed uninterested in continuing a conversation about the problem or in continuing to work with the materials. Many of the counting tasks engendered a great deal of silence such as in the following episode. Amy had previously asked D'Andre, a prekindergarten student, to count by 1s out loud, which he did to 29, skipping 17. Then she had him place number cards in order from 1 to 12. He did this correctly and silently.

She then gave him 5 green cubes and asked him to count them. He began to tap on the table with the cubes like a drum stick.

Amy:	Can you tell me how many you have there?
D'Andre:	*4 seconds silence*
Amy:	Can you count them?
D'Andre	(pushing the cubes on the table like a train) *6 seconds silence*
Amy:	How many cars are in your train?
D'Andre:	(pointing to each cube and whispering). 1,2,3,4,5.

Ultimately, in this episode, Amy persuaded D'Andre to perform the mathematical task that she intended, but the process produced little pleasure for her or for him. In fact, in the face of his disengagement, Amy decided to end the interview early rather than continue to push him.

One exception to the predominance of silence in these counting tasks occurred when Amy asked one of the boys to count plastic dinosaurs instead of bears in a counting task. He continued to play act with the dinosaurs for a few minutes and wanted to take them with him; however, the conversation was not mathematical. In fact, when we observed the boys playing with these dinosaurs later in the classroom they made "families" with them and used them to hunt, but did not engage in either counting or sorting activities.

In these early assessments, Amy alternated playing *Hi-ho the Cherry-O* and *Cadoo* with most children at the end of the assessment. These interactions seemed to produce more interest and conversation on the part of the children than the other traditional academic tasks, although the less familiar children were with the games, the less this was true. Explanations and corrections related to the rules of the games seemed to reduce spontaneous interactions

In *Hi-ho the Cherry-O*, players put ten plastic cherries on their trees. During each turn, players spin a spinner, which has "good" spaces that show one, two, or three cherries and "bad" spaces that show a dog, a bird, and a bucket tipping over. When the spinner lands on a picture of the cherries, the player removes that many cherries from the tree. When the spinner lands on the bird or the dog, the player replaces a cherry from the bucket. The tipped over bucket space requires that the players take all their cherries from their bucket and replace them on the tree. The first player to take off all ten cherries wins.

Unlike in the more school-like tasks, children volunteered information about their counting and thinking when playing games. For example, Candace spoke after each of her turns, exclaiming: "I get to take 3 more off" when she spun the space with three cherries or noting, toward the end: "I'm going to win. See, I got one left." She made this second statement after the spinner landed on the space with two cherries, but before she actually removed them. Thus, she was able to quickly look at the space and recognize that it meant "two," look at her tree and imagine taking two away from the three cherries that remained and realize that only one cherry would remain. This sort of problem solving is very similar to the kind of thinking we were trying to get at in the interview when children were asked to solve problems with cubes, bears, and dinosaurs.

At this time, we are still continuing to search for tasks that afford students with opportunities for genuine talk and engagement with the mathematics and a way of providing more possibilities for us to access mathematical knowledge and for students to learn more about mathematics, particularly problem-solving and reasoning. To this end, we are trying out puzzles and spatial reasoning games in the assessments scheduled for the fall.

Discussion

As a research team, we became persuaded that we had an obligation to make sure that our interactions with children were as productive for them as possible in addition to meeting our goals as researchers. Our initial piloting of the tasks suggests that game-playing and puzzle-completion may provide rich sites for assessing children's mathematical knowledge in ways that are more pleasurable for them and which, therefore, might be more likely to promote positive feelings toward mathematics as well as a desire to engage in more of these mathematical experiences. We plan to continue to experiment with putting more traditional assessment tasks in the middle of game-playing or puzzle events, hoping that we may create an environment where children come to see these other tasks as pleasurable. In addition, we plan to make our assessment materials available to children in their classrooms to see if they choose to use them at other times of the day.

In addition to meeting the needs of the children, we have begun to wonder where assessments embedded in games might be quite useful to us as researchers. Although there is much less standardization in a game and it requires more analysis to identify mathematical thinking, it provides a slightly less formal context for observing children. Many studies about the mathematical knowledge of young children have depended on children's completion of tasks in formal interviews (e.g., Saxe, Guberman & Gearhardt, 1987; Thompson & Van de Walle, 1980). We have begun to wonder whether interviews that are based on games rather than formal tasks might shed new light on what we believe children, particularly non-majority children, can and cannot do mathematically and look forward to exploring this question in our own work and in conversations with others.

Endnotes

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Appendix A

ORIGINAL TASKS USED IN PILOT
Can you count out loud for me as high as you can? By 2s? By 5s? By 10s? Can you count backward from 10? Can you start counting at 6? *Give child 3 number cards. (1,2,3; 2,7,10; etc.)* Can you put these in order from smallest to largest?
Can you write the numbers starting at 1 and keep writing as many as you know?
Place 5 green cubes in a line. How many green cubes are there? *Cover 2 green cubes with hand.* How many green cubes are under my hand?
See these dots. Watch what I do. *Put down 2 dots in a horizontal row on a mat. Cover the mat with a screen. Slide 2 more dots behind the screen one at a time. Give the child a mat and a cup of dots.* Put down dots on your mat so they will look just like the ones behind this screen.
Provide index cards with numerals on them and paper plates with dots affixed to them. How many dots are on this plate? Can you find the number card that matches? Can you find the plate with 6 dots on it? Can you find the matching number card? *Show a number card*. Which plate matches this card? What number is this?
Count how many bear pictures you see in this circle. Can you put the same number of plastic bears in the empty circle?
You can use any of the materials on the table if you need to in order to answer this question. *Have available toy cars, markers, paper, Unifix cubes.* Janeshia had 2 cars and her brother gave her 2 more cars. How many cars did she have then? How did you figure that out? Can you show me?
Can you draw a circle? A square? A triangle? *Give wooden shapes and ask children to identify them.* What is the difference between a square and a triangle? What makes a triangle a triangle?
Give child sheet of circles and non-circles. Can you mark all the circles? Why did you mark this one? Why didn't you mark this one? *Repeat with triangles and rectangles.* *Give child oatmeal container, ball, and box. What c*an you tell me about this shape? Do you know the name of this shape?
Can you solve this puzzle? *(Geometric puzzle)* Can you tell me how you knew where to put this piece?
Would you play this game with me? (Hi-ho, Cherry-O, Cadoo)

COMMUNICATING QUANTITATIVE LITERACY

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Quantitative Literacy (QL) has been described as the skill set an individual uses when interacting with the world in a quantitative manner (Steen, 1999). A necessary component of this interaction is communication. To this end, assessments of QL have included open ended items as a means of including communicative aspects of QL. The present study sought to examine whether such open ended items typically measured aspects of quantitative and/or mathematical communication or of mathematical skill. Results indicated that the proportion of items in QL assessments measuring aspects of mathematical skill was significantly more than items assessing aspects communication.

Introduction

Quantitative Literacy (QL) has become a topic of increased focus at both the national and international level in the past thirty years. Sometimes referred to as quantitative reasoning, mathematical literacy, or numeracy, QL focuses not on an individual's mathematical skills but on their ability to interact with the world around them in a quantitative manner (Steen, 1999). This characterization of QL can be found in many supporting documents in the past 30 years. However, one element of these characterizations often mentioned but seldom discussed is that of communication itself. Further, there is little to no supporting research that examines the communicative aspect of quantitative literacy. However, several well known large-scale assessments have evaluated QL using open-ended items. These open-ended items can be viewed as opportunities to assess communication in QL. In an effort to examine the state of communication in QL, the current study examines how these large-scale QL assessments examine responses to open-ended items for communication.

Theoretical Perspective

Communication in Quantitative Literacy

In 1982, the British government published a report titled *Mathematics Counts*, or the Cockroft Report. This report is perhaps the first major modern report describing the importance of QL, or numeracy as called in Britain. The report provided several statements on the mathematics education in Britain and the level of numeracy of its citizens. One such statement reads:

> We believe that all these perceptions of the usefulness of mathematics arise from the fact that mathematics provides a means of communication which is powerful, concise and unambiguous. Even though many of those who consider mathematics to be useful would probably not express the reason in these terms, we believe that it is the fact that mathematics can be used as a powerful means of communication which provides the principal reason for teaching mathematics to all children (Cockroft, 1982, p. 1).

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Later describing the specific characteristics of numeracy, the Cockroft Report (1982) reemphasized the importance of communication in the definition of what it means to be numerate, or quantitatively literate (p. 11).

The Cockroft Report was followed by other documents in the United States. One of the most influential for mathematics education (and QL) was the *Curriculum Standards for School Mathematics* published by the National Council of Teachers of Mathematics (NCTM, 1989). Unlike the later 2000 standards, the 1989 standards purposefully discussed the concept of QL and identified five goals or requirements for students to obtain a degree of mathematical power in terms of being numerate. Among these was learning to communicate mathematically. Further describing what it means to communicate mathematically, NCTM stated: The development of a student's power to use mathematics involves learning the signs, symbols, and terms of mathematics. This is best accomplished in problem situations in which students have an opportunity to read, write, and discuss ideas in which the use of the language of mathematics becomes natural. As students communicate their ideas, they learn to clarify, refine, and consolidate their thinking.

Within the same year that the 1989 NCTM standards were released, the National Research Council (NRC, 1989) published the report titled *Everybody Counts*. *Everybody Counts* characterized QL as a type of number sense that evolves from “concrete experience and takes shape in oral, written, and symbolic expression” (p. 47). The report further emphasized the need for problem solving approaches to be described orally and in writing.

The excerpts from each of these three reports characterize the communicative aspect inherent in QL. A quantitatively literate person is described as demonstrating the ability to communicate in a quantitative way. By the descriptions from these earlier publications, mathematical communication is an aspect of QL. Logically, it follows that since communication can be considered a demonstrated aspect of QL, then QL can be measured by an individual’s ability to communicate quantitatively.

Following the release of the NRC and NCTM publications, several international and U.S. assessments made strides to measure QL. These studies included the Third International Mathematics and Science Study (TIMSS-95), the National Assessment of Adult Literacy (NALS) in 1985 and 1992, the International Adult Literacy Skills (IALS) beginning in 1994, and the Program for International Student Assessment (PISA) beginning in 2000. While none of these studies limited themselves to QL, each assessed it and as part of their assessments used open-ended items. Including open-ended items encourages written communication on the part of the test taker.

The national and international assessment of QL was accompanied by a host of literature. Similar to the reports published in the 1980’s, the newer literature continued to emphasize the importance of communication in QL. Cobb (1997) characterized QL as requiring “a difficult integration of four very different kinds of thinking. This makes it a kind of cognitive emulsion…that constantly threatens to separate into its more basic forms of thought.” (Cobb, 1997, p. 76). These four forms of QL are: computational/algorithmic; logical/deductive; visual/dynamic; and verbal/interpretive. This last form represents the communicative aspect thus far characterized above. As did NCTM (1989), Cobb outlined communication as one of the main aspects of QL.

De Lange (2003) cited the International Life Skills Study (2000) as characterizing QL as partly being communication. De Lange later identified communication as one of eight competencies needed for an individual to be quantitatively literate. Specifically, mathematical communication was characterized as "expressing oneself in a variety of ways in oral, written, and other visual form; understanding someone else's work" (p. 77). In a later publication, De Lange (2006) referenced the PISA study as being "concerned with the capacities of students to analyse, reason, and communicate ideas effectively as they pose, formulate, solve and interpret mathematics in a variety of situations" (p. 15). Therefore, according to De Lange (2003, 2006), at least two large scale assessments of QL focus on a communicative aspect.

Adding to the descriptions mentioned already, Grawe & Rutz (2009) articulated three components of QL: a set of elementary quantitative skills that the individual has a sophisticated understanding of; the ability to solve problems within context; and the ability to communicate the results after they have solved a problem. Grawe and Rutz note that while it is essential to be able to communicate mathematics to make effective arguments, this relationship is reflexive in that "sophisticated rhetoric is also important for effective [quantitative reasoning]" (p. 3).

Up to this point it may seem that much of the review of literature has been a list of excerpts and quotes from QL literature that makes mention of communication in some form. Such an assessment would be mostly correct, and unfortunately so. Moreover, such an assessment only emphasizes the need for an examination of communication in QL assessments that the present study proposes. Various QL literature makes mention of communication as being an important element of QL but goes little further in providing specifics. Yet the consistent description of communication as an essential component of QL represents its significance. "Taken together, these imply that a numerate person should be expected to be able to appreciate and understand some of the ways in which mathematics can be used as a means of communication" (Cockroft, 1982, p. 11). Yet one of the main ways that QL appears to be measured is by tests of mathematical skill (Steen, 2000).

From Mathematical to Quantitative Communication

The previous section used existing literature to illustrate the implied significance of communication in QL. Yet, very little literature exists that describes communication in QL. Therefore, it is logical to examine aspects of mathematical communication, in which there is a body of literature, and see how such aspects can be applied to examine communicating quantitatively. In mathematical communication, the purpose is to communicate about mathematics itself (Lee, 2006). For QL, the purpose of communicating is inherently tied to its context and is not done specifically for or about mathematics (Steen, 2001). A newspaper article may use statistical polling data to explain why one candidate is winning an election, but the article is written to communicate about the election, and not the mathematics itself. In this and other similar scenarios, mathematics might be viewed as the means of communication. While this last facet is similar to mathematical communication, what might be termed *quantitative communication* involves mathematical means but varying, context-based purposes for communicating about mathematical or quantitative information. Therefore, the difference between mathematical and quantitative communication is the purpose for the communication. Since there is currently no literature describing quantitative communication, it is necessary to identify aspects of mathematical communication that may be present in quantitative communication.

Mathematical communication has been described as being written, spoken, or visually represented (Danesi, 2007). Indeed, these different forms of mathematical communication are sometimes used interchangeably or in place of one another (Steele & Johanning, 2004). Since these are mathematical means of communicating, it is logical that they are aspects of quantitative communication. Therefore, the current investigation examines how various assessments of QL evaluated written, spoken, and visually represented communication.

Research conducted on mathematical writing has identified simplistic, procedural, and conceptual types of writing as areas of focus for examination (Kosko, Wilkins, & Pitts Bannister, 2009; Shield & Galbraith, 1998). While Kosko et al. and Shield and Galbraith used the term descriptive rather than conceptual, the two terms are arguably related. Simplistic communication consists of memorized statements or the answer to a problem/task (Kosko et al., 2009). Procedural communication entails descriptions of procedures or strategies while conceptual communication consists of explanations, justifications, or conjectures (Schleppegrell, 2007). Visual representations have not been identified as being procedural or conceptual, but it is logical to conclude that they could be used in both procedural and conceptual ways, but may only be determined as such through use of either written or spoken communication. Dossey's (1997a) description of measuring QL also uses the terms procedural and conceptual in referring to the different levels an individual engages quantitatively with the world. Given this background in mathematical communication, it is prudent to investigate how the different large scale assessments measuring QL (i.e. NALS, IALS, TIMSS-95, PISA) examine different forms of communication. Therefore, the current study seeks to answer the following research questions for quantitative communication:

1. To what degree do assessment rubrics in large scale examinations of QL examine responses that are simplistic, procedural, conceptual, and/or include a representation?
2. Do item-rubrics assessing mathematical communication (mathematical purpose) differ from items assessing quantitative communication (context-based or quantitative purpose) in the degree to which the rubrics of such items assess simplistic, procedural, conceptual, and/or representation responses?
3. Are there differences between assessment of purpose and response types across the different large scale assessments of QL?

Methods

Measures

The current investigation examines whether the communication assessed by large scale QL tests involved simplistic, procedural, or conceptual communication, or asked for representations (*simplistic; procedural; conceptual; representation*). These codes were assigned so that every item that contained any of these codes was assigned each. Therefore, an item rubric may have been coded 1,2,4 to denote that it looked for simplistic and procedural communication and looked for the creation or extension of a representation as well. This was done to examine the combination of mathematical communications assessed instead of simply the most sophisticated or a particular type.

A separate code was used to assess the purpose of communications in open response questions. A *mathematical purpose* was assigned if the assessment sought out a reply for a specifically mathematical context. A *quantitative purpose* was assigned if the assessment sought out a reply for a context other than, but possibly related to, mathematics. Additionally, a *loose quantitative purpose* was assigned if the assessment's context appeared contrived.

Sample

Public-released items and rubrics from the IALS, 2006 PISA study, 1995 TIMSS study, and 1985/1992 NALS were used as the sample for the current study. These four studies claimed to assess QL / numeracy and each included open response items. Therefore, the publicly released items for each assessment were examined. Information about these items is presented in Table 1.

Analysis

Our first step was to examine all released items from each assessment and select all open response items in the QL assessments. The final sample of rubrics examined is represented in Table 1. Our second step in analysis was to examine each item's rubric to see what type(s) of response was being sought (*simplistic; procedural; conceptual; representation*). Then, the purpose of the response sought was examined within context of both the item and the item's rubric. For example, rubrics in which the purpose of the test-taker's response was to communicate explicitly about the mathematics involved was coded as having a mathematical purpose, but rubrics whose communication purpose was to communicate about the context, based upon the underlying mathematics, was coded as having a quantitative purpose. After all rubrics were coded, Chi-Square analyses were conducted to determine if there were (1) relationships between the different response types (simplistic, procedural, conceptual, representation), (2) relationships between item-rubric purposes (mathematical, loose quantitative, quantitative) and response type, and (3) differences between the QL tests (e.g. IALS, PISA) in terms of response type and purpose. Following Chi-Square analyses, post hoc analysis using a standardized residual method (Z_{Crit}=1.96) was employed to examine specific cell differences in the contingency tables (MacDonald & Gardner, 2000).

Table 1. Number of Items for Large Scale Assessments of Quantitative Literacy

QL Assessment	Total # Items	# Open Ended Items
IALS	15	15
2006 PISA	89	64
1995 TIMSS	23	12
1985/1992 NALS	110	36
Total Items Examined	237	127

Analysis

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standardized residual method (Z_{Crit}=1.96) was employed to examine specific cell differences in the contingency tables (MacDonald & Gardner, 2000).

Results

An examination of differences between each response type across all items indicated a relationship between simplistic and procedural ($\chi^2(1)=8.67$, $p < .01$); simplistic and conceptual ($\chi^2(1)=29.69$, $p < .01$); and simplistic and representation ($\chi^2(1)=22.86$, $p < .01$), signifying that certain response types were found more or less often in combination with other response types. For the other combinations of response types, no statistically meaningful relationships were found. Post hoc analyses found item rubrics that sought simplistic responses were less likely to seek a response that was conceptual ($Z = -2.30$, $p < .05$) or include a representation ($Z = -2.10$, $p < .05$). Non-simplistic items were more likely to seek procedural responses ($Z=2.50$, $p < .05$), conceptual responses ($Z = 4.50$, $p < .05$), and representations ($Z=4.00$, $p < .05$).

Descriptive data related to the second research question can be found in the bottom row of Table 2. No meaningful relationships were found between purpose type and response type. Therefore, rubrics assessing responses for a mathematical purpose, loose quantitative purpose, or quantitative purpose did not appear to differ in the degree to which they asked for simplistic, procedural, conceptual, or representation responses.

An examination of communication purpose across tests found a statistically significant relationship ($\chi^2(6)=33.87$, $p < .01$), signifying that some tests were more likely to assess certain communication purposes over the other tests. Post hoc analysis found that, when compared to the other tests, IALS had more rubrics assessing a loose quantitative purpose than would be expected by chance ($Z=3.30$, $p < .05$). Compared to other tests, NALS was found to have fewer rubrics assessing a loose quantitative purpose ($Z= -2.80$, $p < .05$) and more rubrics assessing a purely quantitative purpose ($Z=2.0$, $p < .05$). Compared to other tests, PISA was found to have more rubrics than expected by chance assessing a mathematical purpose ($Z=2.0$, $p < .05$). Table 2 illustrates this latter finding as PISA is the only test containing items with a mathematical purpose. Also, IALS had a larger proportion of loose quantitative purpose items than any other test. NALS was the only test consisting of rubrics assessing only a quantitative purpose. These findings suggest that in terms of the purpose of communication, NALS items were more oriented towards eliciting a response with a quantitative purpose than any other test. Similarly, IALS was more likely to have loose quantitative purpose items and PISA was more likely to have items evoking responses with a mathematical purpose.

Descriptive data related to comparisons between test and response type can be found in the right-hand column of Table 2. A statistically significant relationship between test type (i.e. IALS, NALS, PISA, TIMSS) and simplistic responses was found ($\chi^2(3)=9.34$, $p < .05$). Post hoc analysis found that TIMSS had fewer than expected rubrics assessing a simplistic response (Z =2.40, $p < .05$). Results also indicated no statistically meaningful relationship between test type and procedural, conceptual, or representation response types. These results suggest that while TIMSS had fewer than expected rubrics assessing simplistic responses than other tests, for the most part, the different tests were found to be similar in regards to response type.

Table 2. Categorization of Assessments for Quantitative Literacy Open Response Items

	Mathematical Purpose	Loose Quant. Purpose	Quantitative Purpose	Total
IALS	Simplistic = 0 Procedural = 0 Conceptual = 0 Rep = 0 Total = 0	Simplistic = 9 (100%) Procedural = 0 Conceptual = 1 (11%) Rep = 0 Total = 9	Simplistic = 5 (83%) Procedural = 0 Conceptual = 1 (17%) Rep = 0 Total = 6	Simplistic = 14 (93%) Procedural = 0 Conceptual = 2 (13%) Representation = 0 Total = 15 (12%)
NALS	Simplistic = 0 Procedural = 0 Conceptual = 0 Rep = 0 Total = 0	Simplistic = 0 Procedural = 0 Conceptual = 0 Rep = 0 Total = 0	Simplistic = 30 (81%) Procedural = 1 (3%) Conceptual = 2 (5%) Rep = 5 (14%) Total = 37	Simplistic = 30 (81%) Procedural = 1 (3%) Conceptual = 2 (5%) Rep = 5 (14%) Total = 37 (29%)
PISA	Simplistic = 5 (63%) Procedural = 2 (25%) Conceptual = 1 (13%) Rep = 0 Total = 8	Simplistic = 13 (81%) Procedural = 1 (6%) Conceptual = 1 (6%) Rep = 1 (6%) Total = 16	Simplistic = 33 (83%) Procedural = 2 (5%) Conceptual = 9 (23%) Rep = 5 (13%) Total = 40	Simplistic = 51 (80%) Procedural = 5 (8%) Conceptual = 11 (17%) Rep = 6 (9%) Total = 64 (50%)
TIMSS	Simplistic = 0 Procedural = 0 Conceptual = 0 Rep = 0 Total = 0	Simplistic = 2 (100%) Procedural = 0 Conceptual = 0 Rep = 0 Total = 2	Simplistic = 3 (33%) Procedural = 2 (22%) Conceptual = 4 (44%) Rep = 1 (11%) Total = 9	Simplistic = 5 (45%) Procedural = 2 (18%) Conceptual = 4 (36%) Rep = 1 (9%) Total = 11 (9%)
Total	Simplistic = 5 (63%) Procedural = 2 (25%) Conceptual = 1 (13%) Rep = 0 Total = 8 (6%)	Simplistic = 24 (89%) Procedural = 1 (4%) Conceptual = 2 (7%) Rep = 1 (4%) Total = 27 (21%)	Simplistic = 71 (77%) Procedural = 5 (5%) Conceptual = 16 (17%) Rep = 11 (12%) Total = 92 (72%)	Simplistic = 100 (79%) Procedural = 8 (6%) Conceptual = 19 (15%) Rep = 12 (9%) Total = 127
Note: Statistics in the total column are for test type and in the total row are for purpose type.				

Discussion

The analysis of open response item rubrics for QL assessments illustrated two things. One, with the exception of TIMSS, all tests' open response rubrics were more oriented towards assessing simplistic responses. Two, item rubrics assessing simplistic responses seldom looked for procedural or conceptual descriptions, or the use of representation. Rephrasing these two points, the majority of open response rubrics looked for answers and did not look for procedural or conceptual descriptions that might accompany them. While many items on these QL assessments may have required the test-taker to think critically and deeply about the mathematics they were using, few items in these tests *assessed* such thinking. Rather, they assessed the answer provided, not the reasoning or the ability to communicate such reasoning. With such findings, it appears that the majority of QL assessments appear to assess quantitative communication as the ability to produce a simple answer, which may or may not be preempted with showing one's work. Too seldom do these assessments examine whether the test-taker can describe what they are doing or why they are doing it.

Wilkins (in press) stated that "although several national and international projects have considered the assessment of quantitative literacy, ultimately the notion of literacy that has been emphasized is one focused solely on mathematical achievement" (p. 20). The results of the rubrics for open items presented in the current study supports Wilkins' statement. While Wilkins' focus on QL was not on communication, but on the incorporation of beliefs and dispositions, his words appear well placed when he stated that "evaluation of learning is often reduced to the use of measures of achievement alone, which only takes into account one component of the overall quantitative literacy construct" (p. 20). As explored in the current study, 79% of open response rubrics sought simplistic responses (see Table 2). A simple calculation will show that 80% of these simplistic responses looked for answer-only types of responses. This means that 70% of all item rubrics sought answer-only responses. Only 6% of items looked for procedural responses, 15% conceptual, and 9% sought a representation. The 79% of items seeking a simplistic response from the test taker were less likely to seek a procedural or conceptual description. Recalling that these were open response items, and not multiple choice/response, these results are disturbing.

The implications of the current findings are straightforward. If communication is to be regarded as a critical and essential element of what makes an individual quantitatively literate, then communication must be assessed in QL assessments. To do this, more than a simple answer should be required for a larger number of open response items. While many items are, necessarily, answer-only, far too many require no more than showing one's work and stating the answer. QL assessments must evaluate whether an individual can reason and argue in a way that demonstrates their QL. If QL assessments do not do this, then they truly are reduced to simply another achievement test that happens to have open response items.

Endnotes

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JUXTAPOSING CHINESE AND AMERICAN MATHEMATICS EDUCATION COMMITMENTS

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Using video data of urban Chinese secondary mathematics classes, supplemented with interviews, collected artifacts, and other data, we juxtapose American and Chinese mathematics education commitments along three cross-cultural dimensions (student, school, and culture) to argue that Chinese excellence on international comparisons may stem as much from these interrelated factors as from actual instructional practice.

Introduction

China is a world leader in student mathematics performance as measured on international comparisons, and its students consistently outperform American students (Fan & Zhu, 2004; Hiebert, et al., 2003; Huang & Bao, 2006; Huang & Li, 2009; Lin & Tsao, 1999; Stevenson, Chen, & Lee, 1993). This study examined underlying educational factors (besides actual instruction) that might contribute to Chinese students' excellence on international mathematics comparisons. We assumed that juxtaposing aspects of Chinese and American mathematics educational efforts could provide answers for how China manages to consistently outperform the United States on international assessments, and perhaps suggest areas of potential improvement for American educators.

This study is a cross-cultural comparison of educational commitments; cross-cultural studies are known to benefit each population studied (An, Kulm, & Wu, 2004; Kaiser, 1999; Robitaille & Travers, 1992; Stigler & Perry, 1988; Wolcott, 1999). In May 2009 we collected data in mainland China by videotaping 11 different mathematics lessons in five schools, conducting post-lesson debriefing interviews with each teacher, and collecting student and school artifacts. We also participated in two versions of Chinese lesson study and interviewed a variety of Chinese educators (university researchers, other teachers, and administrators) and students (Figure 1). We have likewise drawn from the literature on cross-cultural comparisons to inform our analysis. Analysis is ongoing, and we follow Glaser's (1965) constant comparative method. This is an instrumental study (Stake, 1994) that generates theory for broader application.

Results

Juxtaposing Chinese and American mathematics education commitments revealed stark differences; principally, those of the Chinese were stronger, with American commitments weaker (or in some cases, non-existent). We present our results in the form of a theoretical framework that organizes these comparisons along three dimensions: 1) student, 2) school, and 3) culture.

Student Commitments

Chinese students appear more committed to their mathematics education than American students. For example, during mathematics instruction, they are more involved with teacher-posed tasks, are more willing to proffer their ideas during discussions, and spend far more time on their out-of-school mathematics studies than American students (Huang & Li, 2009; Lin & Tsao, 1999; Stigler & Perry, 1988). Chinese students also miss very little school (Stevenson &

Stigler, 1992). The Chinese students we interviewed indicated that mastering material often required the self-assignment of additional homework, and claimed that their courses covered twice as much mathematics content as American lessons. Chinese students often attend cram schools to prepare for examinations.

Some of the interviewed: (*left to right*) Ms. Han, a Chinese teacher who had also taught in a New York public high school as part of an exchange program; an American researcher; Jin, a student; Mr. Leung, the school's principal; Ms. Yu, a new teacher.

Between-class exercises at an urban mainland Chinese school. The school buildings are behind the camera in the foreground; in the background are apartment buildings and businesses.

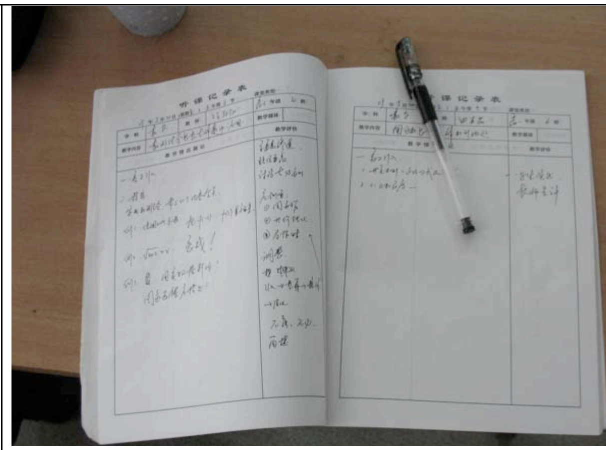
Figure 1. Chinese collective recess (exercises) and four interviewees

Chinese families also appear more committed to their child's education than American parents (Chao, 1994; Stevenson et al., 1990; Tsui & Rich, 2002). Chinese parents emphasize academic achievement more than American parents (Chao, 1994; Lin & Fu, 1990). China's family-size restrictions allow two parents (and four grandparents!) to focus on a single child's education; 90% of parents want their children to enter higher education institutions, and 19% want their children to obtain a Ph.D. (Zhu, 1999). Personal desks and dedicated study times in Chinese households also facilitate students' studies, which are often lacking in American households (Stevenson & Stigler, 1992).

School Commitments

Additionally, Chinese school commitments appear stronger than those in the United States, as measured by teacher, cohort, and professional development factors. For example, Chinese teachers possess a profounder understanding of fundamental mathematics and place a greater emphasis on conceptual knowledge than do American teachers (An, Kulm, & Wu, 2004; Ma, 1999). They connect ideas to prior knowledge, concrete models, or students' lives, and stress students' proficiency with computational skills; they also possess a broader repertoire of questioning strategies, representations, and possible student misconceptions, emphasize abstract and algebraic reasoning, and ask more focused questions to develop better student reflection abilities than American teachers (An, Kulm, & Wu, 2004). Chinese teachers use student errors as a starting-point for further inquiry, promoting a climate of openness to discuss mistakes during the struggle to learn mathematics; because American teachers interpret errors as an indication of a failure to learn, mistakes are discouraged and hid, often leading to student embarrassment (Santaga, 2004; Schleppenbach, Flevares, & Sims Michelle Perry, 2007; Stevenson & Stigler, 1992; Wang & Murphy, 2004). Despite lessons being teacher-controlled and lecture-based; there are student-centered features (Huang & Li, 2009; Mok, 2006). Chinese students' scores (compared to Western students' scores) on the International Assessment of Educational Progress study were proportional to the amount of class time (average minutes per week) spent on mathematics study (Lin & Tsao, 1999). Dropout rates are low in China: less than 1% in primary grades (Liang, 2001).

Ms. Dao's class. A school administrator helps record the lesson. Notice the cup of stimulant: (tea, not coffee), technology, and class size (40 students).



The administrator's notebook where he took notes while observing Ms. Dao's teaching. There were two other Chinese observers, both university professors.

Close-up of the school's camera used to document Ms. Dao's teaching. Notice the quality as compared to my university's cameras (visible in *Figure 3, upper right*).

Ms. Dao's (second from left) office space, shared by other teachers, including Ms. Yu's (far right) office space.

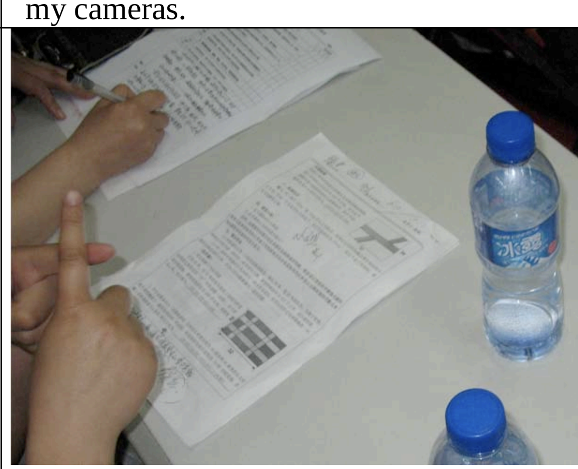
Figure 2. Chinese classrooms, equipment, and offices

Chinese teachers often teach multiple grades as they advance with their students from one grade level to the next (which form multi-year cohorts as they jointly progress), and studies indicate such prolonged contact matters for student achievement, especially with less-experienced teachers (e.g., Park & Hannum, 2001). There is no diversification of curriculum at primary or junior secondary levels, nor ability grouping in China (Liang, 2001; Park & Hannum, 2001), allowing for focused collective effort at the school level.

Mr. Tsao's (*left*) lesson, with observing teachers.

Another view of Mr. Tsao's class. Notice my cameras.

The visitors (other teachers, administrators, a district supervisor, and a university professor) sat on stools along the wall (*left*) and in the back (*3 rows deep*).



Mr. Tsao had distributed copies of his lesson plan to facilitate visitors' observation. They also used this handout during the debriefing session, afterward.

Figure 3. Mr. Tsao's lesson, an example of 'exemplary lesson development'

The school is structured to help with professional development, a required part of teachers' promotional advancement as they participate in school-based research (Huang & Bao, 2006; Ma, 1999). For example, the clustering of Chinese teachers' offices promotes collaboration (Figure 2). Additionally, Chinese teachers "undergo rigorous, multifaceted evaluations of their performance each year" (Park & Hannum, 2001, p. 2). While in China, we filmed two common forms of Chinese lesson study where the collaborative study of teaching material, collective lesson planning, consistent peer observation with debriefing, and re-teaching lessons after refinement helps teachers to grow together (Huang & Bao, 2006; Ma, 1999). In particular, we collected data (see Figures 3, 4) on exemplary lesson development (the *Keli* model) where a teacher allows colleagues, university professors, district leaders, and master teachers to observe their developing lesson, offer feedback, and—after further reflection and revision—repeat the process (Huang & Li, 2009). We also participated in the Chinese practice of inviting a model teacher to teach a model lesson with the school's faculty observing (Figure 5).

We noticed no class interruptions during our visits in Chinese mathematics classes, although security cameras were often at the front of every classroom. Chinese class sizes are unusually large in comparison to American classes (Huang & Li, 2009). Raised lecture platforms are at the front of the room. We also noticed the prevalent teaching laboratories dedicated to professional development, complete with two-way mirrors, (remote-controlled) cameras equipment, in-house editing software, and specialized ceiling microphones.

Cultural Commitments

Besides student and school commitments, China's cultural commitment to education differs from the United States'. Chinese culture is a Confucian culture that places great emphasis on education for the general masses (Zhu, 1999), perhaps best modeled by the Confucius saying *ren yi xue wei shang*—education and learning is the highest priority of the people (Liang, 2001). Confucian culture assumes all are educable and 'perfectible' (Hoyles, Morgan, & Woodhouse, 1999), rather than the Western view that education is the elites' opportunity.

China has made education one of its top national priorities, and plays a role in its emergence on the global scene. The sheer size of China's educational system is staggering: in mainland China alone, there are roughly 600,000 primary schools serving 139 million primary students (Liang, 2001). China has a centralized national curriculum (Huang & Li, 2009; Ministry of Education, 2003). With only 5-6% of students able to attend higher education (Zhu, 1999) and fierce competition for jobs, China has developed an exam-driven system to facilitate acceptance and hiring processes, which culture is continued elsewhere by Chinese immigrants in cram schools (Lin & Tsao, 1999). For the Chinese, education matters: it is also the way out of poverty.

The Chinese also have dedicated teaching-universities apart from their research institutions, with teaching-university professors working in groups with selected schools on professional development; these are non-existent in the United States.

Western mathematics has been historically shaped by Euclid's *Elements*, an axiomatic and deductive work; the Chinese mathematics tradition stems from *Nine Chapters*, a book containing a selection of everyday problems loosely arranged by category into nine sections. From these problems various rules or theories are derived for solving similar problems in a constructive and mechanistic process. The work emphasizes the algorithmic nature and applicability of mathematics (Leung, 2001). China's current culture toward mathematics is a by-product of the ancient pragmatic approach where mathematics was a tool for the common folk serving a utilitarian function through its algorithmic applicability (Leung, 1999), not a highly-regarded academic discipline reserved for the elite. In China, mathematics is for everyone; Leung (1999) argued that the Chinese cultural perception of mathematics as an applicatory and problem-solving skill rather than a philosophical mind training of the elite, has contributed to the notion of mathematics for all (Fowler, 1979). Math-phobia, now an ingrained part of American culture (Paulos, 1988), is not yet apparent with the Chinese.

Conclusion

In trying to understand Chinese mathematics success as measured on international comparisons, we recognize that many factors affect the educational process, actual instruction practice being just one. This study made no judgment about the *quality* of instructional practice in China or the US; rather, we examined the educational commitments underlying Chinese success to present a framework that organizes commitments along three dimensions, each of which is stronger for China: student, school, and culture. Some would argue that Chinese success on international comparisons is due more to these extrinsic factors affecting mathematics education—as opposed to the inherent quality of the mathematics instruction itself (Hoyles, Woodhouse, 1999; Lin & Tsao, 1999). Others would argue that superior instruction is the principal reason China succeeds (Dewey, 1960/1933; Ma, 1999; Stigler & Hiebert, 1999). We feel that student-related, school-related, and cultural factors support instructional practice (whatever that may be) to give China an edge in international comparisons.

Mr. Tsao in the debriefing meeting in the schools' teaching laboratory. Notice the raised (green) lecture platform, the technology, and the two-way observing mirrors in the back.

Later, the teachers form smaller breakout-groups (some groups are in an adjacent room). Mr. Tsao can be seen with back to camera (*far right top*).

Mr. Tsao (*upper-right*) participates in one of the debriefing breakout-groups.

The observing bay of the teaching laboratory for filming, editing and reproducing lessons.

Figure 4. The debriefing of Mr. Tsao's 'exemplary lesson' presentation

These three factors appear to be interrelated and mutually reinforce one another; for example, because Chinese parents are focused on one child's academic performance, they likewise can focus on a single school's—and multi-year cohort's—performance. By contrast, not only are American students, school factors, and culture weaker than in China, but also they appear fractured and disconnected. For example, the structure of American education means teachers can only be committed to particular students (or classes, no cohorts) for a single year, preventing any longer-term investment in a specific group of children's learning by the teacher. Similarly, American schools have no coherent, unified professional development system for their teachers, no centralized curriculum, no mechanism for sharing professional practice, and no teaching-universities working in tandem with US schools. Some American education 'reform attempts' may also be more fad than fact: China's large class sizes and lack of ability-grouping calls into question the American push for small classes and better ability-grouping as advocated by some. Similarly, Chinese teachers' model lessons—taught by an expert teacher in unfamiliar territory (someone else's class)—challenges the common American belief that the 'teacher-must-know-the-student-before-teaching'. In closing, juxtaposing student, school, and culture related commitments between China and the United States suggests that China's commitments are stronger and interrelated. We suggest further research into how other countries can strengthen their own educational commitments by learning from China's educational commitments.

Mr. Lin's (*right*) model lesson. The students are in the first five rows, the observing teachers, administrators, and university professors are behind and to the sides.

Another view of Mr. Lin's model lesson. Notice the observing teachers in the image's foreground.

Figure 5. Model lesson by Mr. Lin

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USING THE SIX INSTRUCTIONAL DESIGN PRINCIPLES TO ASSESS THE MODELLING COMPETENCIES OF STUDENTS IN MATHEMATICS CLASSROOMS

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This paper presents partial findings of a study that documented the development of modelling competencies of students working in groups. Six principles for instructional design were used as a framework for assessing modelling in a holistic way. This paper presents the findings of using these design principles to assess group modelling. In keeping with the theme of the conference, we felt it apt to include modelling since it optimises student learning in mathematics. Furthermore, this paper addresses the need for new assessment approaches in mathematics.

Introduction

Modeling has been identified as a mathematical competence (Blomhoj & Jensen, 2007, p. 47). Modeling involves solving a complex, multifaceted task. It entails working with a multipart real situation and creating a model for that situation. The general definition of a model described by Lesh and Doerr (1998) as a scheme that describes a (real life) system (p. 362). It assists in thinking about that system, making sense of it or making predictions. A model consists of elements, relationships, operations that describe how the elements interact and patterns or rules that apply to the preceding relations or operations. A model focuses on the underlying structural characteristics of a real life system being described. Therefore a model can be a description, explanation or prediction of a real life situation.

Allowing students to model is important in mathematics education. According to Niss, Blum and Galbraith (2007, p. 19) modeling can make 'fundamental contributions' to a student's development of mathematical competencies while Lingefjard (2006, p. 111) affirms that modelling is one of the main and most useful applications of mathematics.

Assessing modelling is difficult since the process students are involved in and products students create when modelling are different from traditional instruction. In an endeavour to assess students overall modeling ability, we envisaged an assessment framework that encompassed the broad aims of modeling. Modelling can be assessed by the various competencies required to model a real situation. The following modelling competencies of groups were identified and characterised in the main study (Biccard, nd) but the same competencies would apply to individuals: understanding, simplifying, mathematising, working mathematically, interpreting, validating, presenting, arguing (Maaß , 2006, p.1360, using informal knowledge (Mousoulides et al., 2008, p. 390), beliefs, 'sense of direction' and 'planning and monitoring' [Treilibs, Burkhardt, & Low, 1980, p. 52).

These competencies occur throughout the modeling process. There is a need however to mesh the assessment of these individual competencies and a holistic *ability to model* in some way. Clatworthy (1989) developed an assessment rubric that was used in a modeling course to assess modeling competence and provided each student with a feedback sheet. He concluded that the development of reliable methods for assessing modeling remains a challenge. The move away from single type assessment towards a dynamic model that could be adapted to each task is closer to what we envisaged. The shortcoming of rubrics is that criteria of achievement and levels of achievement must be re-specified for each task and often before the task commences. We wanted a framework that could be used for reflective assessment also.

English (2007) addressed the cycles of mathematical development displayed by primary school students. She used a purely qualitative approach, transcribed audio and video tapes. This is a valuable approach to assessing modeling holistically. We also wanted a framework that could be used to code any modeling task or student age/experience/ability level. Jensen (2007) suggested a multidimensional approach to assessing mathematical modeling competencies. Three dimensions of competency were explored: the degree of coverage; the radius of action and technical level. Jensen adds that these three dimensions provide vocabulary for discussing quality in performance. It was decided to include the ideas of Jensen and English and to further the discussion on assessing modeling. A qualitative, multidimensional approach informed by instructional principles would provide the necessary vocabulary to discuss quality performance and would be appropriate in evaluating the entire modeling process. We found that rewording the six instructional design principles for modeling created a suitable framework for assessing modeling abilities holistically. The framework allows one to unpack a group's modeling experience and will assist in dealing with the vast amount of data collected. More significantly it will allow one to develop assessment protocols for modeling that teachers can use in the classroom.

Instructional principles and assessment

The practice of assessing modeling is varied and based on different aspects of modeling. Researchers and educational practitioners have different needs and perspectives when it comes to assessment. According to Gijbels, Dochy, Van den Bossch and Segers (2005) "many educators and researchers have advocated new modes of assessment to be congruent with the education goals and instructional principles of problem based learning" (p. 31).

We considered it important when assessing modeling to depart from the instructional principles for modeling. It would then be possible to meet some of the needs of researchers and educational practitioners. Principles in Realistic Mathematics Education (RME) have a strong philosophical base, give better account of the realistic nature of modeling problems and crystallize the interrelationship between instructional design principles and assessment. This allows a better understanding of how mathematics instructional design principles interact with, and inform assessment principles. According to Van den Heuvel-Panhuizen (1994) in RME, teaching and assessment are 'strongly connected' and that "assessment plays a role in all stages of the teaching processes" (p. 31), while Treffers (1987) notes that a description of what is intended by instruction offers support for the evaluation of that instruction. Similarly, the Van Hiele levels of thinking (Treffers, 1987, p. 243) are theoretically based, but manifest in both teaching design and assessment. The levels inform instruction by allowing teachers to focus on which level is appropriate for the students and which activities will support students at these levels. The levels also allow teachers to assess at what level students are working. This duality in principles means that teaching, learning and assessment form a coherent whole informed by theories of learning.

Biggs (1996) also suggests a model of instruction that includes students being placed in situations that are likely to elicit the necessary learning and that assessment tasks address the same performances that are stated in the curriculum. Students have to model to be evaluated on modeling.

An assessment framework should have the following characteristics/features. It should:

- be aligned with instructional principles
- reveal both strengths and weaknesses in student thinking, and
- use authentic tasks in an authentic environment (Baxter & Shavelson in Gijbels et al., (2005, p. 32)

Our framework stems from modeling instructional principles, places students in authentic situations and allows us to gauge student thinking.

The study

Twelve students were selected for the study and worked in three groups of four students in each group. The groups were purposively selected and comprised two groups of students who were stereotyped as 'weak' (Group 1 and Group 3) in mathematics and one group of students who were stereotyped as 'strong' (Group 2) in mathematics. The students were selected based on their school results in traditional mathematics assessment. Part of the results of the full study included comparing the development of modelling competencies in 'weak' and 'strong' students. Student ages ranged from 11 to 13 years. They worked once a week for 60 minute sessions for a period of 12 weeks. They solved 3 modelling problems during this time. These students had not been exposed to modelling problems before. All contact sessions were audio recorded and transcribed. Transcriptions were coded each week in terms of the competencies identified. Transcriptions were also coded in terms of the six principles for modelling instructional design. Students had to complete a number of progress documents each week. Their written work was also scrutinised in terms of developing responses to the questions. At the end of each task students presented their solutions to the other groups. These sessions were video recorded. The following table gives a brief description of the task instructions:

Table 1. Précis of task instruction

Task 1: Big Foot taken from Lesh et al (1992, p. 123). Supporting Material: Example of footprint (size 24). Groups had to model how to find the height/size of this person and also provide a toolkit on how to find anyone's height/size from their footprint.
Task 2: Catalogue Problem taken from Lesh et al (2000, p. 635). Supporting information: Catalogue prices from 1999 and 2009. Groups had to assist another student who had written a letter stating that his pocket money is the same as his sisters' ten years ago. He needed assistance in convincing his parents that he needed more. They also had to provide an amount with suitable reasons.
Task 3: Quilting Problem taken from Lesh and Carmona (2003, p. 85). Groups were given a quilt pattern together with the size of the completed quilt. They had to provide the correct pattern pieces together with cutting and stitching guides.

Six principles for instructional design and modelling assessment

Six principles of instructional design (Lesh, Hoover, & Kelly, 1992) are used in modelling task design to ensure that tasks are indeed model-eliciting. It was found that if these principles were re-worded, they could be used as a suitable framework for assessing modelling. The six principles for instructional design are:

- The Reality principle - requires that the task encourages students to make sense of the situation based on extensions of their own personal knowledge and experiences. Re-worded: *To what extent does the group make sense of the real life situation?*
- The Model Construction principle - requires that the problem must confront students with a need to construct a model. Re-worded: *To what extent does the group construct a model?*
- The Self-Evaluation principle - requires that the task allows students to be able to judge for themselves when their responses are good enough. Re-worded: *To what extent does the group judge that their ideas, responses and models are good enough?*
- The Model-Documentation principle - the task must elicit a response to the problem that requires that students reveal their thinking about the situation. Re-worded: *What is the quality of the documentation that the group produces when modeling?*
- The Simple Prototype principle - requires that the situation is as simple as possible while still creating the need for a significant model. Re-worded: *At what level is the group progressing on a continuum of simple to complex when modeling?*
- The Model Generalization principle - does the model constructed apply only to one situation or can it be applied to a broader situation? Re-worded: *To what extent does the group develop a prototype, generalizable model?*

The instructional design principles can be re-worded so that is it the group or student that is the focus of the teacher/researcher and not the task. The same features that were deemed important for creating modeling tasks are now considered in terms of how groups of students display these features. Moreover, using these principles allows one to focus on what students are achieving holistically in their modeling endeavors. They also allow teachers or researchers to focus on the essential products and processes of modeling. The main study documented the development of modeling competencies of students working in groups. For that reason the design principles were reworded using the term ‘groups’, but the word ‘student’ can also be used when rewording the questions. A brief discussion on each re-worded question follows.

The Reality question - clustered within this question are competencies of the groups’ understanding of the problem, to what extent they use their informal knowledge, their mathematizing skills and interpreting of their model. It is important to gauge whether the ‘students school mathematics’ ability is functioning with their ‘real life sense making abilities’ (Lesh etal., 2000, p. 616).

The Model Construction question - this will cluster group competencies of mathematizing, working mathematically and presenting the solution. Group planning and monitoring will also play a significant role here. It is important to remember that the products of a modeling task include: predictions, explanations, justifications or descriptions (Lesh, Hoover, & Kelly, 1992) Therefore group models should be assessed on a combination of these products and how well they integrate elements of these in their final presentations.

The Self-Evaluation question - this will cluster competencies of their understanding, their ‘sense of direction’ (Treilibs, Burkhardt, & Low, 1980, p. 52), their use of informal knowledge and validating their model. This also means that students will sense that they have sufficiently dealt with the problem, and will not continually ask the teacher ‘is this good enough’ (Lesh et al., 2000, p. 620). It is important to ascertain whether the group detected deficiencies in their initial thinking; compared alternative ideas and judged them according to their use in the problem; integrated strengths from these alternative ways of thinking; extended or refined the interpretations and assessed the adaptations that were made (Lesh et al., 2000). From this we can see that a number of reconceptualization cycles are necessary in modeling.

The Model-Documentation question - this clusters competencies of simplifying or structuring the problem, mathematizing, working mathematically and a sense of direction. In documenting the solution process students will use various forms of representation. These forms of representation allow one insight into student thinking. Lesh and Doerr (2003) remind us that the modeling process leaves an 'auditable trail of documentation' (p. 31). Modeling tasks usually require a product (a letter, a report etc), but in leading up to the product groups will produce other documentation that can be used to judge some of their modeling competencies.

The Simple Prototype question - what progress is the group making in terms of their mathematical thinking when exposed to modeling tasks? Competencies of understanding interrelatedness of the task elements, verifying the model and argumentation are pertinent here. It is important to remember that it is not necessarily complex mathematics that is required or anticipated but rather as Iversen and Larson (2006) explained modeling as complex thinking using simple mathematics, as opposed to traditional teaching which involves simple thinking using complex mathematics.

The Model Generalization question - competencies include arguing, validating and presenting. The extent to which the group can generalize their model to the situation and to other situations is indicative of having mathematized the problem at a significant level. The ability to generalize a model concurs with a shift in the students' thinking, from thinking about the modeled situation, to a focus on mathematical relations Gravemeijer, 2002). According to Gravemeijer (2002), when this happens then the dissimilarity between the model and the situation model dissolves. A group's (or individual's) advance in creating a generalizable model will confer with Freudenthal's idea of modeling as 'organizing'.

The six questions were used over three tasks to gauge each group's competence at modeling in a holistic way. The questions could also be used to assess individual students when working on individual modeling tasks.

Results

Using the six principles for assessing group modeling means that the teacher/ researcher needs to consider both the difficulty level of the task and the age/ability/experience of the students. We were able to make the following conclusions about each group's performance for each task using the six questions after coding transcriptions and the group's written work.

The Reality question: This allowed us to consider exactly how much of their own knowledge are students prepared to bring into a situation and to what extent this assisted (or hindered) them in solving the problem. Each group made better real life sense of Task 2. This may be that the task was the best context match for them. They spent much more time discussing issues of fairness and family relations. We needed to discern between two types of real life sense. There were times when the groups used real life information to assist them (M) while sometimes it was just peripheral information (T). This question allowed us to focus more carefully on how much real life information students were using and what they were doing with the information.

M: You get some people that are short and have big feet (Task 1).
T: nowadays you pay more for the popcorn than you do for the (movie) ticket (Task 2).

The Model Construction question: This enabled us to focus specifically on the model students were constructing. In considering the groups' model constructions, we found that groups produced better models for Task 1 than the other tasks. When the need for a generalisable model was very well specified, we found that the situation model was improved. It is important that tasks are set up with the need for a generalisable model well specified. Group 2 set up a very comprehensive model for Task 2 while the other groups simplified the information given too much, which led to models that did not take enough information into consideration.. This question allowed us to focus on how well the model described or explained the problematic situation and what factors the groups focussed on and not simply if a model was generated or not. The Self-evaluation question: Groups were starting to judge their own responses without calling for the teacher by Task 2. A feature of the weaker groups is that individual members became distracted and unfocussed by each other's input and ideas. They had difficulty detecting deficiencies in their initial thinking. They were largely unsure of what a 'good' idea (G) is or a 'weak' idea (W). The following shows how this group moved from an arbitrary 'plus' or 'times' to making their first measurement.

W: Then, I think you have to plus..... (a little later)... I think we have to times it.
G: Let's measure how tall I am before we do anything.

We found the following type of comments of groups judging themselves more frequently during Task 2 and Task 3.

S: come on people, they want a method.
S: what is this going to help us?
N: That doesn't give us an answer.
N: Ya, that's a good idea.

This question allowed us to track what groups thought of what they were doing throughout the problem solution and not only their final solution. The stronger group displayed more incidents of judging what they were doing for themselves.

The Model-documentation question: Groups' working documents started off being very disorganised and haphazard. By the third task they were producing more structured working sheets with notes indicating what their measurements were about. Students initially only kept their final answers. It was after they needed to find some of their calculations and measurements that they kept more detailed explicit notes. The need to present their solutions at the end of each task was a contributing factor. We were able to observe the types of representations groups preferred to use and the meanings they gave to the representations. This question allowed us to focus on all the written documentation that students produced, not only their final presentation sheets.

The Simple Prototype question: The length of the study made it difficult to assess if students were working in more complex ways. This question is still valuable to use over a longer period of study. We did find that groups used very basic mathematics. Treilibs et al. (1980) remind us that student abilities to apply mathematics to real situations lags by at least three years after their first learning of it.

The Model Generalization question: Both Group 1 and Group 2 created strong models for Task 1 and were able to generalize their model. Strong models are models that take the full situation into account while some groups produced weaker models where they selected only a small part of the given information. Since the tasks were all based on proportional reasoning, strong models included multiplicative reasoning while weaker models included reasoning that was additive which made generalizing very difficult. Group 3's (weaker) model for Task 1:

M: Why don't we say this number (22.5 a group member's foot length) *plus what or times what equals his height?* (They wanted to reach a result of his measured height of 148).
E: 22.5 times 9 is too much…..22.5 times 6 is 136.
G: So, times 6 plus…You times it by 6, and then you plus 12.

For Task 2, all groups worked so deeply in the situation model and were not confident about their situation models that this resulted in little or no attention being given to the generalizable model. For Task 3, the process of creating the situation model largely tied in with the process for creating a generalizable model. Students have to a take elements of their situation model that are the basis of the generalizable model. These elements reflect the essence of the mathematics that is relevant in solving the problem. This question allowed us to focus on what underlying features of the problem did the students detect. It also allows one to focus on more than just the solution to the problem. If groups are able to reach a level of being able to generalize a model to other situations then they have reached much higher levels of thinking.

Conclusion

Using the six instructional design principles as a framework for assessing student modeling has proven useful to us. The questions can be used to describe broader areas of modeling ability than only focusing on individual competencies. Group or individual modeling can be gauged holistically in terms of very important modeling aims. These questions encompass the aims of modeling: that students do make sense of a real situation, that students do judge their solutions for themselves, that they do create 'auditable trails of documentation' (Lesh & Doerr, 2003, p. 31) and that students do construct models for situations. The competencies relevant to achieve positive responses to these questions and successful modeling are on a much higher mathematical thinking level than assessment in traditional instruction. The six principles will allow teachers to focus on the important products of student modeling: that students make sense of a real life problem, that they construct a model, that they judge their responses for themselves, that they document their progress, work towards more complex models and eventually create generalizable models. The questions, each in turn, cover areas such as: reality, construction, reflection, representation, thinking and prediction. The questions should not be considered in some hierarchical way since they cover areas that are distinct but integrated.

Using these six questions not only allows one to assess student modeling in a holistic way, but also allows one to find weaknesses in task construction and formulation, task-group match, student thinking as well as weaknesses in teacher interaction, practice and decisions. These exposed weaknesses, once reflected upon, will allow one to improve modeling experience. The framework provides a suitable way for unpacking of the entire modeling episode as well as providing the vocabulary to do so. The questions are broad enough to be applicable to all modeling problems not only in terms of student assessment but for practitioner reflection.

Endnotes

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VARIABLES INFLUENCING ELEMENTARY STUDENTS'ACHIEVEMENT IN MATHEMATICS

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This study investigates variables influencing student achievement in mathematics using the NCES Early Childhood Longitudinal Study (ECLS). Initial analysis of the results identified that the amount of mathematics discussion, frequency and duration of time that students do mathematics in their classes are significantly correlated with achievement scores. However, no relationships were found between student achievement in mathematics and the amount of teacher course work in teaching of mathematics, teacher employment status, teaching experience in the grade, or certification type. Parental involvement and extracurricular activities were significantly correlated to 3rd graders' achievement in mathematics.

Introduction

The National Council of Teachers of Mathematics (NCTM) and the National Association for the Education of Young Children (NAEYC) highlighted the importance of a well-designed mathematics curriculum on early childhood mathematics. Educators should ''use curriculum and teaching practices that strengthen children's problem solving and reasoning processes… actively introduce mathematics concepts, methods and language through a range of appropriate experiences and teaching strategies'' (2002, p. 4). Wang (2009) also suggests that educators aim to provide young children the opportunity to learn high quality, challenging mathematics, especially accessible to children. The purpose of this study is to investigate variables influencing elementary school students' achievement in mathematics. The study specifically addresses research question, "How are the teacher's attributes and instructional practices associated with student achievement in mathematics?" and "What variables, other than teacher attributes and instructional practices, influence elementary students' achievement in mathematics?" The study used the National Center for Education Statistics (NCES) Early Childhood Longitudinal Study (ECLS) data. The ECLS data provides descriptive data on a national basis of children's status at entry into school, children's transition into school, and their progression through grade 8. The ECLS provides a data set on how a wide range of family, school, community, and individual variables affect early school success (http://nces.ed.gov/ECLS/).

Perspectives

Teacher Qualifications and Students' Achievement

A number of studies reported the significance of teacher attributions impacting on student mathematics achievement. Teacher attributions include a teacher's personal traits, experience, qualifications, instructional approaches, and so forth. In this study, we will focus on teachers' qualifications and their instructional approaches. The No Child Left Behind (NCLB) Act mandated only highly qualified teachers be hired (USDE, 2002). This requires teachers to have minimal background qualifications, such as state certification and a bachelor's degree. Some researchers even argue that the most important predictor of student achievement is teacher quality (Darling-Hammond, 2000; Rockoff, 2004).

Existing studies on teacher attributes have, however, reported contradictory findings. Teacher attributes such as educational attainment, years of teaching experience, and area of certification have been reported both as significant and non-related factors. Xin and his colleagues (2004) found that teacher attributes have no impact on mathematics achievements and argued against using teacher credentials as the standard of selection on the teacher market. They did, however, concede that some problems existed in their data analysis process due to the limitations of the data. Goldhaber and Brewer (1996) found that teachers certified in science and mathematics had a positive influence on student performance scores, but not in History and English. The Darling-Hammond (2005) group studied 4th and 5th grade classrooms and found that certified teachers consistently produce stronger student achievement gains than do uncertified teachers. A study by Mandeville and Liu (1997) assessed the interaction effect of teacher mathematics preparation and the thinking level of 7th graders in their performance solving mathematics problems. Based on this study, teachers have an important influence on the development of students' high-level mathematics thinking skills, but an insignificant influence on low-level mathematics thinking tasks. Sweetland and Hoy (2000) also reported a positive association between school characteristics and student achievement on state test. According to their study, academic emphasis has a strong influence on teacher behavior and student achievement. They also reported that academic emphasis, reasonable goals set by teachers, teachers' beliefs in their students' ability to achieve, or an orderly learning environment, are "significant predictors of between-school differences in student achievement in both mathematics and reading" (p.697).

With higher grade students, teacher preparation and certificate type had a generally positive effect on students' mathematics learning outcomes. Despite such variations in teacher test scores, a number of studies stated that teacher experience positively affects student achievement (Goldhaber & Anthony, 2007; Jepsen, 2005). A study by Nye and her colleagues (2004) also supports, but not significantly, the impact of teacher experience and/or education on student achievement. Comfort et al (2000), however, found a positive relationship between years of teaching experience and teachers' degree and student achievement.

Teacher Instructions and Students' Achievement

Even though a teacher may have strong characteristics and high qualifications, those aspects may not match the importance of a teacher's instruction. Research shows that teachers differently implement tasks, which affect student engagement in different ways (Henningsen & Stein, 1997). Whether a teacher uses teacher directed/prescribed or student-centered instruction, their actual teaching approaches may vary. However, it is necessary to examine which forms of instruction lead to an improvement in student achievement. Examples of such forms come from a teacher's contribution to mathematics achievement, effective instructional strategies, or homework activities (Ai, 2002; Henningsen & Stein, 1997).

Studies on instructional activities focus on various aspects: different types of classroom practices aimed at improving mathematics education; specific types of instructional tasks aimed at maintaining student engagement; the use of problem-centered classrooms aimed at developing mathematics concepts and operations; learning activities at home; and different types of activities, such as worksheets, experiments, and classroom discussion (Henningsen & Stein, 1997). Other examples include: encouraging active student engagement with challenging problems; allowing sufficient time for students to wrestle with problems and to record their solutions; promoting discussion among children as they determine various ways to solve problems; and expecting and encouraging students to explain and justify their solutions to other children in the class. These are all considered as factors that influence student achievement (Thornton, 1995). Previous national and international studies on instructional practices, demographic variables, and their relationships to student achievement are based on data from NAEP and the Third International Mathematics and Science Study (TIMSS), which dealt with 4th, 8th and 10th grades. However, only a limited number of studies considered the relationship between teacher qualification, instructional practice, and student achievement in mathematics with young children. This is possibly due to the shortage of data that link student test scores to the characteristics of their teachers (Guarino, Hamilton, Lockwood, & Rathbun, 2006).

Other Factors Affecting Students' Achievement

There has been continuing interest in identifying factors that affect the improvement of mathematics education. Some studies focus on student-level factors, such as individual family issues, home resources, gender, race, and socio-economic status (SES) (Broeck et al., 2003; Comfort et al., 2000; Ma, 2005; Okpala et al., 2001; Willims & Jacobsen, 1990). Affective and cognitive factors for students are always considered by educators (Ai, 2002; Higbee & Thomas, 1999; Wilkins & Ma, 2002). Parental and peer influence on students are reported as factors with a consistent relationship to growth in mathematics achievement (Hall, 1999; Okpala et. al., 2001; Wilkins & Ma, 2002).

The research done on the influence of school governance or policy related factors on student mathematics achievement has shown mixed results. It seems that the results of these studies show fluctuations and changes in the effectiveness on student achievement in mathematics. Such differences in results may be interpreted by the combinations of factors related to school governance or policy with other variables, such as: students' backgrounds, their experiences outside of school, and teacher attributions, which are all closely interrelated (Teddlie & Reynolds, 2000).

Methods

The ECLS is a nationally representative longitudinal study that follows a sample of U.S. children enrolled in 1,000 kindergarten programs from kindergarten through Grade 5. The ECLS uses a multistage sample, which represents different socioeconomic and racial-ethnic backgrounds. The ECLS began in 1998 and concluded in 2004; 1998–1999 (Kindergarten), 1999–2000 (Grade 1), 2001–2002 (Grade 3), and 2003–2004 (Grade 5). Data on children's cognitive, social, emotional, and physical development were collected in the fall and in the spring from students, parents, teachers, and schools. Data about the children's home environment, educational practices at home, the environment at school and in the classroom, and classroom curriculum and teacher qualifications were collected from children, families, teachers, and schools (http://nces.ed.gov/ecls/). In this study, we used the data on 3rd grade students attending public schools in the four geographical locations, who were assessed in mathematics in the spring of 2002. The data were further filtered to exclude cases for which data were recorded with negative number codes (that indicate lack of relevant information or respondent's refusal to submit information). These data were used to estimate the degree to which variables of interest – teacher attributes (certification, qualifications, experience, etc.), pedagogical methods and tools (discussion time, group activities, use of worksheets, computers, etc.), and student attributes (reading scores, SES, home-environment, etc.), among others – were associated with student achievement in mathematics.

A preliminary correlation analysis revealed both statistically insignificant as well as significant correlations between the selected variables. Factor Analysis with Varimax rotation was employed to extract factors from the correlated set of variables. Using factor loadings that were at least 0.6, factor scores were constructed. These factors along with other independent variables were then used in linear regression procedures to estimate the degree to which they were associated with student achievement in mathematics.

Results

Until now, findings of various regressions of student achievement scores in mathematics showed a number of statistically significant relationships with teacher-, school-, and student-level factors. The ANOVA test result (Table 1) shows that there is a significant relationship between students' mathematics scores and variables including teacher qualification, instructional methods, gender, race, SES, and student reading scores.

Table 1. ANOVA Test summary

	Sum of Squares	df	Mean Square	F	Sig.
Regression	970985.870	20	48549.293	380.698	.000
Residual	898172.794	7043	127.527		
Total	1869158.664	7063			

Table 2 shows significance of the extracted factors on mathematics achievement scores. The p-values show that third graders' mathematics achievement scores were related to the amount of time spent discussing mathematics. This included talking to the class about their mathematical work, discussing solutions to mathematics problems with other children, and working out and discussing mathematics problems that reflect real-life situations. Also, the more often and the longer students do mathematics in class, whether as an entire class, in small groups, or individually, the higher their achievement scores were. However, teacher use of the traditional way of solving problems, testing through quizzes or manipulatives, was not significantly associated with student achievement scores. The amount of time they spend on measurement, fractions, and skill development also did not show any significant associations with student achievement in mathematics.

Table 2. Instructional approaches and students mathematics scores

Instruction	REASONING	***MATH DISCUSSION***	MANIPULATIVES	MATH SKILL	PROBLEM SOLVING	***TIME FOR MATHEMATICS***
Sig.	.360	***.024***	.426	.598	.744	***.008***

Variables related to the self-reported teacher attributes were aligned with categories in Guarino et al's NCES report (2006) – 3rd grade teaching experience in years, certification level (full vs. less than full), employment status (full vs. part time), degree status (receipt of a master's degree), and the number of courses taken in mathematics teaching (0 to 6 or more courses). As can be seen in Table 3, the teacher attributes, such as their certificate type, whether they teach reading courses or mathematics courses, their main assignment at school, or years they taught this grade are not significantly associated with student achievement in mathematics.

Table 3. Teacher qualifications and students mathematics scores

Teacher qualifications	TEACHER'S CERTIFICATION TYPE	TEACHER'S TEACH-READING COURSES	TEACHER'S TEACH-MATH COURSES	MAIN ASSIGNMENT AT SCHOOL	YEARS TAUGHT THIS GRADE
Sig.	.634	.781	.417	.313	.087

Table 4. Students variables and their mathematics scores

Students Variables	*READING IRT SCALE SCORE*	*SES*	*GENDER IS FEMALE*	RACE				*PRIMARY HOME LANG. IS ENGLISH*
				BLACK AND NON-HISPANIC	*ASIAN*	WHITE	HISPANIC	
Sig.	***.000***	***.000***	***.000***	***.000***	***.007***	.072	.594	***.013***

The study also analyzed the relationship between students' attributes and their mathematics achievement scores (Table 4). We find that students reading scores, gender, SES, race, and primary home language are strongly associated with their mathematics scores.

We also used Pearson's correlation to investigate the influence of other variables on students' mathematics achievement scores. In particular the study analyzed factors related to students scores in reading, parental involvement, student's extracurricular activities, and use of computers. Partial results of the analysis are presented in Table 5. As can be noted, mathematics scores are strongly correlated to their reading and general knowledge scores. Factors on parental involvement such as how often parents attend open house, PTA/PTO, and/or school events have small but statistically significant correlation coefficients with mathematics achievement scores. However, how often parents contacted the school or how often parent-teacher conferences occurred were not significantly correlated. Further, student mathematics achievement scores were significantly correlated, albeit with small correlation coefficients, with their non-subject related extracurricular activities, such as if they volunteered, visitations to the library or a museum, attend a play, concert, or show, and participation in athletic events. Visiting the zoo or aquarium, using a home computer, or the frequency of a child's computer use did not show significant correlation with their mathematics achievement. As can be seen in Table 5, variables statistically associated with mathematics scores were also significantly correlated with students reading scores and general knowledge scores.

Discussion

The research questions addressed in this study ask whether or not instructional practices and teacher attributes are related to student achievement scores, as well as what other variables influence on student achievement in mathematics. The findings show that organizational factors have more impact on student achievement than do content-specific factors. Spending more time on mathematics and discussing mathematics were positively associated with mathematics achievement. While we may now assume the significance of time spent for mathematics and mathematics discussion, we are reluctant to do so with other instructional factors (e.g., manipulatives, problem solving, and mathematics skills). This result seems to imply that students may learn better by understanding or through the process of meaning making. That is, after learning concepts or skills with various instructions, student may need time to process and internalize their learning especially through discussions. This suggests the need for a more detailed study of frequency and duration of other forms of instruction and the use of discussions.

The available data do not show how much time teachers actually spent on mathematics skills, problem solving or manipulatives and the combined approaches between those instructions and discussions.

Table 5. Pearson Correlations

(Parental involvement; extracurricular activities; students mathematics scores)

		T-Score		
		Reading	Math	General Knowledge
READING T-SCORE	Pearson Correlation	1	.741(**)	.836(**)
	Sig. (2-tailed)		.000	.000
MATH T-SCORE	Pearson Correlation	.741(**)	1	.699(**)
	Sig. (2-tailed)	.000		.000
GENERAL KNOWLEDGE T-SCORE	Pearson Correlation	.836(**)	.699(**)	1
	Sig. (2-tailed)	.000	.000	
PARENT CONTACTED SCHOOL	Pearson Correlation	-.013	-.027	-.074(*)
	Sig. (2-tailed)	.657	.363	.013
HOW OFTEN ATTEND OPEN HOUSE	Pearson Correlation	.112(**)	.117(**)	.142(**)
	Sig. (2-tailed)	.000	.000	.000
HOW OFTEN ATTEND PTA/PTO	Pearson Correlation	.045	.096(**)	.061(*)
	Sig. (2-tailed)	.131	.001	.040
HOW OFTEN PARENT-TEACHER CONF	Pearson Correlation	-.024	-.013	.006
	Sig. (2-tailed)	.419	.671	.853
HOW OFTEN ATTEND SCH EVENT	Pearson Correlation	.118(**)	.148(**)	.175(**)
	Sig. (2-tailed)	.000	.000	.000
HOW OFTEN VOLUNTEERED	Pearson Correlation	.106(**)	.115(**)	.136(**)
	Sig. (2-tailed)	.000	.000	.000
VISITED THE LIBRARY	Pearson Correlation	-.137(**)	-.130(**)	-.134(**)
	Sig. (2-tailed)	.000	.000	.000
GONE TO A PLAY, CONCERT, SHOWS	Pearson Correlation	-.116(**)	-.098(**)	-.142(**)
	Sig. (2-tailed)	.000	.001	.000
VISITED A MUSEUM	Pearson Correlation	-.081(**)	-.067(*)	-.090(**)
	Sig. (2-tailed)	.007	.024	.003
VISITED A ZOO, AQUARIUM	Pearson Correlation	-.002	-.018	-.018
	Sig. (2-tailed)	.935	.535	.540
HAVE HOME COMPUTER CHILD USES	Pearson Correlation	.(a)	.(a)	.(a)
	Sig. (2-tailed)	.	.	.
FREQ CHILD USES COMPUTER	Pearson Correlation	.068(*)	.058	.059(*)
	Sig. (2-tailed)	.023	.052	.047
PARTCP IN ATHLETIC EVENTS	Pearson Correlation	-.187(**)	-.206(**)	-.275(**)
	Sig. (2-tailed)	.000	.000	.000
TAKES NONENG LNG INSTRUCTION	Pearson Correlation	-.056	-.039	-.028
	Sig. (2-tailed)	.059	.186	.345

** *Correlation is significant at the 0.01 level (2-tailed)*
* *Correlation is significant at the 0.05 level (2-tailed)*

This study also provides us insights on the effects of different types of parental involvement: parental participation in open house, PTA/PTO, school events, or parent teacher conference.

Although the importance of parental involvement in student achievement has been pronounced, some studies have not agreed on the effect of types and frequency of parental involvement on students' achievement (Desimone et al. 2004; Ho & Williams, 1996). We assume that there is a need of considering multiple aspects of diverse parent populations such as language, culture, etc. Specific student factors such as visitations to concerts, museums etc., participation in extracurricular activities, etc., have significant influence on mathematics achievement scores. It is possibly due to students' learning motivation developed from such activities (Stefan, 2004).

The findings show that a teacher's experience teaching 3rd grade impacted on student achievement, whereas no direct relationship was found between courses teachers took or their employment status, and student achievement in mathematics. Unfortunately, the survey questions were not specific enough to discern whether or not a teacher's mathematical pedagogical and content knowledge are associated with student achievement. Based on our results regarding teacher certification type, a more focused and in-depth investigation ought to be done, especially on the question of subject-specific training, such as the time a teacher spends learning mathematics, what they learn or how they were taught. According to Rivkin et al. (2005), the effect of teacher qualifications decreases during the first 3-5 years in the classroom, because a novice teacher's teaching skills improve around the 5th year. However, the effect of teacher experience on student achievement may vary due to teacher qualities or abilities (Kukla-Acevedo, 2008).

The NCES collected the 5th grade data in 2004 and the 8th grade data in 2007 to see how the same group (Kindergarten Class of 1998-99) of student progressed through 8th grade. It would be beneficial for teacher educators to see how instructional approaches and teacher attributes influence student mathematics achievement in middle school by analyzing the 5th and 8th grade data.

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ENGAGING SECONDARY STUDENTS AND TEACHERS IN PROFICIENCY-BASED ASSESSMENT AND REASSESSMENT OF LEARNING OUTCOMES

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According to a recent report published by The Urban Institute, nearly 1.3 million of the roughly four million ninth graders attending public schools each year will fail to graduate (Swanson, 2005, pg. 8). The Consortium on Chicago School Research (CCSR) concluded that students maintaining at least a B average in their 9th grade year have more than a 95% chance of graduating from high school. Conversely, 9th grade students with less than a C average have a greater likelihood of dropping out of school than of graduating. Clearly, 9th grade is a pivotal year for students. The PARLO study targets 9th grade students taking Algebra 1 or Geometry and will research the efficacy of a system that emphasizes students attaining “proficiency or better” on a limited set of high-value, standards-based learning outcomes. Rather than using a traditional letter grade system, PARLO establishes a new proficiency-based approach to learning and assessment. If a student does not attain proficiency on an outcome, he/she will have subsequent opportunities (reassessment) to learn the material that he/she has not yet mastered and demonstrate proficiency. For the purposes of this study, the definition of proficiency consists of three degrees of mastery: proficient (P) when a student grasps the major concepts taught, not yet proficient (NYP) when he/she demonstrates minimal understanding and high performance (HP) when he/she demonstrates an almost flawless understanding of the learning outcome. Learning outcomes are an explicit description of “what a learner should know, understand, and be able to do as a result of learning” (Bingham, 1999, p.4). Proficiency levels along with explicit learning outcomes help to provide teachers with coherent evidence of student mastery.

This study employs a cluster randomized trial of 44 schools. The schools will be assigned to either a: (1) treatment group using the PARLO system, or (2) control group in which students are taught and assessed using a traditional model. Randomization will be done at the school level to simulate a naturalistic setting where an entire school employs this type of assessment system. The research questions under investigation are: (1) Does the use of PARLO lead to increases in students’ achievement and engagement in mathematics? (2) Does PARLO lead to increased student interest in pursuing advanced mathematics courses and/or STEM content? (3) Will teachers’ regular and sustained use of PARLO foster changes in teachers’ conceptions about: (a) how students learn mathematics, and (b) the capacity of students to achieve proficiency?

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RECONCILING STUDENT THINKING AND THEORY: THE DELTA LEARNING TRAJECTORY AND THE CASE OF TRANSITIVITY

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An important aspect of the learning trajectories (LT) construct is their dual theoretical and empirical nature (Confrey et al., 2009). While literature and theoretical perspectives are the basis of LTs, empirical data are necessary to ensure their validity through "iterative cycles of empirical testing and theoretical revision and refinement" (Duncan, 2009, p.607). The Diagnostic e-Learning Trajectory Approach (DELTA) LT validation process began with a synthesis of research articles. Through conducting clinical interviews and teaching experiments, a task design matrix with proficiency levels was created and outcome spaces were written based on student interview behaviors. Both pilot and field test items were created to assess these levels. Finally, rubrics were developed based on an examination of all responses to the field test items. Each step of this validation process led to revisions and refinements of the LT.

This poster presents the process by which the DELTA research team validated a LT for equipartitioning. Each level of the trajectory describes the development of young children's fair sharing of collections to adolescents' formalization of the partitive fractional quotient construct. Through analyzing data from clinical interviews and pilot item responses, this poster charts our progress in achieving greater clarity in how children's abilities to fairly share a single whole using multiple methods relates to their emerging understandings of the equivalence of non-congruent parts. It describes how analyses of pilot item responses revealed multiple ways that children argued for equivalence between two figures, including composition/decomposition and transitivity. These observations were incorporated into a revised LT and additional items were field tested. An analysis of responses to these items during rubric construction revealed that multiple methods are significant if students can justify them and recognize transitivity within and across figures. Each cycle of revisions helped the team to infer about students' thinking and resulted in an incorporation of those inferences in the LT. Ultimately, validation takes time but yields a more robust description of how children's ideas develop.

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USING PORTFOLIO ASSESSMENT IN MATHEMATICS COURSES FOR ELEMENTARY TEACHERS

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This poster display will present all aspects of the portfolio assessment program including course requirements, logistics, sample tasks, rubrics, student work and journal prompts. The intent of using this strategy in these content courses is to indicate to students clearly what is valued in their learning. As Clarke suggests: "It is through our assessment that we communicate most clearly to students which activities and learning outcomes we value" (Clarke, 1989 p.1).

As the students participate in the sequence of courses and create their portfolios they realize that what is expected and what is valued is correct solutions of the problems posed, but more importantly providing evidence of their understanding of the “Big Ideas” in the course and how the problems relate to those ideas. Clear and complete communication is stressed using appropriate mathematical vocabulary and terminology.

Portfolio assessment is not without it’s challenges and concerns and these will be shared and discussed. Evidence gathered from four consecutive years of portfolios will also be shared. This evidence indicates that students are focusing on their thinking and understanding of the mathematics content ideas rather than simply expecting to get a correct answer or do the right procedure. There is a shift toward self-assessment and improved communication and use of appropriate mathematical terminology.

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Chapter 5: Classroom Discourse

Research Reports

Brief Research Reports

Posters

COMPARING INTERACTION AND IMPROVISATIONAL COACTION IN MATHEMATICAL COLLABORATION

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In a recent paper (Martin & Towers, 2009) we introduced the notion of improvisational coactions as a process through which mathematical ideas and actions, initially stemming from an individual learner, become taken up, built upon, developed, reworked and elaborated by others, and thus emerge as shared understandings for and across the group. We suggested that improvisational coactions are a specific kind of interaction, but with particular characteristics. In this paper we both illustrate and elaborate on the distinction between these two ways in which a group can work together and highlight the different kinds of mathematical understandings that emerge from these.

Introduction

The research reported in this paper forms part of our ongoing research program concerned with the nature of mathematical understanding and how it might be theorized and characterized. In recent years our specific focus has been the phenomenon of collective mathematical understanding—the kinds of learning and understanding we may see occurring when a group of learners work together on a piece of mathematics. We have characterized the growth of collective mathematical understanding as a creative and emergent improvisational process and illustrated how this can be observed in action (Martin, Towers, & Pirie, 2006; Martin & Towers, 2009; Martin & Towers, 2007; Towers & Martin, 2009). Here we present data that help to illuminate a distinction between interaction and improvisational coaction.

Theoretical framework

Our research draws on theoretical framings from two main perspectives—mathematical understanding (particularly the work of Pirie & Kieren, 1994) and the study of improvisational group performances in jazz and theatre (e.g., Berliner, 1994; Sawyer, 2003). While on the surface the phenomena of mathematical understanding and group improvisation may appear strange bedfellows, our initial explorations of the nature of collective mathematical understanding (Martin & Towers, 2009; Martin & Towers, 2007; Martin, Towers, & Pirie, 2006) have suggested that the fields of jazz and theatre improvisation have much to offer theoretically to an advancement of our understanding of how interacting groups coalesce on particular ideas that move the action (be it jazz or mathematical problem-solving) forward.

Improvisation is broadly defined as a process of “spontaneous action, interaction and communication” (Gordon Calvert, 2001, p.87). It is a collaborative practice of acting, interacting and reacting that emphasizes participants’ “ability to integrate multiple, spontaneously unfolding contributions into a coherent whole” (Ruhleder & Stoltzfus, 2000, p. 186). Improvisational theorists (Becker, 2000; Berliner, 1994; Sawyer, 1997, 2003) have proposed particular characteristics of the improvisational process and we have adapted and elaborated these in the context of collective mathematical understanding (Martin, Towers, & Pirie, 2006). In a recent paper (Martin & Towers, 2009) we developed the idea of improvisational coactions and its relationship to the growth of mathematical understanding at the level of the group. We chose the term *coaction* as a means to describe a particular kind of mathematical action, one that whilst obviously in execution is still being carried out by an individual, is also dependent and contingent upon the actions of the others in the group. Thus, improvisational coacting is a process through which mathematical ideas and actions, initially stemming from an individual learner, become taken up, built upon, developed, reworked and elaborated by others, and thus emerge as shared understandings for and across the group, rather than remaining located within any one individual.

Methodology and methods

We worked with a number of student teachers who were in their first year of a two-year Bachelor of Education degree program at a university in Canada. They were preparing to be elementary school teachers, and none of those participating were, at the time, specializing in mathematics. The volunteer students were invited, during a series of lunch breaks, to be videotaped as they worked on some mathematical problems. They were encouraged to form their own groups of three or four and to choose tasks from a booklet supplied, which contained nine tasks covering different areas of mathematics ranging from ‘‘Fractions, Decimals and Percentages’’ to ‘‘Mathematical Argument’’ (Teacher Training Agency, 1998). The task booklet was produced in the United Kingdom with the intention of helping elementary teachers determine their development needs in mathematics. We chose this set of tasks for this study as we felt they offered an appropriate level of challenge to the student teachers, and were also presented in a way that would stimulate discussion and collective action. The main purpose of the data collection was to allow us to elaborate the theoretical notion of improvisational coactions through exploring it in greater depth than we had previously, and thus we deliberately chose tasks and participants that would allow us to do this.

The analysis of the video data drew on the approach proposed by Powell, Francisco, and Maher (2003). This is a seven stage process involving (1) simply viewing the tapes in their entirety; (2) describing the video-data through writing brief, time-coded descriptions; (3) identifying ‘critical events’ with regard to our interest in improvisational coaction; (4) transcribing relevant excerpts of data relating to these critical events; (5) analyzing and coding of the critical events using our existing definition and description of improvisational coactions together with elements of the Pirie-Kieren Theory; (6) identifying and constructing a storyline to “discern an emerging and evolving narrative about the data” (Powell, Francisco, & Maher 2003, p. 430); and (7) composing narratives, and for us the linking of those narratives to characteristics of coaction to further develop our theoretical notion of improvisational coactions.

Results

In this paper we are only able to share a few brief excerpts from the video data of two groups of students. However, some of our comments that follow are drawn from the analysis of the larger data set, and not merely these short transcripts. The first group has three students known here as Mary, Shauna, and Kay. The second group, while retaining Mary and Shauna, has Kay replaced by Hillary (this was not a deliberate part of the research design, but a result of Kay being absent).

Extract One

The students in the first group (Mary, Shauna and Kay) have chosen the following task from the booklet, and are trying to determine what the correct answer should be:

Children were asked to find the original cost of an item which had been reduced in a sale by 15% to £850. One child did the following calculation:

$$£850 \times \frac{15}{100} = £122.50$$

$$£850 + £122.50 = £977.50$$

Describe why the child has arrived at an incorrect solution and calculate the correct answer.

Just prior to the extract below, the students have spent a few minutes playing with the question. Initially they thought the child had calculated the £122.50 correctly, but that they should have subtracted this. However, they then realised that the child had incorrectly found 15% of the sale price. Kay has pointed out to the group that 85% equals £850 and that they need to calculate what 100% would be. We join the group as they begin working on this:

1 Kay: I've got to write it down in order to…[*she looks for a pencil in her bag*]
2 Shauna: You can write on there.
3 Kay: I've got to see it. I'm a visual learner. I can't do it all in my head. [*she starts writing out the calculation*]
4 Shauna: Neither can I but I have no idea what to write down.
5 Kay: One hundred equals…[*talking as she writes*]
6 Mary: You know what, all we have to say is why she didn't come to the correct answer, well we calculate the correct amount.
7 Shauna: Yeah.
8 Kay: Eighty-five equals. Eight-fifty. Eighty five x equals…[*she is still writing*]
9 Shauna: It looks like the child's done a great job [*she looks at Mary*]. I would have done the same thing. Because you just go…
10 Mary: I remember doing these and that's how…
11 Shauna: Usually like if a person's gets it wrong thing it's because they go by the information that's in the question.
12 Kay: Okay. A thousand pounds is the answer [*she looks up but not at Mary or Shauna*].
13 Mary: Oh. Yeah?
14 Shauna: Instead of what's on…
15 Kay: I would figure a thousand pounds would be the answer. Now does that work?
16 Shauna: But that's like pretty close to that. Like, did you round up? [*they are looking at Kay's written work*]
17 Kay: No.
18 Shauna: Okay.
19 Shauna: So but are they going to be using x's at this age?
20 Mary: We don't know what grade this is.
21 Shauna: Like what age is this?
22 Kay: That's just how I would do it.
23 Mary: A thousand. So what's fifteen percent of a thousand?
24 Kay: Okay, so in order to work backwards, fifteen percent of a thousand would be hundred and fifty. [*Mary and Shauna are watching Kay as she explains to them*]
25 Mary: So it's not quite.
26 Kay: Fifteen percent of one thousand is one fifty so if you take a thousand minus one fifty you get eight hundred and fifty and therefore it's right.

Although all three students are talking and contributing throughout this episode, there is not a sense of collaborative mathematical working taking place. Kay sets out her need to write down the calculation she is performing and then proceeds to articulate this as she works (lines 5,8,12,15). Shauna and Mary do not engage with her mathematical working—instead there is something of a conversation between them about the problem generally. They seem to be simply waiting for Kay to finish and to tell them the answer. When she does this, there is no great surprise from Mary (line 13) who seems happy to accept Kay's solution. Kay's question (line 15) seems posed to herself rather than to the group and does not seem to invite (or receive) a response. Although Shauna questions her a little (line 16 - thinking that the answer is merely the incorrect solution rounded up – suggesting she doesn't really follow what Kay has done), Kay dismisses this saying "that's just how I would do it" (line 22). There is an attempt by Mary (line 23) to engage the group in checking the answer but Kay quickly gives just the answer, followed by the final statement of "therefore it's right" with no suggestion that she needs the group to agree with her.

Extract Two

In the next two extracts the second group (Mary, Shauna and Hillary) have chosen a task that requires them to classify a number of triangles drawn on 9 pin geoboards by their side length properties and to label them as equilateral, scalene or isosceles. There is a short pause as they seem to consider where to start:

[*Shauna looks at Mary, who nods back.*]
27 Shauna: It's all coming back to me.
28 Hilary: I don't remember scalene or isosceles.
29 Shauna: Isosceles is this, okay? [*She is drawing.*] Where two are equal?
30 Mary: Yeah.
31 Shauna: Equilateral is when they're all equal?
32 Hilary: Hm hm...
33 Shauna: And scalene is?
34 Mary: They're all wonky?
35 Hilary: This must be scalene. [*Pointing to triangle in Figure 1.*]
36 Shauna: Okay.
37 Hilary: When it has one, one sss…[*pause*]
38 Mary: One longer?
39 Shauna: Isos, eq and scale. So the scale, none of them [the sides] are equal?

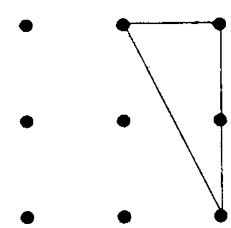

Figure 1

Although the initial statement of Shauna (line 27) suggests that properties of triangles and their associated names are not new to them, none of the three students seems immediately able to recall and restate the definitions for the three different kinds of triangles. No one student is able to offer a complete definition and instead the three students each offer what can be characterised as partial fragments of an image for the scalene triangle. Mary talks of it as being "wonky" (line 34), Hilary and Mary both develop the idea of "one longer" side (lines 37 and 38), whilst Shauna extends this idea to the conception of "none of them [the sides] are equal" (line 39). There is a building on what individuals offer, and the image emerging is a function of the discourse and is not simply being articulated by any one student.

Extract Three

Following the extract above, they begin work on the task, and start to label each of the triangles as equilateral, isosceles, or scalene using their definitions. However, they decide that in the case of some triangles they need to measure the sides to be able to determine their type. But they do not have a ruler, and instead turn to examining the relationship between the dots (what the students refer to as pins or dots or pegs) and the sides of the triangles. Just prior to the extract below they have been using a rule they have generated that states that the distance between any two neighboring pins (even diagonal neighbors) is equal, but they are now unsure whether this is right.

40 Mary: I don't know...
A pause
41 Shauna: But, maybe, yeah...
Another pause
42 Mary: Like, there's something with it going across there that messes it up. Why can't we think of it?
43 Hilary: 'Cos there's always...
44 Shauna: Okay. I got it, I got it. If it's like...
45 Hilary: [*Laughing*] Are you making this up?
46 Shauna: No, I'm just thinking. Say for like a right angle. Okay, it made sense and now I can't say it out loud [*pause*]. I think its because this is obtuse, like...
47 Hilary: Because, wait. Oh my goodness, don't you remember? *a* squared plus *b* squared equals *c* squared?
48 Shauna: Yeah.
49 Hilary: That means these are not. *a* squared plus *b* squared equals *c* squared. So this cannot be equal to two. This is two squared plus two squared, do you see what I'm getting at here? This isn't the same distance as this. [*She is now talking about the hypotenuse and one other side of the triangle in Figure 2 again.*]
50 Shauna: Well, it worked [*laughing*]. It worked!
51 Hilary: So this is not.
52 Shauna: Yeah, cos you didn't take the square root.

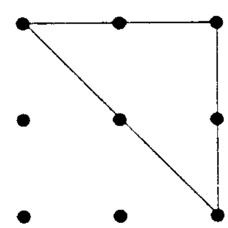

Figure 2

Just prior to the start of the extract above they have realised that their current image and criteria for classifying the triangles are incorrect. However, none of the three is initially able to offer a new image, idea, or starting point, evidenced by the pauses in speech (lines 40-42). Suddenly though, Shauna interjects, and although she cannot easily articulate her thinking she states what seems to be an important idea "say for a right angle" (line 46). Her recalling of this prompts Hilary, who builds on what Shauna seems to be thinking (even though it was not clearly articulated), to recall Pythagoras' Theorem, recognizing that it might be appropriate to use in the context of this problem (line 47). This concept is then brought into their current thinking as another image that they can use in determining the type of triangle at which they are looking. The process of finding, recalling and restating Pythagoras' Theorem is an ongoing one, with individual suggestions and thoughts acting to spur further contributions from the group that collectively build on these.

Discussion

In reference to our first data extract we suggest that while the three students are collaborating and talking together (as they did throughout the whole hour they worked together) a collective understanding does not emerge from this. At a superficial level, the three students do appear to have worked together, and to have participated in the finding of the answer. However it is the nature and form of the collaboration which may (or may not) give rise to the growth of mathematical understanding at the collective level.

We suggest that Mary, Shauna and Kay are interacting rather than coacting and that what we see is a set of individual understandings that, although being compatible with one another, are never truly shared or distributed. There is no sense of collective purpose or any interweaving of partial ideas being offered, picked up and built on by others. In the case of Mary and Shauna this is particularly significant, as we would question whether at the end of the extract their understanding of percentages has grown. Although the 'group' has solved the problem we would suggest that Mary and Shauna's individual understandings are limited and likely to be some kind of memorized, procedural replication of Kay's approach and method of solving the problem. Improvisational literature refers to the notion of a "driver" (a dominant role wherein one member of an improvisational group dominates the emerging direction), the presence of which typically flattens and ultimately halts an improvisational performance. Kay acts as the 'driver' throughout this extract (though this should not be taken as a criticism of Kay, her mathematics, or her intent) and removes the opportunity or need for the kinds of collective interdependence we see in the second two extracts.

In the second two extracts we see something very different occurring. Here, no one student acts as the driver, instead all contribute to a collective understanding that we see emerging from the interweaving of their suggestions and ideas. We characterize Mary, Shauna and Hillary as improvisationally coacting. There is a sense of unpredictability about the pathways their collective mathematical actions, and emerging understandings, will follow, and it is this that contributes to the improvisational nature of the process. They act on the ideas of each other, building on what is offered to collectively work together in order to have (or even re-have) a useable image. This understanding emerges from the way that the offered ideas are starting to intertwine and be developed through the discourse. There is a sense in the extracts that no one student simply wants (or is able) to tell the others what to do, or to merely state a mathematical idea without expecting some response. Equally, those listening to the idea seem to accept their responsibility to do something with what is offered, and not merely receive it (in the way Mary and Shauna do in the first extract). The way in which the second group is able to interweave fragments of each individual's knowing, to allow a shared (rather than taken-as-shared) image to emerge from their coactions, is what enables their collective mathematical understanding to grow and ultimately enables them to successfully complete the task.

Conclusion

Our analysis of the extracts above highlights the distinction we wish to make between interaction and improvisational coaction, and the importance of observing *how* a group collaborates—even when all its members appear to be participating. Coaction emphasizes the notion of acting with the ideas and actions of others as they are offered to the group, and places as much responsibility on those who are positioned to respond to an action as on the originator. Interaction we see as being more of a process of acting on the ideas of another in a reciprocal or complementary way. Such interaction may allow for separate, individual traces of mathematical understanding to be observed, but groups who work in this way seem to lack the capacity (or responsibility) to build upon and add to the emerging collective image and therefore diminish the capacity for collective mathematical understanding to develop and be observed.

As noted earlier, in the first extract we remain concerned about whether Mary and Shauna's individual understanding has grown through the observed episode. In contrast, we suggest that collective image making and the growth of collective mathematical understanding were powerful factors in enabling Mary, Shauna and Hilary to be successful in solving the task in the second extract, and also likely to enable their individual understandings to also grow. Groups that improvisationally coact, as in the case of Mary, Shauna and Hilary, have the capacity to work mathematically, and to solve problems, in ways that individually may not be possible.

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EXPLORING THE RELATIONSHIP BETWEEN WRITTEN AND ENACTED CURRICULA: THE USE OF MATHEMATICAL WORDS

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This paper reports on part of a study that examined the relationship between written and enacted curricula. The purpose of the study was to investigate the use of a discursive framework to explore fidelity of implementation focused on the mathematics presented in the curricula. Both quantitative and qualitative differences in mathematical word use in the curricula were revealed. In particular, opportunities for objectification and using a wide variety of types of fractions were more present in the written curriculum. Here, I focus on the use of "fraction" and fractions themselves to provide an example of the utility of the framework.

Introduction

Since the publication of the first set of standards by the National Council of Teachers of Mathematics (1989) and subsequent development of curricula to support their vision, there have been calls for evidence that the standards and their associated curricula are improving student learning. An important factor in assigning credit or blame for student mathematical learning to the standards or curricula involves documenting the extent to which these curricula are being used in classrooms in ways "faithful" to their original intent. It is only after some level of fidelity of implementation has been established that credit or blame can be assigned. In two key documents that examined studies of these curricula's effectiveness (National Research Council, 2004; Senk & Thompson, 2003), the need for attention to implementation was strongly emphasized. Researchers have documented the enactment of curricula in a variety of ways, including asking teachers which chapters have been completed, observing classrooms for evidence of standards-based practices, and collecting textbook-use diaries from students and teachers (e.g., Tarr, Chavez, Reys, and Reys, 2006). What seems absent from these methods is attention to the mathematics presented in the written curriculum and that which is enacted in the classroom (i.e., the enacted curriculum).

In this paper, I present a small part of a study in which I examined the relationship between the written and enacted curricula in one classroom. The purpose of the study was to investigate the usefulness of a discursive framework for revealing the mathematics. I argue that a detailed analysis of the mathematical discourse revealed relationships between the written curriculum and what a teacher and her students enacted in the classroom.

Theoretical Framework

In this study, I conceptualize the written and enacted curricula as discursive texts, and define *mathematics* as discourse about mathematical objects and *mathematical learning* as a change in participation in mathematical discourse. This allowed me to compare the curricula using a discourse analytic framework, the Commognitive framework (Sfard, 2008). Commognition (created by merging *communication* and *cognition*) treats communication (interpersonal exchange) and cognition (intrapersonal exchange) as two forms of the same phenomenon. It was

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

developed to emphasize the close relationship between these two processes. Commognition suggests a detailed analysis (i.e., a search for patterns) of the use of discursive features of mathematics, including (1) Mathematical Words, (2) Visual Mediators, (3) Endorsed Narratives, and (4) Mathematical Routines. Mathematical words include those that signify mathematical products (e.g., triangles) and processes (e.g., multiplying). Visual mediators are artifacts created for the primary purpose of mathematical communication, including but not limited to algebraic symbols, diagrams, and graphical representations. Narratives include any text, spoken or written, which is framed as a description of objects, of relations between objects or processes with or by objects, and which is subject to endorsement or rejection (i.e., being labeled true or false). Finally, routines are repetitive characteristics of mathematical discourse. A comparison of these features of mathematical discourse in the written and enacted curricula provides information about the fidelity of implementation of the curricula as one would expect to find a relationship between these features in the textbooks and the classroom.

One phenomenon that is particularly sensitive to this type of analysis is objectification. Objectification is defined as "a process in which a noun begins to be used as if it signifies an extradiscursive, self-sustained entity (object), independent of human agency" (Sfard, 2008, p. 412). The process consists of two closely related sub-processes: reification and alienation. Reification is the replacement of talk about processes with talk about objects and alienation is the use of discursive forms that present phenomena in an impersonal way, as if they were occurring of themselves, without the participation of human beings. These two processes, when taken together indicate that what was previously something to "do" becomes a discursive "object". Mathematical and scientific discourses are particularly dependent upon objectification for their successful evolution. The objectification of fractions occurs when an individual reifies the division of one into four equal parts into an object (that can be marked as "$\frac{1}{4}$," "0.25," etc.) and "encapsulates" three of these one-fourths into a single entity, "$\frac{3}{4}$". Objectification in school mathematics is more important today than ever as more students are expected to take advanced mathematics courses and therefore to operate on a larger domain of numbers.

Methods

I used *Connected Mathematics 2* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006) as the written curriculum for this study, in particular, a five-day Investigation, *Multiplying with Fractions*, included in *Bits and Pieces II: Using Fraction Operations* (the second of three Units in grade 6 that address fractions). When "Investigation" is capitalized in this paper, I am referring to this particular Investigation. The union of the teacher guide and the student guide served as the written curriculum. The enacted curriculum analyzed for this study took place in a sixth grade classroom in a rural community in the Midwest on five consecutive days in October 2006. The class was heterogeneous in mathematical ability (i.e., the students were not tracked). The teacher of this particular class was a veteran *Connected Mathematics* teacher. She had used *Connected Mathematics* for 13 years, attended and conducted professional development for *Connected Mathematics*, and verbally endorsed the curriculum. The union of student and teacher discourse, both oral and written, served as the enacted curriculum.

In the larger study, the written and enacted curricula were compared in terms of the four mathematical features included in the Commognitive framework. However, in this paper, I focus only on a comparison of the mathematical word use, and even more specifically, on the use of “fraction.” This includes any derivatives of the word (e.g., fractions, fractioning) and fractions themselves (e.g., “two thirds”, “$\frac{2}{3}$”).

Findings

Uses of “Fraction” in the Curricula

Four common categories of the use of “fraction” and its derivatives (e.g., “fractioning”) emerged from the written and enacted curricula, including fraction as a number, a part, an adjective, and a verb. Table 1 provides examples of each category of use in the curricula.

Table 1. Examples of Categories of “Fraction” Use in the Written and Enacted Curricula

Category	Written Curriculum	Enacted Curriculum
Number	“The strategy introduced in the Getting Ready involves changing the form of mixed numbers and whole numbers so students can operate in the same way as when both factors are **fractions**.” (TG, p. 75)	S: “One thing I don't get is, you know how when we said eight times seven and we showed like the picture. Like, that wouldn't work with **fractions**, would it?” (Day 2)
Part	“What **fraction** of the goal does Nikki raise?” (SG, p. 36)	S: “She, it said what **fraction** of the pan she bought, and she would buy two sixths of the pan because if you split these into thirds you'll get two sixths.” (Day 1)
Adjective	“Sometimes, they have to find a **fractional** part of another fraction.” (SG, p. 32)	T: “You can buy any **fractional** part of a pan of brownies and pay that fraction of twelve dollars.” (Day 1)
Verb	-----	T: “Maybe you're going to have a picture of something and be **fractioning** off parts or whatever.” (Day 4)

1. Dashes (i.e., “-----“) indicate that the category of use of “fraction” is not present in the designated curriculum.
2. “TG” indicates the Teacher Guide and “SG” indicates the Student Guide.

I distinguished between the “number” and “part” categories by determining which word (i.e., “number” or “part”) could replace “fraction” in each sentence. Even though “fraction” is used as a noun in both of these categories, when “fraction” is used as “part,” it is used only in specific types of phrases (e.g., "**fraction of** the whole pan"). Therefore, it is only when "fraction" is used as a number that it is objectified and discussed as if it has a life of its own. The fraction as number example from the written curriculum refers to fractions as factors and factors are objects in their own right. In the fraction as number example from the enacted curriculum, the student is comparing fraction multiplication to whole number multiplication which reifies "fraction." In contrast, the other examples use "fraction" and its derivatives as processes or descriptors which does not fulfill the reification requirement of objectification. Figure 1 summarizes the relative frequency of each category in the curricula.

"Fraction" as number is the most common use of "fraction" in both the written and enacted curricula; however, its use as number represents nearly three-fourths of the uses in the written curriculum compared to slightly more than half in the enacted curriculum. "Fraction" as part is more common in the enacted curriculum than in the written curriculum (42% compared to 15%) which indicates that "fraction" and its derivatives are objectified more often in the written curriculum than in the enacted curriculum.

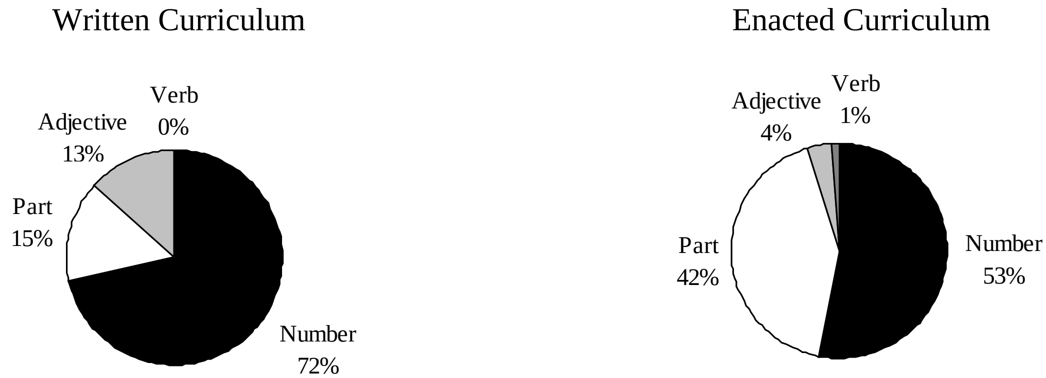

Figure 1. Relative frequencies of categories of "fraction" use in the written and enacted curricula

Fractions in the Curricula

Next, I examined the use of fractions in the written and enacted curricula. Figure 2 summarizes the relative frequencies of proper fractions, improper fractions, and mixed numbers in the curricula.

The relative frequency of the appearance of each type varies substantially. For example, proper fractions represent 57% of all fractions present in the written curriculum compared to 83% in the enacted curriculum, whereas mixed numbers represent 35% of all fractions present in the written curriculum compared to just 13% in the enacted curriculum. This may indicate that in this Investigation, students have more experience with proper fractions and less experience with improper fractions and mixed numbers than intended by the authors of the written curriculum. In terms of objectification, the extension from multiplication of fractions to include improper fractions and mixed numbers may facilitate reification (and therefore objectification) by problematizing the "part of part" discursive pattern. For example, it is possible to speak about "$\frac{1}{6} \text{x} \frac{3}{4}$" as "$\frac{1}{6} \text{of} \frac{3}{4}$" (i.e., part of a part) indefinitely. Therefore, if only proper fractions are provided as examples, the reification of fractions may never happen. However, it is much more problematic to speak about "$\frac{7}{6}$ x $\frac{3}{4}$" as "$\frac{7}{6}$ of $\frac{3}{4}$." This problematizing is important because it may facilitate the reification of fractions.

The five most common proper fractions in the written and enacted curricula are the same (i.e., $\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{4}$). In fact, even their order of prevalence is the same. When taken as a whole, these common fractions make up 42% of all proper fractions that appear in the written curriculum and 70% of all proper fractions included in the enacted curriculum. This statistic indicates that students are having less exposure to a variety of proper fractions than the authors intended.

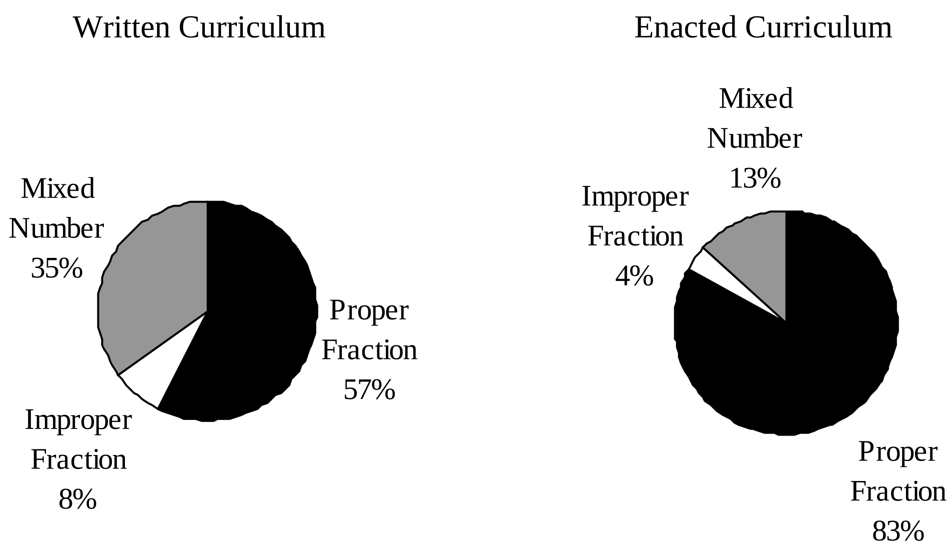

Figure 2. Relative frequencies of the total number of appearances of proper and improper fractions and mixed numbers in the written and enacted curricula

As mentioned previously, reification of fractions is dependent upon the encapsulation of fractional parts (e.g., four one-sevenths) into one number (e.g., "$\frac{4}{7}$"). Evidence of such reification in association with proper and improper fractions and mixed numbers is rare in the enacted curriculum because nearly all fraction discourse is of the form "X of Y" where X is a fraction (e.g., two thirds) and Y is either another fraction or a contextual object (e.g., brownie pan). Examples such as the ones presented here are ubiquitous throughout the five days:

Example 1 (Enacted Curriculum).
S: "Because he wants to buy ***one half of the pan*** *that is two thirds full." (Day 1)*

Example 2 (Enacted Curriculum).
*T: "***One third of a half,*** *right? So what are you starting with, if you* ***have one third of a half,*** *what are you starting with?" (Day 3)*

This prevalence of "X of Y" is also found in the written curriculum in the first problems of the Investigation (Days 1-3); however, the last two problems (Days 4-5) contain less of this use.

In both the written and enacted curricula, estimation discussions seem to facilitate the objectification of fractions. For example, the following problem from the written curriculum provides such opportunities:

Example 3 (Written Curriculum).

Getting Ready for Problem 3.3

Estimate each product to the nearest whole number $(1, 2, 3, \ldots)$.

$\frac{1}{2} \times 2\frac{9}{10}$ $\quad 1\frac{1}{2} \times 2\frac{9}{10}$ $\quad 2\frac{1}{2} \times \frac{4}{7}$ $\quad 3\frac{1}{4} \times 2\frac{11}{12}$

Will the actual product be greater than or less than your whole number estimate?

Here, the question asks "Will the actual product be greater than or less than your whole number estimate?" Numbers are "greater than or less than," therefore, the use of these words promotes the objectification of fractions. This type of language is used in several places in the written curriculum in association with estimation. Another example is included here:

Example 4 (Written Curriculum).

"$\frac{9}{3}$ is 3 and $\frac{1}{2}$ of 3 is $1\frac{1}{2}$, which is greater than 1." (TG, p. 62)

Again in Example 67, $1\frac{1}{2}$ "is greater than" 1 seems to imply the reification of "$1\frac{1}{2}$" as a mathematical object in its own right rather than as a whole number (i.e., 1) and a fraction (i.e., $\frac{1}{2}$). This is not the case for all estimation problems in the written curriculum; however, it is notable that such instances are common in the written curriculum in association with estimation.

Some language used in association with estimation in the enacted curriculum also indicates the possible reification of fractions. The following statement is made by a student that is rounding up $2\frac{9}{10}$ to 3:

Example 5 (Enacted Curriculum).

S: "Because if I was going to round up the nine tenths like Graham did so" (Day 4)

"Rounding up" is language associated with numbers. That is, $2\frac{9}{10}$ (and $\frac{9}{10}$ itself) seems to be used here as a number. Another example comes from a student multiplying $4\frac{1}{2}$ by 1:

Example 6 (Enacted Curriculum).

S: "Well I think you'd have to times it 'cause one times any number is always itself, so I think it'd be about four and a half because" (Day 4)

This student states explicitly that $4\frac{1}{2}$ is a number and therefore multiplying it by 1 would "be about four and a half." The next excerpt, taken from the same day indicates the complexity of determining whether or not reification has occurred. The Question being discussed is "$\frac{1}{2}$ x 2 $\frac{9}{10}$":

Example 7 (Enacted Curriculum).

(1) T: Could you help me with an estimate? If I said I was going to get one half of two and nine tenths, Graham, how could we think about estimating that answer?
(2) S1: Well, nine tenths, that's close to a whole, so -
(3) S2: One.
(4) S3: Yeah, so
(5) T: So what, Graham?
(6) S2: Nine tenths is close to a whole.
(7) T: Okay. So you could round that up into a whole and then two would be a three. So you would call this [points to $2\frac{9}{10}$] about three?

This excerpt is interesting because the student seems to be reifying "$\frac{9}{10}$" because he says "nine tenths is" in Line 6. This is in contrast to saying "nine tenths are" in which the plural verb is used because the nine is plural. In addition, the teacher uses "round that up" in Line 7 which also tends to be used in cases where a fraction is reified because numbers are rounded up. The complexity comes from these indications of reification combined with the use of "whole" in Lines 1, 5, and 6 by both the teacher and the students. Discourse indicative of reification would use "one" instead of "whole" to indicate that "$\frac{9}{10}$," because it is a number, should be compared to another number.

Discussion

The question addressed in this discussion is, "What does an investigation of the mathematical words in the written and enacted curricula allow us to see?" The analysis revealed several potentially significant differences. First, "fraction" is more often used as a number in the written curriculum than in the enacted curriculum (72% of uses compared to 53% of uses). In the enacted curriculum, 42% of the uses of "fraction" were as a part compared to only 15% in the written curriculum. This is important because opportunities for students to use fraction as a number facilitates students' objectification of fractions. Second, 83% of all fractions in the enacted curriculum are proper fractions compared to 57% of the fractions in the written curriculum. The remainder are improper fractions or mixed numbers. Third, the five most common fractions represent 42% of all proper fractions in the written curriculum compared to 70% in the enacted curriculum. The second and third points, when taken together, indicate that students are likely not receiving opportunities to use the wide variety of fractions and fraction types provided in the written curriculum. Finally, the "X of Y" discursive pattern is used extensively on all five days in the enacted curriculum and only on the first three days in the written curriculum. Although this discursive pattern likely serves students well as they develop meaning for fraction multiplication, the fact that the pattern continues throughout the Investigation suggests that students are not be given opportunities to move beyond talking about fractions as parts to fractions as numbers in the context of multiplication. These four examples of mathematical differences were made obvious using the Commognitive framework and likely would not have been brought to light using common methods to document curricular fidelity of implementation.

The claims in this paper are made with several caveats. First, this analysis compares the written text with an enactment of the written text (one of infinitely many possible enactments); this should be kept in mind when reading these statements as some results may be attributed to this difference in curricular form. Second, similarity (and difference) here is through my eyes only. That is, another person (e.g., a teacher, a textbook author) using their own lens may see things quite differently. Finally, the evidence for my claims is gleaned from five days of the written and enacted curricula. That is, none of my statements can be generalized either to the written curriculum as a whole or the enacted curriculum as a whole. Rather, they highlight insights gained through the use of this framework regarding the relationship between the written and enacted curriculum on these five days that may be of interest to teachers, curriculum developers, and mathematics education researchers.

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INTELLECTUAL WORK: THE DEPTH OF MATHEMATICAL DISCOURSE AND ITS RELATIONSHIP TO STUDENT LEARNING

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In this paper we consider the transformative use of language at the micro-level of classroom action – how teachers and students respond to one another moment-by-moment and the resulting knowledge constructed in those exchanges. We focus on a theoretically significant construct of discourse, intellectual work, in order to describe the levels of intellectual work in teacher and student discourse and to determine how differences in intellectual work are related to learning. Using techniques of discourse analysis combined with HLM, we found that although intellectually demanding discourse is rare, there is a large pay-off for student learning when it is present.

Introduction

We take as our premise that learning occurs in interaction. And one of the primary means of interaction in classrooms is discourse; thus, our focus is the mathematical discourse of teachers and students and the relationship of these discourses to student learning. Like Vygotsky, we view language as a mediational tool that can fundamentally transform the potential for action and make new kinds of meaning and understanding possible in classrooms (Vygotsky, 1986; Wertsch, 1991). In this paper we consider the transformative use of language at the micro-level of classroom action – how teachers and students respond to one another moment by moment and the resulting knowledge that is constructed in those exchanges.

By nature of the institutional authority of their position, teachers play a large role in determining discursive norms, but they alone do not determine patterns of interaction. Because of the reciprocal, dependent and unpredictable nature of discourse, teachers and students together co-construct these context-dependent ways of engaging in classrooms. Ways of responding and interacting are negotiated, internalized, and eventually become recognized ways of doing things in the localized mathematics communities we call classrooms. As in any community, classroom events develop a sense of coherence over time where certain types of discourse moves are expected and employed as standard routines of behavior. Through these familiar and repeated patterns of interaction, learning occurs. Thus, differences in discourse patterns (e.g., consistently requesting justifications instead of answers only) can lead to differences in learning.

We are interested in a particular theoretically significant construct reflected in classroom discourse – the amount of *intellectual work* required of students (Hiebert & Wearne, 1993; Pierson, 2008; Stein, Grover & Henningson, 1996; Vygotsky, 1978; Webb, Nemer & Ing, 2006). *Intellectual work* reflects the cognitive work set in motion and required of students within a given turn of talk. Higher levels of intellectual work extend thinking and include discursive moves such as providing justifications, examples, conjectures, explanations, and challenges; making connections across representations; generating problems and scenarios (contextualizing); or requesting these activities from students.

Intellectual work can be thought of in terms of what cognitive activities students are asked to (and do) engage in during real-time, classroom conversations. Over time, if students routinely make sense of mathematics, struggle with complex problems, reason for themselves, generate

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multiple solution paths, and communicate their understandings, the potential for deeper mathematical understanding increases. Higher levels of intellectual work are invitations to do exactly this, to engage deeply with the processes and content of mathematics.

Our research sought to describe the levels of intellectual work in teacher and student discourse and to determine whether and how differences in intellectual work were related to students' mathematics achievement. Specifically, the research questions guiding our study were:

1. What levels of intellectual work are observed in seventh-grade mathematics classrooms?
2. What is the relationship between the intellectual work reflected in students' responses and the intellectual work required by their teachers?
3. What is the relationship between the level of intellectual work in teacher and student discourse in a curricular unit on rate and proportionality and seventh graders learning of mathematics as measured by a validated pre-and post-test?

Methods

Participants and Setting

This study is part of a larger program of research, Scaling Up SimCalc, which investigated the efficacy of a 3-week curricular unit on rate and proportionality using the SimCalc™ Math Worlds technology (see Kaput, 1994) as it was taken to scale across Texas. The broader experimental study investigated the impact of a technology-rich curriculum combined with focused professional development on student learning. Overall main effects were statistically and practically significant showing that students in SimCalc classrooms learned more than their counterparts in the control classrooms ($p<.0001$ and a student-level effect size of .63; see Roschelle, Shechtman, Tatar, Hegedus, Hopkins, Empson, Knudsen & Gallagher, in press).

In our study, we analyzed the discourse moves of 13 teachers and approximately 250 students in seventh-grade mathematics classrooms as they implemented the SimCalc technology and curriculum. All students took a pre- and post-test that assessed basic understandings of rate and proportionality as well as more complex and conceptually difficult understandings including the interpretation of piecewise linear graphs of motion and the relationships between graphical, algebraic (written symbols) and tabular representations of motion. We selected a lesson in the middle of the SimCalc Math Worlds curriculuar unit and videotaped 13 teachers teaching the same lesson. The corresponding transcripts and video footage along with the pre-and post-test scores provide the data corpus for this study.

Coding Scheme

Operationalizing the intellectual work present in moment-to-moment discourse moves is not straightforward. In a broad sense, students can a) give (or be given) information to make sense of (*give* moves) or b) request information or, more likely, have their teacher request information from them (*demand* moves). In give moves, the speaker's role is to do the work of making connections, supplying information, selecting relevant responses, and evaluating the correctness of ideas. Discourse moves in the demand category offer this role to another (typically the student) as the speaker's role shifts from knowledge source to facilitator. The final outcome of discourse routines relying on give versus demand moves is largely the same, an increase of the group's common knowledge, but the responsibility and division of labor in achieving it is different. Our coding scheme for intellectual work takes into account the distinction between give and demand moves as well as the cognitive action required of the participant. We define three levels of intellectual work with relevant examples in Table 1 below (see Pierson, 2008 for a more detailed description of the coding scheme).

Table 1. Intellectual Work Coding Scheme

	Demand	**Give**
Low: Moves that provide basic information such as reading values off graphs/charts, performing calculations, or requesting these activities from others.	How long was the trip?	180 miles.
Potentially High: Moves where claims or conjectures are requested or provided but evidence and justifications are not provided; moves include generating problem scenarios for given graphs, interpreting graphs of motion and making conjectures, or requesting these activities from others.	What can you tell about the trip from looking at and analyzing the graph?	The bus stopped after two hours.
High: Moves facilitating engagement in mathematical argumentation and justification, engaging with another's thinking in a sophisticated say, or requesting these kinds of activities from others.	How do you know that the bus stopped?	Because the distance stayed the same but the time kept going.

Using this coding scheme, we coded each transcript turn by turn indicating the level of intellectual work present, whether the move was a give or a demand, the speaker, and the mathematical correctness of the statement (note that demand moves were not coded for correctness). To establish reliability, a random sample of 10% of the transcript exchanges were selected and coded independently. Interrater agreement was 89%. (Note: Although we present a finalized coding scheme, it was developed and refined through an iterative process, drawing both from the data itself as well as findings from relevant literature.)

Quantitative Analyses and Hierarchical Linear Modeling

Because of the nested nature of the data set and the fact that student test scores were not independent, we also analyzed the data using hierarchical linear modeling. For each measure of discourse (both teacher and student discourse) we specified and ran a two-level hierarchical linear model on student post-test achievement scores, nesting students within classrooms. We constructed each model using the appropriate classroom discourse measure of intellectual work as a predictor at the classroom level and pre-test achievement scores as a predictor at the student level. This allowed us to test the nature and strength of the relationship between teacher and student intellectual work and student achievement.

Findings

A Profile of Teacher and Student Intellectual Work Across Classrooms

In this section we report the results from the analysis of intellectual work of both teacher and student discourse. Figure 1 on the following page aggregates frequency counts for the various categories of Give and Demand moves across all classrooms showing the total number of teacher demand moves, teacher give moves and student give moves for each of the three levels of intellectual work. In these 13 classrooms, we see that low-level give and demand moves dominate the discourse, that teachers rarely demand high levels of intellectual work, and that students provide high levels of intellectual work infrequently (see similar findings in Kawanka & Stigler, 1999; Stein, Grover & Henningson, 1996; Webb et al., 2006).

The Relationship Between Teacher and Student Discourse Moves

To better understand if and how the intellectual work reflected in student discourse was related to patterns in teachers' discourse, we focused on exchanges where a teacher demand move was followed immediately by a student give move. For each teacher we categorized his or her demand moves into one of three categories – Low, Potentially High and High. Within each of these three levels of Teacher Demand we identified and then counted the type of student give move that followed. Table 2 below aggregates counts of demand-give pairs across the data set.

	Level of Intellectual Work		
Discourse Move	*Low*	*P Hi*	*Hi*
Tchr Give	1194	100	16
Tchr Demand	984	181	35
St Give	1371	333	36

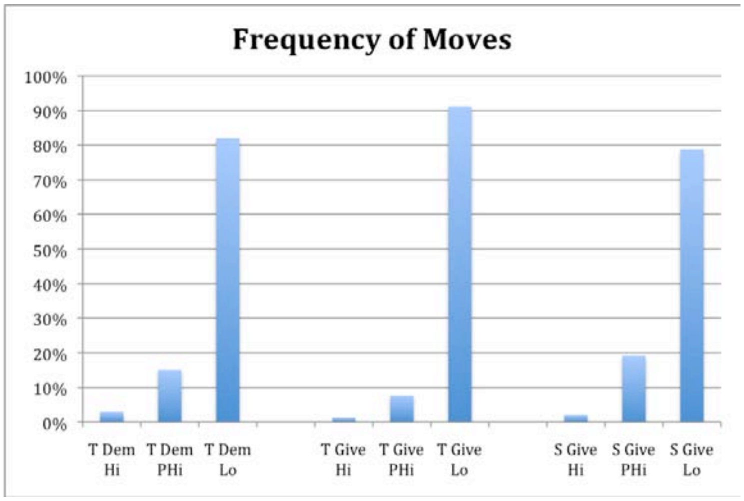

Figure 1. Frequency Counts of Discourse Moves by Level of Intellectual Work

Table 2. Frequency of Demand-Give Exchanges Across Classrooms

	T Demand Low	T Demand P Hi	T Demand Hi
S Give Low	**850 (.908)**	67 (.39)	5 (.147)
S Give P Hi	86 (.092)	**102 (.593)**	9 (.265)
S Give Hi	0 (.00)	3 (.017)	**20 (.588)**

Essentially we see that low demands lead to low gives, potentially high demands lead to potentially high gives, and high demands lead to high gives (see bold cells). To test this relationship more robustly using statistical methods, we first disaggregated these frequency counts for each teacher. In other words, we counted the number of student give moves within each category of demand moves for every teacher. Table 3 provides an example for one teacher.

Table 3. Teacher Profile of Student Give-Teacher Demand Pairs (Frequencies)

Tchr 13611		St Give Low	St Give P Hi	St Give Hi	St Give Average
Categories of Teacher Demand Moves	Demand Low	51	6	0	$.105 = \frac{6(1)}{57}$
	Demand P Hi	16	13	1	$.5 = \frac{13(1)+1(2)}{30}$
	Demand Hi	1	5	2	$1.125 = \frac{5(1)+2(2)}{8}$

Based on these frequency counts we then calculated the average Student Give level for each category of teacher demand (see the last column in Table 3). To have a meaningful metric we used a weighted average, weighting student give moves as follows: Low Gives were weighted as 0, Potentially High Gives were weighted as 1, and High Gives were weighted as 2. We chose this metric because it accounts for both the quality and quantity of student give moves. For the classroom in Table 3 the average Student Give level following a Potentially High Teacher Demand move is given by the calculation

$$\frac{16(0)+13(1)+1(2)}{16+13+1}=.5$$

In other words, the student is equally likely to reply to a teacher potentially high demand move with either a low level give (weight of 0) or a potentially-high give (weight of 1); the average student response lies in between a low give and a potentially high give. Figure 2 visually displays the average Student Give level for each Teacher Demand Level along with the mean Student Give Average across all 13 classrooms.

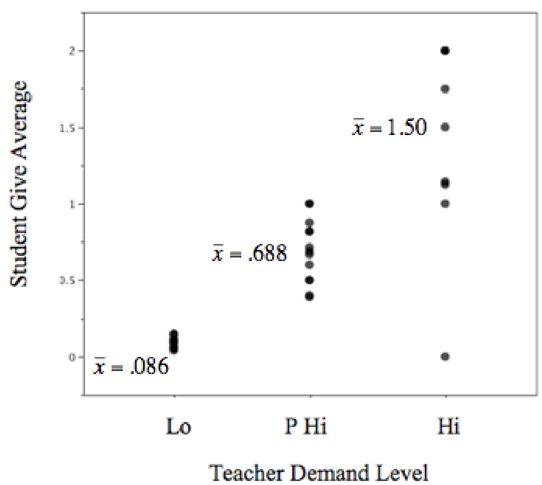

Figure 2. Student Give Weighted Average By Teacher Demand Level Across Classrooms

Notice that the average level of intellectual work for a student response is more variable across classrooms as the level of teacher demand increases. The 13 data points for Demand Level Lo are tightly clumped between 0 and .25, meaning that when a teacher asks a low-level question, students generally respond in kind. However, when teachers ask questions requiring more intellectual work, students respond in a variety of ways indicated by increased variation in the average student give moving left to right. Teacher demand moves act as a ceiling of sorts, limiting but not completely determining the depth of intellectual work in student responses. Moreover, given the relative infrequency of high demand teacher moves (occurring in less than 3% of teacher moves, or 35 times out of 1200 total teacher demands), one would expect larger variance simply due to the small sample size of high demand teacher moves.

We followed this analysis with a MANOVA to specifically test the relationship between teacher demand and student give pairs. This analysis revealed a statistically significant, positive relationship ($p<.0001$) between Teacher Demand Level and average Student Give Level across the 13 classes. Thus, a teacher's Demand Level does predict the average student Give Level.

The Relationship Between Discursive Patterns and Learning

To test the nature and strength of the relationship between discursive patterns and student achievement, we created two metrics – one to measure the level of intellectual work present in the teacher's discourse and another to measure the level of intellectual work presents in students' discourse moves. The metric for teacher discourse is the ratio of high and potentially high demands to low demands. We did not include teacher give moves because variation in teacher give moves did not appear to systematically explain variation in student learning. ($r = .055$).

The metric for student discourse is a weighted average considering both the clarity and correctness of student responses as well as the level of intellectual work. The weighting was as follows: All low-level give moves were weighted 0 regardless of correctness or clarity. Potentially high give moves that were unclear or incorrect were weighted 1. Potentially high give moves that were correct were weighted 2. High give moves that were unclear or confusing were weighted 3. High give moves that were incorrect or technically correct but lacking detail were weighted 4, and high give moves that were correct were weighted 5. Because student questions were so rare we did not include student demand moves as part of this weighted average. The scatter plots and the corresponding regression equations in Figure 3 display the relationship between average gain score and intellectual work in student and teacher discourse.

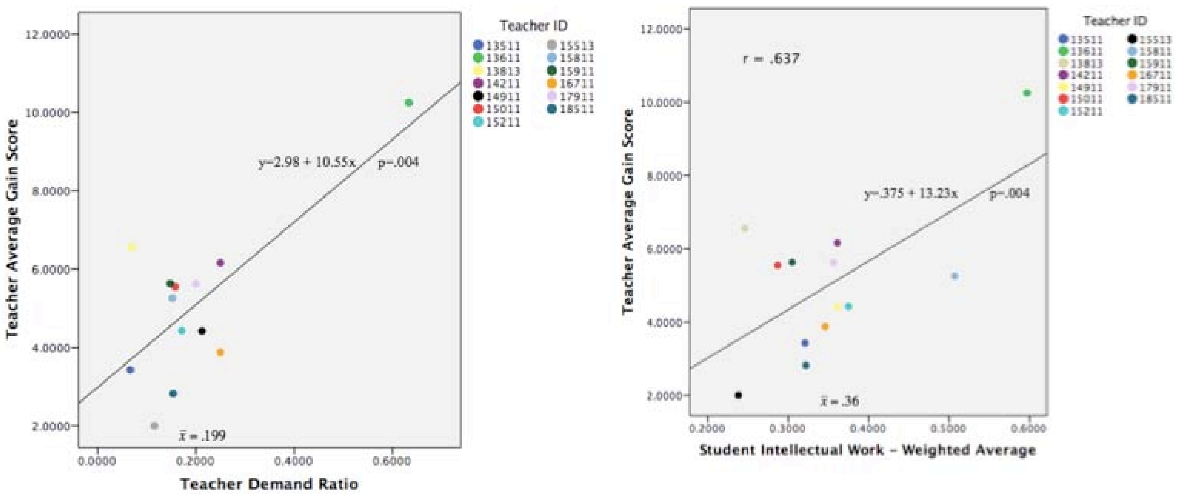

Figure 3. Scatter Plots of Teacher and Student Intellectual Work by Mean Teacher Gain Score

To test for the relationship between intellectual work and student achievement, we examined the coefficient for the group-level (i.e., classroom level) intellectual work slope in both HLM models. The teacher discourse model yielded a parameter estimate that was positive and marginally significant ($\gamma_{01}=6.82$, $p=.09$); the student discourse model parameter estimate was also positive and marginally significant ($\gamma_{01}=13.35$, $p=.11$). Though not statistically significant, we report these findings because 1) they approach significance with a relatively low number of classrooms (level-two units) and therefore low power, and 2) practically speaking, the results are significant. The associated increase in a class's average post-test score with a corresponding increase of .2 in the intellectual work of teacher and student discourse is 1.36 and 2.66 points respectively (out of 18 total). Recall that the teacher discourse metric is a ratio of high-level demand moves to low-level demand moves. In practical terms, if teachers can ask just one more high or potentially high-level question for every 5 low-level questions, the predicted increase in their class's average post-test score is roughly 8%. (Note that the mean demand ratio across all teachers was .20: for every 5 low-level demands there was one high or potentially high demand.)

In addition, we also found patterns in our data related to equitable achievement when inspecting the coefficient for the group-level pre-/post-achievement slope. Consider, for example, the scatter plot in Figure 4. Each student can be represented by an ordered pair where the x-coordinate represents his or her pre-test score and the y-coordinate represents the corresponding post-test score. Ordered pairs of the same shape represent all students in one classroom; thus Figure 4 displays student achievement data for two classrooms and their regression lines. The variable of interest is the pre/post-test achievement slope.

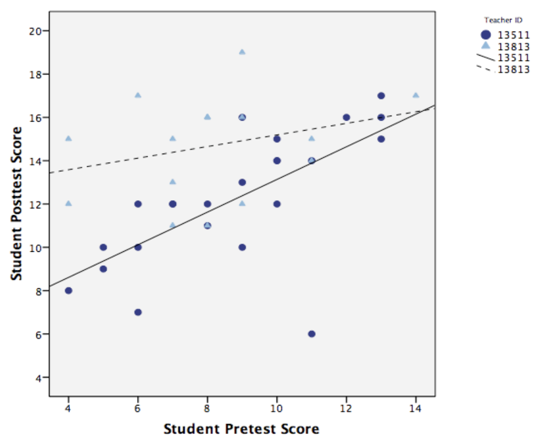

Figure 4. Scatter Plot and Regression Lines for Student Pre/post Scores in Two SimCalc Classes

Some classrooms have steeper slopes and some have flatter slopes. In other words, pre-test scores are stronger determinants of achievement in some classrooms, which manifests in larger pre/post achievement slopes (e.g., see the circles and the solid regression line in Figure 5). Some claim that more equitable classrooms are those in which the pre/post achievement slope is closer to 0. In other words, all students achieve equally regardless of their prior knowledge. Our HLM analyses revealed that 1) the patterns of intellectual work at the classroom level helped to explain variation in pre/post achievement slopes, and 2) that these discourse moves were positively related to more equitable classrooms. The coefficients in the teacher and student discourse HLM models were *negative* and statistically significant (teacher intellectual work: γ_{11}= -.88, p=.017; student intellectual work: γ_{11}= -1.52, p=.04). In other words, classrooms with higher levels of intellectual work moderated the effect of prior knowledge on student learning. An increase of .2 in teachers' (or students') intellectual work metrics corresponded to decreases in their class's pre/post achievement slope of .18 (or .3). Interestingly, intellectual work seems to play out as a factor in more equitable classrooms by decreasing the impact that pre-test scores have on post-test achievement.

Conclusions and Implications

In response to our research questions and based on the findings from the analyses we draw the following conclusions.

1. Low levels of intellectual work in both teacher and student discourse persist despite resources (curriculum, technology, professional development) providing ample opportunities for teachers and students alike to explore deep mathematical ideas. The classroom practices of the teachers and students in this sample seemed to be typical of most middle school mathematics classrooms. Therefore, the low levels of intellectual work observed are likely present in other classrooms. This motivates questions of why these discursive features are relatively rare in seemingly typical classrooms and leads us to re-examine our tacit models of teaching.
2. Using a multivariate analysis of variance we found that the intellectual work of teachers' demand moves are predictive of the level of intellectual work of students. In other words, teachers usually get what they ask for.
3. Correlational analyses and hierarchical linear modeling revealed a marginally significant, positive relationship between the intellectual work of teacher and student discourse and student learning of rate and proportionality. Specifically, the HLM model predicts an 8% and 15% increase in a class's mean post-test score (an increase of 1.36 and 2.66 points out of 18 total) with a corresponding increase of 20% in teacher and student intellectual work metrics. Additionally, classroom communities with higher levels of intellectual work moderate the effect of prior knowledge on student learning – an increase of 20% in intellectual demanding teacher and student discourse corresponds to decreases in the class's pre-post achievement slope of .18 and .3 ($p<.05$).

In general, we found that classroom discourse and normative interaction patterns can guide and influence student learning in ways that both improve and impede achievement. There is tremendous and often unrealized power in the ways teachers talk with their students. We also found that classrooms with high levels of intellectual work are positively related to student learning. However, in order to more fully understand the relationships between intellectual work and student learning we need more data, especially in classrooms with good mathematical discourse. In closing, we are not proposing that there is one "right" model for classroom discourse. The key is for teachers to continually grow, refine, reflect and change – becoming a student of their students. This is a process that continues throughout a career.

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TEACHING MATHEMATICS WITH MULTIPLE STRATEGIES IN MIDDLE AND HIGH SCHOOL

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Although there has been extensive scholarship about the importance of teaching with multiple strategies in the elementary grades, there has been relatively little discussion of what teaching with multiple strategies would or should look like in the middle and high school levels. This paper begins our exploration of this practice by addressing the following questions: (1) Why do middle and high school teachers think that mathematics instruction should include a focus on multiple strategies?; (2) What attitudes and concerns do middle and high school teachers have about teaching multiple strategies for solving mathematics problems?; and (3) How do middle and high school mathematics teachers report teaching with multiple strategies in their classrooms? We examine these questions through interview data collected from experienced middle and secondary mathematics teachers.

Introduction

The practice of teaching children multiple strategies for solving mathematics problems has been recommended in many recent mathematics education policy reports in the US, including the National Council of Teachers of Mathematics' (NCTM) Curriculum Focal Points (National Council of Teachers of Mathematics, 2006) and the National Research Council's (NRC) Adding it Up (National Research Council, 2001). Indeed, the idea that students benefit from generating, comparing, and reflecting on multiple solution methods has been a central precept of mathematics reform pedagogy for at least the past 20 years (Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005). For example, Silver and colleagues note that, "It is nearly axiomatic among those interested in mathematical problem solving as a key aspect of school mathematics that students should have experiences in which they solve problems in more than one way" (Silver et al., 2005, p. 288).

Although there has been extensive scholarship about the importance of and implementation of this practice in the elementary grades, there has been relatively little discussion of what teaching with multiple strategies would or should look like in the middle and high school levels. Should middle and high school mathematics teachers provide instruction in multiple strategies, or are the benefits of this instructional approach limited to the elementary grades? Are there differences in the ways that this practice can and should be implemented in middle and high school classrooms, as compared to elementary classrooms?

This paper begins our exploration of the practice of teaching students multiple strategies for solving mathematics problems, in the context of a professional development institute whose goal was to encourage middle and high school algebra teachers to think more deeply and carefully about the use of multiple strategies in their instruction. The paper addresses the following questions: (1) Why do middle and high school teachers think that mathematics instruction should include a focus on multiple strategies?; (2) What attitudes and concerns do middle and high school teachers have about teaching multiple strategies for solving mathematics problems?; and (3) How do middle and high school mathematics teachers report teaching with multiple strategies in their classrooms? We examine these questions through interview data collected from experienced middle and secondary mathematics teachers.

Teaching with multiple strategies in the elementary grades

Research on the practice of teaching with multiple strategies has been conducted almost exclusively in the elementary grades. Projects such as Cognitively Guided Instruction (CGI) played an early and important role in drawing elementary educators' attention to the issue of multiple strategies (Carpenter et al., 1998). Research on CGI prompted elementary school teachers to think more carefully about students' invented strategies, and encouraged the acceptance of multiple strategies for solving problems in elementary classrooms. Case studies of effective implementation of CGI and related programs describe teachers prompting students to solve problems in two different ways, to compare the similarities and differences between multiple methods, and to discuss and justify why various solution strategies may yield the same answer (Carpenter et al., 1999). Overall, there appears to be a consensus among researchers that there is a developmental appropriateness to accepting multiple solution strategies from young children, in that children are encouraged to begin solving problems using the intuitive mathematical knowledge that they have developed prior to entering the classroom, and to progress to strategies that are increasingly complex and abstract for the same sorts of problems.

In addition to describing models of effective instruction with multiple strategies, the research literature on multiple strategies in the elementary grades has furthermore identified ways that a focus on multiple strategies can go awry. In particular, a commonly identified problem is that attempts to teach with multiple strategies take the form of mere "serial sharing," where students share and present multiple strategies but the teacher does not draw mathematical connections between them (Ball, 2001). Here students miss the crucial opportunity afforded by the presentation of the multiple strategies to investigate, compare, and justify diverse mathematical solutions, methods and algorithms. Without these connections, the pedagogical value of presenting multiple strategies is unclear (Ball, 2001).

Multiple strategies in middle and secondary school

While there has been substantial research regarding teaching with multiple strategies in the elementary grades, there has been significantly less thinking about this practice at the middle and high school levels. In fact, it is not always clear what a focus on multiple strategies means in the secondary grades. For example, should algebra students learn more than one way to solve linear equations and proportion problems? Should students be asked to invent, share, and compare multiple strategies for solving quadratic equations?

In summary, the ubiquitous practice of teaching students multiple strategies for solving mathematics problems has not been given as much attention in middle and high school, and thus merits further exploration. In this paper, we begin to examine this practice by analyzing middle and high school teachers' views about the use of and value of multiple strategies. We focus in particular on the teachers' initial views, before they participated in our professional development.

Method

In this section, we provide a brief description of the professional development project which provided the context for this study. We then describe the types of data collected, and the process by which the data were analyzed.

Setting

The context for this study was a one-week professional development workshop held in July 2009. The goal of the workshop was to prepare participating teachers to implement a set of researcher-developed curriculum materials in their Algebra I courses. These curriculum materials were developed as part of an NSF-funded research project whose goal was to 'infuse' multiple strategies into Algebra I courses.

Participants

The workshop participants were 13 experienced Algebra I teachers from nine middle and high schools in the Boston metropolitan area, representing a mix of urban and suburban schools. Eleven of the teachers taught in public or charter schools, and the other two teachers were from private schools. The mean number of years of teaching experience for the thirteen teachers was 10 (range 3 to 25). Teachers were recruited through print and e-mail advertisements seeking experienced Algebra I teachers who wanted to explore new techniques for teaching algebra.

Data Sources

The data analyzed in this study were collected in introductory interviews that were conducted with the teachers as the first activity of the first day of the workshop. Teachers were interviewed individually; all interviews were audio-recorded and later transcribed. The interviews were conducted by the authors, as well as by other trained research assistants. The interview was semi-structured; we began with a series of pre-determined questions but asked follow-up questions as appropriate. In particular, the teachers were asked about whether they felt it was useful to expose students to multiple strategies while teaching math, what they believed the advantages and disadvantages were to this approach, and whether and how they used multiple strategies in their current teaching. Teachers were also presented with several specific mathematics problems, and asked what strategies they would teach students to use for each of these problems. Each interview took approximately 30 minutes to complete.

Analysis

After transcription was complete, multiple researchers independently listened to the interviews and identified statements that were relevant to the research questions. Meetings were held to discuss emergent themes in the data, after which we revisited the transcripts to confirm or challenge these emergent themes. Finally, meetings were held to discuss tentative findings and evidence supporting these findings.

Results

Our analysis of the interviews sought to answer the following questions: (1) Why do teachers think it is important to teach multiple strategies in middle and secondary school mathematics?; (2) What attitudes and concerns did teachers have about teaching with multiple strategies?; and (3) How did teachers report using multiple strategies in the classroom?

Why is it important to teach with multiple strategies?

During the pre-interview, all thirteen of the teachers indicated that they believed teaching with multiple strategies was useful. When teachers were asked what the advantages were of teaching with multiple strategies, their answers fell into four categories (in describing these categories, we adapt terminology introduced by Leikin and Levav-Waynberg (2007)): (1)

success-oriented, in that this instructional practice affords multiple entry points into a problem for students who learn differently; (2) *understanding-oriented*, in that teaching with multiple strategies facilitates the development of a deeper understanding of mathematics; (3) *affective*, in that this practice increases qualities such as student enthusiasm and interest; and (4) *efficiency-oriented*, in that knowledge of multiple strategies helps students to solve problems more quickly or efficiently. Table 1 indicates the types of responses that each teacher provided; note that some teachers' responses fell into multiple categories. We describe each of these types of responses below.

Success-oriented responses. In explaining why they believed teaching with multiple strategies was important, most teachers responded that teaching with multiple strategies could provide multiple entry points to a given problem for students with different learning styles. In the pre-interview, ten of the thirteen teachers cited this as a primary advantage of teaching with multiple strategies. These comments from Tamara and Bernadette were typical of these responses:

> I think some students will think, "This method, I get it," like that. These students over here, they'll like the other method. And so if you give more than one method, you probably have a better chance of reaching a greater number of students (Tamara). Everybody's brain is different, so if you only teach it in one way, you're going to miss some kids. So the more ways you give them to access it, the more likely the majority of them are going to get it . . . So, the more the merrier, basically (Bernadette).

In such responses, teachers frequently indicated that they believed teaching with multiple strategies could improve their likelihood of "reaching" a greater number of students by showing each student at least one strategy that he or she could understand.

Table 1. Teachers' views on the importance of teaching multiple strategies

Response category	Advantages/Purpose	Jody	Robert	Kelley	Nina	Maxine	Carol	Rachel	Kara	Tamara	Mindy	Julia	Bernadette	Naomi
Success-oriented	Gives multiple entry points into a problem for students who learn differently	x		x		x	x	x	x	x	x	x	x	
Understanding-oriented	Develops a deeper understanding of mathematical processes		x					x	x			x		
Affective	Affective (makes students more engaged and enthusiastic)													x
Efficiency-oriented	One way could be faster or more efficient				x									

Understanding-oriented responses. A second, and much less commonly cited reason for the importance of teaching with multiple strategies was that learning multiple ways of solving a problem could help students develop a deeper understanding of mathematics. Four of the thirteen teachers mentioned this as an advantage. In a comment typical of these teachers, Robert noted, "I think if you can look at two sides, and make a decision of, hey, you know this doesn't work because of this, then you know what you're doing."

Affective-oriented and efficiency-oriented responses. Two additional reasons for teaching with multiple strategies were given, although each was cited by only one teacher. A single teacher, Naomi, cited affective reasons when describing the advantages of teaching with multiple strategies, saying that learning with multiple strategies could help students become "much more enthusiastic and upbeat, more engaged, more interested in learning, just more into it." A second teacher, Nina, was the only teacher to cite speed and efficiency as reasons for teaching with multiple strategies. She explained that learning multiple strategies could help students to use faster and more efficient approaches when problem-solving.

What attitudes and concerns did teachers have about multiple strategies?

Teachers were also asked what they saw as the potential disadvantages of teaching with multiple strategies, and what concerns they had about exposing students to multiple approaches to solving math problems in their classrooms. Their concerns fell into three general categories: (1) *risk of student confusion*; (2) *motivational concerns*; and (3) *teacher constraints* (such as lack of class time and the large quantity of material in the curriculum to be covered). Teachers' responses are summarized in Table 2 and discussed in more detail below.

Table 2. Teachers' concerns about teaching multiple strategies

Response / category	Disadvantages	Jody	Robert	Kelley	Nina	Maxine	Carol	Rachel	Kara	Tamara	Mindy	Julia	Bernadette	Naomi
Risk of confusion	Could confuse some students			x		x	x	x	x	x	x			
	Could confuse low-achievers			x	x					x	x	x	x	
Motivational concerns					x		x	x			x	x		
Teacher constraints	Lack of time, need for extra prep							x					x	
No disadvantages noted		x	x											x

Risk of student confusion. The most common concern cited by teachers was that they would risk confusing some students by showing them multiple strategies for solving problems. Ten of the thirteen teachers mentioned this concern in some form, with six of these expressing concern that low-achieving students specifically could become confused.

Motivational concerns. A second possible disadvantage of teaching with multiple strategies cited by teachers, expressed by five teachers, was student motivation. These teachers cited their students' preference for generating an answer rapidly and correctly, and what they perceived as their students' aversion to thinking in greater depth about a problem.

Teacher constraints. The third category of concerns cited by teachers regarding teaching with multiple strategies was teacher constraints, such as lack of class time, the large quantity of material in the curriculum to be covered and the need for more preparation time. Two teachers mentioned such concerns.

How did teachers report using multiple strategies in the classroom?

All of the teachers in the study reported that they used multiple strategies in their teaching. Yet rather than indicating that they taught with multiple strategies as a matter of course, most teachers indicated that they used this approach only for certain topics, under certain conditions, and as time permitted. Most teachers indicated that any class discussions about the multiple strategies would be brief and would focus on student preferences.

Regarding class discussions of multiple strategies, most teachers stated that any such discussions would be brief and would center around student preferences. Kelley was typical of those teachers who did appear to have a routine for conducting class discussions about multiple strategies; her routine involved polling students as to which method they preferred:

> I'll do one [method], then do the other. And I'll ask them, "Okay, who likes this method?" And then we'll call this Andrew's method. "Who likes this method?" And we'll call that Kayla's method. And then for homework, I'll ask the kids, "Which method do you like, Andrew's or Kayla's?" And then I'll go over it with the method they liked better. (Kelley)

While Kelley did have a routine for talking about multiple strategies, the discussion was not particularly substantive, extending little beyond a show of hands for students' preferred methods.

Furthermore, most teachers said that their class discussions would not address the question of whether one solution strategy was better than another, but rather that they would tell students to use whichever strategy they preferred. Teachers thus emphasized that they would discuss multiple strategies in terms of student preferences, rather than with regard to considerations such as evaluating which strategy worked better for different problems.

Discussion

Implications for mathematics teaching and research on mathematics teaching and learning

A key question that this study sought to explore was the extent to which teaching with multiple strategies may differ in middle and high school from teaching with multiple strategies in the elementary grades. Indeed, the substantial differences between middle and high school compared with the elementary grades, both in terms of students' development and mathematical knowledge and in terms of the content that is taught, might lead us to ask whether teaching practices that are advocated for in elementary grades are not as applicable in the middle and secondary grades. Our data speak to this issue, particularly in the ways that our teachers' views on multiple strategies did and did not align with elementary teachers' views taken from the literature.

On the one hand, the middle and high school teachers who participated in our study did not express the views about multiple strategies that are widely present and held at the elementary level, namely that teachers should accept multiple strategies from students as they progress from intuitive to more abstract methods in a developmentally appropriate progression. Nor did the middle and high school teachers express the view commonly held at the elementary level that the key pedagogical value of presenting multiple strategies is that it allows students to investigate, compare, and justify diverse mathematical solutions, methods and algorithms.

On the other hand, the middle and secondary teachers that participated in our study expressed some ideas about the value of teaching with multiple strategies that we would likely never expect to hear from elementary teachers on this topic. For example, some of the teachers in our study cited speed and efficiency as reasons for teaching students multiple strategies for solving problems; such concerns with speed would unlikely be found at the elementary level.

How can we interpret this disconnect between what elementary teachers say about teaching with multiple strategies and what secondary teachers say? One interpretation might be that the secondary teachers are essentially wrong. In this view, the discussion taking place and the teaching practices being enacted at the elementary level could be considered much further along in terms of adoption and implementation of reform practices, while the secondary grades remain largely traditional in terms of practices and curricula (particularly outside of middle school). By this interpretation we might presume that perhaps in time, teaching and learning mathematics in secondary school will more closely resemble what happens in elementary schools. In this view, it appears that perhaps the disconnect between the views expressed at the elementary and secondary levels may be a temporary artifact of where we are in the development of reform-oriented practices at all levels.

In support of this idea that we are in a transitional period, it is interesting to note that the main reason secondary teachers give for why they think it is important to teach with multiple strategies is essentially that if one keeps showing students more and more ways of looking at something, eventually something will make sense to each student. This is essentially a muddled view of the reform rhetoric from elementary school. As noted above, this is in fact quite different from the reasons why we teach multiple strategies in elementary school, namely in order to support students' developmental progression and to help them to justify, compare and investigate different strategies. In fact, this view that multiple strategies should be taught so that at least one method can be found that appeals to each learner seems to emerge more directly from the research on learning styles, for which many scholars argue there is limited evidence (Pashler, McDaniel, Rohrer, & Bjork, 2008).

Alternatively, perhaps the disconnect between what elementary and secondary teachers say about teaching with multiple strategies may be due to the fact that there are important differences in the mathematics and in students at the elementary versus at the secondary school level that indicate that certain practices are appropriate for one group and not as appropriate for the other. In secondary school, students are generally more comfortable with abstraction, and the strategies that they learn are more complex. The role of invention of strategies does seem quite different at the elementary versus at the secondary level. For example, we would not likely expect that secondary school students would invent new strategies for solving complex problems, such as solving a quadratic equation, in a reasonable amount of time, and even if they could do so, one might question whether the students would get the same value from this exercise as elementary students do from invention.

A complementary argument might proceed as follows: perhaps rather than arguing that some practices are good for elementary but not for secondary and vice versa, instead it might be the case that a practice is good for both but for different reasons. There may still be important reasons for teaching multiple strategies in secondary classrooms, but they perhaps are not the same as the reasons for teaching multiple strategies in elementary school. Perhaps while in the elementary grades teaching with multiple strategies has the goal of building on students' intuitive knowledge in a developmentally appropriate way, in secondary school teaching with multiple strategies may have the goal of helping students to develop a more connected understanding of mathematics, and to develop fluency in mathematics by becoming more flexible and adaptive problem solvers.

In either case, our research indicates that secondary teachers clearly articulate different goals for teaching with multiple strategies than those found in the literature on teaching with multiple strategies at the elementary level. The issue merits further exploration, to think carefully about whether teaching with multiple strategies in some form is useful for secondary students and, if so, what goals this practice might be seeking to address.

A primary limitation of this study is that we did not observe teachers' actual classroom practice, but looked only at their descriptions of their practice. It would be useful to observe teachers' practice in order to see how they implement a multiple strategies approach with students.

Conclusions

Providing instruction with multiple strategies introduces a new challenge to teaching. At the middle and secondary school level, where limited research has been done on what teaching with multiple strategies does or should look like, the challenge is even more pronounced. Teachers' limited idea of the primary benefit of teaching with multiple strategies, that it can offer a greater number of students an "entry point" into a problem, may be interfering with their likelihood of conducting substantial class discussions. For instance, teachers who believe that the primary purpose of multiple strategies is to find a strategy that "works" for every student may feel satisfied ending the discussion once it appears that all students have found some approach that they can use to begin a problem. Yet a discussion limited to this scope does not afford students opportunities to engage deeply with the multiple methods, to probe the similarities and differences among them, to consider why and in what contexts they work, and to think about and justify which strategies they would choose to use in various situations in the future. In light of the National Research Council's (2001) goal that students should be able to utilize multiple approaches and adaptively select strategies appropriate to diverse types of problems, a discussion about multiple strategies whose main goal was merely to ensure that more students could use any single strategy does not seem adequate.

Teachers' interview responses revealed that they generally held positive views of teaching with multiple strategies and had the desire to incorporate multiple approaches in their teaching. All of them were indeed attempting to do so. Yet their descriptions of how they were utilizing multiple strategies in their classrooms indicated that they lacked routines and scaffolding to engage students in substantial thinking about multiple approaches. These findings suggest that providing teachers with curriculum materials that routinely present multiple methods for solving problems as well as specific guidance for conducting rich classroom discussions may help middle and secondary school teachers to realize the potential learning associated with multiple strategies for their students.

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THE EFFECTS OF USING PREDICTION FOR MATHEMATICS CLASSROOM DISCOURSE

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The value of utilizing prediction questions in mathematics classrooms has been recognized as a potential instructional tool helping students make sense of the mathematics they learn. In this paper we describe how the use of prediction questions helped classroom discussions on related mathematical ideas, by comparing a class using prediction questions to a similar class where no such questions were asked. The results suggest that the prediction questions allowed the students the opportunity to interact with the related mathematical ideas, connect to previous learning, and become more engaged and thoughtful in the classroom discussions.

Introduction

We found that prediction is a useful teaching and learning tool for many purposes (e.g., instructional tool, Kim & Kasmer, 2009; means for motivation, Kasmer & Kim, in press b; conceptual understanding and reasoning, Kasmer, 2008; Kasmer & Kim, in press a; Kim & Kasmer, 2007). In this paper, we will compare the qualities and nature of discussions on mathematical ideas between two middle school mathematics classes (treatment and control), taught by the same teacher, using the same curriculum. The only discernable difference was the use of prediction questions during the launch segments of the treatment class. We will highlight how using prediction questions impacted classroom discussion in terms of the kinds of ideas shared, thinking encouraged, and connections attempted. In this study, prediction is defined as reasoning about the mathematical ideas of the lesson prior to instruction by using previous knowledge, or connections to related concepts.

Literature Review

Prediction is considered a valid construct within reading and science instruction. In terms of reading comprehension, prediction questions were found to support knowledge acquisition (Palinscar & Brown, 1984). Other researchers concur that predicting affords students opportunities to connect previously learned knowledge with the new knowledge they encounter. (Duffy, 2003). Prediction also has been found to benefit student learning in science education. Furthermore, Gunstone and White (1981) found the model they developed, Predict-Observe-Explain (POE), to secondary school students to be an important instructional practice to help students make sense of physics concepts. The POE model is used to assess students' understanding by requiring students to perform three tasks. Initially, students must predict the outcome of an event and justify their prediction; students then describe what they see happen, and finally they reconcile any conflict between their initial prediction and the observation. POE has been found to be an effective model in helping students make sense of complex scientific ideas.

The effectiveness of using prediction in the teaching and learning of mathematics has recently been investigated. Battista (1999) established the benefits of students making predictions in geometry lessons. When students resolved the discrepancies between their predictions and actual answers, they were able to build more useful mental models for

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mathematical ideas. Lim, Buendía, Kim, Cordero, and Kasmer (in press) also suggest that predictions presents opportunities for both students and teachers to become aware of their misconceptions or ill-formed ideas. Prediction also aligns with other reasoning aspects such as generalizing, abducting, conjecturing, and visulization (Lim et al, in press).

In Kasmer's (2008) study, where prediction questions were incorporated into the launch of algebra lessons, overall results suggest prediction is a relevant and valid construct with respect to enhanced conceptual understanding and mathematical reasoning. This study demonstrated that prediction provides students opportunities to connect and refine their existing knowledge and reported positive results in developing understanding. The study also revealed that these students exhibited a higher level of engagement compared to a similar class where prediction questions were not used. When predictions were used, students were engaged in sustained conversations that were created by a culture precipitated by the inherent risk free virtue of prediction questions as within the nature of predicting is the absence of certitude. These students had an opportunity to think about the question and record their responses, and as a result were confident in their responses and motivated to share and listen to others' responses.

This paper focuses on the comparison of the nature of a classroom discussion between two classes by using data collected at the midpoint of an 8-month longitudinal study. In doing so, we analyze the kinds of mathematical ideas and reasoning present during the classroom discussion, by using the frameworks described in the following section. We also analyze students' prediction responses in terms of these frameworks, in order to explain possible attributes that influenced the discussion in the treatment classroom.

Table 1. Conceptual Understanding Framework

Code	Conceptual Understanding Indicators
CU1	Represent patterns in tables, graphs, words, and equations
CU2	Understand and recognize patterns as linear, exponential, or something else
CU3	Understand the meaning of a representation (an equation, a table, or a graph) as a whole and parts of it
CU4	Understand and use the relationship among a table, an equation, and a graph (e.g., a constant in the equation is the y-intercept of a graph and the (0,b) in the table)
CU5	Use equations, graphs, and tables to solve problems and relate the answers to problem situations
CU6	Find a pattern (linear, exponential, or something else) in a table/graph and use the pattern to predict for a particular incident
CU7	Identify and compare characteristics of tables and graphs of various algebraic relationships.

Conceptual Undersatnding and Reasoning Frameworks

For the purpose of this study and analysis of data, two frameworks were developed, although it is understood that conceptual understanding, and mathematical reasoning are not mutually exclusive entities. However for the purpose of data analysis these constructs were treated separately. The conceptual understanding framework [CUF] was developed by incorporating ideas found in the current research related to conceptual understanding (e.g., Hiebert & Carpenter, 1992; National Council of Teachers of Mathematics [NCTM], 2000) and the

Connected Mathematics Curriculum that the classroom used (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998).

Conceptual understanding is based on the premise that as the number of connections among ideas and concepts increases, so too does the understanding (Hiebert & Carpenter, 1992). The conceptual-understanding indicators of the CUF (see Table 1) are embedded through the curriculum and align with the NCTM's (2000) *Principles and Standards for School Mathematics* [Standards]. These indicators were used as a framework for data analysis in seeking qualitative evidence of the existence of differences between treatment and control classes relative to classroom discourse and opportunities to interact with the conceptual understanding indicators.

The mathematical reasoning framework [MRF] resulted from synthesizing constructs from the Standards (NCTM, 2000), *TIMMS Assessment Frameworks and Specifications* (Mullis, Martin, Smith, Garden, Gregory, Gonzalez, Chrostowski, & O'Connor, 2001), and the *Connected Mathematics Project* (Lappan, Fey, Fitzgerald, Friel, & Philips, 1998). The MRF framework is comprised of five reasoning indicators, which were utilized to highlight student-reasoning processes and analyze the data (see Table 2). The mathematical reasoning framework is global in nature, and speaks to all grade levels and mathematical content. The conceptual understanding framework specifically aligns with middle school algebra.

Table 2. Mathematical Reasoning Framework

Code	Reasoning Indicator
R1	*rmulate, evaluate, and support generalizations*. Formulating a generalization is defined as making a statement about something true for any case.
R2	*Construct, evaluate, and support/dispute mathematical arguments*. Constructing an argument is defined as making an informal or formal statement about a specific or general case; one form of this is making a conjecture that may lead to a generalization in the end.
R3	*alyze/evaluate a problem situation.* Analyzing and evaluating a problem situation is defined as making information from the problem useful for solution.
R4	*e inductive/deductive reasoning to establish/support mathematical relationships*. Using inductive reasoning is defined as searching for mathematical relationships through study of patterns while using deductive reasoning is defined as utilizing an established mathematical relationship to support a pattern found in a specific case.
R5	*aw and support conclusions in varied topics.* This indicator is about making a statement that summarizes the findings that is not necessarily a generalization or an argument.

Methods

Participants

Two classes taught by the same teacher were selected for the purpose of this study: one class for treatment where prediction questions were purposely posed, and the other class as a control group where there was no exposure to purposely-posed prediction questions. All students in this school are randomly assigned to each of their core class periods. In addition, selection of courses in this school does not affect classroom composition as the master schedule dictates the class periods during which core academic courses were offered and all students are required to participate in music and foreign language classes. As a result, the classrooms were considered homogeneous. Indeed, results from an Independent-Samples, *t* test produced no significant differences relative to the mathematics scores on the state test (Michigan Educational Assessment Program) between the treatment and control groups ($t_{38} = .77$, $p < .448$). Similarly no statistically significant difference in reading scores between the treatment and control groups ($t_{38} = -0.07$, $p < .941$) were apparent.

Data Collection

For this particular aspect of the study, data sources included student written prediction responses (treatment class), teacher interviews, teacher journals, and video segments along with corresponding transcripts (treatment and control classes).

Prediction responses. The classroom teacher posed prediction questions exclusively to the treatment classroom during the launch of each problem in the linear and exponential units. These prediction questions implicitly reflected the mathematical content of the problem, without revealing the essence of the problem. The teacher presented the prediction questions in conjunction with the launch of the investigation. Students recorded in writing their individual responses to each prediction question the teacher posed. The teacher elicited student responses, without commenting on the accuracy of the prediction. Students were also encouraged to comment on others' ideas. The prediction questions and student responses were revisited during the summary segment of the lesson. Students' written prediction responses were collected from the treatment group at the completion of each lesson where prediction questions were posed during the linear and exponential unit. These artifacts documented characteristics of the nature of these responses for the units of study. The teacher posed the prediction questions to the treatment class while launching the problems during the units of study. For this analysis we looked at classroom video and the corresponding prediction responses (n = 19) from one particular lesson of study.

Classroom videotapes. A total of 9 video observations were made during the instruction of the treatment class, and 5 video observations during the control class. The videos analyzed for this particular paper were drawn from the midpoint of the longitudinal study. In these two video observations, students encountered exponential growth patterns for the first time. Prior to this problem students had studied linear relationships for 6 weeks. Students explored the growth of paper ballots as they are repeatedly cut in half. The problem context was:

Alejandro is making ballots for an election. He starts by cutting a sheet of paper in half. He then stacks the two pieces and cuts them in half. He stacks the resulting four pieces and cuts them in half. He repeats this process, creating smaller and smaller pieces of paper (Lappan, Fey, Fitzgerald, Friel, & Philips, 1998, p. 5).

A prediction question posed in the treatment classroom was: "If Alejandro makes 10 cuts; can you predict how many ballots Alejandro might have? What is your reasoning?" Students provided written responses to this question before exploring the problem context, and discussed their predictions and rationale as well as related mathematical ideas as described earlier.

Teacher interviews and journals. Interview data were collected prior to the treatment, at the conclusions of the linear and exponential units of study. The purpose of these interviews was to gather information with respect to the teacher's portrayal of the possible benefits of prediction questions, validate her journal entries and potentially illuminate related issues which may have arisen from viewing the video data. The classroom teacher submitted electronic journals weekly. These journals captured the main mathematical ideas taught during the week, the teacher's perception of the effectiveness of the prediction questions, and provided an opportunity for the teacher to share any concerns she may have about the study.

Data Analysis

Analysis of data comprised of coding the students' written prediction responses with respect to conceptual understanding (CUF) and mathematical reasoning (MRF) (treatment class) and coding the transcripts of classroom discussion for alignment with conceptual understanding (CUF) and mathematical reasoning (MRF) indicators (treatment and control classes). Video data were transcribed. We used each student statement as a unit of analysis, and coded to a CUF or MRF. In situations where no alignment between the student statement and the CUF or MRF was evident, no code was assigned to the statement.

The teacher interviews were transcribed and presented to the teacher to confirm the transcriptions' accuracy and credibility. Interview transcriptions were read to become aware of the teacher's perception of the efficacy of prediction questions in her classroom, and any notable differences she expressed between the treatment and control classes. This teacher artifact was read a number of times in order to uncover surfacing common themes and categories relating specifically to differences between student discussions and student engagement.

The electronic journal entries sought to reveal information related to possible beneficial outcomes of using prediction questions regarding teaching practices, student discussion and student engagement. Analyzing the teacher's journal validated interview responses and provided affirmation and clarification of teacher practices viewed in the classroom-videotaped segments. A similar method to the interview analysis was employed for this data artifact.

Results

In this section, we describe the results of the analysis of the two classrooms' discussion segments, using the data drawn from one particular lesson. However, these results are indicative of the differences between the treatment and control classes throughout the entire study.

To compare the qualities and nature of discussions on mathematical ideas an example of a classroom discussion centered on the same mathematical idea is presented in Figure 1. This vignette illuminates the opportunities students were provided with to interface with the conceptual understanding and mathematical reasoning indicators. This highlighted example, as described earlier, was the students' initial work with exponential relationships. In this problem students probe the effect of repeated cutting in half on the total number of sheets of paper. While in this particular example, the treatment class discussion transpired during the prediction phase of the lesson, the control class discussion occurred during the summary portion of the lesson. For the control class, there was no prior discussion before students began the problem. The teacher read the problem to the students, and they began working in small groups. It appeared that the prediction questions provided students in the treatment class an impetus and focus for classroom discourse, whereas the control class discussion was centered exclusively on the student-generated results of the problem.

It is evident that students in the control class had limited exposure to the potential mathematical ideas that were inherent in the problem but failed to surface in the discussion. Students in the control class merely discussed the results of the problem, and touched upon the two different perspectives of doubling. In fact, the teacher drew the conclusion for the students

("So you guys are looking at the cut [the number of cuts] to double, and you guys are looking at the ballot to double, there's the difference"). The control class discussion demonstrated 4 occurrences of CU1 (represent patterns in tables, graphs, words, and equations) and 3 occurrences of R3 (analyzing and evaluating a problem situation is defined as making information from the problem useful for solution). No other conceptual understanding or reasoning indicators were exhibited in this discussion. That is, conceptual understanding and reasoning indicators were minimally activated. The interactions during the control class were dominated by the teacher, and students' participation was limited.

Conversely, students in the treatment class, to a greater extent were afforded more opportunities to think about, discuss the mathematical concepts of the lesson and extend their thinking. The treatment class discussion demonstrated 7 occurrences of CU1, 5 occurrences of CU2, 2 occurrences of CU3, 1 occurrence of CU5, and 2 occurrences of CU6. With respect to the reasoning indicators 5 occurrences of R2, 2 occurrences of R3, and 3 occurrences of R4 were evident. Moreover, the treatment class conversation expanded the scope of the intended mathematics of the problem. In particular, the specific objectives of the problem were to begin to understand the effect of repeated doubling. The treatment class students also thought about whether this situation (repeated cutting of paper in half) was a linear relationship or something else, although this was never explicitly asked of students to consider in the problem context. The discussions in the treatment class can be characterized as more dynamic compared to the discussion of the control class.

It appeared as though the prediction questions provoked and contributed to a more vibrant discussion on rich mathematical ideas beyond merely getting the answer to the given problem, which provided the conceptual understanding and mathematical reasoning indicators to emerge. In fact, various conceptual understanding and reasoning indicators were present in students' written responses to the prediction question, which is evidenced in the following prediction responses as examples.

If he has one piece of paper and cut it in half he has 2 parts. 10 x 2 =20. 20 ballots.

Yes, I can predict *the amount. If there are 2 halfs* (sic) *and you cut them again, you will double that amount 2 x 10.*

2,048 ballots. Because every time you put something in half you get twice as many. You will keep going up twice as much as the number before it.

Both responses were coded as CU1, CU6, R2, R3, and R4. However, reasoning in these responses illuminates two different arguments, leading to revisiting the concept of linearity. Such an opportunity to synthesize thinking prior to the discussion triggered various aspects of the conceptual understanding and reasoning indicators to help students build connections, and bring these ideas to the discussion. The statements put forth by the students made during these discussions reflected the thinking that students were afforded the opportunity to conceptualize by responding to the prediction questions.

The teacher interview and journal data also suggested from the teacher's perspective that the prediction questions did indeed provide the students with opportunities to interact with the conceptual understanding and mathematical reasoning indicators. These opportunities she felt, contributed to more rich discussions and provided her with a focus to help drive the discussion. The teacher also believed that the students in the treatment class appeared to be more motivated and more engaged in the mathematics of the lessons.

Indicator	Treatment Discussion	Indicator	Control Discussion
	T: If he makes 10 cuts can you predict how many ballots he has?		T: Alright, so you started off with one cut, so I have a piece of paper I cut it, how many ballots do I have 2, cut it how many do I have 4? Cut it again.
CU1, 6	S1: I think it's just 2 to the tenth power		How many of you said 6? Parker, why did you think it was 6 ballots?
CU1, 5, 6	T: You think it's 2 to the tenth power?		S1: I thought it would double, 2, 4, 6, 8.
	S2: I looked at the table on page 5 and it looks like 10 x 2	CU1	[The teacher asks other students who also said 6]
	T: So you think it will be 20?		T: The rest of you got what?
CU2/R4	S3: It would be 20 because it's a linear relationship		S2: 8
	T: Why would you say it's linear?		T: Why did you get 8?
CU1, R2 R3	S3: Because it's just doubling		S2: Because 2 to the third power is 8. If you cut once you get 2
	T: So you would just consider that linear	R3	T: So you were using powers. Thomas what were you doing?
CU1, 2, R2, R3	S4: I think it's linear too, because it's doubling		S3: Well when you divide the sheet it doubles
	T: So you're thinking the same thing, it's linear because it's doubling anybody else?		T: Garrett?
	T: How many of you think it's linear, other than it's doubling	R3 CU1	S4: Doubling it
R2, R4, CU1, 2	S5: It's not linear because it's multiplying by the same number, because each time you cut you're multiplying by the same number it's squaring	CU1	T: So how is that doubling it is different from Parker's doubling?
	T: So you think instead of doubling it's squaring,		S5: You're doubling it from the last number instead of the number of cuts
	Will, what do you think?	CU1 R3	T: So you guys are looking at the cut to double -, and you guys are looking at the ballot to double, there's the difference.
	S6: There's no constant [rate]		At this point students begin working on the ACE problem for homework
	T: There's no constant, so there's no pattern you think		
CU2, 3 R4	S7: If you graphed it, the line would get steeper every time		
CU1,3, R2	T: So you looked at it from a graphing point of view, saying that it's not going to look like a straight line		
R2, CU1, CU2	S8: Every time you ripped the paper, it's doubling what you had before and that's increasing what you have from before. That's the reason I don't think it's linear		

Conclusion

In order to make predictions students must think about important mathematical ideas imbedded in the problem context. The prediction questions helped students draw on prior knowledge and build connections among mathematical ideas. Furthermore, these prediction questions allowed students in the treatment class to explicitly and implicitly interact with the conceptual understanding and mathematical reasoning indicators. This precursory thinking contributes to more rich discussions that are built upon the ideas that students bring to the table. As evident in the comparison of the two classrooms' discussions, prediction questions enabled students to think about related mathematical ideas, rather than mechanically solving the problem. By doing so, prediction questions provoked a dynamic student discussion on mathematical ideas in the process of thinking, not necessarily as an ending product or conclusion.

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BROKERING AS A MECHANISM FOR THE SOCIAL PRODUCTION OF MEANING

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In this paper we highlight three generalizable brokering moves that can function as a mechanism for the social production of meaning. These broking moves, which were identified through analysis of classroom videorecordings from an undergraduate course in differential equations, facilitated the emergence of a complex and sophisticated inscription known as a bifurcation diagram. The three brokering moves we detail each involve different types of efforts to influence the degree of continuity between communities: the broader mathematical community, the local classroom community, and the various small groups that make up the local classroom community. The analysis highlights how various members of the class can function as brokers, with the teacher playing a unique role as a member or peripheral member of all three communities.

Introduction

A pressing concern in mathematics education is to reveal processes by which inquiry-oriented classrooms enable learners to explore and develop their own reasoning powers while simultaneously connecting them with the collected wisdom and conventions of the discipline (Cobb & Bauersfeld, 1995; Lampert, 2001). A teacher's role in this process is one that often comes with considerable tension. For example, in her work with elementary school students, Ball (1993) posed the tension in the following way: "How do I create experiences for my students that connect with what they now know and care about but that also transcend their present? How do I value their interests and also connect them to ideas and traditions growing out of centuries of mathematical exploration and invention?" (p. 375). Research in inquiry-oriented undergraduate mathematics classroom reveals similar tensions regarding the role of the teacher and other students in the social production and uptake of ideas (e.g., Wagner, Speer, & Rossa, 2007).

In this report we address the aforementioned pressing concern by identifying the brokering moves of the teacher and some students in an undergraduate mathematics class that functioned as a mechanism for the social production of meaning. From an individual cognitive point of view, there are well-established mechanisms that describe how individuals build ideas. From a social point of view, however, mechanisms that describe how ideas are interactively constituted are less developed, especially at the undergraduate level. Such mechanisms are significant because they address the complex job of teaching and specify teacher moves that promote the social construction of meaning (Rasmussen & Marrongelle, 2008).

In his seminal work on communities of practice, Wenger (1998) highlights how brokering has the potential to "cause learning" by introducing into one community elements of practice from a different community (Wenger, 1998, p. 109). We adapt Wenger's work to the classroom and consider three different communities: the broader mathematical community, the local classroom community, and the various small groups that make up the local classroom community. The brokers in these communities are the teacher and specific students in the class. A broker, by definition, is someone who has membership status in more than one community. For example, in our case the teacher is a member of the broader mathematics community, the

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classroom community, and a peripheral member of each of the small groups that make up the classroom community.

Methods

Data for the analysis is drawn from classroom videorecordings collected during a 15-week classroom teaching experiment (Cobb, 2000) conducted in an undergraduate differential equations course. There were 30 students in the class, roughly split between mathematics and engineering majors. The brokering moves we identified came from a retrospective analysis of videorecordings from two cameras during three contiguous class sessions. We focused on these three sessions because we recognized that something powerful happened in terms of student learning on these days and we wanted to characterize it. Maher and Martino (1996) refer to such occasions as “critical events.” Such events involve conceptual leaps or significant mathematical progress and hence demand attention and explanation. In our case, the significant event was student reinvention of a bifurcation diagram.

We began analysis by creating complete transcripts of each class session. We then engaged in an iterative cycle of examining the data and writing detailed summaries of the events that transpired. Through this process and reflection on the literature, we refined our understanding of the phenomenon under consideration and characterized the students’ mathematical progress in terms of the emergence and evolution of a classroom mathematical practice (Rasmussen, Zandieh, & Wawro, 2009). Further analysis of the data resulted in identification of various brokering moves that facilitated the emergence and evolution of this classroom mathematical practice. Subsequent synthesis of the various brokering moves resulted in identifying three main types of brokering moves.

Results and Discussion

We refer to the three broad categories of broker moves as creating a boundary encounter, bringing participants to the periphery, and interpreting between communities. Each of these brokering moves is exemplified in the following paragraphs.

Creating a Boundary Encounter

A boundary encounter refers to direct encounters such as meetings, conversations, or visits between communities. Any boundary encounter will involve boundary objects. Boundary objects refer to objects that serve as an interface between different communities. A broker, by virtue of his or her membership in more than one community, is in a position to bring forth boundary objects that can facilitate encounters between communities. Wenger’s examples of boundary encounters all entail direct meetings, conversations, or visits between communities. We adapt this notion to also include indirect encounters between communities. For example, except in rare cases, it is not feasible for direct encounters between the broader mathematics community and the classroom community to occur. Instead, the teacher as broker can offer indirect opportunities for the classroom community to encounter the broader mathematical community. The teacher can do this by offering opportunities for students to engage in the broader discipline practices of modeling, symbolizing, defining, algorithmatizing, or proving (Rasmussen, Zandieh, King, & Teppo, 2005; Schwarz, Dreyfus, & Hershkowitz, 2009). Such encounters allow for the possibility of participation in the authentic practice of mathematics, and hence provide occasions for the local classroom community to indirectly encounter the broader mathematics community.

We highlight two exemplary instances of “creating a boundary encounter” from our analysis. The first example is between the local classroom community and the broader mathematics community, while the second example is between the classroom community and one of the small groups. Our first example focuses on the teacher as broker in his role of selecting and constituting tasks. Because the teacher is a member of both the broader mathematics community and local classroom community, he is in a position to recognize characteristics of tasks that are likely to be productive for enabling newcomers to engage in the discipline practices of mathematics, such as modeling and symbolizing. In the first example, the teacher recognized that a particular task (which we refer to as the fish.com task) might function as a boundary object because he could use the task to engage students in creating a new differential equation to fit a revised set of assumption. In this way, the students indirectly encountered the mathematics community via modeling.

After creating a modified differential equation, the way in which the teacher constituted the remaining portion of the task was crucial in actually creating a further boundary encounter. In particular, the teacher constituted the task in such a way that it opened up the possibility for students to engage in symbolizing. Specifically, the teacher invited students to develop a report to an imagined audience, rather than requesting students to recommend a specific numerical answer. Constituting the task as a report to the new owners, without prescribing what form this report was to take, allowed students to develop their own inscriptions. Thus, the modified differential equation and task to develop a report functioned as a boundary object because it provided the classroom community an opportunity to encounter the mathematics community via participating in the discipline practice of symbolizing.

The second example comes from our analysis of Lorenzo and Kenneth’s presentation of their work on the fish.com task. In inquiry-oriented classroom settings, it is fairly common for particular small groups to present their work on a problem to the entire classroom community. Such presentations represent the opportunity for a boundary encounter between one small group and their local classroom community. In our experience, not all such small group presentations realize this opportunity. For example, a small group presentation that simply reports back to the class what their group did without a substantive exchange of ideas and interpretations leaves the interface between communities somewhat empty. In order to fulfill the potential for a boundary encounter, there has to be some boundary object that leverages differences between the communities and actions by brokers to encourage an exchange of ideas and interpretations between communities.

The report by Lorenzo and Kenneth included an inscription that the class had never encountered. This inscription, which was the product of Lorenzo and Kenneth’s own creative efforts, was a graph of equilibrium values, *P*, versus constant harvesting values *k*. A portion of their report that shows this graph is shown in Figure 1.

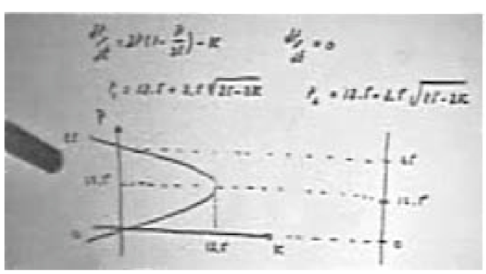

Figure 1. A graph of equilibrium values versus harvesting rate

The P versus k graph presented by Lorenzo and Kenneth's small group functioned as a boundary object because this particular inscription was entirely new to the rest of the class and was the center of a substantive exchange between Lorenzo's group and the whole class. As they presented their report, Lorenzo and Kenneth functioned as brokers as they carefully explained how to mathematically interpret their novel graph of P versus k.

Kenneth and Lorenzo's presentation positioned them as brokers between their small group and the larger classroom community. Their small group also included the teacher as a peripheral member. This is noteworthy because while their carefully explained mathematical connections opened the door for the classroom community to interface with their new inscription, it was the teacher who pushed the door open even further. In particular, the teacher used their presentation to make explicit a connection to a previous small group presentation and to raise questions about how to further interpret the novel inscription. As our two examples of the more general brokering category of creating a boundary encounter illustrates, this particular type of brokering move sets up the opportunity and conditions for the boundary encounter to be realized. Our remaining two categories of brokering moves exemplify different ways that encounters between communities are realized.

Bringing Participants to the Periphery

Broker moves in the second category—bringing participants to the periphery—help or encourage participants to move toward another community along a continuum. This is in contrast to the first category in which participants were set up to encounter the other community in a way that highlighted some difference or discontinuity. In the preceding examples the broker's role was to create the opportunity for participants to engage with a boundary object (task, inscription) that was relatively new to them. This set up a boundary encounter in the sense of being an encounter between discontinuous, distinct communities—those who would already know how to engage the task versus those who would not or those who had created and interpreted an inscription versus those who had never seen it before.

In comparison, bringing participants to the periphery is about moving along a continuum between communities. We tender two examples of this category of brokering move. Our first example of bringing participants to the periphery highlights the interplay between the local classroom community and the mathematical community. During Lorenzo and Kenneth's presentation the teacher layered what is called a flow line (or phase line) on top of the student generated P versus k graph (see Figure 2). Flow lines were familiar inscriptions to these students and were used in Lorenzo and Kenneth's report as well a in the reports from several of the other reports, but they had yet to be related to Lorenzo and Kenneth's novel P versus k graph.

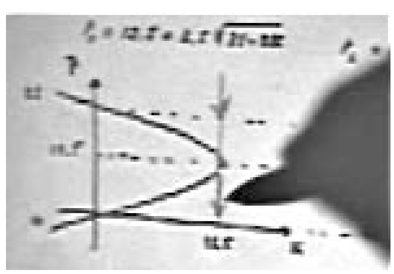

Figure 2. Layering a flow line on the P versus K graph

The teacher then handed over responsibility for layering the flow lines to Kenneth, in a move that allowed the students to take ownership of the embellished graph. The initial layering of a flow line was a brokering move by the teacher that served more as the creation of a boundary encounter. This initial step introduced a new interpretation to the *P* versus *k* graph that brought it closer to the mathematical community's notion of a bifurcation diagram. The initial introduction of something completely new in this way is a brokering move the presents a discontinuity. However, as the teacher encouraged Kenneth to make further interpretations of phase lines for the *P* versus *k* graph this became more of a process of encouraging Kenneth, and the class with Kenneth as their representative, to reason and symbolize in ways that were more consistent with the broader mathematics community.

For example, in Figure 3 we see Kenneth adding additional flow line to their *P* versus *k* graph. As Kenneth did this, he stated that you could drop a phase line "anywhere" on the graph, and chose to do so for $k = 6$.

Kenneth: Yeah, if you were to drop, like, a phase line on top of the graph anywhere, like if we were to drop one here [draws a vertical line at $k = 6$, see Figure 3] you'd see that increasing in there [draws an upward facing arrow on the line just drawn inside the parabola], decreasing there [draws a downward facing arrow on the line above the parabola], decreasing there [draws downward facing arrow on the line below the parabola].

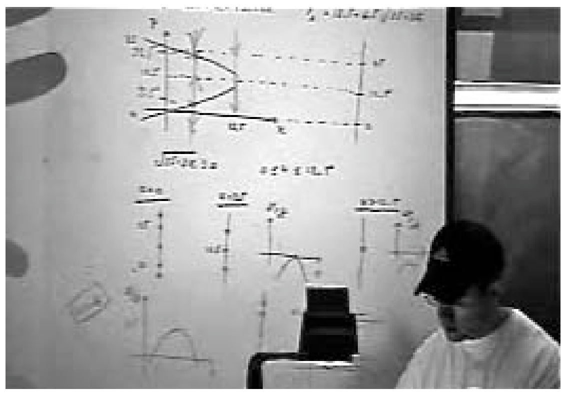

Figure 3. Kenneth adds a phase line on the P versus k graph

As this short transcript illustrates, Kenneth's addition to the graph highlights how a graph of *P* versus *k* can have meaning for regions inside and outside the parabola. The intended meaning of such inscriptions is now much more in line with the expectations of the broader mathematical community. In this way, Kenneth and the rest of the class have been drawn into the community of mathematics.

Our second example of bringing participants to the periphery highlights brokering moves between the local classroom community and smaller groups within this community. The class sessions we analyzed contained many examples of the teacher acting as a broker to encourage members of one community to engage ideas of another community or for members of one community to explain their ideas more fully to another community. In this way the teacher is requesting that participants move toward another community through a continuous periphery. For example, after Brady (student presenter prior to the presentation of Lorenzo and Kenneth) had completed his initial explanation, the teacher said, "Brady, let's pretend that I'm an owner and I make a lot of money, and I'm an executive, but I'm not so sophisticated with tables and stuff. So I need some help understanding that table that you've got up there." This request for clarification asked Brady to extend the ideas of his group in more detail and more clearly to the rest of the classroom community. A few minutes later after further explanation from Brady, the teacher asks the class, “So that’s a question to you all—does that explain what his table is? If you’re the owners, are you understanding what he’s, what information he’s provided?” In this way the teacher requests the members of the classroom community to engage in the ideas of Brady’s group and thus bring themselves toward the periphery between Brady’s group and the classroom community.

Interpreting Between Communities

As the label for this category implies, interpreting between communities is a brokering move in which brokers facilitate the understanding of one community regarding how ideas are construed, notated, related, or labeled by another community. In comparison to the first brokering category, creating boundary encounters, this third type of brokering move occurs when a broker takes specific steps to fulfill or realize the opportunities that the creating boundary encounter moves offered. We provide two illustrative examples of this particular brokering category. The first example involves brokering between the local classroom community and Lorenzo and Kenneth’s small group. In the second example the brokering takes place between the local classroom community and the broader mathematical community.

Our first example of interpreting between communities comes from an episode in which Lorenzo responds to a student’s question about a graph d*P*/d*t* versus *P* shown in the case when the parameter *k* is greater than 12.5. Lorenzo’s response to this question went well beyond an answer to the particular question. In particular, Lorenzo, acting as a representative for his small group, elaborated how the case where $k > 12.5$ fit with the other cases and how their group understood various connections to their novel *P* versus *k* graph. This explanation was significant because it served the purpose of framing how others’ analyses could be interpreted in terms of one or more of these cases.

Lorenzo continued connecting his group’s work to that of the others by explicating various relationships for the case when $k = 12.5$. Specifically, Lorenzo pointed to the vertex of the *P* versus *k* graph (see Figure 4a) and explained that, “If you take *k* to be exactly 12.5...these two graphs of equilibrium solutions meet at this point here.”

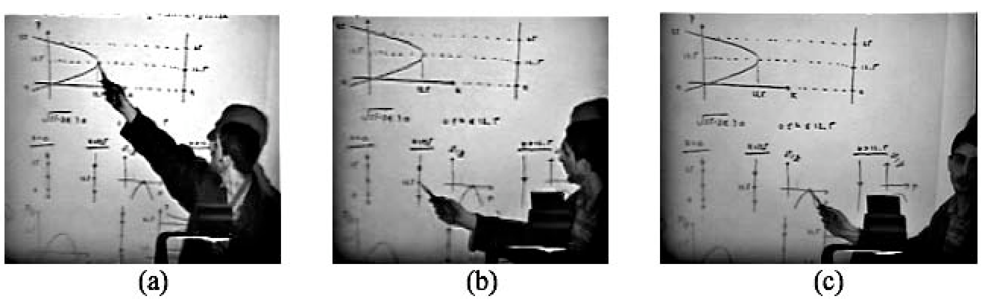

Figure 4. Making connections to the graph of P versus k when k=12.5

Here Lorenzo made an explicit connection to their algebraic equations for d*P*/d*t* = 0 and their novel graph of *P* versus *k*. Lorenzo continued his explanation by purposefully linking the *P* versus *k* graph to two other inscriptions with which the class was more familiar. As he pointed to the node on the flow line (Figure 4b), he stated, “and you have only one equilibrium solution.” He immediately connected this to a d*P*/d*t* versus *P* graph when he stated, “You can also see how if you graph d*P*/d*t*, that, uh, your population is going to be decreasing all the time except at one point,” as he pointed to the vertex of the parabola (Figure 4c). Lorenzo was careful in his

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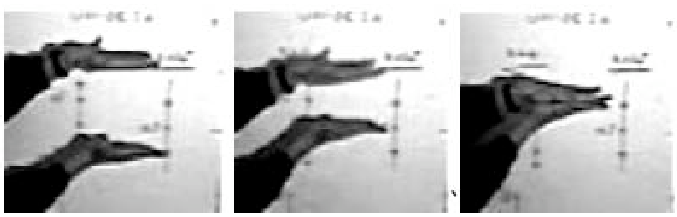
pointing, and through this care, he linked the inscriptions together by explaining how to construe the same information from three different graphical inscriptions.

We see Lorenzo's pointing as a type of linking gesture that facilitated his efforts to interpret between communities. Linking gestures are often used to "provide conceptual correspondences between familiar and unfamiliar entities" (Nathan, 2008, p. 376). Here we see that Lorenzo was able to provide conceptual correspondences from what his classmates were already familiar with to a new, unfamiliar inscription through a careful use of pointing gestures. In relating what was familiar to the class to what was unfamiliar, Lorenzo's use of linking gestures facilitated his efforts to interpret between communities.

Our second example of interpreting between communities highlights the teacher's unique brokering role as the only person who is a member of both the broader mathematical community and the classroom community. Given this unique position, it is the teacher who can (re)interpret the mathematical ideas that are emerging in the local classroom community in terms of the conventional or formal terminology used by the broader mathematical community. In this way, the teacher can infuse formal terminology into the discourse of the classroom community.

In our second example of interpreting between communities, the teacher inserted the conventional term "bifurcation" by using linking gestures to connect the familiar with the unfamiliar. One of the most familiar inscriptions for the classroom community was that of the *P* versus *t* graphs. The term bifurcation was unfamiliar to the classroom community, however the fact that the structure of the solution space is different for different *k* values was becoming increasingly familiar for students. In Figure 5, we see the teacher use a series of gestures that link the changing number of equilibrium solutions to the term bifurcation. In particular, the teacher extended his hands and forearms in a parallel manner to portray the parallel equilibrium solutions on the *P* versus *k* graph, and then he brought his hands and forearms together (Figure 5c), at which point he explained that the "technical" term for the parameter value at which there is a change in the number of equilibrium solutions is "bifurcation value."



(a) (b) (c)

Figure 5. Introducing conventional terminology

Through these moves, the teacher explicitly introduced the conventional or formal term "bifurcation" at a point in the classroom discussion when it served the function of labeling an idea that was an emerging part of students' mathematical reality. We see this as a noteworthy departure from teaching that often starts lessons with formal or conventional terminology because it enables students to see themselves as capable of participating in the cultural practice of mathematics.

Conclusion

In this report, we introduced three generalized broker move categories: creating boundary encounters, bringing participants to the periphery, and interpreting between communities. Each of these brokering categories highlight the view that teaching and learning mathematics is a cultural practice, one that is mediated by and coordinated with the broader mathematics community, the local classroom community, and the small groups that comprise the classroom community. Because these categories were developed out of two days of classroom data, we make no claim that these categories are exhaustive. Furthermore, we contend that both the course content and the timing of the data observed influenced the categories' formulation. Thus, we expect that observing other data sets would result in the creation of additional broad categories or in the facilitation of a sharper definition of the existing categories through the creation of subcategories. It is to this end—observing more data sets for the expansion of the categories as well as for a sharpening of the existing categories—that we anticipate a direction for future research.

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DEVELOPING MULTIPLICATION FACT AUTOMATICITY THROUGH GAMES

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This research study examined the effects of using games to help students gain multiplication fact automaticity. This study compared two fourth-grade classrooms in which one played multiplication games and the other did not. An independent-samples t test indicated that the games had a positive effect on students' learning of multiplication facts. Case study analysis revealed that games helped students develop more sophisticated strategies for figuring out hard to remember facts. These strategies included geometric images, relationships between factors and products, double counting, and the use of related facts.

Introduction

Memorizing multiplication facts has long been a part of elementary school. However, many students seem to have trouble with the automaticity, automatic recall of facts. According to Caron (2007), "That multiplication table, ten by ten columns that we insist with good reason our students learn between third and fourth grade remains a mystery to some" (p. 278). Multiplication fact recall creates a solid basis which students use to further their mathematical thinking as they mature into middle and high school. Indeed, without the ability to quickly retrieve multiplication facts, students cannot perform higher-level tasks such as finding common multiples when adding fractions or solving algebraic equations (Woodward, 2006).

Games are one way to motivate students. Intermediate grade students specifically enjoy games in which they must find and use a strategy to win. At the same time that they are having fun and strategizing, students are also developing a conceptual understanding of patterns and number facts (Olson, 2007). Games in the classroom create an opportunity for students to learn from their peers by comparing their strategies to become better players. Different game choices will likely appeal to multiple intelligences since students are able to choose the game they feel is the most beneficial to them and are encouraged to use those games to achieve automatic recall of multiplication facts. The purpose of this study was to determine the effectiveness of using multiplication games to aid students in multiplication fact automaticity as well as determining how the games helped students develop their mathematical thinking.

Literature Review and Theoretical Framework

Elementary school students are asked to memorize their multiplication facts is so that they can use them as a basis to solve more complex problems (Kilpatrick et al., 2001; Smith & Smith, 2006). The constant reliance on counting fingers, especially for larger products, gets in the way of using more sophisticated strategies for finding the answers to difficult-to-remember facts. Knowing the basic computational facts is theorized to free the brain from figuring out the fact to using the fact in more sophisticated (Donovan, Bransford, & Pellegrino, 2007; Kilpatrick et al.; Woodward, 2006). Students need begin with a conceptual understanding of multiplication before they can have a firm grasp of their multiplication facts (Smith & Smith, 2006). Students can better retain information when it holds meaning for them. Some suggested strategies to help students gain a conceptual understanding include (a) sharing their strategies, (b) using manipulatives, (c) creating problem-solving situations in which students must use multiplicative reasoning, (d) playing games, and (e) carefully sequencing learning activities (Isaacs & Carroll, 1999).

When students communicate their thinking in mathematics, it leads to higher-level learning (Chapin, et al., 2003). Classroom atmospheres should encourage such communication. Students retain more information when they are excited about it and can relate to it. Therefore, it is the teacher's responsibility to provide opportunities for their students to make personal connections to new learning and to give students time to share those connections (Gallenstein, 2003). In addition, Cooke and Buchholz (2005) and Olson (2007) found that providing students with materials, such as games, to explore and asking them mathematical-related questions about their interactions with those materials can generate participation in mathematical discussions. When students are excited to learn, they are more likely to participate in mathematical discussions. Finally, facilitating mathematical discussions allows the teacher to focus on the individual learners in the classroom and use that new knowledge to guide further instruction (Fuson, et al, 2005).

Methods

A quasi-experimental design (Creswell, 2008) using mixed methods were used to investigate whether games support students' learning of math facts. Quantitative analysis allowed us to determine whether the games had a positive effect on student learning. The two groups of students were Author 1's fourth grade students, one from 2008-2009 and the other from the 2009-2010 school year. Qualitative methods using a case-study design with constant comparative and matrix analysis allowed us to describe how games supported students' learning of multiplication facts.

Context and Participants

Greenwood Elementary School (pseudonym) in the Pacific Northwest was in a mixed neighborhood comprising of an upper middle class neighborhood and a low income housing complex. Math fact automaticity was a school-wide goal: 85% of the students would correctly answer 100 facts within 5 minutes by the end of 2009-2010 school year. The participants in this study included two fourth-grade classrooms taught by Author 1. Class A was from the 2009-2010 school year and Class B was from 2008-2009. The students in both classes were randomly assigned a school administrator and then the third-grade teachers balanced the student groups for students with learning disabilities, behavior management plans, high abilities, and English language learners. The demographics of the two classes were similar. We selected three low-scoring students (Trenton, Mandy, and Alyse) who did not qualify for special services for case study analysis.

Intervention

Students in Class A (2009-2010) were asked to study multiplication flash cards at home and they played three multiplication games (Akers, et. al., 1997) a minimum of three times a week. Students in Class B (2008-2009) were also asked to practice multiplication flash cards at home and were given weekly timed tests to determine student growth.

Three multiplication games were used. The first, Product Tic-Tac-Toe, helped students to learn their facts because only a small set of facts was focused on, which helped them gain conceptual understanding by grouping related facts (e.g., 3, 6, 9). Students found the answers to the same problems more than once. This repetition helped them to gain better fact fluency. The second, The Factor Game, further supported the development of a conceptual understanding by creating an opportunity for students to investigate a number and its factors. In turn, this helped them learn their facts by focusing on the actual factors of products. Students were challenged to find all of the factors of given products rather than just a few. The third, Array Flashcards, helped students to learn their facts by providing them with a visual representation of each of the facts they found hard to learn. This visual representation was used to support students who intuitively think about mathematics from a spatial sense rather than numbers and their relationships (Akers et al., 1997).

Product Tic-Tac-Toe. Product Tic-Tac-Toe is a game in which students use a pre-made tic-tac-toe card. Figure 1 shows an example of a tic-tac-toe card for students who are working on their 3's, 6's, and 9's. Each square contains the product, or answer, to a multiplication problem using the factors below the table.

1	3	6
9	18	27
36	54	81

1 3 6 9

Figure 1. Tic-tac-toe card for students working on 3's, 6's, and 9's

To begin, player 1 places a paper clip on one factor below the game board. Player 2 places a second paper clip on another factor below the game board and then puts a marker on the number within the table that represents the product of the two factors. Player 2 then moves only one paper clip to a different factor and then puts a marker on the number representing the product of those two factors. The students continue playing until one of them has three products in a row or a draw is declared.

Factor Game. The Factor Game is one in which students are given a board with products, and each student takes a turn selecting a product on the game board (Figure 2). The partner then finds all of the factors for the selected product and earns those points.

0	1	2	3
4	5	6	8
9	10	Free	12
15	16	18	20
24	25	30	36

Figure 2. Factor Game board for use with factors 0 through 6

Using the game board in figure 2, Player 1 might choose 30 and earn a score of 30. Player 2 identifies the factors of 30 including 1, 2, 3, 5, 6, 10, 15. The sum of these factors becomes his/her score, 42. Then, Player 2 chooses a product and Player 1 selects the remaining factors for their respective score. Once a number has been chosen as a product or a factor, it can no longer be used by either player. The game continues until all factors on the board have been used. The winner is the player with the highest score.

Array flashcards. Students choose the multiplication facts that were the hardest for them to remember. Using centimeter grid paper, students cut out arrays that represented each multiplication fact that was difficult to recall. Students wrote the multiplication problem on one side of the array and the answer on the other side. Then, students either practiced independently or quizzed each other on these facts.

Data Collection and Analysis

To determine whether there were significant differences in the number of facts students could recall between Class A and Class B, pre- and post-test design was used. Students in both classes were given a pre-test in September consisting of 40 randomly selected single-digit multiplication facts with factors between zero and ten to assess their automaticity. The post-test was given in November and consisted of the same 40 facts. The difference between the pre-test and post-test was used to indicate *growth*. An independent-samples *t* test was conducted to evaluate the difference between the mean growth of the two independent groups, Class A and B.

Three data sources were collected to describe how students used the multiplication games and their game preferences. The case-study students were interviewed three times using a semi-structured protocol (Creswell, 2008), daily field notes were made, and students wrote reflections on their learning and game preference. Field notes included detailed descriptions of students while they were engaged in the games, their conversations with each other, and whole-class discussion. Analysis of these data included constant comparative methods to collapse the data into meaningful categories. Matrix analysis allowed for cross-case comparisons to generate aspects of games that supported students' memorization of facts.

Results and Discussion

An independent-samples *t* test indicated that the games had a positive effect on students' learning of multiplication facts, $\alpha<.05$, $t(48)=2.05$, $p=.046$. The number of facts that students learned in the class A ($M=26.56$, $SD=15.59$) was significantly higher than the students who did not use games ($M=18.32$, $SD=12.77$). A weak effect size (0.16) may be the result of using the games for nine weeks and the small number of students (26) in each class. After nine weeks of playing the games, the case students showed growth. The number of facts that Alyse knew increased by 10%, Trenton's growth was 38%, and Mandy's growth was 35%.

Before beginning the study, the case-study students were asked which facts were the easiest for them to remember. All three of the students found the 10's facts were the easiest. Trenton said, "They were easy because all he had to do was look at the number and add a zero to the end of it to find the correct answer." Alyse said she could count by tens easily so it didn't take her long to find the solution. Mandy agreed that she could also add ten to a number quickly to find the answer. Next, we asked the students which facts were the hardest for them to learn or remember.

1	Trenton	Nines. I just don't know them very well.
2	Alyse	Nines. We never really did them last year. I remember a few, but
3		that's it.
4	Mandy	Eights. I don't know them at all.
5	Author 2	What do you do to help you learn the facts that are hard for you?
6	Trenton	I count in my head. Like, ten times five. I can draw ten circles
7		and put five in each, or I can use my fingers.
8	Alyse	I count on my fingers or use counters. It's easier to remember
		that way.
9	Mandy	I count with counters.

It was obvious from our conversations with these students that they needed more exposure to their math facts. Nines seemed to be a main problem (lines 1 & 2). This was not a big surprise to us as many students find it hard to find multiples of nine. Clearly these three students were relying heavily on their fingers and/or counters to use repeated addition to find the answers to multiplication problems (lines 6-9). When students used repeated addition, there are two mechanisms to count.

The first can is referred to as *count all* (Carpenter, Fennema, Franke, Levi, & Empson, 1999). Students use the count all mechanism when they create the required number of groups with a corresponding number of objects in each and then count all of the objects. For example, when multiplying 3 x 4, a student may draw 3 groups of 4 tick marks on his paper. Then he counts each tick mark, disregarding the groups. The second, *double counting,* is a mechanism in which a student counts individual items in a group simultaneously with the number of groups (Steffe, 1994). If a student were multiplying 3 x 4, she may count 1, 2, 3, 4 on her left hand and then hold up one finger on her right hand signifying that she counted one group of four. Then, closing her left hand, she continues counting 5, 6, 7, 8 moving one finger with each number. After counting eight, another finger on her right hand is straightened. Following the same procedure of counting and extending a finger, she reaches 12 with three extended fingers on her right hand. Here, the student kept track of the number of groups with her right hand and on her left hand she counted individual objects in the group. Vergnaud (1988) suggested that this double counting was fundamental to multiplicative reasoning, essential for more sophisticated thinking. These three students relied on repeated addition in which they counted all.

The first game that Author 1 introduced was Multiplication Tic-Tac-Toe. She explained the rules and asked a student to play with her in front of the class. Trenton immediately gave Author 1 advice for one of her next moves, demonstrating his understanding of winning strategies through these suggestions. His enthusiasm was hard to contain with his exclamation, “Oh, I get it!”

After playing the game for several weeks, Trenton described his game strategy. “I put the paperclips on numbers so that I could get numbers on the board. If there was a number I wanted, I tried to get the paperclips to equal that number.” Through his explanation does not clearly articulate how he made his decision, it is clear that he is thinking about different combinations of products that he can make using the factors. While engaged in this thinking, he also recognized that some products could be created in more than one way and indicates a growing level of mathematical thinking. Likewise, Alyse was beginning to select factors to make particular products. She explained, “I had to see what was available on the board and then think what I could do with it. Like if 12 was available, I would try to get the paperclips to 6 and 2 so I could equal it.” In contrast, Mandy focused her attention on blocking rather than trying to get three-in-a-row by claiming the needed product first. A more sophisticated blocking strategy for larger game boards would be to avoid putting a clip on a factor that could be used to create the needed product.

The second week, Author 1 introduced the class to The Factor Game. Again, the excitement was high. This game proved to be more challenging; students had to know the factors of a number in order to be the most successful in the game. However, this did not seem to detour the students from wanting to play. Mandy, Trenton, and Alyse used their fingers to skip count to see if a number was indeed a factor of the product that had been chosen and developed strategies which varied in sophistication while playing.

Mandy’s strategy was to select the highest product on the board. She reasoned that she could boost her score quickly. However, she did not realize that sometimes the highest product could give her partner more points. For example if she chose 24, her partner could identify the factors 1, 2, 3, 4, 6, 8, 12 giving them a score of 36 points. Trenton noticed that prime numbers have only one factor on the board, so he developed the strategy of selecting primes. Alyse struggled to identify factors and found the game laborious. However, she gained an insight; some numbers have more factors than others.

The third week, Author 1 demonstrated the construction of array flashcards. She explained, “You needed to make flashcards for the facts that were hardest to remember.” To Author 1’s surprise, the three case-study students happily made arrays for the facts that they did not know. Even though they had more arrays to construct than their classmates, these three students were busy counting squares, drawing arrays on grid paper, cutting them out, and writing the corresponding multiplication sentence. Alyse, Mandy, and Trenton said that they liked using array flash cards because they knew they needed help with certain facts and it was nice to have a set of cards devoted to only those facts. Alyse eagerly explained that she enjoyed the challenge of memorizing them and couldn’t wait to weed out the ones she had memorized. Only Mandy explicitly used the size of the rectangles as a clue. She explained, “I knew that the bigger rectangles had bigger answers, so sometimes that helped me.”

After teaching the three games, Author 1 noticed that the three students were still counting on their fingers and was curious how the games supported their learning.

11 12	Alyse	I really like them. I like the array flashcards. I made them for my nines. It helped me to count out the squares [on the rows] and find the right answer.
13 14 15	Trenton	I like the Tic-Tac-Toe because I’m getting better at it. It helped a lot with my twos, fives and tens. I got to move up to the green card! (The green card was made for practice with threes, sixes, and nines.*)*
16 17 18	Mandy	I like Multiplication Tic-Tac-Toe. When I see the numbers over and over again, it helps me remember them. And I know if I got one wrong because the number is not on the card.

Alyse preferred array flashcards and seemed to use the geometric shape to help her remember the facts. She also recognized that the rows on the arrays represented repeated addition interpretation of multiplication (line 12), indicating a growing conceptual understanding of multiplication. Clearly, Tenton and Mandy preferred Multiplication Tic-Tac-Toe (lines13, 16). They had a better grasp of their facts and were beginning to recognize the relationships between factors and their products. Trenton recognized patterns between related facts and was able to advance to another set of factors (line 14). Both Trenton and Mandy recognized the value of repetition in gaining multiplication fact automaticity (lines 13 & 16-17). Mandy based her answers on the related facts on the board, indicating that she recognized relationships between factors and related products (lines 17-18).

Twenty of the 26 students said they use their fingers or pictures to count up to the answer with repeated addition. The other 6 students used a related fact to build the unknown pair. Trenton knew the multiples of 2, 5 and 10. When he multiplied 8 x 3, Trenton first found 5 x 3 and then used repeated addition by counting on his fingers to complete his answer. Speaking to himself he might think, 5 x 3 is 15, so I need 3 more groups of 3. That would be 15, pause 16, 17, 18; then 19, 20, 21; then 22, 23, 24. This strategy shows the use of a related fact and double counting. Each pause signified a count of the number of groups. In recent research (Xin, et. al.; Zang, et. al., 2009) on the development of students' with learning difficulties, double counting was found to be an intermediary step essential to transition students from repeated addition relying on fingers to more sophisticated reasoning strategies. It is clear that Trenton was using a more sophisticated strategy than his initial use of counting all. In Author 1's classroom, five other students also used more sophisticated reasoning that relied on both double counting and the commutative property. They would begin with 2 x 3 = 6 and then would then double the answer, which would result in a 12. Next, they would double the answer a third time, resulting in the answer of 24. Mathematically they broke the 8 into its prime factors and then used the commutative property. Symbolically this process is represented as follows: 8 x 3 = 2 x 2 x 2 x 3 = [(2 x 3) x 2] x 2. We theorize that Trenton was on the cusp of developing this strategy in November and that the other two case study students will also advance their thinking by using the double counting process as the school year progresses. The six students who used related facts seemed to have a firmer conceptual understanding of their multiplication facts than the rest of the class since they were able to draw upon knowledge of facts they already knew.

Conclusions

The results of this study suggest that the use of multiplication games did in fact help students to gain automaticity. As noted by Olson (2007), students enjoy the challenge of finding ways to win games while also developing a better conceptual understanding. The games helped the students learn more multiplication facts and there was evidence that one of the case-study students developed a more sophisticated counting strategy. Students enjoyed playing the games and placed greater importance on learning their facts.

Students eagerly learned the rules of the games each time they were introduced and hurried to play the game themselves. Many students in Class A wanted to practice their multiplication facts at home and this desire indicates that learning them was important. They wanted to play the games at a faster pace and also win. The students in Class A identified four ways that they practiced facts at home: (a) computer games, (b) array flash cards, (c) problems written on paper by a parent, and (d) the traditional flash card. In contrast, many students in Class B reported that they really did not practice their multiplication facts at home. As noted by Diaz-Lefebvre (2004), when students believe an activity like learning multiplication facts are important, they are more likely to engage in the necessary practice. Students in Class A were highly motivated to practice their facts because they were excited by the games.

The games helped all of the students in class A develop a better conceptual understanding of multiplication as shown through the diagrams that they drew and use of double counting when they used repeated addition. They were beginning to use more sophisticated strategies that relied on relationships between the different multiples and factors. Students in Class B were asked to memorize the facts in a context that was void of meaning and importance. They saw little to no connections between the fact and the different representations. As suggested by Diaz-Lefebvre (2004), students who do not have a deeper understanding of material are at a loss for retaining it.

Multiplication games had a positive effect in the fourth-grade classroom. Such games can be incorporated into any classroom by merely setting aside a small amount of time each week to devote to the games. It was clear that the amount of time spent playing the games was truly beneficial to students. Without making strategies explicit when teaching the games, the students relied on their own critical reasoning skills. This forced them to think on a higher level than simply using flashcards. This higher level thinking paired with a conceptual understanding may help students achieve greater success in mathematics in the future. Additional research is needed to determine the long term effect of gaining mastery of facts through games and other activities that build conceptual understanding on students' later achievement in mathematics.

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EXAMINING SOCIOCULTURAL CONTEXTS OF CLASSROOMS TO FOSTER STUDENT MATHEMATICAL DISCOURSE AND LEARNING

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Mathematics learning and teaching are optimized in classrooms when reform-oriented culture (ROC) is present. This report presents a case study that illustrates how ROC manifested and influenced mathematical Discourses in one sixth-grade classroom. The data was drawn from a study that addressed the question: How do classroom interactions influence mathematical Discourses? The study used interpretive methodology for analysis. One finding was that classroom boundary interactions either enhanced or hindered mathematical Discourses dependent upon sociocultural context alignments. An implication of this research is when "effective" learning and/or teaching strategies are identified, "effective" implementation may require paying close attention to sociocultural context alignment.

Introduction

In the past, mathematics education reform has been articulated in terms of content (NCTM, 2000), curriculum and assessment (NCTM, 1989), and teaching (NCTM, 1991); and in each of these standards documents are descriptions of sociocultural elements of mathematics classrooms and advice for transitioning classrooms from traditional to reform-oriented culture (ROC). For this study, sociocultural elements include all classroom interactions related to learning and teaching. The purpose of this investigation was to examine classroom interactions closely using a perspective that would offer insights into how mathematical Discourses (more than talk, engagement, and participation) influenced learning and teaching. This report offers a glimpse into a study (Grant, 2009) that examined how classroom interactions influenced mathematical Discourses related to learning and teaching.

The research study took place in a large urban Midwestern school district in the United States. The participants were from three sixth-grade mathematics classrooms from two different schools from within the school district and included teachers, students, and Mathematics Coaching Program (MCP, Erchick & Brosnan, 2005) instructional coaches. The study examined interactions from each classroom and then compared the three classrooms to illuminate the findings to address the research question: How do classroom interactions influence Discourses related to mathematics learning and teaching?

Methodology

The overarching method for the investigation was case study including comparative case study analysis (Stark & Torrance, 2005). Data sources for this study included videotaped mathematics instruction (~1,350 minutes), survey responses from teachers and students, and audiotaped interviews with teachers and coaches. NVivo 8 (QSR International, 2008) qualitative analysis software was used to support the data analysis.

The theoretical model used in this study was inspired by relational perspectives developed by Cobb and several colleagues (2002) The mathematics teaching and classroom practice literature (e.g., Franke, Kazemi, & Battey, 2007) suggested that classroom learning requires social and cultural interactions. A significant problem for the investigation was determining what sociocultural elements to target when studying mathematics classroom interactions. The theoretical model used in this study addressed this issue and focused the observation and analysis using three key constructs – classroom culture (social norms and practices), Discourse (more than talk, participation, engagement, and community), and relationships (that support opportunity for learning). These constructs were prominent in both the relational perspectives of Cobb and colleagues and the mathematics teaching and classroom practice literature.

The theoretical model offered a point of view that was beneficial for interpreting or making sense of the complexities of interactions that occur in a mathematics classroom (Yackel & Cobb, 1996). This theoretical model and the classroom practice literature led to defining a hierarchically organized set of codes that were used to focus the classroom observation and analysis on specific sociocultural elements of the mathematics classrooms targeted in the study.

Data analysis included both descriptive statistics and interpretive analysis (Erickson, 1986). The theoretical model was used serially – each of the three theoretical constructs in turn was used as a lens to code all of the data from each classroom (i.e., the hierarchically organized codes were assigned to specific data). In other words, all data were reviewed and coded at least three times, one pass for each construct. Then descriptive statistics were generated to describe the data and analysis quantitatively – categories with the highest density coding by construct were identified, then inferences were made and those sufficiently warranted by the data across all constructs led to claims and findings.

The study presented three case studies, one for each classroom, and a cross case analysis by construct was done to further elucidate the findings. This paper presents one of the findings that emerged from the study, but was limited in scope in an effort to be concise. The case study and analysis in this paper is from one classroom (Eva) and focused on only one construct (classroom culture).

The classroom culture construct focuses on social norms and processes related to interactions within mathematics classrooms; examples of the hierarchy and codes follow: a) cultural influencer – teacher expectations; b) mathematical process – connections; c) sociomathematical norm – student explaining; and d) social norm – listening.

Findings

One of the major findings from the study was that some classroom interactions enhance while others hinder mathematical Discourses related to learning and teaching; and the sociocultural contexts within the classroom appear to determine whether Discourse emerges or not. These types of classroom interactions that depend on sociocultural contexts are called boundary interactions. Several examples of boundary interactions are described within the case study.

The following case study and discussion are presented to demonstrate how boundary interactions manifest in practice. Three examples of classroom interactions are presented. The sociocultural context alignment in each situation described in the case enhanced the mathematical Discourse. The discussion that follows the case suggests alternative sociocultural context alignments for the boundary interactions that would likely hinder the mathematical Discourse. The alternative sociocultural context alignments were inspired by data from the study.

Eva's Mathematics Classroom

Eva taught sixth-grade mathematics for 90 minutes three times each day during this investigation. During the observation period, all of the mathematics topics in Eva's classroom were related to fractions. Eva's class included a diverse group of students. The class was comprised of slightly more males (12) than females (8), a diverse representation by race or ethnicity included a balance of Black (8) and White (8) students, and there were biracial (2) and other (2) racial or ethnic students. On average, there were 20 students present on each of the observation days.

Eva described this class as a good class, but not her best. During the initial pre-observation interview, Eva described her instructional style as one that was "organized" and "structured" (Feb. 20, 2009). Observation data validated her description; instruction followed a pattern. First, students completed bell work (two or three problems related to the previous day's mathematics) as Eva circulated the room observing student work, answering questions, and taking notes (sometimes written) of who did what and how. Next, Eva reviewed the bell work in a whole-class format that included students' explaining taking 30-45 minutes of the 90-minute instructional period. Then, students engaged in activities, usually in small cooperative groups of two to four followed by students' sharing solution strategies. During the last 5 to 10 minutes of class, Eva articulated a summary review of the days' mathematics or presented new mathematical ideas. This instructional pattern was consistent with little variation on the observation days.

Given Eva's admission of being structured and organized, it was not surprising that the normal desk configuration in her classroom was straight rows facing front. Each day I observed her classroom, prior to children entering the room, she spent time straightening the rows and preparing supplies for children's ready access or for easy distribution at the appropriate time during instruction. However, when she wanted students to work cooperatively, students reorganize desks to accommodate the collaboration and Eva's oversight ensured a timely transition. At an appropriate time following cooperative activity or before leaving Eva's classroom, students returned desks to their original positions.

Classroom Interactions on the Boundary

In Eva's classroom, there were several instances when classroom culture enhanced Discourses and other instances when it served to hinder them. Classroom interactions that hindered Discourses included: a) fact or procedural reproduction; b) low-level questioning; and c) negative social norms. Conversely, classroom interactions that enhanced Discourses included: a) mathematical connections; b) student explaining; and c) listening and respect. Additionally, there were boundary interactions – classroom interactions that sometimes enhanced and at other times hindered Discourses depended on related sociocultural contexts such as: a) collaborative sense making; b) communications; and c) teacher explaining (see Table 1 column 1). These sociocultural contexts were boundary interactions in Eva's classroom, but I cannot conclude they would manifest as boundary interactions in other classrooms.

Boundary interactions listed in Table 1 enhanced mathematical Discourses when sociocultural contexts aligned with enhancers such as those in column two. Conversely, boundary interactions hindered the mathematical Discourses when sociocultural contexts aligned with saboteurs such as those in column three. For example, in Eva's classroom for the boundary interaction teacher explaining (column one) the mathematical Discourse was enhanced when the sociocultural context alignment supported students' sharing ideas (column two). Conversely, for the boundary interaction, collaborative sense making (column one) the mathematical Discourse was hindered when the sociocultural context alignment included students tasks without choices (column three).

Table 1. Boundary Interactions with Examples of Sociocultural Context Alignments from the Classroom Culture Construct

Boundary Interactions	Discourse Enhancers	Discourse Saboteurs
collaborative sense-making communications teacher explaining	tasks with choices students' sharing ideas students' comparing mathematical approaches	tasks without choices no opportunities for sharing ideas prescribed solutions only

Table 1is not an exhaustive representation of boundary interactions and sociocultural context alignments that enhanced or hindered Discourses in Eva's classroom. These boundary interactions and sociocultural context alignments may be valid for other mathematics classrooms, but more research with a broader scope is needed before such conclusions can be made.

There were glimpses of reform-oriented culture (ROC) within Eva's classroom. One example of ROC occurred when two students' perceived they had an opportunity to share an authentic idea based upon their independent thinking. These two boys had each autonomously thought about comparing fractions conceptually instead of using one of the procedural approaches that had been the focus for instruction over the last several days. The students were asked to compare three fractions $\frac{3}{4}$, $\frac{3}{5}$, and $\frac{3}{12}$ and order them from least to greatest. The following classroom snapshot is a descriptive vignette that summarizes the interaction processes.

Classroom Snapshot 1 – Listening and Revoicing

Eva began the class discussion of the bell work by inviting two boys to share their thinking. Eva discovered the two boy's approach as she circulated the room assessing student work and understanding. The boys explained their thinking and approach without Eva interrupting or correcting errors in their explanations. At the end of each explanation and throughout the mathematical Discourse related to the bell work, Eva congratulated each boy. She revoiced what each boy had explained after both explanations were done.

In this vignette, the boundary interaction is communication and the sociocultural context alignment that enhanced the Discourse is students sharing ideas. Eva created the opportunity for student sharing. The two boys' articulated rationale was that each fraction had the same numerator and different denominators, thus all that was needed was to compare the relative sizes of the pieces by using the denominators. The students' sharing led to a broader class discussion and analysis than perhaps would have otherwise emerged had the Discourse been limited to comparing fractions using only the two procedural approaches the class had been practicing. The ensuing Discourse included students' generalizations about relationship between the magnitudes of denominators and the size of the pieces and the importance of assuming all fractions were based on an equivalent whole.

A second example of a boundary interaction is collaborative sense making and the sociocultural context alignment that enhanced the Discourse is a task with choices. The instructional segment started with Eva asking students how to show $\frac{7}{10}$ using a pictorial representation of a fraction bar. Several students contributed and described what to do as Eva drew on an overhead projector. Eva asked, "So, I've got $\frac{7}{10}$ but I have 100 squares. How would I divide this up?" and the following interaction ensued:

Classroom Snapshot 2 – Developing fraction representations

Eva: [waits ~10 seconds, repeats the question several times as she waits, and more students raise their hands] Student A1?

A1: You can make boxes of ten.

Student: no [shouting out]

Eva: So, I'd have to make boxes of ten.You're right. What do I know about the boxes of ten Student A1?

A1: Um

Eva: They need to all look how?

A1: The same.

Eva: They all need to look the same. A1, So, could I go like 1, 2, 3, 4, 5 and could I make my boxes of ten like this? [drawing 5 X 2 rectangular arrays on the overhead displayed grid paper]

A1: Yeah

Eva: You bet I could. How else could I make them? Student A2?

A2: You could take one like, one set of ten, like a row [gesturing with her hands as she speaks] and color it in.

A3: What if you took one bar [10 of the 100-square grid] and colored in 7?

The task called for students to create a pictorial representation of a fraction and yielded two different approaches and an interesting question (Line 12). This student's question was a clarifying question that enhanced the Discourse by encouraging more collaborative sense making to emerge after the initial task was completed. The fraction being represented, $\frac{7}{10}$, could have been easily done using a row of 10 blocks; however, the nature of the task, representing the fraction on a 10X10 grid likely encouraged the student to seek clarity. Nonetheless, students' collaborative sense making Discourse was enhanced because the student's question led to the class having to consider whether the proposed pictorial representation met the criteria of the original task. Using Bloom's taxonomy, the original task was a knowledge question, but the student's question was an evaluation question; the Discourse was enhanced.

The third and final example of a boundary interaction is teacher explaining (as implicit telling) and the sociocultural context alignment that enhanced the Discourse is student comparing mathematical approaches.

Classroom Snapshot 3 – Reflecting and Evaluating a Mathematical Procedure

Eva: by 4's

Students: 4, 8, 12, … [Counting by 4's up to 48].

[Students are unenthusiastic, and lose synch. Eva writes on the overhead sighs, and then runs out of space]

Eva: Do we have to count like this? Seriously guys, what would be the easiest way to do this? We'll be counting forever. What's an easier way to do this Student A?

Student A: You know how we put the numbers at the bottom and circle them? Instead of going through the whole thing.

Eva: OK. OK. Student G?

Student G: Factor tree

Eva: Guys? Factor tree. Awesome. I would say factor tree. You're probably gonna spend less time than if you do it the other way. Let's try it? Let's try factor tree.

Eva appeared to have contrived a situation that caused students to reflect and evaluate the efficiency of a mathematical approach by commenting about running out of space for writing and showing great exasperation during the execution of the mathematical procedure. Her theatrics (Lines 3 & 4), which were not in character for her, were interpreted as teacher explaining, but not telling. She asked in Line 4 if there was a better way to find the least common multiple than counting ALL of the multiples of a number. In fact, Eva asked the question multiple times. This approach had been observed as a way to focus students' attention and to generate wait time for thinking. The result was several students raised their hands to offer ideas, which evidenced several students' compared mathematical approaches and enhanced the Discourse related to learning and teaching.

Discussion

In the case study, examples of how boundary interactions when coupled with sociocultural context alignments enhanced mathematical Discourses related to learning and teaching. In this section, we will examine how those Discourses might have been hindered by Discourse saboteurs. Consider the first example in Classroom Snapshot 1, the process was described in the vignette, Discourse might have been different had Eva not created opportunities for the two boys to share their ideas for comparing the three fractions. Would the class have had the opportunity to reflect on the size of denominators when the numerators are the same? Additionally, the Discourse included thinking about the importance of the whole and the relationship it plays when comparing fractions. How helpful might it have been for a student harboring a misconception about fractional comparisons to hear three different explanations for comparing fractions conceptually followed by a comparison done procedurally?

In Classroom Snapshot 2, collaborative sense making was the boundary interaction. However, how might the Discourse have differed had Eva simply dismissed the student's proposed representation as not correct? That is, offered a task with no choice. Instead, the task was designed to accommodate "What if" questions and students were encouraged to decide whether the representation was appropriate given the task; the task afforded choices. Consider task options that are presented as choices that in practice are not. For example, suppose Eva had provided several representations for students to select the one that was correct. The cognitive rigor of the task must be considered if collaborative sense making is desired. The task that was the focus for this example was not especially rigorous for a sixth-grade class, but the pedagogical approach of valuing and using all students' input enhanced the Discourse.

In Classroom Snapshot 3, teacher explaining was the boundary interaction, even though the teacher explanation or telling was implicit. How might the Discourse been different had Eva explained explicitly that they needed to use a different approach? She could have simply told the students that what they were doing was inefficient and they could get the solution faster by using one of the procedural approaches they had been practicing, i.e., offered a prescribed solution. When Eva created wait time by asking and re-asking questions, had she not, would the same number of students raised their hands to offer their ideas about the question? Additionally, how effective are mathematical Discourses when students choose not to participate?

Eva's teaching actions appeared to be more aligned with developing integrated mathematical knowledge than sharing correct answers. Her sociocultural perspective was aligned toward reform-oriented culture (ROC), which led her to encourage students to share their authentic thinking and ideas. Eva invited and encouraged students to act this way as evidenced by the opportunities she afforded them during instruction independent of whether or not she finished her instructional goals for the day. This type of instructional perspective manifested in the emergence sociocultural context alignments that enhanced Discourses toward mathematical inquiry and sharing of ideas, ROC. The classroom culture in Eva's classroom often led to multiple student approaches for finding correct solutions and mathematical connections; restated, using a traveling metaphor, the journey was valued as much as the destination in Eva's classroom.

One implication for practice from this study is if expected outcomes fall short for new learning and/or teaching strategies, something to consider examining is the sociocultural context alignment. That is, reflect on classroom interactions and consider adjusting related sociocultural contexts such as creating opportunities for student explaining or revising the task for increased student autonomy before concluding that the new strategy was ineffective.

Conclusions

If a goal of reform is to usher in effective mathematical Discourses related to learning and teaching, then those focused on support and implementation must not lose sight of ROC and reflecting on the alignment of sociocultural contexts that enhance or hinder boundary interactions. Simply stated, ROC is about creating opportunities and space within instructional settings for students to be both learners and teachers as their authentic ideas emerge. However, sociocultural contexts must be appropriately aligned to enable authentic mathematical utterances from students as the norm rather than the exception.

Some reform-oriented learning and teaching strategies appear to be promising, but they often emerge independent of consideration for sociocultural contexts extant in today's classrooms. As educators, we must forego rigid plans focused on curriculum coverage and/or strategy implementation, and look for opportunities to allow ROC to emerge and infiltrate classrooms. As supporters of teachers engaged in reform implementation, we must be cognizant of and prepared to help teachers transform ROC stifling classroom cultures. As researchers, we must further define and articulate sociocultural nuances related to recommended reform-oriented learning and teaching strategies to support the emergence of ROC within classrooms.

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MULTIPLE VOICES IN MATHEMATICS CLASSROOM: AN ANALYSIS OF MATHEMATICS TEACHER'S TELLING

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Traditionally, teacher' telling was understood as limited to presenting mathematical facts and procedural skills. However, recently, researchers propose to reconsider teacher's telling to understand its function more fully in the context of teaching and learning mathematics. Adapting this perspective, this research analysed a preservice mathematics teacher's classroom discourse to show that teacher's telling functions not only to present mathematical information. More importantly, it was shown that teacher's telling functions to highlight emerging mathematical voices by diverse agents and position the voices in the context of mathematical argumentation for the collective construction of mathematics. This paper also discussed factors to facilitate the teacher's telling that celebrates the gift of the diversity in mathematics class.

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Introduction

Recent reform movements in mathematics education emphasize teaching and learning of mathematics based on students' active participation (e.g., Cobb & Bauersfeld, 1995; NCTM, 2000). For the purpose, it is recommended that mathematics teachers be equipped with discursive competence to fulfil their role as a facilitator and orchestrator in class (Chazan & Ball, 1999). The emphasis on discourse roles in teaching and learning of mathematics is predicated upon the ideas that mathematics is social practice. When considering mathematics as social practice, an individual's practice of mathematics is always revalorized in the context of the communal practice. An individual's practice of mathematics becomes assigned to a certain position with respect to the culturally shared notion of legitimacy of how to do mathematics. In particular, a teacher plays a role to position individual students' practice of mathematics by deciding what kind of question needs to be raised to whom, whose argument deserves to be referred to for an extended inquiry, and so on. This suggests that teacher's discourse functions as an epistemic device, which is defined as the means whereby actors or institutions establish and negotiate the legitimacy of knowledge and of knowledge production (Moore & Maton, 2001). As an epistemic device, teacher's discourse shapes the practice of mathematics in class by distributing the norms of doing mathematics and assigning a student to a certain position according to the norms. Then, it is of significance to address the features of teacher's discursive practice that is consistent to the teacher's roles recommended by reform movements. In this perspective, this research specifically focuses on teacher's telling to investigate the features of teacher's telling to create a classroom community where students are positioned as authors of mathematics and as arbiters of what counts as truth in the classroom practice of mathematics.

Theoretical Background

Research shows that telling belongs to a teacher's major discourse repertoire (Lobato, Clarke, & Ellis, 2005; Smith, 1996). However, the recent emphasis on the constructivist approach to teaching and learning mathematics carries a negative connotation of teacher's telling. That is, teacher's telling hinders a student's active construction of mathematics. This is based on the traditional notion of teacher's telling which conceptualizes telling as stating information or

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demonstrating procedures. This notion of teacher's telling reflects the assumptions that mathematics is a fixed collection of definite truth and that teaching mathematics is concerned with transiting the knowledge package from a teacher to a student (Smith, 1996). When mathematics is considered as social practice, the traditional notion of teacher's telling needs reconsideration since it is confined to only a partial aspect of mathematics.

In fact, Chazan and Ball (1999) criticized that an exhortation to avoid telling oversimplifies the teacher's role and provides only inadequate guide for teachers. In reform-oriented mathematics class, one of teacher's roles is to support and sustain intellectual ferment by monitoring and managing disagreement among students in meaning negotiation. Telling is a useful discourse move for a mathematics teacher to provide intellectual resource and to steer students' arguments. By telling, a teacher may return an issue to the students in class, to appropriate a student's comments by rephrasing them to other students. Lobato and her colleagues also proposed reformulation of telling as a set of teaching actions that serve the function of stimulating students' mathematical thoughts via introduction of new ideas into a classroom conversation (Lobato, Clarke, & Ellis, 2005).

Since the 1990s, community of mathematics educators adapted the perspective of mathematics as social practice. In this perspective, mathematics includes not only facts and skills but also intersubjective meanings that emerge through the human engagement in the context of doing mathematics. This suggests that mathematics teaching is not merely a process in which a teacher exports facts and skills. Rather, in class, a mathematics teacher is engaged with students to collectively construct mathematics out of emerging ideas in the context of on-going meaning negotiation. Adapting this perspective, this research analyzed a mathematics teacher's telling in class to reconsider its didactic function in the context of teaching and learning mathematics and to investigate the features of teacher's telling to create a mathematics classroom with intellectual ferment.

Research Methods

Setting & Participant

This research has been conducted in the context of teaching practicum. The analysis focused on the case of a preservice mathematics teacher, Lami[1], who had participated in the program for three semesters. In teaching mathematics, Lami encouraged students' active participation into mathematical inquiry to learn mathematics. Lami's class was organized around problem solving and mathematical communication. Lami developed the tasks based on real world phenomena. The task included a sequence of questions to guide students' collaborative construction of mathematics. So the questions were not simply concerned with finding the answer to a given task but also with facilitating students' explanation and justification of their mathematical findings. The students worked on the task collaboratively in small groups of 3-4 and shared their findings in whole class discussion where Lami played a role as a facilitator and orchestrator. As she described in her journal, Lami tried to adapt the social constructivist approach to her teaching and reflected upon her role in classroom discourse through her practicum. Hence, the analysis of her classroom discourse would provide some understanding of the features of teacher's telling that is consistent to the educational vision of recent reform movements.

Data Collection & Analysis

Data for this research had been collected for 3 semesters: from Fall 2005 through Fall 2006. All the class sessions taught by Lami were observed and video-recorded. The video-recordings of her teaching were transcribed for discourse analysis. The classroom discourse analysis focused on Lami's telling in whole class discussion where she consciously played her role in discourse as a teacher. The preliminary analysis of the transcripts focused on the identification of discourse types that occurred in Lami's classroom discourses. All the utterances by Lami and the students were separated sentence by sentence and placed on individual cells of the spreadsheet. In the analysis, both form and function of teacher discourse were considered. So after sorting the utterances by the form of a sentence, the functional aspects of utterances were taken into account. Then, content of teacher discourse (i.e., whether the teacher is talking about mathematics or not) were considered.

A coding scheme was primarily developed based on literature review and then refined through multiple passes through the classroom discourse data. The analysis identified 4 types of the teacher's discourse moves: questioning (Q), telling (T), managing (M), and revoicing (R). All the discourse moves including telling became refined into a set of subcategories since they carried out various functions, which will be described in following section. The codes for the subcategories of the discourse moves were assigned to each utterance. The coded transcripts were analysed both quantitatively and qualitatively. Lami' reflection journals were collected to validate and supplement the analysis of her classroom discourse.

Findings

Functions of teacher's telling

Telling is defined as a discourse pattern that Lami used to declare a statement referring to a mathematical content. The analysis showed that Lami's telling could be distinguished further into subcategories by taking into account of what kind of mathematical content carried. Following table presents the subcategories with the descriptions and the codes.

Table 1. Subcategories of Telling

CODE	DESCRIPTION
T1a	Presenting information concerning mathematical concepts, procedures, representation, methods which are detached from a student's mathematical experience and meaning system
T1b	Presenting information concerning mathematical concepts, procedures, representation, methods in a way to relate with a student's mathematical experience and meaning system
T2a	Explaining the mathematical background of a given task in a way that is detached from a student's mathematical experience and meaning system
T2b	Explaining the mathematical background of a given task in a way to relate with a student's mathematical experience and meaning system
T3	Referring to a student's mathematical position
T4	Referring to a teacher's own mathematical position such as giving a comment on a student's mathematical argument
T5	Evaluating a student's mathematical claim

Table 1 shows that the function of teacher's telling can be diversified depending on types of a mathematical content (e.g., information about mathematical concepts or procedures, explanation of a task, a mathematical position of a class participant, and evaluation). Furthermore, teacher's telling can be categorized according to types of an agent. Specifically, T1a and T2a are telling moves that convey mathematical knowledge by external beings such as textbooks, professional mathematicians, or, in general, the community of mathematics. On the contrary, T1b, T2b, and T3 carry mathematical knowledge authored by a student. T4 and T5 deliver mathematical knowledge by a mathematics teacher. Thus, this diversification of telling patterns suggests that teacher's telling functions more than exporting textbook-like knowledge of mathematics. In telling, a teacher highlights emerging mathematical perspectives and locates mathematical positions by diverse agents such as the community of mathematics, a student, and a teacher) to have the floor in class.

Use of Telling for Positioning Diverse Mathematical Voices in Class

Then, how did Lami use these various types of telling moves in her teaching? What does the pattern of use tell about her teaching practice? Table 2 presents the result of the quantitative analysis of the frequency of each type of telling.

Table 2. Frequency of Telling Subcategories (in Percentage)

CODE	FALL 2005	SPRING 2006	FALL 2006
T1a	2.27	5.66	0.00
T1b	2.77	5.66	2.08
T2a	4.55	9.43	0.00
T2b	2.27	5.66	8.33
T3	40.91	52.83	55.21
T4	36.36	11.32	23.96
T5	11.36	9.43	10.42

Table 2 shows that while T1a and T2a were used slightly more often than T1b and T2b, the frequency of T1b and T2b increases through the period of practicum. While a-type telling refers to knowledge by the external mathematics community, b-type telling refers to mathematical knowledge that is meaningful to students' own mathematical resource. Thus, it can be inferred that this switch is related to the change in Lami's position regarding the legitimate source for mathematics practice. For instance, following is an excerpt of a class transcript from the first semester of her practicum:

How do you think of the problem? It might be difficult but it is about the number of possible outcomes (T2a). So you have to find how many possible outcomes there are (T1a).

In this excerpt, Lami told the students the background principle of the problem and the method to solve it. There was no room left for the students to use their own cognitive resources. Following excerpt is from her reflection journal written in the very early stage of practicum:

"I expected the students to consider all the possible cases and then to choose proper cases. But they seemed to try to find the exact cases without considering the set of all the possible cases. The class couldn't finish the second question because we ran out of time. I regret that I didn't tell them to find all the possible cases first" (September 23rd, 2005).

This journal excerpt reflects Lami's beliefs that are coherent to those in the transcript excerpt assuming that a teacher's knowledge is the only legitimate source for mathematics practice. However, as she observed the strength of the students' ways of doing mathematics in her teaching, she became to change her beliefs about legitimate source for mathematics practice and began to introduce the students' mathematical perspectives in her teaching.

The consistent increase of T3 confirms this change. Since T3 refers to a student's mathematical position, the increase of T3 suggests that Lami became to place the students' mathematical resource at a more central position in the classroom practice of mathematics. The use of T3 had gone through a qualitative transformation in the regard that T3 more often accompanied revoicing[2] (T3-R) over the practicum. The analysis tells that the frequency of T3-R is 22.22% in Fall 2005, 41.51% in Spring 2006, and 50.00% in Fall 2006. T3-R can be distinguished from T3 in the sense that T3-R highlights not only the mathematical position of a student but also the ownership of the position. Then, what is the function of T3-R in the context of teaching mathematics?

1	Teacher:	What is a circle?	Q
2	Student 1:	A ring.	A
3	Teacher:	A ring.	T3-R
4		Can you express it in more mathematical terms?	Q
5	Student 2:	360 degrees.	A
6	Teacher:	360 degrees.	T3-R
7		That means turning around, right?	T3-R
8		How does it turn around?	Q
9	Students:	A center!	A
10	Teacher:	A center.	T3-R
11		There is a center and then how would you draw a circle with a compass?	Q-R
12	Student 3:	An identical length.	A
13	Teacher:	An identical length.	T3-R
14		So a circle is what you draw with an identical length around one spot.	T1b-R

In this episode, instead of giving the definition of a circle, Lami exploited the students' notion of a circle through a sequence of questions. T3-R functions to highlight the students' mathematical ideas that were elicited by the teacher's questions. Often teacher's questioning followed T3-R. Lami highlighted an emerging mathematical idea then guided the students to explore another features of a circle by asking a follow-up question. By this pair of Q and T3-R, Lami could make the mathematical trajectory of the students' mathematical exploration explicit and successfully guide the students to construct the definition of a circle collaboratively.

Another interesting pattern is that the frequency of T4, a telling move that conveys a teacher's mathematical position, decreased in the beginning but increased in the last semester of practicum. In addition to this quantitative change, there observed that Lami's way of talking her own mathematical position had been changed qualitatively over the practicum. Following is a transcript excerpt from the first semester:

1	Teacher:	Why did they use symbols?	Q
2		It is because it is convenient.	A

In this episode, Lami raised a question and then she answered it by herself. So her discourse flew like a monologue which does not invite the students' mathematical voices. On the contrary, Lami integrated her mathematical position with the students' arguments:

1	Teacher:	Why is this an equilateral triangle?	Q
2	Student:	I repeated drawing segments of the same length.	A
3	Teacher:	So this length is the same to this length.	T3-R
4		Wait. Why are these two segments of the same length?	Q
5	Student:	Because the circles have the same radius.	A
6	Teacher:	Because the circles have the same radius.	T3-R
7		Which circles?	Q
8	Student:	These twos.	A
9	Teacher:	These twos?	Q-R
10		How about these? Why do these two have the same length?	Q
11	Student:	Centered on here.	A
12	Teacher:	Which circle do you draw if you construct one centered on here?	Q-R
13	Student:	This circle.	A
14	Teacher:	Then these two become to have the same length. So those have all the same length.	T4

This conversation began with the teacher's question to request justification that a given triangle is equilateral. Lami raised a series of questions to elicit the student's mathematical positions and used revoicing in her telling to make explicit the connection among the positions. Finally, she presented the justification which she collaboratively constructed with the students.

The increase in the use of T4 may indicate that Lami became aware of what a teacher can offer to the class. More importantly, Lami seemed to learn how to talk with her students. She actively listened to the students' mathematical voices as revoicing and negotiated her mathematical position according to the students' ways of thinking and talking mathematics. Thus, her mathematical position could be well-integrated into the on-going mathematical argumentation in class.

Conclusion

This research began with the question of what the features of teacher telling are to maintain and sustain the intellectual ferment in a mathematics classroom. Based on the analysis of a teacher's classroom discourse, diverse functions of teacher's telling were identified. The subcategories of teacher's telling reveal diverse agents that are involved with the classroom practice of mathematics. Moreover, the pattern of change in ways of telling suggests that positioning is the essential function of teacher's telling. By positioning diverse agents as legitimate, a teacher decentralizes the practice of mathematics to create classroom community where diverse voices are heard and to extend the territory of current practice of mathematics in class. The analysis of Lami's reflection journal suggests that a teacher's beliefs about mathematics need to be changed to acknowledge the voices by the diverse agents and the gift of the diversity in mathematics class. In addition, a teacher's pedagogical content knowledge is of significance to support all the agents to go through all of way together to the wonderland of mathematics and, most of all to make the journey be a chance to have the travellers open their eyes to the world that they've never appreciated before.

Endnotes

1. Pseudonym

2. Revoicing is discourse of reuttering another person's speech through simple repetition, rephrasing, expansion, and reporting (Forman, et al.,1998; O'Connor & Michaels, 1993).

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EXAMINING RELATIONSHIPS BETWEEN TEACHER FACILITATION OF MIDDLE SCHOOL MATHEMATICAL DISCOURSE AND STUDENT PARTICIPATION

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Forman (2003) claims that mathematical learning occurs through communication in these communities of practices and that learning is a discursive activity. Investigating and contrasting classroom settings where students appear to participate more and less productively can help us identify instructional practices that appear to be more and less successful with supporting students' productive participation in mathematics classrooms. What is the relationship between teacher facilitation of classroom discourse and student participation in two middle school classrooms?

For this study, two sixth-grade classrooms from the same school were compared and contrasted to identify how the teachers facilitated student participation and how the students participated in the classroom discourse. Teachers at this school used the *Mathematics in Context* (2006) textbook series, which are materials that include rich mathematical problem solving tasks that have the potential to foster dialogue among students. Three video recordings from the fall, winter and spring of each classroom were analyzed for this study. Codes were developed through an emergent process.

Students in Classroom A appeared to engage in more sophisticated mathematics classroom discourse, but more students participated in Classroom B. In Classroom A, a mean of 43% of students participated in whole-class discussions by presenting their solutions at the overhead projector and explaining strategies that they used to solve mathematics problems. After they explained their strategies, students responded to questions from 28% of their classmates about why their strategies worked. In Classroom B, a mean of 69% of students participated by responding with non-elaborated answers to procedural questions posed by the teacher. Teachers in each classroom facilitated discourse differently. The results of this study demonstrate ways in which teachers' explicit and implicit communication of behavioral expectations supported and constrained students' participation in classroom discourse. One teacher's explicit behavioral expectations provided students with the opportunity to share their elaborated problem solving strategies with the whole class. The second teacher limited the nature of student participation (but appeared to increase frequency) by asking students to respond to procedural or computational questions.

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STUDENT-REPORTED LISTENING IN MATHEMATICAL DISCUSSION

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Investigations of mathematical discussion tend to focus on student talk or factors contributing to its quantity or quality. However, listening has also been characterized as an important component of student mathematical discussion (McCrone, 2005). Yet, most of the studies on listening in mathematics focus on teachers' listening to students rather than students' listening to students. The purpose of the current investigation is to provide an initial look at how student listening relates to their mathematical discussion. The present study asked 78 high school geometry students to complete a survey including questions about the quality of their listening actions, frequency they listened to other students, and the frequency of mathematical discussions. Students' listening actions (*listen*) were assessed with four questions on a 6-point Likert scale with higher scores representing "truer" statements. Responses were averaged into a composite score. In order to assess the frequency of other listening and discussion activities, students were asked to rate the following statements on a 4-point Likert scale. Higher scores represent a more frequent activity: Our class has discussions about mathematics (*class disc*); In math class, I have opportunities to talk about math with a partner (*partner*); In math class, I have opportunities to talk about math in a small group (*small group*); When other students are talking about math during class, I listen (*listen freq*); When we have class discussion about mathematics, I participate (*disc. participation*).

Table 1. Pearson Correlation Results for Listening Activity and Frequency

	Listen	Listen Freq	Class Disc	Small Group	Partner	Disc. Participation
Listen	-	.35** *n* = 74	.13 *n* = 72	.04 *n* = 72	.13 *n* = 73	.09 *n* = 73
Listen Freq	.35** *n* = 74	-	.14 *n* = 72	.24* *n* = 72	.36** *n* = 73	.12 *n* = 73
*$p < .05$, **$p < .01$						

The indicate positive relationships between listening activity and listening frequency. Listening activity was not found to have a statistically significant relationship with any frequency of discussion, but listening frequency was found to be positively and significantly related to small group frequency and partner frequency, suggesting that mathematical discussions in small groups and partner contexts have a stronger relationship to more frequent student listening than whole class discussions. Therefore, these results provide evidence that having students discuss math in small group and/or with a partner provides a stronger context for student listening than whole class discussions.

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"YES, AND...": HOW GROUPS BUILD IDEAS IN THE COLLECTIVE

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This poster presentation draws on data from our ongoing study of the nature of collective mathematical understanding and the role of improvisational theory as a framework for interpreting collective actions in a mathematics classroom (Martin & Towers, 2009; Towers & Martin, 2009). Improvisation is broadly defined as a process "of spontaneous action, interaction and communication" (Gordon-Calvert, 2001, p. 87). It is a collaborative practice of acting, interacting and reacting, of making and creating, in the moment, without script or prescription and draws attention to our capacity to integrate multiple, spontaneously unfolding contributions into a coherent whole—a capacity that may be vital for mathematical sense-making in collaborative groups. We suggest that there are specific characteristics of the improvisational process that have much to offer theoretically to an advancement of our understanding of how interacting groups coalesce on particular ideas that move the action (be it jazz or mathematical problem-solving) forward. In this poster we focus on one of these, the notion of "yes, and…", to consider the ways in which a group of learners solve a mathematical problem.

The data we offer here feature three preservice teachers (not mathematics specialists) working on the problem of finding a number with exactly thirteen factors. What occurs throughout the session (and is illustrated in the extracts embedded within the poster) is a continual process of the offering of ideas or innovations and the subsequent adding of something new—a process recognized by Sawyer (2001) as "yes, and"—the "First Rule of Improv"—wherein strong improvisers support the emerging 'storyline' by accepting the previously offered line and adding to it with their own contribution. Through our analysis we identify the ways in which collective mathematical understanding emerges and grows and suggest that the capacity of the group to repeatedly say "yes, and..." to conceptual innovations is a key element in enabling the group to both work productively on the problem and arrive at a solution.

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Chapter 6: Equity and Diversity

Research Reports

Brief Research Reports

Posters

COMPUTATIONAL PERFORMANCE AMONG SIXTH-GRADE STUDENTS IN CHINA: SCHOOL MATHEMATICS REFORM, SCHOOL LOCATIONS AND ETHNICITY

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In this study, a stratified random sample of 4784 sixth graders from a populous province in China was assessed about their computational performance after the implementation of school mathematics reform. The results show that Chinese sixth graders developed their computation skills well with the use of new curriculum materials. However, the results reveal a large gap in students' computation performance between rural and ethnicity schools and those in cities. The results call for further investigations of possible factors contributing to the performance gap, and to inform possible changes in educational policy and practices to improve elementary mathematic education in rural and ethnicity schools.

Introduction

The development of students' computation skills is generally valued in school mathematics. Being able to compute quickly and correctly is taken as an essential skill for students in many education systems, and it is often assessed in many cross-national studies of students' mathematics achievement (e.g., Lapointe, Mead et al. 1992; Mullis, Martin, Gonzalez, & Chrostowski, 2004). However, the tremendous emphasis traditionally placed on students' acquisition of computational skills has been challenged and subsequently decreased in recent school mathematics reform in many education systems including China and the United States (e.g., Leung & Li, 2010; NCTM, 1989, 2000). Students' rote acquisition and practice of computation skills is no longer favored. Instead, the development of students' conceptual understanding and problem-solving ability has been taken as a more important goal. While the proposed changes in school mathematics are generally accepted in many education systems (e.g., Liu & Li, 2010; NCTM, 2000), concerns about possible weakening of students' computation skills are inevitably mounted. There is no exception for the situation in China, where students' computation skills have often been valued highly in mathematics education (e.g., Zhang, 2006). Recent change in educational emphases in China provides an interesting case for others to examine and understand possible impacts of school mathematics reform on the development of students' computation skills.

It is generally documented that Chinese students outperform their counterparts from many other educational systems in international mathematics assessments (e.g., Lapointe, Mead et al. 1992). In these existing international assessments, Chinese students' mathematics performance was measured mainly with students sampled from economically developed areas. The difficulty in accessing students in remote and/or rural areas often restricted the possibility of getting a well represented sample from the vast student population in China. However, few studies that are available suggested that the quality of school education varies dramatically between urban and rural areas, due to very limited resources and instructional support available for teachers and students in rural schools (e.g., Ma, Zhao, & Tuo, 2004). Simply excluding those students in rural schools won't provide an accurate picture about the status of Chinese students' mathematics learning in general, or their computational performance in particular. Thus, this study was developed to empirically investigate, in a relatively large scale to include different types of schools, the status of Chinese elementary school students' computation skill development in the current context of school mathematics reform. In particular, the purposes of this study include the following:

(1) Assessing elementary school students' acquisition and development of computation skills that are required in elementary school education in China;

(2) Investigating possible similarities and differences in students' acquisition and development of computation skills in terms of school locations and ethnicity.

Assessing Chinese Student Computation Skills

Teaching and Learning Computation Skills in China

The development of student computation skills has long been taken as an important goal in school mathematics in China. Back in 1963, the nationwide arithmetic teaching and learning syllabus stated clearly that students are expected to firmly master basic knowledge of arithmetic and abacus, and be able to compute correctly and quickly. Chinese teachers also developed various approaches for teaching computation skills that were found to be effective. For example, many teachers combined the teaching of oral computation, paper-and-pencil computation and abacus, and found that this method was effective for developing student computation skills. Over the years, there have been some changes in specific requirements and approaches for teaching and learning computation skills, such as decreasing the requirement of computing with large numbers in 1986 and deleting the requirement of teaching and learning abacus in 2001 (Zhang & Ma, 2008).

The recent curriculum reform in China has led to the release and implementation of new curriculum standards in 2001 (Liu & Li, 2010). In the new curriculum standards, the requirements for teaching and learning computation skills have emphasized developing students' number sense, students' conceptual understanding of computation concepts and properties, and their application of computation knowledge and skills. Although fostering students' computation ability and skills are still important, the requirements in computations with large numbers and computation difficulty have been further decreased. Students are expected to learn multiple computation methods, including oral computation and estimation. The use of calculators is also introduced in teaching and learning elementary arithmetic. Correspondingly, the expectation of developing students' oral computation skills specified in textbooks has become less specific than previous versions.

Assessing Student Computation Skills: A Framework

In order to provide a comprehensive assessment of students' computation skill development, we propose a three-dimensional framework to characterize and assess student computation skills. The first dimension is from a mathematical perspective. Mathematically, computations with different numbers and symbols are closely related, but they are not the same. For example, the addition of two fractions with versus without the same denominator will require students to follow different procedures, but this will not happen to the case of adding two whole numbers. Student computation skills with whole number addition can not simply be taken as what students can do in carrying out fraction addition. Thus, the assessment of student computation skills requires a consideration of computing with different numbers and symbols.

The second dimension is from a curriculum perspective. Curriculum, as developed and structured to outline students' learning experiences, specifies the requirements in expected performances including computations. Different forms of computations are included in the Chinese curriculum, including oral computation, paper-and-pencil computation, and applications in solving problems. Thus, the assessment of students' computation skills needs to consider different forms of computations.

The third dimension is from a psychological perspective. Although computation itself is a procedural skill, students' learning of computations now requires more than skill acquisition. Students' knowing and application of specific computation concepts and skills are also emphasized in school mathematics in many education systems. In this study, the assessment of student computation skills is thus developed to contain such items that help measure students' understanding and application of specific computation concepts and skills.

Method

Participants

To assess elementary students' acquisition and development of computation skills, the study was designed to focus on the population of sixth graders who have learned all different computations required in elementary school education. The assessment was carried out in one populous province in the western part of China (i.e., less developed region in comparison with the coast region). The summary statistics of educational development in that province published in 2003 was used as a guideline in the sampling process. A stratified random sampling procedure was employed to obtain a representative sample of expected 5000 sixth graders from different elementary schools in that province.

In particular, all elementary schools in the province were first classified into 4 categories: urban, town, rural, and ethnicity. The number of students in each type of school across the entire province formed the base for selecting a proportional number of students in each category. The number of students selected from each participating school was restricted to 60 sixth graders per school (with a few exceptions as required by schools). As a result, a total of 73 elementary schools were randomly selected in 4 types of schools from 15 regions across the entire province. A total of 5100 copies of assessment instruments were distributed, and 4784 were collected.

The Design of Assessment Instrument

In China, there is no commonly acceptable and used instrument for assessing student computation skills. Because this study aimed to assess student computation skills with the implementation of new curriculum standards, the assessment instrument was thus designed to align with the mathematics curriculum standards in China (Zhang & Ma, 2008). In addition to the aforementioned three-dimensional framework, the design of the assessment instrument also adhered to the following principles:

(1) The assessment content should cover basic knowledge and computation skills required in elementary mathematics, including whole numbers, decimals, fractions, solving simple equations, ratio and proportion;

(2) The instrument should assess both students' computational results (correctness) and their skill level (speed and flexibility);

(3) The items are designed with a consideration for large-scale assessments.

The combination of these considerations led to the development of the instrument matrix that specifies content and performance requirements: (a) oral and paper-and-pencil computations in addition, subtraction, multiplication, and division with whole numbers, decimals, percent, and fractions; (b) performance dimension includes knowing, simple computation, complex and mixed computation, computation with flexibility. Based on the matrix, a total of 105 items were developed to assess sixth graders' computation skills that are required in elementary mathematics curriculum. Moreover, all items were designed to ask students to compute, but not multiple-choice items. The 105 items were grouped into two tests (A and B). Test A contains 35 items that are organized into four groups: oral computations, paper-and-pencil computations, mixed computations, and solving equations with all different numbers. The testing time was limited to 10 minutes. Test A was designed to assess the development of sixth graders' expected basic computation skills. The level of sixth graders' computation skill development is differentiated in terms of the percentage of correct computations carried out by students from different types of schools. Test B contains 70 items and completion time is also limited to 10 minutes. Because it is assumed that sixth graders won't be able to complete all 70 computations within 10 minutes, Test B was designed to assess sixth graders' computation competence and speed. Students' computation performance was to be assessed through the percentage of correct computations completed.

Study Procedure

Pilot tests and revisions. To ensure the feasibility of the instruments, pilot testing was used four times. The first three pilot testings were carried out at different types of schools. The students' performance in solving each computation item was carefully analyzed, and compared with teachers' evaluation of their students' expected performance. These pilot testing results were used to adjust the number of computation items included, item difficulty and presentation format. After these adjustments, the fourth pilot testing was carried out at three elementary schools with different achievement levels. The results from the fourth pilot testing helped finalize the test instruments and the procedure for testing administration.

To obtain a measure of the assessment instruments' reliability and validity, both tests were piloted in randomly selected elementary schools. For the reliability measure, both tests A and B were piloted with a total of 439 sixth graders from nine classes in three randomly selected elementary schools. The reliability estimates (Cronbach's Alpha) are 0.55 for Test A and 0.76 for Test B.

For the validity measure, 97 sixth graders were obtained from two randomly selected elementary schools in a big city. These sixth graders were tested two times that were two weeks apart. These students' performance at the end of the fifth grade was also obtained and used as reference criteria. Students' average performance from these two tests were then obtained and used to calculate criterion-related validity measure. The estimates were obtained as 0.71 for Test A and 0.66 for Test B in one school, and 0.80 for Test A and 0.65 for Test B in another.

Data collection procedure. Using the aforementioned sampling procedure (above section), all participating schools were informed about the purpose of tests and relevant instruction. The Department of Education in that province assisted the data collection to provide special permission for this survey. Because relevant instruction was included for both tests, teachers in participating schools were asked to administrate the data collection. Many graduate students majoring in education at a normal university were also dispatched to different schools to provide on-site guidance. Both tests were administrated at all participating schools within three days at the end of October, 2008.

Data analyses. Only two scores (1 or 0) were used to evaluate students' computation. While students' correct computation was given the score "1", a wrong answer would lead to a score of "0". All students' computations were scored by a group of graduate students and teachers. And the results (either 1 or 0) for each student were then entered into a computer and analyzed using SPSS.

Because China did not have such a large-scale assessment of students' computation skill development, no existing criteria were available as references to help interpret students' performance. Thus, an estimate of expected students' performance was also carried out for Test A. In particular, about 40 experienced elementary mathematics teachers were invited as experts to estimate students' expected performance. These teachers were selected from different schools located in urban, town, and rural areas in that province. These teachers' estimations (in fact, many teachers actually tested their students with the instrument) were then used as a possible criterion for the interpretation of our results in this survey: For test A with a total of 35 computation items, students' performance would be rated as "excellent" with 32-35 correct computations, "good" with 26-31 correct computations, and "satisfactory" with 21-25 correct computations. The overall average percentage of expected correct computations would be at or above 85 percent (i.e., about 29 correct computations).

Results

The survey results show that sampled sixth graders did well in their development of computation skills with the use of new curriculum materials. In general, sixth graders reached the expected performance in elementary school mathematics. The following sections are organized to provide the detailed results to answer our research questions in terms of students' performance in Tests A and B.

Chinese Sixth Graders' Acquisition and Development of Basic Computation Skills that are Expected in Elementary School Mathematics – Test A

The development of Chinese sixth graders' basic computation skills. The returned Test A shows that about 97 percent of all 4784 participating students finished all computation items, and the average number of correct computations is about 31 (30.73 based on computation) out of 35 computation items (i.e., 87.8 percent). Because our sampled experts estimated that sixth graders should have an average of 85 percent correct computations for Test A, the survey result suggested that the development of sixth graders' computation skills overall (87.8 percent correct) reached such an expectation. Table 1 shows the number (and corresponding %) of six graders who were correct on various numbers of computation items.

Based on sampled experts' estimation, students' performance on Test A would be classified as excellent with 32-35 correct computations, good with 26-31 correct computations, and satisfactory with 21-25 correct computations. Other data indicate that about 90 percent of all sampled sixth graders performed well (i.e., with 26 or more correct computations) on Test A. 7.4 percent sixth graders had satisfactory performance, whereas 57 percent of sixth graders had excellent performance (i.e., with 32 or more correct computations).

Because Test A required sixth graders to complete 35 computations in 10 minutes, it did place a high requirement on sixth graders to do all computations correctly within the limited time. There is a high percentage of sixth graders (14.4 percent) reached this excellent level of computation performance. Further analyses suggested that these students were mainly from urban or town schools with 15.2 percent and 19 percent sixth graders reached such excellence, respectively. At the same time, there were 179 (3.7 percent) students who did less than 21 computations (i.e., 60 percent items in Test A) correctly. We noticed that these students were mainly from rural schools (146 of 179).

Table 1. The Distribution of Students' Computation Performance in Test A

Number of correct items	Number of students who did correctly	Percentage of students who did correctly
35	690	14.4%
32-34	2036	42.6%
26-31	1525	31.9%
21-25	354	7.4%
0-20	179	3.7%

To examine whether sixth graders have specific difficulties in computations, we took a close look at the percentages of students who performed correctly on each computation item. The quantitative results suggest that the majority of sampled sixth graders can do correct computations for all items. Although there are some variations in students' computation performance across different computation items, it is not the case that sampled sixth graders found any computation item overwhelmingly difficult.

The computation items in Test A were developed to assess students' basic computation skills with different types of numbers. Further analyses revealed that the development of sixth graders' computation skills varied in terms of the types of computation skills assessed. In particular, their computation skills showed a decreasing trend from basic oral computations, paper-and-pencil computations, mixed computations, to solving equations. In oral computations, sixth graders did better with whole numbers and decimals than those involving fractions. The results suggest that sixth graders' oral computation skills were very well developed. Their computation performance was decreased with the increased computation complexity and its combination with other relevant knowledge and skills.

The development of Chinese sixth graders' basic computation skills across school types. Because these sixth graders were sampled from four different types of schools, we compared their computation performance in terms of school types. Students' performance presented a trend of decreased average number of correct computations and increased standard deviations from urban, town, ethnicity, to rural schools. The significance test results indicate that sixth graders from urban and town schools performed significantly better, and with much smaller diversity, than those from rural schools. However, there is no significant difference in students' computation performance between rural schools and those mainly serving ethnicity populations. Further analyses also indicate that the difference between urban and town schools is not significantly different ($Z=1.22$, $p>0.05$). But there are significant differences in terms of the average number of correct computations between town schools and ethnicity schools ($p<0.05$), and between urban schools and ethnicity schools ($p<0.01$).

Chinese Sixth Graders' Computation Speed and Correctness that are Expected in Elementary School Mathematics – Test B

The development of Chinese sixth graders' computation speed and accuracy. The returned Test B shows that the average number of correct computations is about 36 (36.36 based on computation) out of 70 computation items (i.e., 51.94 percent). Because Test B was designed to assess students' speed and accuracy in computations with whole numbers, decimals and fractions, it included 70 computation items and sixth graders were not expected to finish all computations correctly in 10 minutes. Nevertheless, there are still 11 (out of 4784) sixth graders who did all 70 computations correctly in 10 minutes. Moreover, if taking correct computations of 50 or more items as "excellent", the result shows that 9 percent of sixth graders reached this level. If taking the correct completion of half of Test B (i.e., 35 computations) as “good”, 58.51 percent (2799 out of 4784) students were able to reach this performance level. If taking the correct completion of 25 computations (more than one third of 70 items in Test B) as “satisfactory”, 88.73 percent (4245 out of 4784) students reached this level.

The development of Chinese sixth graders’ computation speed and accuracy across school types. As these sixth graders were sampled from four different types of schools, we also compared their computation performance in terms of school types. Students’ performance presented a similar average number of correct computations between urban and town schools, but a lower average number of correct computations for ethnicity schools and rural schools. In particular, rural schools had the lowest performance with the highest level of divergence among students. The significance test results indicate that sixth graders from urban and town schools performed significantly better, and with much smaller diversity, than those from rural schools. However, there is no significant difference in students’ computation performance between rural schools and those mainly serving ethnicity populations. Further analyses also indicate that students’ performance in Test B between urban and town schools was not significantly different. But there are significant differences in terms of the average number of correct computations between town schools and ethnicity schools ($Z=2.315$, $p<0.05$), and between urban schools and ethnicity schools ($Z=2.125$, $p<0.05$).

Discussion and Conclusion

Based on students’ performance in Tests A and B, it is clear that sixth graders developed their computation skills well with the use of new curriculum materials. In general, their computation performance met the expectation of elementary school education in China. In particular, over 95 percent of sixth graders had satisfactory performance in Test A computations, and 57 percent had excellent performance. Moreover, the majority of sixth graders could compute quickly and correctly. At the same time, however, there were about 3.7 percent of students who had difficulties in basic computations and they were mainly from rural schools.

While the assessment results reveal a successful story about Chinese elementary students’ computation skill development in general, it also led us to find out that students in rural schools need tremendous help. Students in rural schools not only had less satisfactory performance in computations than those in urban and town schools, but also had the highest level of divergence among rural students themselves. The results indicate the existence of a large gap in elementary mathematics education between rural schools and those in cities, which is consistent with what other mathematics educators observed and reported (e.g., Ma, Zhao, & Tuo, 2004). The consistency suggests that the gap in students’ computation performance between rural schools and those in cities revealed in our assessment is unlikely due to the implementation of new curriculum materials. Nor did the curriculum reform eliminate the pre-existent gap in mathematics education between rural schools and those in cities. In fact, the continued existence of the gap strengthens a call for further efforts to investigate possible factors contributing to the performance gap. Such investigations should help inform what possible changes in educational policy and practices may be needed to decrease such a performance gap.

It is often perceived that ethnicity students may not have equal chances to learn well through elementary school education. The results from this assessment suggest that this is not necessarily the case. Although students from ethnicity schools did not perform similarly well as the dominant ethnicity group of Han students in cities, they actually had slightly better performance than those Han students from rural schools.

The assessment also helped us to learn possible weaknesses in students' computation skill development. It appears that students experienced relatively more difficulties when computations were related to the use of computation concepts and properties. The results suggest that there is plenty of room for improving students' computations with more training in thinking and the use of computational properties. As there are also some students who exhibited very strong performance in computations, the results suggest the need to further understand what may help those students develop strong computation skills. Learning from successful students' performance and related contributing factors can help inform possible instructional changes to improve other students' computation skill development.

At the same time, because the results are also restricted to students' computation performance in this assessment, much more research is needed to examine possible connections between students' computation performance and their mathematical performance in other aspects such as problem solving. Some other questions have also emerged that need further research efforts in the future. For example, how may students' computation skills relate to the development of mathematics competence in the future? To what extent may the development of students' computation skills be facilitated or hindered by the new curriculum materials? The findings from this survey not only show the unique value of conducting a large-scale assessment of students' mathematical performance to inform the status of educational development with current policy and practices, but also serve as a starting point for developing systematic and in-depth education studies of school mathematics.

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EXAMINING THE NUANCES OF EQUITY PEDAGOGY: WORKING TOWARD DEVELOPMENT

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This study utilized teacher's analysis of student work to reveal teachers' equity pedagogy perspective. Twenty teachers participated in the study, providing written narrative responses for analyses. Data were coded using research-based equity pedagogy codes, intended to reveal the nuances of the particulars of equity pedagogy. Results indicate that teacher change in terms of equity pedagogy may be identified in ways that allow the mathematics teacher educator and the mathematics education researcher to support the teacher in improving upon particular elements of equity pedagogy. Additional themes emerged which have further implications for pedagogy.

Introduction

The vision of mathematics education, put forth in the NCTM *Principles and Standards* (2000) captures research-based directional goals of the field in terms of the quality and type of mathematics that should be taught; experiences students should have; the role of the teacher; expected outcomes in terms of student mathematical knowledge, disposition toward mathematics, and fluency with the mathematical processes. The literature of the field supports that the learning with understanding that is envisioned in the Standards and in the work of mathematics educators across the field is a more complex endeavor than merely knowing the "how to do" of mathematics (Heibert et al., 1997). Scholars support this notion by proposing a view of mathematics understanding that includes a broad and integrated interpretation of procedural and conceptual knowledge (Baroody, Feil, & Johnson, 2007; Star, 2005). Toward the realization of goals of learning with understanding, we recognize the role of the teacher in supporting students' learning mathematics in deep and meaningful ways, in terms of teacher knowledge and teacher pedagogical decisions (Ball, 1993; Ball, Lubienski & Mewborn, 2001; Boaler, 2002; Hill, Rowan & Ball, 2005); and that teachers must deliver high-quality instruction and have ambitious expectations for all students.

We cite these elements of mathematics pedagogy not to suggest that they include of all that the field has to offer. But these elements are examples of what leads to the goal to have *all* students learn mathematics, and as such are also examples of elements of an equity focus in mathematics pedagogy. But, like learning mathematics, knowing and implementing equity pedagogy is a complex endeavor supported by a rich literature where equity scholars have generated theory and empirical research delineating the importance and potential of equity pedagogy. One of many definitions for equity pedagogy comes from Banks (1996), who defines equity pedagogy as a set of practices that utilizes strategies and instruction that reflect the cultural practices and perspectives known to impact student learning. Given the existence of definition, the abundance of literature supporting elements of equity pedagogy, and the collective set of practices that result from work in the field, we propose that the time is right to consolidate these practices into a tool to help us identify and refine elements of practice, and move practice forward to better support equity goals. In the study we report in this paper, we explore a way to identify and describe elements of equity pedagogy; and we use these descriptions to suggest ways to move a practice toward a more equitable mathematics pedagogy.

Theoretical Framework

Over the years, equity has meant many things to many people (Gutiérrez, 2002). A general definition of equity has been given as involving “our judgments about whether or not a given state of affairs is just” (Secada, 1989, p. 68). According to Lipman (2004), equity means “equitable distribution of material and human resources, intellectually challenging curriculum, educational experiences that build on students’ cultures, languages, home experiences, and identities; and pedagogies that prepare students to engage in critical thought and democratic society” (p. 3). The NCTM (2000) names equity as one of its guiding principles, and states, “Equity does not mean that every student should receive identical instruction; instead, it demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students” (p. 12).

Researchers and theorists generate a literature around these and additional characteristics to define a mathematics pedagogy that supports equity. For example, Banks (2003) contends that equitable pedagogy is involved when teachers employ teaching methods that accommodate the differences of diverse students to stimulate academic achievement. This is mainly concerned with interactions between teachers and students and requires a mutual respect in every aspect of instruction. In the mathematics classroom, when teacher-student relationships are fluid and equitable, students learn to collaborate, share tasks, accept criticism, alternate opinions, respect others, construct their knowledge, and be responsible for each other (Dornoo, 2010).

Researchers argue that using a variety of pedagogical strategies such as problem posing, integrating language arts into the teaching of mathematics, small-group work, application of mathematics to everyday real-world contexts that utilizes explicit talk about mathematical meaning, reasoning and mathematical practice, project-based learning, current event analysis and application to mathematics will arouse students’ interest (e.g., Boaler, 2002; Delpit, 1988; Erchick, 2002; Ladson-Billings, 1995, 1997; Learning Mathematics for Teaching, in press). Another important aspect in providing equity pedagogy is to have and maintain high expectations for all students. Moses and Cobb (2001) posit that there is a very high correlation between teacher expectations and student achievement. Further, an equity classroom should be an active classroom in which students are given the opportunity to share their ideas with other students as well as with the teacher. It should be a classroom with a curriculum that utilizes real world examples in specific ways; and teaching that efficiently uses instructional time, emphasizes student effort, sends students the message that *everyone* can do the work, and that arranges for students to work autonomously (Boaler, 2002; 2009; Delpit, 1988; Dornoo, 2010; Erchick, 2002; Ladson-Billings, 1995, 1997; Learning Mathematics for Teaching, in press).

According to the NCTM (2000), teachers need to select activities that grow out of real-world problems relevant to students. Teachers with strong mathematics content and pedagogical knowledge understand the value of problem-solving activities on any mathematics topic, since it “offers opportunity for rich insight into students’ thinking processes and levels of understanding” (Frost & Dornoo, 2006, p. 222). Pedagogy strategies enacted in the mathematics classroom should adopt the pedagogical theories and practices that use mathematics not only as an analytical tool but also as an opportunity for commitment and/or capacity to develop a deep conceptual understanding of mathematics (Dornoo, 2010). Finally, Jo Boaler "argues that investigations into equitable teaching must pay attention to the *particular* practices of teaching and learning that are enacted in classrooms (italics in original)" (2002, p.239). Thus, the tool we propose and utilize in this study can help identify and draw attention to the particular practices of equity pedagogy, and the nuances of how these practices are apparent in one's pedagogy and the particular practices form the set of analysis codes we use in our study[2].

Methodology

Participants and Context of the Study

Participants in this study are certified or licensed teachers credentialed to teach in grades kindergarten through middle school in Ohio. The teachers work in schools enrolled in the Mathematics Coaching Program (MCP) (Brosnan & Erchick, 2007), a program primarily focused on supporting the lowest achievement schools in the state. These schools are almost exclusively urban and rural schools, and share common demographics of traditionally underrepresented groups in mathematics. Thus the equity perspective of this research is an appropriate one to inform the work of the MCP and to contribute to the work on equity and diversity initiatives.

Data Collection

In this study, twenty teachers provided responses to 10 selections of student work, giving explanations of student thinking about mathematics and suggestions for teacher instructional decisions for each selection. Participating teachers responded at the beginning and at the end of the academic year, responding to the same items both pre and post. The number of teacher participants, responses to student items, and the pre- and post-study data collection schedule resulted in 400 qualitative responses for analysis. Teachers submitted written responses in hard copy, and researchers and their support team entered the data into an electronic document.

Data Analysis

We analyzed teachers' extended responses by assigning one or more equity pedagogy codes as listed in the abbreviated codebook of Table 1[1] to each of the content and pedagogy responses. Note that each equity code was designated as an (E)xample or (N)onexample, thus generating 20 codes available for our analysis. Multiple researcher reviews of sample sets of data, collaboration on coding assignments, and comparisons to coding by an outside reviewer helped reach inter-rater reliability of goals in the analysis. Equity codes are supported in the literature and the extended code book provided the research team with narrative description, literature justification, and examples for each code. In addition to interpretation of the qualitative coding, we also conducted a numerical analysis of the resulting codes.

As an example of the research support generated for each equity code, consider the ETR code in Table 1 used to code narrative that represents the ways in which a teacher "**E**xplicitly **t**alks about, or addresses ways of **r**easoning". Ladson-Billings (1997) describes a desired pedagogy that is in contrast to a pedagogy of poverty. The teacher in Ladson-Billings' example questioned students, made suggestions, and used questioning to push students' thinking. She made reasoning and the students' communication of it a responsibility of everyone in the classroom. Likewise, Delpit (2006) offers precepts to transform urban students' experiences. One of her precepts is to demand the critical thinking important for student learning. Through critical thinking students come to experience multiple solutions and understand that mathematics is not a fixed field of knowledge. Both scholars' work supports the notion of expectations of reasoning, and that teachers explicitly pursue these expectations with their pedagogy.

The ETR equity pedagogy code also provides us examples of equity pedagogy coding. For a data point to be assigned an ETR code, it reveals teacher actions toward and/or expectations of student reasoning, elements of argument and proof, and/or thinking. For example, on one item used to generate participant responses, participants analyzed student responses to the problem: "Bobby was given the problem 17 – 9 = ____ and solved it as follows: 17 – 9 = 17 – 10 – 1." In their analysis, participant responses revealed a reasoning emphasis in their analysis of the student's work. Responses of "Bobby might know that 9 rounds to ten and it is easier to subtract 10 than 9. He also might think that since 10 is one more than 9, you need to subtract 1 from 10 to get 9" and "Bobby knew that 10-1 is the same as 9, so he thought that by making a nine on each side he would get the same answer" show how teacher participants considered the student's potential reasoning in their analysis.

Another example comes from a participant response to the following item: How can we arrange 12 tables, with no spaces in between, to allow the *most* chairs to be placed around the arrangement? The item included a student response that "Melva said that the most would be 24 chairs" and participants commented on the student's mathematical thinking. Participants' responses explaining that Melva "realized that if each table was flanked by a table on each side that there would only be two sides left to sit at but she forgot about the extreme ends," "understand that all tables must have at least 2 open sides, that tables surrounded by other tables cannot have chairs," "is confused about area and perimeter" reveal participant lenses to be focused on the student's reasoning.

Table 1. Abbreviated Code Book

Code	Equity Pedagogy Codes*
RWP	Suggests **R**eal-**w**orld **p**roblems or examples
EST	Defines **e**xplicit **s**tudent **t**asks and work
ETL	**E**xplicit **t**alk about the meaning and use of mathematical **l**anguage
ETR	**E**xplicitly **t**alks about, or addresses ways of **r**easoning
ETMP	**E**xplicitly points out strategies and **t**alks about **m**athematical **p**ractices
IT	Quality of **I**nstructional **t**ime spent on mathematics
EDC	**E**ncouragement of a **d**iverse array of mathematical **c**ompetencies
ESE	**E**mphasis of **s**tudent **e**ffort and message that effort will eventually pay off.
AU	**A**utonomous student work **o**pportunities
EE	**E**xpressed **e**xpectation that everyone will be able to do the work.
***** In use, the equity codes include an ending of E (Example) or N (Nonexample)	

Results

We present our findings first in terms of the overall results, then with more details in two teacher profiles, and finally with a comment on emergent results. Andy's and Chris' (pseudonyms) profiles provide particular examples from the data for consideration of implications for equity pedagogy and the emergent results provide additional insights.

Overall Results

Pretest data from the group of 20 participants collectively generated 393 equity code examples and 155 nonexamples. Posttest data generated 408 equity code examples and 134 nonexamples. This small shift in each of examples and nonexamples was positive but not overwhelmingly so. In addition to considering the total pre and post counts, other organizations of the results are of interest. For example, of the 20 possible codes, 6 were represented more than 30 times in the data set (See Table 2). Of those 6, that 5 were examples and 1 was a nonexample, paired with the improvement from pre to post, speaks well of the teacher participants and their growth. Similarly, of the 20 codes, 8 were represented fewer than 10 times in the data set, and more than half of those were nonexamples, again speaking well of the teacher participants.

Table 2. Summary of High and Low Representation of Codes

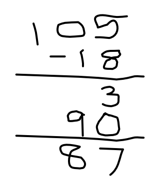
Low Representation (<10) codes			High Representation (>30) codes		
Code	Pretest Count	Posttest Count	Code	Pretest Count	Posttest Count
RWPE	2	0	ESTE	30	30
ESEE	3	3	ETRE	154	174
EEE	8	9	ETMPE	67	77
RWPN	0	0	EDCE	47	35
ESTN	9	8	AUE	40	47
ETLN	4	1	ETRN	61	40
ESEN	0	1			
EEN	3	0			

Alex's Equity Pedagogy Profile

Alex's profile is an example of a consistently equitable perspective (see Table 3). Where a participant may have as few as 9 equity examples, and as many as 15 equity nonexamples, Alex's profile includes a high number of equity examples and very few nonexamples. In the multiple data collections, Alex's responses generated 32 equity examples in one set of responses, and no nonexamples, with half of the equity examples being ETR (Explicitly Talks of Reasoning).

Alex wrote of Mike's alternative algorithm in the following problem:

Students are asked to solve the word problem: *Candy has 105 jelly beans, she eats 18 of them, and how many does she have left? (The teacher walks around and sees a variety of answers, including Mike's to the right.)*



Alex's analysis included that "Mike can see that 105-15=90. That is 3 less than the 18 he is subtracting. So 90-3=87. Mike doesn't need to do the traditional method with this kind of knowledge." We code this as an ETRE because of teacher's recognition of and appreciation of the student's reasoning. Thus we see the teacher as reading the student work through a reasoning lens. Alex also wrote of the student Melva's response to the following problem:

> The teacher asked the class to respond to the following problem: How can we arrange 12 tables, with no spaces in between, to allow the *most* chairs to be placed around the arrangement? Melva said that the most would be 24 chairs.

Alex responded that "Melva may have pictured two people at 12 tables and simply multiplied to equal 24. If Melva drew a picture and put chairs all around, [s]he would see the maximum would be 26 chairs" indicating again a reasoning lens in interpreting the student response.

That Alex's equity pedagogy examples fell so predominantly in the ETR category indicates a kind of modus operandi, a characteristic of this particular teacher that seems to have become a habit of being in practice. Additionally, even though there was some very slight change from the pretest to the posttest, Alex's perspective remained consistent throughout, with most instances being the same equity perspectives from pre to post, and about the same numbers in each.

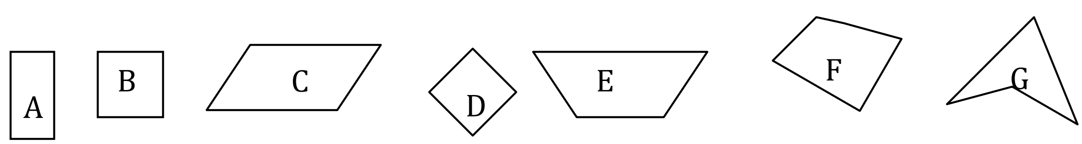
Table 3. Pre and Post Equity Code Counts

Teacher Pre/Post test	RWP		EST		ETL		ETR		ETMP		IT		EDC		ESE		AU		EE		
	E	N	E	N	E	N	E	N	E	N	E	N	E	N	E	N	E	N	E	N	Totals
Alex-Pre	-	-	4	-	1	-	14	-	7	-	-	-	3	-	1	-	2	-	-	-	E=32,NE=0
Alex-Pos	-	-	2	-	2	-	14	-	6	1	-	1	3	-	1	-	-	-	-	-	E=27,NE=3
Chris-Pre	-	-	4	-	6	-	7	2	3	1	-	-	5	1	-	-	6	-	-	-	E=31,NE=4
Chris-Post	-	-	-	2	3	1	6	3	3	1	2	-	2	3	-	-	3	1	-	-	E=19,NE=11

Chris' Equity Profile

Chris' profile is an example of a teacher with a less consistent lens than Alex's consistent ETR lens, and revealing a broader range and more equal distribution of equity lenses across the data set (see Table 3). In Chris' profile, it is in the change from pretest to posttest that we take notice. First, the kinds of equity codes generated, from both example and nonexamples, changed from codes in 9 categories in the pretest to codes in 12 categories in the posttest. Pretest codes were showed fewer kinds of codes but larger number of occurrences (4-7); posttest codes showed a greater number of kinds of codes, but fewer occurrences of each (1-3). Finally, the change from pretest to posttest showed an increase in the number of nonexamples, a situation that suggests a step back of some sort. However, that there were more categories also suggests a broadening perspective. With the growth to more perspectives, it is reasonable that some, less practiced, may start as nonexamples for Chris, and over time develop to equity pedagogy examples.

One category where Chris added 2 equity pedagogy examples in the posttest was a category that had no occurrences in the pretest, the Quality of **I**nstruction **T**ime (IT) category. Many data points were assigned multiple codes, and these 2 samples from Chris' responses are a case in point. Consider this problem in the study: Miss Jones put the following picture on the overhead and asked her students to identify all of the rectangles. Jose picked A, B, and D.



Chris's response was "Proving a concept is invaluable for students. It can help students understand any misconceptions they may have" followed a suggestion to have students make their case for what figures were indeed rectangle, and why. The response was coded as ITE because of the perspective of using instructional time for a thoughtful and rich experience, thus making it an equity pedagogy example. But the data point was also coded with an ETLE, again an equity pedagogy example, but this one for **E**xplicit **t**alk about the meaning and use of mathematical **l**anguage. With arguments made by students centering on the definition, the language would be a critical element in the proofs and misconceptions noted by the participant.

Chris' equity pedagogy examples were distributed across many different equity pedagogy codes indicating a number of things. It may indicate either a range of perspectives or a lack of focus. The increased number of nonexamples in the posttest could, as alluded to earlier, indicate the initial stages of a broadening perspective, where development will improve and strengthen the categories; or it could mean a regression to less equitable perspectives.

Additional Emergent Themes

Qualitative analysis of narratives also revealed an emergent theme suggesting new questions in this work. Consider two examples: Example one concerned teacher analysis of student work on a graphical representation problem. The graph had no labels or numbers, and the student was asked to write a story to match the graph. The student explanation suggested a representation of distance against time, but every teacher in the sample of 20 viewed the graph as representing only speed against time; hence no teacher interpreted the student's choice as correct. Example two concerns teacher responses on a probability problem. Participants who responded with thorough explanations usually revealed an understanding of the mathematics; but most responded with responses that were clearly incorrect or with what we might call "non-answers" circumventing the topic and suggesting little to no knowledge of the content. In any case, most did not know the mathematics well enough to question students through suggested probability experiments or help students come to a mathematically valid understanding.

Whether the graphing case represents a lack of content knowledge or represents a limited view of what graphs can represent is not clear; that few participants understood the probability content is better established in the data. In any case, a teacher with a lack of knowledge or a limited view of the content cannot hear, analyze, utilize or even allow students' explanations; misses equity pedagogy opportunities when the student is providing a diverse perspective to the lesson; and does a disservice to the student who is misdirected as being incorrect. These conditions lead to mis-taught mathematics, an equity issue by any standard.

Closing Discussion

We find the most significant part of this work to be how results of coding can assist mathematics teacher educators and mathematics education researchers in both identifying the nuances of the particulars of a practice of equity pedagogy, and provide guidance to support teacher change. Consider that we see Chris has 3 AUE and 1 AUN. For that AUN, one could suggest a slight adjustment to the instruction that will change the AUN to an AUE. So, for the AUN, a teacher is directing the student on how to do the work, thus taking away the student's opportunity for autonomous learning, a simple adjustment to pursue explanation from the student and expect further attempts, would change the nonexample to an example.

Addressing adjustments to increase particulars of equity pedagogy also could mean small but significant change in equity pedagogy. And for those teachers who believe that substantive change requires curriculum or resource changes over which they might have no control, or additional activity to add to their work, these results and this tool may help them see pedagogy as a place where sometimes seemingly small and subtle actions can make a difference for students.

Endnotes

1. Equity pedagogy codes adapted from Learning Mathematics for Teaching (in press).
2. The project's full codebook includes theoretical and empirical warrants to support each of the analysis codes.

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HISPANIC STUDENTS' MATHEMATICS LEARNING AND MANIPULATIVE USE

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In this study we examine the relationship between student mathematics learning and manipulative use for a national sample of elementary-aged Hispanic students. Data for this study were drawn from the ECLS 1998/2004 database (NCES, 2006). Further, students' home-language and socio-economic status were considered as possible moderators of this relationship. Results indicate that there is a positive relationship between manipulative use and elementary students' mathematics learning; however, home-language and SES were not found to explain significant variation in this relationship.

Introduction

The National Council of Teachers of Mathematics (NCTM, 2000 states that "all students, regardless of their personal characteristics, backgrounds, or physical challenges, must have opportunities to study–and support to learn–mathematics" (NCTM, 2000, p. 12). Therefore, English Language Learner (ELL) students should not be impaired because of language (Khisty, 1995). Many ELL students encounter learning situations in their classrooms in which their limited language proficiency impedes their ability to succeed in mathematics. However, teachers' use of multiple instructional strategies in the classroom may help these students reduce cognitive demand due to their limited language proficiency (Herrell, 2000). Several educators have proposed the use of mathematical manipulatives as a strategy to help ELL students' mathematics learning process (Lee & Jung, 2004; Lee, Silverman & Montoya, 2002), which may help to reduce cognitive demand (Herrell, 2000). As Lee and Jung (2004) suggest, "the focus is on the increased use of manipulatives… to facilitate communication between the teacher and student, helping the student to access key conceptual ideas without being dependent on the language" (p. 271).

However, few studies have analyzed the impact of manipulative use on ELL students' mathematics performance. Posadas (2004) studied the effects of manipulative use and visual cues on ELLs' mathematics learning. Sixty-four Hispanic students who had failed at least four mathematics objectives of the Texas Assessment of Academic skills (TAAS) participated in this study. Students were separated into three groups: two treatment groups (manipulatives or visual cues) and one control group. The objectives assessed in the study were multiplication, division and estimation strategies to solve problems. One day per week, for 5 weeks, students worked on a specific objective, and during instruction either manipulatives or visual cues were used for either one of the treatment groups. Participants' performance was measured one time before treatment and once every week of treatment. Posadas found no significant differences between the treatment groups and the control group performances. One possible reason for these results, as stated by Posadas, was the short period of time that students used the tools. According to Sowell (1989), students need to use manipulatives for a period of time of at least a year in order for their performance to improve. No studies analyzing the use of manipulatives for ELL students for that period of time were found.

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In the current study, we used a national sample of elementary-aged Hispanic students to analyze the relationship between mathematics learning and manipulative use. In particular, the aim of this study is to answer the following questions:

1. Is there a relationship between elementary-aged Hispanic students' manipulative use and mathematics learning?
2. Does student home language and SES moderate the relationship between elementary-aged Hispanic students' manipulative use and mathematics learning?

Methods

Data

This study used the Early Childhood Longitudinal Study 1998/2004 (ECLS Kindergarten class of 1998-1999) database from the National Center for Education Statistics (NCES, 2006). The ECLS was designed to collect data from students in Kindergarten during the academic year 1998-99 and follow these students through eighth grade. The data set contains seven waves of data for the participants: two times during the Kindergarten year, two times during first grade, and one time during each of the third, fifth and eighth grade years. For this study, data from four waves was used to examine the longitudinal growth rates in mathematics achievement of students during their elementary school years. The four waves represent (a) baseline measure, Kindergarten Spring semester 1999; (b) First follow-up, first grade Spring semester 2000; (c) Second follow-up, third grade Spring semester 2002; and (d) Third follow-up, fifth grade Spring semester 2004.

This data included a total of 10,673 students. Of these students, about 18.1% of them were Hispanic. This sample represented a total of 1,927 students. We chose to analyze the data of Hispanic students because if the home language for these students is different from English, it is likely to be the same across the group, i.e., Spanish. In addition, NCES provides Kindergarten and First grade students with the opportunity to use a Spanish version of the assessment instruments in the case of limited English proficiency. Therefore, for this group, students' mathematics scores were less likely to be affected by the language of the assessment instrument and more likely to reflect students' true mathematical understanding.

Measures

Mathematics achievement. The dependent variable was mathematics achievement scores. The mathematics assessment included items related to topics such as number sense, properties and operations, measurement, geometry, spatial sense, data analysis, statistics and probability, patterns, algebra and functions; and the items involved conceptual and procedural knowledge and problem solving (NCES, 2006).

Students' English proficiency was tested at Kindergarten and First grade by using the Oral Language Development Scale. Students who did not have adequate English proficiency and whose home language was Spanish were assessed using a Spanish version of the instrument (NCES, 2006). No translated version of the instruments were used for Third or Fifth grade data collection as it was felt that most students show English proficiency after First grade (NCES, 2006).

Using item response theory (IRT) techniques, the ECLS staff adjusted test scores to make them comparable across the different waves (i.e., grades K, 1, 3, 5). In this study, the IRT scores were used to measure growth in mathematics knowledge, that is, mathematics learning. We created a linear growth trajectory for each student using four achievement scores from each student. Each student's linear trajectory was created by regressing student's achievement on time. The slopes of these lines represent the rate of change over time or learning.

Independent Variables. The primary independent variable was manipulative use. In each fall semester data was collected on the frequency that students used manipulatives in their mathematics classroom. At the Kindergarten and First grade level, two questions measuring this frequency were used. One item asked about student's use of geometric manipulatives, and the other asked about student's use of counting manipulatives. The exact teacher prompts were: *How often do children in this classroom do each of the following math activities: (a) work with geometric manipulatives,* and *(b) work with counting manipulatives.* These items were rated on a 6-point Likert scale (1 = *Never*, 2 = *Once a month or less*, 3 = *two or three times a month,* 4 = *once or twice a week*, 5 = *three or four times a week*, 6 = *daily*). The maximum value of these two items was taken as a students' measure of frequency of manipulative use. For example, if a teacher indicated that a student used geometric manipulatives *once or twice a week* and indicated that the student *never* used the counting manipulatives, in total the student was using manipulatives in mathematics instruction at least once or twice a week.

For the third and fifth grade level, frequency using manipulatives was measured using a 4-point Likert scale (1 = *Almost every day*, 2 = *Once or twice a week*, 3 = *Once or twice a month*, 4 = *Never or hardly ever*). The teacher prompts were: *How often do children in your class engage in the following: work with manipulatives e.g., geometric shapes.* Because the teachers were asked about their students' manipulative use in a different way at different grades, values for each grade were recoded to create comparable measures across all four grades. The new manipulative use variables were coded as low, medium and high frequency use. The initial values and new values are presented in Table 1.

Table 1. New Frequency Measures Relating to Original Measures

New Labels	New values	K-1 grades original values	3-5 grades original values	New Description*
Low	1	1,2	4	Never/hardly ever
Medium	2	3,4	2,3	Between twice and eight times per month
High	3	5,6	1	Almost every day

Note. The new description is based on a combination of the original K-1 and 3-5 descriptions.

Student background variables came from the student questionnaire, including students' socio-economic status (SES), and home language (English or non-English). During Kindergarten Fall semester, student's home language was identified by asking parents if languages other than English were spoken at home. For this study, home language was coded as *non-English*= 0 and *English*= 1.

In this study, we wanted to examine growth in students' mathematics learning through their elementary school years. Therefore, because we considered the first wave as the starting point for that growth, we coded the grade levels as 0 = *Kindergarten*, 1 = *First Grade*, 3 = *Third Grade*, 5 = *Fifth Grade*. Because no data were collected in second and fourth grade, it was necessary to code grades to reflect the appropriate time between data points.

Analysis

The analysis focused on the relationship between manipulative use and the mathematical learning of Hispanic students throughout their elementary school preparation; and how this relationship was potentially moderated by student's home language and SES. A two-level hierarchical linear individual growth-curve model (HLM) (Raudenbush & Bryk, 2002) was employed to examine the relationship between manipulative use and growth in mathematics achievement at the elementary school level. Level 1 of our model was a set of separate linear regressions, one per student. These equations regressed students' four mathematics achievement test scores on time measured according to the grade (Kindergarten = 0, First grade = 1, Third grade = 3, and Fifth grade = 5), frequency of manipulative use (LOW, MEDIUM, and HIGH), and interactions between time and frequency of manipulative use.

In this study, three dummy variables for manipulative use were used. In order to maintain non-collinearity in the regression equation it was necessary to include only two of these variables in the model. In this case, LOW was not included in the model and thus the effects of MEDIUM and HIGH are in comparison to LOW.

Level 1 (growth model). The first level involved the within-student growth in mathematics achievement scores, which was specified by the variables of time (grade), two frequency variables (MEDIUM and HIGH) and the interaction between frequency and grade; LOW was used as the comparison group and thus not included in the model. With this model specification, we examine the effects of manipulative use and its interaction with *grade*. The linear level-1 or within subjects equation was specified as:

$$\mathrm{Y}_{ij} = \pi_{0i} + \pi_{1i}\,\mathrm{grade}_{ij} + \pi_{2i}\,\mathrm{MEDIUM}_{ij} + \pi_{3i}\,\mathrm{HIGH}_{ij} + \pi_{4i}\,(\mathrm{MEDIUM}_{ij})(\mathrm{grade}_{ij}) \\ + \pi_{5i}\,(\mathrm{HIGH}_{ij})(\mathrm{grade}_{ij}) + e_{ij}$$

Where Y_{ij} is the mathematics achievement score for student *i* at time *j*; *grade* indicates the grade at which the measure was collected (*grade* values are 0, 1, 3, and 5); and MEDIUM_{ij} and HIGH_{ij} indicates whether the frequency of manipulative use for student *i* at time *j* was MEDIUM, HIGH or neither of them with values of 0 or 1. The parameter π_{0i} indicates the initial mathematics achievement for student *i* when time is zero (i.e., Kindergarten) and the use of manipulative was LOW.

The parameter π_{1i} is the coefficient for the time variable and represents the rate of growth (i.e., mathematics learning) for student *i* throughout the elementary school grades. As suggested by Ma and Wilkins (2007), it specifically represents the "natural growth" reflecting cognitive maturity of students during elementary school. The parameters π_{2i} and π_{3i} are the coefficients for the medium and high manipulative use frequency variables and represent the overall main effect of those variables on mathematics achievement (these terms are not pertinent to the present study but are included in the model to account for possible variance due to the main effects). The parameters π_{4i} and π_{5i} are coefficients of the interaction between time and medium and high manipulative use. Different from natural growth, they represent the added growth due to medium and high manipulative use, respectively, compared to low manipulative use. To better understand these parameters, assume that student *i* at time *j* has used manipulatives almost daily, that is, HIGH=1 and MEDIUM=0. In this case, the model at level 1 simplifies to:

$$\mathrm{Y}_{ij} = (\pi_{0i} + \pi_{3i}) + (\pi_{1i} + \pi_{5i})\,\mathrm{grade}_{ij} + \mathrm{e}_{ij}$$

In this way, and similar to the models presented by Ma and Wilkins (2007), we can identify the parameters related to growth in mathematics achievement, π_{1i} and π_{5i}. This model, in addition to allowing us to study the growth in mathematics achievement throughout the elementary school years, permits us to investigate the effects of manipulative use on that growth. In this case, π_{1i} is the parameter indicating the natural growth or growth due to maturity and π_{5i} indicates the growth due to using manipulatives more often. If π_{5i} is statistically significant, it means that using manipulatives more often (compared to students using manipulatives less) adds significantly more to growth in mathematics achievement in addition to the natural growth during the elementary school years. Similarly, the parameter π_{4i} is related to the growth added by medium manipulative use compared to low manipulative use. The term e_{ij} is the error term. In HLM models such as the one in this study, the e_{ij}'s are assumed to be independent and normally distributed with mean zero and variance σ^2 (Hedeker, 2004).

Level 2 (between-student model). At the second level we intend to model the between-student variability in the growth terms. We include home language (HOMELANGUAGE) as a potential moderator of mathematics growth. In addition, SES is included as a statistical control. The second-level model, or between-student equation, was specified as:

$$\pi_{1i} = \beta_{10} + \beta_{11}\,\mathrm{SES}_i + \beta_{12}\,\mathrm{HOMELNGUAGE}_i + r_{1i}$$
$$\pi_{4i} = \beta_{40} + \beta_{41}\,\mathrm{SES}_i + \beta_{42}\,\mathrm{HOMELNGUAGE}_i + r_{4i}$$
$$\pi_{5i} = \beta_{50} + \beta_{51}\,\mathrm{SES}_i + \beta_{52}\,\mathrm{HOMELNGUAGE}_i + r_{5i}$$

Where β_{10}, β_{40}, and β_{50} are slope parameters for growth, and interaction between *grade* and the two manipulative use variables (MEDIUM and HIGH), respectively. The parameters β_{11}, β_{41}, and β_{51} represent the effect of the variable SES on the coefficients π_{1i}, π_{4i}, and π_{5i}, (i.e., the overall growth and the two interactions between manipulative use and growth, respectively; the parameters β_{12}, β_{42}, and β_{52} represent the effect of the variable HOMELANGUAGE on the coefficients π_{1i}, π_{4i} and π_{5i} respectively; and the values r_{1i}, r_{4i}, and r_{5i} are random individual-specific errors (Hedeker, 2004). Statistically significant values for the HOMELANGUAGE variable indicate that home language is a moderator of growth, that is, change in student achievement over time is different due to student home language (English and non-English).

The statistical analysis, in this study, has several steps. First, a base model was estimated in which no variables were included at the second level. The goal of this first step was to model the relationship between manipulative use and mathematics learning. Second, the variables SES, and HOMELANGUAGE were included in the second level model. With this model, we wanted to explain the variability in Hispanic students' mathematics achievement growth and the interaction between manipulative use and growth.

Results

The longitudinal model used in this study allowed us to identify the added growth in mathematics achievement due to manipulative use (see Ma and Wilkins, 2007). Results from estimating the base model (see Table 2) indicate that growth was statistically significant for students, adding 14.19 points to mathematics achievement annually (p<0.001). Added growth due to MEDIUM and HIGH manipulative use compared to mathematics learning growth due to LOW is also shown in Table 1. Added growth due to MEDIUM and HIGH manipulative use was statistically significant and greater than the added growth due to LOW manipulative use. On average, MEDIUM manipulative use added 1.12 points (p<0.001) to mathematics achievement, and HIGH manipulative use added 4.14 points (p<0.001) to mathematics achievement, annually. Comparing growth added by MEDIUM manipulative use to growth added by HIGH manipulative use, we found that HIGH manipulative use added significantly more growth than MEDIUM manipulative use (p<0.05). To compare the added growth due to MEDIUM and HIGH manipulative use, we calculated the 95% confidence intervals for the two parameter estimates using the largest standard error (SE = 0.49). The confidence interval for added growth due to MEDIUM manipulative use was from 0.14 to 2.1; and for added growth due to HIGH manipulative use the interval was from 3.16 to 5.12. The lack of overlap between these two intervals indicates a difference between the effects.

Next we used student level variables to model the variability in mathematics learning (π_{1i}), added growth due to MEDIUM manipulative use (π_{4i}) and added growth due to HIGH manipulative use (π_{5i}). The mathematics learning variable was allowed to vary randomly in the model, and the parameters related to growth due to manipulative use were fixed. After controlling for SES and home language, mathematics learning growth due to MEDIUM manipulative use remained statistically significant (β=14.59, p<0.001) and increased compared to the base model (see Full model on Table 2). SES was found to explain a significant amount of variability in mathematics learning. That is, there is a 0.88 increment in mathematics growth per standard unit increment in SES. Therefore, students with high SES tend to have a higher growth rate in mathematics achievement. However, home language was not found to explain variability in students' mathematics learning, indicating that students with different home languages have similar achievement growth.

Table 2. Natural Growth and Added Growth to Mathematics Achievement by Frequency of Manipulative Use for Hispanic Students

	Base Model		Final Model	
	B	*SE*	*B*	*SE*
Fixed Effects				
Grade	14.19***	0.21	14.59***	0.29
SES			0.88***	0.21
HomeLanguage			-0.38	0.33
MEDIUM * grade	1.12***	0.27	1.05**	0.31
SES			0.08	0.23
HomeLanguage			0.36	0.33
HIGH * grade	4.14***	0.49	4.34***	0.71
SES			0.60	0.54
HomeLanguage			0.35	0.90
Random Effect	Variance component			
	Base Model		Final Model	
Intercept	73.66***		57.26***	
Grade	7.24***		7.13***	

*Note. *** p<0.001, ** p<0.01. Base Model in this table can be read as indicating the average growth in mathematics achievement due to maturity (grade) and manipulative use (MEDIUM and HIGH). Final Model in this table can be read as indicating the growth due to maturity or Medium or High manipulative use after controlling for SES and home language.*

Considering growth due to manipulative use, after controlling for SES, and home language, mathematics learning growth due to MEDIUM manipulative use remained statistically significant (β=1.05, p<0.01). Mathematics growth due to HIGH manipulative use also remained statistically significant (β=4.34, p<0.01) and increased from the base model (see Full model on Table 2 above). SES and home language were not found to explain a significant amount of variability in added growth due to MEDIUM or HIGH manipulative use. Therefore, students using manipulatives between two and eight times per month (MEDIUM) have similar growth due to manipulative use, regardless of SES or home language. Similar results were found for students using manipulatives almost every day (HIGH).

Discussion

The purpose of this study was to examine the relationship between elementary school students' mathematics learning and manipulative use. Further, we wanted to examine home language as a potential moderator of this relationship. Using the ECLS data, we examined this relationship using a hierarchical linear growth curve model with a national sample of Hispanic students. By using multiple measures of achievement collected over time we were, in effect, able to model student learning as change over time. In this way, we were able to truly examine the relationship between manipulative use and mathematics learning as opposed to the relationship between manipulative use and achievement measured at a single point in time.

From the longitudinal analysis, in general, we found that there is a positive relationship between manipulative use and student mathematics learning for elementary school students. SES and home language were not found to moderate the relationship between manipulative use and mathematics learning. Based on this result, we conclude that manipulative use can potentially help all students independent of their home language. Students whose home language is not English were found to have similar gains in achievement when compared to their English home language peers. Language proficiency can be a limitation for mathematics learning for ELLs (Khisty, 1995), however, manipulative use has been proposed to help ELLs with their learning in mathematics classrooms (see Cummins, 1998; Herrell, 2000; Lee, Silverman & Montoya, 2002, Lee & Jung, 2004). The findings from this study seem to support this proposal as no differences in mathematics learning growth due to home language were found.

From the previous discussion of the results, we can draw two main conclusions from this study. First, there is a relationship between manipulative use and students' mathematics learning. Student mathematics learning tends to be higher for students who use manipulatives more often during their elementary school years. This result is inconsistent with the results reported in Posadas (2004). However, different from Posadas, the present study investigated manipulative use over a long period of time, which is necessary in order to observe mathematics learning (Sowell, 1989). Second, home language of Hispanic students was not found to moderate this relationship indicating that the relationship between manipulative use and mathematics learning does not vary because of language spoken at home. This finding potentially indicates that manipulative use helps all students learning regardless of their home language; and further, that language issues in mathematics classrooms may be reduced when manipulatives are used.

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ON THE USE OF LOCALE IN UNDERSTANDING THE MATHEMATICS ACHIEVEMENT GAP

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This paper reports on a multilevel regression analysis using the complete 2000 Kentucky CTBS-5 dataset (nearly 50,000 ninth grade math scores) on individual- and school-level variables representing sex, ethnicity, socioeconomic status (SES), and school size. Significant mathematics achievement gaps emerge for most areas and interactions. The same variables are then compared across "rural" and "non-rural" categories. Findings include that rural schools erase the gender gap and diminish the race gap and poverty gap in math achievement for all students combined, and erase the influence of low SES on mathematics achievement for females and minorities.

Introduction

"Achievement gaps" in school subjects, but most significantly, in mathematics, are the focus of much political and professional discourse (White House, 2010; NCTM, 2005). These gaps are observable differences in assessment scores between two disjoint subsets of the population. Most often, gaps are identified according to socially-ascribed characteristics of the student, such as sex and race or ethnicity; or according to the contextual characteristics of the school and/or community including: aggregated gender, ethnicity, and socio-economic status (SES). School-size is used less frequently as a contextual characteristic despite its importance to structuring social aspects of schooling. Locale-type (loosely: rural, urban, suburban) is used rarely as well, though when it is, it is often coded by urban vs. non-urban.

The exclusion of locale in investigating achievement gaps is a fatal omission given that locale reflects one of the principal ways that society and culture is demarcated—by choice of place of residence. People tend to self-sort into locations in service of and in preference to differing social structures (sense of belonging, access to recreation and fitness opportunities, etc.) (Bishop, 2008). When locale is used as a frame for examining achievement gaps, such studies often focus on "urban" locations in ways that unnecessarily conflate location with race and SES ("urban," "Black," and "poor" become nearly synonymous) (Johnson & Kritsonis, 2006). More careful study of the individual variables as they relate to locale could lead to more productive theories of action to address gaps. Large-scale quantitative measures of achievement gaps according to rural locations were not extant in the authors' review of research.

The current study contributes to the understanding of the influence of "the usual suspects" for detecting mathematics achievement gaps by examining the complete dataset from the mathematics portion of the 2000-2001 Kentucky Comprehensive Test of Basic Skills (CTBS-5). The present study is unique in that we include an "unusual suspect"—school size—and then frame the investigation of the same variables outlined above, but comparatively between rural and non-rural locales. Multi-level regression analysis on the complete dataset detected school-level and individual-level significance in mathematics achievement by several variables including interaction variables. The secondary analysis (rural vs. non-rural schools) reduced to non-significance several factors that showed significance in non-rural and complete data. Hence, the study provides evidence of the importance of understanding the possible affordances and constraints of rural vs. non-rural schooling and, more broadly of the socio-spatial and cultural features inherent to locale as they relate to achievement and the processes of schooling.

Background and Relevant Literature

Educational leaders contend with achievement gaps as an equity issue revealed by state and federal accountability systems. Reform efforts of recent decades would suggest that the manipulation of process variables (e.g. teacher qualifications, professional development, curriculum alignment) has proven difficult to implement and has been limited in its effectiveness (Steinberg, 1996). Structural variables, particularly organizational size or scale, offer opportunities that have been largely unexplored by policymakers, practitioners, and researchers.

Achievement gaps are commonly used to describe inequitable distributions of achievement (whether “distributions” is used in the descriptive sense or in the sense of a purposeful, agentive allocation). Reproduction theorists see achievement gaps as evidence of schools’ role in maintaining broader social and economic inequities (Bowles & Gintis, 1976; Bourdieu & Passeron, 1990). According to this theory, it is unreasonable to expect that any educational unit—school, district, state, region—will ever reflect an equitable distribution of academic achievement because inequities are necessary for the reproduction of a stratified society. Alternately, liberal progressivists view school reform as existing primarily to close achievement gaps and equalize educational outcomes (Haycock, 2001).

The inclusion in this study of smaller organizational size as a key structural variable follows research showing how it can be used to mediate the relationship between SES and academic achievement (Friedkin & Necochea, 1988; Howley & Howley, 2004; Johnson, 2007; Lee & Smith, 2001). In general, these scholars find that smaller organizational size is associated with diminished explanatory power of variables operationalizing poverty. This study extends such size-by-SES lines of inquiry to further investigate the possible interaction of (1) size and student gender, and (2) size and student ethnicity. As such, the study also contributes to a considerably smaller literature—and one that is characterized by less consistent findings than those reported in size-by-SES studies (Arnold & Kaufman, 1992; La Sage & Renmin, 2000). While less consistent in their findings, these studies suggest the potential for inquiry into the influence of school size on achievement gaps related to race/ethnicity, and gender.

Methodologically, the current investigation contributes to a limited subset of the school size literature using student level data (e.g., Howley & Howley, 2004; Lee & Smith, 2001). The present study similarly uses individual student data as the foundation for a nested design. It differs from earlier studies, however, in that the current model uses the universe of student-level data for the 2000-2001 grade 9 cohort in Kentucky. Using the entire population rather than a sample eliminates issues related to sampling and weighting, and therefore offers the possibility that findings will more accurately characterize the relationships of interest.

Methodology

The following research questions guided the quantitative investigation in this study:

1. In what ways (strength and direction) is mathematics achievement among students influenced by socially-ascribed characteristics of the student (gender and ethnicity), by the contextual characteristics of the school community (aggregated gender, ethnicity, SES, and school size), and by the interaction of individual and contextual characteristics?

2. To what extent do influences on mathematics achievement vary according to school locale (rural or non-rural) in which the student is enrolled?

Multi-level regression techniques appropriately model the hierarchical structure of educational data and offer several advantages over other statistical approaches such as Ordinary Least Squares regression (Hox, 1995; Iversen, 1991). Moreover, by using independent variables at student- and school-levels, the model can detect possible importance of cross-level interactions. For example, the effects of student ethnicity may vary depending upon the poverty level of the school.

Data set

The dataset was prepared using information obtained from the Kentucky Department of Education (KDE) and the National Center for Education Statistics (NCES). The resulting merged data set comprised 49,979 cases representing all members of the 2000-2001 Kentucky grade 9 cohort with CTBS-5 scores reported. For the purposes of investigating differences related to school locale, we also created subsets of the data comprising (1) students in rural Kentucky schools (n=17,315) and (2) students in non-rural Kentucky schools (n=32,491). Rural and non-rural school designations relied on NCES locale codes (locales 7 and 8 = rural; locales 1-6 = non-rural).

Variables

The dependent variable represents measured student achievement in mathematics, operationalized as scores on the math component of the CTBS. The distribution of values for the dependent variable was reviewed using descriptive statistics (i.e., skewness and kurtosis), histograms, and Q-Q plots, with findings suggesting a relatively normal distribution of values.

The independent variables represent student-level socially ascribed characteristics, school-level contextual characteristics, and cross-level and same-level interaction terms. Student-level independent variables included sex (SEX1) and ethnicity (ETH1). Additional independent variables represent the school-level aggregation of these same variables: SEX2 (the calculated school-level percentage of female students) and ETH2 (the calculated school-level percentage of minority students). The distribution of values for the two school-level variables was reviewed using histograms and Q-Q plots. Values for the ethnicity variable were highly skewed toward lower values (i.e., there were many more schools with small or non-existent populations of nonwhite students). To allow for a useful degree of variability, values for ETH2 were transformed using the natural logarithm. Review of a histogram and Q-Q plots confirmed that the distribution of values for the transformed variable was close to normal.

Two additional school-level variables operationalized school size and school socioeconomic status. The school size variable (SIZ2) represents the Fall 2000 student membership, and the school SES variable (SES2) represents the school-level free and reduced meal rate for the 2000-2001 academic year. Histograms and Q-Q plots suggested relatively normal distributions for these variables.

Following Cronbach (1987), all student- and school-level variables were centered to reduce the collinearity of related independent variables. All student-level variables were centered on the mean of individual schools (Bickel, 2007). All school-level variables were centered by subtracting the state mean from each case value. To account for cross-level influences in the multilevel model, several interaction terms were computed from the centered independent variables:

The eight cross-level interaction terms were implied by model choices—that is, by choices related to fixed and random effects, and to hypotheses about variability in the random intercept and random slopes. The additional interaction term (SES2SIZ2) was suggested by findings from Bickel & Howley (2000).

(1)(SEX1SEX2) student sex X school percent female; (2)(SEX1ETH2) student sex X school percent minority; (3)(SEX1SES2) student sex X school poverty; (4)(SEX1SIZ2) student sex X school size; (5)(ETH1ETH2) student ethnicity X school percent minority;	(6)(ETH1SEX2) student ethnicity X school percent female; (7)(ETH1SES2) student ethnicity X school poverty; (8)(ETH1SIZ2) student ethnicity X school size; (9)(SES2SIZ2) school poverty X school size.

Data Analysis

The research model theorized that 15 independent variables (at both levels and including cross-level interaction terms) exert influence over the dependent variable representing mathematics achievement. Specifically, the model hypothesized that student-level sex (SEX1), and ethnicity (ETH1), school-level proportion of female students (SEX2), ethnicity (ETH2), socioeconomic status (SES2), and school size (SIZ2) constitute a robust set of predictors for mathematics achievement. The model further hypothesized that the influences of the student-level variables are moderated by school-level ethnicity, proportion of female students, size, and SES. The model also hypothesized that the influence of school socioeconomic status is moderated by school size.

In sharp contrast to most empirical research, the dataset includes the entire population: all ninth graders in Kentucky who took the CTBS math test in 2000-2001. Thus, there was no need to make inferences from a sample to the larger population. Relationships that differed from zero were, by definition, “real”. Inferential statistical procedures are therefore both gratuitous and potentially misleading. While statistical significance was then—strictly speaking—immaterial to the study, significance levels are nevertheless reported and can be treated as indicators that an observed relationship might be of practical significance (Bickel, 2007).

Results

Table 1 reports descriptive statistics for the dependent variable (operationalized math achievement score) and the school-level independent variables. Table 2 reports frequency statistics for student level independent variables, ethnicity (ETH1), and sex (SEX1).

The first numerical column of Table 3 below reports the results from the regression model predicting mathematics achievement for the combined dataset from the 15 independent variables described above. Two research questions guided the quantitative investigation grounding this study. The first considered the distribution of individual student achievement among students with differing individual characteristics and across differing school environments. Results reported here indicate that socially-ascribed characteristics of students influenced achievement in ways commensurate with results in the extant literature. Specifically, all else equal: (a) male students, on average, would be expected to score about 6% of one standard deviation higher than female students, and (b) minority students, on average, would be expected to score about one-third of one standard deviation lower than white students.

Table 1. Summary of Descriptive Stats.
For Dependent* Variable and Independent School-level Variables

Variable	N	Min.	Max.	Mean	SD
*Grade 9 CTBS-5 Math Scale Score	48,437	—	—	696.43	52.505
Percent Female Students (SEX2)	48,566	0	100	47.841	6.543
Percent Minority Students (ETH2)	48,566	0	81.25	13.019	15.503
School Percent Poverty (SES2)	48,566	1.5	90.26	35.845	17.632
School Size (SIZ2)	46,757	54	2,006	1009.06	425.589

Table 2. Frequency Statistics for Computed Ethnicity Variable and Student Sex Variable

Variable		Value	Frequency	Percent	Valid Percent	Cumulative Percent
ETH1	Valid	White	41,267	85.9	87.0	87.0
		Minority	6,162	12.7	13.0	100.0
		Total	47,429	97.7	100.0	
	Missing		1,137	2.3		
	Total		47,128	100.0		
SEX1	Valid	Male	25,189	51.9	52.1	52.1
		Female	23,119	47.6	47.9	100.0
		Total	48,308	99.5	100.0	
	Missing		258	.5		
	Total		48,556	100.0		

Table 3. Comparison of Results for Multilevel Regression Model One Analysis for Variables Predicting Student Performance on Grade 9 CTBS-5 Math

Variable	All (N_1 = 46,680) (N_2 = 334)	Rural (N_1 = 16,839) (N_2 = 126)	Non-rural (N_1 = 29,918) (N_2 = 208)
Student Sex (SEX1)	-2.922***	-.650	-2.953***
Student Ethnicity (ETH1)	-16.985***	-14.071***	-17.061***
School Percent Female (SEX2)	.222	-.096	.453*
School Percent Minority (ETH2)	-1.853**	-1.787	-1.923*
School SES (SES2)	-.783***	-.637***	-.831***
School Enrollment Size (SIZ2)	-.0005	.002	-.003
SEX1 X SEX2 interaction	-.071	.131	.096
SEX1 X ETH2 interaction	-2.486***	-.090	-3.038***
SEX1 X SES2 interaction	.071*	.046	.122**
SEX1 X SIZ2 interaction	-.0003	-.002*	.0009
ETH1 X ETH2 interaction	-9.400***	-7.656	-9.642***
ETH1 X SEX2 interaction	-.519	.908	.459
ETH1 X SES2 interaction	.195*	.138	.269*
ETH1 X SIZ2 interaction	-.004	.004	-.006
SES2 X SIZ2 interaction	-.0004***	-3.43E-05	-.0008***

Notes: N_1 = *N at level one (students);* N_2 = *N at level two (schools);* $^{*}p \le 0.05$; $^{**}p \le 0.01$; $^{***}p \le 0.001$.

Results suggest that contextual characteristics of schools exert influence over achievement levels among students—here too consistent with descriptions from the literature. The influence was substantially weaker than the influences of student-level characteristics, however. In particular, the results suggest that, all else equal: (a) increases in percent poverty would be expected to decrease math scores for all students regardless of individual SES, and (b) increases in percent minority enrollment would be expected to decrease math scores for both minority and white students. The influence of school size was nonsignificant for math achievement.

With regard to interaction effects, results indicated that the composite influence of student and school characteristics exerted influence over achievement that was stronger, on average, than the influence of school characteristics alone. These results suggest that contextual characteristics influence students differently according to student sex and ethnicity. School ethnicity exerted nearly one-third more influence over outcomes for female students than it did for all students combined. In contrast to its negative influence over all students combined, school SES exerted a slight positive influence over achievement for girls (i.e., all else equal, girls, on average, would tend to score higher in higher poverty schools). The interaction of school SES and student ethnicity was also positive and relatively weak, suggesting a similar pattern. The interaction of school-level and student-level ethnicity was negative and quite strong, indicating that the negative influence of school ethnicity was more than six times stronger for minority students than for all students combined. The significant interaction of school size and school SES indicated that the negative effects of school SES were mediated by school size (i.e., smaller school size diminishes the negative influence of poverty over student achievement).

The second research question considered whether these influences vary according to school locale (specifically, rural versus non-rural). To allow for comparing results for students in rural schools with other Kentucky students, Table 3 includes coefficients obtained from the same regression using (1) the data set comprising all valid Kentucky cases, (2) a subset comprising all valid rural cases, and (3) all valid non-rural cases.

When results for rural and non-rural student populations were compared, the following results emerged:

1. The influence of student sex on math achievement was reduced to nonsignificance in rural schools (i.e., the achievement gap between male and female students in rural schools is so small that it should not be considered real).
2. Student ethnicity exerted a weaker influence over math achievement in rural schools than in non-rural schools (i.e., white students, on average, outperformed minority students by a smaller margin; the achievement gap was narrower).
3. Among all students combined, the influence of school SES on math achievement was weaker in rural schools than in non-rural schools.
4. Among female students, the influence of school SES on math achievement was reduced to nonsignificance in rural schools.
5. Among minority students, the influence of school SES on math achievement was reduced to non-significance in rural schools.

Discussion

Perhaps most striking of results 1-5 above is that rural locale, in a sense, “erases” the gender gap and “diminishes” both the race gap and the poverty gap in mathematics. Given the attention given to gender and race equity in the past century of educational discourse, this result suggests that a better understanding of the influence of the places and structural elements of rural schools and communities may offer insight into mechanisms to reduce the gender and race gaps in mathematics elsewhere. Minimally, further study could illuminate obstacles for achieving equity in non-rural locales. It should be noted, that this result hardly demonstrates the absence of gender or race discrimination in rural areas. Indeed the importance of this finding remains to be seen in further investigations of schooling in rural areas. Still, it suggests that the relative lack of research on rural education (Arnold, Gaddy, & Dean, 2004) should be addressed, both by researchers and by funding agencies committed to improving mathematics education. Much of the rural education research points to the centrality of community, family, and attachments to the land in rural life (most notably, Theobald, 1997). Further research could determine the extent to which these key characteristics may structure social supports that promote equitable outcomes in mathematics achievement in rural schools, as suggested by findings 1-3 of the present study.

For female and minority students, schools in rural locales again erased the influence of poverty on mathematics achievement that was detected in analysis of the complete dataset. Rural areas in Kentucky, and the United States more broadly, face high rates of poverty (Johnson & Strange, 2009). It is reasonable to surmise that the effects of poverty are, in a sense, more equitably distributed, and its effects therefore, are relatively less pronounced than for other locales where, as previously mentioned, one's ethnicity or gender is a stronger physical marker of poverty than would be the case in a rural locale. Result 5 in particular seems less generalizable to a national picture of rural vs. non-rural mathematics achievement by social or contextual characteristic. The rural central Appalachian region of which Kentucky is a part has a notably different ethnic profile than, say, the rural Southwest, suggesting that this result merits further scrutiny for states or regions where rural demographics differ significantly from that of Kentucky. In most other ways, results from this study of Kentucky could be considered a fairly representative picture of rural America (Johnson & Strange, 2009).

Overall, this study suggests that the current focus of efforts on process reforms such as instructional strategies, same sex grouping, and injections of professional development as part of efforts to "close achievement gaps" ignore potential benefits of focusing on structural variables such as school (district) size, administration, and the like. Understanding these results clearly involves further investigation, but one that is open to the possibility that the values, commitments, and arrangements underlying community and locale may explain mathematics achievement gaps better than process variables such as the qualifications of a teacher. These results may even suggest, as in the work of Paolo Freire, that an underlying community-centered ethic can raise achievement broadly; indeed, that a person's success in a given area may depend on the success of his neighbor. As Cesar Chavez remarked, "We cannot seek achievement for ourselves and forget about progress and prosperity for our community... Our ambitions must be broad enough to include the aspirations and needs of others, for their sakes and for our own."

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PUTTING THE CITY BACK IN "URBAN": HIGH SCHOOL MATHEMATICS TEACHING IN LOW-INCOME COMMUNITIES OF COLOR

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This study reports interim findings from an ongoing research and professional development project at urban high schools located in two low-income, communities of color. The project collaborates with teachers on improving their instructional practices, using a framework of culturally relevant mathematics pedagogy, which is described in this paper. We measure instructional practices by a qualitative and quantitative analysis of data from 49 classroom observations of seven teachers. In particular, we use culturally relevant mathematics pedagogy to describe effective practices of the highest-rated teachers.

Introduction

The preparation and training of mathematics teachers to teach in "urban" contexts is a pressing issue. Although "urban" has taken on a wide spectrum of connotations, its meaning here literally indicates "of or relating to a city." Urban schools, by virtue of being located in densely populated areas, are contained in large school districts, which are typically associated with bureaucratic leadership structures, emphases on standardized testing, high teacher turnover rates, as well as a mixture of certification, induction and mentoring programs (Weiner, 2000). In addition, larger numbers of people are concentrated in geographically smaller spaces in cities, creating a pattern of islands of homogeneity. Therefore, while a city as a whole may be diverse, the neighborhoods are often homogenous in terms of race and socio-economic class. One consequence is that African American or Latino students from the lowest income families tend to be segregated, de facto, in particular urban schools (Lipman, 2004).

The research described in this paper is conducted in two such low-income neighborhoods in New York City. Each neighborhood contains about 130,000 residents: Bushwick's residents are primarily Latino (Puerto Ricans, Dominicans, Mexicans, and Ecuadoreans) and Brownsville's residents are primarily African American. More than half of the families in Bushwick, and more than 2/3 of the families in Brownsville, are in the bottom two quintiles of New York City income (Furman Center, 2008); these neighborhoods are among the ten poorest neighborhoods citywide. We conduct our research at two case high schools; each school enrolls about 400 students largely from their surrounding neighborhoods.

At these two high schools, many, if not most, students enter high school with below proficient elementary and middle school mathematics state test scores, indicating that they likely have weak backgrounds in school mathematics, in crucial areas such as number and operations or ratio and proportion. At the same time, they skillfully navigate a complicated urban environment, and demonstrate that they are sophisticated problem solvers outside of the standardized testing context. As a field, we know little about effective mathematics teaching in situations like this, situations that are typical in schools that serve students of color in low income, inner-city neighborhoods. In this paper, our goal is to share details of an approach to analyzing teaching around a framework of culturally relevant mathematics pedagogy, specifically in a context of these two urban high schools.

Teacher Learning of Culturally Relevant Mathematics Pedagogy

A promising approach to effective mathematics teaching is culturally relevant pedagogy (Ladson Billings, 1995). As Tate (2005, p. 35) explains, “one barrier to an equitable mathematics education for African American students is the failure to ‘center’ them in the process of knowledge acquisition and to build on their cultural and community experiences.” Centering the Teaching of Mathematics on Urban Youth aims to improve teachers’ practice by creating and sustaining a teacher learning community around culturally relevant mathematics pedagogy. We have currently completed seven months of the two-year professional development and research. At this early stage, we can offer a detailed picture of our thinking about culturally relevant mathematics teaching and mathematics learning opportunities for students.

To enable all students to have equitable access to success in mathematics, culturally relevant teaching (Ladson Billings, 1995) is proposed, as instruction that meets three criteria: 1) emphasizes students’ academic success, 2) encourages the development of cultural competence, and 3) facilitates the students’ development of critical consciousness. We extended the ideas of Ladson Billings (1995) and Gutstein, Lipman, Hernandez, and de los Reyes (1997) to formulate *culturally relevant mathematics pedagogy*, which consists of three, tiered pedagogical components. The primary tier is teaching mathematics for understanding, which implicates both its definition as connections between mathematical concepts, procedures, and facts (Hiebert & Carpenter, 1992) as well as the sociocultural definition of understanding as the valued activity of engaging in sense making of problematic situations (Wenger, 1998). Teaching for understanding emphasizes the importance of classroom tasks that enable students to make connections across ideas and procedures, as well as classroom norms that support varied structures for student participation.

One potential mechanism of mathematical understanding is thought to be the use of relevant or meaningful real-world contexts as a regular aspect of mathematics instruction (Moses & Cobb, 2001). Accordingly, the second tier of culturally relevant mathematics pedagogy is that teachers should include local, relevant or familiar contexts in their teaching, and, as Tate (2005) suggests, “center,” at least in part, the instruction on the students’ experiences. One interpretation of “centering” is that mathematics be used to describe or analyze contexts that are “real” to students’ worlds. An additional interpretation is that teachers can create classroom norms or structures for participation that are accessible to students so that they can be central participants in the building of mathematical understanding (Hand, 2003).

The third tier of culturally relevant mathematics pedagogy invites teachers to address local or societal issues of power and fairness (Diversity in Mathematics Education Center for Learning & Teaching, 2008), to develop students’ critical consciousness and mathematical literacy. One way to conceptualize analyzing issues of power is in terms of unequal distribution of power and resources among people across society. For instance, mathematics could be used with students to analyze local demographic patterns, over time, in connection to the growth of low-income public housing “projects”, or to use mathematics to analyze the compound interest rates built into subprime mortgages that have led to high rates of home foreclosure in these neighborhoods. These types of mathematical investigations correspond to the examples presented in the teaching mathematics for social justice literature (c.f., Gutstein, 2003). Developing students’ critical consciousness in mathematics class also includes addressing issues of power between people and mathematics (Skovsmose, 1994). In other words, opportunities can be given to students to be critical thinkers about the mathematics that they study: who creates it, for what purposes, and with what impacts? When mathematics is viewed as a social practice generated by human agency for communication, representation, or efficiency, this shift in perspective inverts the typical power dynamic where mathematics is a gate, guarded by the teacher, to be passed through.

Culturally relevant mathematics pedagogy is a comprehensive approach to mathematics teaching that lends itself to longer-term professional development. The Centering the Teaching of Mathematics on Urban Youth is a two-year project that focuses on improving high school teachers' instructional practices, using the framework of culturally relevant mathematics pedagogy. Participants do mathematics together, collaborate to learn about their students and the socio-historical context of the communities in which they teach, reflect about their instructional practices using the framework of culturally relevant mathematics pedagogy, and analyze the relationship between their instructional practices and their students' participation and learning. The project is comprised of three activity types: two five-day summer institutes over consecutive summers, bi-monthly school-based department meetings, and work with individual teachers through observations, debriefings, interviews and planning sessions. The approach to professional development and details about its content are described elsewhere (Rubel, under review).

In this paper, we focus on our findings after the project's opening semester, after the first summer institute, six group meetings at each high school, and seven classroom observations of seven focal teachers. Our analysis demonstrates culturally relevant mathematics pedagogy as a framework with which to analyze teaching. At the same time, our analysis also offers a nuanced description of mathematics teaching and student opportunities for learning at these two urban high schools.

Methods

Four of five mathematics teachers at Harwood HS and three of four mathematics teachers at Carver HS (both school names are pseudonyms) are full project participants (along with three preservice teachers and seven teachers from four other urban high schools). Three of these seven focal teachers identify as White, two as Afro-Caribbean, one as African, one as African American. Three of the teachers are male, and four are female. The most experienced teacher has eight years of experience, the least experienced teacher is a first year teacher, and the median number of years of experience is six. None of the teachers are local to these particular communities; only two of the teachers were themselves raised in New York City.

In the first semester of the 2009-2010 school year, we visited each teacher for seven observations in three rounds (three consecutive classes in October, two consecutive classes in December, and two consecutive classes in February), for a total of 49 observed lessons. Each teacher's observations were conducted in the same class period, with the same group of students. We took detailed fieldnotes during those observations to complete a Classroom Observation Inventory (COI), in an attempt to measure and categorize how the teaching in that lesson corresponds with the three components of culturally relevant mathematics pedagogy.

The COI notes a variety of dimensions of teaching and learning. First, the COI contains adapted instruments from Kitchen, DePree, Celedón-Pattichis, & Brinkerhoff (2007) and Weiss, Pasley, Smith, Banilower, & Heck (2003) for ratings of: lesson design, implementation, mathematical discourse and communication, intellectual support, and engagement, each on a 1 to 5 point scale. Second, we adapted instruments from Stein (2004) to classify the lesson's main mathematical task in terms of its cognitive demand, using the ranked categories of doing mathematics, procedures with connections, procedures without connections, and memorization (Henningsen & Stein, 1997). Third, we adapted the categories of student participation in Weiss et al. (2003) to create a set of modes of student participation: listening; investigating or problem solving; discussing; reading, writing, or reflecting; using technology; and practicing skills. Each of these categories is further subdivided into more specific categories. We tabulated the duration of student participation for each participation category in each lesson. Finally, we also described if and how teachers used local or other relevant contexts in each lesson observed, as well as if and how teachers drew on students' language or other cultural resources in the lesson.

Results

We begin with results of the COI ratings across the categories of lesson design, lesson implementation, mathematical discourse & communication, intellectual support, and engagement, as a way to demonstrate the range of ratings across the seven teachers. Table 1 indicates each teacher's mean rating for each of the five categories across the seven observed lessons, as well as their overall mean rating.

Table 1. Mean Classroom Observation Ratings by Teacher

	A	B	C	D	E	F	G
Design	1.17	1.11	1.34	1.23	1.86	2.63	3.27
Implementation	1.36	1.26	1.60	1.79	1.93	2.81	3.05
Mathematical discourse and communication	1.29	1.43	1.71	1.86	2.57	3.29	3.29
Intellectual support	1.86	2.14	2.14	2.71	3.57	4.00	3.71
Engagement	1.86	1.86	2.14	2.14	2.43	3.43	3.86
Mean of Five Ratings	1.51	1.56	1.79	1.95	2.47	3.23	3.43

Lessons that had higher ratings in design and implementation also had higher ratings in mathematical discourse and communication, intellectual support, and student engagement (Pearson correlation coefficients of at least 0.703, at 1% significance level). Teachers F and G had ratings that are higher (with statistical significance) than the other teachers, and teachers A and B had ratings that are lower (with statistical significance) than the other teachers. In the remainder of the results section, we focus on these two subsets of teachers. We present our results in terms of three aspects of culturally relevant mathematics pedagogy: a) promoting understanding by using tasks that enable students to make connections between and among mathematical ideas and procedures, b) creating multiple modalities for students to participate in learning mathematics in the classroom, and c) connecting with students' everyday experiences.

Lesson Mathematical Tasks

The main, enacted mathematical task in each observed lesson was identified, described, and then rated according to its requisite level of cognitive demand, using the categories of Henningsen & Stein (1997): memorization, procedures without connections, procedures with connections, or doing mathematics. None of the tasks were rated at the highest level, as "doing mathematics." The majority of the lessons, 34 of 49 lessons, contained tasks that were rated as low-level cognitive demand, either memorization (9) or, more often, procedures without connections (25). The two teachers with the lowest mean ratings presented their students with tasks only at these low levels. Thirteen of the lessons had high level tasks, procedures with connections, but these tasks were limited to the lessons of just three teachers. Twelve of these thirteen tasks were in the lessons of the same two teachers who also had the highest rated lessons in terms of design, implementation, mathematical discourse and communication, intellectual support, and engagement. This correspondence demonstrates a relationship between the task's level of cognitive demand and other aspects of the lesson.

Modes of Student Participation

Throughout each observed lesson, we classified and quantified student participation, according to the following mutually exclusive categories, adapted from Weiss et al. (2003): a) listening; b) discussing, c) investigating or problem solving; d) reading, writing, or reflecting, e) using technology, or f) practicing skills. Across all 49 observed lessons, the most frequent mode of participation offered to students was listening, which accounted for 33% of all instructional time (30% of the total instructional time was spent listening to a teacher presentation). The second most frequent mode of student participation was practicing skills, which accounted for 23% of total instructional time. Less frequent for students were investigating and problem solving (19%), discussing 10%, all in whole group format); reading, writing, or reflecting (4%) and using technology (2%). The remaining 9% was spent on teacher "housekeeping" activities that are not instructional in nature.

Table 2. Distribution of Student Participation Modes by Teacher

	Teachers with lowest rated lessons		Teachers with highest rated lessons	
	A	B	F	G
Participation Categories				
Listening	44%	39%	29%	31%
Practicing skills	45%	39%	6%	14%
Investigating or problem solving	0%	0%	34%	28%
Discussing	5%	0%	17%	18%
Reading/writing/reflecting	0%	2%	1%	5%
Using technology	0%	0%	9%	0%
Teacher "Housekeeping"	6%	20%	4%	4%
Overall mean of ratings	1.51	1.56	3.23	3.43

There is variability across the seven teachers, as shown in Table 2 above. For teachers A and B, who had the lowest mean lesson ratings, the combined categories of listening (to the teacher), practicing skills, and teacher "housekeeping" formed nearly the entire range of modes of participation in their two classes (95% for A and 98% for B). The lessons of teachers F and G, the teachers whose lessons had the highest mean ratings, also devoted substantial amounts of time on listening, practicing, and teacher "housekeeping" (39% for F and 49% for G). However, their lessons also included substantial amounts of time devoted to other types of participation like investigating and discussing (a total of 51% for F and 46% for G). Ms. Kendall (a pseudonym), one of the teachers with the highest mean lesson ratings, stands out in terms of participation structures in that in her class, students about as much time problem solving or investigating (34%) than they do listening (29%) and practicing skills (6%) combined. Ms. Kendall's classes are characterized by her students' enthusiasm to participate and discussions in which students' ideas are explored even when they may not yet correspond with normative mathematics in terms of representation or correctness.

Connections to students' experiences

The two teachers with the lowest lesson ratings selected low-level tasks focused on memorization of facts or procedures and contexts were typically abstract, like a geometric figure or an algebraic equation. In some instances, these teachers' lessons included contexts that could potentially connect to students' experiences but did not facilitate student engagement or increase depth of student knowledge. For example, in one lesson, a teacher listed a set of jersey numbers for members of a basketball team, and a second set of jersey numbers for members of the team who could not play in that game. The prompt was then to find the members of the team who *could* play in the game, all as a way to model the notion of a set and its complement. The set notation does not illuminate aspects of the game of basketball, and conversely, the context of basketball does not illuminate this particular mathematical notation or idea.

Another of the seven teachers, Ms. Tristan (a pseudonym), one of the two teachers with the highest mean lesson ratings, demonstrated a range of ways of connecting to students' experiences as a way to promote student participation and build understanding of mathematical ideas. Ms. Tristan capitalized on geometric features of the urban environment in her mathematics instruction and used neighborhood maps in several ways. These activities extended her experience in our professional development, in which we explored the use of local maps as a mathematical context in a variety of ways, which are described in detail elsewhere (Rubel, Chu, & Shookhoff, under review).

Ms. Tristan used maps in several different ways in her lessons. For one lesson that involved learning the distance formula as a way to compute the distance between two points, Ms. Tristan used a gridded neighborhood map and had the students consider the "as the crow flies" distance from the school to the nearest subway station. In a different lesson, that dealt with perimeter of polygons and unit analysis, Ms. Tristan's students were asked to calculate a precise estimate of the time it would take to walk the perimeter of the neighborhood. Students coordinatized a gridded map and used the Pythagorean Theorem to find the perimeter of the polygon-shaped neighborhood. In a third lesson, dealing with triangle points of concurrency, Ms. Tristan had each student plot his or her own address and that of two friends on the neighborhood map and then calculate a fair meeting location, which was defined as each person traveling the same distance. This lesson hinted at critical thinking about using mathematics to define and negotiate aspects of fairness.

Using neighborhood maps is a literal interpretation of "centering" mathematics on students' worlds. Ms. Tristan demonstrated other interpretations of how to connect instruction to students' everyday experiences. For instance, Ms. Tristan showcased one student's break-dancing abilities to demonstrate the notion of balance point, as a way to introduce the geometric idea of centroid. In another lesson on finding shaded areas within geometric figures, Ms. Tristan presented a series of "optical illusion" images that involved positive and negative space. Students eagerly deconstructed each total shape into positive and negative space, an approach that extends well to shaded area mathematics problems. In this last example, Ms. Tristan did not draw on students' out-of-school experiences, but created within the classroom an accessible, shared experience for students to use as a problem-solving anchor and structured the lesson so that students could be central participants.

Discussion

These results begin to challenge the notion that reform teaching practices that emphasize student communication and problem solving might not be effective with students from low-income communities (Boaler, 2002). Our results demonstrate relationships among ratings of lesson quality as well as between those ratings and categories of task cognitive demand and types of student participation structures. The cognitive demand of a mathematical task sets a range of modalities for student participation. Low-level tasks, requiring only memorization of facts or executing steps of a procedure, lend themselves only to listening and practicing. By contrast, tasks with higher-level cognitive demands, in which students must detect patterns or make connections across mathematical representations, afford opportunities for more varied participation modalities. Ms. Kendall's lessons, with tasks of higher level cognitive demand, also had higher ratings of mathematical discourse and communication and student engagement and had the most varied participation opportunities for students.

These preliminary findings also add nuances to the description of culturally relevant mathematics pedagogy, in terms of instructional practices and in terms of student participation and engagement. Other studies have demonstrated that the use of contexts can be a distraction, (Boaler, 2002), especially for students from low-income communities, most notably when the students have different experiences with the context than the teacher realizes. In our project, we have begun to see examples of teachers using contexts that are, instead, local to the students. Since this is the focus of the professional development activities, we hypothesize that we will see development of such practices among other participating teachers as the professional development continues. We hypothesize that Ms. Tristan's use of local and relevant contexts are related to the high levels of student engagement we have observed in her classes. These contexts authentically connect students' experiences with the mathematics in question.

In this article, we have profiled the instructional practices of two teachers as a way to more fully describe the framework of culturally relevant mathematics pedagogy and how we are measuring it in classrooms in urban schools. These initial findings also function as formative assessment for the ongoing professional development activities. Taking a longer-term perspective, we are hopeful that over the course of this project, teachers will continue to grow, in ways that we have and have not yet anticipated. We anticipate that continued collaboration will lead to sustained growth in challenging mathematical tasks, flexibility and variety in participation, and connections to students' experiences.

Endnotes

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FACILITATING MULTIPLE STRATEGIES USE IN A HIGH POVERTY CLASSROOM THROUGH PROBLEM-BASED LESSONS

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All learners need opportunities to experience mathematics in a rigorous fashion that promotes success and draws upon their prior knowledge. A series of problem-based lessons that encouraged the use of multiple strategies were implemented in a fifth grade classroom. Students completed a pretest, posttest, and follow-up test six months after the intervention. They showed significant gains in their use of multiple strategies after the intervention and a marked increase in their ability to solve word problems. Students reported transferring their use of multiple strategies to their typical classroom and feeling more self-confident.

Introduction

Problem solving is a process standard that has been hotly debated by the mathematics education community and parents (National Council of Teachers of Mathematics [NCTM], 2000; Saunders, 2009; Schoenfeld, 2004). Problem-solving instruction varied across decades. From a focus on content-specific heuristics to transferring experts' use of explicit strategies to novices, and embedding problem-solving notions within lessons centered on a specific mathematical topic (Schoenfeld, 2004). Besides explicit problem-solving instruction, some advocates suggest that problem-based instruction [PBI] engages learners in rich learning experiences they might utilize in the future (Herried, 2003; Pecore, 2009). While debates over modes of mathematics instruction continue, the achievement gap remains too wide (United States Department of Education [USDOE], 2008; 1990-2007). Recent scores from the National Assessment of Educational Progress [NAEP] indicate there are gaps in students' mathematics knowledge related to race and income status (USDOE, 2007). On average, Caucasian fourth grade students outperformed African-American peers by 20 points and similarly, Caucasian fourth grade students scored 21 points higher than Hispanic peers. All children can achieve at high standards and should be given opportunities to learn in a caring environment filled with challenging learning opportunities (Ayers, 2008). Literature from the last decade provides evidence for using student-centered mathematics instruction to facilitate learning for conceptual understanding (Ball, 1993; Lampert, 2001).

Literature Review

Student-centered mathematics instruction

Student-centered mathematics instruction uses learners' "backgrounds and cultural values, as well as abilities" as the starting point for effective instruction (National Research Council [NRC], 2005, p. 14). The National Mathematics Advisory Panel created their own unique definition of student-centered learning that prevented previously mentioned studies from being used as evidence for student-centered learning (Boaler, 2008), however the authors will not make that same mistake. Student-centered learning acknowledges that learning for all individuals may occur in different ways for each child (NRC, 2005). Discourse and activities promoting reasoning are two common features of student-centered mathematics instruction (Cobb, Yackel, & Wood, 1992; Lampert & Cobb, 2003). There are varying degrees of student-centered mathematics instruction, from learner-selected inquiries to fairly teacher-guided discussions about a mathematical topic. Problem-based learning [PBL] uses complex problems as the means for teaching content and problem-solving strategies (Pecore, 2009; Ronis, 2008). Problem-based tasks often employ students' experiences as starting points for investigations (Pecore, 2009). PBL advocates suggest that it (1) facilitates learners reasoning and metacognitive actions while problem solving, (2) develops students' social skills as part of a problem-solving team, and (3) motivates students intrinsically (Hmelo-Silver, 2004). Instructors perceiving mathematics as the study of logical analysis that gives learners tools to "describe, abstract, and deal with the world in a coherent and intelligent fashion" (Schoenfeld, 1982, p. 30), may utilize PBI to provide opportunities to learn content and engage in mathematics as problem solvers.

Problem solving

Problem solving has an extensive history in educational psychology literature and has been a mainstay of mathematics education research (Mayer & Wittrock, 2006; Polya, 2004). De Corte and Verschaffel's (1981) problem-solving framework uses three phases to characterize the cyclical problem-solving process, (1) thinking, (2) execution, (3) verification. The thinking phase involves all behaviors and actions necessary to begin carrying out problem solving. This includes reading the problem, creating internal representations of the task, and considering possible strategies. After considering the task, problem solvers execute one strategy and carry out necessary computations. Verification requires using strategies to verify whether the solution is correct and appropriate for the context. These strategies may not necessarily include overt behaviors.

Verschaffel and De Corte (1997) noticed that many elementary-aged children solve word problems and fail to consider whether the solution jives with their authentic experiences. They employed a teaching experiment to examine whether 10-12 year old boys might benefit from PBI. They provided students with authentic, complex word problems and investigated whether students were able to provide more realistic solutions than their peers receiving traditional instruction. After five lessons, students in the experimental group had significant growth from pretest to posttest in the number of realistic responses to word problems, $p<.0001$. They also showed a significant increase in correctly solving problems requiring transfer, $p<.05$. One month following the instructional sequence, these students continued to provide significantly more realistic solutions to word problems than their peers, $p<.05$. Verschaffel and De Corte indicated that more realistic responses to word problems may lead to solving more word problems correctly and should be investigated. One limitation of this study was that both control and experimental groups were entirely male. Much of the foundational problem-solving literature's evidence stems from work with gifted or older students, learners working in sterile labs, and examining students' work on highly contrived problems (Lesh & Akerstrom, 1982). Lesh and Akerstrom (1982), as well as Lester (1982) suggest that problem-solving research needs to investigate problem solving with younger or non-white students. The present study aims to answer Verschaffel and De Corte's (1981) wondering about solving more word problems correctly. It also aims to answer Lesh and Akerstrom's call for examining what strategies young learners employ when solving word problems and their flexibility with employing multiple strategies.

Representations and strategies

Representations are valuable for their ability to communicate ideas and as a means for learners to discuss knowledge (Greeno & Hall, 1997). Facility with multiple representations influences learners' strategy-use somewhat (Bostic & Pape, in press; Brenner et al., 1997). Students with more knowledge about representations and adequately developed schemas for problem solving have shown enhanced mathematical knowledge on achievement tests (Brenner et al., 1997). Recent studies using graphing calculators (i.e., Bostic & Pape, in press; Herman, 2007) showed that technology-use influences what solution strategies students choose. During one month of instruction on quadratics equations in an Algebra II classroom, students' achievement was higher for those using a graphing calculator that provides better on-screen linking of representations than older graphing calculator models (Bostic & Pape). Students using the newer technology reported that instruction, coupled with the graphing calculator, assisted them in thinking about which strategy was most effective to solve a word problem. Furthermore, students valued learning more strategies to become better problem solvers. One likely step in advancing research on students' facility with strategies is to examine what strategies students select while engaged in PBI.

Summary

This study examined elementary students' problem-solving achievement and facility with multiple solution strategies. PBI lessons that related to students' experiences were developed and implemented. The intent behind the lessons was to enhance students' use of multiple strategies during PBI. Students were interviewed six months following the initial study and their responses were investigated to determine whether the intervention impacted their current problem-solving behaviors and self-confidence. The research questions for this study were (1) in what ways did PBI influence students' ability to solve word problems and problem-solving behaviors and (2) how did students' self-confidence as problem solvers change after the intervention? Both quantitative and qualitative methods of analysis and results will be discussed.

Method

Participants

The state of Florida grades its schools using an A through F scale each year using several factors, summative state-wide tests are key factors determining that score. Louisville elementary school is located in a mid-size city in Florida; its school population during the time of this study was 95% African-American and 96% of the student body received free-and-reduced-lunch. Its highest score in the last seven years was a C; the last three years it scored F. Louisville's faculty and staff are deeply committed to supporting students and making them successful inside and outside of school. The principal invited mathematics education faculty and graduate students to work with students. Fifteen fifth grade students attended a voluntary summer school program and all students were invited to participate in the study. Parents and guardians were given the option of having their child participate in a traditional mathematics program taught by one of the classroom teachers during the intervention. No parent or guardian asked to have their child removed, however nine parents/guardians and students returned completed informed consent and assent forms. One participant was not available to complete the pretest on the first day. In the following academic year, the instructor returned to Louisville to interview participants and administer a follow-up test. Four students from the summer program left Louisville elementary school and were not able to participate in the follow-up study. All participants in the intervention identified themselves as African-American. All names of places and people are pseudonyms.

Instructional program

The instructional program consisted of PBI over four days. During the week, the classroom teacher observed the instructor and students. Problems were adapted from word problems found in National Science Foundation-funded curricula. Problems correlated with Florida's Sunshine State Standards (Florida Department of Education, 2007), incorporated authentic experiences germane to students, and could be solved using multiple representations. One mathematics education professor and one elementary teacher examined the problems for evidence of validity, appropriateness for the grade level, and whether the problems' scenarios were relevant to students' experiences. The validity team suggested that the problems were appropriate, met grade-level expectations, and incorporated students' relevant experiences. A detailed description of the PBI used in this study is presented by Bostic and Jacobbe (2010).

Data collection

Students completed a pretest on the first day, a posttest on the last instructional day, and a follow-up test six months following the intervention. Questions were created using the same resources as those used for the intervention. All assessments consisted of three problems; the items were similar across the instruments except for changing some numerical values and some language (e.g., a dog has four legs was changed to a cricket has six legs). According to the validity team, changes did not affect problem difficulty, grade level appropriateness, or prevent solution strategies from being explored. Fraction operations, combinations, and number fluency were selected as the three topics for the problem-solving achievement tests. Participants took approximately 10 minutes to complete the pretest and 25 minutes on the posttest and follow-up tests.

Six months later, students were interviewed individually in an empty classroom for about 20 minutes each. The interviewer asked participants to reflect on their experiences from the summer, discuss whether they perceived any changes related to their problem-solving capabilities or mathematics achievement, and how their experiences from the summer compare to their current classroom experiences.

The authors used thematic analysis to identify key themes related to effects on students' problem solving behaviors after the intervention (Hatch, 2002). First, the authors read the transcripts and located potential statements for analysis. Statements were lifted from the transcript and copied into a word processing document. This new document contained approximately forty statements. After several iterative readings of the statements, the authors organized the statements under potential thematic headings. Themes were scrutinized and those with a paucity of evidence were not included in the final set of themes. Two themes had substantial evidence and will be discussed here.

Results

Correctness and solution strategies

Participants' responses to each problem were coded as either correct or incorrect. Next, the authors examined students' solution strategies. A solution strategy was considered unique if it was different from a previous strategy for the given problem. For instance, a student might use three distinct symbolic solution strategies to solve a problem. Paired t-tests allowed comparisons of participants' growth during the intervention; one-way interpretations were employed due to prior studies indicating that it is likely students would experience growth in their ability to solve complex word problems after engaging in PBI (De Corte & Verschaffel, 1981; Charles & Lester, 1984; Verschaffel & De Corte, 1997).

Participants demonstrated statistically significant growth from pretest to posttest in solving more problems correctly, $t(7) = -2.69$, $p = .011$. Most students could not solve any problems correctly on the pretest. There was not significant growth from pretest to follow-up test however this is likely due to participants leaving the school and the low power necessary for achieving statistical significance. The five students who participated in the follow-up testing did show statistically significant gains from posttest to follow-up test, $t(4) = -2.138$, $p = .049$. This result may be confounded with students' mathematical knowledge growth and classroom experiences during the academic year after the summer program.

Besides solving more problems correctly, students also used more strategies to solve each problem after the intervention, $t(7) = 2.525$, $p = .020$. Even with low power, students continued to employ more strategies on the follow-up test than the pretest, $t(3) = 4.70$, $p = .002$. Classroom teachers' instruction was somewhat traditional in nature however they also incorporated student-centered aspects such as cooperative grouping and encouraging content-related discussions.

Students' choice of various strategies changed somewhat after the intervention and while not statistically significant, the results point towards a trend to significance. Participants tended to use more pictorial representations on the follow-up test than pretest, $t(3) = 2.324$, $p = .051$. Participants also tended to use more verbal strategies on the posttest than pretest, $t(7) = 1.825$, $p = .055$.

Problem-solvers' current behaviors

Participants suggested that they engage in the problem-solving process in their current classroom as described by De Corte and Verschaffel (1981). Bobbie characterized the three stages in her own words.

> First, I read over the problem. Then go back and if it has numbers, I look at those again. Then I try some strategies like add, subtract, multiple, and divide. If they have that 'how many' or 'altogether' or I forgot the word. But if they [problem] say 'multiply' or 'divide', I do that. And then I work them all our and see if they have an answer and I work it all out and see if I have the answer. Then I have the answer.

When asked about whether she checks her work, Bobbie indicated that she does it explicitly "sometimes" but typically she examines the problem's solution and determines whether it seems reasonable. Bobbie's description of how she solves word problems provides evidence that adds to De Corte and Verschaffel's finding (1981). Like her U.S. peers, Bobbie perceived word problems as problems that are usually solved in one step and have language within the problem that cued her into employing a specific operation. Many students tried to use one of the four arithmetic operations as a means for solving problems on the pretest. These students have developed a word problem schema due in part to the instructional opportunities they had in the classroom prior to the study. Students' use of strategies broadened somewhat after the intervention. Participants indicated they use pictorial strategies in their current classroom more frequently, "I get scrap paper and I'll draw pictures so that way I'll get it better" (Storm). Storm's strategy choice was typical and similarly described by the other participants.

Participants also commented that they valued their peers' use of multiple strategies during problem-solving activities in their current classroom. They commented that listening to their peers' different strategies assisted them while problem solving. "I love seeing how others solve a problem" (Logan). The teachers in the participants' current classrooms created a classroom environment that stimulated students' minds for thinking about multiple strategies at times. Logan described how his classroom teacher provided him an opportunity to examine other's strategies in a whole-class learning environment.

> Well, they [peers] don't have to explain it to me but they can go up there. Mrs. Patricia, she bring [sic] up people. One time she bring up [sic] a girl. We had these dry erase boards and she do [sic] it on the chalkboard, you know, the person who do [sic] it real good. She tell [sic] them to go up and put it on the board and put it there. And I be [sic] like, 'oh that's cool' and I like that and I try to learn it.

Logan went on to elaborate that he did not necessarily hold this view until after working with his classmates during the summer PBI. Their current classroom environment may have contributed to students' statistically significant test scores between posttest and follow-up test. Students were cued into thinking about multiple strategies to solve word problems during the school year. They commented that they continued to employ various strategies during the school year, especially symbolic and pictorial strategies.

Self-confidence

Some participants described feeling more self-confident after the summer PBI. The support in the classroom from the instructor and other students may have influenced students' perceptions. When they perceived the classroom as a community of problem solvers, they might have experienced feelings of confidence and desire to maintain engagement in problem solving. Storm suggested that she

> had more struggles in summer school and because I had a lot of people helping me out and we had more strategies we could use, which is great and now when I'm in there [current classroom], I bring them all in there and I know how to do it and my grades came up.

Storm and her peers talked about grades in class as a factor in their engagement in mathematics instruction and self-confidence. During the intervention, students were challenged with difficult word problems, encouraged to delve deeply into the question, and think critically about viable solutions. Logan's feelings about hard problems and problem solving changed "from there to here. I had a difficult question in the summer and I get [sic] aggravated. But now I kind of get aggravated but I get more into it and work hard" (Logan). Furthermore, they used each other as a resource for support during problem solving. Bobbie felt support from her peers and instructors also influenced her understanding of mathematics topics.

> I had more struggles in summer school and because I had a lot of people helping me out and we had more strategies we could use, which is great. And now when I'm in there [current classroom], I bring them all in there. And I know how to do it and my grades came up…and it got me to a higher level that I can move on from.

The influence of a demanding, student-centered problem-solving classroom may have influenced students' self-confidence and problem-solving behaviors for the long term.

Discussion

Students engaged in PBI experienced positive benefits in their problem-solving capabilities, facility with multiple solution strategies, and self-confidence. As problem solvers, these students not only perceived themselves as more capable but also demonstrated that they were more likely to solve word problems correctly. Participants described feelings of empowerment and self-confidence related to problem solving.

In their current classroom, participants continue to experience instruction that somewhat facilitates learners using multiple solution strategies and examining problems from multiple perspectives. While students' use of specific solution strategies, such as using pictures or verbal representations were not statistically significant, the trend towards significance provides direction for future investigators to consider interventions that support students' use of pictorial and verbal solution strategies. Some older students perceive non-symbolic representations as less mathematically rigorous thus it is imperative that mathematics instruction continues to stress the value of all representations as being useful (Bostic & Pape, in press; Brenner et al., 1997; Herman, 2007). Starting instruction that encourages multiple strategies in elementary grade levels and carrying it through to middle and high school may enable students more options while engaging in problem solving. Without instruction that scaffolds mathematical learners to consider various strategies during the thinking phase, students may continue to use overly simplistic or inappropriate means to solve problems. PBI provides these opportunities for learners to engage in problem solving and build robust knowledge of strategies. By aligning content-focused pedagogy with student-centeredness, problem solving, and facility with multiple strategies, it may be possible to narrow the achievement gap.

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STRUGGLING FORWARD: REFINING A FRAMEWORK FOR TEACHING MATHEMATICS WITH AND FOR SOCIAL JUSTICE

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In this paper I describe the creation, use, and refinement of a framework for understanding the practice of teaching mathematics with and for social justice. This framework emerged out of a self-study of my own teaching practice where I attempted to document the struggles of a mathematics educator learning to teach mathematics for social justice. The refined framework calls for the defining of the student to match the definition of content to be learned, with implications for lesson design and further research.

Introduction

Teaching mathematics for social justice has been presented as a way to address some of the inequities present in mathematics education and the world at large (Gutstein, 2006). Inclusive education has been presented as a means for providing all students, regardless of challenges and abilities, access to engaging content in the classroom (Villa & Thousand, 2005). Combined, these approaches to education can be summarized as teaching with and for social justice (Wager, 2008). Though these approaches have been presented as promising, the changing of practice to enact these approaches in the classroom has been described as a problem for in-service mathematics teachers, specifically teaching mathematics for social justice (Brantlinger, 2007; Gau, 2005; Gutstein, 2007). Schoenfeld (1985) defines a problem as a task that is complex with an unknown method for completion. As a member of the research community, I am interested in describing the problem of attempting to change one's practice towards what is proposed in teaching mathematics with and for social justice. The purpose of this study was to document the struggles of a mathematics educator moving towards this pedagogy of inclusively teaching mathematics for social justice, from an insider's perspective, in order to provide insights to teacher educators and practitioners wanting to promote or engage in this work, which is in direct response to a call for work to be done to "see what teachers struggle with as they learn to teach mathematics for social justice" (Diversity in Mathematics Education, 2007, p. 420) What emerges from this effort is a framework for understanding the practice of teaching mathematics with and for social justice, with implications toward designing instruction.

Theoretical framework

In Lampert's (2001) study of her own teaching, she developed a framework for understanding the practice of teaching as a network of relationships. The teacher has a relationship with the content, the student, and the student-content relationship (See Figure 1). Ultimately, the goal for a teacher is to get the student to "study" mathematics, where studying is described as "any practice engaged in by students in school to learn."(p. 32). Thus, forging a relationship between the students and the intended mathematical content, but how does the problem space change when the task shifts to teaching mathematics with and for social justice? Lampert's framework provides a base with which to overlay the other two perspectives: Gutstein's (2006, 2007, 2009) representation of teaching mathematics for social justice and

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

Udvari-Solner, Villa, & Thousand's (2005) conception of inclusive lesson design, the Universal Design Process.

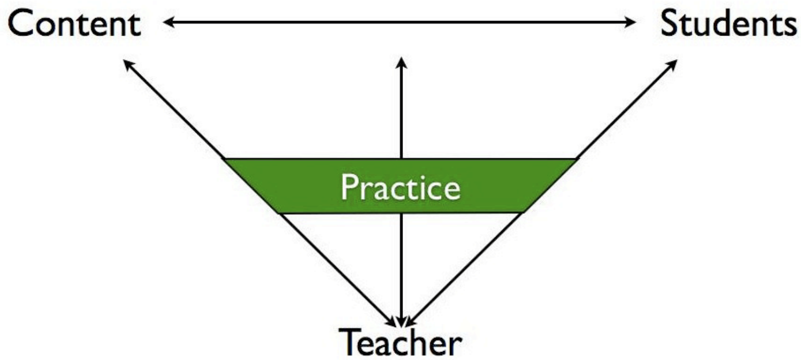

Figure 1. Lampert's (2001) visual representation of the "problem space of teaching practice"

Teaching mathematics for social justice (Gutstein, 2003, 2006) is an approach to mathematics education that attempts to realize the goals of culturally relevant pedagogy (Diversity in Mathematics Education, 2007) to "produce students who can achieve academically, produce students who can demonstrate cultural competence and develop students who can both understand and critique the existing social order" (Ladson-Billings, 1995, p. 474). Gutstein (2006, 2007, 2009) articulates the teaching of mathematics for social justice as simultaneously promoting the use and development of three types of knowledge: classical, critical and community (See Figure 2). Classical knowledge is the mathematical knowledge needed to gain access to advanced mathematics and to excel at high-stakes tests (Gutstein, 2006). Critical knowledge is the knowledge (both mathematical and otherwise) necessary to understand one's sociopolitical reality (Gutstein, 2006). Community knowledge is the knowledge (both mathematical and otherwise) that exists within individuals from the school community context, which may not be understood by those who do not participate in the community (Gutstein, 2006). This component of teaching mathematics for social justice acknowledges the "funds of knowledge" (Gonzales, Moll, & Amanti, 2005) that exist in the community and can provide context and motivation for facilitating the use and development of critical and classical knowledge. Taken together these three domains describe the aims of teaching mathematics for social justice and the challenge of providing instruction that respects the validity and power within each of them.

Expanding on Lampert's notion of content are the components of teaching mathematics for social justice as articulated by Gutstein (2006, 2007, 2009). A teacher engaged in teaching mathematics for social justice is concerned with the student learning the identified mathematical objectives of the unit (classical knowledge), learning how the identified standards can be seen in the everyday reality of the student (community knowledge), and how the mathematical objectives could be used to better understand that everyday reality or impact it for the better (critical knowledge). By expanding the notion of content in the problem space of teaching, all other relationships are now affected. For example, if the mathematical content for the lesson is to be rooted in radically local contexts, then the teacher is concerned with the student "studying" not only the mathematics, but also the community and critical knowledge that situates the mathematics. Thus the entire problem space of teaching is impacted, even though teaching mathematics for social justice is put under the umbrella of *content*.

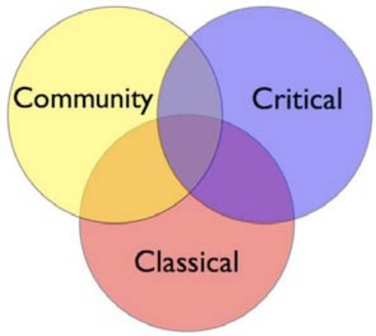

Figure 2. Teaching mathematics for social justice as intersection of knowledge (Gutstein, 2009)

Some of the issues associated with in-service teachers attempting to teach mathematics for social justice has been described as a curriculum or lesson development issue. Gutstein (2007) specifically names the time, resources, and skills necessary to develop a social justice curriculum are not necessarily available to in-service teachers. In her study of teachers attempting to teach mathematics for social justice, Gau (2005) noticed that teachers were unable to balance between teaching mathematics and teaching for social justice and doing one exclusive of the other, rather than in concert.

To address some of these documented struggles, I applied the Universal Design Process (Udvari-Solner, et al., 2005), which is a means for developing lessons that addresses the needs, abilities, and interests of all students who are to engage with the lesson. The Universal Design Process is commonly associated with supporting teachers of inclusive classrooms, where all students, regardless of label and/or ability, are taught together, and the underlying assumption is that "living and learning together benefits everyone" (Falvey & Givner, 2005, p. 5). The intent of the Universal Design Process is important to highlight, because to decide to teach mathematics for social justice, suggests the enactor of such an approach realizes that there are inequities in the mathematics classroom and/or the world at large that can be addressed with such an approach. Having made such a choice to teach mathematics for social justice would also suggest that the teacher would want to provide "a socially just community in which students participate equally" (Wager, 2008, p. 99) or to teach mathematics *with* social justice. It is my belief that the Universal Design Process can help with both of those intentions and assist in addressing some of the problems identified above.

Specifically, the Universal Design Process (Udvari-Solner, et al., 2005) has four components: a) learning about the students in the classroom, b) naming the content that is to be learned, c) deciding how students will engage with the content, and d) how students will demonstrate their learning of the content. Learning about the students calls for "developing positive profiles of students' social and academic abilities, strengths, and learning concerns…"(p. 138). This component goes beyond learning the students' names and interests but actively seeking out the optimum means for delivering instruction, or learning how the students learn best. Udvari-Solner, et al. (2005) recommend using a multiple intelligence perspective (Gardner, 1993) in order to understand the students as learners in the classroom, for the purpose of making appropriate design decisions throughout the process.

By declaring one's intention to teach mathematics for social justice, the dimensions of content have already been defined by Gutstein (2006, 2007, 2009). But in addition, the Universal Design Process calls for the teacher to decide "what level of knowledge or proficiency students are to demonstrate; and what context, materials, and differentiation are necessary to allow all students, including those with disabilities, a point of entry to learning" (Udvari-Solner, et al., p. 141). Another way to consider this design point is to define how the student will relate with the content, or defining the nature of the relationship.

Deciding how students will engage in the relationship with the content, or the "process" component, involves deciding on the "instructional strategies that afford students multiple means of engaging with the curriculum"(p. 143). The students engage with the content through the tasks and environment that the teacher has designed and enacted. This component represents how the students will learn the content of the lesson, given their specific learning profiles. Referring back to Lampert's (2001) framework, this component of the Universal Design Process would equate with the student-content relationship (see Figure 3).

The last piece of the Universal Design Process, or the "product" component, asks teachers to decide "how students will demonstrate and convey their learning"(pp. 145-146), which is the assessment portion of the design and provides an opportunity for students to reflect their learning within a tangible artifact. This is the evidence that the students are "studying" the content. Further, the product can be used as evidence that the process component was effective in facilitating students learning what the lesson was designed to teach (see Figure 3). This evidence of learning, or lack thereof, can also be equated with evidence of success/struggle in attempting to teach mathematics for social justice, which foreshadows the methodology of this study.

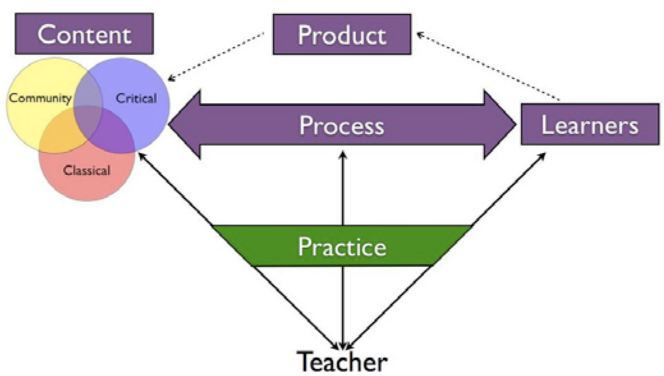

Figure 3. Problem space for teaching mathematics with and for social justice

The resulting framework provides a visual representation of the practice of teaching with and for social justice, with implications towards instructional design and evaluation.

Methods

I chose an insider's perspective (Ball, 2000) in order to articulate the nature of the struggles a teacher may have in attempting to teach mathematics for social justice. I assumed the role of teacher in a public high school mathematics classroom in a rural school district in the Midwest and attempted to teach a mathematics lesson with and for social justice (Wager, 2008). The lesson, that was taught over 6 class periods, was designed with information gathered during a three-month pre-lesson enculturation period, using the framework described previously. Ball

(2000) provides several questions that probe the appropriateness of engaging in this kind of "first-person" methodology.

Ball's (2000) first question asks: "Does the researcher have a conjecture or image of a kind of teaching, an approach to curriculum, or a type of classroom that is not out there to be studied?" (p. 391). My interest lies in describing the struggles experienced by a teacher of mathematics who decides to shift his or her pedagogical practices towards what is envisioned in teaching mathematics for social justice. Finding a teacher with a genuine desire to change his or her pedagogical practices to those described by Gutstein (2006, 2007, 2009), and willing to be studied in the midst of that change would be difficult to find. My other desire was to capture this event in a context that is different from what the literature already contains. Brantlinger (2007) suggests that this approach to mathematics should be used in non-urban environments, such as the rural community where I situated this study. Combined, finding a teacher wanting to change his or her practice to what is described, in the desired context, would be difficult. Thus, necessitating the creation of such a situation.

Next Ball (2000) asks if the researcher has the tools necessary "to be a designer, developer, and enactor of the practice or would an experienced practitioner be a more reliable partner in this construction?" (p. 391). I would describe myself as an experienced teacher of reform mathematics and an inexperienced teacher of mathematics for social justice with a desire try this approach to mathematics education, thus making me "well equipped" to fill the role of teacher in this study. Regarding my position as a researcher, I may not have been the best equipped to develop and enact this study, but given the resources and expertise of those surrounding me in the university that I am associated with positioned me as sufficiently equipped to engage in this work.

Ball's (2000) third question asks: "…is what the researcher wants to know uniquely accessible from the inside or would an outsider be able to access this issue as well, or perhaps better?" (p. 391). From the work of Lampert (2001) and Heaton (2000) it can be seen that there is something to be gained by having a teacher critically examine her or his own practice. And in my time as an in-service teacher I was constantly examining and modifying my practice to better meet the needs of my students and my responsibilities as their teacher. That experience was primarily concerned with identifying areas of struggle and reacting accordingly, and what I added to do this study is the act of documenting the experience. Admittedly, a better situation for this study could be to have a paired examination of the teaching practice with myself examining from the inside and someone else examining from the outside. But given the scarcity of resources and the scale of this study this ideal situation was not possible to realize.

Lastly, Ball (2000) poses: "…is the question at hand one in which other scholars have an interest, or should have an interest, and if so, will probing the inside of a particular design offer perspectives crucial to a larger discourse?" (p. 391). In addition to what I have stated previously, the Diversity in Mathematics Education Center for Learning and Teaching suggests that, "More work is needed in this area to see what teachers struggle with as they learn to teach mathematics for social justice…" (2007, p. 420). If this approach to mathematics is at all fruitful in its attempts to address inequities within the classroom and in the broader world, then understanding the struggles of teachers attempting this approach is a worthwhile and necessary endeavor. In sum, I would argue that capturing the struggle of a teacher trying to engage in a new pedagogy, specifically this approach to mathematics education is a valid candidate for an insider's perspective.

The data sources included student products, transcribed audio teacher journal (Cochran-Smith & Lytle, 1993), and classroom video recordings. Analysis of the transcribed data called on the tradition of grounded theory (Corbin & Strauss, 2008; Emerson, Fretz, & Shaw, 1995). Open coding was used to identify units within the transcript that were associated with perceived struggle within the class period. A second pass allowed for refining and categorizing the specific areas of struggle, with a third pass allowing for themes to emerge from the coded data. I compared the themes from the transcript data with the student product data looking for areas of agreement.

Results and Discussion

…if we are looking at the three C's of classical, critical, and community, I don't think I did that.
Audio journal excerpt from 6.2.2009

The above quote was recorded immediately after my last day of teaching and refers to whether or not what happened in the classroom could be defined as teaching mathematics for social justice and suggests the frustration I had over what had occurred during the six-day lesson. The analysis of the data revealed poorly defined tasks that assumed students would "study" (Lampert, 2001) rather than explicitly designing tasks where students would need to study. The most troubling problem I encountered, was the scarcity of mathematics being used within the student products, thus lack of evidence demonstrating students learning of the mathematical objectives.

Given the problems experienced in this lesson, what improvements can be made for the next attempt? Or what could practitioners or teacher educators take away from this experience if they were to either promote or attempt this approach to mathematics education? I propose adapting the framework to better fit the demands of teaching mathematics with and for social justice and better define the learners of mathematics for social justice.

In the students' final products there was a low level of fidelity between the intended content to be learned and the level of demonstration in the products. Previously in this paper, I proposed that the notion of content within the problem space of teaching be expanded to contain the classical, critical and community components proposed by teaching mathematics for social justice, which is also what is being assessed within the product component. I now propose that the three components extend into other design elements of the framework. Instead of merely developing a learning profile of a student, a teacher should gauge the students' aptitude for the various components of knowledge suggested by teaching mathematics for social justice. What is the collective knowledge about the community context? What perspectives have students considered in thinking about the topic? What positions do students hold? How could mathematics be used to learn more about the topic? These are all questions that when answered would create a classroom profile that would better inform the lesson design, which leads me to discussing process. Instead of defining students as learners from a multiple intelligence perspective (Gardner, 1993) as suggested by Udvari-Solner, et al. (2005), I propose defining students as learners of mathematics for social justice (see Figure 4). This redefining of the student may also serve to address some of the concerns practitioners or teacher educators may have with Gardner's Multiple Intelligence theory, which has been criticized within the literature (Waterhouse, 2006).

Connected to the redefining of the learner is the problem of connecting tasks to a purpose. What I propose to answer that challenge is to be explicit in the process component of the lesson design as to how specific tasks will allow students to "study" the named content, and to be explicit with students over intent of the topic. This is similar to what Harel (2008) proposes, in his "necessity principle", where a well designed problem will create a need to use certain mathematics, only I wish to extend it to include the two other components of knowledge proposed in teaching mathematics for social justice.

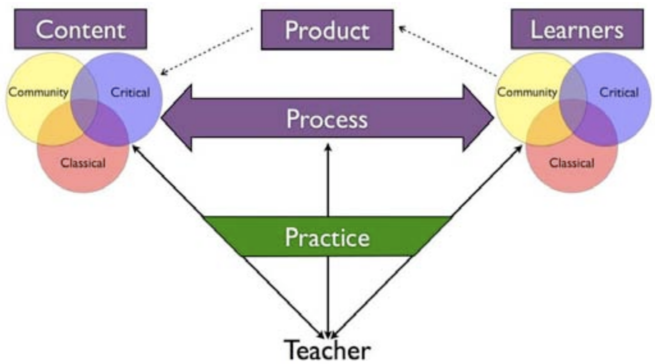

Figure 4. Defining students as learners of mathematics for social justice

Conclusion

This study was designed around an aspiration I had to teach mathematics with and for social justice. Given the problems that existed previously in the literature concerning in-service teachers attempting to teach mathematics for social justice I made an effort to address those problems using the Universal Design Process. Taking an insider's perspective, I documented my experience in an attempt to define some of the problems associated with taking such an approach towards mathematics education. What resulted was a different way of defining students as learners of mathematics for social justice and a retooled framework that utilizes this different interpretation of students and aims to be explicit in how students are to engage with and learn all of the components of content that teaching mathematics for social justice entails. This framework should aid me in future explorations of the challenges of teaching mathematics with and for social justice within contexts not traditionally represented in the literature. It is also my desire that the struggles that I experienced may provide some insight for practitioners and teacher educators wanting to improve classroom practice toward something more just.

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TEACHERS DEVELOPING MATHEAMTICS DISCOURSE COMMUNITIES WITH LATINAS/OS

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In the U.S., more and more teachers find Latinas/os in their classrooms. Yet, few are adequately prepared, both academically and socially, to address the unique strengths and needs of this growing population. This paper depicts the journey of a middle grades teacher as she navigates the complex reality of the mathematics teaching and learning process with linguistically diverse students. The concept of Mathematics Discourse Communities is introduced and elaborated to highlight the critical role of language development in the mathematics classroom, particularly how it conveys beliefs, values, and meanings surrounding students' ability to do and be successful with mathematics.

Introduction

For roughly the past 25 years, there has been much discussion around and research conducted on the role of language in mathematics education (Schleppegrell, 2007). The interest in mathematical talk was accelerated with the inclusion of "Communication" as a standard by the National Council of Teachers of Mathematics (NCTM, 1989). Communication, however, has often been narrowly conceived as "talk about mathematics" (O'Connor, 1998). Recent texts addressing the mathematics teaching and learning of English Language Learners (ELLs) present the role of language in mathematics learning in relatively simple ways that do not reflect the complex social dynamics that impact language use and development (e.g., Bresser, Melanese, & Sphar, 2009). It has become clear that new perspectives are needed on the way that language and literacy intersect with mathematics teaching and learning. Language, or mathematics discourse, can no longer be thought of merely as a mechanism to express mathematical thinking.

At the same time, discussions of Latinas/os in education inherently involve issues of language given their affiliation with Spanish. While many Latinas/os are native English speakers, the majority is not (U.S. Census Bureau, 2005). Therefore, millions of Latinas/os enter schools with linguistic skills that are largely not valued or seen as resources by schools. In classrooms across the country, their "English learning" status is seen as an obstacle, a flaw that requires intense, compensatory attention to "fix" (Ruiz, 1984). Often times, the consequence of this position is to design learning environments that emphasize remedial skills, positioning English learners to fall further behind (Lipman, 2004). A preoccupation with learning English becomes the priority above all else (Gutiérrez, Asato, Pacheco, Moll, Olson, Horng, Ruiz, García, & McCarty, 2002).

The serious issue of underachievement in mathematics among Latina/o students calls for a re-evaluation of our orientations towards Latina/o learners, our conceptions of the intersection of mathematics and language, and the alignment of our mathematical teaching methods with these realities. A focus on how mathematics teachers create and develop Mathematics Discourse Communities (MDCs) has the potential to address all three of these elements.

The primary question I aim to address is: *How do monolingual middle school teachers develop and utilize MDCs with Latina/o students?* More specifically, what issues and challenges (e.g. sociopolitical, institutional, pedagogical, curricular) surround the teachers' development and utilization of MDCs? In this paper, I will elaborate upon the concept of MDCs, describe the context of the study, and provide a vignette of a teacher. The vignette will reveal how the teacher perceives and addresses the unique strengths and needs of bilingual learners in mathematics class. An analysis of the data follows the vignette. Finally, I will outline the implications of these findings and offer recommendations for practitioners and researchers.

Theoretical Framework

Studying the development of MDCs among Latinas/os is important not only to “see” how they are socialized to and within the discipline of mathematics, but also to see how other realms of schooling intersect with the mathematics teaching and learning process. Despite common belief, mathematics is not an isolated field of study that produces an objective skill base with which students should walk away; rather, it is embedded in a complex, multi-dimensional social web of values, ideologies, and modes of operating. The teacher is the primary – though not exclusive – vehicle through which these (dominant) values, ideologies, and modes of operating are brought into classroom and transmitted to the students. However, students are agentive (not passive in the learning process) and unique (when compared to one another), and they, too, will bring unique perspectives forth in the mathematics learning process. Examining MDCs with Latinas/os will illuminate the points of contention and difference amongst students and between the students and teacher and how these contentious spaces lead to (mathematics) learning (Gutiérrez, Baquedano-López, & Tejeda, 1999) or alienation (Gee, 2004).

Understanding that language is unique to a particular social setting, Gee (1996) introduced the notion of Discourses (with a capital D). Discourses are:

> *ways of behaving, interacting, valuing, thinking, believing, speaking, and often reading and writing that are accepted as instantiations of particular roles (or ‘types of people’) by specific groups of people, whether families of a certain sort, bikers of a certain sort, business people of a certain sort, church members of a certain sort, African-Americans of a certain sort, women or men of a certain sort, and so on through a very long list. Discourses are ways of being ‘people like us’. They are ‘ways of being in the world’; they are ‘forms of life’. They are, thus, always and everywhere social and products of social histories* (p. viii).

Discourses, Gee argues, include much more than language and should be appreciated in its social context. Consequently, when investigating the role of language in any context, we cannot focus on language alone.

From this perspective, mathematics classrooms inherently maintain a unique Discourse community, a MDC. When combined with the work done on cultural social practices (e.g. Gee, 2004; Rogoff, 2008), the power of Discourse communities to socialize youth becomes apparent. Given that Latinas/os largely are not finding success in mathematics, examining the MDCs in which they learn becomes all the more necessary and urgent. Schieffelin and Ochs (1986) put forth that, in terms of language, there are two, concurrent socialization processes; “socialization through the use of language and socialization to use language” (p. 163). In other words, we not only learn language through social interactions, but are also socialized into particular communities of practice (Wenger, 1998) through particular language practices, or Discourses. Furthermore, membership in these particular Discourse communities is intimately connected to one’s identity (Wortham, 2006). Martin (2000) writes: “…it is my firm belief that detailed analyses of mathematics socialization and identity – and the multiple contexts that affect them – offer the best hope for understanding long-standing achievement and persistence problems" (p. 186).

The focus of this investigation is on the teacher's role in developing MDCs. MDCs involve ways of being, thinking, and speaking that are unique to a mathematics environment. While MDCs refer to the participants, the setting, and the interactions within the setting and between the participants, the process of being apprenticed into the specialized community can be thought of as socialization. The teacher, being the person of authority, is instrumental in the mathematics socialization process. Each teacher, as a result of their particular beliefs, values, and experiences, initiates the mathematics teaching and learning process in a unique way, and students interact with this socialization process incurring mixed results (in terms of affiliation with mathematics, for example) (Martin, 2000).

Gee (1996) also argues that classroom discourse is not a discrete feature of the classroom experience, but rather is inextricably and intimately connected to the particular ways one ought to act, think, believe, and value, with respect to a particular community. In mathematics, Franke, Kazemi & Battey (2007) contend that "students ways of being and interacting in classrooms impact not only their mathematical thinking but also their own sense of their ability to do and persist with mathematics, the way they are viewed as competent in mathematics, and their ability to perform successfully in school" (p. 226). From this perspective, language is more central to mathematics learning outcomes than previously thought. Accordingly, attention should be focused on how language interacts with other dynamics in MDCs. The need to study mathematics discourse, as it is developed in particular communities of practice (Wenger, 1998), becomes even more critical.

Potential Indicators of Mathematics Discourse Communities

While Mathematics Discourse Communities can be indexed by an infinite array of words, gestures, and actions, it is useful to articulate some of the more discrete indicators that play a major role in establishing distinct MDCs. These indicators are meant only to provide the reader a clearer picture of the way the two participants' view and convey the mathematics teaching and learning process, specifically with Latina/o learners. Table 1 below outlines some of the themes that emerged repeatedly throughout the year and contributed to the respective Mathematics Discourse Communities established by each teacher.

Table 1. Indicators of Mathematics Discourse Communities

Utility of Mathematics	Language Development	Tool Use in Mathematics	Students Positioned as Competent
Habits of Mind	Participatory Patterns	Delivery of Mathematics Content	

Context of Investigation

The data presented here is part of a larger study examining the practices of two teachers, Ms. Hendrix and Mrs. Lenihan, who teach at the same school, Southwest Elementary School. Southwest is located on the southwest side of a very large, Midwestern city. Of the 1300 students, approximately 70% of the students are Hispanic, 15% White, and 15% African American. Of the "White" demographic group, the majority is of Middle Eastern descent. 85% of students qualify for free or reduced-cost lunch, and 30% of students are enrolled in the bilingual or ESL program (though probably 75% of students are native speakers of a language other than English). I have known the two teachers professionally for two years, because I have been working with them in their classrooms and have had them as students in Masters courses at the university.

The data consists of fieldnotes, audio recordings of planning sessions and interviews, and video recordings of lessons. Collaborative planning sessions (the two teachers and I) took place once-per-week. This planning session eventually evolved into sessions where we watched footage from past lessons and discussed what transpired and what might be done to improve upon lessons. The teachers identified two classes on which they wanted to focus. For each teacher, one of those classes was the English Language Learner (ELL) class. Per administrative decision, the students scoring lowest on language proficiency assessments were grouped into one class. In 8th grade, the "ELLs" were prohibited from enrolling in the Algebra class, which is the other 8th grade section on which we focused our attention. The four classes were videotaped about twice-per-week.

For data analysis, I used two approaches to arrive at appropriate and accurate claims about the phenomenon of my investigation: grounded theory and critical discourse analysis. Grounded theory (Corbin & Strauss, 2008) was utilized to isolate the most important themes that surfaced in the data. These themes resulted from an ongoing analysis of the empirical data and are developed in concert with the theoretical lens through which the data was evaluated.

Because of the intimate relationship between language and ideology, created through sociohistorical interactions, discourse needs to be analyzed in a particular way. Critical discourse analysis aims to answer that call. Traditionally, most discourse analyses have only considered the literal value of words; that is, what can be seen or heard in this moment. Gutiérrez and Stone (2000), however, argue for a critical discourse analysis that accounts for the vertical text, or the diachronic history of a text, which considers the sociohistorical context that informs and supplements what is being said.

Vignettes

Mrs. Lenihan teaches mathematics to all of the 7th grade students. Each year, she is also responsible for teaching either Language Arts (essentially, vocabulary development) or writing to her homeroom class. Mrs. Lenihan is in her third year at Southwest, and her fourth year overall. She is a young, White, energetic, monolingual teacher who completed her teacher preparation at a well-known, local private university. She often speaks of trying to create a classroom that is "student-centered." As a result, students' desks are arranged in clusters, and on most days, the students are given an activity in which they are to work with their groups. Though she has access to traditional mathematics textbooks, she chooses not to use them with her classes. Instead, she strives to develop meaningful mathematics lessons that are project-based and grounded, as much as possible, in the lived experiences of her students.

Mrs. Lenihan often remarks about the importance of being able to communicate mathematically. For the past two years, she has attempted to develop her students' ability to write proficiently about mathematics problem-solving activities. Her motivation for this goal is largely rooted in the state's assessment program, which requires students to respond to multiple "extended response" items. Mrs. Lenihan has developed an extended response protocol and regularly draws on past test questions in order to help her students improve their mathematics writing skills.

Also, Mrs. Lenihan tries to cover the 7th grade mathematics topics through projects. For example, she has designed a lengthy project called the Dream Home Project to connect many concepts, such as measurement, area and perimeter of polygons, surface area, proportional reasoning, and cost formulas. Not only does this project activate students' creativity and engage students who might otherwise not fully engage in mathematical tasks, it is successful in tapping into family resources, or funds of knowledge, as students frequently rely on parents and other family members to help them through the complex process of envisioning, designing, constructing, and decorating their Dream Homes. Typically, the project is completed in pairs or groups of three, though working alone is an acceptable option.

The students reported that the Dream Home Project helped them master particular skills. For example, the students overwhelming claimed that this project allowed them to learn well how to measure with a ruler. Surprised that they had not learned this skill in the primary grades, I asked them to share their experiences with measurement in the elementary grades and explain how this measuring in the Dream Home Project was different. One student responded:

> *When we were in elementary school, we really didn't measure things like we measure in 7th grade, like how do you use a quarter inch, or how do you use… We measured small things. The only time that we would measure things was on a test, or we'd just skim past it and that would be it.*

Mrs. Lenihan offered this evaluation of her students' comments:

> *I think what you're getting at is when we measure something now, it has to make sense; it has to all go together. Maybe in previous years or when I gave a quiz, I would just put lines on it [to measure]. You could say, "Oh, that's 1 ¾, or 2 inches, or 3 inches," but [now] it has to make sense. If it didn't [in the Dream Home Project], you knew your measurements weren't right.*

This exchange uncovers an important distinction: teaching to cover topics and teaching for meaning. As it turns out, the students did have experiences measuring with a ruler. However, their experiences were likely limited to two dimensional shapes and test-like problems, activities that did not carry meaning when evaluated in light of their lived experiences. Mrs. Lenihan's Dream Home Project intended to provide a significantly different context, an authentic problem-solving situation. The measurement tasks were embedded in a larger framework and enabled the students to make meaning of measurement as it related to their lived experiences or future lives. As one student put it: "I learned how to construct walls and use creativity and imagine things as though I were a real architect."

Mrs. Lenihan developed another small project in which students identified their daily or weekly activities (in a bound unit of time) in order to examine the various representations of rational numbers (i.e., fractions, decimals, percents). The culminating activity of this project is to construct circle graphs representing the students' various daily or weekly activities. While this idea is not necessarily unique to Mrs. Lenihan's class, her theoretical approach to learning mathematics contrasts sharply from a teacher implementing reform-oriented curriculum through a traditional approach (see, for example, Brown, Pitvorec, Ditto, & Kelso, 2009).

Even though Mrs. Lenihan's lessons often operate within a larger project, some are delivered more traditionally. Below is a depiction of a lesson delivered during the rational numbers project mentioned above:

As usual, the day begins with a warm-up. Mrs. Lenihan has worked hard to establish a routine in which students enter the room and immediately begin work on the warm-up. In order to institute this mathematics class norm, she walks around the room passing out tickets, which is part of a school-wide positive reinforcement program. At times, Mrs. Lenihan will also use tickets to reward students who have completed their homework. The tickets can later be redeemed for out-of-uniform passes, entry to holiday dances, or entered in a drawing for various prizes.

Concurrently, the following warm-up appears on the board:

Put the #'s in order from least to greatest: 8/3, 63/7, 0.25, 12½%

The students struggled with the meaning of 12½%. Many students argued that it equaled 1,250, apparently moving the decimal point two positions to the right. Mrs. Lenihan suggested that this equality did not logically make sense, and she asked the students to consider 12.5% "as a portion of a pizza." The moment Mrs. Lenihan was about to say "pizza," one student finished her thought with the word "whole," but his appropriate use of mathematics language went left unacknowledged. Mrs. Lenihan quickly instructed the students to "discuss in their groups why 12.5% is less than 1." After the two minutes and after she felt she had sufficiently convinced the students that 12.5% was a fractional part less than 1, Mrs. Lenihan offered the following summary:

L: So, 1,250 doesn't make sense; neither does 12.5 pizzas. The decimal has to be less than 1.

Expressing frustration with this hurried process of ordering abstract numbers, one student commented:

St: It's easier to do it on the board than talk about it.

Next, Mrs. Lenihan quickly moved on to the lesson of the day: converting fractional parts of a circle into percentage equivalents (e.g. 6/24 = ¼; ¼(360°) = 90°). She delivers this lesson in 4 or 5 minutes with the work-up from the previous class. Mrs. Lenihan talks very fast in an effort to complete this lesson and the rest of her objectives in the short 40-minute class period. She does not conduct any checks for understanding, as there is no time.

Analysis

While this is certainly only a snapshot of life in Mrs. Lenihan's mathematics class, there are a number of themes that can be gleaned from these interactions. First, from dialoguing with Mrs. Lenihan and observing her class over the years, it is clear that she has wonderful intentions to maximize student engagement, provide plentiful opportunities for students to communicate (both in whole-class discussions and small group collaborations), and generally help students make meaning of mathematics concepts. However, there are institutional pressures that weigh on her (i.e. short class periods, frequent modifications of daily class schedules, obligatory external assessments, pressure to directly prepare students for what is typically on high-stakes assessments in an effort to raise test scores, etc.).

Mrs. Lenihan's good intentions and corresponding instructional practices are compromised by these pressures. For example, the implications of short class periods are observable daily. Classes at Southwest are 40 minutes, with zero minutes to pass between classes, essentially leaving 37 minutes for instruction. In the instance above, when Mrs. Lenihan was about to say "pizza" and the student finished her thought with the word "whole," Mrs. Lenihan may have missed an opportunity to capitalize on a brilliant contribution made by a student, but given the context, it is more likely that she felt hurried to move along. Moreover, Mrs. Lenihan frequently has to end whole-class discussions abruptly, take over discussions for the purpose of making her point quickly, or carry over lessons to the following day. This results in a more disjointed flow of the curriculum (and pedagogy) and places the class behind their ideal pacing schedule.

Second, it has been my experience that ticket distribution for positive reinforcement campaigns and "character development" programs are wide-spread in urban schools. It is hard to criticize teachers' efforts to make clear behavior expectations and class norms, and acknowledge those who successfully accomplish them. However, there seems to be an underlying objective to use the program to curb non-conforming behaviors and nudge less productive students to do more. Thus far, there is no evidence that shows that not giving these students tickets motivates them to change their ways. The implementation of this program raises questions about how we perceive particular students and what ought to be done to change students' academic and behavioral patterns.

Third, the warm-up "problem" chosen by Mrs. Lenihan is decontextualized and illustrates how she has caved to pressures to familiarize students with test questions. By decontextualized, I mean that the "problem" only carries meaning to those with a firm understanding of rational numbers. Furthermore, only those students fluent in the discourse of rational numbers will be able to participate in sharing-out. With her students' "achievement" in mind, she selected the item because she is aware that there is always a problem like this on the state's standardized assessment. This type of "question", however, does not work well if Mrs. Lenihan's objective is for her students to cognitively understand the relationship between the different representations of rational numbers. The task makes it difficult to facilitate the learning of the concept, and as a result, the students struggled to make sense of the activity.

Finally, the student's aside about the difficulty of talking about complex mathematical idea speaks both to the inadequacy of the warm-up problem as well as the lack of opportunities to engage in mathematical communication. While Mrs. Lenihan regularly incorporates opportunities to problem-solve collaboratively and negotiate meaning, communicating mathematically is a difficult norm to establish in a matter of months – especially if students have experienced years of mathematics instruction centered around teacher explanations. To her credit, Mrs. Lenihan continually aims to foster productive group dynamics and mathematical responses from her students that reflect the specialized language of mathematics.

Conclusions and Implications

There is consensus that the traditional methods of teaching mathematics are not sufficient to advance achievement among learners, especially the Latinas/os (e.g., Khisty & Willey, 2008). Given the persistent mis-education of Latinas/os and the corresponding social realities they experience (Gándara & Contreras, 2009), there is an urgent need to evaluate classroom learning from new perspectives. No longer can we examine and view the mathematics teaching and learning process with Latinas/os solely through a cognitive lens, as if learning can be defined by what the teacher says and how well the student's brain absorbs the information. There is an emerging need to foreground particular components of the mathematics teaching and learning process, such as the role of language and the development of specialized discourse communities, the tools used to make sense of mathematics, and students' participatory patterns, among others.

Paying attention to MDCs requires us to focus simultaneously on the overall mathematics classroom environment, embedded in a school context and nestled in a sociopolitical reality, as well as the micro interactions that make up daily activities. Examining MDCs allows us to see the implicit and explicit messages conveyed to students about the discipline of mathematics and the mathematics teaching and learning process. It allows us to identify what norms and mathematical obligations the teachers intend to establish in their classrooms. It will begin to answer Cobb, Gresalfi, & Hodge's (2009) call to "understand not merely whether but why students have come to identify with their classroom obligations, are merely cooperative with the teacher, or are developing oppositional identities (p. 48)."

For teachers, reflecting on the community of practice(s) we establish makes evident to us what we believe about mathematics, how that manifests in classroom instruction and activities, and how easily our underlying beliefs and assumptions are portrayed in what we say and do. It will help us clarify what it is that we want our students to be able to do, important among them being able to seamlessly participate in sophisticated mathematics discourse communities. Utilizing the framework of MDCs will enable us to implement the strategies that promote productive micro interactions, ultimately achieving more of the results we desire. At the same time, it will help us weed out counter-productive assumptions – assumptions that contribute to the macro educational reality Latinas/os endure today.

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TRANSFORMATIONS OF DREAMS AND PERSISTENCE OF MEANINGS: GIRLS ON THE FAST MATH TRACK, TEN YEARS LATER

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Girls on Track is a longitudinal study in its eleventh year, following over three hundred fast math track middle-school girls. Each cohort started with a summer math camp data collection via observations, surveys, content tests, school records, and interviews. Every several years we interviewed study participants about their current relationships with STEM disciplines, family dynamics, and career plans. This paper focuses on the significance of mathematics in the development of plans and dreams of four women from the first cohorts. These case studies are viewed against the backdrop of all study data. Over the years, our models became more centered on girls and their immediate communities and networks, rather than the institutional track. Roles of mathematics in the persistence of deeper meanings of women's lives are significantly more complex than sequential milestones in the linear school-to-career "pipeline" model.

Introduction

This paper centers on four case studies of women who were on the fast math track in middle school ten years ago. One of them is now a math major, two are in applied STEM fields, and one is a journalism major. We compare and contrast the view of STEM as a direct job goal, and education as a series of steps toward that goal through a narrowing, increasingly more challenging pipeline (Barker & Aspray, 2006; Blickenstaff, 2005; Stage & Maple, 1996), to the view of mathematics as a personal strength, an asset in a variety of shifting career roles, and a tool for constructing meaning (Belenky, McVicker Clinchy, Goldberger, & Tarule, 1986; Csikszentmihalyi, 2008; Jungwirth, 1993; Wenger, 1999). This paper is a step toward an integrated model connecting academic track point of view and personal and community significance of mathematics. Such a model can provide a greater diversity of perspectives in STEM education, and ultimately support better solutions of human problems through pedagogical developments.

Theoretical Background

The under-representation of women in science, technology, and engineering careers continues to be a big national concern (Barker & Aspray, 2006; National Research Council, 2001). In the nineties, the differences in elementary and middle school mathematical achievements between girls and boys, pronounced in earlier studies, ceased to be significant, as, for example, data from the Third International Science and Mathematics Study show (Beaton et al., 1996). However, girls do not persist at the same rate as boys in continuing their study of mathematics beyond middle school, taking less rigorous courses and leaving the pipeline leading into science, engineering, and technology field careers, with mathematics serving as a filter or a gatekeeper (Blickenstaff, 2005).

These and other reports cite the importance of rigorous high school mathematics as vital to improving the quality of the workforce for the twenty-first century, and call for increased intervention efforts that encourage girls and young women to select rigorous advanced mathematics courses beginning as early as middle school algebra and continuing through calculus in high school. However, fewer studies are done about the meaning and significance of mathematics in the lives of girls and women (Jungwirth, 1993), and social structures outside of classes, such as "geek circles" (Varma, 2007) that may support STEM careers. This paper is an exploration of the significance of mathematics in the persistence of meaning threads in the first ten years since the middle school.

Four Cases And Their Threads of Meaning

From the pipeline point of view, Cara is the very model of a modern female STEM career woman, studying for a PhD in mathematics. Lorna is an example of someone who switched from a math path to an applied and traditionally female-oriented biomedics path, still staying close to STEM. Hellen selected hydrology, another STEM application, and Kelly dropped out of the fast math track and STEM career path, initially wanting to be a veterinary doctor, but by high school going into journalism. In this part of the paper, we explore the roles of STEM in some examples of personal meaning threads that emerged from these four cases. In the next part, we situate the threads within a model that comes from the larger study.

Cara: The Conceptual Architect

Cara is a poster woman for the math pipeline model. She is currently studying for her PhD in mathematics. In her undergraduate years, she took all available math courses at her institution, and obtained a math major and two STEM minors. In high school, Cara was taking all math courses earlier than the average, and at a more advanced level, which defined her as a fast math track student. She partially attributes the rather surprising fact that she never met a math teacher she did not like to always being in earlier, faster classes that tend to be assigned to better teachers.

However, Cara experienced a switch in future job plans in high school, in a scenario not rare for girls that age (Blickenstaff, 2005): she did not like the math requirements of her chosen career. She wanted to be and prepared to be an architect, taking and enjoying AutoCAD classes. However, conversations with people from the field convinced her that architecture required "the wrong kind of math": a lot of tedious number crunching and memorization for field exams. She examined her strengths and decided to be a mathematics major, because she deemed she was stronger at conceptual understanding. Her current major is a very applied area of mathematical modeling, with prospective jobs in the private high-tech companies or a combination of private and academe research. This is the superficial story, however. The threads of meaning that emerged as interview data categories are the following.

Controlled information and communication. Cara is a family-centric, strongly religious person for whom obeying authority and controlling information and communication sphere is a matter of explicit attention and utmost importance. She would not take a programming class at school because of the immaturity of other students in it, for example. Likewise, she rejects much of social networking: "Everybody uses Facebook, so I would not." For Cara, mathematics is associated with a pure, controlled, conceptually advanced system she can embrace and follow in the company of a selective group of people. This thread is the reason Cara does not want to take an academe path, because she can't control who she will be teaching, and her past experiences clearly demonstrated teaching occasionally becomes rather unpleasant because of the randomness of students.

Close groups of geeky, serious friends became an integral part of this thread. Cara talked at length about following older friends' advice in choosing classes, the academic advisor, and leisure activities. She is an avid gamer, for example, and a path into web site programming, then architecture, then mathematical modeling has been supported for her by gaming since she can remember herself. Gaming with friends, as a part of an exclusive, geek culture REF reinforces STEM orientation for Cara.

"I am very practical." Cara spends a lot of time carefully researching, planning and discussing with her network of family and trusted friends all matters of learning and future career. She selected her graduate advisor based on the fact he hooked every graduate student with a multitude of internships, providing job connections and opportunities from early on. Cara graduated with multiple minors by applying AP courses to her college work. She carefully selects professors for her classes, which, again, leads to "never met a math or science professor I did not like." This thread pulled Cara toward an advisor with a strong business network, and a career in the private sector.

Modeling. Far beyond being a mere future job, or even a career, modeling is a thread of meaning for Cara. In middle school, she and her friends made art, and she was the person creating web sites to host their art projects. She explained how building architectural constructions, using software, grew out of these early pursuits, and then modeling real life phenomena with mathematics was a continuation of the same thread. While architecture and web sites seem hands-on and visual, Cara approached web design from the coding rather than WYSIWYG perspective, and architecture from programming - so her "design" thread has been growing into the increasingly conceptual and mathematical direction of modeling.

Lorna: The Academic Caring

Lorna was going to be a mathematician or a math teacher, but higher-level math courses proved too abstract for her. She describes one class where she struggled and succeeded somewhat, but the next one "started with that proof that square root of three is irrational, and I had no idea how to even start, so I dropped the course and switched my major." She decided to switch to biomedics.

Parenting as teaching. In her interviews, Lorna talked about her mother in detail, tenderly remembering the many advanced, very pleasant academic interactions from her early childhood on. She describes her mother as very smart, despite having to support the family from the age of fifteen as a first-generation immigrant from India, and not getting a college education. When Lorna progressed into math courses beyond those her mother had before, their roles reversed, with Lorna teaching math to her mother as a way to understand it better. This may explain Lorna's success at landing a job at the Sylvan Learning center as a calculus tutor, with "everybody there at least twenty years older than I was." Lorna also sounded indignant (with apologies, catching herself at it) about parents of struggling school or college students not helping them better throughout their academic lives. She sees parenting as teaching, and teaching as so much related to caring as to be, metaphorically, parenting. She tutored relatively advanced math throughout high school, and chose the math major as a continuation of this "caring" thread.

As college mathematics became more abstract, Lorna could not assume this caring role anymore. She described her dismay at being able to understand material with some help from another student, but not having enough mastery over it to help others. While capable of being a math student, she was no longer in the position of "math parent" and therefore decided to change things around. Biomedics supports her image of "parenting" or "caring" through the participation in an advanced and sophisticated field.

There's nothing I can't do. Another defining thread in Lorna's life is her incredible assertiveness. For example, she described several instances of persevering through multiple rejections while landing jobs as a young teen, and most recently finding a biomedics internship that took waiting by doors and multiple calls to five different professors repeatedly saying, "No." Lorna says, "There's nothing I can't do if I put my mind to it!" Her early childhood and teen years experiences with mathematics, especially tutoring, supported that image of assertiveness and power.

But what about dropping off the math track in college? Paradoxically, Lorna claims she wanted to do it to keep her feeling of power, *including* math power. The major switch had her take quite a few advanced science classes in her last college semesters, and she wanted to feel capable and confident, especially about mathematics involved. It was a challenge, but in ways appropriate for her personally. It is Lorna's self-confidence that kept her from continuing with the math major, which, given her previous track, she was likely quite capable of getting, just not in that parenting, helping, assertive manner that was so important for her.

Hellen: The Art Of The Science

Hellen fondly recalls doing math with her big brother when she was young, but back then, she wanted to be an entertainer. "Saturday Night Life people seemed so happy!" Starting from high school, though, she turned toward sciences, and is currently applying to enter a graduate school in hydrology. Hellen's threads of meaning reveal a surprising path toward science for her.

Ownership through creativity. Hellen is an outgoing, friendly person who smiles a lot. She mentioned how she can make others laugh, and how entertaining others was so important that she considered it as a career. But what really hit a chord with Hellen was designing her own science experiments. She specifically recalls a biology camp where she did this over several summers, and a school project about water quality that got her interested in that endeavor. Later, she liked classes where she could be creative about her assignments, making beautiful and meaningful outlines and working hard on presentations. For her, science is about design, it is a creative, almost humanistic endeavor. Theater is still her big hobby. Hellen was thrilled that at her graduate school interview, her writing skills were highly prized as something that will give her an edge as an applicant. She is looking forward to graduate school experiences being more open for that sort of creativity than undergraduate courses. Designing experiments is the thread that pulls Hellen toward STEM.

Multi-generation networks. Family and friends play a big role in Hellen's professional orientation. Her dad went to the science camp that somewhat determined her choice of subject. She mentioned keeping in touch with camp counselors, who were undergraduate students at a time, and following in their steps. Hellen kept connections with many of them, now advising her on the choice of a graduate school. Since her creativity, including scientific creativity, is also audience-oriented, the network thread playes a key role in pulling Hellen into the direction of a STEM career.

Kelly: Stories With A Face

Kelly studies to be a journalist, works as a journalist for several student organizations, and volunteers in various journalist capacities for friend and community projects. In the middle and high school, she wanted to be a veterinarian, but an experience with an internship made her realize it's not a job for her. However, anything and everything about journalism is tremendously satisfying. She wants to present important and interesting stories either through being a news anchor, or through investigative reporting. Are there any threads in Kelly's life related to mathematics? Here are two examples.

"People relate to my stories with a face" - Kelly talks about the importance of information being personal and human, delivered by someone listeners know, recognize, and trust. She thinks it is her particular cognitive strength to give stories a twist people appreciate and understand, and to present information in interesting ways. She works for days and weeks improving her videos, "But when you are done and there is three minutes of it, you can see a strong story." Visual and information literacy at this level, while not directly mathematical, is related to situated STEM skills.

"It takes a village." Kelly is a highly networked woman, she has many friends and is a member and a leader of many student organizations. Mathematics was always challenging for her, took more time, and required more efforts than other subjects. School mathematics was also separate from life mathematics such as finances, and not something ever, ever used. Kelly's family, however, rallied around her, helped her when they could and hired tutors when they could not. Kelly noted the very positive feeling that she was constantly supported in challenging and extrinsic hardships, and able to pass these gatekeepers through collective efforts of her family and her own hard work. In a sense, mathematics was something to rally *against.*

Situated complexity. As a possibility not yet set in stone, Kelly considers going into investigative reporting, and she discusses at length the joys of seeing unexpected twists in stories, playing devil's advocate and otherwise deeply interacting with her daily subject matter. Undoubtedly, this requires logical abilities and very likely at least some statistics and data analysis, but Kelly does not see this as mathematics (Lave & Wenger, 1991). She does acknowledge that her work as a club treasurer involves math, but again, she sees that as very different from "school math" (Hoyles, Noss, & Pozzi, 2001). The use of technology and software she describes in her award-winning short video work is sophisticated too, but mathematics in it is very much applied. Kelly's thread of complexity in everything she does suggest high logic abilities, without an attempt to explicate them or to apply them to any STEM direction.

From The Leaky Pipeline To Persistent Threads of Meaning

Leaky Pipeline

Many studies considering women in STEM careers adopt the "leaky pipeline" perspective, starting somewhere around the first Algebra classes and ending in PhD degrees, with many women not making it all the way through (Barker & Aspray, 2006; Blickenstaff, 2005; Stage & Maple, 1996). We summarized the characteristics of several pipeline models from the literature:

- The pipeline is linear, with no turns, pauses or detours.
- People who drop out of the pipeline are gone forever.
- Milestones within the pipeline, such as classes or internships, are motivated by being prerequisites to further milestones.

Threads of meaning

Yet this picture does not seem to match women's own views on meaning and significance of STEM in their lives, or our analysis of the same when we adopt a perspective centered on women rather than classes and institutions. From case studies, as well as quantitative analysis of larger samples of our longitudinal data, emerge individual threads and thread categories of personal meaning that form each woman's path, and in which STEM has varied significance.

Cara's *controlled information and communication* and *"I am very practical,"* Lorna's *parenting as teaching*, Hellen's *multi-generation network*, and Kelly's *"it takes a village"* threads belong to the category of ***PLNs, personal learning networks*** (Siemens, 2005) that start with families, and develop to include friends and eventually colleagues. These threads also belong to the ***Parent Power*** category. We found that threads from these intersecting categories had a strong impact on STEM tracks of all study participants.

Cara's *modeling*, Lorna's *ownership through creativity*, and Kelly's *situated complexity* are examples from the thread category of ***content-specific ways of knowing*** (Belenky et al., 1986; Lave & Wenger, 1991) These content threads may not tie with particular academic or curricular subjects, but are overarching themes or topics important in each person's life and developing from naive to sophisticated forms as the person matures. The pedagogical practice of cross-disciplinary unit studies (McColskey, Parke, Furtak, & Butler, 2003) is connected to this category.

Cara's *"I am very practical,"* Lorna's *"There is nothing I can't do,"* Hellen's *ownership through creativity* and Kelly's *"People relate to my stories with a face"* are example of women's awareness and focus of their ***personalities*** in career paths. STEM skills have particular, complex relationships with personality traits, a topic women discuss at length in their interviews.

Lorna's *parenting as teaching* and *"There's nothing I can't do,"* and Kelly's *"People relate to my stories with a face"* are examples from the ***caring and helping*** thread category. While studies show that women in the academe are not any more likely than men to find job satisfaction in caring or helping roles rather than research roles within the field (Stage & Maple, 1996), girls in our study integrated caring into career selection and explained how they view their work as helping people and the world.

Cara's *modeling* and Kelly's *situated complexity* belong to the category defined by the **abstract-applied** gradient. The need for grand causes and abstract concepts, or the need to immediately see results of pursuit are expressed in, and supported by, different approaches to STEM.

Conclusions

We would like to finish with a quote about life dreams from Janush Korczak, a Polish pedagogue and a champion of children's rights (Korczak, 1990):

> *When we don't have enough material to reason, there appears a poetic meaning of what little we have. Into a dream we transform the feelings that don't get realized in reality. The dream becomes our life's program. If we only knew how to decipher it, we would see that dreams do come true.*
>
> *If a poor boy dreams of being a doctor and becomes a nurse, he fulfilled his life's program. If he dreams about being rich, but dies on the bare mattress, his dream did not come true only superficially: after all, he did not dream about hard work toward a goal, but about squandering money away; he dreamed about champagne, but drank moonshine; dreamed about salons, but had bar brawls; wanted to throw gold to the wind, but wasted coppers. The other wanted to be a priest, but became a teacher or simply a groundskeeper, but, being a teacher, he's a priest, being a groundskeeper, he's a priest.*
>
> *She wanted to be a terrible queen, and is she not a tyrant to her husband and children, having married a low-level clerk? Another wanted to be a beloved queen, and is she not ruling a folk school? The third one wanted to become a great queen, and is her name not covered in glory, the name of a wonderful, extraordinary seamstress or matron?* (p.112)

While the dreams and plans change and transform, the personal *meaning threads*, and the corresponding *significance of STEM* in threads and thread categories are continuous and persist through time. Examining these meanings and significances at the level of individuals and their networks can prove fruitful in understanding how to support women in STEM careers.

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UNDERSTANDING THE CONSTRUCTIONS OF MATHEMATICS AND RACIAL IDENTITIES OF BLACK BOYS WHO ARE SUCCESSFUL WITH SCHOOL MATHEMATICS

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This study investigated the mathematics and racial identities of Black 5th through 7th grade boys who attend school in a southern rural school division. The data pool consisted of focus group interviews, mathematics autobiographies, review of academic records, and observations. Four factors positively contributed to mathematics identity: (a) the development of computational fluency by third grade, (b) extrinsic recognitions, (c) relational connections, and (d) engagement with the unique qualities of mathematics. For these boys, racial identity in school is connected to perceptions of others' school engagement; this sense of "otherness" led to a redefinition of their own mathematics and racial identities.

Introduction

The success of Black[1] boys in mathematics receives little attention, although there is a vast amount of literature that describes the academic achievement and schooling experiences of Black boys in terms of failure (Thompson & Lewis, 2005). Despite the highly documented underachievement and low-level course enrollment patterns of Black boys, not all of them achieve at low levels. In fact, there are Black boys who stand in opposition to the literature that documents their failure and underachievement. Black boys' mathematics identities are shaped by culture, community, and experiences with mathematics (Berry, 2003 & 2008). The development of a positive mathematics identity is essential towards helping boys sustain an interest in mathematics and develop persistence with mathematics. Examining the perceptions of successful Black boys is critical to identifying the strengths, skills, and factors that promote their success.

This study investigated the constructions of mathematics and racial identities among thirty-two Black 5th through 7th grade boys who are considered successful in school mathematics as measured by high pass rates on the state standardized mathematics assessments and above average grades in mathematics. The purpose of this phenomenological study is to investigate mathematics identity development of Black boys and explore how racial identity interacts with mathematics identity.

Theoretical Perspective

Martin (2009) contends that mathematics learning and participation should be conceptualized as "*racialized forms of experience*" that are structured by the relations of race that exist in the larger society (Martin, 2009; p. 299). This paper considers two constructs of Martin's framework: conceptualization of race and conceptualization of learners. The conceptualization of race focuses on race as a sociopolitical construction that is historically contingent. Conceptualization of learners considers the negotiated nature of identity with respect to mathematics by asking, "What does it mean to be Black in the context of mathematics learning?" We sought to understand how Black boys conceptualized race in the context of learning mathematics and how they negotiated their identities as learners of mathematics. This study defined mathematics identity as one's belief about (a) their ability to do mathematics, (b) the significance of mathematical knowledge, (c) the opportunities and barriers to enter mathematics fields, and (d) the motivation and persistence needed to obtain mathematics knowledge (Martin, 2007).

Nasir, McLaughlin, and Jones’ (2009) reviewed the research on the relations between racial identities of Black students and school-related outcomes to find that the literature indicated three conflicting findings. First, findings indicated that when students hold strong identities as Blacks, their academic achievement suffers and/or academic identification decreases (Fordham, 1996; Noguera, 2003; Osborne, 1997). Second, other findings indicated the opposite of the first finding, suggesting that racial identity is a protective factor for education for Blacks and serves as a buffer for racial discrimination (Chavous et al., 2003; Oyserman, Harrison, & Bybee, 2001; Sellers, Copeland-Linder, Martin, & Lewis, 2006; Wong, Eccles, Sameroff, 2003). The third finding suggested that there is no linear relationship between racial identity and academic outcomes for Black students and that it varies depending on the nature of Black identity (Carter 2005; Chavous et al., 2003, Harper & Tuckerman, 2006; Sellers, Chavous, & Cooke, 1998). Nasir, McLaughlin and Jones (2009) offered two explanations for these mixed findings: (1) researchers have used various definitions of racial identities; consequently, this lack of consensus may contribute to differential findings; (2) difference in racial identity is constructed in local contexts; thus being Black in a southern rural town may be different than being Black in a large urban city. This study defines racial identity as the meanings the boys assign to themselves and the Black racial group.

Research on the mathematics and racial identities of Black boys who are successful with school mathematics suggest intrinsic and extrinsic factors contribute towards the development of Black boys identities. In a study focusing on racial and mathematics identities of middle school Black boys who are successful with school mathematics, Berry and McClain (2009) found three overlapping components that contributed to the development of a positive mathematics identity: (a) motivation to succeed in mathematics; (b) strong beliefs in their mathematical ability; and(c) caring mathematics teachers. Additionally, Berry and McClain (2009) found that parents of Black boys engaged in racial socialization practices designed to help their sons’ manage in a world where racial prejudice and discrimination are likely to be aimed at them. The boys received explicit messages about racism and messages of expectations concerning high levels of mathematics and academic achievement.

Research Questions

1. How do Black 5th through 7th grade boys who are considered successful in school mathematics construct their mathematics identity?
2. How do Black 5th through 7th grade boys who are considered successful in school mathematics construct their racial identity within the context of learning mathematics in a rural school division?
3. What is the relationship between construction of mathematics identity and racial identity amongst Black 5th through 7th grade boys who are considered successful in school mathematics?

Methodology

Participants

The participant pool consisted of 32 rising fifth through seventh grade Black boys who participated in a two-week summer program focusing on algebraic reasoning and problem solving. Thirteen rising seventh graders, 12 rising sixth graders, and 7 rising fifth graders attended the summer program in which they were the only attendees. The boys were selected for the program based on their potential for placement or their current placement in advanced mathematics courses. All of the boys earned or were close to an “advance pass” designation on the state standardized test. All of the boys attend schools in the same school division in a southern state in which Black students made up approximately 12% of the student population.

Data Collection

For this study, we employed focus group interviews, boys’ mathematics autobiographies, review of documents (grades, test scores, and teachers comments), and observations. The purpose of the focus group interviews was to gain insights into the boys’ experiences and perceptions. For example, one of the focus group questions was: “When you think of someone who is good at mathematics, what characteristics do they have?” Fourteen boys participated in one of three focus group interviews (four to five boys per focus group) that lasted about 45 minutes. A focus group protocol was used to maintain consistency across all groups. All focus group interviews were video-recorded and transcribed. The purpose of the mathematics autobiography was to engage the boys in thinking about their experiences with school mathematics, document important mathematical milestones, and gain a sense of how the boys perceive themselves as learners of mathematics. The boys’ student records (courses, grades, standardized test scores, teacher comments, and exceptionality status) were reviewed for placement in the summer program, to get a sense of the boys’ mathematical history, and to verify previously collected data. Informal daily observations during the two-week summer program provided insights into the boys’ interactions with their peers and with their teacher.

Data Analysis

Analysis occurred after the two-week summer program because the authors were also the instructors of the summer program. The video-recording was transcribed and copies of the mathematics autobiographies were made so that memoing could occur within the transcription and the mathematics autobiographies. Memoing allowed the authors to do initial coding. The codes used during the memoing came from the literature. One of the initial codes used was “beliefs about the significance of mathematical knowledge” drawn from Martin’s (2007) research on mathematics identity development. For each code, definitions were created so the codes could be consistently used throughout the analysis of data. Once initial coding was completed, the data was reread and re-coded to verify the initial coding and to ensure consistency. After this, the database was sorted by codes then reread and re-coded. At this point, we looked for themes within each section (code) to see if there were dimensions that required the data to be further discriminated. Through this process, themes emerged from the data. From this categorization and classification of the data, we described the findings.

Findings

Three themes arose from the data and presented according to the three research questions: (a) construction of their mathematics identity, (b) construction of their racial identity, and (c) the relationship between these two processes to redefine their own racial and mathematics identity. Four factors positively contributed to mathematics identity: (a) the development of computational fluency by third grade, (b) extrinsic recognition in the form of grades, standardized test scores, tracking, and gifted identification, (c) relational connections between teachers, families, and out-of school activities, and (d) engagement with the unique qualities of mathematics. For the boys in this study, racial identity in school is connected to perceptions of other students' school engagement. The interaction between the boys' racial and mathematics identity led to a sense of "otherness" and resulted in a redefinition of their own racial and mathematics identity.

Four Factors Positively Contributed to Mathematics Identity

The development of computational fluency by third grade. The boys articulated computational fluency as a characteristic of people who are good at math. During focus groups, several boys described this attribute similarly. For example Tinashe stated, "I think I'm good at math because there are some things that I can get down quickly." Fluency with mental mathematical strategies and computing large numbers was defined by boys as a significant characteristic for mathematics achievement. As the boys wrote their mathematics autobiographies, they described computational fluency as an attribute that contributed positively to their mathematics identity. Their speed and accuracy with mathematics operations initially drew boys to mathematics. For most boys, they recognized computational fluency as a positive factor contributing to their mathematics identity in third grade. Derrell and Jamal's voices represent this recognition: "I was first drawn to mathematics in 3rd grade. My whole grade was learning multiplication, decimals, and fraction…I picked up on it very quickly. I finished my work quicker and faster than everyone so I would have to read" (Derrell, Mathematics Autobiography); "I was drawn to math when I was in third grade. What drew me to math was realizing that I was really good at it. I saw that I could solve math problems faster than I could solve problems in any other subjects" (Jamal, Mathematics Autobiography).

Extrinsic recognition. The boys utilized extrinsic recognition as a factor in contributing to their mathematical identity. Several outside authorities such as grades, standardized test scores, tracking, and gifted identification provided the boys with proof of their mathematical success. For example, Vince stated, "…because every time I answer a question or something, well not every time but most of the time, I get the question right. And most of the time at school or math tests I usually get A's and B's every time." Similarly, Calvon, stated, "Yeah because…in sixth grade we had a lot of tests, and I got A's and B's on them. And I was a really good student, and I had A's on my report card in math." School performance was a defining attribute of their mathematics identity. The boys shared the standard of A's and B's on tests and report cards to demonstrate success in mathematics. Other boys referenced their scores on the standardized state mathematics test.

The boys' awareness of tracking in advanced mathematics classes and gifted identification in mathematics also contributed to their mathematics identity. Zuberi and Eddison's quotes are representative of the boys' reflections about tracking and gifted identification: "I think I'm successful in mathematics because I've been in advanced math classes and I've done a lot of hard stuff…I was in it this year and I think last year and maybe third grade" (Zuberi, Focus Group) and "…because most of the time I'm good with numbers. And the past four years I think I've been in either advanced or advanced honors" (Eddison, Focus Group). The boys realized that being placed in advanced or gifted mathematics courses meant that others recognized them as successful in mathematics.

Relational connections between teachers, families, and out-of school activities. The boys' relationships with their teachers, families, and out-of school activities contributed to their mathematics identity. In their mathematics autobiographies, many boys described their parents as having a significant impact on seeing themselves as successful at mathematics. For example, Geff wrote in his mathematics autobiography, “My mom actually was the first person to tell me I was good at math. I felt good because my mom told me it can lead to a good education.” Marcus described his parents’ support and extension of mathematical learning: “My dad is a math teacher so I learned most of the stuff I know from him. It felt fun because I was learning things I never knew. My dad helped me realize I was good at math. My best math teacher was my mom because I would come home not knowing what to do with my homework and my mom helped me.” These excerpts are representative of the significant role of families in positively impacting boys’ mathematics identity. Families provided positive reinforcement, motivation, and academic support as well as instilling a sense of the significance of mathematical knowledge for the boys’ educations and futures.

Connections with mathematics outside of school often created contexts for boys to construct positive mathematics identities. For example, Derrell described an out-of school context in which his family enabled him to realize his positive mathematics identity: “I realized I was good at math when my mom, brother, sister, or grandparents were doing bills or taxes and needed to know simple multiplication like 8x8=64 and I knew the answer in one second…Also, everyone asked me, ‘How did you know this and that?’ That made me feel very happy.” The out-of school activity of doing bills and taxes provided an opportunity for Derrell to demonstrate his computational fluency. His family’s praise initiated Derrell’ development of his positive mathematical identity. Beyond mere positive recognition of the boys’ mathematical abilities, these contexts provided opportunities for the boys to connect school mathematics with their outside-of-school lives and to experience additional success as mathematicians.

Relational connections with teachers contributed to the boys’ mathematics identity as the boys described individual mathematics teachers who helped them connect with mathematics in positive ways. The following excerpt from Rasheed’s mathematics autobiography is typical of boys’ positive descriptions of teachers:

> My mom and teachers helped me realize I was good at math. My best teacher was my fourth grade teacher, Mrs. Hebblethwaite. She was my best math teacher because she pushed me to the limit. She was a good math teacher. It was fun to be in her class because she always made math fun. She was different from my other teachers because she took time to explain and help me when I had hard work. (Mathematics Autobiography)

The boys described influential teachers, like Rasheed’s, who “made math really exciting” (Calvon, Mathematics Autobiography), “challenged” boys (Jamal, Mathematics Autobiography), “could be fun and could get you to do your work” (Reymond, Mathematics Autobiography), and “actually helped” boys understand mathematics (Vince, Mathematics Autobiography).

Engagement with the unique qualities of mathematics. The boys differentiated between the unique qualities of mathematics and other disciplines by describing the challenge of mathematics and their pride at persevering to completion. Jamal’s portrayal of mathematics is representative of boys’ comments:

> What I like about math is it’s kind of complicated, and I like, I want my work to be complicated so I can actually do better when I get to higher grades. And it feels like I finished something. It’s like when it’s hard, like when we were doing an engineering

> project, I feel like I finished something really good, like I did a really good job with it. (Mathematics Autobiography)

The boys' descriptions of mathematics included words like complicated, complex, challenging, and requiring concentration. The boys also described the distinctive ways mathematics problems engaged their thinking through problem solving, engaging interactively, utilizing multiple strategies, and making connections to other disciplines.

Racial Identity In School Connected to Perceptions of Others' School Engagement.

The boys' construction of racial identity in school was influenced by their perceptions of other students' school engagement. The boys perceived that teachers treated groups of students differently, based on race, gender, or ability. The following focus group conversation is representative of these comments:

> Damitri: Some of the teachers. Like sometimes teachers give other kids more attention than other kids. Well it feels that way.
> Keeshawn: Yeah.
> Damitri: Like different races of kids. Yeah they favor kids. Well in my math class, my math teacher favored a couple of kids over me and a couple of my friends. Well when I'd like raise my hand when she's working with some student and then she'd say, 'I'll come to you in a minute.' And so she'd be like, 'I'll come to you after this student.' And then she'd look at me and then walk to a different student and then go over and help them and then help me.
> Keeshawn: Yeah, it's happened to me but not with race. It's not about like your skin color or anything. It's about like the people who usually get more questions right.
> Damitri: The smarter students or the ones they think are smarter.

Black boys were in the minority in these boys' mathematics classes and schools. As a result, many boys felt isolated. One of the purposes of the summer program was to bring this group of Black boys together because many of them were the only Black boy in their mathematics class. For some boys, the feeling of isolationism created discomfort. Wynn and Kavion's quotes represent the perceptions shared by boys during focus groups: "At my school really it matters what classes I'm in. I was the only Black who was in there the whole year. It's better in the gifted classes because personally I think the teachers are nicer," (Wynn, Focus Group); "I kind of feel uncomfortable in my school cuz…they wear shirts and it has [confederate flags] on it. It just makes me feel very uncomfortable. So when I see a whole group of them in the bathroom, I just don't go. I just go back to my class," (Kavion, Focus Group).

A Sense of "Otherness" and Redefining Their Racial and Mathematics Identity

As the boys connected racial identity with their perceptions of others' school engagement, they made distinctions between other Black males in mathematics at their school and themselves. This sense of "otherness" caused many of the boys to reflect on how they perceive other Black boys and how others perceive them. The following focus group conversation is representative of the boys' descriptions of other Black males in mathematics:

Jamal:	I think some African Americans just give up on math because they say they can't do it and they don't even try to learn. So I think that's part of their parents talking to them.
Mateo:	I see that at my school too.
Jamal:	Yeah, some people are just trying to be cool.
Echoes of yeah, yeah	
Jamal:	And then some people will pretend that they're cool and not nerds and not answer questions.
Mateo:	I see that at my school.
Eddison:	Mostly they're failing. They gave up.
Jamal:	Cuz like most of the black people in my grade they don't have black friends so they would rather be cool cuz they think if you're cool you might get more friends.
Mateo:	If you be yourself, you'll get more friends.
Jamal:	I'm sometimes called nerd in my grade.

The boys described other Black boys as preferring to "show off" (Wynn and Kavion, Focus Group) or "be cool" (Jamal, Focus Group).

The boys also clarified that Black males can be successful in mathematics. Some boys described the perception of Black males' lack of success in mathematics as a "negative stereotype" (Malcolm, Focus Group). For example, Calvon described this when he said, "I think like African Americans are good at math. It's just that like some people like putting them down and like not making them feel good" (Calvon, Focus Group).

Collectively, the boys described people who are successful in mathematics as those who follow directions, persevere, collaborate, want to learn, meet challenges, and are smart. Boys contrasted these attributes with their perceptions of others' school engagement. The interaction between boys' racial and mathematics identity led to a sense of "otherness". Keeshawn described the impact of his perception of others on his own mathematics identity:

> I know that African American males aren't usually, don't achieve too well in math and stuff. But I feel that just because like statistics show that African Americans don't do as well in math, don't achieve more, I still feel that we can do good. It's just statistics say most African American males, so that kind of makes me, that kind of gives me a boast. (Focus Group)

The boys used these perceptions to redefine their own racial and mathematics identity. Tinashe and Zaire's quotes are typical of boys' connections between attributes of mathematics success and their experiences with racial identity in school: "I think [being Black] hasn't affected me because it doesn't really matter what color I am... I'm addicted to math," (Tinashe, Focus Group) and "[Being successful in mathematics] feels ok because some people think we're actually, we're smart," (Zaire, Focus Group). The boys recognized important attributes in themselves that enabled them to be successful in mathematics.

Discussion and Recommendations

The boys in this study attended schools with a small percentage of Black students and most were the only Black male in their advanced mathematics classes. Understanding this context provides a lens of the developed sense of "otherness." The perceptions of "otherness" allow these boys to engage in school differently from their own and others' perceptions of how Black boys engage in schools. The perceptions of "otherness" shifted during the summer program because the boys were surrounded by other Black boys who were identified as smart in mathematics. Many of the boys saw this as an opportunity to engage with other boys by collaborating and challenging one another. The boys often position themselves in postures of confidence and challenge when solving mathematics problems. Teachers must understand the structure of the boys' experiences to appreciate that such postures are not deficit nor defiant; rather, they are transferences from other settings. A study of transferences across settings may be necessary to broaden understandings of Black boys' development of mathematics identity. Teachers' knowledge and appreciation of the unique qualities of mathematics that attracted, motivated, and engaged these Black boys may provide them with a lens to identify mathematics problems that may positively impact the development of the boys' mathematics identity. Also, the complexity and challenge of mathematics was one significant quality of mathematics that should not be ignored. Mathematics should not be simplified or dumbed-down but rather teachers should hold high expectations for their students to solve challenging and complex mathematics problems. This careful balance between creating a learning environment where challenging mathematics is accessible but without negating the challenge appears to be instrumental in promoting the boys' positive mathematics identity.

Endnotes

1. We use the term Black because it represents broadly the boys in this study. Four of the boys are bi-racial (all have African American and White parents) and three boys are African.

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WHAT IS IN ADULT MATHEMATICAL TALK?

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This study investigated the nature of adult mathematics-related talk during free play with children between 18 and 39 month olds and whether the nature of the adult input depends on the child's gender. Thirty-six parent-child dyads participated in a 30-minute free play session. The adult speech was transcribed and coded for mathematically-related input. Our findings suggest that parents talked more about numerical quantities (e.g., quantity words) and numerical relations (e.g., cardinality and transformation of numbers) than on mathematical attributes such as shape and color with their young children during their free play interactions in spite of the child's gender.

Introduction

Difficulties in mathematics are widespread in many industrialized nations such as Canada and the United States. Children who are weak in basic arithmetic may not acquire the conceptual structures required for advanced mathematics, which serves as a gateway to careers in many disciplines. Development of counting is a critical pathway to learning about numbers. Counting weaknesses have been linked to mathematical difficulties later in formal schooling (e.g., Geary, 2003). Additionally, the informal acquisition of number concepts before children enter kindergarten has been found to be related to number knowledge development, which is a strong predictor of arithmetic achievement in first grade (Baker et al., 2002; National Mathematics Advisory Panel, 2008). Moreover, a children's emergent mathematics competence in kindergarten has been found to be a stronger predictor of subsequent academic achievement than emergent reading competence (Duncan et al., 2007).

Much of the research on the development of numerical knowledge has focused on preschool-aged and kindergarten aged children with the objective of designing and setting the standards for early childhood mathematics curricula (e.g., Clements, Sarama, & DiBiase, 2004; Clements & Sarama, 2008; Starkey, Klein, & Wakeley, 2004). However, research in neuroscience has demonstrated that learning in the first three years of life sets the trajectory for a person's subsequent capacity for learning (e.g., Greenough, 1997). Hence, it is important to investigate mathematics learning during the first three years of life in order to better understand factors that influence mathematical development. As such, more research is needed on young children's daily numerical experiences in order to better understand how early home environment influences their acquisition of number knowledge (Sophian, 2009). Unlike language development, very few studies in mathematics development have examined the adult input that children receive at home on number concepts. The few existing studies using interviews and checklists suggest that parental mathematical input is related to four- and five-year-olds' level of number knowledge, such as knowing differences in quantities and that numbers have magnitudes (e.g., Saxe et al., 1987; Blevins-Knabe & Musun-Miller, 1996; LeFevre, Clarke & Stringer, 2002). The current study investigated the nature of adult mathematics-related input (such as the use of count or quantitative words) in the facilitating the acquisition of number concepts between 18- and 39-month-olds in a naturalistic free play session. Furthermore, we examined whether the quality and/or quantity of adult mathematical input depend on the child's gender.

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Theoretical framework

Language acquisition studies have indicated that children's general vocabulary growth is related to the amount of language input they receive (e.g., Hart & Risley, 1992; Naigles & Hoff-Ginsberg, 1995). Moreover, children's understandings of mental states are facilitated by the maternal use of mental verbs such as 'think' or 'know' (Adrian, Clemente, & Villanueva, 2007; Tardif & Wellman, 2000). A recent study of preschoolers by Klibanoff, Levine, Vasilyeva, & Hedges, (2006) found a relation between the amount of mathematical input of daycare teachers and the growth of children's mathematical knowledge during the year. Moreover, developmental studies suggest that early mathematics representations such as numerosity (1 unit or 2 units of something) are linked to mathematics language, for example, the knowledge of count words (e.g, *one* and *two*) (e.g., Huttenlocker et al., 1994; Jeong & Levine, 2005). Children learn the cardinal word principle and the meanings of all the number words within their counting range after acquiring the meanings of *one* by 2.5 years old, *two* by 3.0-3.5 years old, and *three* by 3.5-4.0 years old (Wynn, 1990, 1992). These studies suggest the importance of the amount of mathematical talk children receive in the daily lives during the early years of life and its impact on their acquisition of mathematical language and concepts. Specifically, parents who provide quantitative input to their children by using sentences such as "You need *two* more puzzle pieces" while engaging in a play activity may foster the growth of children's mathematical development. However, we currently do not have a good overview of what kind of mathematical talk young children receive at home.

Furthermore, by analyzing adult input, this study seeks to address the question of whether socialization plays a role in mathematics learning in boys and girls. Gender-stereotyped perceptions about children's mathematics competence by parents have been reported as early as preschool and kindergarten (Lloyd, Walsh, & Yailagh, 2005; Lee & Schell, under review; Manger & Eikeland, 1998). Boys as young as four years old show an advantage over girls in number sense (Jordan et al., 2006) despite the lack of consistent gender differences observed in young infants performing a numerosity judgment task (e.g., Lipton & Spelke, 2003; 2004) as well as in three-, four-, and five-year-old children performing counting and magnitude comparison tasks (Lee & Schell, 2009). This raises the question whether gender differences in early mathematics are related to socialization and motivation such that boys are encouraged more than girls on number tasks, even at an early age (e.g., Aunola et al., 2004) or whether there are cognitive explanations for gender differences in early mathematics (Geary, 1998). By examining the interactions between caregivers and their children in a free play session with a standard set of toys picture books, differences in the quality of talk in terms of introducing mathematical-related words (e.g., caregivers focusing more on how the doll is driving the car to work with girls vs. caregivers focusing more on how many trucks are on the floor with boys) could be identified. The free play session requires video recording young children and their caregivers in order to capture both the verbal and nonverbal ways in which mathematical words are presented to the children. Without the need for questionnaires or checklists that rely on parental memory, this approach allows us to examine the input that occurred in a more naturalistic manner and to obtain the frequency and nature of incidental mathematically relevant input that children receive at home.

Method

Participants

In the current study, 36 families (19 boys and 17 girls) between 18 and 39 months old and their parents (usually the mother) engaged in a 30-minute naturalistic free play session in their home. Seventeen and nineteen children were in the 18- to 28-month-old group and the 29- to 39-month-old group respectively. Nineteen children were enrolled in part- or full-time daycare or nursery program while the remaining seventeen children were looked after by their stay-at-home parents. These families were predominantly Caucasians with average socio-economic status (SES) using maternal education as a proxy for SES since studies such as Catts, Fey, Zhang, & Tomblin (2001) have found that it is a good proxy measure for SES. All but four mothers have attained a college or higher degree. Children and their families were recruited through birth announcements, online advertisements, recruitment flyers and contacts provided by families that participated in this study. Each family received a $5 gift card for their participation.

Table 1. Coding Scheme

Color	• Yellow, blue, green…
Shape	• Circle, square, triangle
Quantity words	• How many, how much, so much, several, some, a few, altogether, most, more, same (only in the context of quantity, and not in the other context such as color), greater than, less than, count, lots, another, all, ones, all, some, little, little bit, a little, any, a couple, bunch… (Note: “match” or “anything else” are not quantity words)
Counting words	• Recite counting words such as ‘one’, ‘two’, ‘three’ (Similar to reciting the alphabets) • Count each word said as one occurrence. E.g., If the caregiver says “One, two, three”, count = 3, and not 1 because there are three count words in the utterance. • Number words such as ‘one’, ‘two’… Not associated with an object or an array of objects.
Counting objects	• Counting objects in an array or in sets (1,2,3,4..)
Transformation of object arrays	• Add, subtract, take-away (e.g., “If you take away from three, how many do you have?”) OR • One object item removed from an array of objects (e.g., 1 bird flew away, how many are left?)
Cardinality words	• Stating or asking for the number of things in a set without counting them (e.g., Point to the child the number ‘2’ and ask him/her to point or show two items/objects or the above example “There are 3 books.”) • The number word (‘one’, ‘two’..) here is associated with an object or a set of objects (E.g., ‘SIX sides of a cube’, ‘THREE trucks’) • No counting involved
Ordering numbers (different from reciting a list of number words)	• “Three, what comes after three?’

Materials

To minimize differences in the immediate environment and in the activities that the children and caregivers engage in, each child-caregiver dyad was provided with a standard set of toys (e.g., five balls, eight car counters, ten animal counters, twelve Lego blocks) which could be used and labeled in a number of ways for free play session as well as two pop-up books on colors, numbers and shapes. Furthermore, the various sets of provided toys could facilitate in eliciting the use of number words during the interactions. Consequently, any observed differences between the verbal labeling and interactive practices can be attributed to inherent differences between individuals, rather than differences in the environment.

Transcription and Coding

Each free play session of the parent-child dyads was videotaped, transcribed and coded using a portable Observer XT system that allows the experimenters to work the two cameras using a remote control in a separate room from the dyad. The parent-child dyad was also asked to wear a wireless microphone each to ensure a high audio quality. Such an arrangement ensured that the free play session was as naturalistic as possible. All the adult speech were transcribed and coded by at least two trained research assistants for the mathematically-related input in the following categories: i. shape and ii. color, iii. quantity words (e.g., how many, less than, more), iv. cardinality (stating or asking for the number of items in a set without counting), v. counting words (e.g., one, two), vi. counting objects in an array or in sets, vii. ordering numbers (e.g., 'Three, what comes after three?'), and viii. transformation of object arrays involving addition or subtraction (e.g., "If one cow went missing, how many do you have now?")

Results

The total word tally for the utterances by parents in the 36 play sessions was 38,512 and 34,587 words for boys and girls respectively. Only 2,434 (4.43%) and 2,175 words (4.12%) were mathematically-related words heard by boys and girls respectively. Thus, our young children in this study heard on an average of 4.28% of mathematical talk produced by their caregivers during the 30-minute play session. In terms of mathematical talk, our results indicate that about 69% of the mathematically-related adult utterances involved quantity words, counting words, counting objects, cardinality, ordering, and transformations of object arrays. Specifically, parents used more words that indicate quantity (31%) and cardinality (25%). In contrast, words about shapes and colors made up about 31% of the total mathematically-related speech (see Table 2).

Table 2. Frequency (%) of words in adult mathematical-related input

	Shape	Color	Quantity words	Cardinality	Countin g words	Counting objects	Transfor-mation	Total
Boys	241 (9.90)	487 (20.01)	701 (28.80)	618 (25.39)	89 (3.66)	297 (12.20)	1 (0.04)	2434 (46)
Girls	304 (13.98)	400 (18.39)	713 (32.78)	553 (25.43)	108 (4.97)	96 (4.41)	1 (0.04)	2175 (54)
Total	545 (11.82)	887 (19.25)	1414 (30.68)	1171 (25.41)	197 (4.27)	393 (8.53)	2 (0.04)	4609 (100)

Note: Ordering numbers = 0 for boys and girls

A series of t-tests were conducted to determine whether there were gender differences in parent mathematically-related input. Based on the frequency or incidence of mathematically-related utterances in the eight categories, our t-tests did not reveal any significant gender differences. Thus, parents did not differ in terms of their use of mathematically-related language with girls or boys.

Discussion

In sum, our findings suggest that though parents talked more about numerical quantities (e.g., quantity words) and numerical relations (e.g., cardinality) than about mathematical attributes such as shape and color, the total amount of mathematically-related talk heard by young children was about 4%. Moreover, we did not find any gender differences in terms of the mathematical talk parents produced and the different aspects of numerical knowledge such as counting objects or cardinality.

The low amount of adult mathematical talk is disconcerting in light of studies such as Klibanoff et al. (2006) indicating a positive correlation between the amount of mathematical input by early childhood educators and the growth of mathematical knowledge of preschoolers. Furthermore, some researchers have suggested that the mapping of non-verbal number concepts or numerosities to number words that represent quantity, and ultimately the acquisition of counting principles is facilitated by the acquisition of mathematical language (e.g., Gordon, 2004). This low amount of mathematical talk could possibly be a reflection of the parents' personal comfort level with mathematics or due to the parents' assumption that mathematical knowledge is something that is taught in school. The latter reason is more consistent with the feedback that we have obtained from our parents during the debriefing session after their free play interactions. Almost all the parents, even those with older children (between 29-39 months old), reported that they have been focusing on naming objects rather than mathematical attributes or numerical quantities and relations.

A limitation of this study is that our free play sessions of the parent-child dyads were overwhelming those of mothers due to the inherent nature of more women are the primary caregivers. For this study, we have specifically requested that the primary caregiver who spent more time with the child to participate to try to obtain as naturalistic interactions as possible.

Nevertheless, this research is poised to make interesting contributions in understanding the acquisition of number words and counting concepts by children younger than 3.5 years old. No studies currently exist that seek to investigate the types and amount of mathematical talk young children receive at home from their parents in a naturalistic setting or examine whether gender of the child is a factor that influences the nature of adult mathematical talk. Additionally, this research, situated in the earlier childhood years (18-39 months), begins to make a novel contribution to the existing research which largely explores later childhood years (3-5 years old). Our findings will help to inform early childhood educators and policy makers of the amount and type of mathematical knowledge children have already acquired when they first enter either a preschool or kindergarten educational program. As an illustration, the small percentage of adult mathematical talk on counting words or counting objects in an array reported in this study would suggest that more time has to be spent to focus on the acquisition of verbal rote counting. Therefore, knowing how parents talk about everyday mathematics would facilitate the design of better early childhood mathematics curricula and intervention programs for children at risk for mathematical difficulties.

At the same time, parents and caregivers should be encouraged to use more counting words as well as to engage in more counting and patterning activities with their children in daily life. This approach of learning mathematical concepts through daily activities could lay a foundation for subsequent mathematics competence for children to give them a head-start once they enter formal schooling as well as prevent some children from gradually learning to avoid things involving mathematics or even developing mathematical anxieties or phobias (e.g., Ashcraft, 2002).

Endnotes

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A BALANCING ACT: MATHEMATICS TEACHER EDUCATION IN PUBLIC INTEREST

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While internationally there is ample interest among mathematics educators in social justice, the literature base on *mathematics teacher education* for social justice is very limited (Gates & Jorgensen, 2009). The Mathematics Education in the Public Interest (MEPI) research project provides preservice elementary and middle school teachers (PSTs) with a mathematics content course, *Math for Social Analysis*, that integrates development of mathematical knowledge needed for teaching with knowledge needed for critical, responsible citizenship. This poster presents various practical challenges that we, as mathematics teacher educators, face in course design and implementation, including: (1) Mathematics content and social critique; (2) Mathematics content and pedagogy; and (3) In-class and out-of-class experiences and learning.

Balancing Mathematics Content and Social Critique: The greatest challenge we face is finding appropriate balance in depth and breadth of emphases on helping PSTs learn mathematics while helping them to critically engage with social issues. Nolan (2009) explained how the "statistics and figures" content approach that takes mathematics as usual and appends social justice concepts will not be enough (p. 207). *Math for Social Analysis* addresses all five NCTM content strands, yet we found number and operations and statistical concepts naturally align with social issues more easily. We struggle to balance time spent examining mathematical concepts for deep understanding with time spent engaging PSTs with social issues.

Balancing Mathematics Content and Pedagogy: We dualistically aim to balance PSTs personal experiences with learning rich mathematics within the context of social issues, while: (1) critiquing their own understanding of mathematics as a discipline in relation to the democratic purposes of schooling and (2) building new pedagogical content knowledge embedded in MEPI principles.

Balancing In-Class and Out-of-Class Experiences and Learning: Our challenges extend outside of the classroom as we strive to balance the in-class and out-of-class experience through service learning. The partnership requires a greater commitment from the service learning organization to address the scheduling and assessment of our PSTs. Their feedback becomes a component of the PST's course grade.

Endnotes

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ARE YOU SURE THIS IS MATH? USING SOCIAL JUSTICE IN MATHEMATICS EDUCATION

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The goal of this research was to further explore the significance and effects of incorporating social justice into mathematics curriculum. My interest in this topic was ignited while reading articles on alternative mathematical instruction discussed in a mathematics publication (Gutstein & Peterson, 2006) and reflecting on past experiences as a math teacher at an inner city middle school based on unmotivated and disinterested mathematics students, students desires to discuss current political and social events, and my struggle to teach curriculum with meaning and teach mandated curriculum simultaneously. Critiquing banking concept of education, I illustrate negative effects of traditional mathematics curriculum on student and teacher practices.

Using Friere's perspective of banking concept in education and critical theory, I analyze the differences between traditional mathematics and justice oriented mathematics. Friere (1998) describes banking concept of education as the teacher putting money into the bank, the student regarded as empty receptacles into which the teacher deposits the knowledge. Dialogue doesn't exist in banking education and therefore critical thinking is not utilized. The absence of dialogue and critical thinking leads to missed opportunities of students to exist as change agents in classist societies. Friere draws upon social and political consciousness which arises from dialogue and critical thinking. This theory lends to the importance of engaging students in justice oriented mathematics which allows students to dialogue and become conscious of social ills which occur in society.

Data was collected over a course of six months (from December 2008 to May 2009) and visits were made three times weekly. Data collection included teacher and student post interviews and pre interviews, observational field-notes, and document collection. To further enhance data collection, I maintained a researcher's journal that included a record of my concerns or feelings that arose during fieldwork.

Much of the research concerning children from poor communities of underrepresented racial groups has aimed to describe their lack of access to socially realistic mathematics. An exploration of the strategies teachers use when engaging in socially just mathematics lessons could help other educators to develop more equitable curricula and teaching practices in the future. This critical study also emphasizes a need for critical examination of mathematics curriculum used in urban communities and represents the beginning of an effort to broaden our understanding of the role that social class plays in educational achievement.

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CULTURALLY RELEVANT TEACHING IN MATHEMATICS EDUCATION: DOES IT ADD UP?

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Although the continued call for the *need* to do culturally relevant teaching in mathematics is an indication that the battle to convince teachers to adopt culturally relevant strategies has not been won, the argument that educators *should* be doing culturally relevant teaching, and that children are, indeed, culturally different from each other, are truths that are uncontested in equity literature. But what made it possible to conceive of children as culturally different? How did culturally relevant teaching emerge as something that could and should be done?

The purpose of this conceptual paper is to apply Foucault's (1977) genealogical methodology to the concept of culturally relevant teaching. First, I will describe genealogy and the value of using a genealogical research design to consider culturally relevant teaching. Then, I will identify the eruption of forces that intersected to make it possible to conceive of students of color as culturally different. I will provide an overview of Ladson-Billings' (1994) theory of culturally relevant teaching, analyze its take up in mathematics education discourse, and propose that the synonymy of culturally relevant teaching with equity prevents us from questioning it as a problematic way of thinking about equity work in schools.

This qualitative project was a Foucauldian (1977) genealogical study of the discourse surrounding children of color and culturally relevant teaching in education research and mathematics education discourses. For Foucault, genealogy was a framework for investigating and thinking about present truths in a way that could disrupt the perception that our present reality is an inevitable result of a linear series of events. Genealogy is useful for tracing the emergence of a belief that is so commonsensical that we might see it as without history, in order to consider how it has become the truth.

This genealogy traces references to culturally relevant education and teaching within the education literature -primarily American education research journals, books, and practioner publications- in order to understand why and how it came to be possible to think of children as culturally different.

Until we can talk openly about the dilemmas within and around this equity strategy, and other attempts to transform mathematics classrooms into more equitable places, the time and space that *is* opened up in schools for equity work may do very little to improve the school experiences of children of color and may be supplanting more critical and effective ways of "doing equity".

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EXPLORING AFRICAN AMERICAN STUDENTS' IDENTITY CONSTRUCTION AND THEIR MATHEMATICS ACHIEVEMENT

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A growing numbers of studies focus on the importance of students' understanding and the mathematical meaning created by the students as they participate in mathematical studies. These studies emphasize the socially and culturally situated nature of mathematical activity with learning viewed as a collective process of enculturation into the norms and discourse practices of the mathematics class that foster students' ability to see themselves as doers of mathematics (Boaler & Greeno, 2000). From this perspective, the development of identity, or the process of identification, is seen as an integral part of students' engagement and participation in the mathematical practices of a community constituted by the students and their teacher in a classroom (Boaler & Greeno, 2000). It is through the social interaction and participation structures that students' identities as learners and as active agents are enacted as they participate in the classroom mathematical practices mediated by cultural artifacts and resources.

This study involved a case study of an intermediate algebra classroom at a low-SES high school. The aim of the study was to investigate African American students' identity construction and note the sense of agency exhibited in the process. I focused on five key informants who were selected using a purposive sampling method and comprised of two high achievers (male and female) and three low achievers (two females and one male). The stories told by these key informants helped illuminate their sense of identity and agency that they developed and enacted within the figured worlds in which they participated. In investigating the students' identities and agencies, I draw on Critical Race Theory (CRT), and Holland, Lachicotte, Skinner and Cain's (1998) framework of figured worlds, positioning and authoring.

The results of this study revealed that all the participants had seemingly positive racial identifications and were aware of the constraints and the social devaluation that face African Americans in the society. However, they differed in their interpretation and negotiation of these constraints and in the sense of agency they exhibited in the process that influenced their opportunities to participate in mathematics and hence their mathematical identities. Additionally, how students positioned themselves and authored their mathematics identities was influenced by how they negotiated the classroom norms and the constraints and affordances in the figured world of the mathematics learning in which they participated.

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FROM THE POLITICS OF REMEDIATING MATHEMATICS: FACTORS IMPACTING STUDENTS' MATHEMATICS LEARNING EXPERIENCES IN UNDERGRADUATE ALGEBRA COURSES

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While equitable access to mathematics has been a longstanding (yet still-emergent) interest among mathematics education researchers, the expanding role of mathematics in students' transitions to public, four-year, research-oriented universities has received little attention. Despite calls for research that "opens the box" and explores students' experiences in critical transitionary courses at the university, there has been no research that explores the ways in which mathematics is intended (e.g., in curricular materials), enacted by teachers and students, or supported by institutional programming and peer networks in these settings.

The present study focuses on the institution-, course-, and individual-level mathematics socialization factors (Martin, 2000) that impact the mathematics experiences of a group of first-year, African American undergraduates---students who were enrolled in the university's lowest-level, non-credit-bearing mathematics course. Conducted over several semesters, the study questions and empirically investigates the role of structural and personal factors that impact students' mathematics learning experiences, particularly mathematics identity, classroom engagement, and institutional policies and practices. Classroom-level participant observation, field notes and classroom interaction charts were used to document students' engagement with mathematics in the classroom. Semi-structured interviews with students, instructors, and university personnel were conducted in order to study the impact of institutional programming, mathematics instruction, peer networking, curricular goals and expectations, and students' personal and mathematics identities. Narrative analytic techniques were used to analyze transcripts of these interviews (Juzwik, 2006). The proposed poster will describe each phase of the study's design and will include findings based on analyses described above. I will outline the factors impacting students' mathematics socialization in the transition from high school mathematics to the four-year university mathematics pipeline, using race-critical and context-sensitive lenses.

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KNOWLEDGE OF HOW CHILDREN LEARN MATHEMATICS: CAN PARENTS USE THIS KIND OF KNOWLEDGE TOO?

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The purpose of our study is to investigate how knowledge of children's ways of learning mathematics can also be used by parents in helping their children learn mathematics. As part of an early project we developed resources for preservice elementary school teachers on how children learn mathematics. The Connecting Mathematics for Elementary Teachers, (CMET), an NSF supported project (DUE 0126882 and DUE 0341217). CMET (Feikes, Schwingendorf, & Gregg, 2008) developed a commercially published textbook and supporting material for prospective elementary teachers and their instructors. Realizing that parents might benefit from similar materials for parents, the project team has piloted CMET materials with parents to see how they might interpret these materials and what changes need to be made.

In this study we provided resources adapted for parents and sought their feedback. The intent of these resources is to impact the mathematical understanding of children by providing parents with in-depth knowledge of how children learn and think about mathematics. One of our goals is to investigate the effectiveness of these resources by examining how parents use knowledge of children's mathematical thinking while working with children.

Data for this study consisted of: a) written reviews of the materials, b) course work in a graduate mathematics education course, c) transcripts from parent focus groups, and d) transcripts from parent interviews. The purpose of collecting this data was to examine how parents might use knowledge of how children learn mathematics in helping children learn mathematics.

Emergent themes that evolved as the result of these beginning research efforts are: 1) Parents want and need more mathematical content knowledge, 2) they are frustrated by their child's struggle with mathematics, 3) parents began to think differently about helping their child as a result of learning about how children learn mathematics and, 4) even after gaining knowledge of how children learn, parents had a different world view of learning that did not match this new knowledge.

Using knowledge of how children learn mathematics can be helpful to parents but this knowledge may need to be more specifically spelled out or presented in a way that is compatible to their view of how children learn mathematics. Parents may have a different mindset or world view as to how children learn in general; this may hinder their use of resources similar to those developed and used.

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UNDERSTANDING OF FUNCTION OF STUDENTS WHO ARE BLIND OR HAVE VISUAL IMPAIRMENTS

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Equity for all students is the first Principle of the *Principles and Standards of School Mathematics* (NCTM, 2000). This presumes that regardless of race, gender, disability, or cultural background, each student should be presented with opportunities to actively be engaged in learning mathematics. This charge can take on a different meaning when the students under consideration have little or no vision. Mathematics is traditionally taught with a heavy reliance on visual context and can therefore be difficult to accommodate for students who are blind or have visual impairments. The research described in this poster presentation is on the current understanding of linear functions of students who are blind or have visual impairments. The following focus questions are addressed.

- What level of knowledge and type of understanding do students who are blind or have visual impairments have of function?
- What factors do high school/college students who are blind or have visual impairments perceive as influencing their development of understanding of linear functions?

Two interviews, the Mathematic Education Experiences and Visual Abilities (MEEVA) survey and the Function Competencies Assessment (FCA), served as the main data collection instruments. The MEEVA follows a standard interview format and is intended to provide demographic information as well as to address the second focus question by providing information on the students' previous educational experiences in mathematics. The FCA is a task-based interview that consists of problems related to linear functions and their applications. This exam was written using the evaluation taxonomy of Wilson (1971) and analysis of these data utilized the four competencies of function knowledge as described by O'Callaghan (1998). These competencies are: a) modeling, b) translating, c) interpreting, and d) reifying. Analysis also utilized the National Research Council's (2001) definition of *Mathematical Proficiency*, which speaks to a student's procedural fluency and conceptual understanding. An emphasis is placed on students' reification of function as well as their knowledge and use of multiple representations.

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"YOU DON'T HAVE ENOUGH DATA TO TALK ABOUT RACE"

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The title of this piece comes from a recent blind review of a piece I wrote about mathematics teaching in a diverse classroom (Parks, in press). The critique was one I'd heard before about different manuscripts. In each case, the reviewers resonded positively to the piece overall, found my evidence about mathematics learning and teaching convincing, and critiqued my identification of race as a factor that impacted classroom interactions.

The purpose of this poster is to open a conversation about current barriers to discussing race and ethnicity in mathematics education. When Lubienski and Bowen (2000) reviewed the mathematics education literature spanning more than 15 years, they found that only 112 of the more than 3,000 articles examined discussed ethnicity. (In contrast, more than 300 articles discussed gender.) In this poster session, I would like to explore the possibility that our theoretical and methodological conceptions of race limit how we as researchers talk about race – in addition to limiting the ways that we allow others to talk about race. As Lubienski and Bowen (2000) point out, many of the pieces that do discuss race focus on the narrowing of the achievement gap in various contexts. In this type of research, race is conceptualized as an independent variable, and large study sizes allow generalizations about populations to be made. However, when race is conceptualized only as a category for large groups of people, little theoretical space is left for alternative ways of thinking and writing about race. Alternative conceptions of race and ethnicity, which focus not on the characteristics of populations, but on the micro-interactions of small groups of humans might offer more freedom in analysis. For example, Omi and Winant (2004) wrote about "the performative aspect of race" (p. 10), where race is seen as continually acted out in various ways by various people. Thus, race is understood as performed rather than as a population category; and, "the enormous number of effects race thinking (and race acting) have produced" (p. 9) are possible objects of study. Exploring this conception of race within mathematics education might mean that researchers can look at the way performances by individual teachers and students are related to race without suggesting that others who might identify as the same race would perform in similar ways. This sort of complexity might bring a richness to many descriptions of classroom interactions. For example, many socio-cultural analyses of discussions, questioning, and problem solving could be made more complex if researchers attended to the ways that race-thinking and acting (along with gender- and class- thinking and acting) influenced the ways that mathematical ideas were taken up or not in classrooms. After all, one might reasonably argue that socio-cultural analyses of classrooms that don't explicitly attend to race, gender, and class don't have enough data to talk about mathematics.

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Chapter 7: Geometry and Measurement

Research Reports

Brief Research Reports

Posters

DEVELOPING STUDENTS' GEOMETRIC REASONING IN A NETWORKED COMPUTER ENVIRONMENT

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This paper presents a new computer environment for students to explore various topics in geometry. In this design, each student in a small group is assigned responsibility for moving one vertex of a polygon so that the group collectively explores changing properties of that shape. This paper investigates a group of four students completing a series of activities about quadrilaterals. We show how certain features of the computer environment as well as the sequencing of tasks helped students move from initial understanding of shapes as wholes toward focusing on the features to noticing hierarchical relationships between different types of quadrilaterals.

Introduction

In the research literature on geometry, the van Hiele theory (1959/1985) has been widely recognized for providing a coherent framework to explain how students develop their geometric understanding. In elaborating five levels of geometric thinking, van Hiele stressed that it is not maturation or age but rather instruction and experience that facilitate students' progress within and across these levels. In this paper, we explain how we developed a new computer-based learning environment and organized an accompanying sequence of tasks in order to support students' geometric reasoning, and use the van Hiele levels as a framework for examining changes in students' thinking over the course of an instructional sequence. This design is one in a family of learning environments we are developing as part of a longer-term design-based research project focused on supporting novel forms of classroom mathematical exploration and interaction through a local network of graphing calculators or handheld computers (e.g. White, 2006; White & Brady, 2010; White, Lai & Kenehan, 2007).

This paper presents results from a single design experiment conducted as a part of this larger ongoing project. In reporting on this study, the present paper focuses on investigating the extent to which certain features of the computer environment, and the particular sequence of tasks students completed, may address challenges in students' learning about geometric concepts. In particular, we focus on students' learning of quadrilaterals, an area in which middle school students have difficulties (Jones, 2000; Yu, Barrett, Presmeg, 2009). Learning quadrilaterals can be difficult for many reasons: 1) students many times have a prototypical image of a shape, 2) shapes have a long list of features, 3) and the categorization of shapes are hierarchical (Fujita & Jones, 2007). We specifically designed three variations of the learning environment and three corresponding tasks to address these challenges. The first set of activities engaged students in dynamic manipulation and open-ended exploration of quadrilaterals in a Cartesian space. The second variation pushed students to focus on the features of the shapes. The third set of activities was intended to help students focus on the relationships among different quadrilaterals. In this paper, we will examine how the design of the computer environment and activities help students progress in their understanding of quadrilaterals.

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The Van Hiele Theory

The original van Hiele theory presented five levels, of which the first three pertain to our study. In the first level, *Visual*, students see shapes as gestalt figures and refer to them using familiar objects. For example, they might say that a rectangle looks like a door. In the second level, *Analytic*, students begin to recognize components of figures and can see that squares have four equal sides, etc. Progressing to the third level, *Abstract*, students begin to understand the hierarchical relationships among various shapes. For example, they can begin to see how squares can be rhombuses. Although there is debate as to exact nature of this progression, researchers continue to refer to the van Hiele theory because it provides a clear framework for investigating students' geometric understanding (Battista, 2007).

Study

Context and Participants

Results reported in this paper come from the first phase of a design experiment (Brown, 1992) we implemented in a charter school located in an urban area. This first phase included a group of four sixth and seventh grade boys who volunteered to participate in the study outside of class time.

Computer Environment

The computer learning environment *NetGeo* uses the *NetLogo* modeling environment and the TI-Navigator 3.0™ graphing calculator network (Wilensky & Stroup, 1999). In particular, the geometry environment is based on the Netlogo program PANDA (Perimeter and Area) Bear (Unterman & Wilensky, 2006, 2007). In the activity, each student logs a calculator on to the network and then uses arrow keys to control an individual point displayed on a shared computer screen. These points are linked with those of other students in a small group via the network to form polygons; in the activities for the present study a group of four students thus jointly manipulated a quadrilateral. The screen also displays the lengths of the sides and the angles at the vertices (Figure 1).

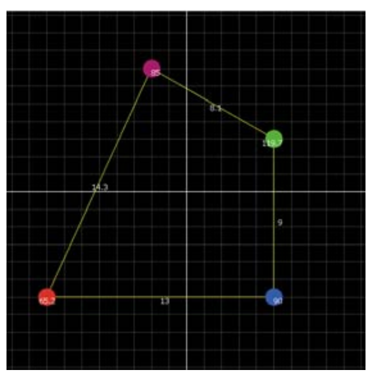

Figure 1. Interface from the *NetGeo* environment

Activities

Instructional sessions (conducted by the first author) spanned six days and included three variations of computer interface and correspondingly different sets of activities (two days each). Days 1 and 2 focused on familiarizing students with the learning environment and the quadrilateral terms and definitions. For these days, the group was given freedom to explore the space and to construct shapes without constraints, and then asked what their definitions were for the various quadrilaterals they constructed. After terms and definitions were clarified if

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necessary, the group continued to work on their construction with the revised definitions. During these two days, students were given a sheet of quadrilateral names (*square*, *rectangle*, *parallelogram*, *rhombus*, *trapezoid*, *kite*, *quadrilateral*) with blank lines so that they could fill out the definition that the group had collectively agreed on.

During Days 3 and 4, the interface was adjusted so that values (angles and lengths) were now displayed on individual students' calculators rather than on the computer screen. In addition, students no longer updated these values by simply moving their points, but instead needed to press a "mark" button for a new quadrilateral to be formed, and for the values to be correspondingly updated on the calculators. A sample task for these days was to ask students to first construct a certain shape—for example, a parallelogram. The researcher then "stamped" the quadrilateral on to the screen, and the group was asked to use the least number of steps to make a square. In the case in which a shape is a special case of another shape, no movement would be necessary. For example, if the group had a square and were asked to construct a rectangle, no movement would be necessary and the least number of steps would 0. This task involved more planning, and the goals of this activity were for the students to 1) focus on the values of the shape's features as displayed on the calculators and 2) begin to notice hierarchical relationships. During these days, the students were also given a sheet with all seven quadrilateral shapes and asked to draw arrows between shapes, in which one shape is "a special case" of another shape.

During Days 5 and 6, the group no longer had information regarding the length and angle values. Instead, there were monitors in the computer environment that indicated certain properties (essential and non-essential) of the quadrilateral: the number of 90 degree angles, number of sets of parallel lines, number of congruent opposite sides and number of congruent adjacent sides. The students were also provided a chart with the names of quadrilaterals displayed by row and the above properties by column, and the students were asked to fill out the chart and leave cells empty if the certain property was not essential to the given shape.

Methods

All sessions were videotaped, and all computer screen states recorded as a video file. Videos of all sessions were transcribed and analyzed with regard to the ways student solution strategies corresponded to different levels in the van Hiele framework. Students were also interviewed before and after the six-day sequence of activities. During the interviews, students were given various cut-out shapes and asked to lay them on a paper, identify them and draw arrows between shapes in which they thought one shape was a special case of another (similar to Jones, 2000). After the sorting quadrilateral task, we asked the students, 1) What is a square? 2) Is a square a special case of a rectangle? 3) What is a rectangle? 4) Is a rectangle a special case of a parallelogram? 5) What is a rhombus? 6) Is a rhombus a special case of a trapezoid?

Results

Pre- and Post-Instruction Interviews

Table 1 summarizes results from the student interviews conducted before and after the instructional sequence. Out of a total of 24 pre-interview responses among the four students, only 10 responses pertained to important features of the shapes and thus would be categorized as van Hiele level 2 or above. In the remaining instances, students typically responded by referring to the shapes in a visual way. For example, for question 2, one student Alan answered, "Yes, because a square can be a rectangle, but a rectangle can not…wait a minute, be a square. It's too big." Another student Brad answered question 4 as, "It is part of a parallelogram because of the shape of it, and how their length and width. And parallelogram is just cut off a little bit on the sides."

By the post-interviews, only one response would be categorized as van Hiele level 1, whereas 21 responses referred to the attributes or properties of the shapes. For example, the same student Alan from above answered question 2, "Yes, cause square has all the properties of a rectangle but a rectangle does not have all the properties of a square." During the post-interview, Brad answered question 4 as, "Yes, because a parallelogram has…a rectangle has four right angles, but parallelogram doesn't have four right angles."

Table 1. Percentage of student interview responses pertaining to appropriate features or properties of shapes (van Hiele level 2)

	Q1	Q2	Q3	Q4	Q5	Q6	Total
Pre	4 (100%)	2 (50%)	2 (50%)	1 (25%)	1 (25%)	0 (0%)	10 (42%)
Post	4 (100%)	4 (100%)	4 (100%)	4 (100%)	3 (75%)	2 (50%)	21 (88%)

Below, we present a series of episodes selected from the activity days to illustrate the different type of activities and to examine how instances of student reasoning about these tasks in this environment corresponded to the van Hiele levels.

Episode 1: Attending to the Shape as a Whole (Day 1: Create a Rectangle task)

This episode occurred during the first day of the activities. The students were able to complete the previous task of creating any rectangle by creating a prototypical rectangle with horizontal sides. The group was then given a task to create a rectangle such that none of the sides were horizontal. The following transcript presents the students' dialogue a few minutes into the task.

1. Alan: You both [referring to Dan and Matt] got to make a 2, 8 length. Hurry up go.
2. [The students move their points around.]
3. Brad: I see what he's doing…
4. Alan: Move Matt, it's you.
5. Matt: Left, right, up, down?
6. Alan: Move over here. Up.
7. Matt: There, there. [referring to Figure 2]
8. Interviewer: Is it a rectangle?
9. Alan: 9.8, 9.8, 2.2, 2.2.. why can't we just make a square?
10. Interviewer: What's the definition of a rectangle again?
11. Brad: Four 90 degree angle.
12. Interviewer: So Dan, is this a rectangle?
13. Dan: [shaking his head] Uh uh..

The above episode shows how the group originally relied on their prototypical image of a rectangle and did not attend to the features–a characteristic of van Hiele level 1 thinking. Alan thought that they had accomplished the task (line 7), but he did not attend to the fact that the angles did not display 90 degrees. After the interviewer asked, "Is it a rectangle?" (line 8), Alan noticed the values of the lengths ("9.8, 9.8, 2.2, 2.2") but still did not properly attend to the values of the angles. The researcher then brought the students back to the definition of a rectangle, which Brad was able to correctly say required four 90 degree angles (line 11). Dan was then able to recognize that they had not created a rectangle. This episode shows how the group originally saw the shape as a gestalt whole, and after some prompting, began to notice the features. Throughout consequent activities, the students began to rely more on the values displayed on the computer screen to complete the tasks.

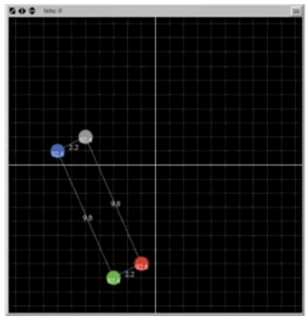


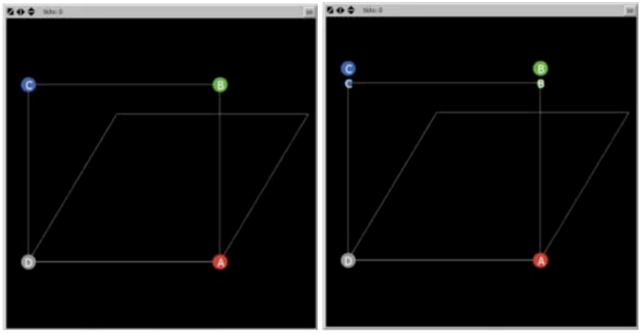
Figure 2. Shape created during episode 1

Episode 2: *Attending to the Features (Day 3: Changing Parallelogram to a Square)*

The following episode occurred during the third session. The students were first asked to create a parallelogram. The researcher then stamped the students' shape on the computer screen, and in the following task the students were asked to create a square using as few moves as possible.

1. Alan: No no…don't. Only those two move [pointing to Brad's and Dan's points—upper points C and D in Figures 3 and 4]. Him and him move a little bit here. He moves over here.
2. Matt: Guys, move over two.
3. [The students move their points around.]
4. Matt: Yeah, buddy. [referring to Figure 3]
5. Alan: Nah, still not the same.
6. Interviewer: How do you make sure it's equal length?
7. Alan: [looking at calculator screen] It's 12, 13.
8. Brad: 12, 13.
9. Matt: 90 degrees.
10. Dan: 90 degree angle.
11. Alan: No, no…but the length aren't the same.
12. Dan: We should've all gone to the corner.
13. Alan: Go one more time up. [referring to Dan and Brad], Him and him. [Figure 4]

The design of Day 3 and 4's computer environment and activities was intended to encourage students to start paying attention to the values of the quadrilateral (angles and lengths) as seen on the individual calculators rather than the computer screen. In this excerpt, after some initial discussion, the students agreed that only two students (Brad and Dan, the ones on the top) should move over to the left. After Brad and Dan had moved to the left and aligned with the bottom points, Matt initially thought that they had completed the task (line 4). Alan disagreed and after the researcher asked, "How do you make sure it's equal length?" (line 6), Alan attended to the values of the lengths as displayed on his calculator and realized that one side was length 12 and the other side was length 13 (line 7). The other group members also chimed in and noticed the lengths and also that the angles were 90 degrees (lines 8-10). This episode highlighted how the group members began to attend to the features of the shapes, which is characteristic of van Hiele level 2 thinking. This progression from simply visually inspecting the shapes in the first day of activities to properly noticing the features of the shapes in Day 3 was part of the design of the computer interface and instructional activities to help students to progress along the van Hiele levels.



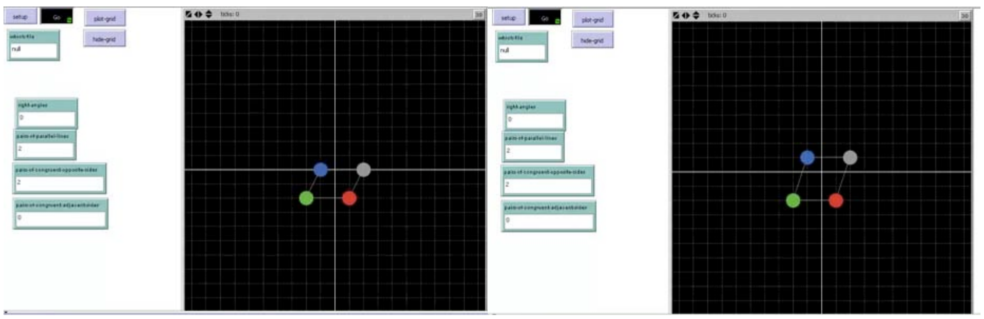
Figures 3 and 4: Quadrilaterals create during episode 2

Episode 3: Beginning to Notice Hierarchical Relationships (Day 6: Creating a Rhombus)

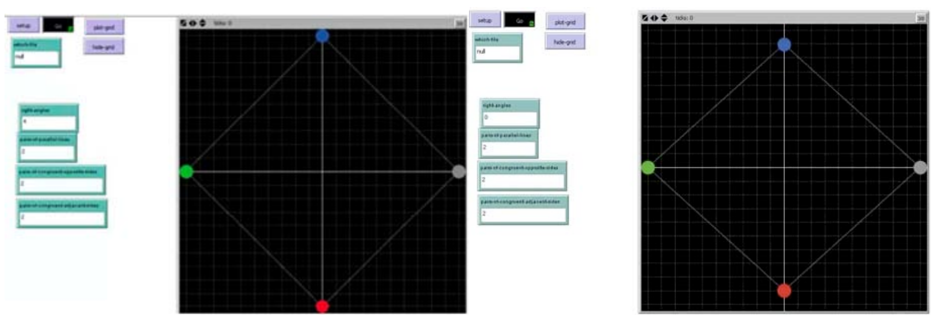
The following episode occurred on the last day of the activities. In the previous task, the group was asked to create a rhombus, which they successfully accomplished by simply creating a square. The researcher followed up by asking the group to create a rhombus that is not a square.

1. Alan: Yeah yeah, okay Dan…Brad, you move to the other side…Here rhombus [referring to Figure 5], but it's not exactly a rhombus I know.
2. Matt: [focusing on monitor on "congruent adjacent sides"] No, because a rhombus has a pair of congruent…2 congruent sides.
3. Alan: No no…Yeah, but it has all the same side.
4. Brad: [referring to the upper points] Matt and Dan, go up. There we go.
5. Matt: It's a rhombus. Check it out. [referring to Figure 6]
6. Alan: It is, it is, because it has the thing…oh no…
7. Matt: It's a rhombus, it's a rhombus…
8. Alan: [noticing monitor] It can not be…It doesn't have two pairs of adjacent sides.
9. [The students continue to talk and move their points around.]
10. Matt: Let's just make a diamond, remember?
11. Alan: Oh wait a minute, let's make the diamond you know from the corners.
12. [The students continue to direct each other and eventually end up with Figure 7]
13. Brad: But it has 4 right angles.
14. Alan: No, wait a minute. Wait a minute. We can make it not a square. Dan [referring to top point], you go down, I [referring to his lower point] go up. Not a square anymore. It still has adjacent sides. We did it. Now it does not have right angles. [Figure 8]

In this Day 6 task of creating a rhombus that is not a square, the students began to display understanding of hierarchical relationship between rhombus and square. Initially, the group tried to create the prototypical rhombus with sides that are horizontal (Figure 5). After Matt noticed that the monitors did not display 2 pairs of congruent adjacent sides and 2 pairs of congruent opposite sides (which makes it all sides equal), the group tried to create another quadrilateral (Figure 6). After some failed attempts at creating the prototypical rhombus, Matt and Alan remembered that they had previously constructed a diamond that was also a rhombus. After they had constructed the diamond (Figure 7), Brad noticed that there were still four 90 degree angles, so it was still a square. Alan then moved up his point and Dan moved down his point to create a shape that still had four sides equal but no 90 degree angles (Figure 8). This episode shows the group negotiating and demonstrating their knowledge of which features are essential for a rhombus and which are not. This is an important part of beginning to develop their knowledge of hierarchical relationships between shapes, which is characteristic of thinking at van Hiele level 3.



Figures 5 and 6. Quadrilaterals students mistook as rhombi



Figures 7 and 8. Quadrilaterals created during episode 3

Discussion and Further Research

This paper presented a computer environment designed to help students to move from simply viewing shapes as prototypical images, to noticing the shapes' features, to understanding the hierarchical categorization of the various shapes. We presented episodes from the three variations of activities to highlight how the different aspects in the technology drew students' attention to particular aspects of the shapes. In the first variation, the students at times still demonstrated views of the shapes as gestalts but gradually began to notice the features. During the second variation, the students began to pay more attention to the values of the angles and lengths as displayed on their individual calculator screens. The last set of activities helped students to move beyond noticing the individual features to beginning to talk about general properties of the shapes. Although this paper focused on students' interactions with those features of the designed environment that were intended to highlight properties of shapes, further research and analysis of these data will also examine the forms of student collaboration supported by this interactive design.

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ELEMENTARY-LEVEL CONCEPTUALIZATION OF VOLUME USING NUMERIC TOP-VIEW CODING WITHIN A 3D VISUALIZATION DEVELOPMENT PROGRAM

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This paper focuses on a learning trajectory that deals with connections between top-view numeric coding and the concept of volume. The study's Year-1 elementary-grades children initially became proficient at top-view numeric coding of 3D block structures. The following year, the same children unexpectedly applied this knowledge to the concept of volume of 3D rectangular arrays. As a result, a refinement of the learning trajectory brought these connections to younger children during Year 3 of this on-going study.

Introduction

The National Research Council's (NRC) report, *Learning to Think Spatially* (2006), identifies spatial thinking as a significant gap in the K-12 curriculum, which NRC claims is presumed throughout the curriculum but is formally and systematically taught nowhere. This research team intends to fill this gap by developing new and modified curricular activities guided by the Spatial Operational Capacity (SOC) framework developed by van Niekerk (1997) based on Yakimanskaya's (1991) work. Using design-research (Cobb, et al, 2003) principles this study's immediate goal is to describe children's mathematical sense making of these activities. It is conducted in a dual-language urban elementary school within one of the largest public school districts in the mid-southwestern United States. This paper focuses on the refinement of learning trajectories to support children's conceptual understanding of the volume formula for rectangular 3D arrays using top-view numeric coding.

Theoretical Frameworks

Spatial Visualization

The National Council of Teachers of Mathematics' Principles and Standards for School Mathematics (NCTM, 2000) prescribes that in their early years of schooling, students should develop visualization skills through hands-on experiences with a variety of geometric objects and use technology to dynamically transform simulations of 2D and 3D objects. Later, they should analyze and draw perspective views, count component parts, and describe attributes that cannot be seen but can be inferred. Students need to learn to physically and mentally transform objects in systematic ways as they develop spatial knowledge. From a purely academic perspective, the importance of visual processing has been documented by researchers who have examined students' performance in higher-level mathematics. For example, Tall, et al (2001) found that to be successful in abstract axiomatic mathematics, students should be proficient in both symbolic and visual cognition; Dreyfus (1991) calls for integration across algebraic, visual and verbal abilities; and, Presmeg (1992) believes that imagistic processing is an essential component in one's development of abstraction and generalization.

Spatial Operational Capacity Framework

The spatial operation capacity (SOC) framework (Yakimanskaya, 1991; van Niekerk, 1997; Sack & van Niekerk, 2009) that guides this study exposes children to activities that require them

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to act on a variety of physical and mental objects and transformations to develop the skills necessary for solving spatial problems. The SOC model (see Figure 1) uses:

full-scale figures, that, in this study, are created from loose cubes or Soma figures, made from 27 unit cubes glued together in different 3- or 4-cube arrangements (see Figure 2);

conventional graphic 2D pictures that resemble the 3D figures;

semiotic representations (Freudenthal, 1991) such as front, top and side views or numeric top-view codings that do not obviously resemble the 3D figures; and,

verbal descriptions using appropriate mathematical language (Sack & Vazquez, 2008).

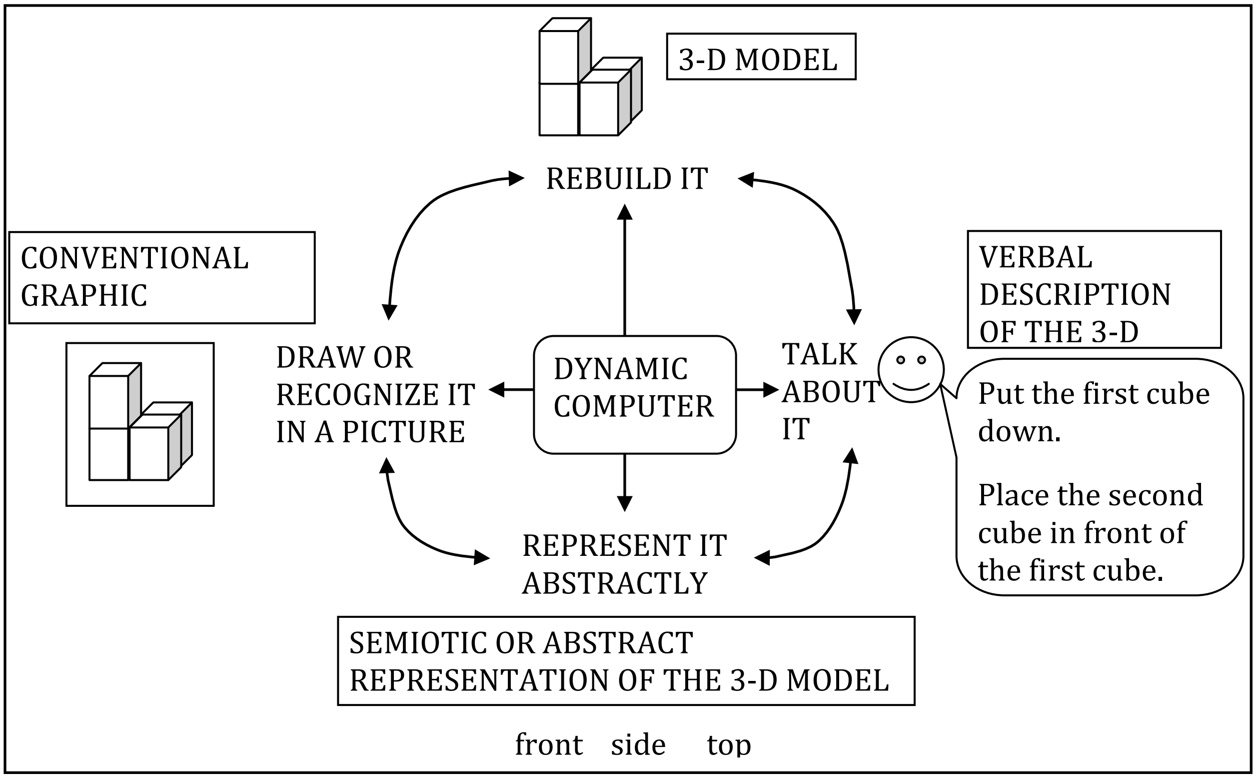


Figure 1. Multiple representations within 3-D visualization

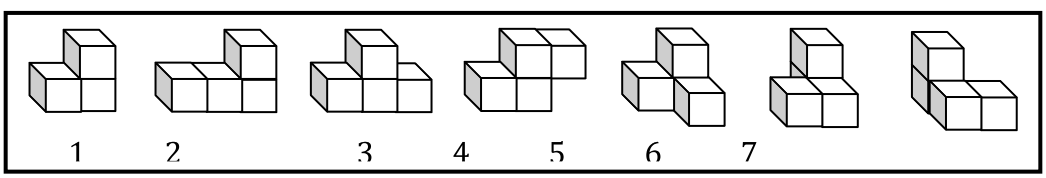


Figure 2. The Soma set can be made by gluing unit cubes together

The study utilizes a dynamic computer interface, Geocadabra (Lecluse, 2005), a tool that was not available when the SOC framework was originally formulated. Through its Construction Box module, complex, multi-cube structures can be viewed as 2-D conventional representations or as top, side and front views or numeric top-view grid codings (see Figure 3). Whereas one can move around a 3-D model to see it from other vantage points, one may see various views of a dynamic computer-generated figure through its ability to be rotated in real time.

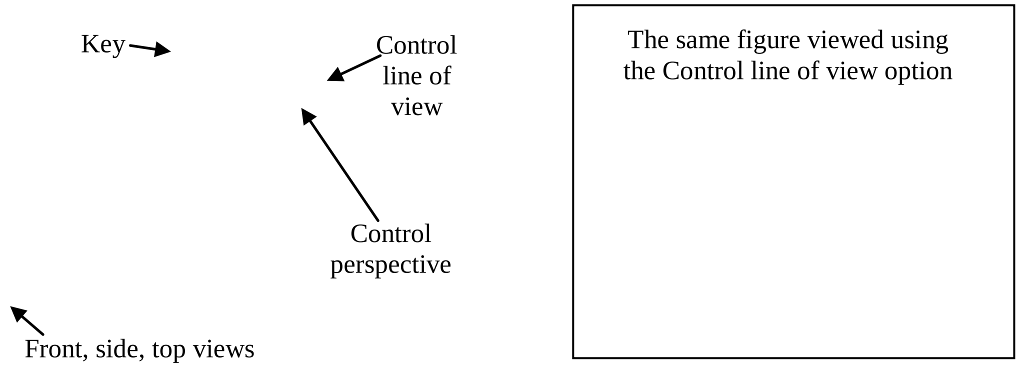


Figure 3. The Geocadabra Construction Box

Methodology and context

The design research methodology (Cobb, et al, 2003) guiding this study's instructional decisions is based on learning trajectories developed from an instrumentalist standpoint (Baroody, et al, 2004). This conceptual and problem-solving approach aims for "mastery of basic skills, conceptual learning, and mathematical thinking" (p. 228). Within this approach, "teachers are concerned about students' understanding and promote the use of any relatively efficient and effective procedure as opposed to a predetermined or standard one" (p. 228). Each lesson is part of a design experiment followed by a retrospective analysis in which the research team determines the actual outcomes and then plans the next lesson. This may be an iteration of the last lesson to improve the outcomes, a rejection of the last lesson if it failed to produce adequate progress toward the desired outcomes, or a change in direction if unexpected, but interesting, outcomes arose that are deemed worthy of more attention.

During the 2007-2008 academic year, a university-based researcher and two teacher-researchers formed the research team that worked with a 3rd-grade and then a 4th-grade group of children weekly (one hour per group) in teacher-researcher, Vazquez' 3rd-grade classroom within the school's existing after-school program. English and Spanish parent/guardian and student consent-to-participate forms were sent home to parents of all after-school 3rd- and 4th-graders. All respondents were accepted into the program. The research team adopted social-constructivist instructional approaches, supporting a problem-solving environment that fostered students' creativity according to readiness and interest. To make sense of student understanding of 3D structures the research team used the following data sources: formal and informal interviews, video-recordings and transcriptions, field notes, student products and lesson notes.

Results

During introductory lessons the children interact with loose cubes and the Soma figures solving problems with the 3D models and with 2D task cards that illustrate combinations of the Soma figures requiring figure identification and classification. By the middle of the second month, the children begin to use the Geocadabra Construction Box to digitally reproduce figures printed in a customized manual as shown in Figure 4. These activities provide the children opportunities to coordinate numeric top-view codings with 2D conventional pictures. Children who need scaffolding may first build the 3D figures using loose wooden cubes.

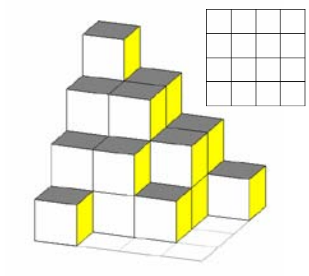


Figure 4. Sample Tasks

During Years 1 and 2, in line with their constructivist leanings, the research team's instructional planning was guided by the children's interest in top-view coding. Using Geocadabra, the children created their own 2D task-cards of structures they had assembled with two Soma figures. Using these task cards, they drew coding puzzles for peers to decode and check. Later, they developed their own ways of coding Soma assemblies that had holes or overhangs and negotiated a non-conventional coding system for the whole class to use (Sack & Vazquez, 2009). The children became very proficient at moving among the 2D conventional pictures, top-view numerical coding and 3D digital dynamic representations on Geocadabra.

During Year 2, an unexpected and interesting event occurred when five Year-1 children returned. They were struggling with the concept of rectangular volume in their regular academic classes where they were required to use the formula. Using an initial learning trajectory based on the work of Battista (1999), the children attempted to solve volume problems by folding nets drawn on grid paper. They struggled to connect the dimensions of the flaps of each net with the height of its corresponding 3D figure. Within the study's problem-solving environment, using a contextual scenario, "Ms. Moral's Shoes," 24 shoeboxes must be shipped to a nearby city. The children, using loose wooden cubes to model the shoeboxes, were expected to find all possible combinations of rectangular arrays with 24-cubic-unit volumes. The research team was surprised to see them record their findings as numeric top-view codings rather than directly with the length-width-height formula. Connections between top-view coding and the volume formula evolved through guided discussion among the teacher and participant children. Figure 5 shows how the children recorded their work.

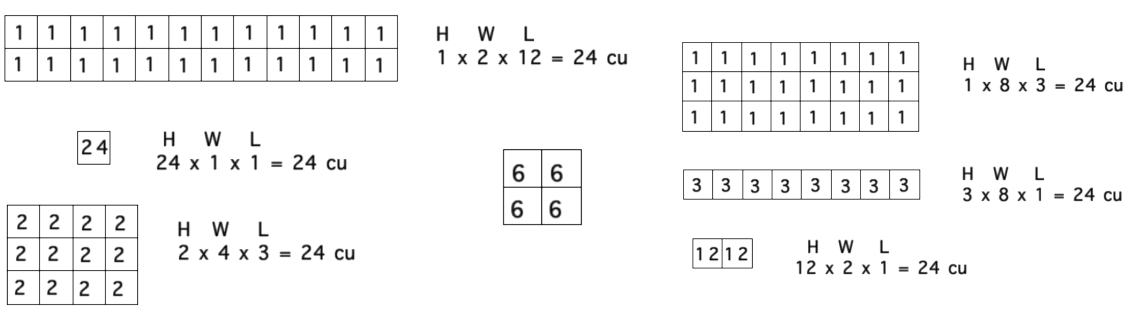


Figure 5. Recordings of 24-cube prisms

Excerpts from field notes, November 5, 2008 (All of the children's names in the following descriptive sections have been changed to protect identity).

Erin built a prism and then proceeded to record it as a top-view coding. She also wrote in the formula and substituted measurements.

She then created other permutations of the same prism by standing it on different faces. [An example of a permutation is shown to the right.]

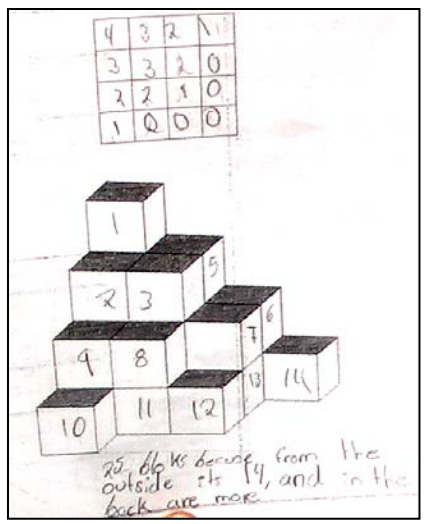
Following Erin's lead, Dani created a 3 by 4 by 2 prism and drew her top-view coding with its associated formula.

This occurrence led to a new Year-3 learning trajectory, more closely aligned to the elementary academic curriculum, to develop pre-formula volume conceptualization for a new group of third-grade children. The objectives were 1) to coordinate the number of cubes in a 2D conventional picture with the sum of the numbers in its numeric top-view coding; and, 2) to discover and integrate the numeric top-view coding representation with the volume formula for 3D rectangular arrays using and reinforcing emergent multiplication skills.

During the children's initial work with the computer and the Geocadabra manual, they were individually challenged to determine how many unit cubes made up the figure in Task 1*f* (see Figure 4). After the research team noticed wide discrepancies in children's interpretations, they asked Evan to use Geocadabra's Construction Box to build the figure in Task 1*f*, which was projected to enable whole-class discussion.

Excerpts from field notes, November 2, 2009:

[The teacher] asked the class how many cubes were in the figure. Most said 24, but some said 25, 13 and 11. The whole class was invited to explain what was going on. Those who said 24 seemed to add the numbers in the [numeric top-view] grid. . . . They pointed out that the top row (of the grid) had 4 + 3 + 2 + 1 = 10 cubes, and then 8 [3 + 3 +2] in front of that [row] and so on until they determined 24 cubes in all. The child who said 25 had enumerated incorrectly, but was able to explain where the 13 came from. He labeled his picture [shown to the left] as follows: 1, 2, 3, 5, 6, 7, . . .(he skipped #4 and got 14 instead of 13).



Of note: He clearly counts each cube rather than visible faces. This is an exceptional step that may be supported by the fact that they have used the actual 3D cubes for the past 6 lessons in conjunction with 2D pictures.

Several children said that 13 came from looking at only the visible cubes in the picture and that there are several more hidden behind and/or below what is visible.

Evan had said 11 and also 24. He showed how he obtained 11 by counting only the grid squares that had non-zero numbers in them [i.e., he counted squares rather than the number of cubes in the squares]. Alan said that 11 came from counting only the cubes with black (top) faces in the picture.

The research team was surprised that none of the children had declared 30 or 20 cubes, based on counting all visible cube faces or all visible lateral faces, since this was found to be a common counting misconception reported in Battista's study on 3D-array enumeration (1999). This activity demonstrates the importance of whole-class discussion. The children were able to hear various explanations dealing with their understanding of the 2D conventional picture and the numeric top-view representation. Those who counted only visible cubes or non-zero grid positions were convinced that there were more cubes when Evan rotated the figure to show the initially-hidden cubes on the left side and back of the figure, demonstrating the impact of the software's dynamic Control-line-of-view function.

Following this discussion and for the next lesson, the children worked at their own computer stations to create figures that had exactly 24 cubes. This activity required coordination of the numbers in the numeric top-view Construction Box grid with the 2D figure each had created on the screen. Some used the Control-line-of-view option to rotate the screen figure to verify the 24-cube sum. These figures were formatted into task cards and were used in the subsequent lesson to reinforce their coordination of the numeric top-view grid with the 2D picture. Each child drew a grid on blank paper and, using a task card with someone else's 2D figure, determined its numeric coding. Then, using only the written coding, each child built the figure on the computer to verify his or her numeric solution. Examples of children's work are shown in Figure 6.

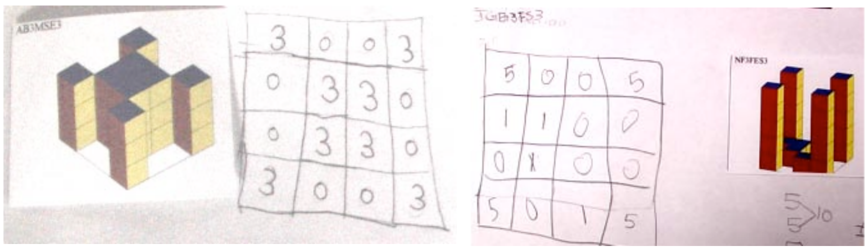


Figure 6. Children's numeric top-view coding of peers' task cards

To address the second objective (integrating top-view coding with the volume formula), the teacher presented a 12-cube rectangular array. Children volunteered to demonstrate how to code different orientations of this model. Then, the class was asked to build a rectangular array using 24 loose cubes. After successfully building and coding one 24-cube array, they were challenged to find as many different 24-cube arrays as possible. Figure 7 shows examples of two solutions that students presented to the class.

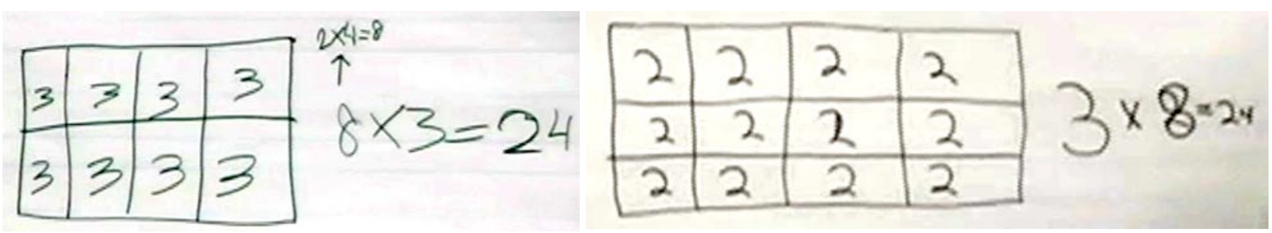


Figure 7. Children's representations and enumeration of 3D arrays

In the following transcript (November 30, 2009), Jose explains the meaning of the 3s in the first example in Figure 7. This interaction was transcribed from a video-recording of a whole-class discussion that occurred when the children were shown the video-clip that was recorded the previous week.

Teacher: What does the 2 mean [referring to the 2 x 4 = 8 above the 8 x 3 expression]?
Evan: The 2 I think are the two lines [unclear] right over there [pointing at the 2-by-4 array projected on the board].
Teacher: Two rows. Okay, and what is the 4?
Evan: The 4 is the groups.
Teacher: Okay, there are 4… Four what?
Evan: Four squares.
Teacher: Four squares in each …?

Evan: Each row.
Teacher: Okay, so that's a 2-by-4 [array]. Do you all agree that that's 8?
Class: Yes.
Teacher: And why do you have … times 3?
Jose: Because there's 3 in each … There's 3 in each square.
Teacher:Three what?
Jose: Three cubes [gesturing layers].
Teacher: In each square … stacked up.

Jose's gesturing indicated that he had formed a mental image of the three layers in the array. Holding his hand flat toward and parallel to the table top, he lifts it up three times to show three stacks or layers.

The children continued to find other combinations of 24-cube 3D arrays including their permutations as shown in Figure 8.

Figure 8. 2X3X4 arrays

Conclusions

Activities designed to help third-grade children move fluently to-and-fro among three visual SOC representations, namely, 3D cube models, their 2D pictures and their more abstract numeric top-view codings, facilitated their conceptualization of the volume formula for rectangular prisms. The initial coding activities that involved non-specific cube structures (e.g., Figure 4) were refined during Year 3 of the study to include a focus on "how many?" Through group discussion, the children described different ways of enumerating the cubes in these figures. In one method, they added the numbers in the top view coding. In others, they used their visual skills on the 2D pictures, by cumulating the number of cubes in each layer or in each stack of cubes. They justified the result by use of an alternate approach. Later, they were able to apply the "how many?" concept to rectangular arrays, fostering their understanding of the volume formula in concert with their emerging multiplication skills (e.g., Figure 8). The 24-cube problem also connected to their emerging understanding of combinations and permutations through concrete (3D) and abstract (semiotic) representations.

The results of this study reinforce other researchers' findings about the importance of imagistic development (e.g., Dreyfus, 1991; Presmeg, 1992; Tall, et al, 2001). Outhred, et al, (2003), referring to the complexity of representing volume measurement compared to rectangular area measurement, state that "the process is more complex because students have to coordinate three dimensions and diagrams cannot show the layer structure clearly" (p, 84). That these young children in this study were easily able to transfer their knowledge of top-view coding to representation and to enumeration of 3D arrays is significant. Using mental imagery, they explain that each number in the rectangular grid represents the height of a stack of cubes on a space, a measurement rather than a numeracy construct. Battista (1999) reports a common miscounting error when children enumerate cubes in arrays shown in conventional 2D pictures. He reports that they may count the same cube three times if it is located on a vertex or twice if located on an edge based on the number of visible faces. When presented figures, such as that in Task 1*f* (Figure 4), none of the children in this study counted all visible faces when asked to find the total number of cubes. This result shows that this misconception may be averted, believed to be due to the Year-3 children's prior experiences with identification and reproduction activities requiring them to move to-and-fro among the SOC visual representations.

Finally, the after-school environment for this project carries particular advantages in that academic day constraints are not imposed on the instructors or children. Accountability testing at the district, state and federal levels has eroded instructional time with additional mandated test preparation and testing across all grade levels in the public school system. This project has enabled the children to experience authentic mathematics learning outside of the testing shroud.

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NAVIGATING GEOMETRIES WITH DIFFERENT METRICS: COLLEGE STUDENTS' UNDERSTANDING OF TAXICAB-GEOMETRY

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Four college students' understanding of taxicab geometry is examined. Written responses to eight questions on various aspects of Taxicab-Geometry (T-geometry) and their verbal explanations in interviews are analyzed. A proof comprehension and explanation question concludes the sessions. The students had a semester of college geometry that included a significant study of T-geometry including T-circles, T- bisectors of segments, isosceles and equilateral triangles, T-ellipses, and T-parabolas with the taxicab metric. Results indicate that the learning of T-Geometry is by no means a straightforward process where students are able to move effortlessly from Euclidean to T-geometry. Congruence, similarity, distance between (parallel) lines, the interpretation of the metric, (Euclidean) constructions of T-circles and T-angles all pose problems of varying degrees to students.

Introduction

In taxicab-geometry (to be designated as T-Geometry or T-G) points, lines and other objects are embedded in the Cartesian (coordinate) system. The metric between two points is the sum of the absolute values of the differences between the x and y coordinates of the two points. The parallel postulate is still valid and angles are constructed and measured as in Euclidean geometry.

In this study we examine how four mature undergraduate students (mathematics) understand taxicab geometry. More specifically, we investigate: how do these students understand the unfolding of the metric in terms of circles, isosceles triangles, congruence, similarity and the use of circles in geometry to solve problems? Do students distinguish between the concepts of lines, segments between points and distance between points? Are students aware of the analogy between solutions as given in Euclidean geometry and similar problems in T-geometry? Do students understand what methods from Euclidean geometry can be used to find T-geometry solutions?

Data collection and analysis processes center around the following three board questions:

1. How do participants use their knowledge to study T-geometry?
2. How do participants understand T-geometry?
3. What are the main problems the participants exhibit in their solutions to specific assigned problems?

Theoretical Framework

The basic perspective for this study is the model for the development of Euclidean geometric knowledge formulated by the Van Hiele couple since 1958 (van Hiele & van Hiele-Geldof, 1958; 1986). In this model Euclidean geometric knowledge generally grows from one qualitatively different level to the next through five (counting from zero) discernible stages. We are especially interested in the third and fourth levels: Level Three, informal deduction, is characterized by an awareness in the student of the role of definitions, the relations between figures and an ability to deduce claims and facts from given properties and known facts. Level four is the level of axiomatic deduction where a student understands the way theorems are proved from definitions, axioms and other theorems (Flores, 1993). The students in our study seem to be able to reason with definitions and properties of objects. In the van Hiele framework they should be at level three at least. We hypothesize that some of the students are still mainly at this level. Moreover, we think that when alternative geometries like taxicab geometry is studied that a new branch (or domain) of knowledge is established that may not fit the formal levels of van Hiele yet, but requires an autonomous path of development from level three or within level three.

The next perspective for understanding the growth of knowledge of T-geometry is the model of dynamical growth of mathematical understanding proposed by Pirie and Kieren (1994). In this model mathematical knowledge is proposed to go through several levels or modes of understanding starting with "primitive" knowing (which can be any existing knowledge). The model has an "image making" mode, an "image having" mode, a property noticing mode (or level), a "formalizing", mode, an "observing" mode, a "structuring" mode, and finally an "inventising" mode. For the finer details of the model we refer to the article referenced below.

What is of importance here is the emphasis of this model on image making, image having, property noticing and the idea that there is a constant *restructuring of previous existing knowledge*. This process can take the form of *folding back* in which one tries to understand some mathematics by going back to an earlier area of knowledge. This action of going back is motivated by a desire to understand some mathematical problem or concept from another, higher level than the one folded back to.

With the van Hiele levels we will have a framework for understanding the Euclidean geometry type of knowledge we assume to be present in our students, while the Pirie –Kieren model of dynamic growth provided us with a framework for understanding how the students dealt with the new T-geometry they were studying, and why certain events and problems were observed in their work. The combination, the shift from the first to the second is necessary to explain what the effects were of the original Euclidean knowledge on the students' success or struggle to understand the theorems of T-geometry that departs considerably from the Euclidean original.

Research Context and Methods

Four mature students (a mathematics BS close to graduation, a teacher completing requirements for mathematics education, a graduate mathematics student, and a graduate student in mathematics education) were interviewed for this study. They were part of a larger group of thirteen students who took a college geometry class at a southern university in the USA in 2009. Ten students participated in a study to investigate the nature of their understanding of the T-geometry part of the course in College geometry. This study reports on four students from that group.

T-geometry was studied for seven weeks in the course. Content covered in that part included T-circles, the T-bisector of an arbitrary line segment in T-geometry, isosceles triangles in T-geometry, some congruence relations, T- ellipses, T-parabolas, angles in T-geometry, and parallel lines . The group of four studied in this paper are the first group and in the future a paper will be presented on the findings of the whole group of ten students. The students were selected on the basis of their willingness to come back to campus after their final examination to work on a problem solving task followed by an interview with one of the researchers who was also their professor. The students will be designated as Alvin, Louisa, Sandra, and Richard (all names are pseudonyms).

Table 1. Results Written Part of Study

	ALVIN	*LOUISA*	*SANDRA*	*RICHARD*
Q1: What are the main differences between Euclidean and T-Geometry?	Metric, Circles, ellipse, parabola different. Constructions different	Metric depends on orientation. Definitions of objects same as in E-G	Metric changes shape of circle	Congruence is different, and shapes of circle, ellipse, parabola different
Q2: What is the shortest line segment that connects A to B?	Horizontal and vertical segments together	Distance (from A to B) is 24 unit	Line segment from A to B. But unsure of response.	Line connecting A and B
Q3: How can we construct isosceles triangles in T-Geometry?	Correct construction using T-circles: center connected to points on perimeter	Triangle with horizontal base, using segment bisector; no consideration of oriented base	Attempt with diagonal base; then horizontal base with Euclidean bisector	Isosceles triangle with Horizontal base
Q4: Parallel postulate in T-Geometry: How many parallel lines can we draw through point P?	Metric does not affect parallel properties. Just one line!	One unique parallel line, bec. rise over run is independent of metric.	Rise over run does not change so only one line.	There is only one line
Q5: How would you define rectangles and squares in T-Geometry?	Identical to E-G definitions	Same as E-G, but no clear image of what rectangle looks like in T-G	Identical to E-G definitions	Identical to E-G definitions
Q6: How would you define congruent and similar triangles in T-Geometry	Exact copies of triangles, taking orientation into account	All sides need to be the same	All sides and all angles should stay the same	Exact copies of triangles, taking orientation into account
Q7: How would you define the distance beween parallel lines in T-Geometry. Shortest distance between par . lines	Vertical or horizontal segments	Vertical and horizontal distances should be equal	T-length of perpendicular segment between lines	T-length of perpendicular segment between lines
Q8: How can we construct an angle bisector in T-geometry	Constructed Euclidean bisector, but wondered if a construction was possible without using Euclidean circles, based only on T-circles and lines	Tried to use midpoint of opposite side to construct angle bisector. It worked for right angles.	Tried to use T-circles to construct angle bisector, analogous to the tangent circles in E-G.	Euclidean construction with Euclidean circles
Proof comprehension: E-squares and T-squares are equivalent. How to move from E-metric to T-metric and back using similarity of triangles	Explained the steps of the proof correctly and identified the main argument in similarity reasoning to compare metrics	Understood first steps of arguments. Did not know how similarity worked in the two metrics	Understood how ratio of T-length and E-length of a rectangle are reproduced through similarity of right triangles	Understood that ratio between T-length and E-length is preserved through similarity of right triangles

Method of Inquiry

The method of inquiry was twofold. First the students were presented with eight questions on T-geometry ranging from introductory to more conceptual questions. The students wrote their responses and solutions on the worksheets provided. After that they sat for a one on one interview with the researcher where they explained how they had responded to the questions and why they solved problems a certain way or not. These interviews lasted an average of 1.5 hours. All interviews were audio-taped and transcribed later. Data consisted of the worksheets on which the students had written their work and the transcribed interviews with the four students. The transcribed interviews were analyzed and coded per question on the basis of the type of student thinking that was evident, and the forms of conceptual obstacles that were identified. A qualitative analysis of the worksheets, and interviews, was conducted and an explanation of the students' understanding was formulated using the Pirie-Kieren model.

Questions

The questions posed to the students were :(1) What are the main differences between Euclidean and T-geometry? (2) Suppose I have a point A and a point B in the T-plane. What is the shortest line segment that connects A to B? (3) How can we construct (Using Euclidean tools) isosceles triangles in T-geometry? (4)Two lines in T-geometry are parallel if they do not intersect. Suppose we have a line *m* and a point *P* not on *m*. How many lines can we draw through a point P (not on the line) parallel to the given line *m*? (5) How would you define rectangles and squares in T-geometry? (6)How would you define congruent and similar triangles in T-geometry? (7) How would you define the distance between parallel lines in T-geometry? (8) How can we construct an angle bisector in T-geometry? This series of questions was then followed by a proof comprehension question (Conradie & Frith, 2000) dealing with a proof that T-squares and Euclidean squares are identical. The main argument of the proof was that every Euclidean rectangle that is not a square is always a T-rectangle that cannot be a square in T-geometry. The students were asked to explain how the main arguments in the proof were grounded and if they were valid.

Results of the Analysis

First we will present the results from the written part of the study in Table 1. The first column of Table 1 represents the problems from the questionnaire. Subsequent columns contain the students' key responses for each question in brief format. Next we will review the results from the interviews where the students explain their written work. In the discussion of research questions we present an analysis of the data.

The Students' Explanations From The Interviews

On question two Alvin, Louisa and Sandra mentioned in their explanations that they were confused on this question. After connecting A and B Sandra said: "…so that was a little tricky." (…indicates here deleted dialogue; "I" stand for the interviewer)

I: "OK. And the trick was what?" Sandra: "In my head!"

I: "In your head? In what sense?" S: "The trick was in my head! Well, because I have to remember that they're the same thing. Going diagonally from A to B is the distance from what you call that horizontally, C- point". (I: She is pointing now to the 90 degree angle, where the horizontal and vertical parts meet). S: "Right. Those---- (pause) it's the same, either going diagonally or if I go across and then up". The student seemed to hesitate between thinking of the segment as a line concept and the segment as a distance related concept. Segments connect points based on the axioms of what lines are. Horizontal and vertical segments are part of the procedure to determine distance between points. The words of the student suggest that the difference is not fully interiorized.

The comments from Alvin on congruence in question six were similar to his written work. Sandra however said: "Congruent! I said since the angle measures were the same, you see, now you blew me out of the water by telling me I can’t do Side-Angle-Side and all that (laughs). Mainly I said that they had the same angle measures; and I can prove that a side has the same distance". After the interviewer tried to focus on the definition of congruence, Sandra continues: "Well, I was kind of thinking proving them by Side-Angle–Side. Or: Angle-Side-Angle, the basic ways of congruency." When reminded again that she was at the theorem stage, but the definition stage was still not completed, she stated: "OK. So I don’t really know about congruency."

Louisa first hesitated how to explain congruence but eventually claimed that sides and angles need to stay equal for congruence to exist. To get this insight the interviewer made her cut a triangle from the worksheet and place the paper copy in a different position on the grid system. She then states "I mean, if somebody shows me how sides have remained the same but they look different, then I would have expected, then I could understand that and agree with that, but I can’t seem to visualize or construct that myself."

As to similarity in question six, Alvin, Sandra and Louisa all stated initially that proportionality of corresponding sides must go with similar triangles. Only after one or several hints did they realize that proportionality of corresponding sides is a Euclidean consequence of similarity, not a basic general property. In the proof comprehension question Alvin and Richard showed a good understanding of the gist of the proof, but Sandra and Louisa both admitted not to understand the crucial part of the proof. The part where the proof moves from showing that two unequal T-distances lead to a claim about two unequal Euclidean distances, using similar triangles, confused them. They didn’t understand the connection with the initial question of why E- squares and T-squares are identical.

Discussion of Research Questions

How Do Students Use Their Knowledge To Study T-Geometry?

The main source for the notions of the students seems to be what they know from concepts in Euclidean geometry. The Euclidean concepts are adapted to the conditions of T-geometry, modifying distance considerations to accommodate the T-metric. Students seem to use a naïve form of transfer to navigate from Euclidean formats to T-geometry. In the case of the shortest segment between points A and B we see a superficial interpretation of the question where the focus shifts from the segment to the distance. In the case of the isosceles triangle construction the segment bisector method seems to be a preferred choice, leading only to a solution if the base is horizontal. The angle bisector methods are transfers from Euclidean geometry based on equal distances to legs that do not guarantee equal angles in T-geometry.

How Do These Students Understand T-geometry?

It seems clear that Euclidean geometry knowledge informs most of their notions on initial T-geometry. They seem to think about T-geometry through the lens of what they already know from earlier studies of Euclidean conditions. This can help or it can hurt their understanding of T-geometry. However all the students are aware that things change in terms of circles, side lengths, congruence, similarity, angles, bisectors etc. They have a focus on definitions to guide them through the new geometry. Congruence and similarity are problematic for two of the four students. Definitions of congruence and similarity and theorems of congruence and similarity are taken as one whole from Euclidean contexts. All four students included proportionality in their concept of similarity of triangles.

To explain these observations we look at both the van Hiele (van Hiele, 1986) and the Pirie Kieren (Pirie & Kieren 1994) models. Both stress the importance of a phase in learning (van Hiele, at level two, the level of logically connecting previously noticed properties, assuming the first level to be level zero, and the Pirie-Kieren level of property noticing in learning concepts) where the learner is able to notice how individual properties of objects relate to each other and to definitions. We assume that the students are relatively versed in Euclidean geometry. These students studied T-Geometry which differs from the Euclidean version in the existing metric but not in axioms like the important parallel postulate etc. Euclidean geometric knowledge is therefore proposed as the Pirie-Kieren domain of Primitive Knowing, meaning that we assume that the person doing the understanding can do Euclidean geometry as their initial basis of knowledge before studying T-Geometry (Pirie & Kieren, 1994). The students need to build their knowledge of T-geometry concept by concept. It seems that this process has to be repeated one by one for circles, for lines, for segments, bisectors (angle and segment bisectors), for parallel lines, for triangles in various positions, distances between lines, and for the whole body of congruence and similarity.

What Are The Problems They Exhibit In Their Solutions To Problems Posed?

One observes that the habit from E-geometry to place diagrams of triangles and lines such that there is usually a horizontal component can impact the students' awareness that in T-geometry, this horizontal orientation amounts to a special case that may not yield a general method of solution for the problem in question. We see this when students try to construct isosceles triangles, where segment bisectors are used with horizontal bases for the initial construction. For the construction of isosceles triangles the students had a choice between a method using T-circles, which was applied by only one student, and a method using generalized segment bisectors, which was not used by any of the students. The segment bisectors in T-geometry were discussed extensively in class, covering all possible orientations of the segment to show the peculiar segment bisectors. The possibility exists that when the question is framed as: 'construct, using Euclidean tools' students are overwhelmed by the question and wonder if *every E-circle must be replaced by a T-circle.* The responses from the interviews suggest that this is the case.

In the Pirie-Kieren model the recursive nature of the learning process is central to the theory. The student will go back repeatedly to their First knowledge, but every time with some added portion of information or some extra mental image or schema. The dynamic *image making and image having processes of learning* in the Pirie-Kieren theory in particular seem to be very relevant in understanding how T-geometry evolves in the mind of the four students. While a student may have a clear idea of how to construct an angle bisector with Euclidean circles, and execute this flawlessly, the same student (Richard) may not understand yet that the T-distance between parallel lines cannot be literally transferred from the E-geometry context by finding the T-distance of the perpendicular Euclidean distance. If the student did not develop her/his own image of the distance between parallel lines in T-geometry, she/he may not have noticed the specific properties involved in the question of identifying the correct distance between parallel lines.

For T-squares we notice a similar gap: no clear image apparently of T-squares. Squares in T-geometry were not represented or understood in their most general T-geometry shape or orientation. From the comments of the interviews it was also clear that students had quite clear ideas about what a definition of rectangles should be in T-geometry, but two students did not see how in the comprehension test for showing that E-squares and T-squares are identical how the ratio between E-length of sides is carried over from E-geometry to T-geometry and back.

Constructing isosceles triangles was not placed in its basic (or logical) context of T-circles by two of the four students. A third student constructed an isosceles triangle Euclidean circles but with a horizontal base. Three of the four students tried to adapt Euclidean methods of construction based on perpendicular bisectors of a segment without sufficient consideration of the specific T-geometry implications.

Geometry is a system of related concepts that need to be understood one by one and in relation to one another. The students in this study had to develop their notions of circles in T-geometry, angle bisectors in T-geometry, segment bisectors, triangle congruence, and then find out how distance and length are related to angles and shapes and what could be transferred from the domain of E-geometry into the domain of T-geometry. In particular, the students needed to develop an understanding of when constructions from E-geometry could be applied or modified into a T-geometry construction. The data suggest that this process was incomplete after the seven weeks of study, for most of the objects and concepts mentioned.

Generalizations in mathematics do not occur without reflection. What we observe in the students’ work with T-geometry in relation to their knowledge of E-geometry is consistent with the findings that a systematic study of various different sources of knowledge may be a precondition for successful transfer of concepts from one domain to the next (Wagner, 2006). This seems to apply even when the analogy between systems seems obvious from the context of the problem posed (construction of isosceles triangles or construction of angle bisectors in T-geometry). However, the students showed signs of creative and diverse modes of thinking: there was not one model of behavior or one single type of student performance. Richard had a strong command of some principles but missed a clear picture of how to express the distance between parallel lines in T-geometry concepts. Alvin had a very good overall grasp of T-geometry concepts but did not see that angle bisectors could be constructed simply with classical Euclidean tools and Euclidean circles. The main problem in the students’ solutions was the lack of consistent and well argued transfers of knowledge from the Euclidean sphere to the T-geometry plane with its different metric.

Conclusion

In light of the van Hiele theory of growth of geometric knowledge and the Pirie-Kieren concept of levels of dynamic growth of mathematical knowledge it seems that the growth of T-geometry knowledge also passes through various phases of development. One trait of this development seems to be that students have to go through a phase of experiencing how to navigate between E-geometry and the corresponding T-geometry concepts and back. Can we draw a Euclidean circle in T-geometry? Such questions may seem trivial to an outsider, but what we observe is that such questions are far from obvious to students of T-geometry when placed in the context of constructions with straightedge and compass in T-geometry. Studying the effects of change of metric in geometry, T-geometry offers a way to understand and clarify this change of axioms that differs from a change as we know it in traditional non-Euclidean geometry. Euclidean geometry and T-geometry are related as mathematical systems. Learning how to navigate from one world to the other is an excellent exercise in mathematical generalization and concept building. College students and future teachers could benefit from a thorough study of the similarities and differences between the two systems (Martin, Towers, & Pirie, 2006) by expanding their experience with generalizing geometric structures through the study of geometries with alternative metrics.

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PROPORTION AND DISTORTION: EXPLORING THE POTENTIAL OF COMPLEX FIGURES TO DEVELOP REASONING ON SIMILARITY TASKS

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Why is it so difficult for students to transcend visual strategies into preproportional or proportional thinking in the context of similarity? This study is focused on identifying and describing potential extenders for visual intuitions about scale. Described here are 11 strategies that students used during clinical interview to differentiate similar and non-similar figures. There is evidence to suggest that distortion-detection is a skill that enables students to reflect upon and evaluate the validity and accuracy of differentiation.

Introduction

Research literature suggests that students struggle to develop abilities to reason proportionally and to make sense of similarity (Lamon, 2007) yet children as young as third grade have intuitions about what it means for two figures to be the “same shape” (Lehrer et al., 2002; Swoboda & Tocki, 2002; Van den Brink & Streefland, 1979). These intuitions are based on visual perception, which Swoboda and Tocki (2002) describe as a “natural” occurrence. The ability, documented by Van den Brink and Streefland (1979), to visually perceive the relational size of objects and extrapolate the size of such objects in a new context is largely unexplained. Also of interest is an answer to why it is so difficult for students to transcend visual strategies into preproportional or proportional thinking in the context of similarity. What is it about these tasks that prevent students from advancing into the quantitative realm, even after they have done so in other contexts for proportional thinking?

The answers to these questions lie in bridging the previous work characterizing similarity tasks with an analysis of where students access them visually, and what barriers exist preventing more sophisticated solution strategies from being used. Furthermore, it would be important to identify how visual perception is related to other preproportional and proportional strategies. Understanding the nature of visual perception and the boundaries of qualitative and quantitative proportional reasoning will provide important information about how intermediate tasks might be designed to support conceptual growth. This study is focused on identifying and describing potential extenders for visual intuitions about scale by analyzing the strategies that students use during clinical interview to differentiate similar and non-similar figures. Results are shared here in answer to the questions:

1. What strategies do students use to differentiate similar figures and what types of geometric and numeric reasoning are indicated by these strategies?
2. How does the complexity of the figure to be differentiated influence student reasoning about proportion?

Theoretical Perspective

This study assumes a constructivist perspective on the inquiry into student conceptions and the modeling process. This has two implications for the study at hand. First, there is the direct implication that without observations of students themselves, no theory can stand apart from the limitations of the mathematical understanding and biases of the researcher (Cobb & Steffe, 1983).

By observing students interacting with the ideas behind the theory, we open the theory up to the unexpected (Cobb & Steffe, 1983). Thus, the method of clinical interview was chosen as the primary method of data collection.

The second implication is in registering the significance of the data that are collected. It is possible to have as a goal the empirical vetting of a theory, marking instances where the predictive power is great and where it is not. However, another goal, responds to Vergnaud's (1987) challenge to "understand better the processes by which students learn, construct or discover mathematics and to help teachers, curriculum and test devisers, and other actors in mathematics education to make better decisions" (as quoted in Confrey & Kazak, 2006, p. 311).

Literature Review

Proportional reasoning has been investigated through two major types of tasks: comparison tasks and missing value tasks (Lamon, 1993, 2007). Lamon (1993) outlined a conceptual progression for the development of proportional reasoning that stemmed from visual and intuitive solutions growing through successful preproportional strategies up into mature proportional reasoning. This progression was useful in describing and organizing the numeric strategies that were used by students during interviews. However, because of the geometric nature of the tasks it was insufficient to focus only on instances of numerical proportional reasoning in student strategies. In order to capture other forms of reasoning that students used on similarity tasks, it was imperative that a geometric lens also be used.

Data Collection

Population

A population of students in a Midwestern, urban school district was identified to target racial, economical, and academic diversity. The inclusion of diversity in the sample for study was not intended to highlight differences between groups of students, but rather to ensure that a broader extent of prior student experience and knowledge is included in the results.

Instrumentation

Revised Similarity Perception Test (rSPT). The rSPT was designed to mark differences in visual perceptions of geometric proportion. It coarsely differentiates students by their ability to visualize geometric proportional growth as well as their likely understanding of scaling and the relationships between similar figures. The part of the test relevant to this paper is made up of 23 yes/no items divided into four sections (1-4) where students are asked if two given images are different sizes of the same shape.

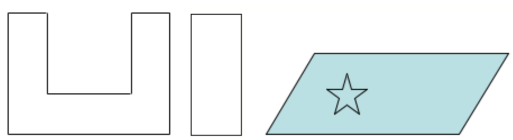

Figure 1. U-shape, Simple Convex, and Complex Figures Used on rSPT Tasks

Three distinct classes of figures are featured on the rSPT items and were included in the subsets given to each student. These distinctive classes are the U-shape, simple convex, and complex figures, which are illustrated with examples in Figure 1. The U-shape is an eight-sided concave polygon with all right angles. All of the simple convex figures are rectangles (as the one shown) or parallelograms. The complex figure class includes figures that were created by setting a second image (stars, cartoon girls, or parallelograms similar to the exterior) inside of a

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

parallelogram. The variation in figure type included in the rSPT items meant that different arrays of characteristics such as lengths within the figure were available to students on each item.

Five questions in Section 1 identify student perceptions of distortion in the general shape. For example, students are shown the letter A in two different fonts. The seven questions in Section 2 identify visual perceptions of distortion in terms of simultaneous horizontal and vertical growth. The seven questions in Section 3 identify visual perceptions of zooming and perceptions of distortion as changing the perspective of the image. These items feature a figure framed by a parallelogram. On some items, such as Item 3.2 shown in Figure 2, different scale factors are applied to the interior figure and exterior frame to give the illusion of zooming in or out on the figure. The last four items in Section 4 identify visual perceptions of continuous all-over growth by varying the scale factor on different components of one figure.

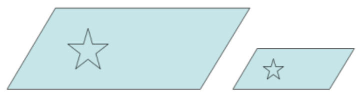

Figure 2. Item 3.2 from the rSPT

The rSPT was administered in entirety to a group of 91 seventh-grade students for sampling purposes. This instrument provided information about students' visual perception of shape, correspondence, and size transformation and helped to divide students into subgroups according to their responses. A stratified purposeful sample (n=21) that included the most common as well as unique response patterns was selected for task-based interviews. This method of sampling intentionally included individuals who exhibited varying abilities, perceptions, and strategies.

Interview Protocol. As a part of a larger interview protocol, students were asked to revisit a subset of the rSPT items from sections 1-4 and think aloud as they responded again to the items. They had just finished a unit of instruction on similarity and the opening prompt was changed to, "Are these figures mathematically similar?" Students were not made aware of their original responses before responding, even if they asked. As time did not permit the review of every item on the rSPT with every student, students were given individually selected subsets of the items. These subsets were chosen carefully to include a variety of items. The goal was to capture the most diverse reasoning possible from each student.

Data Analysis

From the interviews, student responses to rSPT items were transcribed using Transana (Fassnacht & Woods, 2005), a software package used to transcribe and organize data. Student think-aloud responses to rSPT items during the interviews were analyzed using a constant comparative strategy, the unit of analysis being an individual's response to one item on the rSPT. The responses were analyzed for themes, first by student and second by item.

After the framework emerged from analysis, it was used on the data. Each response given by each student was coded according to the strategy type used. In the event that a student used multiple strategies on one item, that response was broken into multiple units so that each received one code. The nonspecific code was only used if a student gave no indication of a strategy at all on an item, thus it is impossible that a student's response could be broken up into non-specific and then specific components. In total, 559 units were analyzed and coded.

Results

Framework

The framework that emerged to describe the types of strategies that students were using to differentiate proportional from non-proportional pairings of figures is oriented toward the types of characteristics that caught student's attention or were useful in justifying a response. It was initially intuitive to include characteristics such as "angle" or "side length" in this framework. In the traditional analytic sense, these are the characteristics by which similar figures are defined. However, other characteristics such as the overall appearance of the figures, variety in the types of lengths students referred to, or types of relationships between the two figures emerged during the analysis of student responses. Four main *characteristic types* encompassed the overall variety that was documented: *Appearance*, *Angle*, *Length*, and *Relationship*. In 39 cases, or roughly 7% of total responses, a student's response was not specific enough to ascertain what characteristics of the shape the student perceived, or how the student arrived at a conclusion about similarity. For example, Elijah responded to an item from Section 2, "*They are the same. I forgot what I was going to say*." A fifth category, *Nonspecific* was added to include all responses that did not reference specific characteristics.

As responses were sorted according to characteristic, it was possible to parse out the strategies that students were using to differentiate the figures. The characteristics that students note are highly intertwined with the strategies that students are using, but they are not necessarily one-to-one; students utilize the characteristics of the figures in very different ways. From those responses where a strategy was evident, a framework of eleven unique strategies emerged from analysis. These are summarized in Table 1.

Appearance-type Strategies. Three types of responses regarding the general appearance of shapes were made: (1) descriptions of cosmetic features of figures such as color or blur, (2) comparisons of shape type such as "rectangle," and (3) the relative positions of subshapes within a figure. These holistic descriptions of the shape did not reference specific components of the figures, but did help students describe their visual perceptions of figures and were indicative of different levels of sophistication even within the broader category of Appearance-type strategies.

Angle-type Strategy. Angle was not a differentiating factor for most of the figure pairs on the rSPT, although it was a factor on Item 1.2, which features non-similar parallelograms. Most students successfully determined that these two figures were non-similar using the non-equality of corresponding angles as evidence. Anna responded, *"I could tell the difference between these two because it's got more of an angle. Like, it's angled more over. This one is up…er…bent over more. I can't really explain it,"* (Anna, 1.2, Non-similar). In the language that students used and methods of angle comparison, there were subtle differences in the ways angles were invoked.

Length-type Strategies. Three distinct length measures were noticed and compared by students while differentiating figures: *primary*, *secondary,* and *gap* lengths. One of each of the three measures are illustrated in the case of the U-shape in Figure 3. Primary and secondary lengths are both measurements of drawn lines within the figure, generally edges. A *primary* length is a length that defines the height or width of the entire figure. In the case of a parallelogram, all four edges were defined as primary because they frame the figure and determine both horizontal width and a "slant height," which most students referred to as "height." All other lengths including remaining edges or drawn lines within the figure are *secondary*. A *gap* length measures the width of a gap in the figure not represented by a drawn line. As with the Appearance-type characteristics that students noticed, students compared and used these lengths in a myriad of ways. Two particular Length-type strategies were documented: *Corresponding Length Comparison* and *The Constant Gap Strategy*.

Table 1. Summary of the Strategies Used by Students to Differentiate Similar from Non-Similar Pairs of Figures

Characteristic Type	Strategy	Description *Two figures are similar/non-similar if…*
Non-Specific (6.98%)	Non-Specific	*(A student's response was not specific enough to ascertain what characteristics of the shape the student perceived, or how the student arrived at a conclusion about similarity.)*
Appearance (26.65%)	Cosmetic	*… they are visually alike/different.*
	Shape Type	*… they are different sizes of the same/different shape type.*
	Relative Position	*… the relative positions of the sub-shapes do not/do change.*
Angle (8.77%)	Angle	*… corresponding angles are of equal/different measures.*
Length (35.24%)	Corresponding Length Comparison	*… one length is longer or the same as another.*
	Constant Gap	*… there exists/does not exist consistency of a space length within two figures.*
Relationship (22.36%)	Constant Difference	*… there is/isn't a constant difference relationship between pairs of corresponding lengths.*
	Constant Ratio	*… there exists/does not exist constant ratio between corresponding lengths.*
	Qualitative Relationship	*…there is/is not a constant qualitative relationship in the components of two figures.*
	Dilation	*…one can be transformed into the other through some act of expansion or dilation.*
	Tiling	*…one can be transformed into the other using tessellation or 'tiling'.*

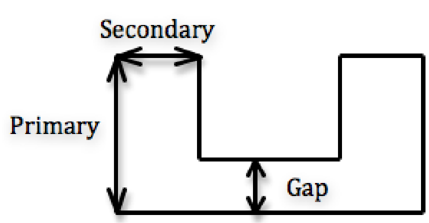

Figure 3. Illustration of Length Subtypes

Corresponding Length Comparison indicated that a student made a decision about similarity based on the comparison of two (and only two) lengths. Arguments that students used that indicate this strategy sound like "these figures are similar, one is just taller than the other." In most of the cases, the lengths to be compared were primary or secondary lengths. In the case of the *Constant Gap Strategy*, it is not. For some students, there is a conceptual difference between a gap length and one that is represented by a drawn line. The difference lies in the fact that a drawn line can be perceived as an object that has properties such as length. A space length is not as easily perceived as an object, although it can be represented as the unmarked distance between two points. In this sense, space lengths can be classified by students differently from drawn lengths, and may not be expected to scale in the same way. The constant gap strategy was based on the expectation that while lengths may get longer (or even scale multiplicatively), gaps remained constant.

Relationship-type Strategies. Three other strategies that relied on a student's perception of relationships within the pair of figures were identified in this study. Two of these strategies were numerical in nature (one additive and the other multiplicative), while the third relied on a qualitative assessment of the relationships. The *Constant Difference* strategy is often seen characterized as a misconception or immature numeric reasoning. Students using this strategy are not yet focused on proportion as a multiplicative concept. The *Constant Ratio Strategy* is familiar, based on numeric reasoning and often taught explicitly in the classroom. Its use by a student named Tom in this study is noteworthy.

Tom was one student who consistently used extrinsic ratios of primary and secondary side lengths to determine similarity. On most items, he chose to compare between two and four pairs of lengths. Tom used this strategy on almost all items yet came to some incorrect conclusions. For example, Tom's method was unreliable on items featuring U-shapes. Tom would often compare the ratio of two pairs of corresponding sides. On one particular item he compared three ratios and when questioned, Tom made a statement that unintentionally explained why his method was unreliable:

Interviewer: You coordinated three things there: the width, then the width of this leg thing, then the height. Is that because this is a different kind of shape? How come three things?

Tom: On the other ones you've chose, you have…There's really not much you can…You can go like that [compares the overall height gesturally] which I kinda did. You, but, you can't really measure…*The more complicated the shape, the different you...the more different ways you could change it to look similar but it could be different (Tom, 2.4, Similar).*

Students using transformative strategies (*dilation* and *tiling*) decided if two shapes were similar/non-similar based on whether one could/could not be transformed into the other. Both strategies were used by students who perceived a dynamic relationship between the two figures in the item and imagined one as transformed from or into the other either by dilation or by tiling. For example, David used a dilating action radiating from the lower corner to describe the relationship between two parallelograms. As he drew, David said, "*I tried to fit that in that and make it bigger. And see…kinda picture it,*" (David, 1.3, similar).

David uses qualifying language such as "like" or "kinda" (kind of) with a tentative tone, as if he is searching for a way to describe what he imagines and is trying words out. Students used common phrases like "fit it in" or "filling up," and prior experiences became tools by which to identify specific perceptions. It was also common for students to make sketches of one shape inside of the other, as David did above with the parallelograms. To highlight the use of many of these tools in a slightly more difficult context than the parallelograms above, the following transcript was taken from David's description of why he thought two U-shapes were similar. David interpreted the larger and smaller shapes as related by dilation.

David:	I think I said these are the same. Like, these are parallel. Just the same shape as it. Kinda like just tried to fit [the small leg] in [the large leg].
Interviewer:	How does this one fit in there?
David:	It kinda doesn't.
Interviewer:	What do you imagine?
David:	I kinda just imagined like...[David draws a sketch.] I try to picture it as it getting bigger and fitting.
Interviewer:	Filling it up?
David:	Yeah! This, like, this corner go into this corner. (David, 2.4, similar)

Although the language used is not formal language typically associated with dilation, it is not lacking in sophistication. David used drawn dots to describe the destinations within the larger U-shape of specific points within the smaller U-shape. This illustrated a very sophisticated conception of correspondence and continuous all-directional growth, even if this conception did not translate into a strategy that was easily verbalized or uniformly applied.

Patterns in Student Strategies

Even students who were capable of analytic reasoning used visual judgment to mediate their responses to items. The use of this visual judgment, particularly to identify distortions between figures, supports Swoboda and Tocki's (2002) hypothesis that students regard distortion as a property of shapes in this context. Two findings are particular supportive: (1) there is preliminary evidence that whether two figures are similar or not impacts student strategy choices, and (2) the presence of different types of distortion has a further impact on student strategies when figures are non-similar. Due to space limitations, only the first finding is explicated here.

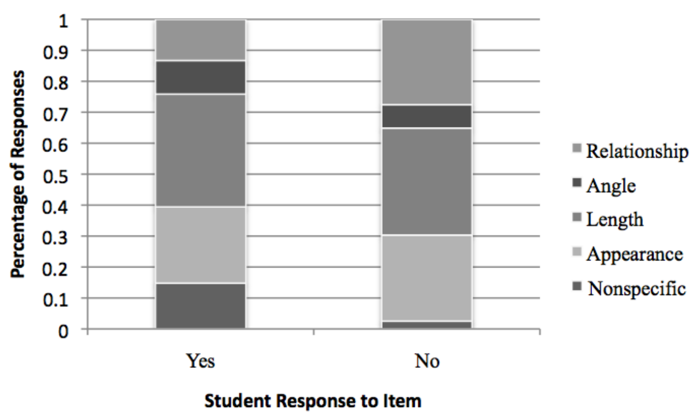

Figure 4. Comparison of Strategy Types Used on Perceived Similar and Non-similar Pairs

Discussion

In Figure 4, the types of strategies used by students are organized. 203 units were 'Yes' responses where a student had decided that the two shown figures were similar (whether or not this was the mathematically correct response). The remaining 356 units were 'No' responses. It is interesting to note that 14.78% of ‘Yes’ responses were Nonspecific, while only 2.53% of ‘No’ responses were Nonspecific. There is also a significant gain in the percentage of Relationship-type strategies used when a student did not think the figures were similar compared to when they thought they were. There is evidence to suggest that distortion-detection is a skill that enables students to reflect upon and evaluate the validity and accuracy of differentiation. On simple differentiation items such as those incorporated in the rSPT, students’ decisions were more evidence-based on items where they perceived distortion than when they perceived none. Of the 39 non-specific responses, almost six times as many were related to figures students perceived as non-similar.

Taking a purely analytical or numerical approach to differentiation tasks can be limiting and in some cases debilitating, especially when the task is to prove that two shapes are similar. When proving two complex figures or U-shapes are similar, a student taking a purely analytical approach must compare each and every possible length including primary and secondary edges, but also space lengths. It is difficult for students to know when enough is enough. In the case of non-similar figures, a student could make multiple comparisons and still miss the one pair that deviates from the pattern. Visual judgment can greatly reduce the analytical workload by indicating to a student where to look to find likely counterevidence to similarity and help support the student in determining a sufficient argument. For example, instead of constructing all possible ratios of corresponding lengths to test for equality, Tom could have used visual judgment to detect distortions in particular components of the U-shape before deciding which ratios to compare.

David was remarkable in his ability to explain his Transformative strategy of imagining the small figure slowly filling up the larger while maintaining its shape. David’s understanding of continuous all-directional growth is quite well developed and is evident in his visualization. The correspondences that he saw indicated that he was, indeed, visualizing dilation. He concluded with a dynamic hand gesture as if he was holding a beach ball as it filled with air. This gesture could also be interpreted as a method of describing continuous all-directional growth. David’s description did not include analytic reasoning, yet it illustrated deeper understanding than Tom’s accurate but incomplete use of the Constant Ratio strategy.

This is not to elevate all visual judgment to extreme levels of sophistication. Certainly there were examples of visual strategies that alluded to more basic conceptions. Nor is this argument intended to deny the importance of analytic reasoning. Visual judgments provide structure for mathematical description, but are not themselves numerical descriptions. In order completely mathematize and abstract the act of classifying or scaling figures, analytical reasoning is required. However, what has been observed in this study suggests that visual perception is not entirely guess-related and primitive, that the consideration of visual perception as a powerful indicator and supportive extender of conceptual understanding in this area might be warranted.

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THE KEY COMPONENTS FOR MEASUREMENT TASKS

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This report describes a collection of instructional tasks developed and piloted with students from second through fourth grade as part of a four-year longitudinal study. These tasks were developed to help students progress to higher levels of thinking in measurement. In addition to describing the tasks and their purpose, we provide student responses to these tasks from individual interviews with eight students. The process of developing and piloting these tasks has been beneficial for us as a research team and has caused us to reorganize our thinking about the teaching and learning of measurement.

Introduction

The technical skills needed to measure accurately and precisely are important in many aspects of our everyday lives. We as teachers should strive to help students develop these necessary skills so that they can function well as adults. However, to focus only on the procedure of measurement is not sufficient. The conceptual understanding required to comprehend the foundations of measurement is imperative to developing a coherent and flexible notion of a unit, which applies not only to length, area, and volume but also to less physical measurements including rates and ratios in general. Unfortunately, many students struggle with even the most basic of measurement tools, the ruler (Lehrer & Schauble, 2000, Bragg & Outhred, 2004, & Cannon, 1992), that suggests a disconnect between numerical measurements and the unit iteration process. This disconnect should be seen as a warning to researchers to “question the depth of students’ understanding of coordinate systems, function graphing, locus problems, and theorems about lengths or distances” (Battista, 2007, p. 902).

Although this report discusses a set of measurement tasks and student responses to these tasks, our main purpose is to describe a framework for developing measurement tasks based on reflective analysis of student responses. The tasks we present here have been developed as the result of our work testing and modifying a set of hypothetical learning trajectories for measurement (Clements & Sarama, 2009). According to Simon (1995), a hypothetical learning trajectory consists of three components: “the learning goal, the learning activities, and the thinking and learning in which the students might engage” (p. 133). We used the learning trajectory to focus our attention on key ideas in measurement, such as unit, comparison, quantification, and ratio, as we designed tasks.

Methodology

The learning activities described in this report were developed as part of a teaching experiment (Steffe & Thompson, 2000). The teaching experiment consisted of multiple teaching episodes in which the research team developed a set of tasks for the student, made predictions

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

about how the student would respond to the tasks and checked our predictions against student responses. This cycle of the teaching episode was repeated multiple times over several years making up a teaching experiment. The data reported here reflect our teaching experiments with eight students as they moved from second grade through fourth grade.

Length

Bent Path Comparison Task

In the Bent Path Comparison Task, we provided students with a sheet of paper containing two separate bent paths with a start and ending point (Figure 1). We asked the students to imagine a bug walking along each path from the start to the end before prompting them to compare the length of the paths. Students were asked to respond to this task without touching the paper and without a measurement tool. This comparison was intended to be a qualitative comparison with the student predicting that one path was longer. After the student predicted which path was longer, we asked if they could check this. We anticipated that the students would use a ruler to measure the length of each path and subtract to find the difference.

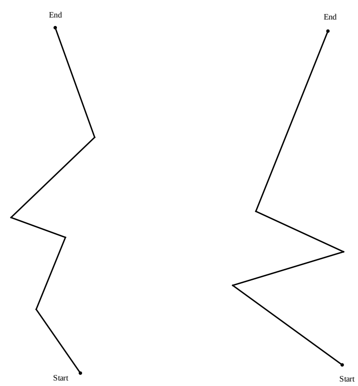

Figure 1

This task was designed to explore two misconceptions, making a comparison based on the number of turns or confusing distance traveled with displacement. This task also allowed us to determine how students would deal with partial units because the paths were composed of segments made up of non-integer lengths when measured in inches. We also wanted to see how the students would visually compare the path lengths and if they could understand and deal with the fact that the sum of the lengths of the individual parts was the length of the entire path.

Student responses. For the initial comparison, several students relied on a mental straightening or stretching of the paths. Drew said “I'm thinking just about that if you stretched this one out in a straight line”, at the same time pretending to hold and pull two ends of a string apart. Sara, who typically struggled more than the others, based her comparison on the number of turns in the paths. Surprisingly Ryan verified that the longer path was indeed longer by reconstructing the two paths using the ruler. He copied each individual section of the path and translated it off to the side, adding the next part so that the pieces formed a straight path. Using this method, the student was able to reduce the task to a simple direct comparison between the two reconstructed straight paths. We felt that using this method allowed the student to avoid any measuring because he never assigned a quantity to any of the lengths.

Wrap Comparison Task

For the Wrap Comparison Task, we began by demonstrating to each student that a pipe cleaner could be bent to fit perfectly around the four edges of a 1-inch square tile. Then we build another shape out of a collection of 1-inch tiles and asked the student to compare the length of the path around the set of tiles to the length of the pipe cleaner. For example, if the distance around one square tile was called 1 pipe cleaner, then the distance around the image in Figure 2 would be two pipe cleaners. At times we directed students into dealing with equivalent fractions by asking them to report their measures in different ways. For example, if a student reported that the length of the path around a set of tiles was two and a half wraps, we asked them to tell us how many half wraps this would be. We hoped that students would report things like the path length is two and a half wraps or five half wraps.

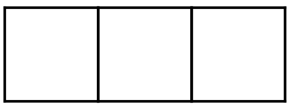

Figure 2

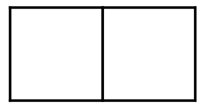

Figure 3

The purpose of this task was to present students with a measurement task that would require the use of a composite unit. In this case, one “wrap” around the square tile was made up of four “sides.” We envisioned this as a way for students to experience one-fourth as a unit with four fourths composing one unit or a wrap. This way of introducing students to a fourth allows them to avoid decomposing a unit into four equal parts to create a fourth. This task also lends itself to proportional reasoning (i.e. four sides is to one wrap as 12 sides is to three wraps) and equivalent fractions.

Student responses. Generally speaking, students responded very well to this task once they understood what was being asked of them. Initially, some students interpreted the task incorrectly and found the perimeter of the set of tiles by counting the sides rather than counting the number of wraps or pipe cleaners. Once the task was clear, most students counted the number of sides around the set of tiles, grouping sides into fours as they went. Some students, however, eventually started counting up the total number of sides and dividing by four to calculate the number of wraps that it would take to be as long as the path around the collection of tiles. Although we felt this was an important move towards proportional reasoning, we found the students’ ability to deal with equivalent fractions in this context to be far more exciting.

While working with the collection of tiles in Figure 3, if a student responded that it would take one and a half wraps to be as long as the path around the set of tiles, then we asked them, “How many half-wraps would it take to be as long as the path around the set of tiles?” To our surprise, most students were able to respond that it would take three half-wraps. As we extended this idea, students were able to tell us that the path around the set of tiles was as long as one and a half wraps or three half-wraps or six quarter-wraps, essentially building equivalent fractions. Although we are not certain that students completely understood all of the mathematical consequences of their responses, we were pleased with their ability to change their unit of measure.

Area

Construction Paper Cut Out Comparison Task

The students were given two figures at a time and asked to compare the areas. The figures were cut from construction paper prior to the interview. The first pair was a square and the triangle formed by cutting a congruent square in half along its diagonal (Figure 4). The second pair was the same square and the rectangle formed by cutting a congruent square in half vertically (Figure 5). Then the students were given the two halves to compare (Figure 6). This process was repeated with a triangular quarter and square quarter of the original square (Figure 7).

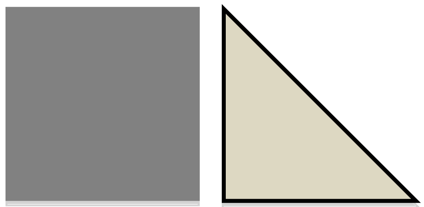

Figure 4

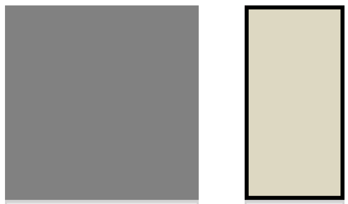

Figure 5

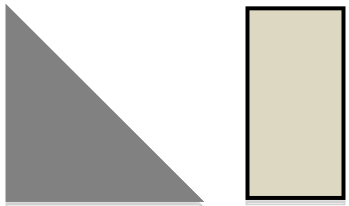

Figure 6

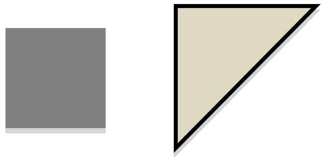

Figure 7

This task lends itself to three different approaches: reasoning numerically that two shapes that have half the area of a third shape necessarily have equal areas, making an intuitive visual comparison between two different shapes, or mentally decomposing and recomposing one half onto the other. The purpose of this task was to explore student reasoning and to determine which of the three techniques students would use to explain their answers.

Student responses. Surprisingly only two students, Anselm and David, determined that the two halves in Figure 6 had the same area. One reasoned numerically while the other relied on decomposing and recomposing. Anselm easily recognized that the two halves were equivalent, stating that because they were both halves of the same whole, the rectangle and triangle would have the same area. He gave a similar argument for the two quarters. In contrast, David compared the two halves (and the two quarters) by lining up the corners and talking about how the uncovered parts could be rearranged so that both figures were the same and therefore had the same area. Most students did not see that the two halves (and the two quarters) had the same area. Instead, they thought that the triangle's area was larger based on an intuitive visual comparison.

Compare Marked Rectangles

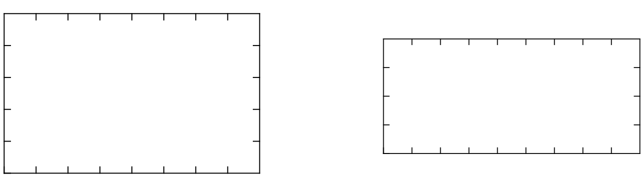

Figure 8

For this task, students were presented with two rectangles, one 5 x 8 inches and the other 4 x 9 inches (Figure 8). Both rectangles had tick marks along the sides revealing the dimensions, but neither rectangle had numerals labeling the dimensions. The students were asked to compare the area of the two rectangles and to explain how they determined which rectangle had the larger area.

This task was designed to force students to structure the empty space inside the rectangles to compare the areas. The textbook series that these students use provides structured square units for almost all area tasks, thus reducing the task to one of counting a set of discrete objects. The rectangles for this task were designed so that the perimeters were equal but the areas were not. We were also interested in exploring how students dealt with the presence of two different units, length and area. The tick marks along the side of the rectangle were designed to draw students' attention to the units of length around the perimeter of the rectangles, but they needed to use those length units to construct area units inside the rectangles.

Student responses. The students demonstrated unique counting strategies for this task. Some seemed to invent rules. For example, Drew counted the tick marks around the figure and did not count the corners. He could not give a clear answer for why, but seemed like he was attempting to avoid double counting the square units in the corners. When the interviewer asked Drew if his response of 22 squares would fill the entire rectangle, he realized there would be some unfilled part in the middle. To take care of that, he said he would add half of 22 to get the inside. Drew's final answer for the area was 22+11. He seemed to have an idea of space-filling, but his spiraling concentric circle method was not accurate.

Others had inefficient counting techniques for determining length, which in turn caused errors when computing area. Anselm and David both made their counting errors when determining side lengths of a figure when only tick marks were present. Several students confounded perimeter and area concepts for this task. These students incorrectly reasoned that figures with the same perimeter necessarily have the same area.

Volume

Cube Comparison Task

In the Cube Comparison Task, we presented students with two rectangular prisms built out of different sized cubes. The first prism was 2 x 2 x 4 made out of 1-inch cubes; the second prism was 2 x 3 x 4 made out of 1-centimeter cubes (Figure 9). We told the students that another child said the figure made out of the 1-centimeter cubes had a larger volume because it was made up of more cubes. The students were asked if they agreed with this decision. Visually, it was clear that the structure made out of 1-inch cubes had a larger volume even though it was made from fewer discrete cubes.

The purpose of this task was to determine our students' understanding of the word volume. We were concerned that our students misunderstood the volume of an object to be a function of only the number of cubes it would take to build the object without any consideration of the size of the individual cubic units. If this was the case, then students would be willing to agree that the volume of the smaller prism was larger based on a count of the number of cubes.

Student responses. Most of the students recognized that the structures were made out of different sized cubes. If the student thought that the figure with more blocks had a larger volume, it gave us an opportunity to introduce another way of thinking about volume. Ryan was convinced by the "other student" saying, "This makes sense because the cubes are way tinier, even though the yellow cube only makes up about two cubes, it's still way more 'cause they're just tons of tiny cubes put together." We asked the student to imagine both structures made of ice and to imagine that the ice melted. Then we asked which pile would turn into more water. With this illustration of volume, the students could get beyond the idea of counting cubes in order to compare the volumes.

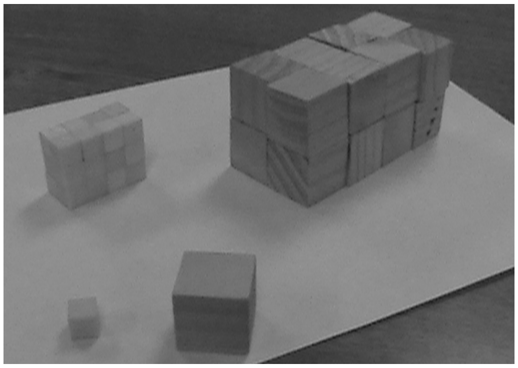

Figure 9

Besides comparing the volume of the figures, many students also compared the units. Ryan realized that the unit size mattered when he said, "Come to think of it, on the yellow brick, is only about two or three cubes on that one [larger structure made of 1-inch cubes]. This only makes up two or three wooden cubes." The student placed two of the wooden cubes next to the entire centimeter structure, and the student continued, "If all this melted, and two cubes melted, it'd be about the same amount of water." The student was making a comparison between the centimeter cube structure and two 1-inch cubes, saying they were about the same volume.

Drew recognized that one cube had a height and width of one inch, and the other had a height and width of one centimeter. It was important that he noticed this characteristic of the cubes. We decided to call them "inch cubes" and "centimeter cubes." Owen immediately focused on the size of the units and did not agree with the "other student" at all. It was also interesting that Abby and Owen related the 24 1-centimeter cubes to two of the 1-inch cubes without prompting. They used this relationship to justify that although the 1-inch cube structure was made up of fewer cubes, it had a greater volume.

Prism Volume Comparison

For this task, we presented students with two rectangular prisms with capacities in a ratio of one to two (Figure 10).

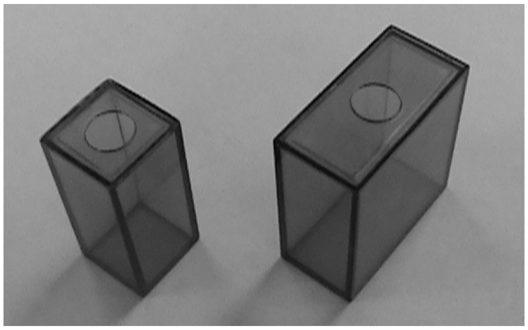

Figure 10

We asked the students to compare the volume of the larger prism to the smaller. If the students did not understand the wording, we asked them how many times they would have to fill the smaller prism in order to fill the larger prism. After the students made a prediction, we demonstrated that the smaller prism could be filled and emptied into the larger prism twice. Next, we shifted our attention to the larger prism and asked the students to fill this container with water so that it would fill the smaller prism exactly once.

Our goal for this activity was to explore student thinking about volume for a task when cubic units were not present. Essentially, we were asking students to compare the volume (capacity) of the two containers, first with the larger prism as the unit and then with the smaller prism as the unit.

Student responses. Most students knew which prism had the larger volume but did not always know how to answer a question with the word compare. One student Ryan said, “It looks like two of these would make up this one [larger].” Some students were not able to think about the containers themselves as units. Instead, they tried to compare the volume of the prisms by imagining them filled with little cubes, essentially comparing the volume of the two objects indirectly by using a third object as the unit. Drew, for example, provided inconsistent responses until we let him fill the containers with water over a sink. Other students, however, were able to cope with the shift in the assignment of the unit, correctly stating that when the smaller object was the unit the larger had a volume of two, and when the larger object was the unit, then the smaller had a volume of one-half.

Discussion

As a research team, we have been engaged in designing and implementing tasks focused on measurement. Through refining and reflecting on this process, we have come to re-organize our thinking. We have created a framework for ourselves that guides the generation of measurement tasks. These tasks help our students develop a solid foundation of the concepts in measurement. Our basic outline for a measurement task has two main components.

In the first step, we ask students to compare two objects by some identified attribute. Here we anticipate only a qualitative response that will help us to determine if the student is able to abstract the correct attribute for comparison. This idea is best illustrated by the *Cube Comparison Task.* When a student was asked to compare the volume of the two prisms in Figure 9, if they did so based solely on the number of cubes, then we would have revealed a misconception that would need to be addressed before prompting the student for a quantitative comparison.

If the student was able to provide a correct comparison of the two objects based on the identified attribute, our second step would be to ask the student how much more of this attribute is contained in the “larger” object. This quantitative comparison would require the student to identify and use a unit to produce a ratio. At this point, the student would have to decide if they would select one of the two objects as a unit or a third object as a unit to quantify the comparison between the two objects. In the *Prism Volume Comparison* task, for example, we wanted the students to select one of the objects as the unit and to produce a ratio between the two objects based on the amount of liquid that the two prisms could hold. In contrast the *Bent Path Comparison* task was better suited to the selection of a third object as a unit. In this case, most students selected, as anticipated, an inch as their unit and constructed two separate ratios by measuring the lengths of the two bent paths. From these two separate measurements, the students could then provide a quantitative comparison. In the form of the task presented here, we did not ask the students how much longer the longer path was. This allowed one student to avoid quantifying the lengths by reconstructing the bent paths into straight paths and making a simple direct comparison. Although this technique allowed the student to answer the question posed, we felt we missed an opportunity by not asking how much longer the longer path was.

Conclusions

As we move forward, continuing to think about the teaching and learning of measurement, we will use our two steps described above to guide our development of tasks and encourage other educators to do the same. By presenting students with two different objects and asking them to compare the objects by an identified attribute and then prompting the student to provide a quantitative comparison (How much longer…), we feel that students' attention will be directed to some of the major ideas in measurement. These ideas include comparison, attribute identification, unit, and ratio. We anticipate that this process will develop students' general understanding of units and ratios by laying a solid foundation for proportional reasoning.

Although we have only discussed measurement of length, area, and volume, we anticipate that these ideas can be applied to less physical measurements such as rates. We do not claim that all measurement tasks should be designed based on these two steps but rather these two steps are helpful in directing attention to major ideas in measurement. We are also aware that the tasks presented in this report do not all follow these two steps and in some cases this allowed our students to avoid the intention of the task.

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THE MAINTAINING DRAGGING SCHEME AND THE NOTION OF INSTRUMENTED ABDUCTION

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Research has shown that the tools provided by dynamic geometry systems (DGSs) impact students' approach to investigating open problems in Euclidean geometry. This paper presents results from a study on how a particular dragging modality, maintaining dragging (MD), can be used for generating conjectures in open problem situations. The study served to test and refine a model describing a utilization scheme (the MDS) associated with MD. In particular the paper discusses the abductive nature of the MDS leading to the new notion of instrumented abduction. Moreover the paper provides evidence that solvers can "free" the MDS from the physical dragging-support, developing a powerful psychological tool.

Introduction

Mathematics education supervisors and leaders have been encouraging the use of technology in the classroom to foster mathematical habits of mind (Noss & Hoyles, 1996; NCTM, 2000, 2006; Mariotti, 2006; Cuoco, 2008). Several studies in the teaching and learning of geometry (for example, Noss & Hoyles, 1996; Mariotti, 2006) have shown that a dynamic geometry system (DGS) can foster the learners' constructions and ways of thinking, and how, thanks to the dragging tool, a DGS can be powerful for explorations in an open problem situation (Laborde et al., 2006; Arzarello et al., 2002; Lopez-Real & Leung, 2006).

Research carried out by Arzarello et al. (2002) and Olivero (2002) led to the description of a hierarchy of dragging modalities, classified through an *a posteriori* analysis of solvers' work that can be observed while a solver is producing a conjecture in a DGS. A key moment of the process of conjecture-generation is described in Arzarello et al.'s model as an abduction that seems to be related to the use of *dummy locus dragging*. The study we report on in this paper (Baccaglini-Frank & Mariotti, 2009, in press) was designed to shed light onto this delicate moment; we proceeded by elaborating (from Arzarello et al.'s classification) four dragging modalities – in particular *maintaining dragging* (MD), developed from *dummy locus dragging* – and developing and testing a model that describes a process of conjecture-generation.

The study makes use of two more notions present in the literature: *abduction*, and *instrument*. Peirce was the first to introduce the notion of abduction as the "inference, which allows the construction of a claim starting from some data and a rule," (Peirce, 1960). Recently, there has been renewed interest in the concept of abduction, with a number of studies focused on its various uses in mathematics education (for example, Simon, 1996; Cifarelli, 1999; Ferrando, 2006), and generating new definitions. In particular, Magnani describes abduction as:

> *the process of inferring certain facts and/or laws and hypotheses that render some sentences plausible, that explain or discover some (eventually new) phenomenon or observation; it is the process of reasoning in which explanatory hypotheses are formed and evaluated* (Magnani, 2001, pp. 17-18).

In the current study we consider "dragging" in a DGS after the instrumentation approach (Vérillon & Rabardel, 1995; Rabardel & Samurçay, 2001), as has been done fruitfully by other researchers (for example, Lopez-Real & Leung, 2006; Leung, 2008; Strässer, 2009). A particular way of dragging, in our case MD, may be considered an artifact that can be used to solve a particular task (in our case that of formulating a conjecture). When the user has developed particular utilization schemes for the artifact, we say that it has become an instrument for the user. We will call the utilization schemes developed by the user in relation to particular ways of dragging, "dragging schemes".

The general experimental design was articulated in two phases, a pilot study followed by a refinement and revision phase that preceded the full-blown study. For both the pilot study and the full-blown study subjects were high school students in Italian "licei scientifici", 9 (3 single students and 3 pairs) students for the pilot study and 22 (11 pairs) for the final study. Since according to the literature (Olivero, 2002), spontaneous use of dummy locus dragging does not seem to occur frequently, first, we introduced four dragging modalities to the subjects during two lessons. We then interviewed students while working on open-problem activities. Data collected included: audio and video tapes and transcriptions of the introductory lessons; Cabri-files worked on by the instructor and the students during the classroom activities; audio and video tapes, screenshots (using the software "HyperCam") of the students' explorations, transcriptions of the task-based interviews, and the students' work on paper that was produced during the interviews.

This paper presents, through a first paradigmatic example, the maintaining dragging scheme (MDS) developed in our model, and through a second example it sheds light onto difficulties which can arise for a solver trying to making sense of what may appear during an exploration when MD is performed. This leads to an introduction of our notion of instrumented abduction, and finally to our third example that shows how the MDS may become a psychological tool.

Formulating A Conjecture Using The Maintaining Dragging Scheme (MDS)

This section illustrates, through a paradigmatic example, how the main ingredients of the MDS described in the model we developed come into play in the analysis of two students' exploration. James and Simon were given the following open-problem activity:

"Construct three points A, B, and C on the screen, the line through A and B, and the line through A and C. Then construct the parallel line l to AB through C, and the perpendicular line to *l* through B. Call the point of intersection of these last two lines D. Consider the quadrilateral ABCD. Make conjectures on the kinds of quadrilaterals can it become, trying to describe all the ways it can become a particular kind of quadrilateral."

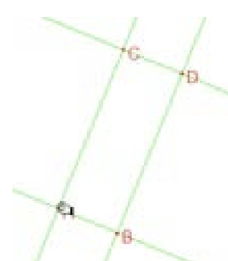

Figure 1

The solvers followed the steps that led to the construction of ABCD, as shown in Figure 1, and soon noticed that it could become a rectangle. Simon was holding the mouse (as shown by his name being in bold letters in the excerpts below), and followed James' suggestion to use MD to "see what happens" when trying to maintain the property "ABCD rectangle" while dragging the base point A. In such situation the selected property "ABCD rectangle", according to our model (Baccaglini-Frank & Mariotti, in press), is called *intentionally induced invariant*. As Simon was focused on performing MD, James' attention seemed to shift to the movement of the dragged-base-point, as shown in Excerpt 1 below.

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

Excerpt 1

1 I: James, what are you seeing?
2 James: Uhm, I don't know...I thought it was making a pretty precise curve...but it's hard to ...to understand. We could try to do “trace”.
3 **Sim**: trace!
4 James: This way at least we can see if...

James seems to be looking for something, which he describes for the time being as a “pretty precise curve” ([2]). This intention seems to indicate that James has conceived an object along which dragging the base point A will guarantee that the intentionally induced invariant is visually verified. This is what we call a *path* (Baccaglini-Frank & Mariotti, in press). Moreover he is trying to “understand” ([2]) what such path might be. In other words he is searching for a *geometric description of the path*. To do this he suggests activating the trace tool.

The solvers activate trace on A and Simon performs MD again. James seems to be searching for a geometric description of the path by interpreting the trace mark on the screen (Figure 2).

Excerpt 2

15 I: and you, James what are you looking at?
16 James: That it seems to be a circle...
17 **Sim**: I'm not sure if it is a circle...
18 James: It's an arc of a circle, I think the curvature suggests that.
…
24 James: Ok, do half and then more or less you understand it, where it goes through.
25 **Sim**: But C is staying there, so it could be that BC is...is
26 James: right! because considering BC a diameter of a circle…

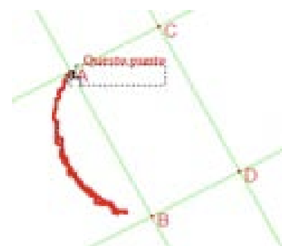

Figure 2

They construct the circle and drag A along it, and then they write the conjecture: “ABCD is a rectangle when A is on the circle with diameter BC.”

In this Excerpt James seems to be searching for a geometric description of the path and identifies some regularity in the movement of the dragged-base-point, “a pretty precise curve” ([2]), then “a circle” ([16], [17]) “considering BC a diameter” ([26]). Moreover, there seems to be the intention of looking for something, which we interpret as an attempt at “making the path explicit”. This can lead to perceiving a second invariant, that we call it the *invariant observed during dragging*, as a regular movement of the dragged-base-point. Both invariants are perceived within the phenomenological domain of the DGS, where a relationship of “causality” may also be perceived between them. Our model refers to this first relationship of causality as a *Critical Link*. Of course such relationship can be formulated within the domain of Euclidean Geometry as a *Conditional Link* between geometrical properties corresponding to the invariants, provided that the solver gives an appropriate geometrical interpretation.

Making Sense Of The Findings Of An Exploration

Using MD the perception of a second invariant, the invariant observed during dragging, can occur in a rather “automatic” way. As a matter of fact, when MD is possible, the invariant observed during dragging may automatically become “the regular movement of the dragged-base-point along the curve” recognized through the trace mark, and this can be interpreted geometrically as the property "dragged-base-point belongs to the curve". In the previous excerpts James and Simon seem to behave in this automatic way, that is, the solvers proceed smoothly through the perception of the invariants and immediately interpret them appropriately, as conclusion and premise in the final conjecture. We refer to solvers who exploit the MDS like James and Simon as *experts*. From the perspective of the instrumental approach (Vérillon & Rabardel, 1995), the MDS may be considered a utilization scheme for expert users of the *MD-artifact* thus making MD an *instrument* (the *MD-instrument*) for the solver with respect to the task for producing a conjecture.

However reaching expert behavior is not trivial, as shown by the fact that many solvers we interviewed did not seem able to make sense of their discoveries even when they appeared to be using MD in a way that seemed coherent with our model. In particular, even when invariants are properly perceived (Baccaglini-Frank et al., 2009) it seems that their simultaneous perception does not guarantee the interpretation of such phenomenon in causal terms. The Excerpt below shows a case in which two solvers have used MD maintaining the property "ABCD rectangle" as their intentionally induced invariant dragging A, they have provided a geometric description of the path and perceived the invariant "A on the circle". However they do not seem to make sense of what they have discovered, and they are not able to reach a conjecture linking the invariants.

Excerpt 3

1 Val: Move A on the circle.
2 **Ila**: You look to check that it stays...
3 Val: There, it remains, it remains a parallelogram.
4 Val: Yes, I mean a parallelo...it remains a rectangle.
5 **Ila**: a rectangle.
6 Val: Yes, more or less.
7 **Ila**: Yes, ok. But...
8 Val: Ok....why?
9 **Ila**: Because...
10 Val: Why?
11 Val: So...I know that, uh, so
12 **Ila**: But B has to always be in that point there.
13 Val: Where?
14 Val: So I think...this remains a rectangle...when AB is perpendicular to DC, ok but in this case it would also be BA is equ, perpendicular to CA.

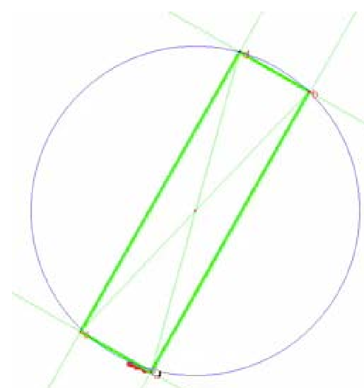

Figure 3

The solvers have constructed a circle corresponding to their geometric description of the path, and seem to clearly perceive the simultaneity of the invariants perceived ("A on the circle" [1] and "it remains a parallelogram" [3]). However the solvers do not seem to be able to make sense of their findings: they repeatedly ask themselves "why" ([8], [9]) and resort to an explanation which has nothing to do with the invariant observed during dragging, but that instead goes back to the definition of a rectangle as a quadrilateral with four right angles ([14]). Note that in the rest of this exploration the solvers will never produce a conjecture using "A on the circle".

Having to base our assumptions exclusively on indirect evidence, it is impossible to say whether the solvers have conceived a Critical Link between the invariants, in the phenomenology of the DGS, but we can definitely say that they are not able to reach a Conditional Link by interpreting their findings geometrically. It is unclear whether the difficulty for them lies in perceiving the invariant observed during dragging as a "cause" of the intentionally induced invariant within the phenomenology of the DGS (so in conceiving a Critical Link), or in interpreting the Critical Link geometrically as a relationship of conditionality between geometrical properties (so in conceiving a Conditional Link).

The Notion of Instrumented Abduction

Unlike Ila and Val, expert solvers seem to withhold the key for "making sense" of their findings, which seems to be conceiving the invariant observed during dragging as a "cause" of the intentionally induced invariant within the phenomenology of the DGS, and then interpreting such cause as a geometrical "condition" for the intentionally induced invariant to be verified. In other words, the solvers establish a causal relationship between the two invariants generating – as Magnani says (2001) – an *explanatory hypothesis* for the observed phenomenon. Moreover, as soon as they decide to use MD to explore the construction, experts seem to "search for a cause" of the intentionally induced invariant in terms of a regular movement of the dragged-base-point. This idea is key; it seems to lie at a meta-cognitive level with respect to each specific investigation the solvers engage in, and possessing it seems to allow complete exploitation of the MDS, culminating in the formulation of the conjecture. Moreover, as mentioned above, the process of conjecture-generation through MDS seems to become "automatic" for expert solvers.

Automatic use of the MDS seems to condense and hide the abductive process that occurs during the process of conjecture-generation in a specific exploration: the solver proceeds through steps that lead smoothly to the discovery of invariants and the generation of a conjecture, with no apparent abductive ruptures in the process. Thus our research seems to show that, for the expert, the abductive reasoning that previous research described as occurring within the dynamic exploration (Arzarello et al., 1998, 2002) occurs at a meta-level and is concealed within the MD-instrument. We introduce the new notion of *instrumented abduction* to refer to the inference the solver makes through the MDS, leading to formulate a conjecture.

MDS as A Psychological Tool

We now take our reflections on the MDS one step further and consider expert use of the MD. We have found evidence that experts may use the MDS as a "way of thinking" freeing it from the physical dragging-support. In the following excerpts we will show how the MDS guided the process of conjecture-generation of Francesco (F) and Gianni (G) even though they were not able to reach an invariant observed during dragging through MD. The solvers were working on the following open-problem activity:

"Draw a point P and a line *r* through P. Construct the perpendicular line *l* to *r* through P, construct a point C on it, and construct the circle with center in C and radius CP. Construct the symmetric point of C with respect to P and call it A. Draw a point D on the semi-plane defined by *r* that contains A, and construct the line through D and P. Let B be the second intersection with the circle and the line through P and D. Consider the quadrilateral ABCD. Make conjectures on the kinds of quadrilaterals can it become, trying to describe all the ways it can become a particular kind of quadrilateral."

The solvers have chosen "ABCD parallelogram" as their intentionally induced invariant.

Excerpt 4

1 G: and now what are we doing? Oh yes, for the parallelogram?

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2 **F**: yes [as he drags D with the trace activated] yes, we are trying to see when it remains a parallelogram.
3 G: yes, okay the usual circle comes out.
4 **F**: wait, because here…oh dear! where is it going?
16 **F**: So, maybe it's not necessarily the case that D is on a circle so that ABCD is the parallelogram.
40 **F**: Because you see, if we then do a kind of circle starting from here, like this, it's good it's good it's good it's good, and then here… see, if I go more or less along a circle that seemed good, instead it's no good…so when is it any good?

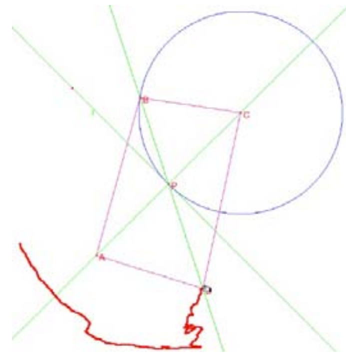

Figure 4

Francesco and Gianni seem to have conceived a geometric description of the path ([3]) that does not coincide with the trace mark they see on the screen as Francesco performs MD ([4]). This leads the solvers to reject the original description ([16]) and search for a new condition ("when" [40]). However they are not able to reach such condition using MD because of manual difficulties they encounter as the exploration continues. This leads Gianni, who is not trying to perform the MD, to conceive a condition in his mind, as shown in the following excerpt.

Excerpt 5
43 G: eh, since this is a chord, it's a chord right? We have to, it means that this has to be an equal cord of another circle, in my opinion with center in A. because I think if you do, like, a circle with center
44 **F**: A, you say…
45 G: symmetric with respect to this one, you have to make it with center A.
46 **F**: uh huh
47 G: Do it!
48 **F**: with center A and radius AP?
49 G: with center A and radius AP. I, I think…
50 **F**: let's move D. more or less…
51 G: it looks right doesn't it?
52 **F**: yes.
53 G: Maybe we found it!

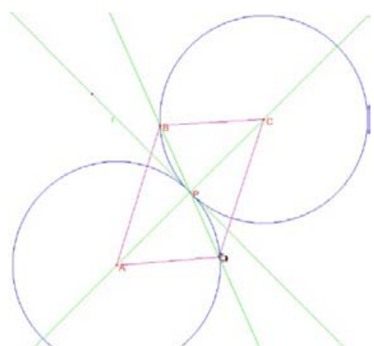

Figure 5

The solvers' search for a condition as the belonging of D to a curve defined through other base points of the construction is now complete, as they construct the circle with center in A and radius AP and proceed to link D to it in order to check the Conditional Link. The solvers seem quite satisfied and formulate their conjecture immediately after the dragging test, proceeding coherently with the model.

Although the "search for a cause" through use of MD with the trace activated failed, the solvers are able to overcome the technical difficulties and reach a conjecture by conceiving a new geometric description of the path without dragging-support. In other words the solvers seem to have *interiorized* the MD-artifact to the extent that it has become a psychological tool which no longer needs external support. Moreover the abductive process supported by MD in the case of an instrumented abduction now occurs internally and is supported by the theory of Euclidean geometry (BP and PD are conceived as chords of symmetric circles). Taking a Vygotskian perspective (Vygotsky, 1978, p. 52 ff.), we can say that the MD has been internalized and the actual use of the MD-artifact has been transformed becoming internally oriented.

Conclusions

We have seen how the model of the MDS seems appropriate for describing the processes of conjecture-generation when MD is used, providing evidence to a correlation between the introduction of dragging modalities, and MD in particular, and a specific new (with respect to those in literature) cognitive process described by the model. We have referred to such process as a form of *instrumented abduction*, a new notion that we hope can be generalized to other contexts in which abduction is supported by another instrument. Furthermore, we seem to have captured the key idea that may lead to complete appropriation of the MDS, that is searching for a cause, and described how it resides at a meta-level with respect to each specific exploration in which MD is exploited as an instrument for conjecture-generation. Finally we described how for expert solvers the MDS might be transformed into a way of thinking. In this sense it may lead to the construction of fruitful “mathematical habits of mind” (Cuoco, 2008) that may be exploited in various mathematical explorations leading to the generation of conjectures.

When such a way of thinking is developed through internalization of the MD-artifact (and is therefore freed from the dragging-support) the abductive reasoning has the advantage of involving geometrical concepts, like in the case of Francesco and Gianni. The geometrical concepts that emerge in this case can become “bridging elements” with respect to the proving phase, since they can be re-elaborated into the deductive steps of a proof. On the other hand, expert use of the MDS still supported by dragging seems to lead to conjectures in which no geometrical elements arise to “bridge the gap” between the premise and the conclusion. In other words, although expert use of MD seems to offer the possibility of generating “powerful” conjectures that solvers might have trouble reaching without the dragging-support, generating conjectures “automatically” through the MDS with the dragging-support, may hinder the proving phase in which these “bridging elements” are essential.

Endnotes

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EXAMINING THE GEOMETRY CONTENT OF STATE STANDARDIZED EXAMS USING THE VAN HIELE MODEL

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In this work we catalogued the content of multiple-choice geometry items on the Ohio Achievement Tests for Grades 3, 5 and 8 according to the van Hiele model of development of geometric thought. Using statewide data from 1,418 students, responses on each question were analyzed to trace performance level at different grade levels. The results indicated that the majority of the items at each grade level focused on Levels 1 and 2. Student performance declined as the question level increased.

Introduction

Usiskin (1987) raised public concerns regarding the geometry knowledge of children in US schools and argued that except for the knowledge of shapes (something learned even before first grade), the geometry knowledge of students at the end of elementary school remains minimal. Referencing the results of the 1982 National Assessment, he pointed out that student performance remained low at all levels. He proposed that both the teaching and content of geometry taught in schools must be reconsidered so as to assure that children’s learning is not hindered as a result. Nearly three decades later, results of national and international studies that measure mathematical performance of children at various grade levels continue to highlight the fragile nature of geometry knowledge of children in US schools. These results are disheartening. Geometry is a major connection between informal and formal mathematics, serving as a critical factor in student success in future mathematics classes (Duval, 1999).

In response to the disappointing results of the students’ performance on norm-referenced examinations (achievement tests), some have argued that the results of these tests should not be given much weight since the test items may not be reflective of what students should know and be able to do. This argument is widely used in places where students’ performance on high-stakes tests is used to gauge teacher effectiveness and school ranking. In many states across the country, including Ohio, student results on standardized exams have political and financial ramifications for schools and districts. Schools are evaluated annually according to whether they have met proficiency standards and placed on “emergency” status if they fail to show progress in students’ results in three consecutive years. Ultimately, schools may be shut down if progress, as measured by standardized exams, is inadequate or insufficient. Therefore, the content of exams and validity and reliability of items used to assess knowledge are of great concern. Of particular interest in the study was the quality of geometry knowledge tested as well as student achievement on standardized exams since the subject remains one of the most overlooked mathematical areas in the US.

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Purpose of the Study

The goal of the study we report here was threefold. First, we aimed to examine the content of the geometry items used on the Ohio Achievement Tests at Grades 3, 5 and 8 according to the van Hiele model to determine what level of geometry knowledge was expected of children. The goal was to see whether the content of the tests agreed with the learning theory.

Second, using data from the performance of 1,418 students from 11 schools across the state of Ohio on each of the items tested, we planned to establish a profile of geometry knowledge of the students. Lastly, by analyzing the common response choices students made on multiple-choice items, we hoped to identify factors inherent in the test items that could have contributed to students' performance. This report is part of a research project in which children's mathematical development from 85 schools across Ohio is traced over the course of three years of involvement in a statewide professional development model (Brosnan & Erchick, 2010).

Theoretical Consideration

Concern about geometry learning and teaching is not a recent development and dates back to the 1950s with the pioneering work of two Dutch teachers, Pierre van Hiele and Dina van Hiele-Geldof. The van Hieles proposed a framework that accounts for the development of geometric reasoning in order to explain how people grow in their geometry knowledge. They identified five different levels of understanding through which an individual passes when learning geometry, including visual, descriptive/analytical, informal deductive, formal deductive, and rigor. According to this model, these levels are not dependent on maturation, but on instruction. While this model has been under revision by some scholars in recent years (Battista & Clement, 1992, 2007) and criticized by some scholars for its inability to trace "in between levels of reasoning" (Burger & Shaughnessy, 1986), it is still widely used in curricula implementation in mathematics classrooms today. For these reasons we opted to use the framework as a lens for our analysis of both the tests and students' performance. A brief description of each level is presented below.

At the visualization level (Level 1), a learner identifies, names, compares and operates on geometric figures, such as triangles, angles, and parallel lines, according to their appearance. At this level, students may see the difference between triangles and quadrilaterals by focusing on the number of sides of the polygons but may not be able to distinguish different quadrilaterals such as parallelograms and trapeziums.

At the analysis level (Level 2), students can recognize components and properties of a figure. However, they cannot see relationships between properties and figures, nor can they define a figure in terms of its properties. At this level a learner analyzes figures in terms of their parts and the relationships between these parts, establishes the properties of a class of figures empirically, and uses properties to solve problems.

At the informal deduction level (Level 3), students can recognize interrelationships between figures and properties, and they can justify these relationships informally. The learners can understand and use precise definitions and are capable of "if-then" thinking but not producing formal proofs.

At the deduction level (Level 4), students can reason about geometric objects using their defined properties in a deductive pattern. Students at this stage could construct the types of proofs that one would find in a typical high school geometry course.

At the highest level, rigor (Level 5), students can compare different axiom systems. Learners establish theorems in different postulation systems and can analyze and compare these systems.

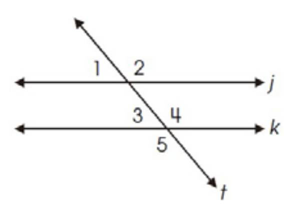
Participants and Methods

The database for the study consisted of the results of 1,418 students who had completed the Ohio Achievement Tests in May 2009. The sample consisted of 471 third-grade, 644 fifth-grade and 303 eighth-grade students' responses to the latest released math exams in their grade level, i.e. third grade test in 2005, fifth and eighth grade test in 2006. The students were from 11 low-performing elementary and middle schools. School ranking was designated by the Ohio Department of Education (ODE) according to the percentage of students who had failed to meet the "proficiency" level criteria set by the state. All participating schools were involved in a statewide professional development program which aimed to raise the mathematical knowledge of both teachers and learners in these schools. Student results were compiled by each school and submitted to the research team online. Student and school identities were removed.

Students' responses to each multiple-choice question on each of the three grade level exams were formatted in Excel spreadsheets. Results were then compared and analyzed. Across grade levels the percentage of correct answers on questions in each van Hiele level was used as an indicator of students' progress. In this work we considered only the tests' multiple-choice questions and students' responses to them in order to avoid potential inconsistencies that could exist when scoring the tests' open-ended response items, of which only one for each grade related to geometry. Since the student responses were not scored by the research team, we were not in a position to assure inter-reliability ratings of open-ended responses.

Analysis of the Tests

We identified the van Hiele level of each question by examining its content. We agreed that although higher levels of reasoning can always be adopted to solve lower-level questions, this did not change the level of difficulty of the question in itself. Hence, we used the highest van Hiele level of geometry reasoning required to solve the problem as a means to rank the level of the question on each of the tests. Additionally, we acknowledged that distinguishing one-step reasoning from identification was not always possible since this kind of reasoning is usually a deduction from definition to its sufficient and necessary condition. Nevertheless, the van Hiele level of thinking is based on experience and the instruction received (Crowley, 1987). For instance, if students are taught to identify parallel lines by using corresponding angles, a question concerning this relationship may be viewed as a Level 2 item; whereas if other ways of identifying parallel lines had been taught and the congruence of the angles was then shown to be a consequence, then this question may be classified as a Level 3 item. Therefore, without information on the actual experience of students, judging the level of question according to the van Hiele model (solely on the basis of its content) may not be adequate. However, in our analysis no subcategory between the levels was defined. Therefore, in places where disagreements occurred among the research team regarding the level of a question, we identified the question to be in the "closer" level by intuition. For example, consider the following example from the eighth-grade test:



(Eighth grade) In the figure, lines j and k are parallel. Which angle is congruent to 1?
A. 2 B. 3 C. 4 D. 5

While it could be argued that the above question measures Level 3 thinking since the congruence of 1 and 3 is a conclusion following the statement of parallel lines, i.e. "If j and k

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are parallel, then 1 and 3 are equal", asserting then that reasoning from lines to angles is involved. Nevertheless, it is also legitimate to argue that congruence of 1 and 3 can be viewed as an identifier of parallel lines; hence the question can be ranked as an "identification" task instead of reasoning that leads to the solution. In this case, we classified the question as Level 2 since we believed one-step reasoning to be "closer" to description than to relation.

Table 1 offers a blueprint of the test items at each grade. As illustrated, all but one question on the third-grade test are in Level 1. These items measure knowledge of triangles, quadrilaterals and the number of their sides and angles. The only Level 2 question on the third-grade test refers to a basic identifier, i.e. convention to describe locations in the coordinate grid. No other properties of the grid, such as the parallel or perpendicular lines, are involved in the question.

In 1992, Clements and Battista concluded that, in geometry, students are extremely unsuccessful with formal proof (upper Level 3 and Level 4). We noticed that (as shown in Table 1), tests were mostly measuring low levels of knowledge. More than 85% of the test items are designed to measure Level 1 thinking in Grade 3, and a large percentage of questions on fifth- and eighth-grade tests were also ranked at Levels 1 and 2 (75% and 67%, respectively).

Table 1. Questions grouped by and van Hiele level

Grade	Level 1 # of questions	Level 2 # of questions	Level 3 # of questions	Total number of questions
3	6 (86%)	1 (14%)	0	7
5	2 (25%)	4 (50%)	2 (25%)	8
8	1 (17%)	3 (50%)	2 (33%)	6

The van Hiele levels of questions used on the fifth-grade test were higher than those on the third-grade test. The two Level 1 questions were about three-dimensional shapes. The figures are more difficult to visualize than two-dimensional figures when drawn on paper. Therefore, the questions are of a higher Level 1 compared to the third-grade items. The four Level 2 questions asked students to find the relative location of lines, symmetry of a circle by its diameter, and geometric properties in the coordinate. These concepts are more difficult than the concepts of sides and angles since they are described by more advanced geometric concepts. The two Level 3 questions measured students' understanding of the angle sum formula of a triangle. Students were asked to find the degree measure of an angle given the measures of the other two angles.

On the eighth-grade test, only one Level 1 question was noted and it required three-dimensional thinking. The three Level 2 questions measured knowledge of parallel lines' angle-related properties, identification of figures and their transformation in the coordinate plane. The two Level 3 questions were also more sophisticated than Level 3 questions in the fifth-grade test. One question tested students' perception of the similarity of triangles and required calculating the index of proportionality of figures. The second question required the use of algebra in calculating the area of a triangle.

As the blueprint illustrates, the van Hiele levels of the questions asked on the three tests increased according to grade level, though not at the same rate. Therefore, the design of the geometry questions on achievement tests we studied was consistent with the developmental sequence of geometry reasoning according to the van Hiele model. Nonetheless, the significant portion of each test at each grade level focused on low levels of geometry reasoning. We will address the significance of this issue in the discussion section of this article.

Analysis of Students' Performance

Tables 2-5 summarize the percentage of students who chose the correct response on each geometry question on each of the achievement exams. A decrease in the percentage of correct responses to higher-level questions compared to correct lower-level responses within every grade level is detectable. In the van Hiele model, as with most developmental theories, a student must proceed through the levels in order, and to function successfully at a particular level, a learner must have acquired the strategies of the preceding levels (Crowley, 1987). Therefore, students who reached higher levels would be fewer than those who only achieved a lower level. Thus, the decrease in performance is consistent with the sequence of development in van Hiele's theory.

Table 2. Third-grade students' percentage of correct responses to each geometry question

Question Level	L1	L1	L1	L1	L1	L2	L1
% of correct responses	87%	26%	87%	76%	40%	51%	50%

Table 3. Fifth-grade students' percentage of correct responses to each geometry question

Question Level	L2	L1	L3	L2	L3	L2	L1	L2
% of correct responses	46%	44%	28%	28%	32%	36%	47%	32%

Table 4. Eighth-grade students' percentage of correct responses to each geometry question

Question Level	L2	L3	L1	L3	L2	L2
% of correct responses	51%	50%	67%	24%	48%	48%

Table 5 illustrates the average percentage of correct responses for questions from each level at each grade accordingly.

Table 5. Average percentage of correct responses by grade and van Hiele level

Grade	Level 1	Level 2	Level 3
3	61%	51%	NA
5	46%	36%	30%
8	67%	49%	37%
All	55%	44%	32%

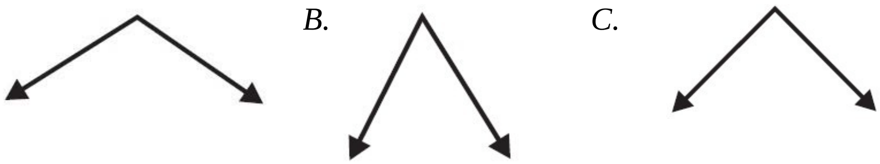
The third graders' average percentage score for the 6 items that were ranked at Level 1 of geometric reasoning was 61%. This average score decreased to 51% for Level 2 questions. Similarly, the fifth graders' average percentage scores on Levels 1, 2 and 3 items were 46%, 36% and 30%, respectively. Lastly, eighth graders' average scores declined from 55% to 44%, and 32% as the level of question increased from 1 to 3. For the entire sample, the average percentage score on Level 1 items was 55%. This number declined to 44% for all items ranked as Level 2. Finally, the average percentage score for Level 3 items was 32%. The results indicate low student performance in all three levels. More importantly, as a group, the performance also declined as the level of geometric thinking increased on the tests. In order to better understand the results, a close examination of items and factors that could have influenced children's choices of wrong answers was conducted. After having done so, we propose three conjectures regarding students' performance based on the language and form of the questions used.

Conjecture 1: Inferences can be drawn based on past experiences and concept images developed during instruction.

As an example, let us consider an item from the third-grade test. The question asked students to identify a picture that showed a right angle with the following images provided:

(Third grade) Which picture shows a right angle?

A. *B.* *C.*

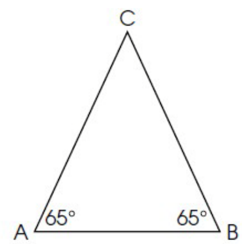


Among the 471 third graders, 171 chose “A,” 174 chose “B” and 123 students chose “C” as their answer. Small differences among the total students who made each choice indicate they were confused since none of the choices met their image of a right angle. We argue that since teachers usually draw a right angle in their instruction as the intersection of a vertical ray and a horizontal one, the students failed to see the picture turned by 45° as the same figure.

Another similar example was noted on the fifth-grade test. The problem asked students to identify the measure of the missing angle in the image when the measures of two other angles were given, as shown below.

(Fifth grade) Triangle ABC is shown. What is the measure of angle C?

A. 50° *B. 65°* *C. 90 °* *D. 180 °*



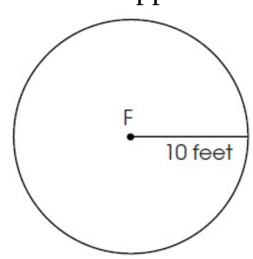
The most common choice was “B”, selected by 202 of 644 (31%) fifth-grade students whose responses were examined. It is plausible that the students assumed the figure represented an equilateral triangle and every angle of it should be equal accordingly. This image is consistent, again, with what is frequently used in class in teachers’ demonstrations of concepts. The imagery could have evoked strong concept images, directing students to incorrect conclusions.

Conjecture 2: Inferences might be drawn due to linguistic clues, i.e. appearance of numbers more than once in the same problem context.

In considering the fifth graders’ common response to the test item that asked them to identify the length of the diameter of a circle given the measure of the radius, 34% of the 644 students selected “10” as their response option. Note that in this question the numeral “10” appeared in both the figure and also in the option “A.”

(Fifth grade) Point F is the center of the circle shown.
What is the diameter of this circle?

A. 10 feet *B. 20 feet* *C. 30 feet* *D. 100 feet*



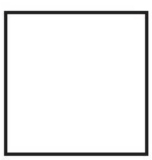
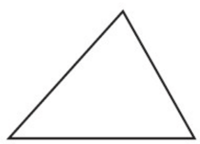
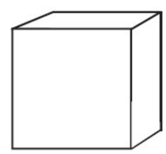
When considering the eighth graders’ responses to the question below which asked them to find the ratio of the areas of two circles with the radius of one circle twice the radius of the other, the common response option “B” selected by 42% of the sample can be explained in a similar manner. In this question "twice" appeared both in the condition and in option "B."

(Eighth grade) Circle A has a radius that is twice the length of the radius of Circle B. Which is an accurate statement about the relationship of the areas of Circles A and B?
A. The area of Circle A is four times the area of Circle B.
B. The area of Circle A is twice the area of Circle B.
C. The area of Circle A is one-half the area of Circle B.
D. The area of Circle A is one-fourth the area of Circle B.

Conjecture 3: Inferences might be drawn due to linguistic clues or meanings students attached to words from personal experiences.

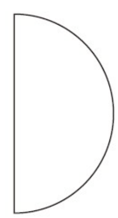
In elaborating on the influence of language on students' choices, let us consider two examples from third- and fifth-grade achievement tests, as shown below. Note that in response to the first example (third-grade test: "Which shape is three-dimensional?"), 53% of the third graders selected "B" (the triangle) as the response option. In retrospective, students could have associated the "three" in "three-dimensional" with "three" as in "three-sided figure." In this context, students could identify the triangle (option "B") as the object with "three" sides. This knowledge was previously tested with the use of another question on the same test that 87% of the sample answered correctly. The language of the text could have provided the wrong hint for selection of the response. The recall from that context could have certainly influenced the children's choice in this problem space.

(Third grade) Which shape is three-dimensional?

A.  *B.*  *C.* 

In reading and interpreting mathematical problems, students draw from multiple resources including their own experiences from real life and how terms are used in their daily lives. An example of such an influence is evidenced in the fifth graders' popular response to the problem below.

(Fifth grade) Malcolm needed to measure the distance across a circular tablecloth. He folded the tablecloth in half as shown. Malcolm measured the length of a folded side. Which part of the circular tablecloth did Malcolm measure?



A. center B. circumference C. diameter D. radius

The most commonly selected response option to this question was "A" (center). This option was chosen by 222 (34%) of the fifth graders in our sample. The student might have interpreted the folding of the circle in half as finding the center of the circle. That is, the folded side is in the middle of the circle, dividing the tablecloth into two equal parts; hence, it is the center of the tablecloth. We conjecture that such contextual interpretation may have contributed to the students' choice, assuming middle to be the center.

Our data are certainly limited both in quality and quantity to permit conclusive inferences regarding children's thinking or influences that could have impacted their choices. We agree that more detailed, descriptive data might provide some direction in better understanding the contributing factors to the children's choices.

Discussion

Analysis of students' performance at three different grade levels on geometry items used on Ohio Achievement Tests clearly speak to problems associated with school geometry learning. Students in all three grade levels had difficulty with visual identification of geometric concepts. Indeed, as specific features were added to a figure, students' performance declined by a much larger margin. Collectively, the students' perception of geometric concepts was underdeveloped at both the visual and descriptive levels. Data further indicated that students had difficulty recalling mathematical terms and definitions.

There is no doubt that student performance on standardized exams is influenced by a variety of instructional and non-instructional factors. From the point of view of instruction, there is always the potential for existence of a mismatch between what is tested and what is taught in the classroom. It is certainly plausible that performance on items that test children's knowledge of basic facts may not be as high when instructional focus is on inquiry and conceptual development. Hence, while we are cautious about making general instructional recommendations based on the findings from this study, we posit that considering that we examined geometry knowledge of the sample using a well-established theory of learning geometry, the results should be considered independent of specific instructional contexts.

Analysis of the test items make explicit the need for a careful consideration of what is included on tests, both in language and content, in order to adequately measure student development. This is particularly important since a "teaching to the test" framework can limit children's mathematical experiences in classrooms, focusing teachers' attention to only lower cognitive level tasks. Analysis of the test items and student achievement also highlight the need to devote greater attention to how geometry is taught in schools.

Endnotes

We concur that regardless of how carefully we may define the levels, debates on their borderlines always exist. As a result, findings of the study should be considered with respect to these flexibilities.

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NON-EUCLIDEAN GEOMETRIES: PRESENT IN NATURE AND ART, ABSENT IN NON-HIGHER AND HIGHER EDUCATION

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This analysis begins with an historical view of Geometry, ending in the Parallel Postulate. Next, are presented new geometric worlds beyond the fifth postulate, discovered as a result of the flaw that many mathematicians encountered when they attempted to prove Euclid's Parallel Postulate. After this brief perspective, a reflection is made on the presence of Non-Euclidean Geometries in Nature and Art and some philosophical implications. Finally, it is analyzed the study of Geometry in Portuguese Secondary Education and the absence of Non-Euclidean Geometries in Higher Education curricula in Portugal.

Introduction

Geometry: From its Origins to Euclid

It is impossible to follow the early evolution of Geometry but it is assumed that, from the observation of Nature, Man created the concept of forms, figures, bodies, volumes and distances.

It was in Greece that geometrical concepts have acquired a scientific form, achieving its splendor with Euclid. Euclid's *Elements*, one of the most important works of all time, contains the teachings which are still the basis of modern Geometry. The *Elements* is based on five logical postulates. Four of these postulates seem to come accordingly to our experience. However, the 5th Postulate does not sound so obvious and intuitive, and is considered a Euclid's own invention.

This is the point of controversy that led to centuries of discussion in the scientific community and served as the motto for this particular work.

New Geometric Worlds Beyond the 5th Postulate

Several mathematicians tried to prove the correctness of Euclid's 5th Postulate for a long time. Although they could get close to real conclusions, they failed, as its primary objective was to prove the Postulate, and not conclude that this could be false (Saccheri – Acute Angle Hipothesis, Legendre, Farkas Bolyai, Gauss). As Greenberg said, it is delightfully instructive to observe the mistakes made by capable people as they struggled with the strange possibility that they or their culture might not accept their conclusions.

It was finally with the courage and determination of János Bolyai and Nikolai Lobachevsky that the Non-Euclidean Geometries were finally acknowledged in Mathematics. Although working separately, both mathematicians developed a new Geometry that exists in spaces of constant negative curvature, which became known as Hyperbolic Geometry.

Later, Bernhard Riemman also developed a similar Geometry to the one of Bolyai and Lobachevsky, creating the concept of Elliptic Geometry, which exists in areas of constant positive curvature. Now, let us try and define each one of these Geometries.

The Hyperbolic Geometry exists in spaces of constant negative curvature. The fundamental difference between this and the Euclidean Geometry is that the Parallel Postulate is replaced by the Hyperbolic Axiom that says that for a point outside a given line there is a multitude of straight lines parallel to it. It is also known that in the hyperbolic surface the sum of the internal angles of a triangle is always less than and there are no squares or rectangles, since the sum of its interior angles is less than .On the other hand, if the surface has constant positive curvature, i.e., is spherical, and since the lines are geodesics on the surface (such as the globe's longitude lines), they always meet in two points (the poles), with no place for parallel lines. Again falls to the ground the validity of the Parallel Postulate. In these surfaces the sum of the internal angles of a triangle is always greater than.

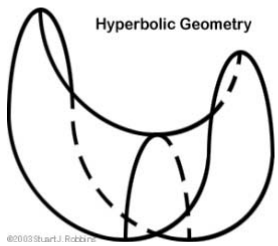

Figure 1. Hyperbolic surface

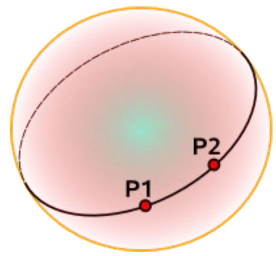

Figure 2. Elliptic Surface

The Presence of Non-Euclidean Geometries in the Real and its Absence in the Study of Geometry

We are now arrived to the last part of this work were will be analysed the presence of the Non-Euclidean Geometries in the real and its absence in the study of Geometry.

We are faced by many phenomena in Nature considered as authentic forms of Art.

A crucial difference between human Art and animal Art is the intention with which it is created. The animals create what seems to be more practical and functional to them. Humans create it just for aesthetic enjoyment.

There are names in Art History that are synonymous of the combination of Art and Science. As an example, one can refer the great Renaissance artist, Leonardo Da Vinci. Da Vinci produced thousands of drawings where artistic and scientific investigations meet.

Figure 3. Cobweb

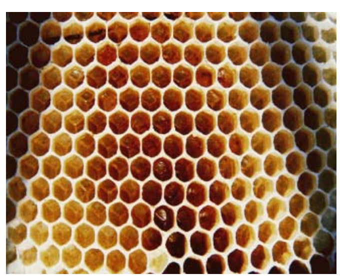

Figure 4. Honeycomb

Figure 5. Circle limit IV by M. C. Escher

Another unavoidable figure is M. C. Escher which became par excellence the creator of impossible worlds. In particular, the regular division of a surface facilitated his approach to infinity. This surface is no more than one model of representation of the hyperbolic disk. Escher, during his work, developed a true glorification of reality, interpreting mathematically patterns and rhythms of Nature forms. But of all forms of Art, the one which is more closely related to Geometry is undoubtedly Architecture. The architectural structures are the result of geometric principles and the final result may come to be a solid figure. Despite the great speed of advancement of societies and technology and the demand for new architectural styles, Euclidean Geometry is still present in most buildings. However, newer concepts from the Non-Euclidean Geometries began to contribute to changes. Antoni Gaudí can, in this field, be considered the great master of Organic Architecture that is the Architecture which is based on principles and forms based on Nature. Gaudí will forever be an icon of world Architecture.

Geometric Structures Independent of Human Intervention

Taking into account the knowledge of common sense, easily we assume that the Geometry of Nature is Euclidean. However, there is no basis for this, partly because concepts such as infinity have no place in the real world, so that in itself brings us to abstract concepts.

There is in Nature an infinite number of elements that cannot be defined by traditional Euclidean Geometry. Examples of that are some trees, clouds, mountains, rivers, the system of blood vessels, nerve structures, etc., which have the name of fractals.

Now, a figure of Mathematics and Physics that cannot be forgotten is Albert Einstein. Since the pioneer work of Einstein, we now have a different view of space and time. For Einstein the Universe was not flat nor time was absolute and both were combined in a curved space-time. The particularity of his work was to have added to the already known three dimensions (width, height and depth), the dimension of time.

Let's now consider a problem, perhaps already known to all : "From a certain point on Earth, a hunter walked 10 km to the South, 10 km to the East and 10 km to the North, thus returning to the starting point. There he found a bear. What colour was the bear?"

At first glance it might seem impossible. However, since it is known by all that the Earth is not flat, the solution is imminent. If the three movements of the hunter are perpendicular to each other and he returns to the starting point, he can only be found in one of Earth's poles. If he encountered a bear, it can only be white and the location is the North Pole.

We can now say that there is no doubt about the presence of Non-Euclidean Geometries in the world. Also notable is its use by various artists. For this reasons, one tried to investigate the "state of play" on the study of Geometry in Secondary Education and of Non-Euclidean Geometries in Higher Education in Portugal. The general perception that students are required to devote more time to Mathematics is a false one. Rather, the syllabus steadily decreases both the amount of subjects and their difficulty. For its part, the study of Geometry suffers the consequences, as it is repeatedly put on hold and, when studied, there is no depth or consistency and, therefore, no logical understanding by students. As a result, it appears that students don't have proficient skills of autonomy in their work, preventing them from acquiring a more accurate and comprehensive learning. In addition, in University Education students don't have, in most cases, any knowledge of Geometries other than Euclidean. The reverse situation could positively influence the way students comprehend the world around them, by broadening their geometrical horizons.

Figure 6. Planet Earth

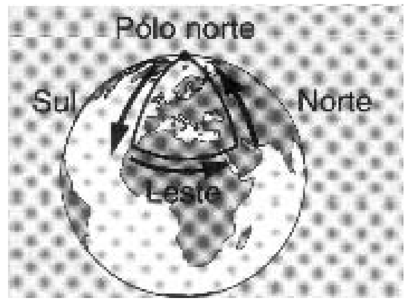

Figure 7. Hunter's Path

Concerning the study of Non-Euclidean Geometries in Higher Education, there was done a demonstrative analysis, although not exhaustive, of the presence of Non-Euclidean Geometries in the curricula at senior level in Portuguese Universities. In about two dozen of courses, only five contained references to the study of Non-Euclidean Geometries. It is a bit unacceptable the absence to its study even in courses of Mathematics, as occurs in the University of Coimbra and the University of Minho.

And it is also unacceptable that in all the courses analyzed, only seven had references to the study of traditional Geometry. This study is absent in courses that could benefit greatly by its presence, as is the case of Arts and Civil and Environmental Engineering.

In a simple and clear way, I tried to demonstrate that, despite the popularized idea that Euclidean Geometry is the only current Geometry in the world, the Non-Euclidean Geometries are present in numerous events that surrounds us. It is expected, therefore, the generalization of knowledge on these issues so that we can look at the reality in a more complete way.

According to the philosopher Immanuel Kant, the reality is synthesized in accordance, not with the real world, but with our mind and its limitations. But as Russell said, Mathematics is the discipline that we don't know what we are talking about nor whether what we say is true...

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SPATIAL STRUCTURING THROUGH AN EMBODIED LENS

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In much of the spatial reasoning literature, concrete activities are assumed to develop into more sophisticated and purely mental concepts. In this paper, we offer a contrasting viewpoint of one spatial reasoning construct–spatial structuring–by examining the intersection of two different data sets (i.e., preschoolers and fifth graders) through an embodied cognition lens. Our analysis and discussion of the students' work reveal the inter-actions of the body are integral to mathematics knowing-not precursors to more sophisticated mental conceptions. In this manner, our paper attends to the complexity of learning by highlighting an ecological perspective.

Introduction

Spatial structuring is described in the literature as "the mental operation of constructing an organization or form for an object or set of objects" (Battista, 1999, p. 418). The construct, like many in the spatial reasoning literature, assumes children's concrete activities such as arranging, building, counting and drawing develop into more sophisticated *mental* concepts that do not require physical movement (Battista & Clements, 1996; Battista, Clements, Arnoff, Battista & Borrow 1998; Mulligan, Mitchelmore & Prescott, 2005; Outhred & Mitchelmore, 2004). In this paper, we provide a contrasting viewpoint and contribute to the understanding of children's spatial reasoning by examining the intersection of two contextually different data sets through an embodied cognition lens. In the first set of data we examined young children's (ages 4-5) two-dimensional spatial structuring in extending and drawing a growing rectangular pattern. In the second data set we explored fifth grade students' three-dimensional spatial structuring in response to a pyramid building patterning task. Our analysis and discussion articulates a view of spatial reasoning whereby inter-actions of the body are integral to mathematics knowing—not simply precursors to more sophisticated mental images or concepts. In this manner, our paper attends to the complexity of learning by highlighting an ecological perspective of individual and collective knowing as it occurred within and across time and space.

Theoretical Framework

Our research draws on contemporary studies in cognitive science and mathematics education which reveal knowing to be a dynamic, contextually contingent, and body-centered phenomenon (Lakoff & Núñez, 2000). Such a theoretical perspective conceives our cognitive structures and processes to extend throughout our bodies and further into the biological, social and cultural environment in which our bodies are embedded. "[C]ontext-dependent know-how [is viewed] not as a residual artifact that can be progressively eliminated by the discovery of more sophisticated rules, but as, in fact, the very essence of *creative* cognition" (Varela, Thompson & Rosch, 1991, p. 148). Learning, then, is not assumed to be a process of converging onto pre-established truth, but a divergent, recursive process of expanding possibilities for action (Davis, 2004).

From an embodied perspective, cognition occurs "all at once" and not on a continuum from concrete to abstract or from particular to general (Maturana & Varela, 1992; Varela, et al., 1991). Knowing is dialectically abstract|concrete, general|particular and contextually contingent (Roth

& Hwang, 2006). Actions with objects are not assumed to potentially lead to knowledge but instead, are acts of knowing in and of themselves (Varela, et al., 1991). Thus, children as competent beings, bring forth worlds of significance through reasoning and using all of the perceptual resources situationally available to them (Roth & Thom, 2009). The unit of "analysis of learning [moves] beyond a narrow focus on individual and 'internal' cognitive processes" (Núñez, Edwards & Matos, 1999, p. 46) to the biological and experiential contexts brought forth in moments of being.

Modes of Inquiry

The analysis of the two data sets relied heavily on the videotapes such that we attended to the students' verbalizations, gestures, actions with objects, interactions with others, and their drawing processes. Analysing both sets involved viewing the video data repeatedly while comparing and verifying our conjectures with each other and against the collected artifacts and relevant theoretical literature. We identified the children's physical sense making activities, their efforts to verbalize their actions, their spatial and numerical structures, and also the relationship(s) among these aspects (Pirie, 1997). In particular, we analyzed and interpreted the data for the following themes related to spatial structuring:

- *Modes of physical activities.* Analysis involved attention to the particular ways that the children physically acted and interacted in response to the geometric tasks.
- *Geometric, arithmetic, and/or geometric-arithmetic structure.* Analysis entailed the spatial and/or numerical conceptualizations that the students demonstrated while working on the geometric tasks.

Data Sources

Data Set 1: Polygon Pictures

Situated at a learning centre, children (ages 4-5) in pairs were asked to watch carefully as we built the first three images of a growing rectangle (i.e., 1x2; 2x2; 3x2). The children were prompted to build the next image and later draw the two-dimensional images of all four pattern elements. Each child responded in different ways to the extension and drawing prompts provided by the researchers. Three exemplars were selected illustrating three children who produced the same 4x2 structure with the tiles, but through closer examination of their physical interactions, gestures and process of drawings revealed different conceptualizations of the geometric structure produced.

Seina (see Figure 1) watched closely as the first three pattern elements were built. Although she did not respond to the prompts (e.g., "What comes next?", "How do you know?"), she built two rows of four tiles moving from left to right. When asked, "How many tiles", she swept her hand over her last set of tiles and answered "eight." Seina's drawings reveal a shift from providing details of individual tiles (elements 1 and 2), to drawing a horizontal line cut with vertical lines (element 3), and finally to a generalized rectangular form. Each drawing indicates what she attended to and how her priority of attention changed as the rectangles grew.

Wasif (see Figure 2) and Breanne sat together at the centre. Wasif constructed the fourth element by placing four tiles (bottom two then top two), followed by individual tiles top-bottom-top-bottom. He too started drawing with details of two tiles followed by growing images of generalized rectangles. After both children completed their drawings they were asked if what they made was a pattern. Breanne answered, "No … this (element 1) has two and these (elements 2, 3) have more. Wasif, disagreed and said, "Yes." When asked why he thought so, without counting he said, “There are two, four, five, six,” and with his hands he gestured that the rectangles had become progressively longer.

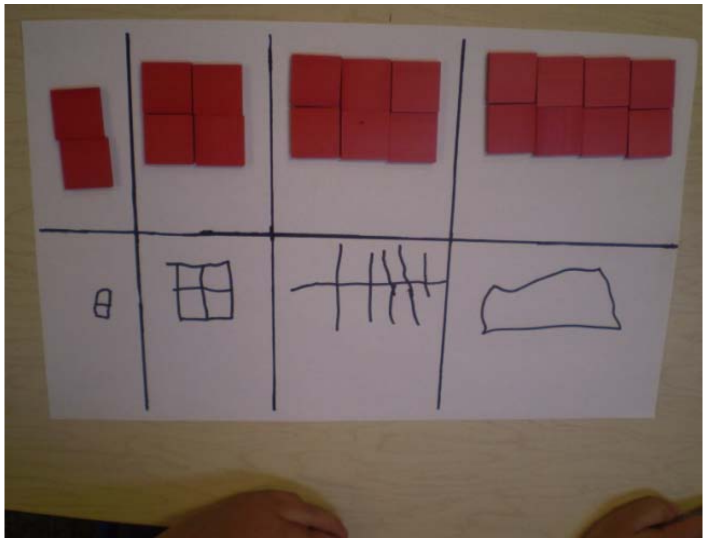

Figure 1. Seina

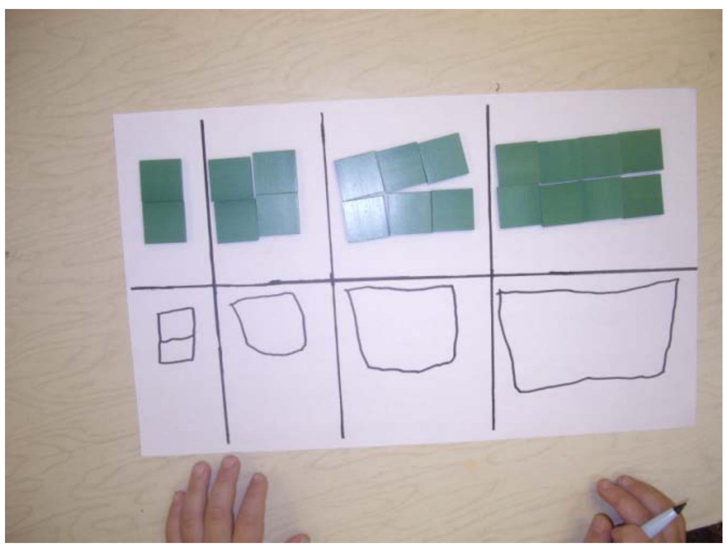

Figure 2. Wasif

Finally, Hunter (see Figure 3) appeared to anticipate the task, because once the tiles were laid out, he interrupted the researcher and pointed to each element, “two, four, six.” When asked what would be next, he responded physically by building. Like Seina, he built two rows of four, but seemed less concerned about the organization of the tiles. “Okay, eight here.” When asked what would come next, he pointed to each element and whispered, “two, four, six, eight” and said “ten” loudly. When drawing his first image he said, “It’s a rectangle, but …[and drew a line splitting the rectangle].” For the next one he said, “It’s a square, but divided into a window.” At this point in the task he started singing and ceased commentating. His third image began with an outer rectangle and he drew six squares inside of it; and the fourth image he sketched each tile, using at least one side connected to another tile. The last two tiles were drawn quickly as loops, likely for speed and efficiency rather than accuracy.

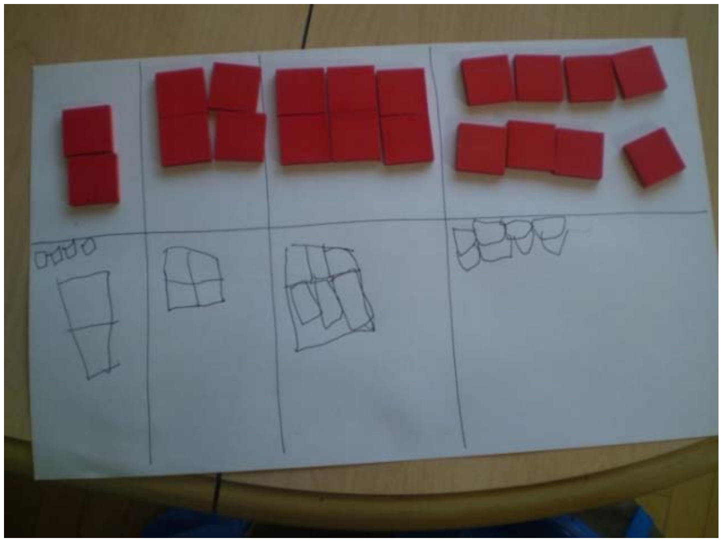

Figure 3. Hunter

Data Set 2: Corner Pyramids

Three fifth grade students worked in a group to solve a series of three-dimensional geometric tasks. Presented with the first three "corner pyramids" (see Figure 4), the group determined how many cubes were in the first pyramid and how many *more* cubes were required for the second through eighth, thirteenth, and twenty-ninth corner pyramid. After the researcher introduced the first three pyramids and tasks to the students, they were left to work on their own for the remaining prompts.

Figure 4. The first three corner pyramids

The students worked individually and together building the pattern elements, counting cubes in them, and producing the same solutions. However, there were distinct spatial conceptualizations that specified exactly how they built and counted each pyramid.

Using multi-link cubes, Allan and Veronique created the fourth pyramid by assembling *horizontal layers* of cubes and counting the 10 'steps' on its front side (Figure 5). William built a vertical *rod* of 4 cubes, attached another 16 cubes to it, and counted the cubes on its bottom (see Figure 6). The students agreed that the two pyramids were identical even though they had built them differently.

Figure 5. The fourth corner pyramid constructed by Allan and Veronique.

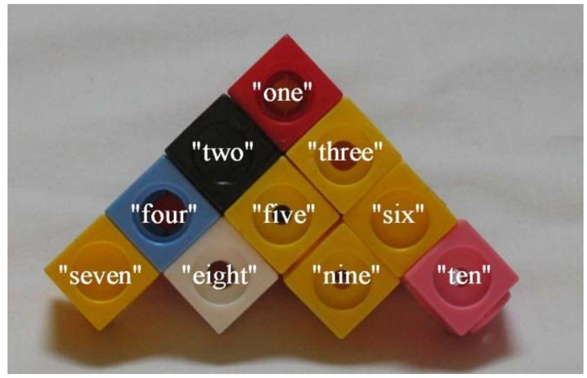

Figure 6. William's counting strategy for the 4th pyramid.

For the fifth pyramid, Veronique took the third pyramid, placed it behind the fourth pyramid and as she examined the *outer layer*, stated, "the more cubes are the ones that you can see." She proceeded to "add five" imaginary cubes onto the fourth pyramid's base and counted on from five with her finger on each of the existing steps to arrive at "fifteen" additional cubes (see Figure 7). William replied that it would be simpler to build the fifth pyramid, turn it over, and count the cubes in its bottom.

Figure 7. Veronique's counting strategy for the 5th pyramid

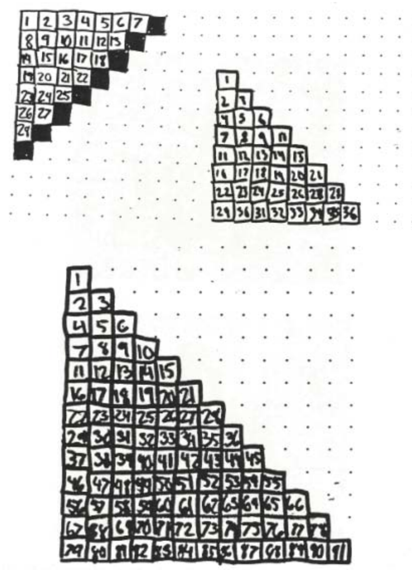

Figure 8. Allan and Veronique's drawings of the imagined bottoms of the 7th, 8th, and 13th pyramids

As the students moved on to the seventh, eighth, and thirteenth pyramids, Veronique and Allan produced three drawings of what they imagined the bottoms of the pyramids would look like (see Figure 8).

The group began building the seventh and eighth pyramids. Veronique and Allan built the seventh pyramid by attaching two rods of six cubes each onto two adjacent sides of the bottom cube of a vertical rod with seven cubes (see Figure 9) and then attached 65 more cubes. William created the eighth pyramid with a vertical rod of seven cubes and attached to its bottom cube, a

36 cube base (see Figure 10) that resembled Allan's drawing of the eighth pyramid's bottom. William then attached the rest of the cubes as steps, constantly turning the pyramid as he built it.

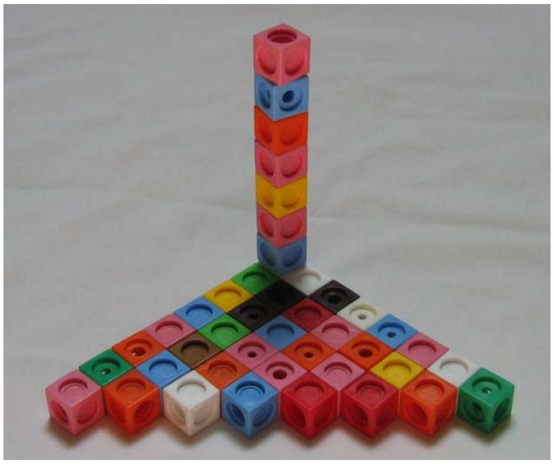

Figure 9. Veronique and Allan's frame for building the 7th pyramid

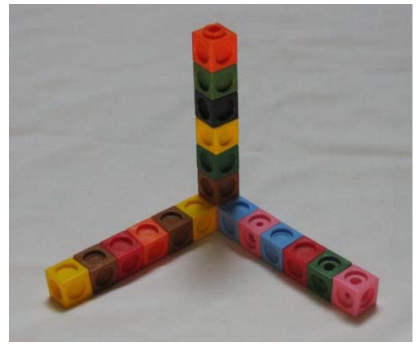

Figure 10. William's frame for building the 8th pyramid

Results

While the children worked in these sessions, their gestures and conversations predominantly directed attention towards their physical activities. This physical work accomplished by the children both individually and collectively emerged as: *building, arranging, drawing,* and *counting*. The two data sets and the examples within each set illustrate for us, the diversity of the children's conceptions of patterns, pattern growth, and the spatial structuring of patterns, as well as the embeddedness of these structures in their physical work. Furthermore, even within the examples we noted that the children conceptualized pattern growth in more than one way. The children's conceptualizations recursively flowed into how they then built the next element in the pattern—not towards mental abstraction, but continuously contingent on the immediate context.

The growth of the children's ability to visualize was very clear in their work. We contend however, that physical activities such as counting, drawing, building, and arranging should neither be considered merely as a means of testing or checking—nor should these actions serve as indicators that children are at a less sophisticated level of spatial reasoning simply because they are not purely mental acts. These children's physical activities were combined with a flexibility of moving between different modes and numerical and spatial images that revealed mathematical sophistication and not its converse. In the episodes, the children's physical activities may appear on first glance, to be low level ways of knowing, acting, or being mathematical but upon further examination into *how* the children actually built, arranged, counted, and drew, very sophisticated and complex spatial-numerical structuring is evidenced.

Our review of previous studies on spatial structuring emphasized counting, building and drawing actions as indicative of children's level of ability. However, we feel that simplicity of mathematical actions is not a clear indicator of what these authors consider to be low level spatial structuring.

Differently and notably observed in the exemplars is that the children's geometric and arithmetic structures did not exist separately from one another. Further, the children's thinking neither progressed in a manner from concrete to abstract nor did it exist as a mental product of their physical actions. Given this, the children's physical activities cannot be viewed as merely material *re*-presentations of categorical mental knowings or understandings. Rather, the children's physical actions in unpredictable yet recursive manners *were* the children's spatial and numerical conceptualizations of the two- and three-dimensional pattern growth. As such, their very physical and always contextually grounded conceptualizations must be considered to be fully embodied events; events that emerged, evolved, and grew in complexity as the children engaged in yet other physical activities of building, counting, arranging, and drawing.

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SYMMETRY AND CONFIDENCE: THE POWER OF CHILDREN'S MATHEMATICAL AESTHETIC

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Mathematics is an aesthetic subject and researchers are beginning to take an interest in what this should mean for education. This paper looks at examples of how an aesthetic sense guides fourth grade students in their search for, and evaluation of, geometric tessellations. A child's aesthetic is different from a mathematician's, but the child's aesthetic is well-suited for leading the child to generative paths of inquiry. As an example, I show how children's aesthetic attraction to symmetry gives them greater confidence and mathematically fruitful ideas when searching for tessellations.

Introduction

The ancient Greeks understood mathematics (and not just its applications to art) as the height of aesthetics. This idea was lost by the time of the Enlightenment, but regained in the late 19th century, when mathematicians again recognized that aesthetics is at the foundation of pure mathematics (Sinclair & Pimm, 2006). Recently there has been a growing awareness of the need to pay explicit attention to mathematical aesthetics in education. The NCTM Standards (2000) call for inclusion of aesthetics in the teaching and learning of mathematics. Nathalie Sinclair (2008) has recently shown how an aesthetic lens can be used to understand certain aspects of classroom interaction.

In this paper, I will look at how children use aesthetics in mathematical inquiry to direct their own search for solutions and to evaluate their own findings. I will show how the aesthetic consideration of symmetry guides children's thinking when exploring open-ended tasks involving geometric tessellations. In particular, the students' awareness of different symmetries adds significantly to their confidence that a path of exploration is likely to be fruitful or that a solution is valid.

Background

What is mathematical aesthetics?

Mathematics is often perceived as being a rigorous, logical subject with no room for subjectivity. This is not the perception of mathematicians, most of whom understand mathematics to be an aesthetic subject (Burton, 2001). What is often overlooked in the popular perception of mathematics is that theorems, proofs, and applications are only one part (often the final result) of mathematical work. Before a system of theorems is built up, an axiomatic foundation has to be agreed upon, and this can only be by aesthetic criteria, such as elegance and simplicity. A search for theorems or new areas of study may involve considerable exploration, which is guided by a sense of aesthetics.

Sinclair, in her book *Mathematics and Beauty* (2006), describes aesthetics as a way of knowing. An artist uses a sense of aesthetics to know how to use colors. In a similar way, mathematicians use aesthetics to know how to explore mathematics. Sinclair defines aesthetics as "A pleasurable sense of fit, which speaks about context and surroundings as well as attributes of the situation in question" (p. 41). She proposes a tripartite model for the role of aesthetics in mathematics covering the motivation of the choice of problems, the generation of conjectures, and the evaluation of solutions. In short, aesthetics is involved throughout the inquiry process.

Importance of aesthetics to mathematics education

Sinclair (2009) has pointed out that many see the consideration of aesthetics for mathematics education as either frivolous or elitist. Those who see it as frivolous believe aesthetics to be peripheral to mathematics rather than the foundation on which mathematics is built. Many see the purpose of mathematics education as the learning of skills needed for life and other coursework—the mere application of what mathematicians have developed. Aesthetics, it is argued, is at best a concern for mathematicians alone. However, Sinclair and others (Sinclair, 2006; Upitis, Phillips, & Higginson, 1997) have convincingly argued that aesthetics is important for meaning and sense-making for students. Others (e.g. Hiebert et al., 1996) have emphasized the importance of incorporating inquiry in mathematics education, which cannot be done without a sense of aesthetics as a guide. Brown, Collins, & Duguid (1989) have argued that learning should be situated in an authentic context. But if students are to experience such an authentic cognitive apprenticeship, their learning must include genuine inquiry guided, not by heuristics, but by aesthetic criteria.

The view that mathematical aesthetics for education is elitist comes from a Platonic view that aesthetics is an inherent property of mathematical objects, independent of culture (Sinclair, 2009). As such, it is believed that aesthetics can be properly evaluated only by experts. Various researchers have shown that students do not always share the same aesthetic appreciation for solutions as mathematicians; such evaluation is therefore believed to be beyond children's abilities. However, it has been pointed out (Burton, 2001) that aesthetics is in some respects subjective and that different mathematicians have different criteria for what they appreciate as aesthetic in mathematics. Whereas certain aspects of aesthetics are shared among mathematicians, others are variable and contribute in different ways for different mathematical tasks, sometimes even leading to different solutions. Children's mathematical aesthetics is somewhat different from that of experts, but may serve their needs better than expert aesthetics by focusing on the aspects of mathematical learning that is important to students (Sinclair, 2006) while still leading them to understand powerful mathematical ideas. This paper is an example of how fourth grade children's sense of aesthetics guides them in their search for solutions to open-ended geometry tasks.

Tessellations

A tessellation, or tiling, is a complete covering of the plane by non-overlapping shapes, typically polygons. Tessellations make an ideal context for exploring the aesthetics of geometry with children. The NCTM *Curriculum Focal Points* has recommended that tessellations be studied in the fourth grade in order to learn transformations (NCTM, 2006).

Some educators have found that tessellations can be powerful tools for teaching geometric ideas. Dina van Hiele-Geldof used tessellations to teach 12-year-old children geometric concepts, such as the congruence of corresponding and alternate interior angles, by having her students explore patterns formed by lines in tessellations of rhombi (Fuys et al., 1984). After a fourth grade unit on tessellations, Upitis, Phillips, & Higginson (1997) found evidence of sophisticated visual understanding which transferred to units outside of geometry.

Symmetry

Most commonly studied tessellations have patterns that repeat throughout the plane; that is, they have translation symmetry in at least two independent directions. There are exactly 17 types of patterns with such translation symmetry, as categorized by the orders of the other types of symmetry that a tessellation can possess: reflection symmetry, rotation symmetry, and glide reflection symmetry. It is not required, however, that a tessellation possess even translation symmetry—a tessellation can be aperiodic (never repeating), though such tessellations are complex and have only been studied by mathematicians since the 1970s (Gardner, 1977).

Symmetry is an obvious aesthetic consideration and has been extensively studied in art as well as mathematics. All children perceive symmetry naturally. Sinclair (2006) already noted an example of how a sudden perception of symmetry enabled a child who was stuck on a problem to regain motivation and use the symmetry to find a solution. Children's perceptions of symmetry may not be highly sophisticated and they may not be able to explain the perceived symmetry mathematically, but symmetry still may be attractive to them and guide their thinking towards generative mathematical ideas.

In my work with fourth grade children, I found several aesthetic considerations in children's thinking, such as alternating rows and "3D" patterns. These considerations both inspired a sense of wonder and enjoyment in the task and directed their thinking along generative paths. In this paper I will focus on how students' awareness of symmetry affected their mathematical inquiry.

Methodology

Population

I conducted exploratory clinical interviews (Clement, 2000) with eleven fourth grade children from two schools: a demonstration school for a large state university serving students from underprivileged families, and a small, multilingual private school with accreditation from the French government. Three of the children were French; the rest were American.

Six of the children were interviewed in pairs, the others individually, in whichever language they were comfortable with. Students were given a variety of tiling tasks, usually in the form of "Is it possible to tile a floor with this shape?" The immediate goal of the interviews was to explore how children conceptualized tilings before and after they attempted to create them. Tasks were adjusted according to student interest, performance, and time, so not all students had the same tasks.

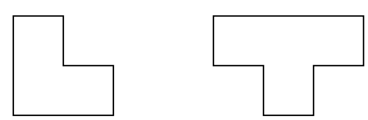

Figure 1. The L-tromino and T-tetromino

Tasks

Several tasks were conducted, three of which will be described in this paper. First, after several tasks that explored tilings with individual pattern blocks, students were given all the squares and triangles in the set of pattern blocks and asked if it were possible to create a tiling with those two shapes together. In another task, students were given dot paper and asked if they could tile by drawing only L-tromino shapes. In their last task, they were asked if it was possible to tile by drawing only T-tetromino shapes on dot paper. (The L-tromino is a shape made with three squares, and the T-tetromino with four squares. See Figure 1.) In the two dot paper tasks, I drew the initial shape and then asked if they could continue the tiling with the same shape.

The tiling with squares and triangles was the most complex task. Though there was no requirement that edges and vertices be matched up, all children attempted to create such tilings in every task. Of particular interest to mathematicians are two semi-regular tessellations with squares and triangles. (*Semi-regular* means that every vertex has the same pattern of regular polygons surrounding it.) The easiest one is simply alternating rows of squares and triangles. (See Figure 2a.) Each vertex is surrounded by the pattern triangle-triangle-triangle-square-square, sometimes referred to as 3.3.3.4.4. No one constructed this tiling, though some did organize their tilings partly or completely in rows of some sort. The other semi-regular tessellation is the 3.3.4.3.4 tiling (or *snub square* tiling; Figure 2b) whose pattern is triangle-triangle-square-triangle-square around each vertex. This tiling is visually complex, but some children found patterns close to it. If we relax the condition of semi-regularity, there are infinitely many possible tessellations with squares and triangles. In this study, no two students found the same pattern.

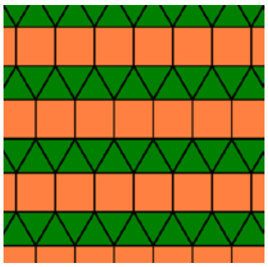

Figure 2a. The 3.3.3.4.4 semi-regular tiling

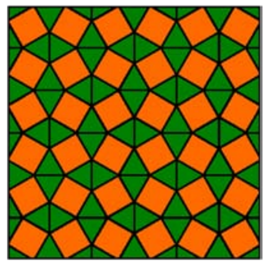

Figure 2b. The 3.3.4.3.4 (snub square) semi-regular tiling

Data Analysis

The primary motive during the interview was to discover how the children were conceptualizing tiling. There was no deliberate probing concerning the aesthetics of each task. Observations concerning aesthetics were only considered after the data were collected. The interviews were video recorded. Detailed notes of each video were made and then analyzed by coding various factors such as perceived tiling possibility, pattern found, initial confidence, final confidence, solution symmetry, and reasons that justified the tiling. These data were then analyzed using an aesthetic lens as an interpretive framework (Sinclair, 2006).

Results

I will focus on the results from the two dot paper tasks and the squares and triangles tiling task. I will draw from the responses of three students: Arlene, Don, and Alicia (pseudonyms). Don and Alicia worked together; Arlene worked alone.

L-Trominoes

Nine children were asked to tile with the L-tromino on dot paper. I drew the first shape and then asked them to draw the other shapes to make a tiling. All but one student believed it was possible to tile with such a shape, though two expressed some hesitation. Six of the children made rectangular units of the shape. The other three created apparently aperiodic tilings which they either denied could work, or expressed hesitation about.

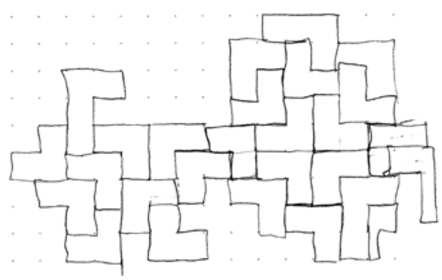

Figure 3. Arlene's initial attempt at left; symmetric attempt at right

Arlene did not seem to know at first if such a tiling was possible. She was one of the three children who attempted to make a tiling by drawing shapes

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seemingly at random. After drawing eight tiles, she decided that the tiling would be possible. Her explanation depended on the fact that the L-trominoes could be placed symmetrically, though her current drawing displayed no symmetry. When challenged to explain further, Arlene attempted to draw a new pattern with two-fold reflection symmetry. (See Figure 3.) Even though she was not completely successful, the fact that she could place the trominoes somewhat symmetrically, together with the fact that she was able to place 27 trominoes without creating a hole, convinced her that a tiling was possible. In other words, her certainty drew partly from the empirical evidence that the tiling was working so far, and partly from the possible symmetry of a successful tiling.

Don and Alicia quickly saw that a tiling was possible with the L-trominoes by putting two trominoes together to make a rectangular unit. Because the rectangle was easy to tile, a tessellation could quickly be made with these units. Wheatley & Reynolds (1996) have also found that many students solve such tiling tasks by unitizing. After drawing the first rectangular unit, Don and Alicia explicitly used a reflection to create a second unit symmetric to the first. (See Figure 4.) They claimed that further reflections could be used to create an infinite rectangular tiling. Simple translations would have been sufficient for tiling their rectangles, but for some reason, perhaps aesthetic, they preferred using reflections to complete the tessellation.

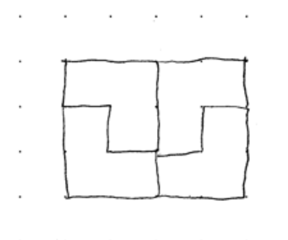

Figure 4. Don & Alicia's tromino tiling

T-Tetrominoes

Six students were given the T-tetromino tiling task. Two believed the shape could tile; two believed it could not tile; and two were not sure. Only Don successfully created a mathematically valid tiling. Most of the children placed tiles at random in an apparently aperiodic pattern. Arlene was not given the T-tetromino task.

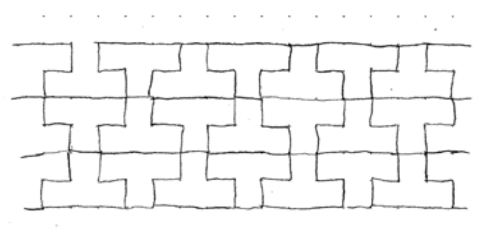

Figure 5. Don's T-tetromino tiling

At first, both Don and Alicia were confident that a tiling could be made with the T-tetromino and they tried to make a pattern together. After reaching a few impasses where no new piece would fit, they lost their confidence that the tiling was possible. They continued to work separately on two different patterns. Don succeeded in making a nice row pattern, effectively demonstrating that the tiling was possible. (See Figure 5.) Notice that he used reflection symmetry to create subsequent rows from the first row. Alicia continued with a seemingly random pattern of 26 T-tetrominoes which seemed to fit nicely together. Because the tiling was working so far, Alicia decided that this empirical evidence was enough to conclude that her tiling might work, but she was not confident of her answer.

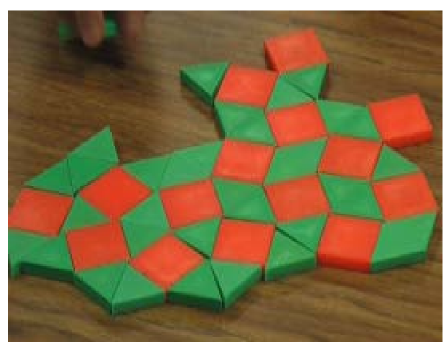

Figure 6. Arlene's final pattern

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Squares and Triangles

Nine children were asked if it was possible to tile with squares and triangles together. Students created various patterns and most concluded that such a tiling was possible, though only four students succeeded in making patterns that were clearly extendable in all directions.

Arlene successfully made the snub square tiling up to eight squares and sixteen triangles. She described her pattern as repeatedly combining a square and a diamond (two triangles). Perceiving this pattern helped her to place the pieces—always putting two triangles together next to a square. This was possibly not a case of true unitizing because it is not easy to visualize how such units tile, though there was unitizing of the two triangles as a "diamond." She then put the seventeenth triangle where a square should have gone without realizing it. After reaching a point where her pattern clearly broke down, she was unsuccessful in locating the source of the problem and decided (without conviction) that the tiling would probably not work after all and that somewhere she would eventually face an arrangement where no tile could fit. (See Figure 6.) She was not sure if the problem was simply that squares and triangles cannot work together, or if she had made a mistake somewhere, though she suspected the former.

Don created a tower of widening rows—each row was one unit wider than the row above it. Though apparently finite, Don said that the rows were meant to continue indefinitely in each direction. The finiteness of the tower width may have been a result of trying to create a pattern with left-right symmetry, an innate aesthetic response.

Alicia and Don then worked together on another pattern that they put together one tile at a time without any clear preconceived structure. With the addition of the fifth piece, the small structure gained simple reflection symmetry. The symmetry was lost for a moment and then regained with the eighth piece. With the thirteenth piece, they managed to form a pattern with two-fold symmetry. Don commented, "I see a star and an X," pointing to the symmetry of the shapes. (See Figure 7a.) From this point on, they were very careful to plan the structure in a way that conserved the two-fold symmetry, though parts were left undone due to lack of tiles. (See Figure 7b.) They tended to clump the triangles together in regions separated by rows of squares without any clear idea of exactly how the structure would continue. But the pattern was clearly symmetric, simple to understand locally, and aesthetically pleasing. This seemed sufficient to assure them that they would be able to continue some sort of pattern forever.

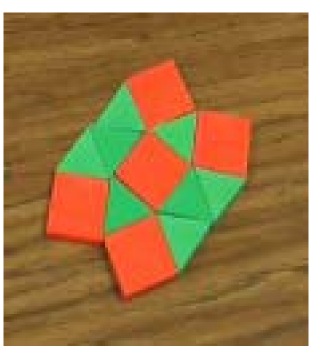

Figure 7a. "I see a star and an X."

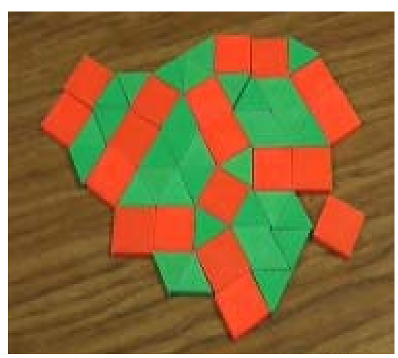

Figure 7b. Don & Alicia's final construction

Discussion and Conclusions

The Power of Symmetry

Finding symmetry enabled children to believe that a tessellation was possible. They did not have the mathematical skills for investigating aperiodic tilings, but symmetric, repeating tilings could be seen to continue indefinitely. Translation symmetry (finding a repeating pattern in both dimensions) is the simplest kind of symmetry and one that is necessary for mathematical certainty of tiling at these children's level of understanding. More interesting is when the children found reflection and even rotation symmetry. Arlene began with a random, aperiodic tiling, and then suddenly saw that it must be possible to arrange the L-trominoes symmetrically. Though she did not quite succeed, she remained convinced that symmetry was possible and therefore a solution was possible. This generative thinking was mathematically correct. With more time, if she had continued to try (perhaps with a little help) to arrange the trominoes with a two-fold symmetric pattern, she could have succeeded.

Children were often convinced by empirical evidence that a tiling was possible, even when a pattern was not found. Alicia's empirical evidence for the T-tetromino was not sufficient to convince her fully because it lacked symmetry. Don and Alicia were fully convinced by their empirical evidence for the squares and triangles tessellation because of its multiple symmetries. In both cases, the children failed to find a solution that was mathematically complete, though Don's and Alicia's symmetric construction was closer to a solution and could have led to one with time and more pattern blocks. Alicia's sense of aesthetics led her to be dissatisfied with a construction that lacked symmetry because it could not lead her to a fruitful idea for tiling. A mathematician, with a somewhat different aesthetic, might have found this apparent lack of symmetry intriguing and explored the possibility of a sophisticated, aperiodic solution. But Alicia's aesthetic correctly led her to evaluate the types of solution that would be fruitful for her mathematical abilities at this time. Aperiodic tessellations are mathematically difficult to work with. Thus the aesthetics of symmetry indicated to these children which attempts were mathematically generative for them.

Symmetry was not always necessary in the solution, but it gave a more appealing answer, such as Don's use of reflection in his T-tetromino rows. The use of reflection symmetry may make the answer clearer, more convincing, and aesthetically appealing.

Symmetry is not a heuristic that inevitably leads to correct solutions. The left-right symmetry of Don's tower of squares and triangles was irrelevant to the tiling he was trying to visualize and could have led him astray, as in fact it did with two other children who made similar structures. Don and Alicia's reflections to create an L-tromino tiling were not any mathematically better than simple translations (though not worse either, and perhaps more pleasing). Symmetry serves, not as a logical indicator of the correctness of a tiling, but as an aesthetic device that lends confidence to the children's solutions by making extensions of their tilings easier to visualize. Even when they could not visualize the extension, the presence of symmetry served as an indicator that they were on the correct path towards a valid solution, an aesthetic conclusion that was more likely correct than not.

Moreover, this aesthetic sense of symmetry can lead to an awareness of mathematical structures expressed in the tiling (cf. Piaget, 1971). Don's perception of "a star and an X" is the precursor to a more sophisticated understanding of mathematical symmetry.

Looking at how the aesthetics of symmetry guides children's thinking may seem like a small consideration. But symmetry is only one aspect of aesthetics. There are many other aesthetic considerations which guide both mathematicians and children in mathematical inquiry. We need to begin to look at all the ways aesthetics, and not just logic and heuristics, guide children's mathematical thinking.

Children's aesthetics is not mature and could be improved through enculturation in the mathematics classroom. Sinclair (2008) has already observed how such enculturation is taking place implicitly in some classrooms. Children's aesthetics could be more finely developed if teachers gave more explicit guidance in the aesthetics of mathematics.

By becoming more aware of the ways aesthetics guides thinking, especially in exploration and evaluation, we can become better able to give explicit guidance to students on the aesthetic aspects of mathematics. More research is needed in this area in order to understand better how aesthetics guides children's mathematical thinking and how this sense of aesthetics can best be developed in an educational environment.

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WHEN IS SEEING NOT BELIEVING: A LOOK AT DIAGRAMS IN MATHEMATICS EDUCATION

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This paper seeks to complexify a relationship between students' geometric reasoning and diagrams used in geometric instruction. The van Hiele levels are used as a frame through which classroom task diagrams are examined. Possible student interpretations of the diagrams biased on different van Hiele levels are compared with the intended meaning of the diagrams in the task. Future research is recommended to examine this issue.

Introduction

In elementary school textbooks and during enacted lessons, students are shown a picture of a shape such as that in Figure 1 and are expected to name it based on the visual representation. For example, if the quadrilateral does not appear to have equal-length sides, the student is supposed to conclude that it is not a square. The student is not expected to describe the shape in Figure 1 as a parallelogram. Instead, students are expected to pay attention to the shape as a whole and match the shape to others they have seen before.

Figure 1. What is the name of this shape?

In contrast, in high school geometry classrooms, when given the same diagram (Figure 1), students are expected *not* to claim that the shape is a rectangle. In fact, many geometry teachers, frustrated why students would assume the angles of the shape are right angles, instruct their students not to "trust" geometric diagrams by appearance. For example, students are told that just because two sides look parallel doesn't mean that they are actually parallel, unless markings or a direct statement is provided that indicate that the sides are indeed parallel.

In this paper, we explore this change of how geometric diagrams are used to convey meaning (visual rhetoric) in textbooks from elementary school to high school. We argue that a fundamental shift of the diagram as *the shape* (an image) versus the diagram as a *representation of a shape that is not shown, but imagined* occurs in textbooks. We suspect that some students make this transition, while others do not, creating the situation where the diagram is meant to mean one thing (a quadrilateral in which we do not know anything special about the sides or angles), while the student sees another (a rectangle). In this paper, we explore examples of visual rhetoric from textbooks and discuss their potential meanings. Finally, we conclude with recommendations for future study.

Theoretical Framework

The van Hiele framework is a hierarchal framework that describes different levels of students' reasoning in geometry. The five levels used in the framework describe different ways that

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students could reason about geometric topics, from visualization and analysis at lower levels to abstraction, deduction and rigor at higher levels (Burger & Shaughnessy, 1986). The van Hiele framework explains that different individuals (such as the teacher and a student) can use the same words (such as "rectangle") with different intension, and thus essentially talk past each other. For example, a teacher might want to talk about a quadrilateral with four right angles, but the student might instead think of the image of a rectangle and not even recognize that it has four right angles. Thus, for classroom discourse to be effective, the ways in which topics are discussed in the classroom should coincide with the van Hiele levels that students are reasoning at. For example, discussions about the definitions and properties of shapes may not be effective if students have only begun to reason about shape by comparing pictures of shapes to real world objects that look similar.

This paper uses the van Hiele levels as a lens through which to compare the visual rhetoric used in elementary textbooks compared to that found in high school textbooks. In the same way that the effectiveness of words can be hampered by differing van Hiele levels, we look at how the geometric diagrams used in mathematics at different grade levels might be affected by differing van Hiele levels. To do this, we focus on the three lowest of the van Hiele levels, denoted here as levels 0, 1, and 2. Burger and Shaughnessy (1986) identified different indicators that suggest a student may be reasoning at a particular. For the purposes of this paper, we included descriptions of some (but not all) of the indicators at these three lowest levels.

Level 0

Students recognize shapes by their by appearance alone. The figure is perceived as a whole, and the properties of the shape are not recognized.

Level 1

Students are able to sort shapes by single attributes and are able to recognize properties of the shapes, but do not recognize relationships between the shapes.

Level 2

Students are able to define shapes and recognize that different criteria may be used to define shapes. Students can sort shapes by a variety of mathematically precise attributes and can recognize relationships between shapes (such as understanding that a square is a special rectangle).

Task Descriptions and Student Reasoning

So what are the different ways that geometric diagrams are used in mathematics textbooks to convey information? In this section, we examine three different tasks adapted from actual prompts found in a variety of United States texts across different grade levels. For each, we discuss the diagram(s) provided in the task, as well as how it might be perceived by students of different van Hiele levels.

van Hiele Level 0 Reasoning Task

The task in Figure 2 was adapted from an activity in a text intended for a kindergarten classroom. As part of the task, students are presented with pictures of real world objects. We note here that the pictures do not include any additional markings added to indicate additional information about the objects. The only information students have to rely upon are the visual images of the objects. Students are asked to identify the objects that are circles as well as rectangles.

Directions: Have students mark an X on the objects that are circles. Then have them put a circle around the objects that are rectangles.

Figure 2. A kindergarten task adapted from a United States math textbook

This task seems to fit very well with a type of reasoning about geometry that is described in the Level 0 of the van Hiele framework. Here, students are expected to use the visual qualities of the pictures of the objects to identify and sort the shapes. These prototypes of the shapes are being used to help students pay attention to certain characteristics of the objects. Thus, the student is expected to identify the dinner plate as a circle and the clipboard as a rectangle based on appearance. However at this point, students might not recognize the framed picture as a rectangle.

While this task seems fairly straightforward for someone who is reasoning at Level 0, how might students at other levels of understanding of geometry interpret these shapes? For students at Level 1, who begin to pay attention to the attributes of shapes, they might be critical of the CD and the dinner plate, measuring the several different diameters to check that it really is a circle and not an oval. These students might also argue that the briefcase and the clipboard do not have straight sides, so therefore they are not rectangles. However, students reasoning at Level 2, who begin to use mathematical definitions and pay attention to mathematically precise attributes, may require additional information about the shapes before they would classify the objects. They may want to know if the sides of the clipboard are really parallel, or if the framed picture indeed has equal-length sides.

In each group of triangles, circle the one that doesn't belong. Explain how you know that it doesn't belong.

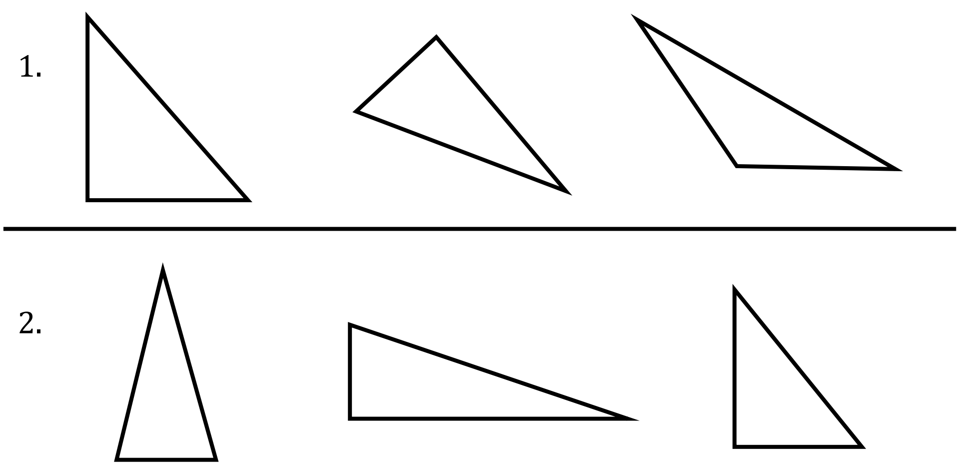

Figure 3. A second grade task adapted from a United States math textbook

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van Hiele Level 1 Reasoning Task

The task in Figure 3 was adapted from a textbook intended for second grade mathematics. Students are given a set of three triangles and asked to circle the one that “doesn’t belong.” Again we note that the images of the triangles are only the three sides of the triangles, no additional information about the triangles are included.

This task seems to be aimed at those students who are reasoning in ways similar to the ways described in van Hiele level 1. Here students may be sorting shapes based on a single attribute, while ignoring other attributes. Indeed in each line of this task, two of the triangles contain apparent right angles, while one triangle contains angles that appear to measure 90°. This task seems to expect to have students identify the non-right triangle as the figure that “doesn’t belong.”

Next, we consider how students who reason at a Level 0 might use these diagrams. In the first line, students might identify the middle triangle as the one that does not belong since its base is not horizontal, a common way that the image of triangles are portrayed. A student reasoning at Level 2 might require additional information about the triangles, and will not assume that angles that appear to measure 90° do.

Classify quadrilateral ABCD using the given information.

1. $\overline{AC}$ and $\overline{BD}$ bisect each other.
2. $\overline{AC}$ and $\overline{BD}$ bisect each other, and $\overline{AC} \cong \overline{BD}$.
3. $\overline{AC}$ and $\overline{BD}$ bisect each other, and $\overline{AC} \perp \overline{BD}$.
4. $\overline{AC}$ and $\overline{BD}$ bisect each other, $\overline{AC} \perp \overline{BD}$, and $\overline{AC} \cong \overline{BD}$.

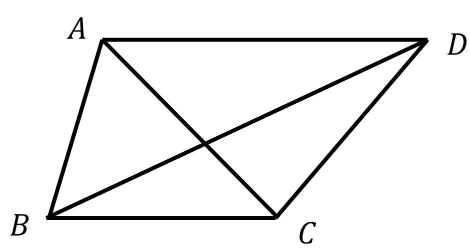

Figure 4. A high school geometry task adapted from a United States math textbook

van Hiele Level 2 Reasoning Task

The task in Figure 4 was adapted from a textbook intended for secondary school geometry. In this task, students are given one diagram for several questions. In each question, certain relationships between parts of the figure are stated. From these criteria, students are expected to name the shape. It is noted that the diagram used does not appear to satisfy any of the criteria of any of the questions. Also, while the questions all refer to the same figure, some criteria are given for certain questions, while excluded from other questions.

This question appears to require reasoning that is similar to the kind described in van Hiele level 2. Students are expected to pay attention to the definitions of different shapes, and explicitly reference them. They are also expected to sort shapes according to their mathematically precise attributes. This prompt seems to expect students to pay attention to the relationships described in each question, and not the visual shape of the diagram, in determining the different shapes.

Consider how someone reasoning in a manner similar to Level 0 would engage with this task. For this student, they have not yet begun to consider formal mathematical definitions of shapes, or consider the implications of describing parts of the figure as bisecting each other. Instead they might look at the diagram and conclude that the shape does not look like any of the shapes they have special names for, regardless of the verbal information printed in the text. Someone reasoning at a Level 1 might not be able to consider a definition of figure in terms of its diagonals, instead of its sides.

Discussion

An alignment between student reasoning of geometry and reasoning about a diagram is, undoubtedly, preferred. However, several interesting things can occur when there is a non-alignment in the ways that students are reasoning about geometry and the reasoning required to use a diagram in its intended way. If a student is at a lower van Hiele reasoning level than what the diagram is requiring, students may end up using empirical evidence from the diagram in their justifications of naming the shape. The diagram may not be intended to be used in this manner, especially if it is not drawn to scale. As stated previously, this may occur when students try to claim a figure is a rectangle, while a problem may be set up only to indicate that the figure is a quadrilateral.

The mismatch of visual rhetoric for different van Hiele levels can also occur in the other direction as well. Students may be at a higher van Hiele reasoning level than the reasoning intended to use a diagram. In this situation, students may be intentional trying to avoid using visual, empirical evidence from a diagram. The question, however, may be intending for students to generalize based on only visual clues from a diagram.

There is a transition in school geometry of which students and teachers may not be explicitly aware. Early elementary geometry often relies of visual evidence of figure, wanting students to generalize from pictures. High-school geometry often relies on logical conclusions drawn from an axiomatic structure. In this system, diagrams are not utilized as justifications. There is a transition here between these two extremes that still remains unstudied.

This research identifies the potential confusion that can exist when the meaning of a geometric diagram requires a different level of van Hiele level than that of the student. This appears to be an area where further research would not only be fruitful, it would also be important. Students often struggle with geometry and reasoning around diagrams could be involved. How aware are teachers of the reasoning necessary to use a diagram? And how might teachers use diagrams in the classroom during lessons or on assessments? How does student reasoning around diagrams change as students progress through elementary school and into high school? How do we support students who might not have adopted new reasoning levels? How is this change of visual rhetoric related, or unrelated, to changes in verbal rhetoric? These are just a sample of questions that future research can attempt to address.

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YOUNG CHILDREN'S INTUITIVE KNOWELDGE OF REFLECTIVE SYMMETRY

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The goal of this study is to investigate young children's intuitive knowledge about reflective symmetry concept and their performance in reflective symmetry tasks. The study was conducted with 15 second graders. The result indicated that 2 out of 15 second graders were able to create a reflective symmetry design successfully. The study suggested the idea of a mirror image appears to be of a higher developmental level than the translation of shapes. We found that knowing the orientation of the shapes seemed to be a more important predictor than finding the distance to the axis to perform reflective symmetry tasks.

Introduction

Geometry and spatial thinking are important ideas to develop because they involve the space where the children live, breath and move (National Council of Teachers of Mathematics – NCTM-, 1989). Clements and Brista (1992) addressed spatial reasoning, which is the ability to "see" and reflect on spatial objects, images, relationships, and transformation as the fundamental constructions in geometry thoughts. Transformation includes the operation of images; such as rotation, reflection and sliding of shapes and symmetry as one of the concepts that are connected to transformational geometry. Children's beginning concept of geometry is not just about shapes, but also about symmetry, congruence, and transformation (Clements & Sarama, 2007). Research studies that addressed young children, stated that they develop preference of symmetry as early as 12 months (Clements & Sarama, 1992; Vurpillot, 1976).

Despite young children's cognitive development of symmetry, little work has been done regarding the intuitive knowledge, which children have when they are presented with geometric tasks. Almost no research study has been conducted to investigate the relationships between elementary students' intuitive knowledge and reflective symmetry tasks.

The authors of this paper investigate the intuitive ideas of symmetry of second grade students. Knowing how children think about geometry will help us understand solutions to a variety of problems in mathematics by using pictures, a common aid children used to solve different kinds of problems.

Theoretical Framework

Palmer and Hemenway (1978) documented that there seem to be two phases in recognition of symmetry. The first stage is to select potential axis of symmetry in all orientations simultaneously defined by mirror-similar parts, and then perform a detailed evaluation of symmetry about the selected axis by comparing the two halves for mirror-identity. With respect to the symmetry axis, many studies indicate that bilateral (or "mirror-image") symmetries which axis is a vertical line are perceptually most prominent, horizontal on the other hand is less dominant, and the diagonal line of symmetry is the most difficult (Hoyles & Healy, 1997; Palmer & Hemenway, 1978; Vurpillot, 1976).

Meanwhile, Vurpillot (1976) reported that mirror images were particularly difficult for young children to distinguish compared to other types of transformations. The task of his study was to distinguish figures between simple rotation (e.g. 90 degree rotation) in a single plane from

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inversions or rotations in vertical/horizontal presentations. He also reported regardless that they were mirror images or not, the transformation from top toward bottom is easier for the children than the change from left to right.

As with symmetry, the children were asked to compare two objects which were similar but not physically identical. Vurpillot (1976) stated that many young children judged congruence based on whether the objects are more similar than different. Another study (Rosser, 1994) documented that orientation was a significant feature for the kindergarteners to judge whether the shown pairs of figures (some congruent but all rotated) were different. Rosser's study indicated that in addition to the congruency of shape and sizes, and understanding the rotation of a shape is a critical feature in reflective symmetry. However, research showed that children's concept of equivalence may develop as young as 18 months but the consideration of its spatial relationships gradually develops later (Vurpillot, 1976).

Hoyles and Healy (1997) analyzed students' thinking-in-change of reflective symmetry. They stated that it is important to know the property of symmetry in order to understand the concept of symmetry. The study noted the property as the reflective image having the same size and shape as the original, it is at the same distance away from the mirror (symmetry line), and is "opposite" or "reversed". Holyes and Healy indicated that without clear understanding of this property students are more likely to make an error when they creat symmetrical shapes.

The goal of this study is to investigate second graders' intuitive knowledge about the concept of reflective symmetry and their performance in creating reflective symmetry tasks. We chose this subject for two reasons. First, limited research has focused on elementary students' intuitive knowledge of this concept. Second, the concept of symmetry is fairly simple and can be readily learned by young children, but it is not typically taught until upper elementary grades.

The research questions are:

1. What are young children' intuitive knowledge about reflective symmetry?
2. What types of knowledge do children need to learn in order to understand this concept?

Method

Participants

The subjects for this study were 15 second grade students (eight girls and seven boys) from three classrooms (five students per each classroom) in two different schools. These schools are located in an urban area of the Southwest of the United States and are currently participating on a National Science Foundation (NSF) professional development project for elementary mathematics. More than 90 % of the student population of both schools is Hispanic and receiving free lunch. Under the district policy, the classes were grouped by English proficiency level. Students' English proficiency level was a selection criterion because communication was necessary to investigate the students' thinking process through interviewing. All participants were Hispanic.

Data collection

Two types of data were collected in preparation of this paper: 1) individual interviews with students using paper and glue to create symmetrical designs, and 2) paper and pencil tasks.

Tasks

Each student was asked to participate in three different tasks. The first task included interviewing the student about what he or she saw when creating symmetrical designs with glue while folding paper. The interviewer drew odd shapes including a dot, a curvy line and a straight line with liquid color glue on the left side of a paper. Next, the interviewer folded the paper in half to produce the symmetrical shape on the right side (see Figure 1).

The interviewer opened the paper carefully after asking the student to predict what would happen, and asked the student following questions about the new design: What do you think about this shape? Can you explain more about this? Can you describe it for me? Can you compare the two halves? How do you know these are the same? Can you find other shapes which are the same? Are there any differences between these two? How can you know those are different?

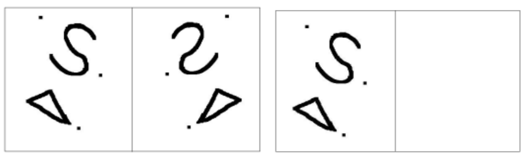


Figure 1. Example of task 1

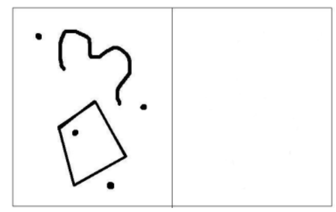


Figure 2. Example of task 3

The second task was similar to the first, however, children were asked to draw a shape by their own and fold the paper in half to produce the symmetrical shape, and observed the result. After folding and unfolding the paper, the interviewer encouraged the student to describe what they saw and tell why they thought these shapes were the same or not. When the students had a difficult time explaining, the interviewer asked similar questions as the ones listed above.

In the third and final task the students were instructed to draw with a pencil on the right side of the paper showing what s/he expected to see if the paper was folded (see Figure 2). During this task, the students were asked to explain how and why they expected such a figure. Students were allowed to use mirrors, pop-sickle sticks, or strip papers to help them measure in order to aid themselves to accomplish their goal.

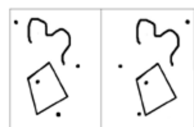
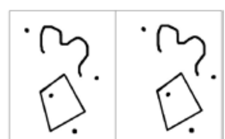
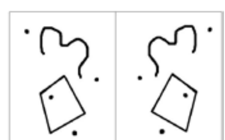
Interview

The authors interviewed the second graders during the month of May, which is at the end of the school year. Since each interviewer observed students' mathematics classroom regularly for the entire school year as part of the NSF Project, they were familiar with the interviewers. Each child was interviewed individually for approximately thirty minutes. The entire interview was audio recorded and transcribed for the analysis. Each student's work from the three activities was collected. The interviewers took notes as the students were solving the tasks. If the interviewer did not understand what the child explained, she asked the student to explain further until it was understood what method she or he was using to solve the task.

Data Analysis

Children's explanations were coded in terms of whether they were correct or not, and if their drawings for the third task were right or wrong. According to the children's explanations and drawings, their thinking about symmetry was analyzed. Children's symmetry understandings were categorized by 1) size, 2) shape, 3) distance, 4) copy, and 5) mirror image. Each category is explained in the table 1 below.

Table 1. Explanation categories, definitions, and examples

Identification criteria	Definition	Examples
Sizes	Focus on the fact that the size of figures on two sides are identical or not	"I think they are the same because they are the same size"
Shapes	Focus on the fact that the shapes on two sides are identical or not	"I think they are the same because the shapes are the same" "they are kind of different because they're facing different"
Distance	Focus on the distance relation but lack of reflection of the figure	
Copy	Lack of both the distance relation and reflection of the figure	
Mirror	Recognize mirror relations including both: reflections and distance	

Results

The careful analysis of the interviews and the drawing resulted in the discrimination of the following categories: size, shapes, distance, copy and mirror image. The "size" category shows if children identified the figures on each side of the symmetry by axis or by flipping lines to discover if the figure was the same size or not. In the "shape" category, the information collected tells us if children realized the shapes were the same on each side of the axis, or if they failed to realize that due to the flipping nature of one of the halves. In the category for "distance", the authors coded those students that placed the figures on the right spot relative to the symmetry side, but they failed to "orient" the figures, not obtaining a mirror image. Dots are a special case inside this category and will be discussed separately. As one can imagine, under "copy" fell all of the drawings made by the children that failed to place the figures in the right position relative to the axis, and that were not flipped either, but that as a whole, the children drew it as a copy of the original, what could have been a translation of a vector with a magnitude equal to half the page. Finally, in the "mirror image", we found those students that succeeded in drawing a symmetric picture of the original design they were presented with.

Considering the categories mentioned before, the following results were obtained:

Table 2. The result of students' understanding of symmetry

Student	Size	Shape	Distance	Copy	Mirror Image
1	yes	yes	no	no	yes
2	yes	yes	no	yes	no
3	yes	yes	no	yes	no
4	yes	yes	no	yes	no
5	yes	yes	no	yes	no
6	yes	yes	no	no	no
7	yes	yes	no	yes	no
8	yes	yes	no	no	yes
9	yes	yes	yes	no	no
10	yes	yes	no	yes	no
11	no	no	no	no	no
12	yes	yes	no	yes	no
13	no	no	no	yes	no
14	no	no	no	yes	no
15	yes	yes	yes	no	no

Size and Shape

When children were presented with the symmetric design, in almost 100% of the time they said that the two halves were "the same". The students who focused their attention on the shapes of each side said that what was presented on the left side was also in the right half. Eighty percent of the students agreed that the sides were "the same". In one case, student number three, pointed out that the sides "look the same, they are the same", to explain how the two halves compared to each other. In another case, student number five, was prompted to describe the differences between the two sides of the first design with glue and responded by saying: "everything is the same". This idea that the two sides were "the same", even though they didn't exactly look that way, is linked with the category "copy" and will be further discussed in the coming section.

Another aspect of the glue design according to the second graders was the size of the shapes. In most cases, 80% again, they told the interviewers how shapes and lines were, or should be, the same size. On the third task, where the students were asked to make a symmetric drawing, they showed a conscious effort to reproduce the figures and lines by making sure they had the same size as the original. Students took measures with the provided paper rulers, their fingers, and sticks. In some cases they made sure that the interviewer knew that even though their work didn't reflect the same size, it should have, and told how they were trying to reproduce the same measures without a graduated ruler.

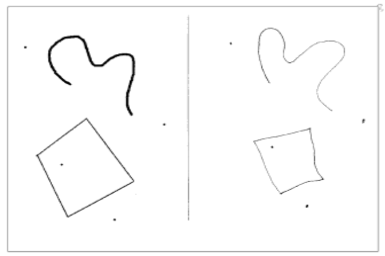


Figure 3. Student's work example of copy

Copy

On task three, 9 out of 15 students copied the original, to the other side of the axis or flipping line. Interestingly, this reproduction of the original matches what they described in task one: they mentioned both sides of the first glue picture were the same, although they weren't since once was a mirror image. On task three, by making a copy of the given design, they made sure both sides were actually the same. Students who made a copy of the last activity, and that were asked if they wanted to fix or change any detail of their work, focused again on the size the figures they drew, not on the fact that they were not mirror images.

Distance

Only two cases out of 15 students fell into the Distance category. These two students, who mentioned that both sides of the figure were the same, and that the respective sizes were equal as well, placed lines, figures and dots in the relative correct spot; however, they failed to "flip" the figures or lines. On the other hand, the dots they reproduce on the third task were perfectly symmetrical to the original ones.

In the presence of this data the authors make two hypotheses regarding this category and how children behave when trying to reproduce a mirror image of a given figure.

First hypotheses: Children are able to reproduce a mirror image when they are not distracted by lines or orientation. Children were correct almost 100 % of the times in their prediction when they were asked to "guess" where one dot of glue was going to "appear" if we closed the paper and the two halves touch.

In two cases, when two different second graders were prompted to talk about how they knew where the dot was going to be, they said that they imagined a "straight line" and showed with their hands a line perpendicular to the axis, from the original to the symmetric side.

Second hypotheses: Children who are able to draw objects and place them in the correct relative position of the paper (distance), but not flipping, are one level closer to being able to obtain a mirror image, compared to those children who did a copy.

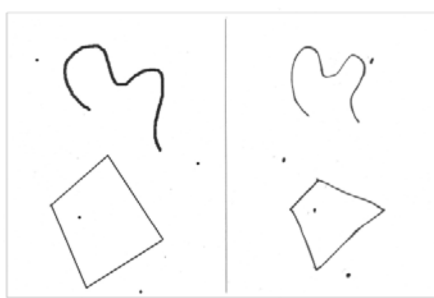


Figure 4. Student's work example of distance

Student 15 copied all the shapes as the other students but while explaining to the interviewer, he corrected himself. He said he was wrong because the dots were not going to touch each other if he folded the paper. He wanted to do it again and figure 4 is the result of his second trial. The placement of the dot was correct but the orientation of other shape remained the same including the inside dot.

The results indicated that most students were able to predict where the symmetrical of one single dot was going to be, but none of the second graders were able to do this with a more complicated design, nor did they place a set of dots when the dot was presented with or inside the figures.

Mirror Image

When the interviewers prompted the students to talk about the differences, they learned that when looking at both sides of the glue designs, one student mentioned that one of the sides was “flipped”, but this student was not able to reproduce this idea on the third task. Student number three, said: “This is the same... but... kinda like (motioning a rotation with her hand) the same but don't.” This student also failed to reproduce a mirror image on the last task. What is interesting in this case, as well as in another two cases, is that students appeared to realize that the two halves were not actually the same, but lacked the vocabulary to describe what they saw. Being this the first time they are asked to do so, and considering these students were in their majority considered English Language Learners (ELL), the authors questioned what kind of information would have been shared if they could have explained it in their native language.

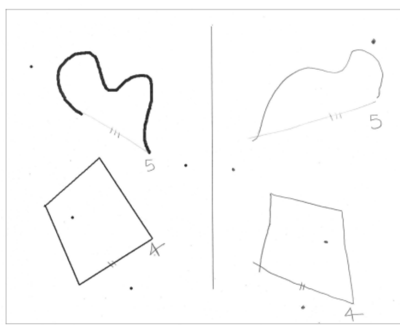


Figure 5. Student’s work example of mirror

Two out of the 15 students made a better attempt, to draw a mirror image, than the rest. In the case of student number one, the work was totally successful, and in case of student number eight, the work was partially successful, with some problems keeping the size of the figures equal to their originals. This student had pointed out that the sizes were the same, but it is assumed that due to his motor skills not being refined yet, the drawing came out smaller than what he was observing in the original.

Student number one, whose drawing is shown left, (see Figure 5) started his work as a copy, and then taking a second look exclaimed “Oh! It had to be the other way” and asked for a new piece of paper to draw what was his final work. The numbers that appeared in the picture were numbers that the student wrote to symbolize how the distances between two sets of points were the same. In this example, he said that if the distance between the extremes of the line were 5, no unit given, then on the other side, on his drawing, it had to be 5 as well. And his second example was one side of the quadrilateral: he told the interviewer how that side had to be the same in the corresponding figures. The student explained that if the side of the original was 4, then it had to be 4 on his side of the drawing as well.

Conclusion and Discussion

The result of this study indicated two features of young children's intuitive knowledge about reflective symmetry concepts. The first one is that the idea of a mirror image appears to be of a higher developmental level than the translation of the shapes. The majority of second graders did very good copies of the original, sliding the shapes and almost achieving a translation of them, but only two students were able to create a mirror image.

Another finding revealed that the orientation of the shapes seemed to be the most difficult task for the young children to produce reflective symmetry successfully. Although they considered perpendicular lines to the axis to locate the figures on the other side, and they were successful locating single dots, the images of figures didn't have the right orientation. When the children had to think about the distance to the axis of single dots, they were much better able to predict the reflective symmetry position. It is generally assumed that the distance to the axis is critical in learning reflective symmetry, because finding the same distance from the axis is the first step to draw a symmetrical design. Nevertheless, this study indicated that the orientation of shapes is a more difficult concept for young children than finding the distance.

The implications for future practice involve teachers prompting students to explain what they mean by "the same" and work more deeply on what the differences are in two symmetric designs, original and image. The idea that the figures appeared "flipped", in respect to one another, should be highlighted and explored further in the classrooms with young children. Teachers should work in depth with the idea of "orientation" of figures, and also with the notion of distances to the sides and the middle, in our case, the axis.

Future research should focus on how labeling "the same" in the symmetric design influenced the production of a copy instead of a mirror image. The researchers also recommend that future projects observe if children can find the symmetry of a single dot, a dot embedded in a design, and a dot that is a part of a figure, for instance, a vertex.

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DYNAMIC GEOMETRY IN HIGH SCHOOL MATHEMATICS CLASSROOMS

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This four-year research project funded by the National Science Foundation compares effects of an approach to high school geometry that utilizes Dynamic Geometry (DG) software with standard instruction that does not make use of computer exploration/drawing tools. The basic hypothesis of the study is that use of DG software to engage students in constructing mathematical ideas through experimentation, observation, data recording, conjecturing, conjecture testing, and proof results in better geometry learning for most students.

Building upon previous studies, this study will seek to answer the following research questions: 1) How do students in the experimental condition perform in comparison with students in the control condition on measures of geometry standardized test and geometry conjecturing-proving test? 2) How does the DG intervention affect student beliefs about the nature of geometry and their beliefs about the nature of mathematics in general? 3) How does the DG intervention contribute to narrowing the achievement gap between students receiving free or reduced price lunch and other students? and 4) How is students' learning related to the fidelity and intensity with which the teachers implement the DG approach in their classrooms?

The research study follows a mixed methods, multi-site randomized cluster design. The population from which the participants of this efficacy trial are sampled are the 10th grade geometry teachers and their students at all high schools in Central Texas at which 50% or more of the students are eligible for free or reduced lunch. For determining the sample size, a power analysis has been conducted. Taking an attrition rate (20%) into consideration, 76 teachers are randomly selected from that population for the study.

Randomized assignment is used, with teachers as the unit of randomization. The 76 teachers selected are randomly assigned to two groups. Each teacher is represented in the study with measurements from only one classroom of students, and the classroom and teacher unit of analysis will overlap, yielding the design where the students are nested within teachers/classrooms, which are nested within schools. Teachers in both treatment and control groups receive relevant professional development. Fidelity of implementation for the experimental treatment is monitored carefully.

The implementation plan for the project is: Year 1: Preparation (All research instruments, DG instructional materials, recruitment and training of participants, etc.); Year 2: The first implementation of DG treatment, and related data collection and initial data analysis; Year 3: The second implementation of the DG treatment, and related data collection and continued data analysis; Year 4: Careful and detailed data analysis and reporting.

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GEOMETRIZATION OF PERCEPTION: VISUAL CULTURE AND MATHEMATICAL VISUALIZATION FOR PROFESSIONAL TEACHER DEVELOPMENT

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Visualization has been considered as an essential factor in learning and teaching mathematics. Thus, the issue of visualization in the teaching of geometry has been a main research focus for several groups, as they recognize its importance for student understanding of mathematical and geometrical knowledge. However, bringing up such questions implies putting together concepts related to the construction of geometrical knowledge and primacy of perception. This research project focuses on the analysis of one way of looking at and representing military plans, architectural designs and military engineering treaties. The empirical work will be done on military plans or architectural designs of the American fortifications in the seventeenth and eighteenth centuries, in accordance with the teachings and rules found in the treaties of military engineering. Theoretical assumptions come from the reflections on the construction and representation of militarized space in mathematical activity, to understand how the operation of perception has become geometrical and how they have created and used mathematical knowledge to represent with the technical perspective theory. This theoretical study is based on Foucault (1989, 2000), Certeau (2007), Flores (2007) and Meneses (2003). As a possible result, mathematical knowledge applied to historical practice can be an exercise in the professional teacher development as a means to design activities related to visualization in the teaching and learning of mathematics.

Endnotes

1. This post-doctoral research project was developed at the College of Education of the North Carolina State University in Raleigh, USA, in collaboration with Professor Paola Sztjan, with support from CAPES / Brazil.

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GEOMETRY ACHIEVEMENT PREDICTORS: PROPOSED REGRESSION MODEL

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This study sought to identify the predictors of the Geometry Achievement (GA) of the Grade 10 Science & Technology students of Charles Herbert Flowers High School (CHFHS) of Prince George's County Public Schools (PGCPS), State of Maryland, United States of America (USA).

The predictors in this study were the entry competence (verbal comprehension, verbal reasoning, figural reasoning, quantitative reasoning and the overall entry competence ratings), middle school achievement (non-mathematics courses, mathematics courses, Middle School Assessment, geometry pretest and overall middle school grades) and grade 9 achievement (non-mathematics courses, mathematics courses, High School Assessment and overall grade 9 grade). The criterion variable was the Geometry achievement.

Multiple correlation analysis of all predictor variables yielded a significant large positive correlation of 0.599. The aforementioned predictor variables if taken simultaneously explained 30.68% of the variance in the Geometry achievement. This suggested that there might be other variables aside from those included in this study, which predict Geometry achievement. The strongest predictor variable was the overall entry competence rating.

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Chapter 8: Methodology

Research Reports

Brief Research Reports

Posters

CONCLUSIONS WITHIN MATHEMATICAL TASK ENACTMENTS: A NEW PHASE OF ANALYSIS

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This study builds on the mathematical tasks framework developed during the QUASAR project by considering the ways in which enacted mathematical tasks are concluded. Though the framework originated from a cognitive tradition, this study takes a sociocultural perspective and reinterprets task phases through the lens of activity structure. Based on observations of 84 middle-school class periods and 4 detailed task conclusion analyses, a modification of the mathematical tasks framework to include a separate conclusion phase is proposed. Implications for analysis of cognitive demand are discussed, as is the relationship between task conclusions and characterizations of the nature of mathematics.

Introduction

Students' engagement with mathematical tasks is widely recognized as a central component of mathematics education at all levels. For the middle school level in particular, Stein and her colleagues (Stein, Grover, & Henningsen, 1996) built on the work of Doyle (1988) and earlier cognitive science to develop the mathematical tasks framework (see Figure 1). Within this framework, the definition of a mathematical task is based on the earlier concept of an academic task but is broadened in duration to align with the mathematical idea under consideration. In other words, a new mathematical task has not begun until the mathematical idea being focused on has shifted, which means that a single mathematical task may contain several academic tasks. The mathematical tasks framework utilizes the key notion of cognitive demand, the kind of thinking processes entailed in solving the task, which has been found to correlate with student performance (Stein & Lane, 1996). The framework becomes valuable as a research tool because its phases allow the level of cognitive demand to be tracked throughout the enactment of the task. For example, during the set-up phase, teachers may cast a task written to be procedures-with-connections as procedures-without-connections; or during the implementation phase, a doing-mathematics task may descend into unsystematic exploration or nonmathematical activity. Overall, we know that it is difficult for teachers to enact mathematical tasks in ways that authentically engage students in the thought processes of mathematics, even when a task is written with a high level of cognitive demand (Henningsen & Stein, 1997).

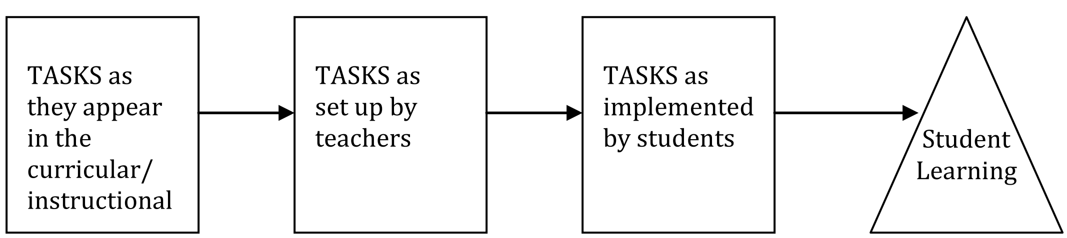

Figure 1. The mathematical tasks framework (adapted from Stein & Smith, 1998)

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

More recent work in mathematics education has brought to the foreground the sociocultural aspects of mathematics teaching and learning. This view implies that mathematical task enactments involve not only students' cognitive processes but also their participation in the activities and discourses of mathematics (Chapman, 2003; O'Connor, 2001; Otten & Herbel-Eisenmann, 2009). The current study views task enactments through the lens of classroom activity structures for the purpose of examining the alignment between the existing mathematical tasks framework and the nature of the classroom activity. In particular, I examine middle school level task enactments, as did the framework developers, and argue that this activity lens reveals a distinct phase of task enactments—the task conclusion—that has thus far been unrecognized by the framework. As is shown below, this conclusion phase is significant from an activity perspective but also has implications with respect to trajectories of cognitive demand. Thus a revised framework is proposed.

Theoretical Perspective

Within the broad realm of sociocultural theory, I employed Halliday's (Halliday & Matthiessen, 2003) theory of systemic functional linguistics (SFL) to examine discourse practices in middle school mathematics classrooms. SFL recognizes three metafunctions of language—ideational, interpersonal, and textual. Language is used to make sense of experience and in so doing serves the *ideational* metafunction; that is, it provides cues and clues regarding the meaning of *what* is being talked about. Language is also a means for acting out the social relationships of those *who* are using the language, thus serving the *interpersonal* metafunction. The *textual* metafunction refers to aspects of the organization of the language itself. Any use of language involves all three metafunctions, though not always to the same degree.

Within SFL, language is viewed as a system in which various sets of options exist with different meaning-making potentials. When someone is producing a text—written, verbal, or otherwise—they make many (possibly unconscious) choices about their language use, each of which influences subsequent choices and the potential ways in which the text will be construed. This language-in-use simultaneously shapes and is shaped by the context of the interaction. For example, a teacher closing the door and addressing the class in a full voice is serving the textual metafunction of marking the beginning of a lesson, but the local context (e.g., the bell ringing, students taking their seats and expecting a lesson to begin) are simultaneously influencing the teacher's discourse. This is closely related to Lemke's (1990) discussion of *activity structures*:

> [A classroom lesson] has a pattern of organization, a structure… In real life you never know for sure what is coming next, but if you can recognize that you are in the midst of a patterned, organized kind of social activity, like a lesson, you know the probabilities for what is likely to come next. (pp. 2–3)

Thus the various language options noted by SFL are restricted in a patterned way when the language is occurring within an activity structure such as lecture, going over homework, or small-group work. Furthermore, activity structures can be recognized by analyzing discursive cues and their textual functions marking the activity structure.

Method

Data for the present study consisted of task enactments drawn from a larger project involving eight middle grades (6–10) mathematics teachers engaged in literature study groups and cycles of action research around classroom discourse.[1] The videotaped class periods in which these task enactments occurred come from the baseline year of the larger project, before any interventions with the teachers had taken place. The present study is focused on the middle-school level, so two of the eight teachers were excluded because they taught high-school content. This left 84 classroom observations from six middle school mathematics teachers as the full corpus for the present study. (Note that there were less mathematical task enactments than class periods because many tasks continued over multiple class periods, and I excluded any tasks whose enactments were not entirely captured within the observed class periods.)

I reviewed all transcripts and watched video clips from the 84 selected observations. I took notes on discursive markings of the task phases (e.g., “teacher addresses class from the front of the room”), general features of the classroom events (e.g., "students worked in pairs on a handout for 20 minutes"), mathematical content of the tasks (e.g., "area of triangles"), and general impressions of what I perceived to be the task conclusion (e.g., "short conclusion with the teacher emphasizing main points"). Using these notes, I then selected four tasks to undergo detailed analysis. The selections were made based on several criteria (i.e., varied teachers, varied mathematical content, varied curriculum materials, varied features of the task conclusion), all intended to lead to a variety of task conclusions in the detailed analysis phase.

The four selected task enactments were analyzed using three different tools from SFL. First, discourse markers and their textual functions were coded with respect to the activity structures of the task enactment. This analysis is the centerpiece of the current study, which focuses on the alignment between the activity structures and the original enactment phases. Second, lexical chains (Christie, 2002) were created to map the semantic progressions of the task conclusions. Third, various linguistic analyses (e.g., theme analysis, amplification analysis, agency analysis; Martin, 2009) were performed with each of the four task conclusions to better understand their discursive nature. These latter analyses are only peripherally reported on here.

Results

Within the full corpus of examined task enactments, a strong alignment existed between the set-up phase of the mathematical tasks framework and the activity structure of “setting up” in-class work. This phase was characterized by the teacher addressing the entire class, usually from the front of the room, and directing student attention to the written task (often in a textbook, but also appearing on hand-outs, overhead transparencies, or the board). Examples of teacher talk marking the beginning of this phase are the following: “Take a look at the three triangles on the top of page fifty-eight…”; “OK, the first part of the assignment I’m handing out to you says…”; and “When you get this worksheet, I would expect to see…”. This phase continued with teachers maintaining the floor (Edelsky, 1993) of the interaction as they describe or clarify the written task and communicate what the students will be expected to do. The students during this activity are typically expected to sit quietly, listen, and possibly ask clarifying questions (e.g., “Are we going to turn this in today?”).

There was also a strong alignment between the start of the implementation phase and a shift in activity structure. In particular, the shift from the set-up phase to the implementation phase was cued discursively by the teacher, who indicated that the students should begin working on the task (e.g., “So go ahead and get started”), and by the students, who began working (often after physically rearranging themselves into pairs or small groups). The teacher relinquished the floor and the discourse separated into several smaller interactions (in the case of partner or group work) or relatively few interactions at all (in the case of individual seatwork). Within this activity structure, the teachers typically circulated the room, listening and observing students as they worked and also talking quietly with students if they had questions or if the teacher had a question for them. Students were expected to be actively engaging with the task and, when appropriate, interacting with one another around the mathematical ideas.

These results show that, as was expected, the set-up phase of the mathematical tasks framework is strongly aligned with the classroom activity structure at that point in the task enactment. Moreover, the beginning of the implementation phase corresponds with a marked shift in activity structure. Further analysis reveals, however, that an additional activity structure shift takes place later in the task enactment that does not correspond with a phase of the mathematical tasks framework. Specifically, an activity structure that may be described as a *conclusion phase* of the task enactment occurred throughout the corpus. I now turn to the results regarding this phase, first describing general trends before turning to the detailed findings of the four selected task enactments.

The beginning of a task conclusion is generally marked by the teacher's reclamation of the floor. Physically, teachers who have been circulating the room or working at their desk return to the front of the room. They also may turn on the overhead projector or write something on the board, drawing student attention from their own work back to the front of the room. Discursively, teachers raise their voice and address the entire class, typically communicating that work time on the task is complete, either explicitly (e.g., "OK, it's almost time to go, so please pass your papers to the left") or implicitly by indicating that something else is about to happen (e.g., "We're going to go over a few of them now"). The activity structure is teacher-led (though students may be called on to give extended explanations) but is distinct from the activity of the set-up phase since students are expected to answer questions, share solutions, or ask questions about mathematical ideas (rather than just about classroom procedures or task expectations).

The functions of the task conclusion and the relationship of the conclusion to the written task are also distinct from both the set-up phase and the implementation phase. The full corpus examination revealed that, within the conclusion activity structure, many different instructional functions existed. In many task conclusions, the teacher presented the answers to the written task, either verbally or on the board, and solicited questions from the students over any part that was not clear. Alternatively, teachers called on students to present their answers. In addition to this answer-presenting function, task conclusions often involved the sharing of various solutions. In some cases, this activity went beyond mere sharing into comparing or connecting different strategies. The task conclusion was also a place where the teachers highlighted or summarized main ideas and possibly asked students to reflect back on their work. There were instances of teachers prompting for generalizations of the solution strategies and of teachers using the task conclusion as an opportunity to practice previously developed skills. In a small number of cases, the task conclusion was the site of concept formalization or mathematical conjecture. In other cases (typically at the end of a class period), the task conclusion was brief, consisting only of the teacher asking the students to write their name on their work and "pass it forward."

This overview of functions of the conclusion phase points to differing relationships between the enactment phases and the written task. In the set-up phase, the teacher is typically clarifying the written task and describing what the students are expected to do in the implementation phase. In other words, the set-up phase consists of talk about what *will be happening* in the implementation phase. In the implementation phase, the students are actually working on the mathematical task, that is, the work is *happening*. In the conclusion phase, the teacher is typically leading the class in a consideration of *what happened* during the implementation phase.

There were cases of students continuing to work solving the written task during the conclusion phase, but generally such activity ceased in favor of sharing, reflecting on, comparing, or connecting solutions.

To provide more insight into the nature of task conclusions, I now turn to a more detailed examination of four task enactments. I focus elsewhere (Otten, 2010) on the mathematical and discursive functions and content of these four enactments, but focus here particularly on the discursive evidence with respect to activity structure pointing to the separability of the task conclusion as a phase.

Mr. Nicks, an eighth grade teacher in an urban school district, enacted a task over the course of two class periods involving the determination of perimeters and areas of various polygons. The set-up phase and implementation phase occurred on the first day of the task enactment. The following day of class began with Mr. Nicks asking five students to write on the front blackboard their work for one of the task items each. Thus the students were no longer working to solve the task items but were beginning the task conclusion. After the students at the board had finished writing and everyone had taken their seats, Mr. Nicks addressed the class, “So you should have a pen out; you should have that green worksheet out.” Mr. Nicks then called out item numbers (e.g., “Number one. Number one comes out to be…”) to focus the students’ attention as they went over the handout. When Mr. Nicks arrived at items that had been written on the board, he explained the solutions, gesturing to the student work, and asked for any questions from the class. The activity structure of this conclusion phase was distinct from the implementation phase in that Mr. Nicks took a leading role from the front of the classroom and the students were expected to be sitting quietly, unless called upon, with their desks arranged in separate rows rather than together in pairs. The task conclusion lasted approximately 20 minutes and 30 seconds, slightly longer than the set up phase and implementation phase combined (or 53% of the entire enacted task). The conclusion phase ended when the students passed in their work and Mr. Nicks asked them to open their textbooks to a new section.

Ms. Doss, a sixth grade teacher in a rural school district using a reform textbook series, enacted a task over the course of three class periods involving the use of fraction strips to determine the filled portions of fundraising thermometers. On the first day, Ms. Doss set up the task and had the students work on it, and on the second day she reprised her set-up and had the students continue to work. Ms. Doss then used the entirety of the third day for the task conclusion. She began the task conclusion by addressing the class, “OK, then get out problem one-point-three from yesterday. (…) You should have problem one-point-three out, ready to go. That’s the one we did with thermometers, OK?” She then said that they would “get started summarizing problem one-point-three.” Ms. Doss continued, “We’re going to work through each of those thermometers.” She then asked for student volunteers to come up to the overhead projector, where the thermometers were displayed, to “explain how they found” the fractions for each thermometer. Ms. Doss pushed not only for the students’ answers but for descriptions of how they arrived at those answered. She also used the task conclusion to emphasize the main ideas of numerators and denominators and what they meant in the context of the task. This conclusion phase was distinct from the implementation phase in that Ms. Doss was positioned near the overhead projector, which was turned on, and the students were not working at their tables but were expected as a class to attend to Ms. Doss or the student speaker. The task conclusion lasted approximately 31 minutes, nearly an entire class period (37% of the entire enacted task). The conclusion phase ended when Ms. Doss asked the students to put their work into their mathematics notebook and take out their homework assignment from the previous day.

Mr. Ewing, a sixth grade teacher from an urban school district, enacted a task over the course of two class periods involving the naming and ordering of decimal numbers. The context of the task was that the students were serving as baseball managers in developing a batting order based on fictional players' statistics. The task conclusion took place during the latter part of the second day of the task enactment. Mr. Ewing, who had been circulating amongst the groups, returned to the front of the room and gathered whole-class attention by asking the students to move their desks back into rows. He said that they would "take some time and talk about our batting orders." He further signaled the beginning of the conclusion by turning on the overhead projector and displaying a blank transparency. Mr. Ewing proceeded to solicit batting orders from the groups, recording them on the transparency. He then asked the students which players were easiest to place in the order and which portions of the lineup were open to managerial discretion. The task conclusion lasted approximately 12 minutes and 35 seconds (26% of the entire enacted task). The conclusion phase ended when the dismissal tone sounded and Mr. Ewing asked the students to "pass [their] papers up, please."

Ms. Tibilar, a seventh grade teacher in the same school district as Ms. Doss, enacted a task over the course of two class periods involving the determination of various triangles with the same area. Relative to the previous three cases in this study, the task conclusion in Ms. Tibilar's case was more difficult to analyze. The difficulty was not because the task conclusion lacked a clear beginning or ending, but because it contained the set-up and implementation of a sub-task within itself and thus had a complex activity structure. During the second day of the task enactment, Ms. Tibilar began the task conclusion by addressing the class from the front of the room, "OK, so what we have to do then is just go ahead and talk about a couple of them on there, and see if everybody's looks the same or if we've got some different ones." She continued by stating that she did not want to go over every part of the written task, only to "talk about a couple of things that I think are important from it." Ms. Tibilar then led a whole-class discussion focusing on the students' work creating different triangles with an area of 15 square centimeters. The relationship between the (fixed) area and the (variable) perimeter was uncovered and explored. This phase was distinct from the implementation phase because the students were no longer actively working on the written task but were instead discussing the work they had done during the previous class period and using it as a basis for making sense of the mathematical ideas of area and perimeter. The task conclusion ended with Ms. Tibilar evaluating the discussion ("Absolutely amazing discussion") and directing the students to place their written work in their journals. The task conclusion lasted approximately 60 minutes and 25 seconds (78% of the entire enacted task), though approximately 21 minutes of the task conclusion were spent working on a sub-task. (The sub-task was a refinement of the original task that was designed to answer a concern that had arisen in the conclusion discussion. Since it focused on the same mathematical idea as the broader mathematical task and fed back into the original task conclusion, I have characterized it as a sub-task rather than a separate task.)

Discussion

This study examined mathematical task enactments at the middle school level using the sociocultural lens of activity structure (Lemke, 1990) as a means of validating and extending the mathematical tasks framework of Stein and her colleagues (Stein, Grover, & Henningsen, 1996). The set-up phase of task enactments were found to align strongly with a "setting up" activity structure, and the shift to the implementation phase corresponded with a shift in activity structure as the students began working to solve the written task. This analysis revealed, however, an additional conclusion phase in which the teacher reclaims the discursive floor and leads the class in a “looking back” or “sharing out” of work done during the implementation phase. The phase is also distinguished by its relationship to the task itself which is typically no longer being worked on but instead is being looked back upon, forming a sort of symmetry with the set-up phase wherein the work on the task lies ahead. This identification of the conclusion as a separate phase is corroborated by other research such as that of Shimizu (2006) who found that teachers in countries across the globe engage in the activity of “summing up” the main ideas of their lessons or connecting between mathematical concepts, utilizing public talk for this activity. (It should be noted that Shimizu’s work is on the lesson-level rather than the task-level and does not connect to the mathematical tasks framework.) Figure 2 presents the revised mathematical tasks framework, including the conclusion phase.

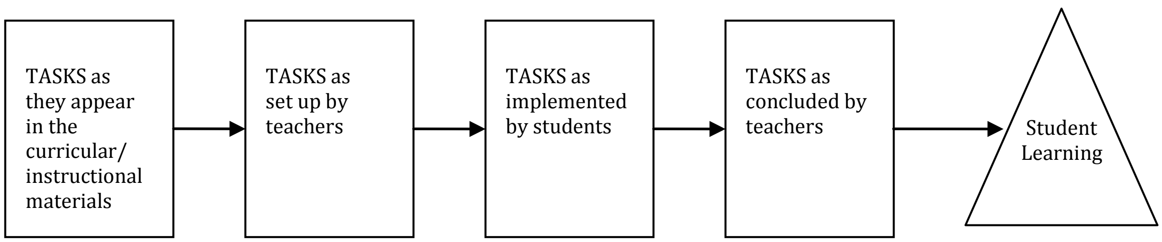

Figure 2. The mathematical tasks framework with a conclusion phase

Although it is clear from an activity structure perspective that the task conclusion is distinct from the implementation, the question remains of whether it is necessary to make this distinction with respect to cognition. I argue that the inclusion of the conclusion phase in the framework will also add value in this respect for at least two reasons. First, the original mathematical tasks framework was deeply concerned with both the thinking processes that the students engaged in and the ways in which those processes characterized the students’ mathematical experiences. As noted by Henningsen and Stein (1997), “the nature of tasks can potentially influence and structure the way students think and can serve to limit or broaden their views of the subject matter with which they are engaged” (p. 525). Thus we must consider not only the processes themselves but how the students’ interpret and reflect upon those processes, forming a notion of what it means to “do mathematics.” It is reasonable to suppose that reflection and characterization of the thinking processes from the implementation phase takes place during the conclusion phase, perhaps not exclusively, but substantially nonetheless.

Second, though space did not allow me to report it here, there were instances in this data corpus of what seemed to the level of cognitive demand shifting in the conclusion phase. For instance, Mr. Ewing’s task enactment mentioned above involved procedures-without-connections during its implementation but descended into nonmathematical activity in the task conclusion as the baseball context became the sole focus rather than the decimal numbers (Otten, 2010). In another task conclusion, the level of cognitive demand could be argued to have descended from procedures-with-connections to procedures-without-connections as the teacher used the conclusion to emphasize only the answers rather than the variety of solution strategies. Alternatively, cognitive demand may increase during the conclusion phase. Another task enactment examined from this data corpus exhibited an implementation phase that seemed to be at the procedures-without-connections level but a conclusion phase that consisted of a rich discussion increasing the cognitive demand to procedures-with-connections. Furthermore, although I did not see it in the course of this study, it is conceivable that a task set-up and implemented at the unsystematic exploration or procedures-without-connections level of cognitive demand could be modified by a masterful teacher into the doing-mathematics level as the students use their previous work as a basis for conjecturing or generalization in the conclusion phase. Therefore, since one of the primary benefits of the mathematical tasks framework is its capacity to illuminate cognitive demand trajectories, it seems that a fuller understanding of these trajectories would be possible by considering task enactments through to their conclusions.

Endnote

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DEVELOPING AND ASSESSING STUDENTS' CONCEPTUAL UNDERSTANDING IN INTRODUCTORY STATISTICS

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The traditional curricular materials and pedagogical strategies have not been effective in developing conceptual understanding of statistics topics and statistical reasoning abilities of students. The statistics education reform movement has placed an emphasis on helping students develop conceptual understanding rather than a focus on mechanical calculations. This study investigated the conceptual understanding of measures of spread in an introductory statistics course centered on deemphasizing computational skills and focused rather on development of conceptual understanding. Open ended questions were developed to assess students' conceptual understanding. A detailed analysis of the students' responses is presented to reveal the range of students' conceptions.

Introduction

The wide spread consensus among statistics educators and statisticians has been that the traditional curricular materials and pedagogical strategies used in introductory statistics courses have not been effective in developing conceptual understanding of statistics topics and statistical reasoning abilities of students. Much of the changes proposed by statistics education research and the reform movement over the past decade have supported efforts to transform teaching practices to include an emphasis on students' development of conceptual understanding rather than a focus on mechanical calculations (Chance & Garfield, 2002).

In recent years, major research interest in the statistical understanding of students seemed to be focused on the measures of centers in distributions (Shaughnessy & Ciancetta, 2002). This research study was partially motivated by the future research suggestion made by Watson et al. (2003) to investigate older students' understanding of spread. Even though there are investigative research studies available on the understanding of variation among university students in the United States by Meletiou & Lee (2002) and in Mexico by Sanches & Mercado (2006), these type of studies are considered to be very limited (Inzuna, 2006). Despite some very recent research studies investigating conceptual understanding of variability among community college students in an introductory statistics course, there are still a lot of unanswered questions regarding how we, as statistics educators, can help students develop conceptual understanding of measures of spread at community college level (Budé, 2007; Slauson, 2008). The purpose of this paper is to investigate students' conceptual understanding of measures of spread and provide learning experiences and environments to support students in developing conceptual understandings of measures of spread.

Theoretical Framework

The guiding theory for statistics education reform is based on a learning theory arising from earlier ideas and writings of Jean Piaget on cognitive development and is referred to as constructivism (Wheatley, 1990; Garfield, 1995; Von Glasersfeld, 1995). The basic tenets of constructivism which view learning as an active process and learners as cognitively active human agents who construct their knowledge through interaction with the environment are in stark contrast with and pose a strong challenge to the earlier conceptions of learners as passive beings whose empty minds are to be filled with information transmitted directly from and controlled by a central knower. Constructivism along with a Vygotskian perspective which takes into account social interactions and historical or cultural influences on learning have become the guiding theory for reform efforts and much research in mathematics, science, and statistics education, as well as for this study (Gordon, 1995).

The changes proposed by statistics education researchers to reform introductory statistics education can be gathered in the following four categories: changes related to content, pedagogy, technology, and assessment. The traditional content and approach to teaching statistics as a sequence of linearly and hierarchically ordered disjoint topics padded with a series of techniques is being challenged widely in the literature. Meletiou-Mavrotheris & Lee (2002) stated that presenting statistical content as a linear hierarchically ordered list of topics might lead students to view statistics as a collection of fragmented formulae and procedures taught in isolation without any interconnectedness established among the various topics. This fragmented view of statistics, created and promoted in a large part by the traditional content and approaches, tends to impress upon our students an image of statistics as a collection of specific, factual and behavioral objectives (Begg, 2004).

The content related changes, which were suggested by the joint curriculum committee of the ASA and the MAA, and approved by the ASA, advocated exploring and producing data, and statistical concepts (Cobb, 1992; Moore, 1997). Since introductory statistics courses are the first, and only course for many students especially at community college level, these courses should offer opportunities for students to work with real, not just realistic, data arising from real problem settings. Interpretations of graphics, developing strategies for explorations of data, and informal inference need to be brought to the forefront of the course.

The traditional pedagogy used in teaching introductory statistics courses is, to a large extent, based on the lecture-and-listen model. We, as educators, need to remember that “we overvalue lectures” (Moore, 1997, p.125). The most common factor among pedagogical changes suggested is a move away from the traditional lecture-and-listen model toward activity-based courses which promote and support active participation, and interaction among all participants. Garfield (1993) argued that since small group cooperative learning activities might be designed specifically to allow students opportunities to actively and individually construct their own knowledge, as opposed to copying down knowledge transmitted, the use of small group activities was aligned with the constructivist theory of learning.

Among the several forms of technology related changes to support the development of conceptual understanding and reasoning abilities of students are the use of computers, graphing calculators, Internet, statistical software packages and Java applets. Using computers or calculators merely to generate statistics, to follow algorithms or to produce graphs of data are very limited views of technology use in statistics education reform. Technology use in that sense becomes a tool for doing statistics not a tool for learning statistics (Moore, 1997; Meletiou-Mavrotheris, Lee, and Fouladi, 2007).

Using technology to help students visualize concepts and understand abstract ideas is considered to be far more important than using technology solely to automate messy statistical computations (Garfield & Ben-Zvi, 2009). Technological tools such as simulations and applets have become increasingly common in introductory statistics courses to illustrate abstract concepts, simulate data, and build understanding of statistical concepts (Zieffler & Garfield, 2007; Chance & Rossman, 2006). The applets have been gaining an increasing importance because of their effectiveness in illustrating various statistical concepts visually (Glencross, 1988; Chance & Rossman, 2001; Chance & Rossman, 2006). However, while software may provide the means for a rich classroom experience, computer simulations alone does not guarantee conceptual change. delMas et al. (1999) made a strong case for using technology with simulations in a predict-and-test environment to create cognitive dissonance in developing students' understanding of sampling distributions.

The conceptual change theory had been used previously in the field of science education and is being used lately by other researchers in social studies education (Posner, Strike, Hewson, and Gertzog, 1982; Houser, Parker, Rose, and Goodnight, in review). According to the conceptual change theory of learning, students who have preconceptions need to experience an anomaly to create cognitive dissonance between their preconceptions or predictions and observed outcomes.

The traditional method of assessing student learning consists of module tests, generally with multiple-choice questions, designed for ease of grading. The traditional exam questions place a strong emphasis on the procedural or computational aspects of statistics and do not evaluate high-level cognitive and conceptual understandings of students (Cobb, 1993; Garfield, 1994; Zieffler et al., 2008). The connection of the traditional testing methods to the actual statistical practice is an unexamined assumption. The efforts of statistics education reform in assessment, which is sometimes referred to as "authentic assessment", starts with challenging and questioning this assumption. The statistical education reform efforts have responded to this challenge and critically evaluated the traditional testing methods to offer some alternative forms of assessing students' statistical reasoning and conceptual understandings (Garfield, 1994; Gal & Garfield, 1997; Chance, 1997; Garfield & Gal, 1999). The traditional dichotomous relationship of teaching and assessment is considered to be too narrow and too specific to provide useful information about student learning (Garfield & Gal, 1999).

The GAISE recommendations indicate that an emerging view of assessment, as an ongoing evaluation of students' learning over the course of the semester with constant gathering of information and providing feedback, can be very valuable in informing our teaching. Among some of the alternative forms of assessment were cooperative group activities, computer lab exercises, portfolios, projects/reports, presentations, essay questions, journal entries, and open-ended writing assignments (Garfield, 1994; Chance, 1997).

Methodology

There were no general hypotheses tested in this research study. Thus, the study was primarily exploratory and qualitative in nature. A questionnaire format was developed as the main data gathering technique. The subjects of the study were community college students enrolled in two different sections of an introductory statistics course taught by one of the authors. The course required a TI-83/TI-84 graphing calculator and was taught with a teaching philosophy centered on deemphasizing manual computational skills but focused rather on conceptual understandings throughout the course. A total of 41 students, from two sections of the course, participated in the study. This particular study analyzed the responses of students and reports the results from only one of those sections. The study was, therefore, based on a final set of 20 subjects consisting of 11 female and 9 male students. The student participant group was comprised primarily of business majors with only a few students declaring mathematics as their major. Only a few student participants had studied mathematics at the pre-calculus or higher level.

An open-ended questionnaire, developed as a research instrument as well as an assessment tool, was administered after the descriptive statistics portion of the course. There were no penalties for giving wrong responses. All the students present in class on the day of administration of the questionnaire chose to participate. The students were issued a 30 minute time limit to respond to the items on the questionnaire. Some of the items of the questionnaire were designed by one of the authors specifically for this research project. Some others were adapted and modified slightly from previous research studies such as Meletiou & Lee (2002), Inzuna (2006), and Delmas & Liu (2005).

Results

A simplified version of the coding scheme based on the research of Watson, Kelly, Callingham & Shaughnessy (2003) was devised to conduct an analysis of data collected. The data were coded according to the degree of mathematical and statistical correctness of the response and/or appropriateness of rationale expressed. Initially a number of different categories of responses were assigned for the data generated from each questionnaire item. Based on further analysis, common characteristics among categories were identified resulting in a consolidation of categories into fewer and more general categories of responses.

Items 7 and 8 were designed to determine if the students had an understanding of the relationship between the size of the standard deviation and the size of the spread that the data points have around the mean. The data resulting from Item 7, for example, was initially coded as five distinct categories of responses. These five initial categories were consolidated into the following more general categories: (i) appropriate responses, correct answer and appropriate rationale, such as *0, 0, 10, 10; I get the largest standard deviation because these are spread out the furthest*, (ii) inappropriate responses, including no response and no rationale, wrong answer and no rationale, and misinterpretation of question, such as *0010, 0009;* no rationale; and (iii) semi-appropriate response, correct answer and inappropriate or invalid rationale. An appropriate response was coded as 2 (60% of responses), a semi-appropriate response was coded as 1 (35% of responses), and an inappropriate response was coded as 0 (5% of responses).

For Item 8, initial six distinct categories of responses were consolidated into the following more general categories: (i) appropriate responses, correct answer and appropriate rationale, such as *1, 1, 1, 1; these numbers allow for the smallest standard deviation because they are closest to the midpoint*, (ii) inappropriate responses, including no response and no rationale, wrong answer and no rationale, or wrong answer and inappropriate rationale, such as *0, 1, 1, 1; lower the numbers are the lower the standard deviation*; and (iii) semi-appropriate response, correct answer and inappropriate, invalid or rote memorization rationale, such as *1, 1, 1, 1; when all the numbers are the same the standard deviation is 0*. An appropriate response was coded as 2 (20% of responses), a semi-appropriate response was coded as 1 (50% of responses), and an inappropriate response was coded as 0 (30% of responses).

Table 1. Percentage in each coding category for the questionnaire Items 7 and 8

	Coding Category		
Item	0	1	2
7	5%	35%	60%
8	30%	50%	20%

As one can observe from Table 1, the percentage of appropriate responses, which was coded as 2, was drastically reduced for Item 8. The reason for this decline was attributed to the fact that there were seven responses with a correct answer and rote memorization rationale, such as *when all the numbers are the same the standard deviation is 0*, and these responses were coded as 1.

Items 1 and 2 explored the students' understanding of spread shown in a pair of box plots in each item. The coding for data resultant from Items 1 and 2 initially produced several categories of responses which were then generalized into the following two categories: (i) appropriate responses, correct answer with or without appropriate rationale; and (ii) inappropriate responses, including wrong answer, no response, no rationale, and misinterpretation of question. An appropriate response was coded as 1 and an inappropriate response was coded as 0.

As one can observe from Table 2, the percentage of appropriate responses, which was coded as 1, was somewhat reduced in Item 2.

Table 2. Percentage in each coding category for the questionnaire Items 1 and 2

	Coding Category	
Item	0	1
1	0%	100%
2	15%	85%

The reason for this decline was attributed to the fact that the second box plot in Item 2 was created based on a data set in which the maximum and the third quartile were the same numeric value. This fact caused some confusion among students, such as *"does not have both tails"*, *"unspecified maximum", and "it is from 3 to ∞."* Prompted by this confusion, a further analysis of the given rationales for Items 1 and 2 was performed. Based on the nature of the rationales, there were four types of rationales identified: (i) a numerical rationale, such as *"spread min to max or the difference between the max and min"*, (ii) a graphical rationale, such as *"tails extending longer"*, iii) a combination of both numerical and graphical rationales; and (iv) idiosyncratic or unclear comments, such as *"larger the median is more spread"*, *"don't know what the spread is because does not have both tails"*, *"starts at 3 but has no stem"*, *"second box plot from 1 to an unspecified max"*, *"bottom one is not even on the graph"*, and *"bottom one is missing a whisker."* Table 3 shows the resultant summary of the analysis of rationales given by the students for Items 1 and 2.

Table 3. Percentage in each rationale type for the questionnaire Items 1 and 2

	Type of Rationale Given			
Item	Numerical	Graphical	Both	Idiosyncratic
1	40%	35%	15%	0%
2	40%	15%	10%	35%

Conclusions

This study investigated a group of community college students' understanding of spread in a descriptive statistics setting. Although the conclusions from the study are based on a single, small group of students, the study revealed some possible common student preconceptions which should be examined closely and taken into consideration in order to promote students' development of understanding of the concept of statistical measures of spread.

In general, the results of the analysis of student responses, which included a wide variety of rationales, justifications, and strategies, provided evidence for the students' ability to organize concepts of spread in a way that is meaningful to them individually. It is also further evidenced by the students' responses that despite the fact that students are taking the same course, from the same instructor using the same textbook, they do build different understandings. These findings support the need to provide environments rich in opportunities for student to construct their own understandings. Teaching methodologies more open to problem solving and supporting student construction of knowledge should be used in the teaching of statistics.

The students' responses to Items 1 and 2 were encouraging in that the students were able to demonstrate a good grasp of spread based on box plots. Only a few (three students or 15%) students showed some preconceptions which were not aligned with the generally accepted conception when one of the data sets had the same numeric value for the third quartile and the maximum.

The students' responses to Items 7 and 8 were not greatly encouraging in that the students mostly were unable to demonstrate an understanding of the association between the size of the standard deviation and the size of the spread that the data points should have around the mean. Mostly students were able to produce the valid arguments for their conclusions, but provided only a rote memorization rationale, such as "*when all the numbers are the same the standard deviation is 0*."

As evidenced in the literature, there seems to be a great research potential in investigating how the students' understandings of graphical representations develop. There is also a need for research on how students can be helped to improve their ability to recognize the difference between what the horizontal and/or vertical axes represent in a variety of different graphical representations, such as histograms, box plots, stemplots, and scatterplots.

A much more detailed and deeper analysis of the results from a much larger group of students could be performed using Structured Observed Learning Outcome (SOLO) Taxonomy in conjunction with a Rasch Item Response Model analysis as implemented by Watson, Kelly, Callingham & Shaughnessy (2003) to produce a variable map of student performance and item difficulty on a single scale. It is also possible to conduct in-depth, follow-up interviews of a subset of participating students. The subset of students may be identified based on either extremely weak or extremely strong reasoning abilities exhibited in the written responses to the survey.

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INVESTIGATING SECONDARY STUDENTS' EXPERIENCES OF STATISTICS

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Data analysis, probability, and statistics have received increased emphasis in school curriculum. While much attention has been given to teaching and learning statistics at elementary and undergraduate levels, very little research has focused on secondary learners. The research discussed in this report investigates the association between secondary students' definitions of learning and their conceptions of statistics. Using the phenomenographic approach, students in college preparatory introductory statistics courses were interviewed and observed. The findings show a hierarchical relationship between the meaning assigned and methods used by the students to acquire knowledge about statistics.

Introduction

Data analysis, probability and statistics have been researched mainly in the areas of elementary, undergraduate, and graduate levels (Groth, 2003; Watson & Callingham, 2003). The statistics literature on secondary learners has not extensively analyzed how these students tend to process and acquire meaning about the subject. Since data analysis and probability have been an integral part of the elementary curriculum, as put forth by guidelines from National Council of Teachers of Mathematics (NCTM) and the American Statistical Association, students enter high school with prior knowledge, past experiences, preformed attitudes, and beliefs that influence how they learn and what they learn (Gal & Ginsburg, 1994). With these countless variations among learners, it is unrealistic to assume that all students will understand course content in the same way, even if they tend to answer questions correctly and solve problems successfully.

We can infer that students exposed to the same instruction process the content differently, and hence they experience each course of study differently (Marton & Pang, 1999). Allowing secondary students in a college preparatory statistics course to describe their experience of learning renders some explanation as to how they assemble prior knowledge, past experiences, attitudes, and beliefs in order to develop an understanding of what is taught in regards to data analysis, statistics and probability. By investigating the role of human experience in knowledge acquisition, a collection of each person's descriptive account of their experiences is formulated using phenomenographic methodologies. Phenomenography aims to explore how people learn and why some people learn particular subjects or domains better than others. Therefore, the objectives of the study were to: 1) categorize the various ways secondary students define statistics, 2) categorize the various ways secondary students define learning or knowledge acquisition, and 3) investigate the relationship between the students' definition of learning and the meaning they assigned to statistics.

Theoretical Framework

The theoretical framework for the study is phenomenography, which is sometimes confused for phenomenology. Although both share the same object of research, which is human experience, they have distinct differences. Phenomenology aims to develop a single theory from the researcher's perspective of the world based on his/her experiences. In contrast, phenomenography focuses on the perspective of others to devise a collection of categories that constitute the variations in experiencing the world (Marton & Booth, 1997). The learner-phenomenon relationship is examined to determine the process of conceptualization, but learning is not considered to be a mental representation nor a cognitive structure (Andretta, 2007; Uljens, 1996). Instead, learning is defined as perceiving something in a new way by discerning it from and relating it to a context (Marton & Pang, 1999; Pramling, 1996).

In addressing the question, "why do some people learn things better than others", Marton (1986) conducted a study on the different ways learners understood a passage of text. Based on analyses of the learning processes used by the students, two approaches emerged: deep and surface learning. In work by Marton and Booth (1997), deep and surface learning are defined. A surface approach to learning is when the student's overall strategy for learning merely focuses their attention on signs or prompts during instruction to gauge or elicit a reaction. In contrast, a deep approach to learning is when the student moves far beyond the surface level approach to more interpretive, comprehensive learning processes. Marton (1986) concluded that learning, from the phenomenographic perspective, involves experiencing both the content to be learned (referred to as "what") and the act of learning (referred to as "how"). The "what" and "how" aspects of learning frame the study's focus on investigating what statistics is (content) and how one goes about learning it (act).

The "what" aspect of learning involves assigning meaning to the content, and discerning the interrelationships between parts of the content in how they structure the whole (Marton & Booth, 1997; Uljens, 1996). The statistics content used in the study is drawn from school curriculum developed using the Guidelines for Assessment and Instruction in Statistics Education (GAISE, 2005), and the NCTM (2000) Data Analysis and Probability Standards. The "how" aspect of learning has two parts: (a) the capabilities or skills the learner seeks to obtain, and (b) the approaches used to acquire knowledge about the content (Marton & Booth, 1997). The NCTM (2000) Process Standards, along with assessment and technology use recommendations for statistics education (Garfield & Chance, 2000; Vellemen & Moore, 1996) were incorporated into the statistics courses the participants took at the secondary school.

The outcome space of a phenomenographic investigation is a collective categorization of the "what" and "how" aspects of learning. Arranged in a hierarchical structure, the outcome space diagrams the logical link between variations in the meaning of the phenomenon and how knowledge is attained about the phenomenon. According to Runensson and Marton (2002), even though there are infinitely many ways to describe something, the outcome space relevantly accounts for differences in student achievement, by uncovering conditions that facilitate the transition from one way of thinking to a different way of thinking. By encouraging students to vary the perceptions, approaches, and skills they use to learn will increase the observable dimensions of the outcome space (Marton, 1986). The outcome space that emerged from the data collected for the study constitutes the range of variation described by the participants. By replicating the study, it is conceivable that more categories would arise.

Methodology

The study was conducted at a suburban secondary school in the southeastern United States. The nine participants were recent graduates from the school who had taken an introductory statistics course during either their junior or senior year. Data collection efforts focused on methods that got students to reflect upon and explicitly describe their experience of learning statistics. A questionnaire was disseminated to collect demographic information. A sixty minute semi-structured interview with each participant was audio-recorded and transcribed. A range of questions were asked with regards to perceptions of statistics, perceptions of learning, and reflections on the experience of learning statistics. The participants also kept a two week journal in which they were asked to solve and reflect upon tasks completed in the class they took that covered data display and analysis, probability, and inference. The interviews and reflection journals represent information rich cases, in which selective quotes meeting the criteria of relevance were coded using processes adapted from phenomenography (Bowden, 1996; Marton & Booth, 1997). This iterative process continued through the deduction of meaning about the phenomenon, and was concluded when the narrowed categories of description were deemed relatively stable. Trustworthiness of data was achieved using four criteria: credibility, transferability, dependability, and confirmability. To ensure credibility, triangulation in data collection, prolonged engagement with participants and member checking were used. Thick, rich description was used to convey findings, ensuring transferability. An audit trail and securing records from the study ensured dependability and confirmability.

Results

The first objective of the study was to categorize the different ways students define statistics ("what" aspect of learning). Four definitions of statistics emerged from the group, as illustrated in Table 1.

Table 1. Ways of Defining Statistics

Experienced as:	Discerned attributes:	Statistics is:	Sample Quotes
Facts/Algorithms	Maintaining basic mathematics skills	A course focused on stating terms, evaluating expressions, solving equations, and graphing.	Andrew: A graphical and numerical way to represent numbers. Just like any other math[ematics] class, you basically deal with equations and graphs.
Concepts About/ Procedures for Handling Data	Techniques for collecting, representing, and analyzing data	The use of established procedures to collect and interpret data.	Mary: The process of gathering information on a specific topic and looking at the possible outcomes.
Inference / Prediction	Processes for making estimates.	The use of procedures to generalize or predict attributes of populations.	Rebecca: Statistics [are] a group of data that is gathered, combined, and analyzed in such a way that it can objectively represent the aspects of a particular population.
Restructure Knowledge/ Seek Meaning / Inform Decisions	Acquiring knowledge to improve quality of life, and to monitor ethical practices of research.	A set of norms for quality research and a set of evolving rules to inform decision making; a change agent.	Sally: I define statistics as the discipline of interpreting data and determining its relevance to our lives. People do studies to find out about our world, and statistics helps make sure that the studies are accurate.

The first column lists the ascending categories of description based on the quality and complexity. Structural aspects of statistics the students tended to focus on are presented in the column entitled *Discerned Attributes.* In relating the discerned attributes to a holistic view, the meaning students assigned to the content is shown in the *Statistics is* column. The *Sample Quotes* column contains actual statements, selected from several that conveyed similar conceptions, and were used to code and describe the category. The rows of the table summarize the category. The first row, for example, should be read as follows: students experiencing statistics as *Facts/Algorithms* tended to focus on maintaining basic mathematics skills, and they viewed statistics as a course in which one states theorems, evaluates expressions, solves equations, and makes graphs.

Table 2. Ways of Learning: Skills or Capabilities Sought

Experienced as:	Discerned attributes:	Learning statistics means being able to:	Sample Quotes
Recall information and duplicate procedures	Information or tasks regarded as important to instructor	State terms and perform simple routine task.	Aimee: If you know statistics you can read and understand the terms on a pretty basic level…and like if you have a school assignment that involves statistics you remember the steps and equations and how to do surveys.
Understand information and procedures	Finding connections among course objectives	Establish relationships among course objectives to assign meaning to concepts and to develop an understanding of how and why routine tasks are performed.	Cara: …when I fully understand how the conclusion was brought about and can reproduce the techniques used…I know I paid attention to it and…can re-use the methods to understand how they are used.
Teach / Explain	Being able to collaborate with others validate conjectures and self regulate.	Communicate understanding by teaching or explaining the concepts to someone else.	Sarah: …when I can fully explain it and teach it to somebody without looking at my notes…I don't have to look at my notes and I can answer their questions or show them how to do problems.
Apply, relate and integrate information and procedures	Reflecting on how concepts of statistics have been encountered or how the concepts can be applied to objects of interest to validate or restructure understanding	Authenticate concepts of statistics by relating and integrating them with personal interest or real life to refine the existing structure of knowledge.	Angela: …when I can apply it outside of class or the learning experience. If I can use any information I have been taught and apply it to a different situation with some ease or without reviewing, then I know I've learned something.
Adapt and assimilate knowledge about analyzed surroundings	Exploring how concepts of statistics can be applied and adapted to phenomenon encountered in surroundings	Adapt and synthesize concepts of statistics with objects experienced to uncover attributes, see the object in a new way, and formulate theories.	Sheldon: …the most important thing I was taught was to not just accept information and that thought can be applied to whatever I do in the future to make sense of stuff I don't understand.

The second objective of the study was to categorize the various ways students define learning. This was investigated in two ways: 1) defining learning or specifically the skills or capabilities students sought when learning, and 2) identifying approaches to learning. Table 2 shows the qualitatively different ways the participants defined learning in the statistical context. The column headings parallel those described in Table 1. The rows may be interpreted as such:

students experiencing learning statistics as *Recalling Information/Duplicating Procedures* tended to focus on informational cues or tasks they perceived important to the instructor, and they sought skills needed to state terms or perform simple routine tasks. The least complex capability sought by students aligns with what Marton (1986) calls surface learning. The more elaborate capabilities require deep learning to be able to adapt and assimilate knowledge after interpreting statistical results.

The second part of the "how" aspect of learning is described in Table 3. The six approaches to learning that emerged depict activities participants engaged in when they wanted or needed to learn concepts for the course. Interpreting the first row of the table, learning about statistics through observation, means to passively receive information about the content by listening to instruction, and by watching or recording classroom events. All of the participants referred to at least two of the approaches to learning in either the interview or reflection journal.

Table 3. Approaches to Learning: The Act of Learning

Experienced as:	Discerned attribute:	Learning by …
Observation	Passively receiving information	Listening to instruction, watching events, and recording information in the classroom.
Study Skills	Individual academic history of successful study habits or routines	Using personalized study skills outside of the classroom that have proven successful in the past.
Reflection	Data presented in context, multiple representations of concepts, examples relevant to personal interest	Considering how personal interest and experience relate to concepts and skills being taught.
Self regulation	Individual inventory knowledge, test depth of understanding	Doing a personal assessment and evaluation of knowledge.
One-way information exchange	Individualized delivery or receipt of instruction	Assisting a classmate or receiving help from a classmate.
Collaboration	meaningful social interaction, productive collaboration, group learning goals for product construction	Working in collaborative groups to investigate concepts or produce a product of work; assisting and being assisted by peers and teacher.

The final objective of the study was to investigate the learner-phenomenon relationship. In Figure 1, the rows consist of the ways of defining statistics from Table 1, and the columns present the ways of learning statistics from Table 2. The cells within the table cross-reference the number of participants describing their experience by the row and column categories, along with the types of activities they used to learn, as described in Table 3. Referring to the Facts/Algorithm and Recall cells, two participants fell into these categories for meaning of statistics and capabilities sought. Furthermore, both students said they acquired knowledge about statistics through observation and study skills. In cells where there are more than two approaches to learning listed, one or more were indicated by the participants represented in the cell.

Discussion

Overall, the four conceptions of statistics that emerged from the study are consistent with findings from similar phenomenographic research. Petocz and Reid (2003) found that statistics majors' conceptions could be characterized in three ways: a focus on technique, a focus on data, or a focus on meaning. From research conducted with undergraduate psychology majors in a service learning statistics course, Groth (2004) categorized five conceptions of statistics: no meaning, process or algorithms, mastery of concepts and methods, tools for getting real-life results, and critical thinking. Although the research on secondary students understanding of statistics is not extensive, the findings of the study support the notion that these learners conceptualize statistics in ways similar to undergraduate students.

A 2006 study by Tempelaar investigated the relationship between statistical reasoning ability, personal background, and attitude towards learning. Introductory statistics students who deliberately planned rigorous strategies for learning content developed statistical reasoning abilities significantly faster than students without a plan. In the results illustrated in Figure 1 above, participants seeking skills categorized in lower conceptions of learning statistics (Table 2) expressed the least descriptive definitions (Table 1). Also, students in the upper left region of Figure 1 used fewer strategies for acquiring knowledge compared to students in the lower right region. Both studies suggest that students who take surface level approaches to learning statistics tend to conceptualize the content more restrictively than those taking deep level approaches. By emphasizing students' responsibility to organize their learning processes so that they include rigorous approaches, better ways of thinking emerge and expand the outcome space.

Meaning		**Skills**				
		Recall	Understand	Teach/ Explain	Apply/ Relate	Adapt / Assimilate
Facts/ Algorithms	Count	2	1			
	Approach	1,2	1, 2			
Concepts/ Procedures	Count	1			1	
	Approach	1, 2, 5			1,2,3,4,5,6	
Infer/ Predict	Count				1	1
	Approach				1,2,3,4,6	1,2,3,6
Validate/ Meaning	Count			1	1	
	Approach			1,2,3,4,5,6	1,2,3,4,6	

Figure 1. Association between the "how" and "what" aspects of learning statistics

The various forms of assessment (tests, projects, journal writing), the use of technology, and implementing process standards (multiple representation, communication, reasoning, problem-solving) are components of the course the participants took. These strategies were used to encourage variation in perception, approaches and skills, as described by Marton (1986) to inform the hierarchical structure of the outcome space. While the outcome space for the study categorizes the conceptions of nine participants, similar secondary level statistics courses would expectedly observe similar as well as new categorizations. The outcome space may aid educators in developing strategies that foster the transformation from a limited way of thinking about statistical concepts to a more expansive way of thinking. Phenomengraphic research on conceptualizing statistics must continue to broadening the dimensions of the outcome space and reveal the dynamics of learning statistics.

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THE IMPLEMENTATION OF A DECISION-MAKING CURRICULUM

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This paper reports on a preliminary evaluation of the implementation of a new secondary level mathematics curriculum which uses engineering contexts, titled Mathematics for Decision-Making (MDM). Because the MDM curriculum is not NCTM standards-based, the authors established their own expectations for an appropriate learning environment. The research team compared data from classroom observations and teacher feedback forms to discern the level of implementation fidelity among a sample of teachers piloting the curriculum. Of the ten teachers sampled, three teachers' implementations showed evidence of supporting the authors' expectations and three teachers showed evidence of opposing them.

Introduction

After the release of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), extensive funding from the National Science Foundation encouraged significant curriculum development in the 1990's. With the development of these conceptually oriented materials (known as standards-based curricula), researchers, policy makers, administrators, and teachers began asking the question, "Do these new materials work?" (Stein, Remillard, & Smith, 2007). The era of accountability (e.g. *No Child Left Behind Act of 2001*, 2002) placed further pressure on developers to prove the effectiveness of standards-based curricula. To determine if these standards-based curricula "work", researchers looked beyond student performance and began to consider the teachers' level of implementation fidelity (NRC, 2004). That is, researchers began studying teachers' faithfulness to the curriculum authors' intents as an important factor in the success of the curricula.

Additionally, most curriculum evaluators depended on the benchmarks of the *Principles and Standards for School Mathematics* (NCTM, 2000) to assess whether teachers created standards-based learning environments while implementing standards-based curriculum (e.g. Tarr, et al., 2008). If a curriculum was not standards-based, this may not have been the most appropriate guide to measure teachers' implementation fidelity and success of the materials. In this paper, we discuss the implementation of a non-standards-based curriculum and investigate how closely the teachers' implementation aligns with the curriculum authors' intentions. In particular, we will answer the question: how do high school teachers' implementations of a decision-making curriculum align with the authors' pedagogical intentions?

The Development and Implementation of a Decision-Making Curriculum

The NSF-funded MINDSET (Mathematics INstruction using Decision Science and Engineering Tools) project is a five-year collaboration between mathematics educators and engineers at North Carolina State University, Wayne State University, and the University of North Carolina at Charlotte (Young, Keene, Norwood, Chelst, Edwards, & Pugalee, 2009). The objective of the MINDSET project is to create, implement, and evaluate a new high school curriculum that integrates mathematics and engineering concepts. The curriculum is intended for high school seniors who have completed two years of algebra and one year of geometry.

The MINDSET course content consists of contextual problems from business as well as from students' lives that employ multi-step problem-solving tools and result in multiple solutions that require interpretation in the context. The MINDSET curriculum is not specifically based on the *Principles and Standards for School Mathematics* (NCTM, 2000), although its tenets do align in many ways. Instead, the emphasis of the curriculum is for students to make decisions concerning contextual problems using mathematical methods; hence, the curriculum is called "mathematics for decision-making" (MDM). High school teachers in North Carolina and Michigan are currently piloting portions of the draft curriculum. Teachers implement different portions of the MDM curriculum into an existing senior level mathematics course so that the MINDSET project team has a sampling of all materials.

Implementation Fidelity

For this paper, fidelity of implementation is defined as the extent to which a teacher implements a curriculum as the author intended, allowing for adaptation, supplementation, and improvisation by the teacher (Dietz & Holstein, 2009). As mentioned above, the goal of similar previous research was to investigate the implementation fidelity of standards-based curricula (e.g. Lloyd, 1999; Remillard & Bryans, 2004; Schoen, Cebulla, Finn, & Fi, 2003; Tarr et al., 2008). Thus, these researchers took into account the types of discourse, teaching methods, assessment techniques, and curricular materials expected in a standards-based learning environment.

Standards-based curriculum evaluators utilize various instruments to measure implementation fidelity. Three instruments in particular are frequently used: classroom observations (e.g. Tarr et al., 2008), surveys (e.g. Schoen et al., 2003), and teacher interviews (e.g. Remillard & Bryans, 2004). Some additional methodologies include homework assignments (e.g. Cai & Moyer, 2006), chapter evaluation forms (e.g. Thompson & Senk, 2001), textbook-use diaries (e.g. Tarr et al., 2008), and table-of-contents records (e.g. Tarr et al., 2008). Researchers use these instruments to measure several categories of implementation fidelity, including use of curriculum, teacher instructional practices, teacher background information, and teacher beliefs.

However, using implementation fidelity instruments in isolation can be unreliable (Porter, 2002). For example, teacher interviews are susceptible to self-reporting bias and classroom observations are subject to researcher bias. Since this unreliability is inherent, most researchers combined instruments to triangulate data when examining standards-based classroom environments. Research confirms that using multiple methods to measure teachers' implementation fidelity yielded results that are more valid.

Because the MDM curriculum authors did not expect teachers to create a standards-based learning environment, the authors chose to complement this work and derive their expectations from a source other than the *Principles and Standards for School Mathematics* (NCTM, 2000). Specifically, the authors chose to utilize the Productive Pedagogy framework from the Queensland Department of Education and Training (K. Makar, personal communication, October 7, 2009; The State of Queensland, 2002). The framework outlines four overarching foci for effective pedagogy: intellectual quality, supportive classroom environment, recognition of difference, and connectedness. The MDM curriculum authors adapted focal points connected to these foci to study implementation of piloting teachers. In particular, the authors expected teachers to emphasize real-world contexts, interpret and synthesize application problems, utilize multi-step reasoning, be mathematically accurate, use technology effectively, promote student exploration, and foster an improvement in student attitudes.

Methods

Using the Productive Pedagogies framework (The State of Queensland, 2002), the authors of the MDM curriculum (one of the three researchers in this proposal is a primary author and the other two researchers are closely connected to the project) established their pedagogical intentions. In order for the teachers to implement the MDM curriculum faithfully, the authors expected the teachers to show evidence of these focal points (see Table 1 for examples of the focal points).

Table 1. Productive Pedagogies Rating

Focus	Description
Intellectual Quality	
Knowledge as problematic	This involves an understanding of knowledge not as a fixed body of information, but rather as being constructed, and hence subject to political, social and cultural influences and implications
Supportive Classroom Environment	
Academic Engagement	Students are engaged and on task. They show enthusiasm for their work by raising questions, contributing to group activities and helping peers.
Technology	Students use appropriate technology (e.g., Microsoft Excel, graphing calculators) correctly. Reflection encouraged during and after technology use. Instruction incorporates interpretation of results
Connectedness	
Background knowledge	Opportunities are provided for students to make connections between their own background knowledge and experience and the topics, skills and competencies they are studying and acquiring.
Connectedness to the world	This describes the extent to which the lesson has value and meaning beyond the instructional context, making a connection to the wider social context within which students live.

Approximately 40 teachers in North Carolina and Michigan piloted parts of the MDM materials during the 2008-2009 school year. The teachers implemented at least two chapters from the first semester of materials in the fourth-year mathematics course they were currently teaching (e.g. Discrete Mathematics, Advanced Functions and Modeling, or Introduction to College Mathematics). The teachers only piloted the first semester, or deterministic semester, of materials because the authors had not yet written the second semester, or probabilistic semester, of materials. During the 2008-2009 school year, ten of these teachers (all in North Carolina) were observed by at least one member of the MINDSET research team as they implemented deterministic semester materials. Some teachers were observed multiple times during their pilot, but no teacher was observed for the entire time of their pilot due to lack of resources. The observer took fieldnotes describing the learning environment.

In addition, after piloting a chapter, most teachers completed a feedback form. In this feedback form, the teachers gave detailed descriptions of the daily implementation of the chapter.

They also described whether the level of the materials was appropriate for their students, how students and teacher used technology in the classroom, and any feedback given by the students regarding the reading level.

Analysis consisted of a comparison of the chapter feedback responses to the observation fieldnotes in order to see how the information aligned. For example, a teacher wrote that he used direct instruction to cover specific material on a given day. An observer was in the classroom that same day and noted how the teacher used student-led class discussion as a way for the students to unpack the specific content. This discrepancy in the data caused us to reconsider what the teacher meant by direct instruction in his feedback form. Once we studied the feedback and the observation fieldnotes, the findings were evaluated according to the Productive Pedagogies framework to determine the ten teachers' implementation fidelity levels.

Results

In general, three of the ten teachers showed preliminary evidence of supporting the focal points in the Productive Pedagogies framework, three teachers showed evidence of non-support, and there was insufficient evidence to make claims about the remaining four teachers. We use three of the focal points from Table 1 to illustrate the difference in teachers' implementation fidelity: Academic Engagement, Technology, and Connectedness to the World.

Academic Engagement

The MDM curriculum authors expected teachers to encourage academic engagement in their classrooms. Based on observation fieldnotes and teacher feedback forms, two teachers (Jim and Sharon) showed clear evidence of supporting student engagement in their classroom while one teacher (Caroline) strayed from this focal point. We briefly describe the results from Jim, Sharon, and Caroline.

We observed Jim on the day he first introduced the linear programming materials. Jim used a combination of class discussion and small-group work encouraging his students to ask questions and to offer ideas regarding the contextual problem. In fact, when a student gave an idea that did not necessarily contribute to the current conversation, the teacher listed the idea on the side of the board so that the idea would not be neglected. This technique allowed the teacher to acknowledge the student and recognize the validity of the idea without spending time on an unnecessary or off-topic point. Jim fostered an open and engaging classroom this observation period. Moreover, the teacher's chapter feedback aligned with the observer's fieldnotes indicating that the single observation was not an anomaly. Thus, Jim's method of student engagement corresponded with the MDM curriculum authors' expectations of academic engagement.

Sharon was observed six times (five times with one member of the research team and once with another member) while she taught two linear programming chapters. For each observation, the observer noted students working in groups at some point during the period. Further, the teacher formally assessed students' knowledge using a group project. The feedback provided by the teacher verified this teaching method. However, the observers found that Sharon tended to use short-answer recall questions, rather than in-depth conceptual questions, during class discussion. This type of questioning did not encourage student engagement. Nevertheless, we classify Sharon's students as having a high level of academic engagement because they spent a large portion of class time in groups and a small portion of class time in class discussion.

Student responses on a survey provided further evidence to support the observations of high academic engagement in Sharon's classroom. After piloting the MDM materials, Sharon gave her students a survey to discern their attitudes and beliefs about the materials. When asked, "Would you recommend that [the teacher] teach these modules in [Advanced Functions and Modeling] in the future and why?" Almost all students (21 out of 23) said "yes." One student claimed, "[The problems] were so much fun, and [they] drew our groups closer together!" Another student stated, "Yeah, it was fun and different because I could actually see where this could be used in real life situations." These comments, in addition to the observation data and the feedback forms, demonstrated how Sharon's level of academic engagement aligned with the intentions of the MDM curriculum authors.

Contrary to Jim and Sharon, Caroline did not provide evidence of Academic Engagement. We observed Caroline twice while she piloted a linear programming chapter. In general, Caroline told her students answers rather than having the students discuss, explore, or conjecture on their own. For example, Caroline defined vocabulary words for the students instead of providing an opportunity for students to discuss possible definitions. At one point, the teacher completed a scaffolding worksheet at the overhead herself without any input from students to complete on their own. This contrasts to the goal of the MDM authors who created this worksheet with the intention that students would complete it in small groups. According to Caroline's feedback form, she used "class discussion" and "group work," but we classify the "class discussion" as lecturing to the class and "group work" as the teacher writing the answers on the overhead while many students were off task. Thus, the teacher feedback conflicted with the classroom observation fieldnotes. It is difficult to draw any definite conclusions about Caroline's academic engagement because only two of her 16 piloting days were observed. Even so, for the two observation periods, the students were not engaged in the materials. The observer noted that Caroline's students did not seem enthusiastic about the topics and rarely asked questions, which was a stark contrast to the behavior of Jim's and Sharon's students. Therefore, Caroline's implementation of the MDM curriculum did not engage students academically as the authors intended.

Technology

One distinguishing factor about the MDM curriculum is the role of technology. Technology is not a main objective of the curriculum but is instead a means to organize, solve, or analyze real-life applications. Consequently, the MDM curriculum authors emphasize reflection and interpretation of a contextual problem during and following technology use. Using spreadsheet software, students are expected to analyze and interpret how values and solutions change and reflect on why these changes occurred, a process known as *sensitivity analysis*. According to observation fieldnotes and feedback forms, two teachers (Amelia and Sharon) encouraged technology reflection and interpretation through sensitivity analysis while one teacher (Bob) did not.

Two different observers visited Amelia's class. The first observation occurred while Amelia taught sensitivity analysis and the second observation was five days after the first, during which time students had mastered sensitivity analysis. The first observer noted that Amelia lectured and used short-answer recall questions. Therefore, one may conclude that students did not have sufficient opportunity to interpret and reflect. However, the second observer noted that the students seemed to have a deep understanding of linear programming. When the second observer asked questions, students easily explained and interpreted the results of the problem. In the time between the two observations, students were given time to reflect on the problem and analyze the sensitivity of the numbers; hence, the teacher provided opportunities for the students to use technology to deepen their understanding of the context. Thus, careful observation and analysis shows that Amelia was faithful to the authors' intentions by encouraging reflection and interpretation during and after technology use.

Another teacher, Sharon, was observed the day before sensitivity analysis was taught and then again a couple days afterwards by the same observer. The observer noted that she saw a large difference in the students' understanding of linear programming as indicated by the way the students talked about problems. During the first observation, the students struggled to set up the spreadsheets and interpret the problem context. After learning sensitivity analysis, the observer noted that the students effortlessly set up spreadsheets and interpreted the constraints and objective functions. In general, they understood the problem at a much deeper level due to their use of technology. Therefore, Sharon, like Amelia, gave a faithful implementation of the technology piece of this chapter.

Some teachers felt sensitivity analysis was too difficult for their students to learn. For example, we observed Bob after teaching linear programming. Due to lack of time, limited technology access, and teacher-perceived student ability, Bob chose not to teach sensitivity analysis. When the observer asked the students asked about their experiences with the materials, the students said they did not like typing in so many values into a spreadsheet. The students also did not remember most of the problems in the text, even though the observation occurred approximately two weeks after Bob taught the materials. We concluded that because the students were not exposed to sensitivity analysis, they did not interpret and analyze problems in a meaningful way. Consequently, they had a shallow understanding of the content as demonstrated by their lack of familiarity with problem contexts. Therefore, Bob was not faithful to the authors' desire to have students reflect and interpret during and after technology use.

According to observation fieldnotes, Amelia and Sharon's students had a deeper understanding of linear programming than did Bob's students. We discerned that one reason for this discrepancy was whether the teacher taught students how to analyze and interpret the sensitivity of the data when using spreadsheet software. Teachers who used technology to reflect and interpret contextual problems, such as Amelia and Sharon, had a high level of implementation fidelity regarding the technology focal point.

Connectedness to the World

The authors of the MDM curriculum wrote all problems in context. Specifically, the authors presented a real-life application followed by the mathematics needed to solve the problem. Therefore, the authors expected teachers to teach the materials in context. If a teacher focused on the mathematics out of context, the authors would consider this an unfaithful implementation of the curriculum. One teacher (Jim) taught in context while another teacher (Janice) instead emphasized mathematical procedure only.

As mentioned before, Jim was observed one time while teaching linear programming. Throughout the observation, he kept the discourse about an introductory linear programming problem in context. In fact, Jim avoided complex mathematical jargon and focused on students' thinking, conjecturing, and interpreting the problem. After the observation, the observer asked Jim why he did not introduce vocabulary such as "feasible region" or "constraint" during the lesson. He responded that his students learned linear programming in Algebra II and he did not want students to jump to the procedures they knew from before. By avoiding previously learned vocabulary, Jim felt that the students remained focused on the context of the problem; thereby, providing students with a fresh outlook on linear programming. Jim's description of his class on his feedback form aligned with those characteristics observed in the classroom.

Janice was observed four times, three times by one observer and once by another. Both observers noted that Janice would "go over" the context but then emphasize the procedure. For example, when teaching shortest path problems, Janice introduced the context (transporting medical supplies) but then discussed only the procedure for the majority of the lesson. While she did not always teach algorithms out of context, this technique was typical for all four observations. Janice's responses on her feedback form were too vague to discern whether they supported the observation fieldnotes. For example, when discussing the lesson on the shortest path, Janice noted, "Together, we found the shortest path." This comment did not reveal how she approached the problem in class and whether she kept the problem in context. Consequently, without additional sources to validate the observation field notes, we were unable to draw a definitive conclusion about Janice's level of implementation fidelity regarding teaching in context.

Because the authors of the MDM curriculum wrote contextual problems based upon real-life applications, they expected teachers to emphasize the context of the problems throughout the lesson. Observers noted that Janice would introduce the context of a problem, but then proceed to teach in an algorithmic manner with no reference to the context. Jim sustained the context of a linear programming problem in a lesson, but sacrificed the formal vocabulary to do so. Nevertheless, we discerned that Jim had a high level of implementation fidelity regarding the connectedness to the world focal point because his students learned about linear programming using a real-life application, not as an isolated mathematical procedure.

Discussion

We provided examples of teachers who showed evidence of either supporting or straying from the focal points established by the MDM authors. Interestingly, teachers who supported a specific focal point tended to support all of the authors' focal points, and teachers who strayed from one focal point tended to stray from most focal points.

We note that teachers who are faithful to the authors' intents are not necessarily better or more effective teachers than those who stray from the focal points. The purpose of the analysis was only to evaluate how teachers implemented the curriculum compared to how the authors envisioned the implementations, not in teacher effectiveness or student achievement.

Unlike authors of many standards-based curricula, the MDM project team has the opportunity to determine and express to teachers exactly what types of pedagogy they wish to see in the classroom because the researchers and authors have personally met the teachers implementing the materials. However, at this time, the authors had not yet explicitly informed teachers of their Productive Pedagogy focal points. Hence, we view our preliminary results regarding implementation fidelity as positive since some teachers seemed to support the focal points without any formal training on how to do so.

As this preliminary research is continued, the research team needs to examine further exactly what they expect to see in the classroom and to explain explicitly these expectations to the teachers. Additional classroom observations need to take place in order to examine whether the teachers are faithful to the authors' intent. Moreover, the discrepancy in the data needs to be addressed; specifically, the information provided by the teachers in their feedback forms did not necessarily align with the information gathered by the classroom observations. Therefore, the research team will need to gather more information to verify teacher-reported data using instruments such as teacher interviews and surveys.

Endnote

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STRUCTURES OF FINNISH AND ICELANDIC MATHEMATICS LESSONS: A VIDEO-BASED ANALYSIS

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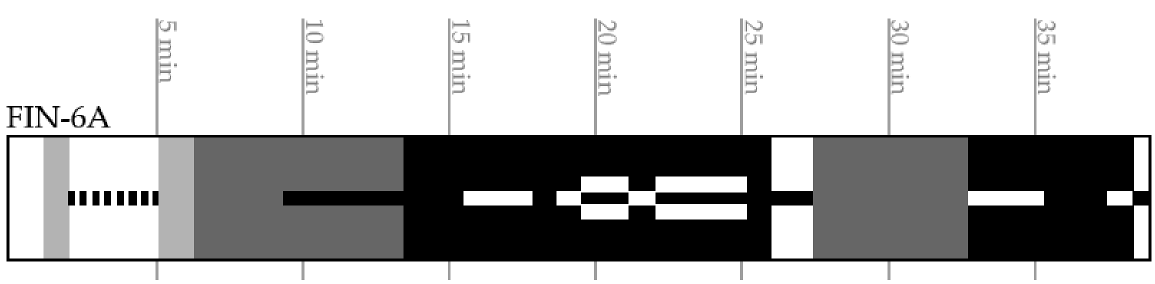
For the purposes of a dissertation project, structures of 40 Finnish and Icelandic mathematics lessons were analyzed. The method of lesson structure analysis, which was developed specifically for this project, offers a means to investigate the different forms of classroom interaction teachers use to achieve their pedagogical goals. The method requires two coding passes. The first pass is inspired by the TIMSS 1999 Video Study and is used to distinguish the main pedagogical functions of lesson elements. The second coding pass, which uses ideas from the Learner's Perspective Study as well as TIMSS, focuses on the forms of classroom participation. The coding categories are sensitive to the sample.

Analysis of videos from Finnish and Icelandic mathematics classrooms demonstrates the coding method for lesson structure. These countries were chosen in part because of their performance in the PISA studies; Finnish students have excelled in all three PISA studies, while Iceland is the only country where the girls have significantly outperformed the boys in mathematics. The recordings—two lessons from ten randomly chosen mathematics teachers of 14 and 15-year-olds in each country—were collected in 2007.

Based on this sample, there are differences in how Finnish and Icelandic mathematics lessons are structured. More than one half of the Icelandic lessons in the sample exemplify the "*einstaklingsmiðað nám,*" or *individualized learning*-strategy, the nationally endorsed pedagogical philosophy. Public instructional discourse can be missing entirely from these lessons, and, instead, the teacher tutors each student independently. This is in stark contrast with the Finnish lessons, where teacher-lead activities in which often the whole class is involved are emphasized. Based on the video evidence, Finnish teachers are rather traditional and pedagogically conservative. Their classroom practices often include a substantial social component, while many students in Iceland are getting used to learning independently, without significant collaboration with others.

The presentation of the method of lesson structure analysis as well as the results of the video study would fit well into the poster session format. The two coding passes can be explained in some detail on a poster. Statistical charts and lesson diagrams, such as the one shown below this text, can also be effectively displayed. Furthermore, copies of the finished dissertation and an article based on the study can be made available for the participants who want to learn more.



Chapter 9: Number and Number Sense

Research Reports

Posters

A DEVELOPING FRAMEWORK FOR CHILDREN'S REASONING ABOUT INTEGERS

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By considering the ways that children reason about integers and the difficulties they face in extending their numeric domains from whole numbers (nonnegative integers) to negative numbers, we seek to problematize the rich and nuanced understandings we in the mathematics community hold of integers. In this paper we document instances of how children's thinking has paralleled that of mathematicians of the past, highlight ways in which children have displayed expert-like reasoning about integers, and provide possible explanations why some children are able to reason about integers in reasonably sophisticated ways.

Introduction

As adults we proficiently operate with numbers, specifically negative numbers, without the need for deep thought or reflection. Much of what we do, like rewriting 5 – (x + -3) as 5 – x + 3 is automatic and beyond the level of conscious awareness. Moreover, we have a variety of understandings, metaphors, and contexts we bring to problems that allow us to think of and use numbers in multiple ways. For example, consider the number -5:

- We can interpret -5 as a directed magnitude (like owing money or debt);
- We can interpret -5 as the location on a number line (coordinate plane, etc.) 5 units to the left of or below 0;
- We can interpret -5 as the number before -4 and after -6;
- We can interpret -5 as the number that, when added to 5, results in 0;
- We can interpret -5 as an action of removing 5 from a set;
- We can interpret -5 as an action of moving 5 units left or down;
- We can interpret -5 as the difference or comparison between two quantities;
- We can interpret -5 as an element of the equivalence class [(0,5)] which also includes (1,6), (-100, -95), (-2,3) and all other ordered pairs of integers satisfying the equivalence relation (a,b) + (c,d) = (a+c, b+d) (where we define (a,b) to mean a – b).

In this paper we seek to problematize the rich and nuanced understandings we in the mathematics community hold by considering the ways that children reason about integers and the difficulties they face in extending their numeric domains from whole numbers (nonnegative integers) to negative numbers. By considering the ways that children reason about integers we can enrich our own understanding and help students develop the flexibility to move back and forth between these different conceptions of negative numbers.

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A Brief Historical Summary

For over 1000 years mathematicians pondered, struggled with, operated on, rejected, and eventually came to accept negative numbers. Not unlike their problematic, unwanted, and barely acknowledged cousin – the irrational numbers – negative numbers were the source of great consternation, confusion, and controversy. "Negative numbers troubled mathematicians far more than irrational numbers did, perhaps because negatives had no readily available geometric meaning and the rules of operation were stranger" (Kline, 1980, p.118). Negative numbers were theoretically plausible yet practically impossible. However, with Fermat and Descartes's creation of analytic geometry as well as the need for more formalized and generalizable solutions to problems, mathematicians saw the rise of algebra and a corresponding acceptance of negative numbers (Freudenthal, 1983; Kline, 1980). This acceptance, though, was not without strong objections from some of the best mathematicians of their day. The philosopher Thomas Hobbes critiqued John Wallis's book on the algebrazation of conic sections for applying algebra to geometry calling it a scurvy book and a "scab of symbols" (Kline, 1980, p. 124). Negative numbers too were rejected and described as "fictitious" (Jerome Cardan, 16th century), "absurd" (Michael Stifel, 16th century), and "false because they claim to represent numbers less than nothing" (Rene Descartes, 17th century). (See Gallardo, 2002; Hefendehl-Hebeker, 1991; Kline, 1980) The English mathematician Francis Maseres rejected negative roots in his 1759 Dissertation on the Use of the Negative Sign in Algebra saying,

> *… they serve only, as far as I am able to judge, to puzzle the whole doctrine of equations, and to render obscure and mysterious things that are in their own nature exceeding plain and simple … it were to be wished therefore that negative roots had never been admitted into algebra* (as quoted in Kline, 1980, p.119).

Why were mathematicians so slow to accept negative numbers? Certainly the lack of a physical, concrete, or geometrically meaningful solution contributed to the problem. Fibonacci, Chuquet, and Descartes (among others) objected to the lack of a tangible, physical or realistic interpretation of negative numbers. For example how can one have a negative number of monkeys (Bhascara I) or buy a negative amount of cloth from a merchant (Chuquet). These "impossibilities" were only accepted if a reasonable explanation for the negative solution could be generated in the given problem context (Gallardo, 2002; Hefendehl-Hebeker, 1991; Kline, 1980). In other words, a negative solution must have a magnitude-based, or positive, interpretation in order to be accepted. Second, the existence of negative numbers made the routine interpretation of addition as the joining of subsets to create a larger set problematic. Hence, the reason for Diophantus's characterizaton of the equation 4 = 4x + 20 as "absurd" since the four units as the result of the summation should not be less than the addend of 20 units (Gallardo, 2002). And, finally, the attempt to remove something from nothing was seen as nonsensical. When describing the problem 0 – 4, Pascal said, "I know people who do not understand that when you subtract four from zero, what is left is zero" (Gallardo, 2002, p. 171).

Epistemologically, these reasons might point to conceptual barriers students today continue to face in understanding and operating with negative numbers (Herscovics, 1989; Thomaidis & Tzanakis, 2007). In this paper we hope to identify some of the challenges students have in understanding negative numbers and understand why those confusions exist. Moreover, we hope to identify the tacit and intuitive understandings children do have of negative numbers so that teachers, researchers and curriculum writers can better build on those strengths.

Methods

Our primary research questions were a) what are students' conceptions of integers and operations on integers, and b) how might these conceptions develop over time. As our study is ongoing, we will present findings primarily addressing research question (a). Participants in our study include 26 elementary school students from Texas and California. Twenty of these children are in kindergarten, first, or second grade, and the remaining six are fifth graders. We have also interviewed 3 specialized adults with expertise either as mathematicians, math educators, or teachers. In addition, we have done a historical analysis of the literature to identify ways of reasoning and specific problems that puzzled mathematicians as well as the ways in which the discipline was able to overcome these problems.

Based on our historical analysis as well as our review of the relevant literature we developed sets of tasks specific to integer reasoning to be used in problem solving interviews with children and our specialized adults. Because of the range in age and understandings of our participants, we have three different interviews, one for K-2, one for 5-8, and another for specialized adults. Although many of the questions are different, the underlying constructs we sought to assess were the same across interviews. Our assessment comprises seven categories that assess children's understandings of:

1. Number (e.g., numbers are ordinal – for counting and ordering; numbers are cardinal – they tell you "how many") and zero (e.g., zero is nothing, zero is a place on the number line, or zero is the additive identity).
2. Number when solving counterintuitive problems (i.e., problems that contradict notions limited to positive numbers such as "addition makes larger" or "subtraction makes smaller"). The following problems are seen as counterintuitive if one's numerical domain is only positive numbers: $3 - 5 = \square$, $6 - \square = 8$, or $7 + \square = 3$
3. Negative numbers as actions or processes (subtracting, removing, comparing, finding distance, and motion)
4. Negative numbers as objects (inverse, directed magnitudes, elements of an equivalence class).
5. Number comparisons: assessing how children think about larger and smaller when comparing negative numbers. (e.g., do they order numbers or do they reason in terms of magnitude.)
6. Symbolic proficiency: does a child recognize "–" can be used to indicate a negative number as well as the operation of subtraction, and does a child know when to appropriately use which interpretation?
7. Contexts and tools in shifting or limiting their reasoning about negative numbers.

See the Appendix for examples of interview questions.

The interviews themselves were conducted at the children's school sites, during the school day. They lasted between 30 and 50 minutes and were videotaped. Our analysis of the videotaped interviews began with a process of open coding (Strauss & Corbin, 1998) focusing on children's solution strategies as well as their underlying ways of reasoning about number and operating with numbers. We used principles of grounded theory (primarily the constant comparative method) to identify emergent, distinguishing themes and features of students' reasoning about integers (Strauss & Corbin, 1998). We balanced emergent codes from the data itself with historical and theoretical findings from the literature. As we are in the beginning stages of the larger study, our analysis is still ongoing and will likely evolve. What we report here is a beginning framework for children's thinking about integers and the different strands "integer sense" might encompass.

Findings

In the following section, we describe our findings with respect to children's conceptions of and thinking about integers. We document instances of how children's thinking has paralleled that of mathematicians of the past, highlight ways in which children have displayed expert-like reasoning about integers outlined in the paper's introduction, and provide possible explanations why some children are able to reason in reasonably sophisticated ways.

First, we have found parallels in young children's thinking about negative numbers with that of mathematicians of the past. These parallels are described in Table 1 below.

Table 1. Conceptual Challenges of Negative Numbers and The Parallels Between Mathematicians and Children's Thinking

Challenges	Mathematicians' Responses	Children's Related Responses
Removing something from nothing	"I know people who cannot understand that when you subtract four from zero, what is left is zero." –Pascal Negative roots rejected as "false" because they claim to represent something less than nothing. –Descartes	"Three minus five doesn't make sense because three is less than five" (Niki, 1st grade). "Three minus five is zero because you have 3 and you cant' take away 5 so take away the 3 and it leaves you with zero" (Callie, kindergarten). "Zero is nothing but negative is more nothing" (Rachel, 2nd grade). A number is "how you know how much something is. Like, this is 2 (holds up two counters)" (Rosie, 2nd grade).
Lack of a tangible, concrete, or realistic interpretation for negative numbers	The Indian mathematician Bhascara I explained that "people do not approve of a negative absolute number." Thus negative solutions were "incongruous." Fibonacci, Cuquet, and Descartes did not accept negative solutions unless the result could be interpreted as something positive.	"Negative numbers aren't really numbers because we don't really count with them in school. And there's no negative one cube (holds up a unifix cube)" (Rosie, 2nd grade). Lacy just solved 3 – 5 = □, answering -2. Lacy: That number line actually helped me a lot. Interviewer: Did these (points to cubes) help you? Lacy: When I used cubes, I mean what could they help me with this? How am I gonna do it? (Lacy, 1st grade)
Counterintuitive situations involving routine interpreta-tions of addition and subtraction	Diophantus claimed the equation 4 = 4x + 20 was "absurd" because the 4 was less than the 20 units that were added.	"4 + □ = 3 is not a real problem. It's not true" (He then crossed out the problem). "Four minus one would equal three." (BradLee, 1st grade) In response to the problem 6 + □ = 4 … "What's that plus for? Isn't it supposed to be a minus?" (Brett, 1st grade)

Table 2. Examples of Children's Expert-like Reasoning About Negative Numbers

Interpretation	Example
1. Negative number as directed magnitude	"Would you rather have positive 6 dollars or negative 23 dollars? So you would of course want 6 dollars ... You wouldn't want to owe a store 23 dollars b/c your credit card doesn't have the money on it. Then you would have to pay 23 dollars b/c you wouldn't have the money right now … it's like I said before if you have 3 dollars and the sandals that you want are 4 dollars you would owe them a dollar. (Brittany, 5th grade)
2. Negative numbers as well-ordered	To solve the problem $3 - 5 = \square$, Jake used a counting back strategy. He counted backwards starting at three saying, "Three, two (put up one finger), one (put up second finger), zero (put up third finger), -1 (put up fourth finger), -2 (put up fifth finger). The answer's -2." (Jake, 1st grade) When asked about numbers smaller than zero, Niki replied, "There is some but I forgot the word for them. Like something 1, something two, something 3, something 4." And when asked to count backwards from five she used the notation displayed in Figure 1, using an "S" to stand for "something." Note that she added a "something zero," or S0. (Niki, 1st grade)
3. Negative number as an additive inverse	When solving $16 + \square = 0$ Tanner answered -16 and justified it saying, "Since this is a positive 16 you can cancel it out with it's opposite which is -16. And then I just checked it with subtraction 16 minus 16 is zero. (Tanner, 5th grade)
4. Negative number as the action of movement (to the left or down)	To solve $3 - 5 = \square$, Lacy initially made stacks of 3 and 5 unifix cubes but quickly turned to a number line. She put a pointer at 3 and moved it 5 spaces to the left while counting aloud (1,2,3,4,5). She answered 02 (her invented notation for -2) and said, "That number line actually helped me a lot." (Lacy, 1st grade)
5. Negative number as the difference (comparison) between two quantities or locations	Rosie is solving the "Train Problem" which says: You are taking the Polar Express from your home to the North Pole. You have one stop to make at *X* Elementary (insert child's school name). Some people are on the train, but we don't know how many. Today the train stopped at *X* Elementary. Four people got on the train and seven people got off the train. Does the train have more, fewer, or the same number of people when it leaves *X* Elementary as when it left your house? How many more/fewer people? "There would be less because if four kids get on that would be 4 more. But if 7 kids get off that would be 7 less. So 4 minus 7 would be something negative. So it would be less." She then writes $4 - 7 = -3$ and when asked what -3 means she says, "That means 3 less than we had on the train before." (Rosie, 2nd grade)
6. Negative number as subtraction and the equivalence between subtraction and adding an inverse (or addition and subtracting an inverse)	The following exchange occurred while Brett solved the problem $6 + \square = 4$. *Brett:* What is that plus for? Isn't that supposed to be a minus? *Interviewer:* Is there any kind of number you can add to make it smaller? Brett: A negative number! I think, hmmm. Oh it's a negative number. (He picked up the number line and moved from six to four.) Two, I mean negative two. *Interviewer:* How are you getting that? *Brett:* Well six minus two would equal four so six plus negative two would equal four. *Interviewer:* Those sound different, one is six minus and one is six plus. How do you know they are equal? *Brett:* Plussing a number, plussing a number, a negative number, would be minusing instead of plussing. (Brett, 1st grade)
7. Negative number as an element of an equivalence class	While answering the Happy & Sad Thoughts Question (see appendix), Tanner said "Tuesday was a sad day, the same as Monday." When asked to explain he said, "Because you get rid of, for this one you get rid of two (points to the 2 smiley faces and 2 sad faces he crossed out) and you have five sad thoughts left over. And on Tuesday you get rid of one (points to the 1 smiley face and 1 sad face he crossed out) and there are five sad thoughts still left over. I'm talking about what she has left over. Each one has 5 sad thoughts. (Tanner, 5th grade)

This suggests that these are natural confusions for children to have and are to be expected given their early experiences with number. Here we see that young children can, at times, resemble early mathematicians in their struggle to understand, accept and operate with negative numbers. But this is only part of the story. We have found, perhaps surprisingly, that children as young as six can overcome these conceptual challenges that many of their peers and even mathematicians of old were unable to.

Below, we present some of the ways in which children articulated many of the “expert-like” ways of thinking about integers described in the introduction of this paper. We are not proposing that the types of reasoning outlined in Table 2 are what one would expect from an expert, but that they are precursors to reasoning about numbers in more sophisticated ways that are developmentally appropriate for children this age.

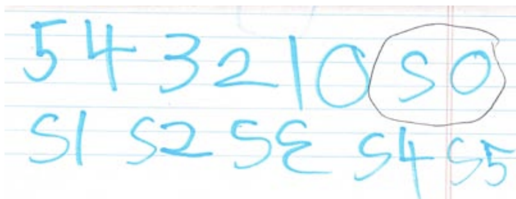

Figure 1. Niki’s Representation of the “Somethings”

Discussion & Implications

Why are some children able to exhibit this kind of reasoning and others not? We posit that different ways of conceiving of and using numbers more generally, might explain children’s ability to reason in relatively sophisticated ways and overcome many of the aforementioned challenges in Table 1. On the basis of our analysis of interviews, we found it useful to distinguish four ways of reasoning about numbers in general that seem linked to children’s thinking about negative numbers:

1. numbers as countable, tangible quantities or amounts,
2. numbers as well-ordered,
3. numbers as a position or location, and
4. numbers in a formalized sense.

More formally, the first three ways of thinking about numbers can be called a cardinal use of numbers, an ordinal use of numbers, and a nominal use of numbers. We believe these ways of reasoning about numbers in general have implications for how children might approach and understand negative numbers. For example, consider the problem 3 – 5 = $\square$. If numbers are viewed as tangible objects (see the examples in Table 1), one could think of a context where he or she had three of something and promised five to a friend. After giving the three she had, she would still owe two. Although this idea can be represented symbolically as -2, one cannot represent -2 in a tangible way. Rosie reminds us of this when she explains why negative numbers are not really numbers in her mind: “We don’t really count with them in school, and there’s no negative one cube.” (Note that as a 2nd grader she has not been introduced to a chip model yet.) What does it mean to take something from nothing or to have -2 cubes? From a cardinality only viewpoint, these problems really do not make sense.

Yet, one could approach the same problem in a number line or motion context where -2 is a position; it is the place one lands when starting at 3 and moving left 5 (or starting at -8 and moving right 6). In this view, negative numbers are seen as ordinal and sequentially related to other numbers, but not necessarily representing an amount or quantity. This way of reasoning about number is also seen in Jake's counting back strategy. For him, there did not seem to be anything particularly special about negative numbers, they just happened to be on the other side of zero. This is in contrast to students who conceived of numbers smaller than, or below, zero but not in ways that allowed them to compare, order, and operate on numbers. We refer to this as a nominal use of number. Although not included in Tables 1 and 2, several students, when asked to identify places to the left of 0 during the Number-Line Game were unable to distinguish unique and well-ordered places to the left of zero numerically (see Appendix for activity description). BradLee initially called every place left of 0, "zero". When asked where he would go if his teacher asked him to go to zero, he replied he didn't know and then renamed -1 as "none" and -1 as "no numbers". Teddy called every place to the left of 0 "negative" and described the entire left side of the number line as the "negative hallway". He later revised his labels calling -1 the "negative classroom", -2 the "negative playground", -3 the "negative cafeteria" and so on until ending with the "negative principal's office." These children saw places to the left of zero as locations and even gave them names (negative classroom, no numbers, etc.), but they were unable to order these places numerically.

However, some children were able to approach and use numbers in a more formal, algebraic way, conceiving of negative numbers as additive inverses, noting the equivalence between subtraction and adding an inverse (or addition and subtracting an inverse), and comparing differences of differences in order to see integers as elements of equivalence classes (though their language is much less formal). We are not proposing that all children should use only a formal, algebraic way of reasoning about number, but that children should be able to approach numbers (and, also, negative numbers) from a variety of views and flexibly move back and forth between these ways of reasoning.

We found that young children can reason in powerful ways about negative numbers given the right tools and opportunities. We suspect that certain types of problems and tasks might be particularly rich sites for children's learning about negative numbers. Moreover, these kinds of problems can provide opportunities for children to construct new views of negative numbers. As our work is ongoing, we anticipate identifying these types of generative tasks as well as developing trajectories of integer reasoning that children could pass through on their way to more sophisticated and flexible ways of reasoning.

Endnotes

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Appendix

Specialized Adult Expert Sample Interview Questions

1. How do you conceptualize negative numbers? How is it similar to the way you conceptualize positive numbers? How is it different?
2. What are your thoughts about why kids seem to struggle when operating with negative numbers? If students learned integers well, what would they understand or be able to do?
3. For each, assume x, y are integers, not equal to 0. Write <, =, >, or ? (can't determine).

a.)	-5 + 7 – 3	-5 + 7 + 3	b.)	- (-12 + 14)	12 – 14
c.)	$\lvert x\rvert + \lvert y\rvert$	$\lvert x\rvert - \lvert y\rvert$	d.)	-x	x
e.)	x + 1	x	f.)	x + x	x
g.)	(-x)(y)	(x)(-y)			

K-2 & 5-8 Sample Interview Questions

1. Open Number Sentences

a.) 6 – 11 = ☐ b.) 12 + ☐ = 4 c.) -11 + 6 = ☐ d.) 5 – ☐ = 18

2. Happy and Sad Thoughts

Courtney has happy thoughts and sad thoughts. One sad thought cancels one happy thought. On Monday Courtney had 2 happy thoughts and 7 sad thoughts:

☺ ☺☹ ☹ ☹ ☹ ☹ ☹ ☹

Did Courtney have a happy day or a sad day? How happy or sad was Courtney? How would you write that?

On Tuesday, Courtney had 1 happy thought and 6 sad thoughts:

☺ ☹ ☹ ☹ ☹ ☹ ☹

Was Tuesday a happy day or sad day? How happy or sad was Courtney? How would you write that? Which day, Monday or Tuesday, was sadder?

3. Number Line Game

In this game, children drew an action card (+ or -) that told them which direction to move on the number line and a magnitude card (1, 2, 3, and so on) which indicated how far to move. The number line was labeled starting at 0 and extending to the right (1, 2, 3, and so on) but had unlabeled marks to the left of zero. During the course of the game, students will eventually land on a place to the left of zero, and are then asked to name these places.

ADDITION AND SUBTRACTION WITH NEGATIVES: ACKNOWLEDGING THE MULTIPLE MEANINGS OF THE MINUS SIGN

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In order for students to use negative numbers in operations, they must make sense of the binary, symmetric, and unary interpretations of the minus sign. However, little is known about the extent to which children reason about these three interpretations prior to integer instruction. Through interviews, I explore the strategies 22 second graders use to solve addition and subtraction problems with negative numbers, the meanings (binary, symmetric, and unary) they assign to minus signs, and their understanding of the operations. Analysis of students' solutions and language suggest that the second graders think comparably to students who have had integer instruction.

Introduction

Algebra is increasingly recognized as a gateway to higher mathematics and as a required skill for employment; consequently, one goal of middle school mathematics is to help students build a foundation of algebra readiness skills (EdSource, 2009). Even with the intensified focus, one reason why students still struggle with algebra concepts is they have difficulty understanding and working with negative numbers (Gallardo, 2002; Vlassis, 2004). Tatsuoka and Baillie identified 89 different rules that students use to solve integer addition and subtraction problems (e.g., subtract the smaller absolute value from the larger and add the sign of the first number); although, these rules have varying levels of success (Tatsuoka, 1983). In particular, students have trouble making sense of the changing role of the minus sign (Vlassis, 2008); they solve problems such as 8 - - 4 by ignoring the duplicate sign (Murray, 1985) or ignoring the first minus, solving, and adding the sign at the end (Fagnant, Vlassis & Crahay, 2005; Schwarz, Kohn, & Resnick, 1993; Vlassis, 2004); they claim that two minuses in a problem mean a plus (e.g., -1 + -5 = 6) (Vlassis, 2004); and they answer problems such as 6 + -2 by adding 2 and then subtracting 2 (Murray, 1985).

Children's learning of whole numbers and whole number operations relies heavily on the use of physical manipulatives and having extended experiences with whole numbers from their toddler to mid-elementary years (Sarama & Clements, 2009). On the other hand, manipulatives for negative numbers are contrived, and according to the California mathematics content standards, students are introduced to negative numbers in fourth grade and are expected to add and subtract them the following year (California Department of Education, 1999). Some of students' struggles with negative numbers might arise because a brief introduction to them a year before learning integer operations is not long enough or because the instruction they receive is not sufficient. If this is the case, we would expect some of their same misconceptions to also arise with students who have not had instruction. The purpose of this study was to closely explore how students solve addition and subtraction problems involving negative numbers and explain their thinking, prior to instruction. More specifically, the goal was to ascertain to what extent students before integer instruction and students post-integer instruction exhibit similar reasoning.

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Theoretical Framework

Gallardo and Rojano (1994) proposed three levels of conceptualization of negative numbers which are further elaborated by Vlassis (2008). These levels include interpreting minuses as binary, symmetric, and unary functions. The *binary* nature of the minus sign corresponds to the subtraction operation (e.g. 5 - 3 = 2). With negative numbers, the binary interpretation is also associated with the changing meaning of subtraction as addition when subtracting a negative number. The *symmetric* nature of the minus sign is associated with “taking the opposite of a number” (Vlassis, 2008, p. 561-2), so – (3) = –3 and – (–3) = 3. Finally, the *unary* nature of the minus sign aligns with the traditional notion of negatives as a new class of numbers to the left of zero on the horizontal number line, with decreasing quantities as one moves away from zero. Students’ descriptions of how they solve integer arithmetic problems provide insight into the functions that they assign to minus signs across problem types. While we know students who have been exposed to negative integers utilize these three functions (Gallardo & Rojano, 1994; Vlassis, 2008), we need more information on younger students’ conceptions to determine if acknowledgement of the three functions arises out of instruction or not.

As Van Oers (2002) points out, “it is highly relevant to study the processes involved in symbolizing activities in a detailed way, as well as to find out the dynamics…that may influence the development of this symbolizing capacity” (p. 30). As children learn new signs, they reflect on the signs and their possible meanings. Based on his framework, students form pseudo-concepts as they take their current understanding of symbols, such as the minus sign, and apply them to new situations, such as integer problems. They also start to use language introduced by adults, although they may still attribute old meanings to the new expressions (Van Oers, 2002). The extent to which younger students acknowledge the three functions of the minus sign and the language they use to describe solutions to integer problems have implications on the timeliness of instruction students should receive on negative numbers.

Research Questions

The following questions guided my research:

1) What do second grade students’ solution strategies indicate about how they make sense of the changing nature of the minus sign? To what extent do the students acknowledge the three roles of the minus sign?

2) What do their solutions and language indicate about their understanding of addition and subtraction with negative numbers?

Methods

Subjects and Site

This exploratory study was conducted at an elementary school in a district in northern California which has a low percentage of English language learners (only 2.3% of the students compared to 25% statewide and 9% nationwide) (Education Data Partnership, 2010). It was important to interview children with a strong grasp of English to get more detailed descriptions of students’ solution explanations without confounding the difficulty of explaining problem interpretations with the difficulty of using the language. A total of 22 second graders (8 male, 14 female) from two classrooms agreed to participate in the interviews.

Materials and Data Collection

Individual interviews lasted for 16 minutes on average. During the interviews, students answered 18 negative number addition and subtraction problems presented one at a time and--

except for the first two questions--in random order. Before moving on to the next question, students had to explain how they came to their answer. Siegler (1996) found that this method is accurate and reliable.

Table 1. Items students completed during interviews

Form A			Form B
6 - 6			6 - 6
4 - 9			4 - 9
-8 + 4	or	$\boxed{-8} + \boxed{4}$	-8 + 4
-9 + 5	or	$\boxed{-9} + \boxed{5}$	-9 + 5
-3 + 1	or	$\boxed{-3} + \boxed{1}$	-3 + 1
-4 + 2	or	$\boxed{-4} + \boxed{2}$	-4 + 2
9 - -1	or	$\boxed{9} - \boxed{-1}$	2 - -4
6 - -4	or	$\boxed{6} - \boxed{-4}$	1 - -2
7 - -2	or	$\boxed{7} - \boxed{-2}$	3 - -5
5 - -3	or	$\boxed{5} - \boxed{-3}$	4 - -6
-4 - -4	or	$\boxed{-4} - \boxed{-4}$	-3 + -7
-7 - -7	or	$\boxed{-7} - \boxed{-7}$	-2 + -5
-3 - -3	or	$\boxed{-3} - \boxed{-3}$	-4 + -8
-8 - -8	or	$\boxed{-8} - \boxed{-8}$	-3 + -6
-2 - 5	or	$\boxed{-2} - \boxed{5}$	-3 - 2
-3 - 8	or	$\boxed{-3} - \boxed{8}$	-7 - 4
-1 - 6	or	$\boxed{-1} - \boxed{6}$	-2 - 1
-2 - 7	or	$\boxed{-2} - \boxed{7}$	-6 - 3

Based on the position of the signs and numerals in the problems there are 32 possible integer addition and subtraction problem types. Two test forms were created in an effort to cover more of the problem types without overwhelming students with too many questions; students' first two questions were the same across forms and four other questions were repeated on both forms. Aside from the first two problems, the order of the questions was randomized on each test, and students were randomly assigned to test form. Students taking Form A saw either the first or second half of their problems with boxes around the numerals to see if this format changed the ways they talked about the negative signs (see Table 1 above).

Analysis

Students' solutions were coded for accuracy, and their verbal reports were transcribed and coded for each question. Solutions were coded for the strategies students used to set up the problem (e.g., ignored all negatives, reversed the numerals, changed the operation), strategies students used to compute the answers (e.g., recall, count the distance between numbers, use a related fact), and manipulations of the answer (e.g., added a negative sign). Student responses that reflected one of the three meanings of the minus sign (binary, symmetric, and unary) were identified based on the codes. Students' responses were also coded for references to their descriptions of addition and subtraction in comparison to how they solved the problems.

Results

Three Meanings of the Minus Sign

All students correctly solved the first question, 6 - 6, so this problem was dropped from analysis. Overall, the average number correct score was 3 out of 17 (20%); however individuals' scores ranged from 0 to 9 out of 17 (0% - 53% correct). Among the group of 22 second-graders and 17 problems, students acknowledged all three roles of the minus sign. Together, they used the binary function on 198 out of 374 questions (53%), the symmetric function on 74 out of 374 questions (20%), and the unary function on 38 out of 374 questions (10%).

Table 2. Second grade students' binary interpretations of the minus sign

Pseudonym	Solution	Explanation	Code
Gretchen	-9 + 5 = 5	Take away the nine and then you add a five. (Student solved 9 - 9 + 5)	Subtract: Self
Paul	-9 + 5 = 14	It says zero minus nine, nine minus zero, and so it would be nine…and then I did nine plus five. (Student solved 0 - 9 as 9 - 0, then did 9 + 5)	Subtract: New Number; Reverse Order
Nicole	9 - -1 = 7	It could be seven because minus minus <points to both "minuses">. (Student solved 9 - 1 - 1)	Subtract: Again
Mollie	7 - -2 = 9	I have a master and he taught me you just cross that <points to the two signs> and do plus. (Student solved 7 + 2)	Equivalent
Bryan	-4 + 2 = -2	I was subtracting two, so I subtract two of these <points to the negative four>. (Student solved -4 - -2)	Equivalent
Monica	5 - -3 = 2	It says five minus three, is two. I did that on fingers. (Student solved 5 – 3)	Ignore

Binary. On average, students used the minus sign as an operator or binary function on 9 of the 17 problems (53%), but the range was large: two students used the binary interpretation on only 2 of 17 problems (12%) and two used it on all 17 problems (100%). However, students frequently ignored negative signs in order to subtract or interpreted negative signs as minuses in interesting ways. Table 2 provides examples of students' binary responses and their corresponding codes. Notes in "< >" describe students' actions during the interview and those in "( )" describe my interpretation of their solutions.

The problem 4 - 9 presents an interesting case because all students used the binary interpretation of the minus sign, but 15 of the 22 students (68%) reversed the numerals and solved 9 - 4 = 5. Surprisingly, none of these fifteen students explicitly stated that you cannot subtract a larger number from a smaller one, a claim which often persists through middle school regardless of whether students solve other negative problems correctly (Ball, 1993; Murray, 1985; Vlassis, 2004). Instead, they commented that it was easier to think about the problem with the larger number first or said they just knew the answer. Three additional students (14%) said there were not enough to subtract and produced answers of zero (Peled, Mukhopadhyay, & Resnick, 1989), and four students (18%) crossed the zero point and reached negative answers.

Symmetric. While all students acknowledged the binary nature of the minus sign, only 8 out of 22 students (36%) used the symmetric format of the minus sign. However, when they did use the symmetric format, they did so for an average of nine problems. Students' symmetric responses fall into one of two categories: 1) solving problems as positive and adding a negative sign to the answer because the problem contained negatives and 2) drawing comparisons between the positive and negative forms of the problem, calculating using positive numbers, and adding a negative sign. This distinction may be subtle but represents advancement in students' thinking. Amelia belongs to the first category; after solving -7 - -7 = -0, she explained, "It has negatives in it, and seven minus seven is zero." In contrast, Bryan changed the original problem, 9 - -1, into a new problem, -9 + -1, and solved it by reasoning, "Nine plus one equals ten, so if it was minus nine plus minus one, it would equal minus ten," which illustrates category 2.

Unary. Finally, 8 out of 22 students (36%) used the unary interpretation of the minus sign and talked about and operated on negative numbers as separate from positive numbers. These students used the unary interpretation for an average of five problems. For instance, when solving -2 - 1, one student, Emily said, "It's one less than negative two, so it'd be negative three." While the unary interpretation was not used as often when students solved problems, 14 of the 22 students (64%) read negative numbers using language that distinguished them from positive numbers, such as "minus five," "below zero five," or "negative five."

Overall, 7 out of 22 students (32%) used all three meanings of the minus sign, 1 out of 22 students (5%) used the binary and symmetric formats, 1 out of 22 students (5%) used the binary and unary formats, and 13 out of 22 students (59%) used only the binary format. Additionally, use of the boxes around the numbers on Form A did not make a significant difference on their strategies.

Interpretation of Addition and Subtraction

Especially for students who acknowledged the unary or symmetric interpretations of the minus sign, the language they used to describe their solution processes reveals conflicting conceptions of addition and subtraction. When operating with natural numbers, addition results in a number that is larger than either addend and that is further to the right on the number line. With negatives, however, adding a smaller positive to a larger negative number (e.g., -8 + 4) results in a larger number further to the right on the number line but which has a smaller absolute value. Likewise adding a negative to a negative number (e.g., -2 + -5) results in a number with a larger absolute value but which is further to the left on the number line.

While all students who took form B of the test got a larger absolute value for problems such as -4 + -8 (answers were -12 or 12), responses to problems such as -8 + 4 show that several students were uncertain whether to focus on moving to the right on the number line or on reaching a larger absolute value. Furthermore, when using the binary meaning of the minus sign, students either ignored the negative signs and/or reversed numbers so they could do positive number subtraction or they wavered between moving left on the number line and getting a number with smaller absolute value. Table 3 provides examples of how students' answers changed between number line and absolute value-based answers for addition and subtraction.

Table 3. Students' conflicting explanations for addition and subtraction problems

Pseudonym	Solution	Explanation	Meaning of Operation
Robbie	-8 + 4 = -12	Eight plus four equals twelve and negative is just plussing but going lower.	Adding: Larger Absolute Value
	-4 + 2 = -2	Negative two…it's just uh, going up, so negative two is less…is more.	Adding: Going Up/Right
Emily	-7 - 4 = -3	You would go from seven, negative seven and then minus and take away four and go from seven, six, five, four, three, and that would be negative three.	Subtracting: Smaller Absolute Value
	-2 - 1 = -3	It's one less than negative two, so it'd be negative three.	Subtracting: Going Left or Getting Fewer

Discussion and Conclusions

The results of this study have a few implications toward curricula and task design in the broader frame of optimizing student understanding. First, merely looking at students' number correct scores on integer tests will not provide sufficient information on how students interpret the negative signs or order and quantity of the negatives. Second graders in this study supplied incorrect answers on many of the problems but demonstrated insightful and complicated reasoning about them. For example, when solving -8 - -8, one student, Mollie said, "Well, if that one wasn't negative <points to the first negative>, then it would have been sixteen, but it's negative, so I think it's eight," which is 8 less than the 16 it would have been. Additionally, some of the students obtained correct answers through incomplete reasoning, which would not be captured through written tests alone. Therefore, we need to pay more attention to the answers students provide and select questions which will supply the most contrast and highlight how they are thinking about the problems. Furthermore, the interviews revealed that students are trying to make sense of how operations with the negative numbers align with their previous ideas of addition and subtraction, so future measures should aim to uncover how students think about these concepts together with the multiple meanings of the minus sign.

Brown and Borko (1992) explain, "Learning involves making connections between new information and existing systems of knowledge; teaching should facilitate making these connections by helping students to relate new knowledge to knowledge they have already developed" (p. 211). Children in second grade are already forming ideas about the multiple roles of the minus sign or what problems like 4 - 9 mean, and in many cases, they use the same strategies identified by researchers who have interviewed students after they have had instruction in negative numbers (e.g., Fagnant, Vlassis & Crahay, 2005; Murray, 1985). Since over half of the children in the study were willing to explore more than the binary function of the minus sign, negative numbers can be introduced earlier, and curricula could address their emerging and conflicting conceptions so that children have more time to become familiar with negatives. One area which instruction could target is illuminating the lack of the commutative property holding for subtraction and guiding students to acknowledge the possibility of solving problems like 4 - 9 without feeling the urge to reverse the numerals. Students should also have the opportunity to explore the multiple meanings of the minus sign. Earlier and extended experiences with the concepts of negatives may help prepare students for operations with them later.

Aside from instructional implications, these results also suggest future areas of study. In the case of students who ignore all negative signs and operate with only the binary meaning of the minus sign, we should explore what experiences influence them to acknowledge the other functions of the minus sign. Further studies should also compare how students approach the three meanings of the minus sign and the changing nature of addition and subtraction both before and after instruction to determine if current instruction helps students progress in their thinking on these topics. Finally, we should further explore the language teachers use to talk about whole number addition and subtraction and investigate its impact on students' reasoning when adding and subtracting with negatives.

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DIFFERENT LEVELS OF REASONING IN WITHIN STATE RATIO CONCEPTION AND THE CONCEPTUALIZATION OF RATE: A POSSIBLE EXAMPLE

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In this study I studied the nature of the conception of within-state ratios. Fourteen prospective elementary and secondary mathematics teachers participated in two, one and one half-hour long written sessions followed by an hour- long clinical interview. This paper presents one of the prospective elementary mathematics teacher's, Mark's use of strategies and justifications behind those strategies in the context of ratio. Data on Mark's justifications showed different levels of reasoning in within-state ratios at the beginning of the interview as compared to the end of the interview. Whether or not different reasoning patterns indicate learning or only a reorienting to something previously learned, data might be an example of a student who possibly had the understanding of particular ratios [internalized ratio] at the beginning of the interview, and came to an understanding of rate [interiorized ratio] by the end of the interview.

Introduction

Noelting (1980b) claimed that a student might approach a proportion such as $\frac{a}{b} = \frac{c}{d}$ by comparing the first set of ratios $\frac{a}{b}$ and $\frac{c}{d}$ or the second set of ratios $\frac{a}{c}$ and $\frac{b}{d}$. In the first case, he called the ratios $\frac{a}{b}$ and $\frac{c}{d}$ as *within (state) ratios,* where the ratio represents the original quantities, within one state, and the quantities come from two different measure spaces such as distance and time in uniform motion. In the second case, he called the ratios $\frac{a}{c}$ and $\frac{b}{d}$ as *between (state) ratios*, where the ratio represents quantities between two situations and so the quantities come from the same measure space such as the comparison of distance to distance or time to time in two situations of the uniform motion (Noelting, 1980b). Noelting and some other researchers emphasized that these two types of ratios originate from different cognitive processes (Karplus, Stage & Pulos, 1983b; Noelting, 1980b; Schwartz, 1988;Vergnaud, 1994). In this study, the focus was on the cognitive processes that make one pay attention to within state ratios at different levels of reasoning.

Kaput & West (1994) defined the particular ratio of "3 pounds per 4 dollars" as a particular substitution of x and y values, meaning a particular purchase; on the other hand, defined the rate-ratio as the linear function defined between two measures (Kaput & West, 1994) ---the function that determines the relationship between the two quantities. In addition, researchers argued that experience with the particular ratio (internalized ratio) situations as a mental operation, relating a particular pair of quantities and realizing the homogeneity, must yield to the conceptualization of rate intensive quantity, rate-ratio (interiorized ratio) (Kaput &West, 1994; Thompson, 1994). This study also focused on possible conceptualization processes involved in the understanding of rate- ratios building on particular ratio situations. Several researchers have studied different conceptions of ratio. Simon & Blume (1994) introduced ratio-as-measure as an intensive quantity that is a measure of the attribute in the situation. Thompson (1994) focused on the conceptualization process for the abstraction of a specific case of ratio as measure, speed. His analysis for the conceptualization process of speed proposed different stages of knowing. Those stages included 1) an image of two quantities covarying, 2) an image of two segments, one for distance and one for time, partitioned proportionally according to units of time, and 3) “an image of segmented total distance in relation to a segmented total amount of time” (Thompson, 1994; p. 204). Heinz (2000) studied identical groups conception in which ratio is understood as a representation of association between two extensive quantities. Similar to the conceptualization process claimed by Thompson (1994) for speed, Heinz (2000) proposed a detailed sequence that students would go through in conceptualizing ratio as an intensive quantity. The sequence is also aligned with the building- up processes, a precursor to the understanding of rate conception (Heinz, 2000; Kaput & West, 1994; Thompson, 1994).

Purpose

This study expands previous research on the conception of ratio (e.g., Heinz, 2000; Kaput &West, 1994; Simon & Blume, 1994; Thompson, 1994) in two important ways: First, in light of the data, using different tasks, reasoning patterns involved in within state ratio conception as an intensive quantity are compared to the reasoning patterns involved in an understanding of ratio as a representation of the association between two extensive quantities. In addition, a student’s use of different representations in those different reasoning patterns in within state ratio situations was documented. Second, whether data showed learning on the student’s part or only a reorienting to something previously learned, an example of a student who *possibly* had the understanding of particular ratios [internalized ratio] at the beginning of the interview, and came to an understanding of rate [interiorized ratio] by the end of the interview was shown. Such work is likely to shed light on the learning of ratio as a measure of an attribute (Simon & Blume, 1994) that researchers claimed to develop on the building-up processes students use in particular ratio situations (Heinz, 2000; Kaput & West, 1994; Thompson, 1994).

Methodology

This study embraces the idea that knowledge is not independent of the knower; researchers can only create models of knowledge of others by observing their behavior (e.g., Clement, 2000). In this study several tasks in the context of ratio were given to fourteen prospective mathematics teachers to investigate the nature of the understanding of within state ratios. Observations were made by asking the students to justify their reasoning in two, one and one half-hour long written sessions. Then, a one hour-long clinical interview was conducted with the students, using both the same tasks and some additional tasks. Some of the tasks were adaptations of tasks used in previous research. And I created several other tasks such as The Mosaic Problem (adopted from Vergnaud, 1994), the Hair Color-2, the Mixture and the Ice-Cream Problem (presented in results section) by taking into consideration previous research results on ratio (e.g., Heinz, 2000; Simon & Blume, 1994; Thompson, 1994; Kaput & West, 1994). For instance, I knew that students without the necessary knowledge base in ratio might revert to primitive stages of knowing when reasoning on problems with numbers whose magnitudes are relatively close to each other (Kaput & West, 1994) and also students with the necessary knowledge base in ratio can reason through the problems regardless of the difficulty of numbers they encounter (Heinz, 2000). Based on these results, I included non-integer multiplies and some difficult numbers in some tasks to distinguish different levels of reasoning in within state ratios. I also paid attention to the fact that the problem solving process assumes the integration, elaboration and the entailment of the previous states into the current state of mathematical knowledge such that necessary previous states of knowledge can be called upon into the current knowing actions (Kieren &Pirie, 1991).

In both written problem solving sessions and interviews, I examined Mark's strategies, solution processes and justifications with the goal of determining what underlying conceptions of ratio he might be revealing through the strategies, processes and justifications. To make sure that hypotheses and conclusions drawn from the data were plausible, other researchers were consulted to challenge the conjectures and /or to affirm their reasonableness. Also, previous research results were visited frequently. As shall be seen in the following sections, discomforting data were also under scrutiny such that based on researchers' opinions on the initial hypotheses about the student's thinking, either the alternative hypotheses were created or more than one interpretation of the data were suggested. To do this, I used the results of an important study done by Tzur & Simon (2004). These researchers introduced two stages of abstraction: participatory stage and anticipatory stage. Participatory stage refers to the understanding level such that the student has access to the knowledge only if participating in the context of the activity through which the knowledge is developed. They asserted, "In the context of the activity means either the learners are engaged in the activity or are somehow (e.g. chance, social interaction) prompted to use or think about the activity" (p. 15). Anticipatory stage refers to an understanding level such that the student is able to call upon the knowledge without being in the context of the activity through which it is developed.

The following sections present excerpts from transcripts of the interview conducted with Mark. Data showed different reasoning patterns and use of representations revealing different levels in the understanding of within-ratios at the beginning of the interview compared to the end of the interview. Though, Mark's initial thinking on the Mosaic Problem showed the same patterns as the beginning of the interview, after he reasoned through an alternative solution to the Mosaic Problem, his reasoning about the ratio as measure situations changed significantly. In particular, his understanding of ratio at the beginning of the interview showed characteristics of an understanding of ratio as a representation of two extensive quantities; whereas, the line of reasoning at the end of the interview showed characteristics of the understanding of ratio as an intensive quantity measuring the attribute in the situation.

Results

Mark's Understanding of Within-State Ratios

As mentioned, Mark's reasoning and use of representations about within state ratio situations changed drastically after his engagement in the Mosaic Problem. For that reason, each section is labeled either "Beginning of Interview" or "End of Interview". Beginning of Interview refers to before Mark reasons about the Mosaic Problem, and End of Interview refers to Mark's reasoning about the Mosaic Problem and after.

Beginning of Interview

At the beginning of the interview, when Mark reasoned through The Hair Color-2 Problem and the Mixture Problem, his explanations and representations resembled an understanding of ratio as an expression that represents the association between two extensive quantities (Heinz, 2000). That is, ratio is made up of two extensive quantities.

> ***The Hair Color-2 Problem:*** *Kelly likes to color her hair with a mixture of red and browndye. When her hair was shorter; she used 15 grams of red and 17 grams of brown dye. Now, her*

hair is longer. She knows that the original mixture will not be enough to color all of her hair. She intends to add 7 grams of red dye to the original amount of red dye and some amount of brown dye to the original amount of brown dye. How much brown dye does she need to make sure that she has the original color?
In a group of seventh graders, the following two solutions were given:
a) The amount of brown dye Kelly needs to add to the original amount is 17(22/15).*
b) The amount of brown dye Kelly needs to add to the original amount is 22(17/15).*
Explain how each answer fits or does not fit the story? Depending on the choice of your answer, explain what (22/15) represents and/or what (17/15) represents.

In the written sessions, Mark had solved the Hair Color-2 Problem using cross multiplication procedure instead of explaining how the answers fit the story and what the ratios represented. At the beginning of the interview, I asked Mark to account for the solutions provided to him. Mark explained what 17/15 ratio represents and he said “So, for every 15 grams of red dye, she needs 17 grams of brown dye”. Then, I asked him to comment on that more:

R: Can you show to me you can put into mathematical terms or, is there a way you can show me for every 15 grams of red there will be 17 grams of brown dye?
M: For every gram of red dye it is going to be 1.15 for brown.
The excerpt shows that Mark knows that the ratio 17/15 represents the relationship between red and brown dye as “for every 15 grams of red, there would be 17 grams of brown.” He also offered that the result of the division tells how many grams of red dye are needed for 1 gram of brown dye. Then I asked him to show 1.15, the result of the division of 17 by 15, using any kind of graphs, diagrams or shapes, he drew two bars and said:
M: This for every one gram of red there is going to be one a little bit more brown [He made two bars, one of them is for R and the other is for B where Brown is bigger than the Red one]

Figure 1. Mark’s reasoning in within-ratios at the beginning of the interview

The excerpt shows that Mark knows that 17 grams of brown dye and 15 grams of red dye are related to each other such that when he partitioned 17 grams of red into 15 equal parts, the result would be “for every gram of red, there will be 1.15 grams of brown” and he can represent the result of division as two different bars because he is thinking that the result of division, 1.15, represents two quantities: 1 gram of red and 1.15 grams of brown. That is, Mark engaged in approaching the Hair Color-2 Problem using a per-one strategy [unit-factor approach]. Heinz (2000) stated that students with the understanding of ratio as a representation of extensive quantities (identical groups conception) would utilize per-one strategy to associate the quantities in the original ratio situation.

Then he commented on what he wrote above as R*15=15 and B*15=17.

M: Right, yeah, and then there is 15 of those whatever you need for red, and then 15 of those will be what you need for brown, or if for brown that is bigger, so you need 15 here and 17 of there.

The excerpt shows that Mark knows that the two original quantities of 15 grams of red dye and 17 grams of brown dye can be partitioned into 15 equal parts such that the result of the partition is 1 gram for red and 1.15 grams for brown. He knows that he can accumulate those 1 grams to make up the original 15 and at the same time he has to accumulate 15 portions of 1.15 grams of brown because of the specific combination they make up, i.e. the original color. Thus he knows that "1.15 grams of brown dye and 1 gram of red dye" have the same color as "17 grams of brown dye and 15 grams of red dye." Another important point is that, Mark is able to think about how many times both quantities need to be iterated. In other words, since he knows that 1 gram of red dye becomes 15 grams of red dye at the end and 15 is 15 times bigger than 1, and he needs to iterate both quantities at the same time, he knows that the other quantity needs to be iterated the same number of times.

Also, Mark's thinking on the "b" option and "c" option of the Mixture Problem showed the same characteristics of his thinking on the Hair Color-2 Problem.

The Mixture Problem: *A group of fifth graders will mix two types of juice mixture and will put them into jars so they can serve them to the kindergarten students during the week. In the first jar, they put 36 grams of lemon juice and 32 grams of lime juice. In the second jar, they put 20 grams of lemon juice and 15 grams of lime juice. To label the jars, the fifth graders wanted to use one number to accurately represent the lemon-lime flavor in that jar. They considered the following three ideas for labeling the jars:*

a) There is 4 more lemons in the first mixture so put 4 on the first label. There are 5 more lemons in the second mixture, so put 5 on the second label.

b) The ratio of 36/32 for the first type of mixture and the ratio of 20/15 for the second type of mixture.

c) 1.125 (the result of dividing 36 by 32) and 1.33 (the result of dividing 20 by 15).

Which idea(s) would accurately indicate the lemon-lime flavor? Why? Provide an explanation for the reason (s) that each of these options would or would not indicate the mixture's lemon-lime flavor.

Mark said, "That is also another ratio, because that is like if you take 36 to 32, 1.125 means for every 1 gram, 1 ounce or whatever of lime juice there will be 1.125 lemon juice in the mixture. 1 and 1.125 so whenever you have like 2, 2 times 1.125, 3 will be 3 times 1.125, four whatever and when you get to 32 then it will be 32 times 1.125". His initial thinking on the Mosaic Problem also showed the same characteristics of his thinking on the Hair Color-2 problem. [To be shared during the conference]

End of Interview: Mark's Reasoning on the Mosaic Problem

This section provides data on Mark's reasoning on an alternative solution of the Mosaic Problem provided to him during the interview. It is important to reemphasize that Mark's initial thinking on the Mosaic Problem resembled his reasoning on the Hair Color-2 Problem and the Mixture Problem.

The Mosaic Problem*: Two sisters, Shari and Melissa, decided to create a mosaic design on their table top from broken pieces of tile. They worked for sometime, and then they had a break. When they had the break they realized that, it took 16 minutes to finish an area of 40 cm square. If they worked at the same rate, how much of the tabletop could they finish in 36*

> *minutes? Can you provide multiple solutions to this problem? Explain how your solutions make sense in terms of the problem context.*

During the interview, I provided the following to Mark "Somebody solved the problem as follows: They divided 36 by 16, 36/16=2.25, and multiplied that by 40, so 2.25*40=90. Can you account for the solution? What does 2.25 represent? Why does it make sense to multiply it with 40?" Mark explained his reason for how he was able to multiply 2.25 with 40 by saying "they worked 2.25 times as long". Then, to make sense of what 2.25 represented he engaged in adding the amounts of 16 minutes and 40 square cm repeatedly to reach the result. At this moment the interview became much more interesting because of Mark's line of reasoning.

M: They originally worked for 16, this is 16 minutes, and they worked then 40 cm, so if you know 36 over 16 is, going from 16 to 36, would be like one, two and 2 and two five so they are staying at the same rate so it will be 40, 80 and a fourth of that"
R: A fourth of that
M: Because point 25 is one fourth, so like 16, 32 and a 4^{th} of 16 and so then 40, 80 and a fourth of it is 10 so it is
R: Fourth of what
M: The area
R: The fourth of which one, like 16, 32.
M: 40, 80 and 90
R: A fourth of what
M: fourth of the 40
R: Then are you going to add that to and you know you can get 4^{th} of 40 because
M: Because you went from 16 to 32 and 4^{th} of 16 is 4 to get 36 and then 40, 80 and 4^{th} of 10, 4^{th} of 40 is ten so then you have the 80 and then 90.
R: So you know that the remaining 4 are 4^{th} of 16 why you are using that.
M: Because it is 2.25
R: Why are you allowed to find 4^{th} of 16 in the context of the problem, why does it make sense?
M: You can use a 4^{th} because 36 is not a multiple of 16 it is more than that so it is, it is not like 32 and then only a fraction of that so that is why you can use one 4^{th} for 16 and get 4, it was just like it was 32 minutes so 16 minutes to 32 minutes would be 40 cm to 80 cm since this is 36 it is going to be a 4^{th} a 4^{th} more, a 4^{th} as much so from 16 to 36 it will be 40 to 90, but without using ratios, you can wrap all these
R: How?
M: [The student draws a graph] so it starts here at 16 over here you are at 40 so it is about there and than 36 it is about 90, [inaudible] rate, it is going to be the it is going to be the constant line.
R: What does that line represent?
M: It tells you how much they will do in a time period, so instead of doing proportions, it would be like if they worked 10 minutes it is going to be about 30 around or it will be around 25 because 2.5 you can't tell from the graph but you can tell it is going to go up it is like rise over run so 2.5 over 1 right so there will be 2.5 cm square.

Mark's reasoning and use of representations seems to be more sophisticated after the

Mosaic Problem. Mark's description of a line graph contrasts with his earlier use of a bar graph to show the relative size of two quantities.

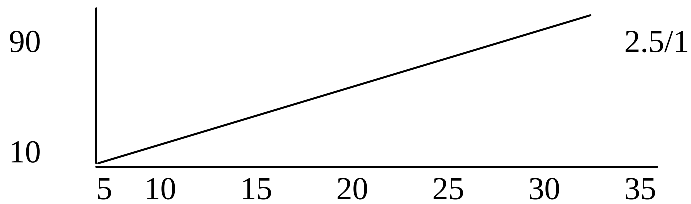

Figure 2. The graph Mark drew for the Mosaic Problem

Thompson (1994) claimed that "as soon as one reconceives the situation as being that the ratio applies generally outside the phenomenal bounds in which it was originally conceived, then one has generalized that ratio to a rate…rate is (from my point of view) a linear function that can be instantiated with the value of an approximately conceived structure" (p.192). In that sense, data show that Mark is able to use the line representation as a representation of constant ratio which represents the invariant relationship, "2.5 square cm for 1 minute," among all the cases outside the phenomenal bounds of the Mosaic Problem. Thompson further claimed "…To say that an object travels at 50 miles/hour quantifies the objects' motion, but it says nothing about a distance traveled nor about a duration traveled at that speed (Schwartz, 1988, cited in Thompson, 1994). However, conceiving speed of travel in relation to an amount of time traveled produces a specific value for the distance traveled" (p.192). In that sense, the last excerpt shows that Mark could think of 10 minutes as worth 2.5 times as much work, which is 25 square cm. Thus, he was able to think of 10 minutes and 25 square cm of work producing a specific value for the invariant relationship of 2.5 square cm per minute among all the cases with the same rate. After this point, when I asked him to comment on the problem below, the Ice-Cream Problem, he could reason about it using within state ratios.

The Ice-Cream Problem: *In an Ice-Cream factory 12.7 liters of milk and 10.5 pounds of sugar is needed for one gallon of ice-cream. The producers realized that they had 11.4 pounds of sugar for the new gallon and wanted to produce the same Ice-Cream with what they have. How much milk do they need to use?*

Kelly solved the problem above and found the answer as 13.7 liters of milk. She then claimed that the ratio of 12.7 to 10.5 is the same as the ratio of 13.7 to 11.4. Then, she wrote:

12.7/10.5= 13.7/11.4

Why should I believe her that this is a true statement? Can you convince me that this is a true statement?

Mark reasoned through that problem using the same argumentation and the graph [To be shared during the conference]. That is, Mark calculated 12.7 divided by 10.5, made a graph to communicate his thinking and claimed the following:

R: What does that line tell you, what is that line?
M: That is, it will be 1.21 over divided by 1. for 1.21 liters of milk there is one pound of sugar.

The above excerpt shows that the line is the relationship between the two quantities that make up the ratio, such that no matter what amount constitutes the mixture, the relationship between the quantities does not change. The invariant relationship is a multiplicative relationship such that the quotient, the result of division of the quantities in within-ratios, represents that.

Mark's Understanding of Between-State Ratios

Data showed Mark's limitation on the Hair Color-2 Problem, option "a", at the beginning of the interview. When I asked Mark to reason about option "a", the meaning of the ratio of 22/15, he said "…I do not know that one…" . However, he was able to reason about the alternative solution to the Mosaic Problem, which involved a between ratio solution process, depended on the quotitive division. Tzur & Simon (2004) in their postulation of *participatory* and *anticipatory* stages of understanding provided a way to make sense of Mark's different responses to the two problems. At the participatory stage of understanding, the student needs to be able to be thinking about or be involved in the activity through which the understanding developed. For Mark, he was able to solve the Mosaic Problem because it cued him to his activity of solving missing-value problems by multiplying each quantity by the same number, which also cued him to use quotitive division. At the anticipatory stage, the student no longer requires a focus on the original activity. He is able to call on the understanding when needed. Thus, for Mark, the Hair Color-2 required an anticipatory level of understanding, because it did not cue him for the activity of solving missing-value problems by multiplying each quantity by the same number.

Conclusion and Discussion

The line of reasoning data corresponds with ratio as being the expression of specific situations, particular ratios, and rate as being the constant ratio, which is reflectedly abstracted from ratio, through generalized situations (Thompson, 1994). Building on Thompson's (1994) analysis for the conceptualization process of speed and Heinz's (2000) hypothetical sequence, I propose the following as an explanation for how Mark came to an understanding of rate building on particular ratios. Mark already had gone through the accumulation of the quantities of 16 minutes and 40 square cm until he reached the targeted quantity of 36 minutes. His focus was first on the specific amounts of 16 minutes and 40 square cm, 32 minutes and 80 square cm and 36 minutes and 90 square cm. Once he made the segments of 16 minutes until 36 minutes – between state ratios---and 40 square cm until 90 square cm—between state ratios---, his focus was on both the whole segment and the specific amounts of that segment, such as 16 to 40, and 32 to 80 and 36 to 90—within state ratios. Mark knew that *each of* those pairs of quantities have a particular multiplicative relationship. Since he already knew that 16 and 40 are related to each other by 2.5 square cm for 1 minute, this made him realize that *each of* those specific quantities, 16 to 40, 32 to 80, 36 to 90, are all related to each other by the same relationship of 2.5 square cm per one minute (an image of an accumulation of two quantities that carries the image that the values of both quantities can vary, but only in constant ratio to the other). At that moment, he claimed that, "You can wrap all these," because he realized that the ratio is a constant ratio, invariant for all cases with the same property. His use of the line graph and explanation of it demonstrated his reasoning about an interiorized ratio or rate.

In sum, Mark's reasoning and use of representations in within state ratio situations at the beginning of the interview was different from his thinking at the end of the interview. More interestingly, Mark's reasoning on the Mosaic Problem presented somewhat different levels of reasoning about within state ratios. Particularly, Mark used a unit-factor approach to reason about the problem, at first. Later he was able to think about the within state ratios as an intensive quantity that quantifies the quality of interest in the situation. Also, at the beginning of the interview, Mark's reasoning on the Hair Color-2 Problem "a" option contrasted his reasoning on the alternative solution to The Mosaic Problem at the end of the interview. It seems that Mark's activity with the Mosaic Problem led either to learning or to a reorienting to something previously learned. In either case, his explanations and representations were more sophisticated following his work on this problem.

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FROM EQUIVALENCE TO RATIONAL NUMBERS: THE CASE OF MEREDITH

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This study presents an analysis of a nine-year-old student's mathematical activity constructing and extending fraction ideas in a fourth-grade classroom teaching experiment. The researchers, supported by the students' teacher and principal, facilitated forty-eight open-ended sessions over the course of the school year. The students, working in pairs, invented strategies and constructed models of their solutions to problems involving fractions and rational numbers. Solutions were shared and questioned by the students as they justified their ideas. Results provide important documentation that students can demonstrate deep understanding of complex ideas before their introduction in the traditional curriculum.

Introduction

This study is part of a larger National Science Foundation funded research project[1] based on data collected from a year-long teaching experiment in a rural/suburban public school. The mathematical ideas introduced in the fourth grade classroom took place before procedural ideas about equivalence and operations with fractions were introduced in the school's curriculum. The study included forty-eight sessions implemented throughout the 1993-1994 academic year. This paper focuses on a six-week interval early in the year and includes three consecutive sessions on September 27, September 29, October 4 and a fourth session several weeks later on November 10. During the first three sessions the students were presented problems about fractions in reference to a particular unit comparison of fractions and equivalence. They were asked to model their solutions to each problem with Cuisenaire rods and develop justifications for their reasoning that could be shared with the class (Steencken, 2001; Yankelewitz, 2009). The conceptual understanding demonstrated by the students concerning the meaning of fractions as numbers rather than simply as operators and the subsequent existence of an unlimited number of equivalent names for each fraction (Steencken, 1998) served as a foundation for extending equivalence notions to integers and rational numbers in the fourth session, November 10, 1993. In that session, the students, based on their work with their partners, were invited to place positive and negative numbers on a large number line written on the dry erase board at the front of the class. As they shared their reasoning about each entry, the class debated a variety of issues and disagreements about what was posted (Schmeelk, 2010). The four sessions analyzed and reported in this research build from the first time the nine and ten year old students encountered two distinct names for the same fraction. They concluded with the students' consideration of where to position those names for positive and negative integers and proper and improper fractions on a number line. In particular, students express concern and engage in discussion about the referent unit on the line for each rational number being considered. Our case study specifically examines Meredith as she builds solutions to the problems and interacts with her classmates and the researcher/facilitator over the six-week interval. The research is guided by the particular question: "How does Meredith's understanding of equivalent fractions extend to rational number?"

Theoretical Framework

The notion of students building and restructuring their thinking based on previous experiences has been discussed in detail by Davis (1984). In the process of building personally meaningful experience, the learner assimilates earlier knowledge into new frames. This process is referred to by Davis as an *assimilation paradigm*. Davis and Maher (1990) refer to these new frames as a set of internal mental functions where a learner sees a new experience to be "just like" or as "similar to" earlier experiences. The properties that make an activity serve as an assimilation paradigm, according to Davis, deal with new ideas for which students have "powerful representations" and "reliably accurate isomorphic images" for all operations that apply. These images, according to Davis, are what guides learners to build solutions to new problems. He suggests, further, that the frames that are built and applied to new problems make building the solutions "simple" for the student. This suggests that learning with meaningful experience in which properties can be recognized and assimilation paradigms can be built constitute conditions for the growth of knowledge.

According to Davis and Maher (1990), the process by which knowledge is assimilated is cyclic (p. 65). It includes building a representation, carrying out memory searches to retrieve or construct a representation, constructing a mapping between the data representation and the knowledge representation, checking the mapping for correctness, and using technical devices associated with the knowledge in order to solve problems. The powerful representations described by Davis (1984) help students to construct mappings between the external and internal representations.

The Rational Number Project (RNP), over more than two decades, has provided seminal foundational research about how students think about fractions and rational numbers (Behr, Harel, Post & Lesh 1992; Behr et al., 1993; Lesh, Post & Behr, 1988). More recent RNP research studies, such as Cramer, Post and del Mas (2002) and Cramer and Henry (2002), show that students, when encouraged to construct their own conceptual understanding of rational numbers, relied less on rote procedures when solving mathematical tasks. These studies showed that the students who emerged from the RNP curricula scored significantly higher and exhibited a better quality of thinking and estimation on tasks given during interviews than comparison group students who did not use the RNP materials. Further, Cramer and Henry (2002) examined the role of using manipulative models to build number sense for addition of fractions and noted that teachers often transition to symbols from manipulatives too soon. Their results show evidence that the RNP students developed an understanding of fraction size as they successfully ordered fractions and were able to estimate answers to problems and verbalize their thinking. Overall, the Rational Number Project materials when used correctly appear to improve characteristics of students' number sense.

Freudenthal (1983) describes equivalent rational numbers in terms of representing the "same thing." He gives the example, "2/3 = 4/6 = 6/9 = …" and states that each fraction in the example is an alias for the same rational number. Freudenthal uses whole number aliases as a metaphor to describe rational number aliases. He states that, "One prefers for the number 5 the expression 5 rather than 3+2, 10-5 and so on, though the others are equally well admissible" (p.133). Steencken and Maher (2003) describe the students' process of gradually replacing their physical models constructed from the rods with symbolic fraction number names as they built solutions for fraction problems, exemplifying Davis's idea of an assimilation paradigm. "Eventually they referred to the comparisons without physically using the rods" (p.130).

Context of the Study

The study took place in a suburban New Jersey district over a one-year period with a heterogeneously grouped fourth grade class of twenty-five students aged nine and ten. The group included fourteen girls and eleven boys. All sessions were videotaped with one to three cameras during extended classroom sessions lasting from fifty to seventy-five minutes. The teacher was present throughout the sessions as well as other observers including: graduate students, the principal, graduate student observers, and the camera crew.

Data and Methodology

Triangulation of data[2] occurred with video data and their full transcripts, observation notes, student overhead transparencies produced for sharing and other student work. The data combination (Pirie, 1996; Lesh & Lehrer, 2000) was designed to optimize the possibilities of examining in detail the students' mathematical activities while reducing human and technological biases (Powell, Francisco & Maher, 2003).

The data were analyzed based on components of the Powell, Francisco and Maher (2003) method, evolved from the longitudinal research video studies of the Robert B. Davis Institute for Learning (RBDIL). Their method includes several stages for the analysis of video data including: viewing, transcribing, coding, writing analytical commentaries and summaries, and developing a narrative. We elaborate on four main phases that were used. Firsts was *viewing* each of the CDs multiple times, which was followed by *transcribing and verifying* each of the CDs. The verification process entailed having another viewer watch the CD to verify that the CD was properly transcribed. Transcription consisted of four components including: line number, time code, speaker, spoken words and actions. The line numbers were used for reference. The time stamp usually was noted in five minute intervals. Transcriptions then listed the speaker's name followed by both spoken words and actions that are not verbally 30 expressed. Particular camera views also were noted within the session transcriptions. The third phase, *coding, was where critical events were flagged, traced, compared and categorized.* The codes were placed on the transcript in a new column to facilitate scanning of the transcript while analyzing critical events. During the last phase, a summary of the findings and a narrative emerged. The narrative tells the story of what took place during the sessions and references the actual critical events in the transcript by line number.

Earlier Research Results

During the first three sessions, September 27, 1993, September 29, 1993, and October 4, 1993, students, for the first time, encountered different number names for the same fractions. A comprehensive list of the tasks during the first three sessions is in Table 1. A summary of the fraction ideas that were investigated by the children who built a variety of models using rods and drawings of rods to represent their solutions is also given. Details about the reasoning of the students and the processes by which they challenged each other and built on the ideas of others are given elsewhere (Yankelewitz, 2009). The collection of tasks and the fraction ideas that were considered by the children are central to better understand the sequence of ideas that evolved. The students offered models to support their solutions and made revisions in order to accommodate the input of others. Consequently, their sharing and arguing about ideas generated multiple justifications that were backed with multiple models.

Table 1. List of Earlier Research Tasks and Fraction/Number Ideas

Session	Tasks	Fraction Ideas
9-27-1993	If I give this orange rod a number name one, what number name would I give to white?	Fraction as number
	If we are calling the orange rod the number name one, what would you call the number name for the red rod?	Fraction as number
	I'm calling the orange one, what number name would I give to two whites?	Fraction as number, Equivalence
	I'm going to call the orange ten…and I'm wondering if you could tell me the number name for white.	Fraction as number
	If I take the same orange rod we've been working with, but I change the number name again. This time I'd like to call it… fifty. I'm wondering if anybody could tell me the number name for yellow.	Fraction as number
	What number would I give to one white rod if we're still calling the orange the number name fifty?	Fraction as number
	Which is larger, 1/2 or 1/3, and by how much?	Comparison of fractions
9-29-1993	Is 1/5=2/10?	Equivalence
	What other number names can we give to one half of a candy bar?	Equivalence
	Which is larger, 1/2 or 1/3, and by how much?	Comparison, Operations with fractions
10-4-1993	Which is larger, one half or two thirds, and by how much?	Comparison, Operations with fractions

On September 27, when asked by the researcher, "If we are calling the orange rod one, the number name one, what would you call the number name for the red rod?" Gregory stated, "one-fifth." When asked why, Gregory replied, "Because five of the red blocks equal up to one orange block." The researcher then asked, "What number name would I give to two of the whites? Two white rods?" Erik exclaimed, "Two of the white rods? That's a red rod." The researcher called on Mark who responded, "One-fifth." Andrew explained why they had chosen one-fifth. Meredith stated, "I think it's two-tenths." Meredith explained, "When we did it before, we said that, we said that orange has ten whites, it measures up to ten whites. And if you put the whites up it would have ten. And two of ten is two-tenths." Two minutes later, the researcher asked the class, "What do the rest of you think of that?" The class remained quiet.

On September 29, the researcher asked the class, "Is one-fifth equal to two-tenths?" Meredith explained that ten white rods equal up to an orange rod so two white rods would be two-tenths. She, then, showed that five red rods equals up to an orange rod, so one of them would be one-fifth. She finished by saying that if you take one of the red rods and put it above two of the white rods, they're equal. The researcher asked other students in the class what they thought. Brian and Erik agreed.

On October 4, the students were asked which fraction, one-half or two-thirds, is bigger. They were asked to build models to support their ideas. After the students had finished building their models, the researcher asked the students again which fraction is bigger. The students stated that two-thirds is bigger. The researcher asked the students how much bigger two-thirds is was than one-half; they replied one-sixth. The researcher asked the class if anyone had made a second model, to which Meredith eagerly exclaimed, "Oh! Oh!" Meredith, then, went to the overhead projector and placed twelve white rods beneath the two dark green rods. She, then, showed that two white rods (two-twelfths) was the difference between two purple rods (two-thirds) and one dark green rod (one-half). Michael exclaimed, “No. They can’t do that because the two-thirds are bigger than the half by a red; so, they can’t use those whites to show it.”

Results

The earlier sessions provide a background for the flow of events that occurred during the fourth session, November 10. The activity ranged from a discussion of Meredith’s five lines on the number line, which she had constructed to parallel her rod constructions, to marking fractions on a big number line on the dry erase board at the front of the class. The session opened with a discussion of five stacked lines Meredith had drawn to where each line represented a different aspect of the number line as seen in Figure 1.

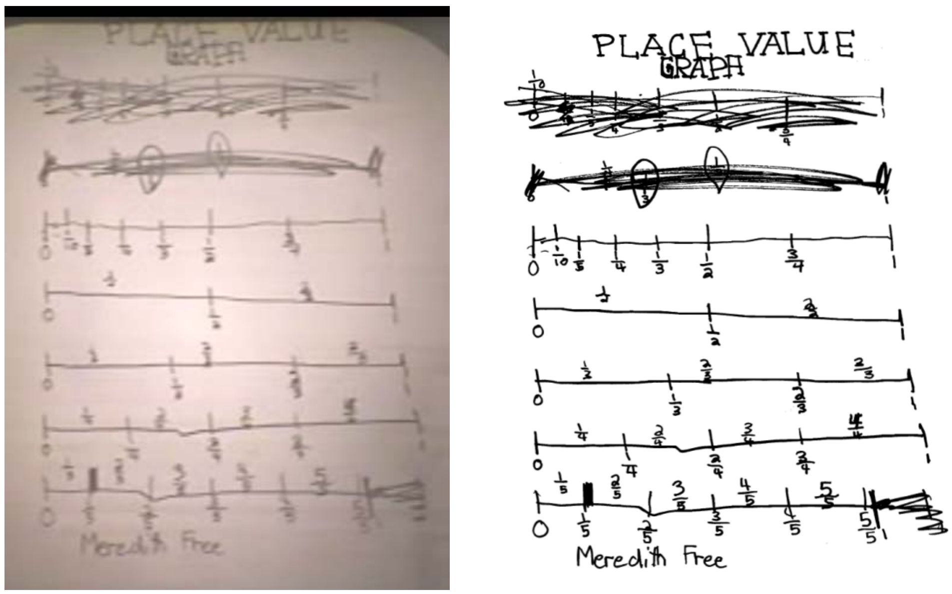

Figure 1. Meredith’s number line

Specifically, the top line consisted of a complete number line: the second line showed two half regions and the number one-half labeled: the third line showed three thirds regions with both numbers one-third and two-thirds labeled: etc. Meredith had labeled the number fractions under the line and the region fractions above the line. Some students expressed concern about the numbers Meredith had placed above the line; for example, Michael stated, “Why are you calling two-thirds, a half? It is not half. It is bigger than half, two-thirds is bigger than half.” Meredith replied that she knew but was unsure “where to put it.” The class debated the two ways to represent fractions (operator and number) for approximately fourteen minutes. Later, students worked on marking numbers on a number line on the overhead projector. A discussion about how many third could be fit in the interval between zero and one triggered the question about where four-thirds would go. The researcher asked to hold the question about the placement of four-thirds; however, Meredith exclaimed that the interval would need to be six-thirds. Meredith continued, “If you made it to two, you would have six-thirds and, then, you could place four-thirds.” The class discussed this idea as Meredith worked on her own at the overhead projector. Two minutes later, the researcher asked to see what Meredith was writing on the overhead projector. The camera captured Meredith’s line where thirds were marked on the interval between zero and two. Under the number named one, Meredith had written in the corresponding fraction three-thirds. Under the number named two, Meredith had written in the corresponding fraction one and three thirds.

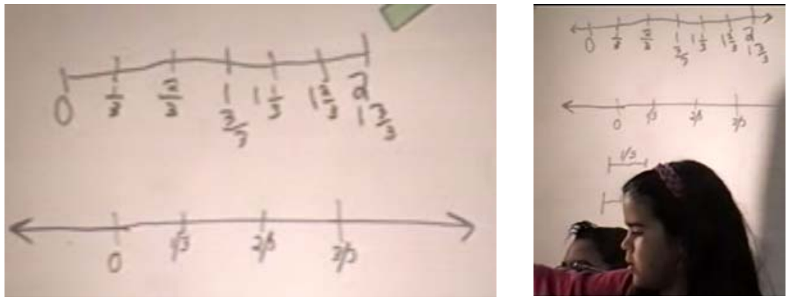

Figure 2. Meredith places thirds on the number line

Fifty minutes into the session, the class was working on placing numbers on a big number line on a dry erase board at the front of the room. The students were asked to select a number to place on the large number line. The first student, Audra, placed the number one-half under the number named zero. A lively discussion resulted about the placement of one-half, where arguments pro and con were offered; it apparently focused on whether the entire line should considered as the unit and the fraction as operator. Later, Meredith addressed this issue by disagreeing with the way the boys would have labeled fourths, operating on the entire line, and, one-fourth, two-fourths, three-fourths, four-fourths, one and one fourth, one and two fourths and one and three fourths correctly. The researcher asked Meredith about the earlier placement of one half by another student, to which Meredith replied that that is not how the line would be divided.

Two minutes later, the researcher stated that she was curious where Meredith would place negative one-fourth, negative two-fourths, negative three-fourths and negative four-fourths. Meredith walked up to the board and correctly placed the numbers between zero and negative one. A student in the class exclaimed, “She is like cutting the number line”.

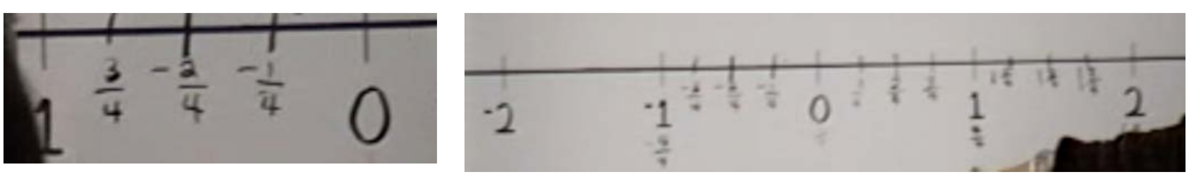

Figure 3. Meredith places fourths on the number line

Conclusions

During the first three sessions, Meredith exhibited evidence that she had some understanding for fraction equivalence as she made multiple models to support a claim that one-fifth equaled two-tenths and that one-sixth equaled two-twelfths. In the fourth session, Meredith presented five number lines on a unit interval where each line showed a different aspect to the complete number line. Her lines suggested that she understood and could work with fractions as operator and as number. Later in the fourth session, Meredith continued to demonstrate her understanding of equivalence between rational numbers by correctly labeling the point named one also as three-thirds, adding verbally that this number could be named four-fourths, five-fifths, or even 100/100.

After extending the number line beyond the unit point, Meredith labeled the number named two as one and three thirds and, verbally, labeled it six-thirds. Meredith correctly extended equivalence ideas to negative numbers as she placed fourths between the interval zero and negative one. For the number named negative one, Meredith correctly labeled the number negative four fourths. Meredith showed that she could correctly use fraction, mixed number and improper fractions to alias rational numbers.

Meredith, over the six week interval considered in the study, built on her understanding of equivalence ideas and extended the idea of having multiple names for number to negative rational numbers recognizing that there are multiple names for representing rational numbers.

Discussion

The study supports the findings from Cramer et al. (2002) and provides a trace of how one student, along with her classroom community, extended her understanding of equivalence of unit fractions to integers and rational number ideas. Meredith thoughtfully engaged in communication with other students and the researchers, offering support for her reasoning which provided her with a strong foundation to swiftly extend equivalence notions to more challenging fraction ideas. We can continue to learn form the story of Meredith and her engagement with her classmates.

Meredith and her classmate's stories are captured on videos that have made possible the tracing of Meredith's journey with her classmates from an understanding of equivalence of fraction ideas to rational number ideas. Studying these videos can provides insight into how these ideas about fractions can be built by students in a classroom setting. There is potential to provide a valuable set of tools for professional development, especially in contexts such as Lesson Study where teachers have opportunities to take these ideas directly to their classrooms. The session videos of Meredith and her classmates along with accompanying transcripts, student work, and other metadata (i.e., data about data) are available on Rutgers Video Mosaic Repository[2], Interested teachers and educators are invited to visit the website.

Endnotes

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2. The video data episodes can be found on the Rutgers Video Mosaic website located at http://www.video-mosaic.org/

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MATHEMATICAL STRUCTURES AND MENTAL STRUCTURES

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The title of our presentation comes from the conference in which Piaget first realized the isomorphisms between mathematical and psychological structures. Our presentation introduces a new psychological structure isomorphic to a mathematical group. "The splitting group" contributes to a better understanding of how students develop measurement concepts, especially for fractions. We designed written assessments and performed statistical analysis to test our model of a related structure and its formation among sixth-grade students. Results affirm the validity of the model and indicate the roles partitioning and iterating play in students' development of measurement concepts.

Introduction

As the name implies, PME conferences seek to bring psychologists and mathematicians together to share ideas and deepen our collective understanding of psychological processes at play in mathematics education. In 1952, a similar conference (whose title we borrow here) brought together psychologist Jean Piaget and mathematician Jean Dieudonné. The result of that meeting provided a logico-mathematical foundation for structuralism (Piaget, 1970b), as recalled by Piaget:

> *Dieudonne gave a talk in which he described the three mother structures [algebraic, topological, and order]. Then I gave a talk in which I described the structures that I had found in children's thinking, and to the great astonishment of us both we saw that there was a very direct relationship between these three mathematical structures and the three structures of children's operational thinking* (Piaget, 1970a, p. 26).

In our paper, we focus on algebraic structures, but note that various coordinations of the three mother structures produce additional structures—both mathematical and psychological.

Algebraic structures, such as groups, rely upon a set of elements that can be composed with one another under particular rules, or axioms. In mental structures, the elements are operations (internalized actions). For example, partitioning operations enable students to mentally break a continuous whole into a specified number of pieces. Piaget (1970b) has provided examples of several group-like structures in children, but he noted that some of these structures lack associativity. The mathematical term for such a structure (one that satisfies the properties of a group, except that it may lack the associative property) is *loop*.

Psychological loops are isomorphic to mathematical loops: There is an identity element, which does not transform the other elements; elements can be composed so that the results of composition are also elements of the structure; and each element in the system has an inverse element, with which composition yields the identity element. Unlike mathematical groups, loops are not necessarily associative, and, sometimes, tautology (A+A=A) precludes associativity because, "if we write A+A–A, where we put the parentheses makes a difference in the result" (Piaget, 1970a, p. 28). Piaget (1971) described the psychological/mathematical connection in the following way:

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

> *Operations are actions coordinated into reversible systems in such a way that each operation corresponds to a possible opposite operation that renders it void…. Identity is then to be considered a product—the product of the composition of direct or reversed operations—and not a point of departure. The group of such transformations is therefore the source of conservation principles; and identity (or more precisely the "identical operation") is merely one of the aspects of this group system, an aspect inseparable from transformations themselves* (p. 36).

The purpose of this paper is to present a new loop—the splitting loop—that might contribute to a better understanding for how students develop meaningful understanding of fractions, if our model of the loop can be validated. The loop consists of three operations (partitioning, iterating, and splitting), which we discuss in the next section. We then share methods and results for quantitatively testing our model of the splitting loop: Do statistical analyses support the validity of the splitting loop as a model for understanding students' development of fractions knowledge? Finally, we describe some implications for teaching and research. Among the research implications is a fresh consideration of the three mother structures and their coordination.

Theoretical Model for the Splitting Loop

Partitioning

Several researchers have affirmed the value of engaging students in activities involving sharing or partitioning sets of objects, in support of students' construction of initial fractions concepts (e.g., Kieren, 1980; Lamon, 2007; Mack, 2001). Steffe (2002) described the corresponding interiorized action—the partitioning operation—as the mental projection of a composite unit (e.g., 5 as five 1's) into a given, continuous whole, which enables the child to imagine how to share the whole equally among, say, five friends. However, research has indicated limitations in fraction conceptions that rely too heavily on partitioning. For example, Mack (2001) found, "students' informal knowledge of partitioning did not fully reflect the complexities underlying multiplication of fractions" (p. 291). Subsequent research has indicated that understanding of fractions as measures (Kieren, 1980) requires iterating operations as well (Olive & Vomvoridi, 2006; Steffe, 2002; Tzur, 2004).

Iterating

Iterating in a continuous context involves mentally repeating a given length or area to produce a connected whole that is *n* times as big as the given part. In two separate teaching experiments, Tzur (2004) and Olive and Vomvoridi (2006) demonstrated that teachers can support students' conceptualizations of fractions as measures by bringing forth students' iterating operations. Tzur (2004) found that fourth-grade students could learn to iterate unit fractional parts to produce other proper fractions. For example, students can produce 3/8 as three iterations of a 1/8 part.

Splitting

Once students have constructed partitioning and iterating operations, two new constructions become possible: the equi-partitioning scheme and the splitting operation. Steffe has described the equi-partitioning scheme as the sequential application of partitioning and iterating, for the purpose of creating equal parts within a continuous whole: The partitioning operation serves to create the equal parts, and the iteration operation serves to test whether one of those parts reproduces the whole when repeated the appropriate number of times (Steffe, 2004). In contrast, the splitting operation involves simultaneous application of partitioning and iterating: “In the splitting operation, the child’s awareness of a multiplicative relation between a whole and one of its hypothetical parts is produced by the composition of partitioning and iterating. In other words, they are realized simultaneously” (Steffe, 2002, p. 303). Figure 1 illustrates tasks designed to elicit children’s use of the splitting operation.

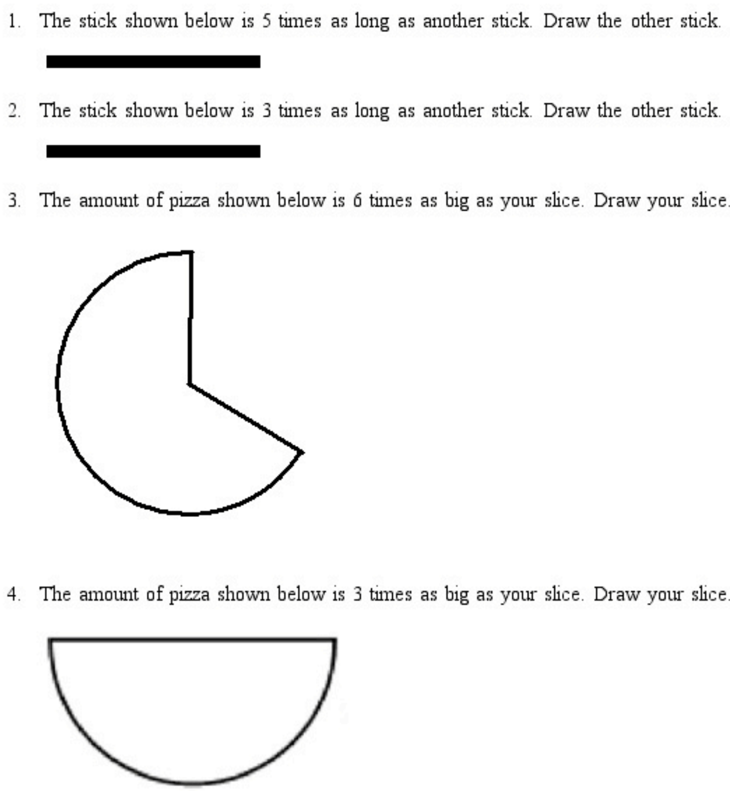

Figure 1. Tasks designed to elicit splitting

Each task in Figure 1 requires the student to use partitioning in service of an iterative goal—finding a part that, when iterated *n* times, produces the given stick or pie. Students who have not composed partitioning and iterating will often iterate the given stick or pie piece instead (Steffe & Olive, 2010).

Composition

In the splitting loop, partitioning and iterating are inverse operations. Splitting is both the identity element of the group and the group’s fundamental aspect. To understand this, consider that splitting composes partitioning and iterating so that each operation reverses the other before any action is carried out. This aspect enables children who split to comprehend and solve the tasks shown in Figure 1. Now, until students can reconcile the products of recursive partitions or recursive iterations—which Hackenberg (2007) attributes to the construction of three levels of units coordination—compositions of two partitions or two iterations are governed by tautology. In other words, partitioning each part of a partition remains indeterminate beyond being another partitioning, until students can coordinate the units involved. Thus, the loop is initially non-associative and contains only three elements: the operations of partitioning, iterating, and the splitting identity.

The splitting loop resembles one Piaget described for partitioning in discrete contexts: "The initial operation is, then, in effect, dividing or sectioning, while the assembling of parts or pieces in a total object is its inverse operation" (1974, p. 30). Thus, partitioning and uniting operations are described as inverses (Steffe, 2004). "This division and reconstruction of the unit (whole) is a central feature of fraction knowing" (Olive, 1999, p. 281). However, if one of the parts is disembedded from the others, a student cannot reproduce the whole from that part by simply reassembling parts; the single part must be repeated, or iterated. The equi-partitioning scheme (EPS) (Steffe, 2002) involves carrying out those actions sequentially, in order to produce a piece that, when iterated a specified number of times, reproduces the whole. Thus, the role of the equi-partitioning scheme as an intermediate between partitioning and iterating, on the one hand, and splitting, on the other, fits naturally. Figure 2 illustrates the model and includes path coefficients representing the relationships between the different operations and schemes. In the next two sections, we describe our methods for obtaining and interpreting these values.

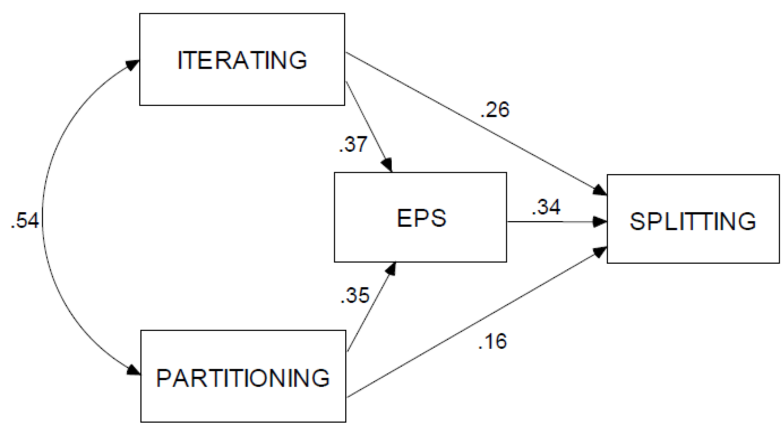

Figure 2. Model of the splitting loop

Methods

In noting the prevalence of theoretical models (like the one we just shared) in mathematics education research, Kilpatrick (2001) lamented the fact that most of these models remain untested by quantitative analysis. We began to address the issue in a previous paper (Norton & Wilkins, 2009), in which we shared methods and initial results from a quantitative study designed to test hypotheses about students' fractions schemes and operations. We have extended those methods to the splitting loop by designing a cluster of four written items to assess each of the schemes and operations in the model (see Figure 2). The items for the splitting operation are those shown in Figure 1. We share a single item from each of the other clusters in Figure 3. Additional items within each cluster varied in the same way as the splitting items (numerically and contextually).

We administered the 20 randomized items to all sixth-grade students in a rural Southeastern school (n=66) and independently assessed student responses to determine whether or not to attribute each scheme or operation to each student. We used Cohen's kappa to measure inter-rater reliability for each scheme or operation and achieved the following results: Perfect agreement between the two raters for the partitioning operation (K = 1.00); "almost perfect agreement" for the equi-partitioning scheme (K = .83, p < .05) and the splitting operation (K

= .93, $p < .05$); and "moderate agreement" for the iterating operation (K=.57, p<.05) (Landis and Koch, 1977). Overall the evidence suggests high reliability between the raters. We discussed any differences in ratings and reconciled these differences to create one rating for each student.

A. Suppose the stick shown below is a piece of candy. Show how someone could share it equally among 5 friends.

B. Make a stick that is 7 times as big as the one shown below.

C. What fraction is the smaller stick out of the longer stick?

Figure 3. Sample assessment tasks for partitioning (A), iterating (B), and equi-partitioning (C)

In order to quantitatively test the theoretical structure of the splitting loop and students' development of it, we created contingency tables for pairs of schemes and operations. We measured associations using gamma statistics and tested for directionality using exact binomial tests. We then conducted a series of path analyses to test the model illustrated in Figure 1. An indirect effect represents a relationship between two variables that is mediated by another variable. Such an indirect effect is represented in the model (Figure 2) by the paths leading from both ITERATING and PARTITIONING through EPS (the equi-partitioning scheme) to SPLITTING. A reduction in the direct effect of ITERATING and PARTITIONING on SPLITTING as a result of adding EPS to the model would provide evidence of the mediating effects of EPS. In addition, finding a statistically significant indirect effect of ITERATING and PARTITIONING on SPITTING (through EPS) would further indicate the mediating effects of EPS.

Results

Table 1 shows a strong directional association between the equi-partitioning scheme and the splitting operation, with equi-partitioning developmentally preceding splitting. Similar tables show strong directional associations (all with Exact Binomial *p*-values of less than .01) indicating that partitioning and iterating precede both equi-partitioning and splitting. In fact, assessments indicated that none of the 66 students constructed the splitting operation without first constructing both partitioning and iterating operations. Path analyses provide for further understanding of these relationships.

Table 1. EPS and Splitting

	Splitting		
EPS	0	1	Total
0	23	3	26
1	14	26	40
Totals	37	29	66

Note. Gamma = .87, $p < .0001$, one-tailed; Exact Binomial p (one-tailed) = .006.

In the first path analysis, we regressed SPLITTING on ITERATING and PARTITIONING to determine the direct effects of iterating and partitioning on the construction of splitting without including EPS in the model. ITERATING and PARITIONING were found to be positively and statistically correlated ($r = .54$). In addition, controlling for each other, both ITERATING ($b = .38$, $p < .05$) and PARTITIONING ($b = .28$, $p < .05$) were found to have a statistically significant positive effect on SPLITTING. These effects can be interpreted in terms of standard deviation increases. For example, on average, for one standard deviation increase in student iterating scores there was an average increase of 0.38 of a standard deviation in the average splitting score, controlling for partitioning scores.

When we included EPS as a potential mediating variable between ITERATING, PARTITIONING, and SPLITTING, both ITERATING ($b = .37$) and PARTITIONING ($b = .35$) were found to have a statistically significant positive relationship with EPS ($p < .05$); and EPS ($b = .34$, $p < .05$) was found to have a statistically significant positive relationship with SPLITTING. After including EPS in the model, the effect of ITERATING ($b = .26$, $p < .05$) and PARTITIONING ($b = .16$, $p > .10$) on SPLITTING were reduced in magnitude when compared to their effects without EPS in the model, and in the case of PARTITIONING, the effect was no longer statistically significant.

In the model, EPS is hypothesized to mediate the effects of ITERATING and PARITIONING on SPLITTING. These indirect effects are calculated as the product of the individual direct effects (e.g., 'path coefficient for the direct effect of PARTITIONING on EPS' x 'path coefficient for the direct effect of EPS on SPLITTING' = 'indirect effect of PARTITIONING on SPLITTING,' or .35 x .34 = .12). PARTITIONING ($b = .12$, $p < .05$) and ITERATING ($b = .12$, $p < .05$) were both found to have a statistically significant positive indirect effect on SPLITTING. These statistically significant indirect effects provide evidence of the mediating effects of EPS on the construction of splitting.

Conclusions

A loop is a set of elements closed under composition, such that one of the elements serves as an identity and each element has an inverse. The results from our study support the idea that splitting forms a loop of three elements under composition: splitting, partitioning, and iterating. In that loop, partitioning and iterating are inverses of each other and, as the simultaneous composition of those two operations, splitting serves as the identity. We demonstrate the claim by reviewing its theoretical basis in the literature, and by using the path analyses shown in Figure 2 to argue that splitting closes the loop.

In our study, and particularly in our task design, we adopted Steffe's (2002) definition of splitting. He identified this operation through his teaching experiments with children, during which some children seemed able to carry out operations of partitioning and iterating simultaneously (splitting), while others only carried out these operations sequentially (equi-partitioning). For example, one pair of fourth-grade students, named Jason and Laura, could carry out iterations of a unit fraction once they had produced the unit fraction through partitioning a whole, but, initially, neither student could posit a hypothetical part to satisfy the kinds of tasks illustrated in Figure 1. Over the course of the yearlong teaching experiment, both students began to solve such tasks, apparently by carrying out the two operations simultaneously.

> *There appears to be a discontinuity between the equi-partitioning scheme and splitting operations. In the former, partitioning and iterating are operations that are more or less*

> *sequentially performed, whereas in the splitting operation, the child's awareness of a multiplicative relation between a whole and one of its hypothetical parts is produced by the composition of partitioning and iterating* (Steffe, 2002, p. 303).

The results illustrated in Figure 2 support Steffe's view by demonstrating that equi-partitioning does, in fact, meditate the construction of splitting from partitioning and iterating. Furthermore, none of the students in our study constructed splitting operations without first constructing both partitioning and iterating. Whereas students with equi-partitioning schemes coordinate iterating and partitioning in order to test the accuracy of their partitions, splitting completes (closes) the system so that, for each operation, the other "renders it void" before it is even enacted (Piaget, 1971, p. 36). As such, splitting constitutes one of those psychological structures that Piaget found isomorphic to mathematical structures known as loops.

Implications

Initially, students may treat the composition of two partitions or two iterations as tautological: A partition of a partition is just some other partition; likewise for iteration. Thus, the composition of two partitions and one iteration (or vice versa) might depend upon the order in which the student carries out the pair-wise compositions. The splitting loop will lack associativity until the student can anticipate the product of two partitions or two iterations. Steffe (2004) has termed the first of these coordinations "recursive partitioning" and posits that such coordinations require the student to create a unit of units of units. For example, determining the product of ¼ and 1/3 requires the student to distribute one partitioning across each of the other partitions so that each third, say, is partitioned into four parts. Then the student must coordinate the three levels of units involved: One twelfth is one of four parts, which collectively form one-third of the whole (or, conversely, the whole as three equal parts, each of which is made up of four equal parts).

Hackenberg (2007) has identified three levels of units coordination as a necessary development—beyond splitting—in the construction of iterative fraction schemes, which enable students to treat improper fractions as "numbers in their own right" rather than mixed numbers (p. 27). Thus, we can think of the splitting loop as a mathematical group when students have conceptualized all fractions, including improper fractions, as multiplicative size relations with the whole. However, units coordination has its own structure—of order relations (Piaget, 1970b).

Just as Dieudonné (and the Bourbaki) did for mathematics, Piaget identified algebra, topology, and order as the three mother structures for psychology. Furthermore, he described a meta-structure, called the INRC group, which governs the synthesis of mental structures. In particular, "measurement is constructed by synthesizing partition and order relations" (Piaget, 1970b, p. 66). Our study contributes to a better understanding for how students might develop measurement concepts (especially for fractions). Students can develop partitioning within the splitting loop, where the operation is composed with its inverse—iterating. Synthesis with order relations becomes possible once students are able to coordinate the three levels of units involved in recursive partitions, or recursive iterations. At that point, improper fractions might become measures for students, and students' splitting loops become splitting groups.

When we consider the nature of mathematical thinking, it is not so surprising that mathematical loops and groups appear in psychological studies of students' mathematical development. As Piaget (1970b) noted, the essential property of logico-mathematical operations—both psychological and formal—is their reversibility. "Because every operation is reversible, an 'erroneous result' is simply not an element of the system" (p. 15). Loops and groups are very basic reversible systems. However, identifying mathematical structures, like the splitting loop, may help researchers focus on mental operations fundamental to students' mathematical development. In one direction, we begin to understand simple operations, such as partitioning and iterating, as part of a larger closed system of compositions. In the other direction, we begin to unpack more complex operations, such as splitting, into their constituent parts. The loop model may also inform us as to the completeness of the system and help us identify which additional operations, such as units coordination, students need to develop in order to grow their structures into larger systems for operating.

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PROPORTIONAL REASONING: HOW TASK VARIABLES INFLUENCE THE DEVELOPMENT OF STUDENTS' STRATEGIES

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This study explores the development of students' strategies going from primary to secondary school when solving proportional and additive problems. The goal is to identify characteristics of the development of proportional reasoning and how the use of integer and non-integer ratios and the discrete or continuous nature of quantities influence this development. The findings indicate that primary school students used systematically the additive strategy in all problems and that secondary school students present a wider variety of strategies that are also used systematically in the solution of all the problems. The type of ratio and the nature of the quantities influenced differently the development of these strategies.

Introduction

In recent years, proportional reasoning has been considered synonymous with the ability to solve proportional missing-value problems (Cramer, Post, & Currier, 1993); that is tasks that include three quantities of a proportion and the fourth quantity is unknown and has to be computed. Recently researches (De Bock, Verschaffel, & Janssens, 1998; Fernández, Llinares, & Valls, 1998; Modestou & Gagatsis, 2007; Van Dooren, De Bock, Janssens, & Verschaffel, 2008) have suggested that proportional reasoning does not imply only the ability to solve missing-value proportional problems but also the ability to discriminate proportional from non-proportional situations. For example, Modestou and Gagatsis (2010) suggested the existence of a new model of proportional reasoning where proportional reasoning does not coincide exclusively with success in solving a range of proportional problems, as routine missing-value and comparison problems, but it also involves handling verbal and arithmetical analogies, as well as the awareness of discerning non-proportional situations from other situations.

The present study focuses on the development of proportional reasoning when moving from primary to secondary school, analyzing the strategies used by students to handle proportional and non-proportional situations. In particular, the main aim of this study is to explore the influence of the different types of ratios (integer or non-integer) and the nature of quantities (discrete or continuous) on this development.

Theoretical Framework

A proportion is "a second order relationship that implies an equivalent relationship between two ratios – and can be expressed as $a / b = c / d$ –" (Christou & Philippou, 2002). According to Freudenthal (1983), the ratio is a function of an ordered pair of numbers or quantities of magnitude. There are two kinds of relationships among quantities: "within" relationships (internal ratio), which are relationships between quantities of the same nature, and "between" relationships (external ratio), which relate quantities of different nature. Take, for example, the problem "Six kilos of potatoes cost €2. If you want to buy nine kilos of potatoes, how much will you pay?" If we relate the weight with the price, we obtain a "between" relationship (2 euro / 6 kilos), whereas if we relate the first weight with the second weight we have a "within" relationship (9 kilos / 6 kilos). Vergnaud (1983) considered two measure spaces (in our example, euro and kilos) and the transformations can be carried out within or between the variables-measures. These transformations reflect the different methods-strategies that students can use to solve the problems (Karplus, Pulos, & Stage, 1983; Modestou & Gagatsis, 2009).

The literature on proportional reasoning reveals a broad consensus that proportional reasoning develops from qualitative thinking, to build-up strategies, to multiplicative reasoning (Inhelder & Piaget, 1958; Kaput & West, 1994; Karplus et al., 1983; Noelting, 1980). These strategies represent different levels of sophistication in thinking about proportions. The qualitative thinking is characterized by the use of comparison words, such as bigger and smaller, more or less, to relate to the quantities of the questions. Build-up reasoning is an attempt to apply knowledge of addition or subtraction to proportion. In this strategy students note a pattern within a ratio and then iterate it to build up additively to the unknown quantity. For example, in the above problem, "3 kilos cost 1 euro, then 3+3+3 kilos cost 1+1+1 euro". Strategies based on multiplicative approaches are based on the properties of the linear function: $f(a+b) = f(a)+f(b)$ and $f(ma) = mf(a)$ taking into account the transformations between or within ratios. Inhelder and Piaget (1958) regarded students' mature use of proportional reasoning as indication of the formal operational thinking in which students observed the consistency of a covariational relationship between and within variables (measure spaces).

A characteristic in the development of proportional reasoning is the difficulty of students in distinguish proportional from non-proportional situations revealed by the erroneous use of additive strategies in proportional situations and by using erroneous proportional strategies in non-proportional situations (Fernández, Llinares, Van Dooren, De Bock, Verschaffel, 2009; Modestou & Gagatsis, 2007; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005). The erroneous additive strategy used in proportional problems consists of the use of the difference between the numbers in the ratio and then the application of this difference to the third number to find the unknown quantity (Hart, 1984; Tourniaire & Pulos, 1985). For example, in the proportional problem "Peter and Tom are loading boxes in a truck. They started together but Tom loads faster. When Peter has loaded 40 boxes, Tom has loaded 160 boxes. If Peter has loaded 80 boxes, how many boxes has Tom loaded?" the erroneous additive strategy would be "As the difference between the boxes loaded by Peter and Tom is 160-40 = 120 boxes, then Tom has loaded 120+80 = 200 boxes. On the other hand, the use of erroneous proportional strategies in additive problems can be exemplified by the following non-proportional problem that is modeled by $f(x) = x + b$, $b \neq 0$ "Peter and Tom are loading boxes in a truck. They load equally fast but Peter started later. When Peter has loaded 40 boxes, Tom has loaded 160 boxes. If Peter has loaded 80 boxes, how many boxes has Tom loaded?" An erroneous proportional method used to solve this problem is, "as Tom has loaded 4 times more the boxes loaded by Peter (160:40=4), then Tom has loaded 80×4 = 320 boxes."

Method

The sample of the study consisted of 755 primary and secondary school students; 65 fourth graders, 68 fifth graders, 64 sixth graders, 124 seventh graders, 151 eighth graders, 154 ninth graders and 129 tenth graders. The participating schools were situated in different cities and pupils came from different socio-economic backgrounds.

Students were given a test consisting of 12 missing-value word problems, four of which were proportional (P), four additive (A) and four buffer problems. Additive situations referred to situations reflecting $f(x) = ax + b$, $b \neq 0$. Buffer problems were included so as to create a variation of the given tasks and avoid the effects of learning as well as stereotyped replies.

Discrete situations (D; e.g. loading boxes) and continuous situations (C; e.g. skating a certain distance) were used. For each situation, proportional and additive problems were created by manipulating only one sentence. For example, where in the proportional situation (P) the sentence “They started together but Tom loads faster” was used, in the respective additive situation (A) the sentence “They load equally fast but Peter started later” was used. In a second step, two different versions were considered, one with integer relationships between quantities (I, P1) and another with non-integer relationships between the quantities (N, P2). In this way a total of eight problems were obtained. Table 1 shows the problems used and their characteristics.

Table 1. Examples of word problems used in the test

P1-Peter and Tom are loading boxes in a truck. They started together but Tom loads faster. When Peter has loaded 40 boxes, Tom has loaded 160 boxes. If Peter has loaded 80 boxes, how many boxes has Tom loaded? (P-D-I)	P2-Peter and Tom are loading boxes in a truck. They started together but Tom loads faster. When Peter has loaded 40 boxes, Tom has loaded 100 boxes. If Peter has loaded 60 boxes, how many boxes has Tom loaded? (P-D-N)
P3-Ann and Rachel are skating. They started together but Rachel skates faster. When Ann has skated 150 m, Rachel has skated 300 m. If Ann has skated 600 m, how many meters has Rachel skated? (P-C-I)	P4-Ann and Rachel are skating. They started together but Rachel skates faster. When Ann has skated 80 m, Rachel has skated 120 m. If Ann has skated 200 m, how many meters has Rachel skated? (P-C-N)
P5-Peter and Tom are loading boxes in a truck. They load equally fast but Peter started later. When Peter has loaded 40 boxes, Tom has loaded 160 boxes. If Peter has loaded 80 boxes, how many boxes has Tom loaded? (A-D-I)	P6-Peter and Tom are loading boxes in a truck. They load equally fast but Peter started later. When Peter has loaded 40 boxes, Tom has loaded 100 boxes. If Peter has loaded 60 boxes, how many boxes has Tom loaded? (A-D-N)
P7-Ann and Rachel are skating. They skate equally fast but Rachel started earlier. When Ann has skated 150 m, Rachel has skated 300 m. If Ann has skated 600 m, how many meters has Rachel skated? (A-C-I)	P8-Ann and Rachel are skating. They skate equally fast but Rachel started earlier. When Ann has skated 80 m, Rachel has skated 120 m. If Ann has skated 200 m, how many meters has Rachel skated? (A-C-N)

Students had approximately 50 minutes to complete the test. They were allowed to use calculators but were asked to write down all the operations they had conducted, so as to be able to follow the problem solving path they used.

Students' strategies were analyzed and categorized into five groups, by taking into account the way students handled the relationships between the given numbers in each situation. Table 2 shows these categories and some examples:

Table 2. Categories of students' strategies

Strategy	Example
Additive Strategy (SAdd)	"As the difference between the boxes loaded by Peter and Tom is 160-40=120, then Tom has loaded 120+80=200 boxes" (P5)
Building-up method (SBU)	"As 40+40+40+40=160 boxes, then Tom has loaded 80+80+80+80+80= 320 boxes" (P1)
Use of ratios (internal or external, SR)	"As 160/40=4 (external ratio) then Tom has loaded 80×4=320 boxes" (P1)
Rule of three (SRT)	"80×160=12800; 12800:40=320 boxes" (P1)
Other incorrect strategies (SOth)	Strategies without sense

The use of a strategy in a problem was codified as 1 and the absence as 0. In this way, in each problem the presence of the five categories of strategies was encoded, and therefore 40 variables arose (eight problems × five categories of strategies). An Implicative Statistical Analysis with the use of the computer software CHIC was conducted (Gras, Suzuki, Guillet, & Spagnolo, 2008*)*, first with primary school students' data and then with secondary school students' data. The similarity diagrams that arose from this analysis allowed the arrangement of the strategies into groups according to the homogeneity, indicating the way they were applied by students.

Results

The similarity diagrams of the strategies used by primary school students and secondary school students are presented first. A comparison of these similarity diagrams follows, in order to identify possible changes in the strategies used.

Primary school students

Figure 1 shows the similarity diagram of the strategies used by primary school students in the different problems of the test. The similarity diagram was generated only with 23 variables clustered in three similarity clusters (i.e., groups of variables). Cluster A consisted of variables referring to the use of additive strategies in all additive and proportional problems. It indicates that primary school students used this strategy systematically, without being able to discriminate proportional or additive problems. Furthermore, the way in which the use of the additive strategy in the different problems has been grouped suggests that the nature of quantities has an influence on its use. Two sub-groups appear: one with the problems including discrete quantities (SAdd1, SAdd2, SAdd5, SAdd6) and other with the tasks with continuous quantities (SAdd3, SAdd4, SAdd8).

Cluster B referred to the use of other strategies in discrete situations (SOth1, SOth2, SOth5 and SOth6), and in the additive problem with continuous quantities and integer ratios (SOth7). This way of grouping suggests that the discrete nature of quantities had also an influence on the use of other strategies

Finally, Cluster C grouped three strategies, the category “Other strategies” in the problems with continuous quantities (SOth3, SOth4 and SOth8); the use of ratios (internal or external) in problems with integer multiplicative relationships between quantities (SR3, SR5 and SR7) and the use of building-up methods in problems with non-integer ratios (SBU2, SBU4 and SBU6). This grouping indicates, firstly that the integer multiplicative relationship between the given numbers implied the use of the ratios regardless of the additive or proportional character of the problems, and the presence of non-integer multiplicative relationships implied the primary school students’ use of building-up strategies. This fact can be easily explained as for primary school students the relation 160:40= 4 is easier than the relation100:40. In this sense, when the multiplicative relationship was easy, primary school students were more prone to use it (ratios), but when the multiplicative relation is non-integer, they derive to use build-up methods.

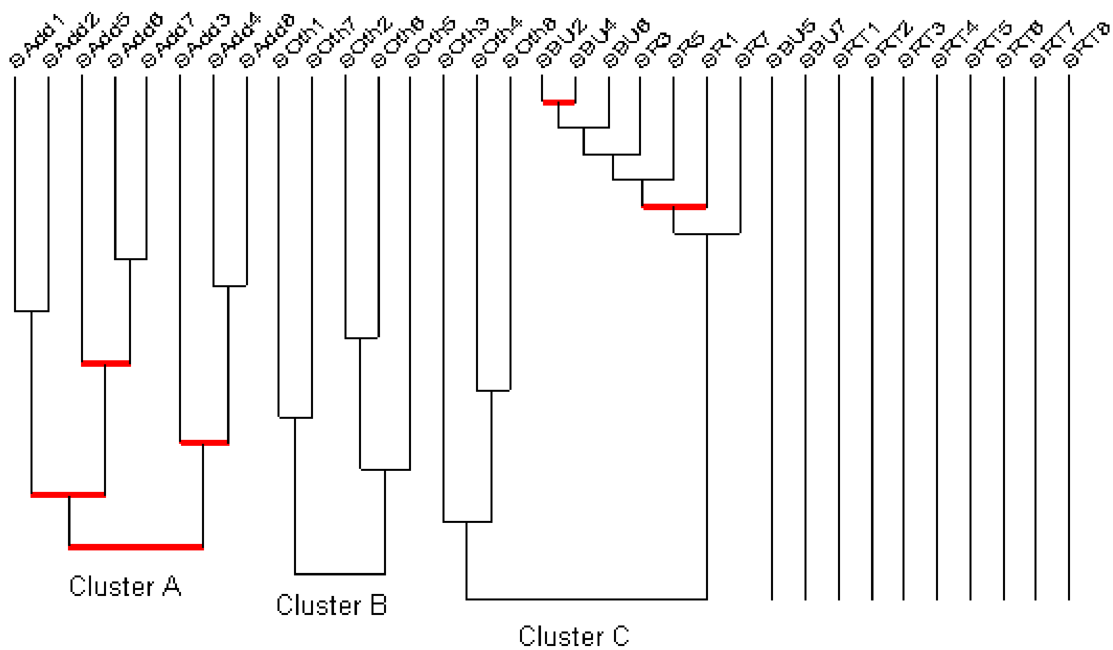

Figure 1. Similarity diagram of the strategies used by primary school students

Finally, the way in which these clusters have been formed indicates that the similarity groups are formed based on the strategies used and not on the proportional or additive character of the tasks. In addition, the formation of the sub-groups (defined by the use of strategies) is mainly based on the nature of quantities (discrete or continuous) or on the number structure (integer or non-integer ratios) and not on the proportional or additive character of problems.

Secondary school students

Figure 2 shows the similarity diagram of the strategies used by secondary school students in the different problems of the test. In this case, six similarity clusters grouped all the 40 variables. Cluster A grouped the use of additive strategies in all problems, where Cluster B grouped the use of other strategies. The use of building-up strategies in problems with integer multiplicative relationships were grouped in Cluster C, and the use of building-up strategies in problems with non-integer multiplicative relationships were grouped in Cluster D. Finally, the use of ratios (Cluster E) and the use of the rule of three (Cluster F) were grouped in two other distinct clusters.

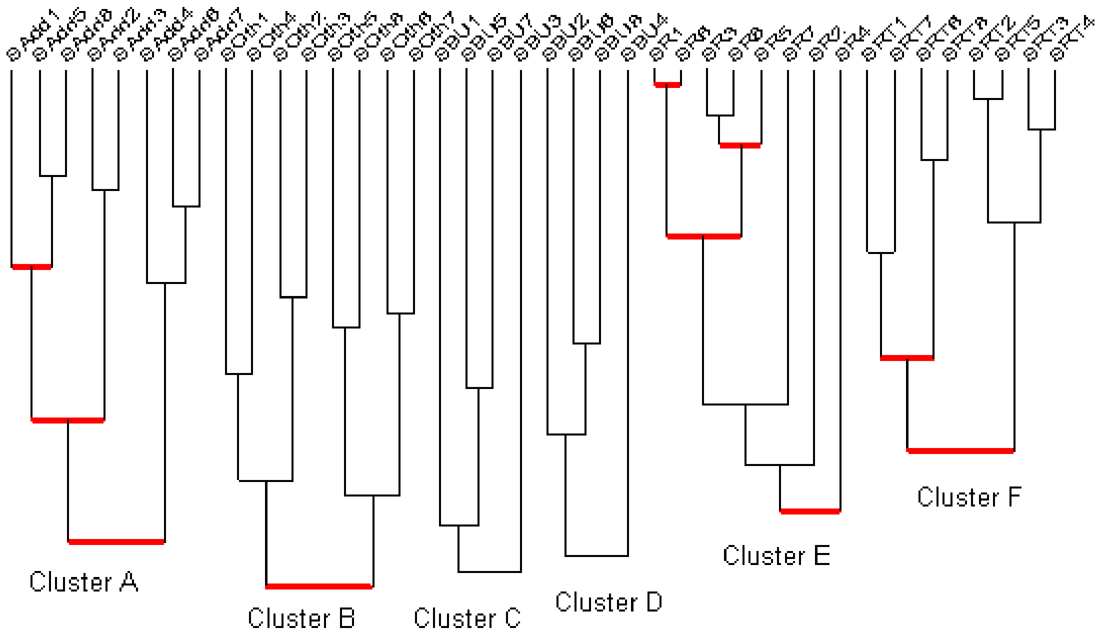

Figure 2. Similarity diagram of the strategies used by secondary school students

The similarity groups are formed in the same way as in the case of the primary school, based on the strategies used and not on the additive or proportional character of the tasks. This data suggest that secondary school students were systematic in the use of strategies in each type of problem. That is to say, secondary school students that used additive strategies (Cluster A), they used them in all problems. The same pattern appeared for all the other different strategies. This data indicate essentially that secondary school students do not differentiate between additive and proportional situations, and handle them in the same way.

Only in Cluster B, the type of problem (additive/proportional) appears to have a minor affect on the use of other strategies. Cluster B is formed by two groups. Firstly, the group formed by the use of other strategies in proportional problems (SOth1, SOth2, SOth3, SOth4), and secondly the group formed by the use of other strategies in the additive problems (SOth5, SOth6, SOth7, SOth8). This grouping suggests that the type of the problem (additive or proportional) influence on the approach adopted by the students when they did not use additive or proportional strategies. Regarding Cluster C and D jointly, it shows that the use of building-up strategies were grouped considering the integer and non-integer ratios in the problem. Cluster C grouped the use of building-up strategies in problems with integer ratios (SBU1, SBU3, SBU5 and SBU7) and Cluster D grouped the use of building-up strategies in problems with non-integer ratios (SBU2, SBU4, SBU6 and SBU8). These data suggest that the type of ratio affects the use of the building-up method.

Cluster E grouped the use of the ratio approach in proportional and additive problems. However, the way of grouping does not provide a clear indication about the influence of the nature of quantities, the type of ratio or the additive or proportional character of the problems. Finally, Cluster F is formed by two groups. One grouped the use of the algorithm “rule of three” in the proportional problem with discrete quantities and integer ratios (SRT1), in the additive problems with continuous quantities (SRT7 and SRT8) and in the additive problem with discrete quantities and non-integer relationships between quantities (SRT6). The other group is formed by the use of the same algorithm in the proportional problems with continuous quantities (SRT3 and SRT4), in the proportional problem with discrete quantities and non-integer ratios (SRT2) and in the additive problem with discrete quantities and integer relationships between quantities (SRT5). However, the way in which the groups are formed in Cluster F indicate that the type of ratio (integer or non-integer) determined the first sub-group (P-D-I and A-C-I form a group, and A-D-N and A-C-N the other group), but the nature of quantity (discrete or continuous) determined the second sub-group (P-D-N and A-D-I form one group, and P-C-I and P-C-N the other group).

Comparison of the primary and secondary school students' data

When the two similarity diagrams (Figure 1 and 2) are compared, differences appear. Firstly, primary school students' similarity diagram grouped only 23 variables, while the secondary school students' similarity diagram grouped all the 40 variables. Clusters were always formed in a way that they grouped the use of one strategy in all problems. This indicates that primary and secondary school students were systematic in the use of strategies independently of the additive or proportional character of the problem. On the other hand, the variables' nature of quantities and the presence or absence of integer or non-integer ratios affected more primary school students' use of a particular strategy than secondary school students' decisions. In fact, the nature of quantities does not seem to affect secondary school students, whereas the number structure only affects the use of proportional strategies.

Discussion

The main goal of the present study was to explore the development of students' strategies when solving proportional and additive problems along primary and secondary education and identify how number structure and nature of quantities affect this development. This was put forward through the identification of similarities or disparities along different age groups (primary and secondary), when solving additive and proportional problems with different characteristics (integer and non-integer multiplicative relationships and discrete or continuous quantities).

Primary school students used additive strategies and building-up methods to solve proportional and additive problems and secondary school students also applied other strategies as the use of ratios and the rule of three. The additive and proportional character of tasks does not seem to have an influence on primary and secondary school students' strategies because they used systematically the same strategy to solve all type of problems. This study supports the idea that proportional reasoning not only implies the ability to solve missing-value problems but also the ability to discriminate proportional from non-proportional problems (De Bock, Verschaffel, and Janssens (1998); Fernández, Llinares, and Valls (1998); Modestou and Gagatsis (2010); Van Dooren, De Bock, Janssens, and Verschaffel (2008), in the sense that there are students who solve correctly proportional problems but solve incorrectly additive problems.

A characteristic of students' development of strategies is that primary school students use the ratios (internal or external) in proportional problems with integer ratios but also in additive problems with integer multiplicative relationships between quantities. However, these students use the building-up method to solve the non-integer versions (proportional and additive). On the other hand, secondary school students do not change the strategy when solving different problems. On the contrary they systematic use the same strategy. Therefore, the effect of number structure on the use of different strategies by secondary school students was lower and there was not a strong effect of the nature of quantities.

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EXTENDING THE DISCUSSION OF INTENSIVE QUANTITIES: THE CASE OF FRACTION MULTIPLICATION

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In this presentation, my aim is to discuss how fraction multiplying schemes produce intensive quantities and extend the categories that Schwartz (1988) suggested in relation to the forms of intensive quantities to include the special case of fraction multiplication. Even though the results of fraction multiplying schemes are the same type of units presented in the proposed situations (e.g., 1/7 of 3/5 of a candy bar is 1/35 of the candy bar)—because of which the fraction multiplication might be thought as a referent conservation operation so extensive quantities are produced—the mathematical operations carried out (by the participants of the study) imply that the production of results of fraction multiplication is no less advanced than the production of intensive quantities that is discussed in the literature. There is a co-varying relationship between two quantities of the same type (such as 1/7 of each 1/5 of a liter (5/5 of the liter) results in a thirty-fifths of a liter when seven mini-parts per part is distributed) unlike the co-varying relationship between two different types of quantities as discussed in the literature by Schwartz. In addition, we need to pay attention to children's construction of fraction multiplication schemes since it is not a straightforward process, as one would expect in the construction of extensive quantities. Extensive quantities are usually produced by the arithmetical operations [addition and subtraction] on same type of quantities. Therefore, the view of, and emphasis on fractions and fraction multiplication/multiplying schemes should naturally be related to intensive quantities in early grades. As Schwartz (1988) suggested, we should not assume intensive quantities, such as "the concept of derivative," can play a role only in the upper school mathematics curriculum or in algebra. I will provide data and analysis of the necessary mathematical operations with which a pair of 8th graders constructed a fraction multiplying scheme in order to discuss how intensive quantities are rooted in the construction of fractions and fraction multiplication

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HOW STANDARDS-BASED CURRICULA AND CONVENTIONAL CURRICULA INFLUENCE STUDENTS' LEARNING OPPORTUNITIES

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This study examined four sets of curriculum materials on the topic of multiplication and division of fractions: two conventional textbooks--*Scott Foresman-Addison Wesley* (Charles, Crown, & Fennell, 2006) and *Prentice Hall Mathematics* (Buice, 2006) and two standards-based textbooks--*Connected Mathematic* (Lap pan, Fey, Fitzgerald, Friel, & Phillips, 2004) and *Middle Grades MathThematics* (Bill stein & Williamson, 2005). Although each curriculum developer claims that their mathematics textbook attempts to develop conceptual understanding, there is not much research that examine whether standards-based curricula provide learning opportunities that are substantially different from traditional textbooks. We analyzed problems used in each of the textbooks according to four criteria: (1) Number of steps required, (2) contextual feature, (3) response type, and (4) cognitive expectation.

Analysis revealed that regardless of textbook type, a large portion of problems in all four textbooks required single step procedure for launching solution. With respect to the contextual feature criterion, more than 60% of problems in *Connected Mathematics* were presented in illustrative contexts whereas more than 70% of problems in other textbooks were presented in pure mathematics context. Relative to the response type criterion, a larger portion of problems in all four texts required numerical answer only. However, *Connected Mathematics* had higher percentage of problems requiring explanation than any other textbooks, which implies that *Connected Mathematics* provided more opportunities for students to explain and justify their solutions. With respect to the cognitive expectation of tasks, similar to other findings, *Connected Mathematics* had higher percentage of problems that required problem solving, representation, and mathematical reasoning whereas a large portion of problems in other textbooks required use of procedural knowledge. Interesting to note here is that although *MathThematics* is considered as standards-based curricula, we did not notice any differences between *MathThematics* and other traditional textbooks. This study provides implications for teachers, curriculum developers and researchers.

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MIDDLE SCHOOL STUDENTS' UNDERSTANDNG ABOUT PRIME NUMBER

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The goals of this study are to inquire middle school students' understanding about prime number and to propose pedagogical implications for school mathematics. As a fundamental concept in number theory, the concept of prime number is introduced in the early stage of secondary mathematics curriculum in Korea. However, despite of simplicity of its definition, literatures highlight the difficulty in grasping the idea about prime number (Davis, 2008; Zazkis & Liljedahl, 2004). This research investigated characteristics of students' understanding about prime number.

This research focused on the following questions: how do students define about prime and composite numbers and how do students understand their relation? Written questionnaire were given to 198 Korean seventh graders to explore their understanding of concepts and procedures relating to prime numbers. They were asked to provide explanations for all their answers. After the written questionnaire, 20 students among them participated in individual in-depth interviews.

The analysis showed that 97 out of 198 students defined prime number following their textbook definition saying that a natural number greater than 1 is prime if its only natural number divisors are itself and 1. 36 students defined prime number as a number having exactly distinct two natural number divisors. On the other hand, 37 students defined composite number as natural number greater than 1 that is not prime number. 76 students defined composite number as a number having more than 3 natural number divisors. In defining prime and composite numbers, the students focused on distinguishing one from another. However, they hardly recognize the mathematical connection between prime and composite number based on the multiplicative structure of natural number. In determining primality of 31×47, most students performed actual multiplication and long divisions to conclude that it is a prime number. Furthermore, many students did not notice that prime numbers hardly appear among big natural numbers. This finding indicates that the mathematical meaning behind a procedure was disregarded. Thus, the results of our study suggest that it is necessary to emphasize the conceptual relationship between divisibility and prime decomposition and the prime numbers as the multiplicative building blocks of natural numbers.

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UNDERGRADUATES (MIS)UNDERSTANDING OF PERCENTAGES

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People of all ages have difficulty with rational number representations in decimal and fraction formats (Humberstone & Reeve, 2007). This is despite the fact that every day, people are inundated with rational numbers all around them. Using multiple representations for a math problem aids both the learning and teaching of mathematical concepts (Gelman, 1986). Rittle-Johnson & Star (2007) improved seventh-grade students' conceptual and procedural abilities by having them compare alternative algorithms. This study looks at undergraduates' performance on problems dealing with representations of rational numbers that they see in everyday life– the proportional increases or decreases in the cost of goods. We explore the following question:

Does representational format influence accuracy on rational number problems?

- Is training with a specific kind of rational number format more helpful than training with other rational number formats?
- Is there transfer from training on one rational number format to problem solving in another?

Twenty seven undergraduates (25 Females) at Rutgers University participated in the study. Each subject received 7 questions in the pretest, a training task, then 7 questions in the posttest. A logistic regression was used to evaluate the participants' overall performance (proportion correct) and the effect of the training models. Responses were binary; they were coded as *Incorrect* or *Correct*. There was a main effect of Question Representation (p = .003). Subjects performed better on bar graph questions than on other question types across both the pretest and the posttest. There was a main effect of Training (p = .049); subjects differed in overall performance depending on whether they received decimal versus fraction training. When presented with a single percent increase (or decrease) in a problem, the direction did not affect performance (p = .731). When there exists more than one process within a problem, participants encountered a significantly greater deal of trouble (i.e., less than 25% of these problems were answered correctly). When given 1 type of question each in the pretest and posttest dealing with multiple processes, the level of previous math knowledge did not seem to matter. It was the case that 2/3 of subjects in both math groups failed to answer *any* of these problems correctly.

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Chapter 10: Problem Solving

Research Reports

Brief Research Reports

Posters

INFERRING IMPULSIVE-ANALYTIC DISPOSITION FROM WRITTEN RESPONSES

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Impulsive disposition refers to one's proclivity to spontaneously proceed with an action that comes to mind without checking its relevance. Analytic disposition refers to one's proclivity to analyze a problem situation and establishes a goal to guide one's actions. An instrument, called the likelihood-to-act survey, was developed to measure students' impulsive-analytic disposition. In this study, we sought to test and refine this instrument by analyzing 92 participants' written responses to open-ended questions that were adapted from items in the likelihood-to-act survey. We found relatively strong correlations between participants' disposition scores for written responses and those from the likelihood-to-act survey.

Introduction

In solving mathematics problems, "doing whatever first comes to mind … or diving into the first approach that comes to mind" (Watson & Mason, 2007, p. 307) is commonly observed among students. To demonstrate, consider the following problem: *Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?* Cramer, Post and Currier (1993) observed that 32 out of 33 pre-service teachers solved the problem by setting up a proportion such as $9/3 = x/15$. Instead of analyzing and reasoning through the problem situation, these students had "blindly" applied the proportional strategy familiar to them. Lim, Morera, and Tchoshanov (2009) use the term *impulsive disposition* to refer to students' proclivity to spontaneously proceed with an action that comes to mind without checking its relevance.

A contrast to impulsive disposition is *analytic disposition*. When one approaches a problem with analytic disposition, one "analyzes the problem situation and establishes a goal … to guide one's actions" (Lim, 2008, p. 45). We have developed a survey instrument that seeks to measure one's problem-solving disposition in the impulsive-analytic dimension. This instrument, called the *Likelihood-to-Act* (LtA) survey, consists of six-point Likert items where participants are asked to indicate how likely they are to respond to a given mathematical problem in the described manner (Lim, Morera, & Tchoshanov, 2009). In order to increase the reliability of the instrument, we have increased the number of LtA items from 18 to 32. A study was recently conducted to investigate the validity and reliability of the 32-item version of the LtA survey. Three other instruments—an open-ended questionnaire, a classification test, and a multiple-choice mathematics test—were developed and administered in conjunction with the LtA survey. This paper focuses on participants' written responses to questions in the open-ended questionnaire and a question that asked participants what they thought the LtA survey was trying to measure.

Theoretical Background

Students are impulsive because they tend to approach a problem with conceptual tools that are familiar to them. In the famous water jar experiment (Luchins, cited in NRC, 2000), subjects, after solving many problems using one approach, spontaneously used the same approach to solve other problems that could have been easily solved using a different approach. This phenomenon

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of solving a given problem in a fixated manner even when a better approach exists is called the *Einstellung* effect. In the context of mathematical problem solving, the Einstellung effect is described by Ben-Zeev and Star (2001) as a *spurious correlation*, which involves a two-phase process: A person first conceives an association between a problem feature and an algorithm for solving the problem, and then uses the algorithm upon perceiving the feature in another problem. In their study, Ben-zeev and Star found that students not only relied on surface-level features in solving problems but "also generate[d] and use[d] correlations between irrelevant surface-level features and solution strategies" (p. 272). Even experienced students were found to be susceptible to this tendency. We call the tendency of making spurious correlations impulsive disposition.

Einstellung effect, spurious correlation, and impulsive disposition refer to the same phenomenon but emphasize different aspects. Whereas the Einstellung effect refers to a mental fixation and spurious correlation refers to a process, impulsive disposition refers to a proclivity. We chose to focus on impulsive disposition because it can be conceived as a negative *habit of mind* (Cuoco, Goldenberg, & Mark, 1996) or an undesirable *way of thinking* (Harel & Sowder, 2005) that can be changed to analytic disposition. Our research is motivated by the intent to create an awareness of impulsive disposition among mathematics teachers so that they can be mindful in teaching their students and avoid propagating a culture where "doing mathematics means following rules laid down by the teacher, knowing mathematics means remembering and applying the correct rule when the teacher asks a question, and mathematical truth is determined when the answer is ratified by the teacher" (Lampert, 1990, p. 31).

Spurious correlations tend to lead to errors, which Radatz (1979) would classify as "errors due to incorrect associations or rigidity of thinking" (p. 167). Impulsive disposition is generally inferred from such errors, examples of which include overgeneralizing proportionality in solving non-proportional missing-value problems (Lim, 2009) and misapplying a procedure for solving linear equations to solve linear inequalities (Tsamir & Almog, 2001). Examples of the latter include $x^2 < 16$ implies $x < \pm 4$ and $\frac{2x-2}{x+1} < 1$ implies $2x - 2 < x + 1$. We also infer impulsive disposition when students use an inefficient strategy to solve a problem that could be solved using a simpler method (i.e., when the Einstellung effect occurs).

We regard an instrument that can assess one's impulsive-analytic disposition to be a viable way to motivate teachers and students to progress from impulsive disposition to analytic disposition. The LtA survey was developed with this purpose in mind. To investigate the validity of the LtA items and to improve them, we developed an open-ended questionnaire to find out the initial steps participants would take to solve a mathematics problem. Our objectives were two-fold: (a) to investigate the validity of the 32-item version of the LtA survey, and (b) to improve the LtA items.

In this paper we address the following research questions: (a) What solution strategies were mentioned in participants' responses to the items in the open-ended questionnaire? (b) How well did the impulsive-analytic disposition scores for the open-ended questionnaire correlate with their impulsive-analytic scores in the LtA survey? And (c) What did the respondents think the LtA survey was trying to measure?

Method

Data Collection

Two groups of participants completed the open-ended questionnaire. The first group consisted of 27 in-service teachers and 10 pre-service teachers who were enrolled in a program for improving mathematics and science education in El Paso. The second group consisted of two mathematics classes for pre-service EC-8 (Early Childhood to Grade 8) teachers: 33 and 22 students. A convenience sample of pre-service and in-service teachers was used because the purpose of this round of data collection was to investigate the validity of the instruments rather than to test hypotheses. Since the LtA survey was the prime instrument that we sought to investigate, we administered it prior to the open-ended questionnaire.

Likelihood-to-Act Survey. The 32-item survey that was administered is comprised of the following categories: algebra, word problem, fraction, and non-mathematically-specific description. Each of the four categories includes four impulsive and four analytic items.

A1i $(x-5)(x-8)=0$. When asked to solve for x, how likely are you to multiply out the terms (i.e., FOIL) and then solve $x^2-13x+40=0$ using the quadratic formula? [impulsive]

A1a $(x-7)(x-4)=0$. When asked to solve for x, how likely are you to study the equation and predict the solution? [analytic]

A2i 30 workers took 8 hours to complete Project P whereas 20 workers took 3 hours to complete Project Q. When asked to determine which project was bigger in size, how likely are you to compare rates (e.g., comparing 30/8 to 20/3)? [impulsive]

A2a 26 workers took 10 hours to complete Project ABC whereas 18 workers took 7 hours to complete Project XYZ. To determine which project was bigger in size, how likely are you to visualize the two scenarios and predict the answer without doing any computation? [analytic]

B6i $\frac{55}{95} \div \frac{11}{95}$. When asked to find the answer without using a calculator, how likely are you to use the invert-and-multiply rule, obtain $\frac{55}{95} \times \frac{95}{11}$, and then simplify the answer? [impulsive]

B6a $\frac{44}{82} \div \frac{11}{82}$. When asked to find the answer without using a calculator, how likely are you to study the two fractions and predict the answer? [analytic]

Three measures can be derived from the LtA survey: (a) the *analytic subscale* is based on the 16 analytic LtA items, (b) the *impulsive subscale* is based on the 16 impulsive items, and (c) the *analytic-impulsive difference* is computed based on the difference between the analytic score and the impulsive score for each pair of items.

Opinion-seeking Question. Appended to the end of the LtA survey is a page with the following question: "In your opinion, which aspect(s) of problem-solving disposition do you think the 32-item survey is trying to quantify (i.e., measure)?" The intent of this question was for us to get a sense of what the participants thought the survey was about.

Open-ended Questionnaire. In this questionnaire, six open-ended questions were posed to uncover participants' initial approaches for solving selected impulsive items in the LtA survey. Two versions were created to cover the 12 mathematically-specific impulsive items (i.e., the four non-mathematically specific impulsive items were excluded). Of the 92 participants, 47 took Version A and 45 took Version B. Below are three of the 12 open-ended questions.

A1 What are the first few actions that you would take when asked to solve $(x-5)(x-8)=0$ for x?

A2 30 workers took 8 hours to complete Project P whereas 20 workers took 3 hours to complete Project Q. What are the first few actions that you would take when asked to determine which project was bigger in size?

B6 What are the first few actions that you would take when asked to find the answer for $\frac{55}{95} \div \frac{11}{95}$ without using a calculator?

Data Analysis

The authors and a graduate student coded all the responses in Version A. Another team of a full-time research assistant and a final-year doctoral student coded all the responses in Version B. Members from both teams met to analyze the responses in a training set. Five responses per item were selected for training. They included typical responses as well as challenging responses. The purpose was for both teams to establish consistency in coding and to agree on a set of guidelines such as analyzing a response in its entirety instead of automatically assigning a code based on the presence of a particular strategy or keyword.

Each written response was first analyzed in terms of solution strategies. Strategies that were similar were grouped together to form a category. The response was then assigned a code according to whether the response had a strong or a weak indication of analytic disposition (A+ or A-) or impulsive disposition (I+ or I-). In situations where a response was incomplete or irrelevant and we could not decide one way or the other, we coded the response as unsure (U). The inter-rater reliabilities for the two teams were 0.89 and 0.96 respectively. The codes, I+, I-, U, A-, A+, were later quantified using a five-point scale for statistical analysis.

We also attempted to quantify participants' depth of mathematical understanding. However, because the participants were only asked to describe the initial steps that they would take instead of actually solving the problem, we found it difficult to infer depth of understanding. Hence, we used a two-point scale: "0" for a response that contained major error(s) and "1" for a response without any major error. The inter-rater reliabilities were 0.93 and 0.98 for the two teams.

Participants' comments on the LtA survey were analyzed in two rounds to find common themes. In round one, the first author noted specific descriptions within each response in order to generate categories which were then collapsed. In round two, the second author used these categories to code the data. Since the purpose was to get a sense of how participants perceived the survey, we did not proceed to establish inter-rater reliability.

Results and Discussion

Solution Strategies

Table 1 shows the distribution of strategies from most common to least common for the 12 open-ended items. Strategies were not the same across items. For example, the top four strategies for Item A1 were: (a) using the FOIL method; (b) setting the two factors, $x – 5$ and $x – 8$, equal to 0; (c) using a guess-and-check approach; and (d) analyzing the problem or following a proper procedure; whereas the top three strategies for Item A2 were: (a) dividing 30 by 8 and 20 by 3 or comparing two ratios; (b) setting up a proportion and/or using cross-multiplication; and (c) commenting that more information was needed in order to solve the problem.

The most common strategy (Strategy 1) for all the items, except B6, in the open-ended questionnaire was consistent with the act described in the corresponding impulsive LtA item. For example, the most common strategy among respondents for Item A1 was using the FOIL method and the act described in the LtA Item A1i was "multiply out the terms (i.e., FOIL) and then solve $x^2 – 13x + 40 = 0$ using the quadratic formula." The top two strategies for Item A5 (using a proportion and finding unit cost) were equally common and both of them were depicted in its corresponding impulsive LtA item. The correspondence between the most common strategy and the act described in the impulsive LtA item suggests that the act described in the impulsive LtA item was appropriate for assessing impulsive disposition.

Table 1. Percent Distribution of Solution Strategies for Each Open-ended Item

	A1	A2	A3	A4	A5	A6	B1	B2	B3	B4	B5	B6
Strategy 1	57%	60%	79%	45%	30%	68%	73%	38%	58%	78%	33%	30%
Strategy 2	23%	21%	11%	9%	30%	13%	24%	38%	20%	13%	27%	25%
Strategy 3	6%	6%	8%	6%	26%	9%	2%	9%	7%	4%	24%	18%
Strategy 4	6%	-	2%	6%	4%	4%	-	4%	4%	-	7%	7%
Others	6%	13%	-	34%	11%	6%	-	11%	11%	4%	9%	23%

Note: The percentages for each item may not add up to 100% because of rounding.

The top three strategies for Item B6 were: (a) immediately cancelling out the common number 95; (b) inverting and then simplifying; and (c) inverting and then multiplying across. In terms of impulsive-analytic disposition, the difference between the first two strategies is subtle. For several students who “cancelled” out the 95s, we found it difficult to determine whether or not they spontaneously applied the invert-multiply strategy (i.e. invert the second fraction prior to canceling the 95s). We ended up replacing items B6i and B6a by a different pair of items in the revised LtA survey.

Coding for Impulsive-Analytic Disposition

Table 2 shows the distribution of disposition codes assigned to responses for all 12 items in the open-ended questionnaire. The frequency of I+ is highest for all the 12 items. This result might be a consequence of participants’ working on the LtA survey before working on the open-ended questionnaire. Participants’ initial exposure to the LtA items might have influenced their subsequent written responses for the open-ended items.

In general, the disposition code (I+ = strong indicator of being impulsive, A+ = strong indicator of being analytic) was dependent on the solution strategy associated with the response. Take responses to Item A1, for example. Out of 31 responses that were categorized as impulsive (I+ or I-), 27 mentioned using the FOIL method to arrive at a solution. These participants said that they would use the FOIL method probably because they could do something to those factors rather than analyze the form of the equation. Some participants commented that they would combine like terms and then solve for x.

Table 2. Percent Distribution of Disposition Code for Each Open-ended Item

	A1	A2	A3	A4	A5	A6	B1	B2	B3	B4	B5	B6
I+	62%	75%	81%	66%	60%	94%	76%	58%	78%	87%	76%	61%
I-	4%	11%	-	15%	2%	-	-	-	-	-	-	32%
U	6%	6%	-	11%	4%	-	-	2%	2%	-	-	-
A-	6%	6%	13%	2%	6%	-	-	2%	-	-	-	-
A+	21%	2%	6%	6%	28%	6%	24%	38%	20%	13%	24%	7%

For example, the response in Figure 1 contains a minor computational error when adding $-8x$ and $-5x$ and a major error in assuming that the quadratic equation, with three terms, could be solved by isolating x. We inferred impulsive disposition when there was evidence of spurious correlations such as associating the FOIL method with the factored form and associating isolating *x* with solving an equation for *x*. We are aware of the inaccuracy of inferring students' problem-solving disposition based on written responses alone, especially given the lack of detail in the descriptions. A more reliable way to infer impulsive-analytic disposition is to conduct task-based interviews (see Lim, 2008).

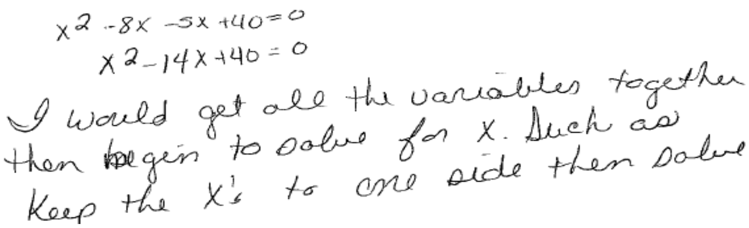

Figure 1. Using the FOIL method and combining like terms

Correlation between Coded-Disposition Score and LtA Subscales

Table 3 shows for each open-ended item, the mean value (1 = strong indicator of being impulsive, 5 = strong indicator of being analytic), its correlation with the Impulsive LtA subscale, the Analytic LtA subscale, and the difference between the two subscales. All the mean values are below three which is the midpoint on a five-point scale. This result suggests that the solution strategies were more indicative of being impulsive than being analytic. The mean value for each open-ended item was negatively correlated with the Impulsive LtA subscale and positively correlated with the Analytic LtA subscale. Seven of the 12 open-ended items were significantly correlated with the impulsive LtA subscale, but only three items were significantly correlated with the analytic LtA subscale. These results might be due to the items in the open-ended questionnaire being constructed based on the impulsive items in the LtA survey.

Table 3. Relating Coded-disposition Score and the LtA Subscales

	Mean Value (five-point scale)	Correlation with Impulsive LtA Subscale	Correlation with Analytic LtA Subscale	Correlation with Analytic-Impulsive Difference
Item A1	2.23	-0.21	0.25	0.31^{*}
Item A2	1.51	-0.11	0.20	0.22
Item A3	1.64	-0.40^{**}	0.45^{**}	0.58^{**}
Item A4	1.68	-0.41^{**}	0.21	0.41^{**}
Item A5	2.40	-0.16	0.17	0.22
Item A6	1.26	-0.43^{**}	0.50^{**}	0.63^{**}
Item B1	1.98	-0.48^{**}	0.44^{**}	0.62^{**}
Item B2	2.62	-0.39^{**}	0.17	0.39^{**}
Item B3	1.89	-0.47^{**}	0.21	0.47^{**}
Item B4	1.53	-0.48^{**}	0.19	0.46^{**}
Item B5	1.98	-0.13	0.14	0.18
Item B6	1.56	-0.17	0.08	0.17

$^{*}p < .05$, $^{**}p < .01$.

Eight of the 12 items have significant correlations with the analytic-impulsive difference score. Items that did not have strong correlations (A2, A5, B5, and B6) were considered for refinement or replacement. For example, after studying items A2i and A2a critically, we modified them to highlight the number of hours worked by *each* worker. The revised version for item A2i now reads "Project P took 30 workers, each working 8 hours, to complete. Project Q took 20 workers, each working 3 hours, to complete. When asked …?"

Item B6 was the least correlated with the analytic LtA subscale score. The supposedly-analytic act described in B6a might be considered *likely-to-act* for someone who would spontaneously invert the second fraction, cancel the 82s, and predict 4 as the answer.

Hence, in the revised version items B6i and B6a were replaced with the following pair of items:

- 3875.4 + 367.9 – 875.4. When asked to find the answer without using a calculator, how likely are you to begin by adding 3875.4 and 367.9 and then subtract 875.4?
- 1545.9 + 694.8 – 545.9. When asked to find the answer without using a calculator, how likely are you to study the decimals and obtain the answer almost instantly?

Table 4. Correlations between Coded Scores and LtA Subscales

	Analytic Subscale	Impulsive Subscale
Coded-disposition Score	0.373**	-0.488**
Coded-correctness Score	0.256*	-0.205*
*$p < .05$, **$p < .01$.		

Each participant took either Version A or Version B of the open-ended questionnaire. We added the disposition scores for all six items in each version to produce the Coded-disposition score for each participant. We added the six correctness scores to produce the Coded-correctness score. Table 4 shows the correlations between the two coded scores and the two LtA subscales. The strong correlations between the coded-disposition score and both the LtA subscales strengthen our confidence in the validity of the LtA survey.

Opinion on Purpose of LtA Survey

Of the 92 participants who were asked to write what the survey was designed to assess, three participants did not respond. From the remaining 89 responses, 48 categories were initially generated and subsequently collapsed into 13 categories. Because more than one category could be assigned to a response, we ended up with a total of 200 counts. Table 5 shows the number of responses for each category and its rank based on frequency count.

The first three categories are related to analytic disposition and the next three categories are related to impulsive disposition. The other categories are not directly related to impulsive or analytic disposition, including the highest-count category (e.g., "find out the way you would answer a given problem situation") and the third-highest category (e.g., "measure critical thinking skills in problem solving").

Predicting was the fourth-highest category and this might be a consequence of the word "predict" being used in 11 of the 16 analytic LtA items. We intended "predict" to be interpreted as an analytic act, but only 9 of the 17 respondents who thought that the survey was about predicting viewed predicting in the manner we intended (e.g., "predict and think about a problem rather than solving it immediately). Five respondents viewed predicting in opposition to analytic disposition (e.g., "automatically begin to predict a solution before analyzing a problem"). The remaining three respondents viewed predicting in a neutral manner (e.g., "what problems they would predict rather than solve"). Knowing that "predict" might be misinterpreted as *guessing without analyzing*, we revised several items. For example, the original version of an analytic LtA item reads: "Given that 6 bottles of mineral water cost $2.10. When asked to find the cost of 30 bottles, how likely are you to notice a relationship and predict that the answer is $10.50?" The revised version reads "… how likely are you to notice that 5 times $2.10 will give you the answer?" In addition, we reduced the number of times "predict" was used from 11 in the original version to 6 in the revised version.

Table 5. Number of Responses and Rank for Each Category about the LtA Survey

	Count	Rank
Analyzing or identifying relationships	30	2
Interpreting or understanding the problem	13	6
Finding an easier way, a shortcut, or alternative ways	7	10
Following procedures	14	5
Acting quickly without thinking	8	9
Assessing fastness in solving problems	7	10
Finding out how one approaches a problem	43	1
Assessing knowledge/skill/problem-solving	26	3
Predicting	17	4
Commenting about teaching and learning	13	6
Visualizing or mentally computing	11	8
Assessing competence in specific math topics	6	12
Others	5	13

Note: Total count = 200 > 89 because the 13 categories are not mutually exclusive.

Conclusions

In this paper, we reported our analysis of students' written responses to open-ended questions with the intent to investigate the validity of the items in the likelihood-to-act survey, which was designed to measure students' problem-solving disposition along the impulsive-analytic dimension. We found that in each question, except Item B6, the most-mentioned strategy was consistent with the act described in its corresponding impulsive LtA item. We also found significant correlations, for 8 of the 12 open-ended questions, between the assigned disposition code for written responses to these questions and the analytic-impulsive difference LtA score. In addition, the Coded-disposition score for the open-ended questionnaire was positively correlated to the analytic subscale and negatively correlated to the impulsive LtA subscale, both with $p < .01$. Less promising results were taken as opportunities to improve the LtA survey items. We are currently administering and testing the revised LtA survey to more than 400 pre-service teachers.

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INDICATORS OF MULTIPLICATIVE REASONING AMONG FOURTH GRADE STUDENTS

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Building upon the genre of research focused on understanding the development of multiplicative reasoning, this study examines the thinking of 14 fourth graders and concludes that the levels of multiplicative reasoning can be further defined in terms of indicators and refined in terms of student strategies.

Introduction

The difficulties encountered by students in their transition to advanced mathematical thinking may be explained by a lack of understanding of many concepts taught in early school years, especially multiplicative reasoning (e.g. Confrey, 1994; Dreyfus, 1991; Harel & Sowder, 2005). By its nature, advanced mathematical thinking relies on a cumulative foundation of prior mathematical experiences. Students cannot comprehend advanced mathematical topics such as differential equations unless they understand underlying concepts, such as differentiation. Differentiation requires conceptual understanding of the idea of functions and that assumes the student understands variables. Understanding variables is dependent upon the students' understanding of number, which is dependent upon understanding of quantification, which requires comprehension of serial correspondence. In other words, there is a structural order with each previous topic serving as a foundation for the next levels of mathematics and therefore we focus here on investigating children's multiplicative reasoning.

Multiplicative reasoning

The concept of unit with respect to addition is quite different than the concept of unit with respect to multiplication. Addition and multiplication involve hidden assumptions. The hidden assumption in addition is that the unit is one, which children readily understand; whereas, the hidden assumption in multiplication is that the unit is one as well as more than one simultaneously (Chandler & Kamii, 2009). Splitting, Confrey's (1994) contribution to understanding multiplicative reasoning, speaks to conditions under which reunitizing occurs after the split in multiplication. Park and Nunes (2001) suggest that children's concept of multiplication originates in their schema of correspondences and not in the concept of addition. Under this definition the concept of multiplication is defined by a constant relationship between two quantities known as ratio and is a core meaning of multiplicative reasoning. The ratio or rate is the constant unit that is called the multiplicand and acted upon by the multiplier. Children employ the schema of correspondence in order to represent fixed relationships between variables and solve multiplication problems.

Clark and Kamii (1996) identified five developmental levels that can be systematically observed in participants progressing from additive to multiplicative thinking. We propose an expansion of the multiplicative reasoning markers listed in Table 1 to a more detailed list of markers. Our focus is to identify observable behaviors of fourth grade students to expand the ideas found in Table 1. Towards this goal we considered the following questions:

1. What are the indicators of multiplicative reasoning among fourth grade students?
2. What strategies do fourth grade students utilize in solving word problems that require mathematical reasoning?

Theoretical Framework

Beginning with the work of Clark and Kamii (1996) we identified five levels of reasoning. The five developmental levels described below emerged from a review of the literature and provided the framework for this study. The multiplicative reasoning levels, described by Thorton and Fuller, 1981; Karplus and Lawson, 1974; Clark and Kamii, 1996, are summarized in Table 1 and provide a framework for this study.

Table 1. Multiplicative Reasoning Levels

1. Spontaneous Strategy	Not yet additive/guessing
2. Additive Strategy	Derives answer utilizing addition or subtraction
3. Multiplicative Strategy (w/o success)	Cannot make transition from additive to multiplicative thinking, but understands additive is not sufficient
4. Multiplicative Strategy (w/ success)	Uses multiplicative reasoning successfully with time to reflect in describing the relationship between the numbers
5. Proportional Strategy	Introduces a new quantity as a unitizing factor then successfully completes problem through multiplication or division

(Adapted from Thorton & Fuller, 1981; Karplus & Lawson, 1974; Clark & Kamii, 1996)

Methodology

Participants

In cooperation with the administration of a large urban school district in North Carolina, the participants were recruited from the fourth grade at a Chapter I school. The participants were age nine or ten and were recommended for study by the school's math specialist as being among the strongest math students in fourth grade. Overall fourteen participants of varying ethnicity, nine of whom were female and five of whom were male, were engaged in the study.

Instrument

The participants were asked to engage in a dialog while being interviewed. Because the computer can provide an environment that can enhance children's own construction of multiplicative reasoning via interaction with the teacher (Olive, 2000), the computer was employed as a research tool. An instrument consisting of ten questions was devised to invoke varying levels of multiplicative reasoning. Items were adapted from paper and pencil instruments developed by previous researchers (Thorton & Fuller, 1981; Karplus & Lawson, 1974; Clark & Kamii, 1996). The test instrument was developed in *Microsoft Visio* such that the participants were able to drag and drop green fish into the larger yellow fish to demonstrate their understanding of the multiplicative reasoning problems presented (see Figure 1). Also provided were manipulatives that included a magnetic aquarium containing green and yellow fish, two pizza pans for test item nine, and manipulatives such as paper clips, buttons, and stick figures for participants to demonstrate their understanding of test item ten.

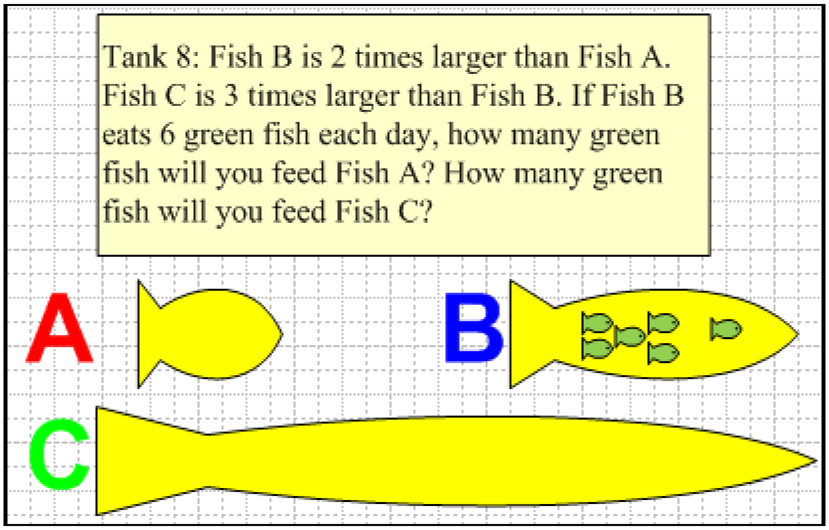

Figure 1. Sample Multiplicative Reasoning Problem

Data

The researcher employed two video cameras, one focused on the computer keyboard, computer screen, and hands of the participant, and the other focused on the participant's written workspace. In addition to the video recordings, the participant responses were collected by the Microsoft Visio program. The participants also provided written work supporting their thinking during the interviews. Transcriptions of the participants' verbal responses were also made. By reviewing the video recordings, in conjunction with the Microsoft Visio files, the written work of the participant and the transcriptions, the researchers explored indications of the emergence of participants' multiplicative reasoning.

Analysis

The data were examined for indications of the emergence of multiplicative reasoning utilizing the following techniques. Participants' data were first coded with respect to the framework introduced in Table 1 by observing the manner in which the participants responded to each question. Coded selections were then reexamined for key words, similarities, and differences within each of the five levels. When analyzing the participants' data, it became apparent that there were indicators not found as descriptors in the original five levels identified in Table 1. These indicators were inserted into Table 2 and then the participants' data were reanalyzed using this expanded framework.

Results

The researcher placed the participants' key words, transcripts, utterances, written work, schemes, and drawings for each question into one of the twelve strategies based on the following criteria found in the indicators column in Table 2. If the participant exhibited non-preservation of the quantification of objects, then the participant was placed at Level 1 Non-quantifier. For example, if the participant placed 32 fish into a fish that could clearly not hold 32 fish, this would be an example of non-preservation of the quantification of objects. If the participant arrived at the answer through guessing, then the participant was placed at Level 1 Spontaneous Guesser. For example, a demonstration of Level 1 Spontaneous Guesser behavior was exhibited when the participant noted that a particular fish should be fed 7 fish because 7 was his favorite number.

Table 2. Strategy Table for Level Mastery

Level	Strategy	Indicator	Reference
Level 1	Non-quantifier	Exhibits non-preservation the quantification of objects, i.e. A < B < C, meaning that 7 can = 8, can = 9	Clark & Kamii, 1996
Level 1	Spontaneous Guesser	Arrives at answer through guessing	Clark & Kamii, 1996
Level 2	Keyword Finder	Derives answer from keywords such as times and applies the associated algorithm	Sowder, 1988
Level 2	Counter	Enumerates objects with a one-on-one mapping with the whole number system	Dienes & Golden, 1966; Steffe, 1988
Level 2	Adder	Derives answer utilizing addition or subtraction regardless if strategy leads to success	Nunes & Bryant., 1996; Steffe, 1988; Thompson & Saldanha, 2003
Level 2	Quantifier	Makes use of the fact that A < B < C	Clark & Kamii, 1996; Lamon, 2006
Level 2	Measurer	Exhibits an understanding of when measurement should be linear or curvilinear and that each measurement has a starting and ending point without overlap or gap between unit measures	Kaput & West, 1994; Karplus, Pulos, Stage 1983
Level 3	Repeated Adder	Demonstrated an understanding that multiplicative answers can be achieved through repeated addition	Fishbein, Deri, Nello, and Marino 1985; Nunes & Bryant, 1996; Piaget, 1965; Steffe, 1994; Vergnaud, 1983, 1988
Level 3	Coordinator	Demonstrates limited ability to coordinate objects, numbers and operations	Park & Nunes, 2001
Level 4	Multiplier	States a multiplication sentence and demonstrates fluency with respect to coordination of objects, numbers and operations	Clark & Kamii, 1996
Level 4	Splitter	Utilizes concept of cutting and halving to indicate the need for division	Confrey, 1994
Level 5	Predictor	Predicts the measure of an object in 1 system given the measure of a proportional or similar object in another system	Kaput & West, 1994; Lamon, 2007

When the participant specifically indicated a need to multiply because the problem had the word “times” (or some other keyword such as twice), the participant was placed at Level 2 Keyword Finder, because the participant derived the answer by invoking the multiplication algorithm. When the participant counted the fish and uttered answers where it was easily observed that the answers were related to a one-on-one mapping with the whole number system, the participant was placed at Level 2 Counter.

Table 2 presents the strategies employed by the participants on the research instrument. The reference column in Table 2 identifies experts in the field who agree that the indicators in column three signify the emergence of multiplicative reasoning. Table 2 assists in answering research question one: what are the indicators of multiplicative reasoning among fourth grade students?

Participants who arrived at their answers via addition, and plainly indicated so by writing or saying that they were adding more fish for each fish, were placed as a Level 2 Adder. Quite often, because participants had success utilizing additive strategies, they tended to utilize additive strategies when they did not have a plan, perhaps because they had been successful in the past. However, utilizing an additive strategy demonstrated more understanding than simply guessing because their answers corresponded to the size of the fish. When participants made good use of the fact that $A < B < C$, then the participants were placed as a Level 2 Quantifier. When participants could not find the answer via multiplication they sometimes would measure the relative size of the fish, thus measuring the fish on the computer screen or the manipulatives and derive an answer through measurement. When the participants derived their answers via measurement, then the participants were placed as a Level 2 Measurer.

Sometimes the participants understood that additive strategies were not successful and attempted, but did not succeed, at utilizing multiplicative reasoning strategies. Often participants utilized repeated addition to obtain the correct answer. They would demonstrate this by writing or uttering $3 + 3 + 3 = 9$, for example. When such repeated addition was utilized, the participants were placed as a Level 3 Repeated Adder.

In many cases the participant may have understood that additive strategies were not sufficient and succeeded at utilizing multiplicative reasoning by demonstrating a good ability to coordinate the objects, numbers and operations defined within the word problem. Such participants often obtained the correct answer but did not or were not able to articulate the method or schema utilized to achieve the correct answer. Such demonstrations provided support for the participant being placed as a Level 3 Coordinator.

If the participant articulated an adequate mathematical sentence, either on paper or verbally, which fully described the mathematical relationship between the fish, the numbers, and the operations, then the participants would be placed as a Level 4 Multiplier. Additionally, some participants spoke or wrote the word “cut” or a similar word to indicate the need for division. In such a case, the participants were placed as a Level 4 Splitter.

When the participants provided a new quantity as a unitizing factor, articulated an adequate mathematical sentence, and successfully completed the problem using either multiplication or division, then the participants were placed as a Level 5 Predictor. In the case of Question 10, the participants exhibiting Level 5 Predictor should have stated something similar to “6 is to 4 as y is to 6 and since 6/4 reduces to 3/2 which equals 1.5 (this is the new unitizing factor) then 1.5 x 6 = 9”.

The literature provides a clear baseline for understanding multiplicative reasoning outlined in Table 1. Yet the participant data reveal that a more detailed analysis can further identify the building blocks underlying the learning of mathematics for fourth grade students. Table 2 identifies multiplicative indicators of these fourth graders along with additional strategies that expand the ideas in Table 1. These expanded strategies and indicators are then employed to re-analyze each of the 14 individual participant’s performance on the instrument. The columns in

Table 3 explain the frequency of the strategies used by the participants. The rows indicate the individual participant's use of strategies. This analysis addresses our second research question concerning strategies.

Table 3. Participant Strategies Frequency Table

Participant	Level 5 Predictor	Level 4 Splitter	Level 4 Multiplier	Level 3 Coordinator	Level 3 Repeated Adder	Level 2 Measurer	Level 2 Quantifier	Level 2 Adder	Level 2 Counter	Level 2 Keyword Finder	Level 1 Spontaneous Guesser	Level 1 Non- quantifier
1		2	6	1				1				
2			1		2	2		2	1		2	
3	1		5		1			1			2	
4			3				1	3	1	1	1	
5			3			1		2	1	2		1
6			3		2	2				3		
7			5		1	2				2		
8		1	3	1	1		1	2	1			
9			4	1		3	1	1				
10		1	5			2		2				
11	1	1	7			1						
12			2		1	2		4				1
13	1		5	1		1				2		
14		3	2	1		1				2		1
Σ	3	8	54	5	8	17	3	18	4	12	5	3

Table 4. Multiplicative Reasoning Strategies

Pre-Multipliers	Emergent	Multipliers
(7 times out of 10)	(5/5 or 4/6)	(7 times out of 10)
Participant 2	Participant 3	Participant 1
Participant 4	Participant 7	Participant 11
Participant 5	Participant 8	
Participant 6	Participant 9	
Participant 12	Participant 10	
	Participant 13	
	Participant 14	

Examining the results reported in Table 3, we observed the frequency of individual participant's strategy use. It was interesting to note the spread of strategies and indicators occurring over these fourth-grade participants. Therefore we sorted the participants' strategies into three categories. Those who were consistent in their choice of level 4 strategies or above (more than 70% of the time) we label as multiplicative reasoners. Pre-multiplicative are those participants who consistently chose strategies below level 4 in their attempts to solve the problems. Those participants who used a range of strategies we label as emergent. Emergent participants are considered non-multipliers. Table 4 summarized these findings.

Summary and Implications

The indicators of multiplicative reasoning are specific writings, utterances of keywords, and behaviors of these participants as they engaged in problem solving. From these indicators 12 strategies were identified that these fourth grade participants utilized in solving multiplicative reasoning word problems. It was interesting to note that each of the 12 strategies was used at least three times. Separating the participants into three categories provided additional insights into the vast spread of multiplicative understandings present in Chapter I schools in fourth grade. In the future, we propose to investigate the strategies and indicators of older children and younger children with the research instrument to provide validation for our findings reported here.

The practical implication of our findings is that they provide teachers with another tool to diagnose and assess students' understanding of multiplication. Teacher observations and informal assessment techniques can be used in conjunction with the list of indicators generated in Table 2. This list is another way for teachers to assist students' progress towards understanding multiplication. The list of indicators and strategies is also useful in determining the trajectories of students' multiplicative reasoning so that teachers can remediate and develop steps towards additional understanding (Ball, 2003; Richardson, Berenson, and Staley, 2009).

Endnotes

1. We wish to recognize the contributions and guidance provided by Kerri Richardson, Heidi Carlone, and Richard Morgan. The views expressed here do not necessarily reflect their views.

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MATHEMATICAL TASKS USED IN CLASS AND RELATED HOMEWORK: AN ANALYSIS OF COGNITIVE DEMAND LEVELS

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This research examined the relationship of the cognitive demand levels of mathematical tasks done in class over the course of a school year and related tasks assigned for homework. In total, 66 mathematical tasks were evaluated using to assess the level of cognitive demand. Results from this research showed that approximately two-thirds of the time mathematical tasks assigned for homework were different levels with those explored in class. Implications for student learning, achievement, classroom practice, and for further research will be discussed in light of these findings.

Introduction

From very early on in formal schooling most children are asked to do some form of homework. The recent *Trends in International Mathematics and Science Study* (TIMMS) (Mullis, Martin, & Foy, 2008) reports that the amount of time spent on mathematics homework by children varies significantly between countries and grades. On average, the amount of time devoted to mathematics varies between countries from two to three times per week with no more than 30 minutes at any one time, to three to four times a week with more than 30 minutes homework at one any one time. While the time spent on homework varies, these results show nevertheless that a considerable amount of time at home is spent by students on mathematics homework. According to Trautwein, Niggli, Schnyder, and Lüdtke (2009), there are three primary functions of homework. First, homework permits teachers to extend time learning well beyond that of the regular classroom hours. Second, homework enables students to rehearse what they have learned, extend their knowledge, and even exercise their working and long term memory. Third, homework enhances student motivation and self-regulation.

Results from empirical studies show mixed results between homework and achievement that have been related to numerous factors including parental involvement (Patall, Cooper, & Robinson, 2008; Tam & Chan, 2009), teacher implementation and beliefs (i.e., assessed versus not assessed, taken up in class, etc.) (Simplicio, 2005), learning disabilities (Sheridan, 2009), and student motivation and self-regulation (Hong, Peng, & Rowell, 2009). Research about homework points to the fact that context, pedagogical, socio-cultural, economic, psychological, and so forth, all contribute differently and perhaps uniquely to homework efficacy.

While some studies compared the types of homework assigned in relation to the types of tasks experienced in classrooms (e.g., Fife, 2009), surprisingly, no studies were found that analyzed the cognitive relationship between student learning and mathematical tasks done in class and those assigned for homework. While the types, duration, and relation to achievement of homework is interesting and indeed useful, an understanding of the way in which the homework is cognitively aligned to classroom learning may be more pertinent to supporting students in rehearsing prior learning at a remediation level, rehearsing new learning, extending new learning independently, and increasing achievement.

As a result, our goals in this research were as follows: (1) to examine the cognitive demand levels of mathematical tasks done in class and related mathematical task assigned for homework, (2) to contemplate the implications of compatibility or discord in cognitive demand levels for student learning and for classroom instruction, (3) to make preliminary conjectures about student achievement in relation to and cognitive demand levels, and (4) to recommend potential research directions in light of our analysis.

Theoretical framework

Mathematical Task

This research draws extensively on the theoretical assertions posited by Stein and colleagues (Stein, Grover, & Henningsen, 1996; Stein, Smith, Henningsen, & Silver, 2000) about mathematical tasks and cognitive demand levels. According to Stein, Grover and Henningsen (1996), a mathematical task is defined as a classroom activity intended to focus students' attention towards a particular mathematical idea, where the activity is ongoing until such time that the "underlying mathematical idea toward which the activity is oriented changes" (p. 460). Stein and colleagues' suggest that a mathematical task may be (and likely is) comprised of multiple related activities rather than just one singular activity, and that the multiple related activities are ongoing until such time that cognitive change occurs on behalf of the student.

Stein et al. (1996; 2000) claim that a central aspect about mathematics tasks the potential for them to be transformed and changed during three phases: "first, as curricular or instructional materials; second, as set up by the teacher in the classroom; and third, as implemented by students during the lesson" (p. 460). Implementation at any of the phases of the mathematical task may shift the intended goals or benefits of the mathematical task. That is, the mathematical task is highly influenced by factors such pedagogy, content knowledge, prior knowledge, social and psychological contexts of learning, and so forth (Henningsen & Stein, 1997).

The potential for mathematical tasks to be transformed has important implications for student learning. As Boston and Smith (2009) have pointed out, over a decade of research has demonstrated that high quality learning environments that are sustained throughout instruction are most effective in occasioning increased student achievement (Boaler & Staples, 2008; Hiebert et al., 2004; Stigler & Hiebert, 1999). Stein et al.'s (2000) framework only considered transformation of the mathematical task during instruction and not transformations that may take place at the homework level which may have differing and unique implications for student learning.

Consequently, it is our assertion that there is (at least) one additional phase should be included in Stein and colleagues' (1996; 2000) framework: student implementation during homework. This fourth phase reflects the fact that while some students may have a shift in their mathematical idea orientation during the lesson, others may not until they revisit and rehearse the material independently during homework. As Stein et al., claim a mathematical task may change once "unleashed in a real classroom" (p. 460). The same may be true during homework.

Cognitive Demand Levels

Stein et al. (1996) define cognitive demand "as the cognitive processes in which students actually engage as they go about working on the task" (p. 461). The authors make the distinction between mathematical "tasks that engage students at a surface level and tasks that engage students at a deeper level by demanding interpretation, flexibility, the shepherding of resources, and the construction of meaning" (p. 459.) Stein and colleagues (1996; 2000) distinguish between low and high cognitive demand levels of mathematical tasks. Low cognitive demand levels are those that predominantly involve memorization and or engagement in mathematical processes in the absence of connections between mathematical ideas, whereas mathematical tasks with high cognitive demand make connections between mathematical ideas and require “doing mathematics” such that students are engaging in self- reflection, -regulation, and so forth.

Previous studies, such as Stein and colleagues’ (1996) research, have focused on mathematical task implementation at the classroom level. Taking Stein’s definition of mathematical tasks as those that continue until such time that the underlying orientation to the mathematical idea changes, research on cognitive relationship between mathematical tasks assigned for homework to provide rehearsal, emphasis, and so forth, those in the used in the classroom is critical and necessary to provide a fuller picture of the implications to student learning and to classroom instruction.

Methods

This research took place in the third author’s eighth-grade classroom and spanned the full academic year (i.e., from September to June). The school in which the research took place was located in an economically, socially, and culturally diverse urban setting. Duane’s class consisted of 28 students, 14 male, and 14 female. All students were between 13 and 14 years of age. Data for this study was drawn from a larger body of data collected from this class exploring the relationship of homework in mathematics to classroom practices.

In previous years, Duane had attended numerous professional development sessions held through his school board and provincial mathematics associations where he had learned about teaching mathematics through problem-solving in order to develop deeper thinking and understanding about mathematics (Twomey Fosnot & Dolk, 2001). Lessons were predominantly structured around mathematical tasks. All of the mathematical tasks analyzed in this research took place in small group settings (i.e., two to four students). Some of the mathematical tasks were structured around a real-world context, while others were not. Often the exploration of the mathematical tasks took place over two or more consecutive days.

Over the course of the school year, Duane used 33 mathematical tasks in class (approximately one per week) to explore five mathematical strands: data management, geometry, measurement, patterning and algebra, and number sense and numeration. At the beginning of each week, Duane would distribute a homework sheet to each student. The homework sheet consisted of at least one mathematical task that was intended to parallel the mathematical task explored during class, plus additional problem sets related to the mathematical strand.

All coding was done by the first and the third authors. The first step of the data coding involved explicitly linking the intended parallel homework question from the assigned homework to the related mathematical task explored during class. Related here implies (a) the use of similar mathematical processes between the mathematical tasks (e.g., reasoning and proving, communicating, connecting, reflecting, selecting tools and computational strategies, etc.), and (b) the same mathematical strand (i.e., data management, patterning and algebra, number sense and numeration, geometry, or measurement). Therefore, a total of 66 mathematical tasks (33 paired mathematical tasks) formed the primary basis of the data set. Also included for the analysis were the mean achievement levels of the students (n = 26, since two left the school mid-year) in the class for each mathematical strand.

Once the mathematical tasks were paired, we each independently evaluated each mathematical task using as our instrument a modified *IQA Academic Rigor: Mathematic Rubric* *for the Potential of the Task* from Boston and Smith (2009) (see Appendix A). This instrument was used by Boston and Smith (2009) to investigate mathematical task implementation of secondary school mathematics teachers following professional development. The rubric is modified from the two levels proposed by Stein et al. to four levels for a more nuanced analysis. At the conclusion of our independent evaluations we compared our results and discussed differences until consensus on the evaluation of the mathematical task according to our rubric was achieved.

Descriptive statistics were computed to summarize cognitive demand levels of class and homework mathematical tasks, differences between the cognitive demand levels of the mathematical tasks, and means of student achievement across mathematical strands. The *Wilcoxon matched-pairs signed ranks* test was conducted to evaluate whether the median level for classroom and for homework mathematical tasks differed and was significant. Also, *Spearman's Rho correlation coefficient* was calculated to examine the predictive relationship between class and homework mathematical task levels, and achievement means from the mathematical strands from the students in the class.

Results

We hypothesized at the onset of this research that there should be a relationship between cognitive demand levels of class and homework mathematical tasks, and that the cognitive demand level would overall be predominantly high (i.e., level 4) given the classroom context (i.e., significant focus on thinking, reasoning, problem-solving). Descriptive statistics revealed that 75.7% of mathematical tasks selected and implemented in class were at a cognitive demand level three or four. In comparison, 94% of mathematical tasks assigned for homework were at a cognitive demand level of three or four (see Table 1). However, mathematical tasks at a cognitive demand level of four that were planned for class instruction occurred 18% more than those assigned for homework.

Table 1. Cognitive demand level of mathematical task (%)

	Class	Homework
Level 2	8(24.2)	2 (6.0)
Level 3	7 (21.2)	19 (57.6)
Level 4	18 (54.5)	12 (36.4)
Total	33 (100)	33 (100)

No significant differences were found between the median levels of the mathematical tasks in-class and those assigned for homework (Wilcoxon, n = 33, Z = -0.052, p = 0.958, 2-tailed). Mean level (standard deviations in parentheses) of cognitive demand for both classroom and homework mathematical tasks was 3.3 (0.847, 0.585, respectively), with more variation between cognitive demand levels for the classroom mathematical tasks. Noteworthy is the finding that 30.3% of mathematical tasks assigned for homework were lower in cognitive demand level than the paired mathematical task from the class (see Table 2). In contrast, 36.4% of homework mathematical tasks were higher than the classroom mathematical tasks. These results suggest that, at least for this data set, almost two-thirds of the homework questions did not represent a sustained cognitive demand level into the homework phase, although approximately one-third did represent an increase in cognitive level demand.

Table 2. Frequency of differences of levels of paired mathematical tasks, n = 33 (%)

No difference	Class higher	Homework higher
11(33.3)	12 (36.4)	10 (30.3)

The mean (with standard deviations in parentheses) for achievement across strands was 74.26 (0.98779). There was asymmetry across the mathematical strands in the number of tasks per strand. Number sense by far had the most amount of mathematical tasks (n = 19), compared to geometry (n =2), which was the lowest. There was also asymmetry across the cognitive demand levels within the strands. Some strands had cognitive demand levels that were the same between classroom and homework mathematical tasks, higher classroom mathematical tasks, and higher homework mathematical tasks. The asymmetrical distribution of mathematical tasks and cognitive demand levels across the mathematical strands make it difficult to assess from the current research the implications of cognitive demand levels on mathematical achievement. Further and different research is needed to assess this relationship.

Spearman's Rho correlation coefficients were calculated to examine the predictive relationship between mathematical strand, mean achievement on mathematical strands, as well as classroom and homework mathematical tasks. As anticipated, the mathematical strands were moderately correlated (r = 0.53) to the mean achievement in the mathematical strand at the level of 0.01. The correlation between cognitive demand levels of classroom mathematical tasks and homework mathematical tasks was negative and not significant (r = -0.03, *n.s.)* These results suggest that there does not appear to be a predictive relationship between the cognitive demand levels and the mathematical tasks.

Discussion

The findings from this research suggest that while the mean and median levels cognitive demand levels were found to be consistent for both classroom and homework mathematical tasks, approximately one-third of the time, assigned homework in this class had a higher, lower, or parallel cognitive demand level with the paired mathematical task from class. Additionally, the cognitive demand level of the mathematics task in class was not predictive of the cognitive demand level of the homework mathematical task. While these findings cannot be generalized because of the sampling methods of this research, they nevertheless raise important and interesting questions about homework and the relationship to classroom practices.

If Stein's (1996) proposed phases at which mathematical tasks can be transformed are extended to include the homework phase, then there are numerous pedagogical considerations that must be contemplated. If the pedagogical goal of homework is to extend the instructional time from the classroom to home so that students can rehearse learning that occurred during class time then the same cognitive demand level between the classroom mathematical task and that assigned for homework may be appropriate. If the pedagogical goal is to extend learning that was started during classroom instruction, then perhaps a higher level than the cognitive demand level between the classroom mathematical task and mathematical task assigned for homework would be more appropriate. A higher cognitive demand level in the homework mathematical task than that of the classroom mathematical task could present additional challenges for a student who may not be able to independently work through a more difficult mathematical task and may not have home support to rely upon (Patall et al., 2008; Pezdek, Berry, & Renno, 2002).

We contemplated whether there would there be any case where the homework level should be lower than the level of the classroom mathematical task? Our collective conclusion was no, and we further suggest that cognitive demand levels of homework that are consistently lower than those of the classroom may foster adverse effects to student motivation towards homework.

Implications for Education and Research

Given that sustained high cognitive level instructional environments have been shown to benefit student learning, it may be useful and necessary to utilize a cognitive demands framework for structuring classroom learning and related homework to be sure that are cognitive demand levels sustained into the homework phase. We found that cognitive demand levels between classroom and homework mathematical tasks differed more than two-thirds of the time. This may pose serious implications for learning.

A limitation of this research is in the assumption made that the level of cognitive demand of the classroom mathematical tasks remained untransformed throughout classroom instruction (Stein et al., 2000). It may be that this was not the case. While the transformation of the mathematical tasks was not the focus of this research, we recognize that this remains an important area for further research that takes into explicit consideration the final phase mathematical task implementation proposed in this research, that of homework. In light of our results, our research raises more questions: *Provided that a cognitive demand level is sustained during classroom instruction, what might the implications to learning of a different cognitive demand level at the homework phase? Are there learning occasions that would warrant a difference in cognitive demand level between the three phases of instruction and the final proposed phase during homework completion? What are the implications of cognitive demand levels of class and homework mathematical tasks and mathematics achievement?*

Endnotes

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Appendix A

IQA Academic Rigor: Mathematic rubric for the potential of the task (Adapted from Boston & Wolf, 2006; Matsumura et al., 2006)

4	The task has the potential to engage students in exploring and understanding the nature of mathematical concepts, procedures, and/or relationships such as: • Doing mathematics: using complex and non-algorithmic thinking (i.e., there is not a predictable, well rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example). Or • Apply the procedures with connections: applying a broad general procedure that remains closely connected to mathematical concepts. The task must explicitly prompt for evidence of students' reasoning and understanding. For example, the task may require student to: • Solve a genuine, challenging problem for which student's reasoning is evident in their work on the task; • Develop an explanation for why formulae or procedures work; • Identify patterns and form generalizations based on these patterns; • Make conjectures and support conclusions with mathematical evidence; • Make explicit connections among representations, strategies, or mathematical concepts and procedures; and • Follow a prescribed procedure in order to explain/illustrate a mathematical concept, process, or relationship.
3	The task has the potential to engage students in complex thinking or in creating meanings for mathematical concepts, procedure, and/or relationships. However, the task does not warrant a "4" because – • It does not explicitly prompt for evidence of students' reasoning and understanding; • Students may be asked to engage in doing mathematics or procedures with connections, but the underlying mathematics in the task is not appropriate for the specific grouping of students (i.e., too easy or too hard to promote engagement with high level cognitive demand); • Students may need to identify patters but are not pressed for generalizations; • Students may be asked to sue multiple strategies or representations, but the task does not explicitly prompt students to develop connections between them; and • Students may be asked to make conjectures but are not asked to provide mathematics evidence or explanations to support conclusions.
2	The potential for the task is limited to engaging students in using a procedure that is either specifically called for or its use is evident based on prior instruction, experience, or placement of the task. There is little ambiguity about what needs to be done and how to do it. The task does not require students to make connections to the concepts or meaning underlying the procedure being used. The focus of the task appears to be on producing correct answers rather than developing mathematical understanding (e.g., applying a specific problem-solving strategy, practicing a computational algorithm). OR The task does not require students to engage in cognitively challenging work' the task is too easy to solve.
1	The potential of the task is limited to engaging students in memorizing or reproducing facts, rules, formulae, or definitions. The task does not require students to make connections to the concepts or meaning that underlie the facts, rules, formulae, or definitions being memorized or reproduced.
0	The task requires no mathematical activity.

PRODUCTIVE FAILURE IN LEARNING THE CONCEPT OF VARIANCE

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In a study with 140, ninth-grade mathematics students on learning the concept of variance, students experienced either direct instruction (DI) or productive failure (PF), wherein they were first asked to generate a quantitative index for variance without any guidance before receiving direct instruction on the concept. Whereas DI students relied only on the canonical formulation of variance taught to them, PF students generated a diversity of representations and formulations for variance but were ultimately unsuccessful in developing the canonical formulation. On the posttest however, PF students performed on par with DI students on procedural fluency, and significantly outperformed them on data analysis, conceptual insight, and transfer items. These results challenge the claim that there is little efficacy in having learners solve problems targeting concepts that are novel to them, and that direct instruction alone is the most effective approach for teaching novel concepts to learners.

Introduction

Proponents of direct instruction bring to bear substantive empirical evidence against un-guided or minimally-guided instruction to claim that there is little efficacy in having learners solve problems that target novel concepts, and that learners should receive direct instruction on the concepts before any problem solving (Kirschner, Sweller, & Clark, 2006). Kirschner et al. (2006) argued that “Controlled experiments almost uniformly indicate that when dealing with novel information, learners should be explicitly shown what to do and how to do it” (p. 79). Commonly-cited problems with un-guided or minimally-guided instruction include increased working memory load that interferes with schema formation (Tuovinen & Sweller, 1999), encoding of errors and misconceptions (Brown & Campione, 1994), lack of adequate practice and elaboration (Klahr & Nigam, 2004), as well as affective problems of frustration and de-motivation (Hardiman et al., 1986).

Klahr & Nigam’s (2004) often-cited study compared the relative effectiveness of discovery learning and direct instruction approaches on learning the control of variable strategy (CVS) in scientific experimentation. On the acquisition of basic CVS skill as well as ability to transfer the skill to evaluate the design of science experiments, their findings suggested that students in the direct instruction condition who were explicitly taught how to design un-confounded experiments outperformed their counterparts in the discovery learning condition who were simply left alone to design experiments without any instructional structure or feedback from the instructor (we will return to this study in more detail in the discussion section). Further experiments by Klahr and colleagues (Chen & Klahr, 2008; Strand-Cary & Klahr, 2008), and others as well have largely bolstered the ineffectiveness of discovery learning compared with direct instruction (for a review, Kirschner et al., 2006).

Be that as it may, the above findings do not necessarily imply that there is little efficacy in having learners solve novel problems, that is, problems that target concepts they have not learnt yet (Schmidt & Bjork, 1992). To determine if there such an efficacy, a stricter comparison for direct instruction would be to compare it with an approach where students first generate representations and methods on their own followed by direct instruction. Expectedly, the generation process will invariably lead to failure, that is, students are rarely able to solve the problems and discover the canonical solutions by themselves. However, this very process can be productive for learning *provided* direct instruction on the targeted concepts is subsequently provided (Kapur, 2008; Koedinger & Aleven, 2007; Schwartz & Bransford, 1998; Schwartz & Martin, 2004). As a case in point, I present evidence from an on-going research program on *productive failure* (Kapur, 2008, 2009) in mathematical problem solving.

Designing for Productive Failure

There are at least two problems with direct instruction in the initial phase of learning something new or solving a novel problem. First, students often do not have the necessary prior knowledge differentiation to be able to discern and understand the affordances of the domain-specific representations and methods underpinning the targeted concepts given during direct instruction (e.g., Schwartz & Martin, 2004). Second, when concepts are presented in a well-assembled, structured manner during direct instruction, students may not understand why those concepts, together with their representations, and methods, are assembled or structured in the way that they are (Schwartz & Bransford, 1998).

To overcome these two problems, a learning design should focus squarely on first engaging students in processes that serve two critical cognitive functions, which in turn, prepare students for subsequent direct instruction: a) activating and differentiating prior knowledge in relation to the targeted concepts, and b) affording attention to critical features of the targeted concepts.

Productive failure (PF) is one such learning design. It comprises two phases—a generation and exploration phase followed by a direct instruction phase. In the generation and exploration phase, the focus is on affording students the opportunity to leverage their formal as well as intuitive prior knowledge and resources to generate a diversity of *structures*—concepts, representations and solution methods—for solving a complex problem; a problem that targets concepts that they have not been formally taught or learnt yet. Research suggests that students do have rich *constructive resources* (diSessa & Sherin, 2000) to generate a variety of structures for solving novel problems (Schwartz & Martin, 2004). At the same time, research also suggests that one cannot expect students, who are novices to the target content, to somehow generate or discover the canonical representations and domain-specific methods for solving the problem (Kirschner et al., 2006).

However, the expectation for the generation and exploration phase is not for students to be able to solve the problem successfully. Instead, it is to persist in generating and exploring the affordances and constraints of a diversity of structures for solving the problem. This process functions to not only activate but also differentiate prior knowledge (as evidenced in the diversity of student-generated concepts, representations and methods). Furthermore, a comparison and contrast between the various structures also affords opportunities to attend to critical features of the targeted concepts (more on this in results section). Consequently, the generation and exploration phase provides the necessary foundation for developing deeper understanding of the canonical concepts, representations, and methods during direct instruction.

Empirical evidence for the above theoretical conjectures embodied in the PF design come from several studies that suggest that conditions that maximize performance in the shorter term are not necessarily the ones that maximize learning in the longer term (Schmidt & Bjork, 1992). Examples of such studies include VanLehn's (2003) work on *impasse-driven learning*, Schwartz and Martin's (2004) work on *inventing to prepare for learning*, diSessa's (1991) work on *meta-representational competence*, Koedinger and Aleven's (2007) work on the *assistance dilemma*, as well my own work on *productive failure* (Kapur, 2008, 2009). Collectively, these research programs support the argument for designing conditions for learners to persist in the process of solving novel, complex (from the learners' perspective) problems without instructional support structures initially. Even though such a process invariably leads to failure in the shorter term, the extent to which this process affords learners opportunities to generate and explore multiple representations and methods for solving the problem, the process can be germane for learning. The purpose of this paper is to report findings from an on-going, classroom-based research program in Singapore on productive failure.

Method

Participants

Participants were 140, ninth-grade mathematics students (14-15 year olds) from an all-boys pubic school in Singapore. Students were almost all of Chinese ethnicity. Students were from four mathematics classes; three classes taught by one teacher (teacher A), and the fourth class by another teacher (teacher B). Students had no prior instruction on the concept of variance, although they had learnt the concepts of mean, median, and mode in grades 7 and 8.

Research Design

A pre-post quasi-experimental design was used with two classes (n = 31, 35) taught by teacher A assigned to the 'Direct Instruction' (DI) condition, and the other two classes (n = 35, 39), under teachers A and B, assigned to the 'Productive Failure' (PF) condition.

First, all students took a five-item pretest (α = .75) on the concept of variance. Not surprisingly, not a single student demonstrated canonical knowledge of the concept, and there was no significant difference between the four classes either, F(3,136) = 1.665, p = .177. Next, all classes participated in five, 55-minute periods of instruction on the concept as appropriate to their assigned condition. Finally, all students took a six-item posttest (α = .74) comprising items on procedural fluency, data analysis, conceptual insight, and transfer.

In the DI condition, the teacher first explained the concept of variance and its canonical formulation as the square of the standard deviation ($SD^2 = \frac{\sum_1^n \left(x_i - \bar{x}\right)^2}{n}$) using a data analysis problem. Next, the teacher modeled the application of the concept by working through several data analysis problems, highlighting common errors and misconceptions, and drawing attention to critical features of the concept in the process. The data analysis problems required students to compare the variability in 2-3 given data sets (e.g., comparing the consistency of performance of three soccer players). Thereafter, students worked face-to-face in triads on more data analysis problems. The teacher then discussed the solutions with the class. After each period, students were given similar data analysis problems for homework, which the teacher marked and returned to the students, usually by the following period.

The PF condition differed from the DI condition in only one important aspect. Instead of receiving direct instruction upfront, students spent two periods working face-to-face in triads to solve one of the data analysis problems on their own. The data analysis problem presented a distribution of goals scored each year by three soccer players for a twenty-year period. Students were asked to generate a quantitative index to determine the most consistent player. During this generation phase, no support or scaffolds were provided. Following this, three periods were spent on direct instruction just like in the DI condition. Note that because students in the PF condition spent the first two periods generating an index for variance, they solved fewer data analysis problems overall than their counterparts in the DI condition. To make this contrast even sharper, PF students did not receive any data analysis problems for homework.

Hypothesis**.** The hypothesis tested was that productive failure will be more effective than direct instruction in learning the concept of variance. That is, expecting to replicate earlier work on productive failure (Kapur, 2008, 2009), I hypothesized that students from the PF condition will be able to generate and explore various representations and methods for generating an index for variance (diSessa et al., 1991), but will not be successful in developing or discovering the canonical formulation on their own (Kirschner et al., 2006). However, this seeming failure would be integral for: a) engendering the necessary prior knowledge differentiation (evidenced in the diversity of student-generated structures), and b) drawing attention to critical features of the concept of variance (evidenced in the comparisons between the student-generated structures), which may help students better understand the concept when presented by the teacher during direct instruction subsequently. This better understanding would result in better procedural fluency, data analysis, conceptual insight, and transfer.

Process Results

Process data included group-work artifacts produced on A4 sheets of paper. These provided a rich source of data about the nature of problem representations and methods generated by the students in the PF and DI conditions. In the PF condition, groups produced four major and *progressively sophisticated* categories of methods and representations. The four categories were: a) central tendencies, b) qualitative methods, c) frequency methods, and d) deviation methods.

Category 1: Central Tendencies. Groups started by using mean, median, and in some cases, mode for data analysis. This was not surprising because students had been taught these concepts in the earlier grades. However, relying on central tendencies alone, it was not possible to generate a quantitative index for variance because the problem was designed in a way to keep the central tendencies invariant.

Category 2: Qualitative methods. Groups generated graphical and tabular representations that organized the data visually and were able to discern which player was more consistent. The visual representations (see Figure 1) afforded a qualitative comparative analysis between the players, but did not provide a quantitative index for measuring consistency even though the ideas of spread and clustering are quite evidently important qualitative conceptual underpinnings for the concept of variance.

Category 3: Frequency methods. Groups built on the qualitative methods to develop frequency-based measures of consistency. For example in Figure 2, groups used the frequency of goals scored within certain intervals to argue that the player with the highest number of goals in the interval containing the mean was the most consistent. Other groups counted the frequency with which a player scored above, below, and at the mean. Frequency methods demonstrated that students could quantify the clustering trends that the qualitative representations revealed.

Category 4: Deviation methods. Figure 3 presents some examples of the deviation methods. The simplest deviation method generated was the range (Deviation method 1, or simply D1). Some groups calculated the sum of year-on-year deviations (D2) to argue that the greater the sum, the lower the consistency. Among these, there were those who considered absolute deviations (D3) to avoid deviations of opposite signs cancelling each other—an important conceptual leap towards understanding variance. Finally, there were some groups who calculated deviations about the mean (D4) only to find that they sum to zero. For both the D3 and D4 categories, some groups further refined their method to consider not the sum of the deviations, but the average (D5).

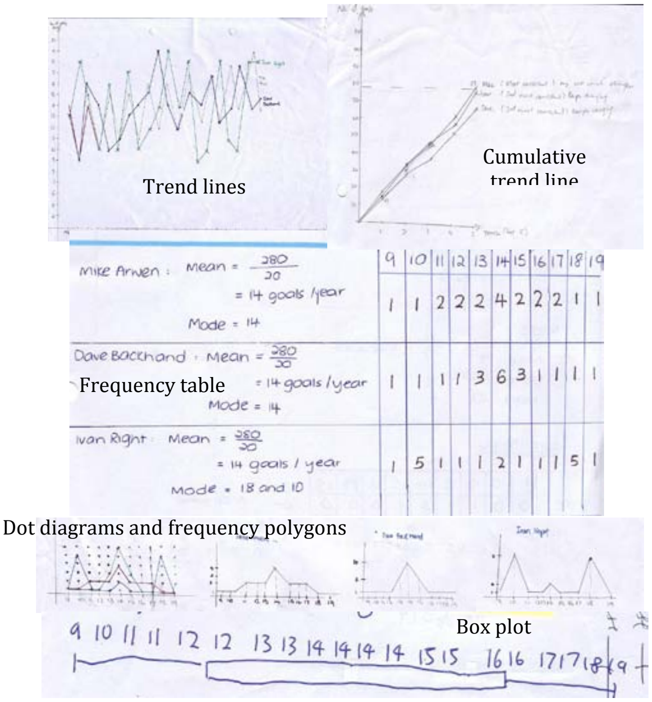

Figure 1. Qualitative methods

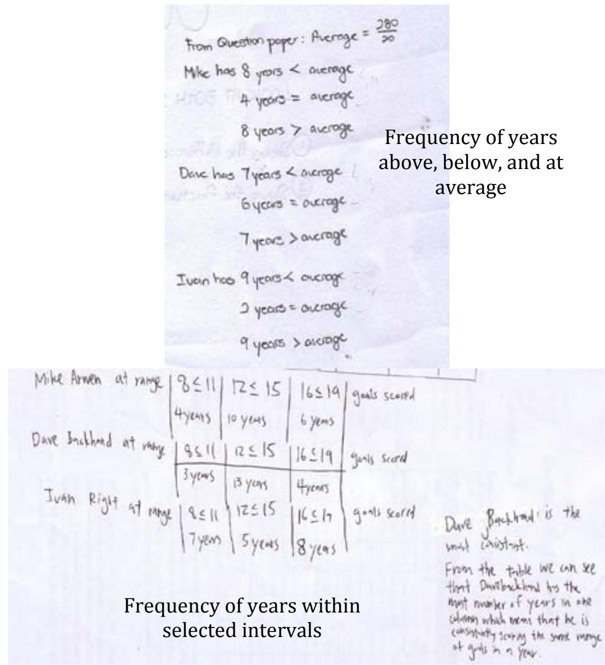

Figure 2. Frequency methods

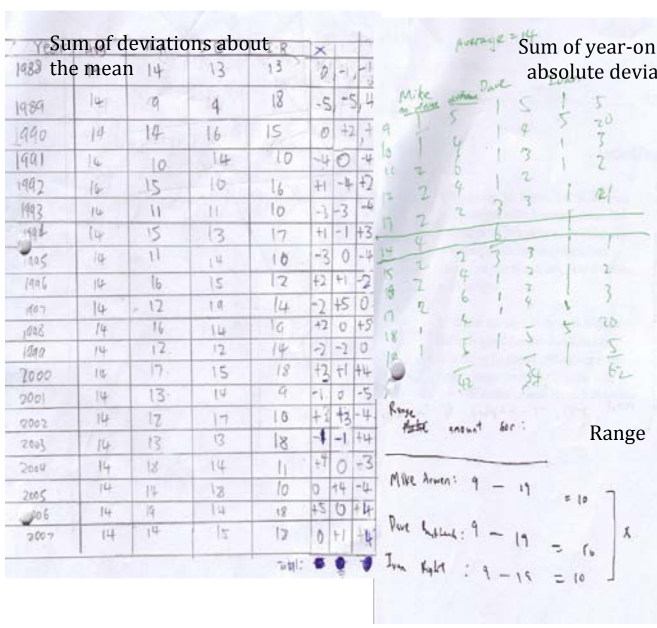

Figure 3. Deviation methods

In both the PF classes, <u>all</u> groups demonstrated representational competence at the Category 3 level or greater. Only 2 groups from PF-A and 1 group from PF-B did not reach Category 4.

Consistent with the hypothesis, none of the groups were able to develop let alone use the canonical formulation on their own. More importantly, note that these structures evidence the hypothesis that students will in fact be able to generate a rich diversity of structures to solve the problem without having first learnt the targeted concept of variance, and that comparisons between these structures will afford students the opportunities to attend to deep conceptual features of the concept. The latter needs more elaboration:

i. Comparing central tendencies with qualitative representations afforded an opportunity to attend to the feature that central tendencies alone cannot convey information about variance, and that different distributions with the same mean can have different variance.
ii. A comparison between the frequency methods and the qualitative methods afforded the opportunity to attend to the quantification of qualitative data into a mathematical index that returns a value for consistency.
iii. Because the deviation methods consider the relative position of a data point, a comparison with the frequency methods afforded students the opportunity to attend to the feature that, for consistency, it is not only important to count a point but also consider its position in relation to other points.
iv. Comparing the Range (D1) with other qualitative or deviation methods afforded students the opportunity to attend to the feature that considering just the extreme points may not be a good measure of consistency, because it tells us nothing about the distribution in the middle.
v. A comparison between D2 and D3 afforded students the opportunity to attend to the feature of why deviations must be positive. The comparison clearly shows that when deviations are left with their signs intact, positive and negative deviations cancel out resulting in a case where the variance could be highly underestimated.
vi. A comparison of D3 and D4 methods afforded students the opportunity to attend to the feature of why the reference point must be a fixed point (e.g., the mean), or else the index is sensitive to ordering of data. If the reference point for the deviation is not a fixed point, then a re-ordering of the data will result in a different value of consistency for the same formulation.
vii. A comparison between the D5 and D3/D4 methods afforded the opportunity to attend to the feature that dividing by the n helps compare samples of different sizes.

In the DI condition, all students relied *only* on the canonical formulation to solve data analysis problems. This was not surprising given that the canonical formulation is relatively easy to compute and apply, and was corroborated further with data from their homework assignments. The average performance (i.e., percentage of problems solved correctly) on the homework assignments was high, M = 92.3%, SD = 7.4%. These process findings serve as a manipulation check demonstrating that students in the PF condition experienced “failure” at least in the conventional sense, whereas DI students demonstrated successful application of the canonical formulation to solve several data analysis problems.

Results

Post-test. The six-items on the posttest comprised: a) one item on procedural fluency (calculating SD for a given data set), b) two items on data analysis (comparing means and SDs of two samples; these items were similar to the data analysis problems covered during instruction), c) two items on conceptual insight (one item dealing with sensitivity to ordering of data points, and another with outliers), and d) one item on transfer (item requiring the development of a normalized score for comparing incommensurable distributions. Note that normalization was not taught during instruction, and therefore, students needed to flexibly adapt and build upon what they had learnt.).

Maximum score for each item was 10; two raters independently scored the items using a rubric with an inter-rater reliability of .96. Performance on the four types of items formed the four dependent variables. Controlling for the effect of prior knowledge as measured by the pretest, $F(4, 134) = 1.890$, $p = .112$, a MANCOVA revealed a statistically significant multivariate effect of condition (PF vs. DI) on posttest scores, $F(4, 134) = 16.802$, $p < .001$, partial $\eta^2 = .33$. There was no significant difference between the classes *within* the PF or DI conditions, nor was there any significant interaction between prior knowledge and experimental condition.

i. On the procedural fluency item, there was no significant difference between the PF condition, $M = 7.66$, $SD = 3.97$, and the DI condition, $M = 7.98$, $SD = 3.89$, $F(1, 137) = .819$, $p = .367$.
ii. On the data analysis items, students from the PF condition, $M = 14.11$, $SD = 4.20$, significantly outperformed those from the DI condition, $M = 11.38$, $SD = 4.86$, $F(1, 137) = 10.290$, $p = .002$, partial $\eta^2 = .07$.
iii. On the conceptual insight items, students from the PF condition, $M = 16.40$, $SD = 6.41$, significantly outperformed those from the DI condition, $M = 8.20$, $SD = 6.15$, $F(1, 137) = 51.359$, $p < .001$, partial $\eta^2 = .27$.
iv. On the transfer item, students from the PF condition, $M = 4.13$, $SD = 3.47$, significantly outperformed those from the DI condition, $M = 3.17$, $SD = 2.35$, $F(1, 137) = 3.218$, $p = .075$, partial $\eta^2 = .02$.

Discussion

These findings are consistent with previous studies on productive failure with other mathematical topics and profile of students (Kapur, 2008, 2009), and also with other studies (e.g., Schwartz & Martin, 2004; Koedinger & Aleven, 2007; Van Lehn et al., 2003). Notwithstanding the limitations of what can be achieved in a single study carried out within a particular domain, context and classroom-based setting, implications arising from the findings are simple and significant: There is indeed an efficacy in having learners generate and explore representations and methods for solving problems on their own even if they do not formally know the underlying concepts needed to solve the problems, and even if such un-supported problem solving leads to failure initially. The process analysis showed that this seeming failure was integral for: a) engendering the necessary prior knowledge differentiation (evidenced in the diversity of student-generated structures), and b) drawing attention to critical features of the concept of variance (evidenced in the comparisons between the student-generated structures), which may help students better understand the concept when presented by the teacher during direct instruction subsequently (Schwartz & Bransford, 1998).

This study contributes to the ongoing debate comparing the effectiveness of direct instruction with discovery learning approaches (e.g., Kirschner et al., 2006; Klahr & Nigam, 2004); discovery learning being often epitomized as the constructivist ideal. It is perhaps worth clarifying that a commitment to a constructivist epistemology does not necessarily imply a commitment to discovery learning. Simply leaving learners to generate and explore without consolidating is unlikely to lead to learning, or at least learners cannot be expected to "discover" the canonical representations by themselves as indeed our findings suggest. Instead, a commitment to a constructivist epistemology requires that we build upon learners' prior knowledge. However, one cannot build upon prior knowledge if one does not know what this prior knowledge is in the first place. It follows that at the very least the burden on the designer (e.g., teacher, researcher) is to first understand the nature of learners' prior knowledge structures; the very structures upon which the claimed "building" will be done. Designing for productive failure presents one way of doing so.

Interestingly, one could argue that Klahr & Nigam's (2004) study supports the above contention although it is often cited as a stellar example of the superior effectiveness of direct instruction over discovery learning. A careful reading of the study suggests that before assigning students to either a direct instruction or a discovery learning condition, Klahr and Nigam conducted a baseline assessment where they asked students to design four experiments on their own. As expected, only 8 out of the 112 students were able to design four un-confounded experiments, that is, the success rates before any instruction on the control of variables strategy (CVS) were very low. Students who were subsequently assigned to the discovery learning condition simply continued to design these experiments but without any instruction on CVS or any feedback. However, for students in the direct instruction condition, the instructor modeled and contrasted the design of both confounded and un-confounded experiments with appropriate instructional facilitation and explanation to make them attend to critical features of why CVS helps isolate the effects of a factor whereas confounded experiments do not. It was not surprising therefore that Klahr and Nigam found direct instruction to be more effective than discovery learning as described earlier in this paper.

From the perspective of productive failure however, the baseline assessment in Klahr and Nigam's (2004) study seems to function very much like the generation and exploration phase where students generate their own structures (in this case, experiments) to solve a problem that targets a concept (in this case, CVS) that they had not learnt yet. If so, the very effects that Klahr and Nigam attribute to direct instruction *alone* seem more appropriately attributed to a generation and exploration phase (their baseline assessment) followed by direct instruction. Therefore, much as Klahr and Nigam set out to show, in part, that there is little efficacy in students exploring and solving problems requiring concepts they have not learnt yet, their findings can be reinterpreted to support precisely the contrasting contention that such exploration can in fact be efficacious provided some form of direct instruction follows, for without it, students may not learn much (as indeed the performance of the students in the discovery learning condition revealed).

Findings from productive failure are also consistent with Schmidt and Bjork's (1992) review of psychological science research on motor and verbal learning. They argued that under certain conditions, introducing "difficulties" during the training phase, for example, by delaying feedback or increasing task complexity, can enhance learning insofar as learners engage in processes (e.g., assembling different facts and concepts into a schema, generating and exploring the affordances of multiple representations and methods) that are germane for learning. Thus argued, designing for a certain level of failure (as opposed to minimizing it) in the initial learning phase may well be productive for learning in the longer run. Future research would do well not to (over)simplistically compare discovery learning with direct instruction, but instead understand conditions under which these approaches can complement each other productively.

Endnotes

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EARLY ELEMENTARY STUDENTS' REPRESENTATION CHOICE AND USE DURING WORD PROBLEM SOLVING

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The current study investigates early elementary children's preferences and repertoire for representing their solutions during problem solving. Participants completed a series of grade-level basic operations story problems to evaluate their problem-solving behaviors. The first- and second-grade students used external representation a high rate overall, but differences emerged in the type of external representations, both within and across grades. The differences follow a predicted shift from more concrete to more abstract external representations. Most participants, especially in the second-grade cohort, used more than one type of external representation. Potential gender effects are also considered.

Introduction

An assumption embedded within common mathematics education practices, and in both Piagetian and Vygotskian perspectives on development, is that children should be aided to move from concrete representations of quantities to more abstract (National Council of Teachers of Mathematics, 2000), especially in the early childhood and early elementary years (Ball, 1992). Learning mathematics can entail not merely *dual* representations (e.g., DeLoache, 1995), but multiple forms of mathematical expression (Gentner & Ratterman, 1991). Many representational tools and symbols are introduced and available to these novice mathematics learners, but much is yet to be understood about how children use and select these symbols. The current study investigates which representations children prefer and how they use these representations.

Offering a choice of materials (concrete counters and blocks, paper and pencil for drawings, tallies, or other written symbols, a 100 chart, and a number line) to express their ideas creates the opportunity for children to use any one of the available tools or none or to use multiple representations (e.g., Fennell & Rowan, 2001) to express their answer and for the researcher to study children's choice and use of multiple symbolic forms. The open-ended story problems used, modeled after the Cognitively Guided Instruction program (Carpenter et al., 1999) do not specify a solution strategy, facilitating examination of children's strategic thinking (e.g., Verschaffel & Decorte, 1993).

The present investigation also addressed potential gender-based differences in problem solving. Carr and Jessup (1997) suggested that first-grade girls prefer physical representation more often than boys who tend to use retrieval strategies and no external representation more often. Siegler's (1999) work implies that with a higher confidence interval for offering an answer, girls may have higher rates of correct answers.

Method

Participants.

Participants all attended the same suburban Midwestern primary school and reflected the diversity of that school. Twenty kindergartners, 22 first graders and 30 second graders participated in the sessions. The preliminary results come from the sessions for 15 first-grade students (average age at first test = 88 months; 7 boys and 8 girls) and 16 second-grade students

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(average age at first test = 98 months; 7 boys, 9 girls). All children attended the same school and received instruction with the same National Council of Teachers of Mathematics reform-based curriculum and thus could all be assumed to have some familiarity with word problems.

Procedure

A longitudinal, cross-sectional, quasi-microgenetic (e.g., Siegler, 2002) design was used. Children participated in 3 sessions: the first two approximately one month apart in Spring, with a delayed follow-up session in late autumn. Each child answered 6 grade-level word problems (Carpenter et al., 1999) per session; [See Appendix A examples]. Children participated in individual videotaped sessions. The Time 2 problems paralleled those from Time 1 except for the numbers used; the Time 2 problems were used again at Time 3 (delayed follow-up). The experimenter read each problem two times and repeated it as many times as requested. After time to solve the problem, either the child volunteered and answer or the experimenter prompted for one and then asked, “Can you tell me how you figured that out?”.

Materials

The interviewer informed the students that could use any of the provided materials and any methods of their choice. The students were asked for their answer and the method they used for each question. The materials provided by the interviewer included: 100 charts, pencil and paper, base 10 blocks, unifix cubes, a number line from 1 through 25 and wooden cubes. The students were well acquainted with the materials from their classroom experience, ruling out the possibility of novelty.

Prior to beginning the problem solving, the experimenter told the participant: “I brought some math tools that you can use if you’d like. There’s are paper and pencil and markers if you want to write anything or draw anything, there’s 100 charts, there’s a number line that goes from 0 to 25, and three types of blocks: the base-ten pieces, the unifix cubes, and the wooden cubes. You can use any of these if they help you solve the problem, but you don’t have to. If you don’t want to use any of them, that’s fine too. It’s totally up to you. Solve the problems in whatever way makes the most sense to you.” The materials were all placed within the child’s reach but the experimenter never encouraged or prompted a child to use them.

Coding

The use of external representations was coded from video, noting the use or creation of any representation during problem solving and subsequent explanation. Gesture was credited when a student counted on fingers or held up fingers to keep track of counting or during the explanation. On-paper representations were categorized as either child-created number expressions (e.g., 43-17 = 26) and drawings (including tally marks or more iconic drawings, such as a tree with birds in it) or use of the preprinted two-dimensional representations, either the number line or the 100 chart. Use of concrete manipulatives was credited for any non-play use of the blocks during problem solving.

Results

Preliminary results are reported herein on a subset of the data. Analyses of the entire set will be conducted with categorical data analysis, chi-squared tests or where appropriate, Fisher exact tests, depending on the expected frequencies.

Students' solutions

Few participants used only one type of representation during a session (first grade, x =5.67, n = 15; second grade, x = 2.33, n =16). Using Fisther exact 3x2 tests, rates did not differ by time period or grade. Additionally, this provides an index that children, especially those in the second-grade cohort, used multiple external representations rather than favoring one representational type.

Examining students' problem-solving actions over the set of problems completed per session reveals a high rate of use of external representations. Note that the first-grade set included 6 problems for 15 children at each time point (and thus 90 problems), while the second grade included 6 problems for 16 children at each time point (and thus 96 problems).

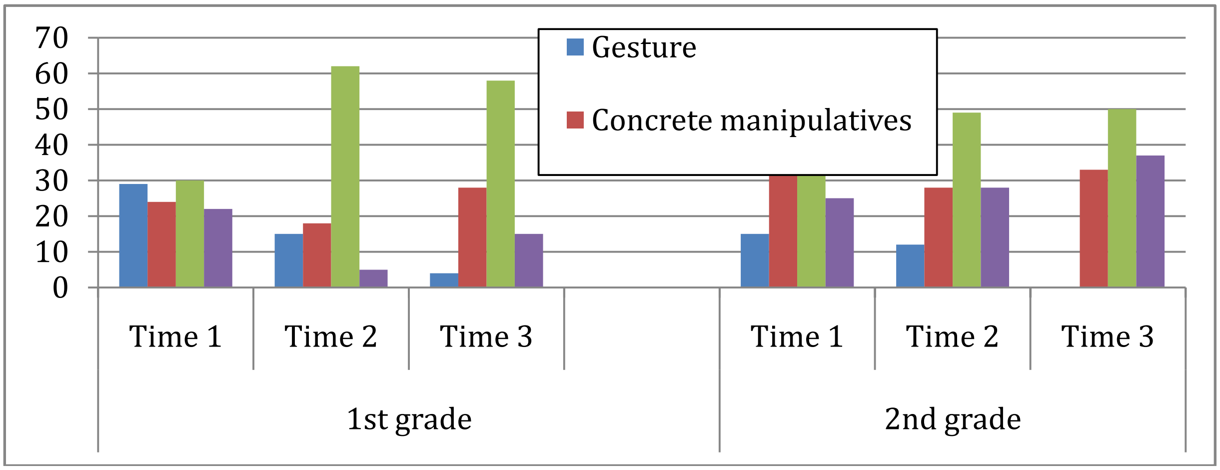

Figure 1. The rate of students' use of external representations during problem solving and explanations

Every child used or created a symbol on at least one of the problems in each session (See Appendix B for sample comments regarding representations). Participants in both grades decreased their use of gesture over the sessions and significantly increased their rate of drawings or written equations. Second-grade participants increased their use of pre-printed 2D materials, particularly the 100 chart, perhaps stemming from increased classroom experiences with this form of number sequence representation for arithmetic problem solving.

The next analysis examined the frequency of students' use of multiple external representations on any one problem. Significantly more 2nd graders used multiple external representations for any single problem than 1st graders (Figure 2). Note that although 4 children used multiple external representations on at least one problem at Time 1, Time 2, and Time 3, these were not necessarily the same children. Across the three sessions, 8 different students used more than one external representation on a single problem. Similarly, all but 2 of the 2nd graders used a combination of external representations on at least 1 problem. This result provides evidence that this cohort of 2nd graders may have had more confidence and fluency in coordinating multiple external representations.

Examining these results together, a subset of children from each grade could be considered consistently eclectic in their use of external representations, using multiple forms across the 6 problems for a session, but many of their peers used multiple external representations only occasionally, raising the question of whether their use of multiple representations was situational (e.g., induced by a struggle on a particular problem), rather than a characteristic preference for the child.

Lastly, on the analyzed subset of data, the observed gender-based trends were in the predicted direction: Boys tended to use counting and retrieval strategies more often than girls, and girls used more direct modeling strategies, on average, than boys. These trends will be further explored with the full data set.

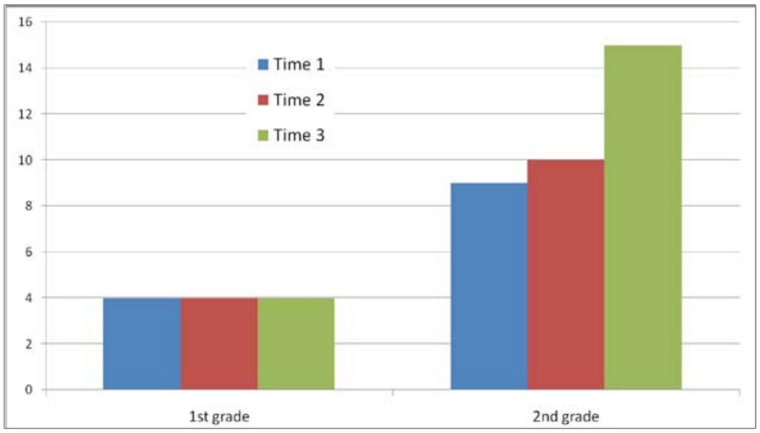

Figure 2. The number of students who used more than one external representation type during their problem solving

Discussion

The children spontaneously opted to use external representations, but displayed differences in the type and frequency of their use, and changes from session to session in favor of creating two-dimensional representations and referring to manipulatives less. These results recall a hypothesized evolution of children's symbol use in mathematics from concrete to abstract (e.g., Carpenter et al., 1999). Given that signs and symbols may mediate learning (e.g., Uttal, 2000), this investigation addresses children's preferences for symbolizing quantities and has potential implications for instructional interventions.

The use of concrete objects to stand for quantities does not guarantee that a child will identify the intended meaning of these tools (e.g., Baroody, 1989, Clements, 1999). Given the widespread use of concrete objects as problem solving and problem representation tools, it is important to document how individual learners understand and use these objects. Then, even given successful use of manipulatives for problem solving, the mapping to more abstract representations, such as writing an equation, cannot be assumed to be an easy or transparent process for a young learner. As noted by Uttal, Scudder, and DeLoache (1997) "Connections must be established for both numerical symbols and operation symbols. (Wearne & Hiebert, 1988, p. 372)" (p. 48). Young learners' mappings or translations between representations must be supported.

Educational Implications

Teachers or other adults cannot take the intended symbol mappings as transparent for the children (e.g., DeLoache, 1995; Hughes, 1986), and by observing their representation choices can gain insight into which external representations students find the most representationally clear. Teachers should offer explicit scaffolding in the use of external representations to increase the likelihood that the representation will be used as intended (e.g., Uttal, Scudder, & DeLoache, 1999). Educators' observation and assessment of learners' preferences for types of external representations may facilitate the scaffolding of their use of other types and recognition of the similarity in intended referents of that other representation type. In particular, when moving from concrete to abstract, learners need to be scaffolded to see the relations between elements such that, for example, the ten base-ten block maps onto a line drawn on paper while the unit piece maps onto a dot drawn on paper. Understanding how students prefer to represent their thinking during solutions and explanations can enable educators to support the learning of the multitude of symbols used in mathematics.

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Appendix A: Example problems

Kindergarten:
Shelly had 9 jellybeans. Some are red and some are blue. 4 of the jellybeans are blue. How many jellybeans are red?
1st grade: *David has 3 bags of candies. There are 4 candies in each bag. How many candies does David have altogether?*
2nd grade: *Ms. Baker's class has 29 students. 15 of the students are boys. How many are girls?*

Appendix B: Examples of Children's Spontaneous Comments about External Representations

- 2nd grader EY: "This is how we learned to do it in class. A line stands for ten and a dot stands for one and that way we don't have to draw them out like the base-ten blocks (…) Sometimes it's hard to keep track when I trade a ten for ones."
- 1st grader CD on using the 100 Chart, "Ugh, I forget which number I started on."
- 2nd grader JM: "I don't use any of that stuff. I just do it in my head." but then he went on to use both written and concrete representations on some problems. On the ones that he solved only in his head, he had some difficulty keeping track of his numbers.
- 2nd grader AO on the base-ten blocks: "These are easier than those (the unifix snapcubes) because those you have to count by one, but these you can just take 1 piece and say "10"."
- 1st grader NC: "I think I'm going to use these (the unifix connecting cubes) cubes right here because they're easiest to work with."
- 1st grader BH, working with paper and pencil: "This is hard." E: Can you think of a different way to figure it out? BH: Yes! (takes out unifix cubes) (...) I knew it was wrong and I thought the cubes might help.

IS IT POSSIBLE TO IMPROVE THE STUDENTS' CRITICAL THINKING DISPOSITIONS THROUGH TEACHING A COURSE IN PROBABILITY?

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The purpose of this initial study was to explore whether teaching our specially designed learning unit would enhance the students' critical thinking dispositions. The unit "Probability in Daily Life" was taught to a group of tenth-grade students, with the purpose of encouraging critical thinking dispositions such as open-mindedness, truth-seeking, self-confidence and maturity. The teacher encouraged class discussion and planned investigative lessons. The students completed a pre and post CCTDI test. A minor improvement was detected, but we believe that these initial results are the first step in showing that it is possible to train students' critical thinking dispositions.

Introduction

Our ever-changing and challenging world requires students, our future citizens, to go beyond the building of their knowledge; they need to develop their higher-order thinking skills, such as system critical thinking, decision making, and problem solving (Profetto-McGrath, 2003; Riddell, 2007; Sezer, 2008). There have been significant changes in the past decades in the field of education. Whereas earlier the teacher was at the center and the emphasis was put on *what* to teach, today's education involves teaching *how* to think, and in particular, how to be a critical thinker. Critical thinking is necessary in every profession, and it allows one to deal with reality in a reasonable and independent manner (Harpaz, 1996,1997; Lipman, 1991, McPeck, 1994).

Critical thinking has been investigated largely in terms of thinking skills that involve the cognitive domain. For decades, promotion of students' thinking has been the focus of educational studies and programs (Facione, 1996; Facione & Facione, 2000). Each of these programs has its own definition of thinking and/or of skills. Some use the phrase 'cognitive skills' while others refer to 'thinking skills' (Resnick, 1987; Zohar & Dori, 2003), but they all distinguish between higher- and lower-order skills. Resnick (1987) maintained that thinking skills resist precise forms of definition; yet, higher order thinking skills can be recognized when they occur.

Thus, there seems to be no clear consensus as to what exactly critical thinking is. Some see it as simply "everyday, informal reasoning" (Halpern, 1998), whereas others feel differently. Lipman considers it to be different from ordinary thinking because it is both more precise and more rigorous, as well as self-correcting. It has also been described by Halpern (1996) as being 'purposeful, reasoned, and goal-directed'. There are taxonomies which set out a list of the reasoning skills involved in critical thinking: for instance, Ennis' taxonomy proposes twelve skills according to (Ennis, 1987).

Nonetheless, it seems evident at this point that the ability to think critically is not something that we are born with but a learned ability that needs to be taught. Our research is underpinned by several questions raised by Passmore (1980): what do we mean by teaching a student to become a critical thinker? How can this be accomplished successfully? Does it merely involve giving the necessary facts or rather nurturing the student's disposition, molding the personality, or is something else involved? Is being a critical thinker a matter of habit? Passmore does not have answers to all these questions, but in his article he does claim that in the process of developing critical thinking we have to stress the student's “natural disposition.”

Facione and Facione (1994, 2000) propose a taxonomy of dispositions that includes such elements as cognitive maturity, searching for truth, open-mindedness, systematicity, analyticity, self-confidence, and curiosity. They developed the California Critical Thinking Disposition Inventory (CCTDI) which was originally intended to be used to measure critical thinking dispositions in college students but has been successfully adapted to be used in high school. Another question then arises as to whether dispositions are something that can be changed or improved. Through our research, we hope to shed some light on these questions.

Methodology

The pilot study reported here was conducted over the period of one academic year in the framework of a larger PhD research and involved one group (n=30) of tenth-grade students in regular high school. The group consisted of thirty students, all high achievers, aged fifteen to sixteen. The teacher was one of the researchers.

The learning unit “Probability in Daily Life” (Lieberman & Tversky, 1996) comprised fifteen lessons, each lasting ninety minutes, and was modified by the researchers to combine critical thinking skills with the probability topics it initially contained (Aizikovitsh & Amit, 2008). This unit in probability studies, which is part of the formal mathematics curriculum of the Ministry of Education was chosen because of its rationale of "*learn[ing] issues relevant to the daily life, which include elements of critical thinking*” (Liberman & Tversky, 2001, p.3). In the unit the student is required to analyze problems, raise questions and think critically about the data and the information.

The purpose of the unit is to teach the students not to be satisfied with a numerical answer but to examine the data and their validity. In cases where there is no single numerical answer, the students are required to know what questions to ask and how to analyze the problem qualitatively, not only quantitatively. Along with being provided with statistical instruments, students are redirected to their intuitive mechanisms to help them estimate probabilities in daily life. Simultaneously, students examine the logical premises of these intuitions, along with misjudgements of their application. This unit examines the term “probability” regarding everyday problems. The uniqueness of the original Lieberman and Tversky’s unit lies in the fact that it allows one to learn interesting subjects of everyday relevance through mathematics. This involves critical thinking elements such as: tangible examples from everyday life, questioning the reliability of information, accepting and dismissing generalizations, rechecking quantitative data, doubting, etc.

The Uniqueness of the “Probability in Daily Life” Learning Unit Modified for Infusion Approach Teaching

The main characteristic that distinguishes the new (intervention) learning unit "Development of Critical Thinking by Means of Probability in Daily Life" modified by the researchers (Aizikovitsh & Amit, 2009) for infusion approach teaching (Swartz, 1992) from the original one is the innovative, ‘different’ teaching method that lies at the basis of this unit. Teaching in the infusion approach necessitates a different structure of teacher-student interaction in class that diverges from the traditional one used in most ‘traditional’ math lessons. This method is known in literature as "dialogical teaching" or "negotiating knowledge" and is characterized by classroom discussions between the teacher and the students and among the students themselves during the lessons, so that the teacher won't be the only person speaking in class (Nystrand, 1997; Pape, Bel, & Yetkin, 2003).

The teacher's role is to encourage the students to make changes in their systems of perceptions and concepts, to convince them that such changes should be made, and to help the students to make these changes (in particular, by allowing the students to talk the topic over among themselves and discuss it together in small groups, in a less ‘threatening’ form than doing it in the larger forum of the whole class).

To fulfill this function, the teacher can use various teaching methods at his/her disposal: oral explanation, texts, experiments, demonstrations, videos, computer programs, the students' own work, group discussions, etc. The method of negotiating knowledge in the classroom emphasizes that the use of any teaching method or tool must be accompanied by dialog between the teacher and the class, and among the students themselves, i.e. by classroom and group discussions. Such discussions may have different purposes (according to specific situations). During such discussions the students may, for example, express their opinions about the topic currently studied, present the insights they acquired as a result of different learning and teaching strategies, ask questions, make comments, argue about interpretations, and so on. It is important to emphasize that the main characteristic of these sessions is a meaningful, authentic dialog where the students feel free to express their original thoughts instead of the ideas the teacher expects them to learn. We believe that this teaching method creates in class an optimal atmosphere in which the students are most likely to follow their dispositions towards critical thinking and thus there is a chance that these dispositions will be strengthened and the students will learn to apply them in a more consistent, aware and systematic way.

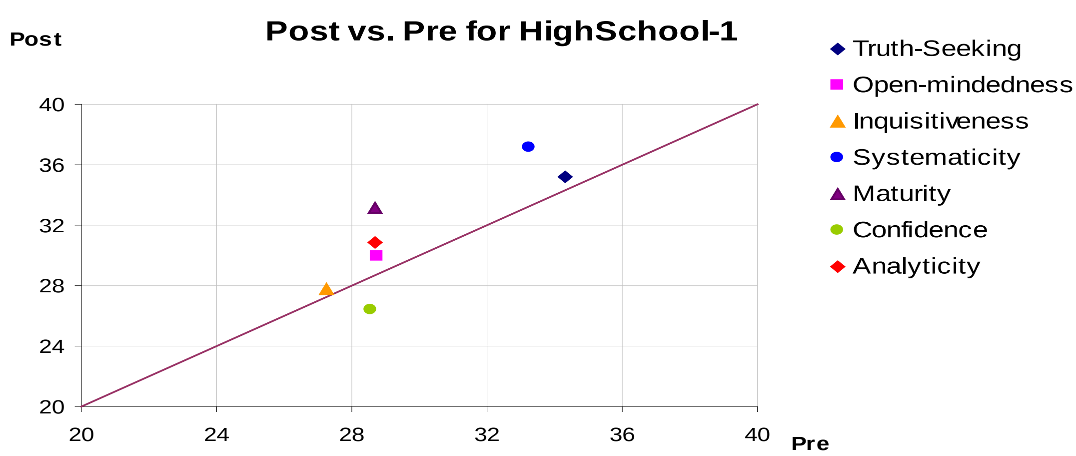
The California Critical Thinking Dispositions Inventory (CCTDI)

CCTDI is a Likert test based on the seven positive aspects of the disposition for critical thinking. It was designed to measure general dispositions profile of the students. CCTDI is divided into seven sub-tests: truth-seeking (sub-test T), intellectual openness (sub-test O), analyticity (sub-test A), systematicity (sub-test S), self-confidence in critical thinking (sub-test C), inquisitiveness (sub-test I) and maturity (sub-test M). The following descriptions of the sub-questionnaires are based on Facione et al. (1996) and Facione (2000). The sub-test T deals with the inclination to investigate in order to arrive at the fullest and most adequate information possible in a given context, raising questions in a courageous way, as well as honesty and objectivity in searching for information, even if this information goes against the investigator’s personal interests or opinions. For instance, those who incline for truth-seeking will not agree with the following claim: “All people, including myself, always present claims that follow from their personal interest,” or, “If there are four reasons ‘pro’ and one ‘contra’, I will be in favor of the four.” The sub-test O deals with the inclination for intellectual openness and tolerance towards other opinions, while remaining aware of the fact that one’s own opinions are different. People who are not tolerant towards views different from their own will agree with the claim “It’s important for me to understand what other people think on various issues.” The sub-test A focuses on application of cause and proof, awareness of problematic situations and an inclination to predict outcomes. For instance, students are asked to answer “agree” or “disagree” to the following statement: “People need reasons for disagreeing with someone else’s opinions.” The sub-test S checks organization, order, focus, and commitment to investigation, and uses test statements of the following kind: “My opinion on controversial issues depends mostly on the last person I have had a conversation with.” The sub-test C measures the person’s confidence in his/her own thinking process, and uses test statements such as “I do better in exams that demand thinking, not only memorizing," or "I take pride in my ability to understand other people's opinions." The sub-test I measures the keenness to acquire knowledge and find explanations, even when this knowledge does not seem immediately applicable. Representative test-statements are: "No matter what the topic, I am keen to learn more about it," or "Learn all you can, you never know when you may need to use this knowledge." Finally, the sub-test M measures the person's inclination to be critical about his/her own decision-making. A mature person who thinks critically can be defined as a person who approaches problems inquisitively and makes decisions while knowing that some problems are inherently poorly constructed, and others have multiple solutions. The CCTDI total score is a measure that estimates one's overall disposition toward critical thinking. A person may be positively and strongly disposed toward seeking to solve problems and address questions using reflective judgment, that is, critical thinking; alternatively, they may be ambivalent toward that, or even negatively disposed and hostile toward that approach. The total score is based on all 75 items. Facione (2000) report correlations that support the simultaneous validity between scores in CCTDI sub-tests and psychological tests.

Results

Chart 1 represents the post vs. pre average CCTDI test sub-scale scores for "High School 1" (the exact t-test values are presented in full in Table 1). Both the chart and the table reveal that in the first round there was a significant improvement in the sub-scales of systematicity, maturity and analyticity, whereas there was no improvement in the other sub-scales. Points on the diagonal represent cases where the pre and post scores were equal. Consequently, the area above the diagonal is where an improvement was observed. Table 1 of the paired t-test results reveals that in the first iteration there was a significant improvement in maturity, whereas there was no improvement in others parameters.

Chart 1. Dispositions for Critical Thinking in the High-school Group



Findings of Interviews with Students

The following two examples are representative of many more similar answers. One student replied as follows to our question regarding the importance and practical function of critical thinking: "I think CT is important when you study mathematics, when you study other topics and when you read the paper, but it is most important when you deal with real life situations, and you need the right instruments in order to do so (deal with these situations)." Another student was asked about applying important components of critical thinking that were capitalized on during the last few classes, and she answered: “First we should check the information source’s reliability and despite all the numerical data, I don’t accept the researcher’s conclusion." Additional data, consistent with these two examples, suggest that infusion of CT into the formal curriculum in mathematics can sharpen the students’ critical thinking dispositions and make them more aware of the nature and importance of these dispositions and their applicability to a wider range of disciplines as well as daily life situations.

Table 1. Disposition Towards Critical Thinking in the High-school

Sub-scale	***Pre-test***		***Post-test***		t value
N=30	*Mean*	*SD*	*Mean*	*SD*	
Truth-seeking	34.30	6.10	35.20	4.36	0.71
Open-mindedness	28.77	5.68	29.97	4.77	1.08
Inquisitiveness	27.27	7.31	27.73	4.71	0.33
Systematic	33.23	7.15	37.10	5.14	2.47(*)
Maturity	28.70	7.10	33.10	5.83	2.43(*)
Confidence	28.57	7.42	26.40	5.62	-1.51
Analyticity	28.70	5.35	30.83	3.14	2.72(*)
CCTDI Total	29.93	4.28	31.48	1.81	**1.90**
(*)=difference significant at the .05 level					

Discussion

In much of the literature on critical thinking we see that there is no significant improvement in all the sub-tests during one year of learning (Barak & Dori, 2009; Ben-Chaim, Ron, & Zoller, 2000). In the sub-tests of our group we did see a marked improvement in systematicity, analyticity, and maturity. On re-examining these sub-tests we noticed that they consisted of a group of questions involving a certain experiment where all the information was presented in the form of tables. During this learning unit we repeatedly worked with tables and we therefore came to the conclusion that the familiarity of the students with the tables enabled them to deal with the statements made and answer these specific questions more easily. The improvement in maturity was more difficult to analyze. At this stage of our research we feel that we don't have sufficient data to explain why there was an improvement at all, or why. Also, the deterioration that occurred in confidence still remains to be explained.

As this paper was being written, the study was expanded to six additional groups. We are eagerly waiting to see the results of these groups which will hopefully shed light on our findings presented here. The interviews that we conducted with the students also gave us information as to how the unit had affected the students. We saw that the students' approach to solving problems started to change. While in the past they would often accept data given to them at face value, now they started questioning their validity. Earlier they would be satisfied with manipulating numbers and arriving at "the answer", whereas now they began searching for other possible answers. The students started thinking about critical thinking.

Conclusion and Implementation

The pilot-scale research described here constitutes a small step in the direction of developing additional learning units within the traditional curriculum. It is apparent that if a teacher makes a decision to focus on improving higher order thinking and perseveres over time, the chances are good that the teacher will succeed. This study showed that it is possible to incorporate into regular schools activities that will develop the students' critical thinking dispositions. The subject matter was part of the high-school curriculum; therefore it does not take time away from the regular syllabus. It also does not take the teachers extra time or effort in order to prepare the unit. What is most essential is the teacher's understanding of the importance of developing the critical thinking abilities in their students.

The general educational implications of this research suggest that we can and should leverage the intellectual development of the student beyond the technical content of the course, which can be done by creating learning environments that foster critical thinking, and which will, in turn, encourage the student to investigate the issue at hand, evaluate the information and react to it as a critical thinker. It is important to note that, in addition to improving the dispositions discussed above, the students also gained intellectual skills such as conceptual thinking during this course and developed a class culture (climate) that fostered critical thinking. To conclude, this research shows the possibility of working in the direction of developing critical thinking dispositions in the framework of the established contents of the mathematics curriculum.

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THE DEVELOPMENT OF NOTATIONAL SYSTEMS—A CREATIVE ENDEAVOR

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The role of symbol systems in learning mathematics has received a significant amount of attention recently in mathematics education. However, relatively few studies have traced out how individual student's systems of notation evolve across teaching sequences. The purpose of this paper is to trace out the evolution of an eighth grade students system of notation as he solved combinatorics problems. The problems were part of a sequence of teaching episodes designed to help students develop a formula for the sum of the first n whole numbers. The teaching episodes were part of a three-year teaching experiment with middle grades students.

Introduction

In 1991 Kaput noted that the following question had been under researched in mathematics education: "How do material notations and mental constructions interact to produce new constructions (p. 55)?" Meira (1995) echoed Kaput's sentiments stating that, "not many studies in the mathematics education literature treat children's production of material representations (e.g., on paper) as the central issue in any significant way" (p. 270). Since that time, a small number of studies have tracked the evolution of students systems of notation in the context of their problem solving activity (e.g., Brizuela, 2004; Izsak, 2000; Lehrer, Carpenter, Schuable, & Penner, 2000; Meira, 1995). These studies have contributed to researcher's understanding of how students learn to produce and use systems of notation in powerful ways, but the issue remains a significant and under researched problem in mathematics education (Yackel, 2000). The purpose of this paper is to trace out the evolution of one eighth grade student's system of notation for solving combinatorics problems. In doing so, I demonstrate how a student's system of notation became progressively more advanced in the context of his problem solving activity. The following research question guided the study:

(1) How do students develop and refine systems of written notation, and in this process how do their systems of notation become more advanced?

Theoretical Framework

Elsewhere I have situated students' symbolizing activity in a radical constructivist framework for communication (see Thompson, 1999 for the communication framework), and I have examined how symbolizing activity functions in student-teacher communication aimed at learning (Tillema, 2010). In doing so, I defined symbolizing activity as any type of material component that a person produces during the functioning of his or her schemes (cf. Ernest, 2006; Hoffman, 2005; Kaput, Blanton & Moreno, 2008; Nemirovsky & Monk, 2000; Radford, Bardini, & Sabena 2007; Sfard, 2007).

For example, consider the following problem:

> *The Outfits Problem:* You have three shirts and four pairs of pants. An outfit consists of one shirt and one pair of pants. How many outfits can you make?

To solve the Outfits Problem a student might label the shirts “1”, “2”, and “3”, and label the pairs of pants “A”, “B”, “C”, and “D”, and say aloud, “there is A1, A2, and A3; there is B1, B2, and B3” etc., while writing “A1, A2, A3, B1, B2, B3”, etc. In this case, the student’s symbolizing activity would include both the verbal expression and notation. More generally, symbolizing activity includes verbalizing, notating, as well as gesturing, and I consider symbolizing activity to be an expression of the schemes (and mental operations that make up these schemes) that a person uses to constitute experiential situations.

For the purposes of this paper, I focus my discussion exclusively on the evolution of a single student’s notating activity. Throughout the paper, I use the terms *notation* and *notating activity* broadly to mean any type of graphic record that a student produces during the functioning of his or her schemes (cf. Kaput, Blanton, & Moreno, 2008, p. 29; Lehrer, Schauble, Carpenter, & Penner, 2000).

Methods and Methodology

The data presented in this paper is from a three-year constructivist teaching experiment (Confrey & LaChance, 2000; Steffe, & Thompson, 2000) with middle grades students, which was conducted in North Georgia from October 2003 to May 2006. In the experiment, four pairs of students were taught during their sixth, seventh, and eighth grade years twice weekly in 30-minute episodes for two to three weeks, followed by a week off. I and one other researcher formed the core of the research team for the three-year experiment, with other researchers joining for shorter time periods. All teaching episodes were videotaped with two cameras, one focused on the students’ work, and one focused on the interaction. The two video files were then mixed into a single video file for analysis. In the rest of this section, I provide details about the student participants in the experiment, my procedures for data analysis, and the design of the teaching episodes reported on in the data analysis.

Participants

In September and October of 2003 the research team conducted 20-minute selection interviews with 20 sixth-grade students out of a pool of approximately 100 students, all of who had the same classroom mathematics teacher. A member of the research team observed four of this teacher’s five mathematics classes, and she consulted with this teacher to identify students to interview. The intention of the research team was to select three pairs of students who were each at different levels of multiplicative reasoning, and one pair of students who were pre-multiplicative reasoners (see Hackenberg & Tillema, 2009 for information about the different levels of multiplicative reasoning). During the selection interviews, we used whole number and fraction tasks to make initial assessments of students’ current multiplicative concepts. Upon completion of the interviews, we consulted again with the teacher, the school counselor, and the principal in order to select students who attended school regularly and were likely to remain in the district for the duration of the experiment. This article reports on one student during the first four teaching episodes of his eighth grade year. During the selection interviews, he exhibited the second level of multiplicative reasoning, and during his eighth grade year he was in an Algebra class—the highest course offered at the school for eighth grade students.

Analytic Procedures

In teaching experiment methodology, data analysis occurs in two phases—ongoing and retrospective analysis (Steffe & Thompson, 2000). The purpose of ongoing analysis is to develop working models of students' reasoning, and to plan tasks for each teaching episode. During ongoing analysis, the research team met between each teaching episode. I documented these meetings, which included a written record of the problems presented to students, changes to the problems presented, and reasons for these modifications as well as my initial impressions of the students' reasoning about and notation for problems.

The purpose of retrospective analysis is to create second order models of students' ways of operating and changes to these ways of operating (Steffe & Thompson, 2000). A second order model is a researcher's constellation of constructs to account for another person's way of operating, which in this study focused on a student's notating activity. Constructing second-order models involved me in repeated viewing of video files, creating notes, and then notes on notes (Cobb & Gravemeijer, 2008), in order to identify moments that captured (1) regularities in the student's system of notation, and (2) changes to this system of notation.

Design of the Teaching Episodes

The four teaching episodes reported on in this study was designed to investigate how two students would use combinatorics problems to construct and evaluate sums of whole numbers, and in this context how they would develop a system of notation that accounted for ordering outcomes. I planned to begin the set of four teaching episodes with the Outfits Problem, mentioned earlier. The goal of posing this problem was to see how students would solve and create notation for a "basic" combinatorics problem that did not involve ordering. I planned to introduce the students to three different ways of notating the solution to this problem—using lists (e.g., A1, A2, etc.), tree diagrams, and arrays. However, rather than present students with these ways of notating the problem, I planned to ask the students to solve the problem without giving specific direction about how to notate their solutions. Once the students solved the problem I planned to introduce them to other ways of notating their solutions.

I planned to continue to pose problems similar to the Outfits Problem during the first two teaching episodes, and during this time to ask the students to explain and interpret each other's notating activity as well as to use multiple types of notation for their solution of a single problem. The purpose, then, of the first two episodes was to familiarize students with the three ways of notating problems like the Outfits Problem.

During the third teaching episode, I planned to pose the Flag Problem.

> *The Flag Problem*: You are the President of a new country. You need to design a flag that has two stripes. You have 15 colors to choose from. How many possible flags could you make?

The Flag Problem differs from the problems from the first two episodes in two ways: (1) the statement of the problem only refers to one type of quantity (i.e., colors) that students could use to make flags, as opposed to two distinct types of quantities (e.g., shirts and pants) that students could use to make outfits; (2) students' solution of the Flag Problem holds the potential for them to establish an order for the stripes (e.g., a top and bottom stripe).

During the fourth teaching episode, I planned to present the Handshake Problem.

> *The Handshake Problem:* There are 10 people in a room. Each person wants to shake hands with all of the other people. How many handshakes would there be?

I expected that the Flag Problem would provide a foundation for solving the Handshake Problem because the Flag Problem held the potential for students to order the outcomes. I anticipated students might solve the Handshake Problem by ordering the handshakes, which would enable them to count each handshake twice (e.g., count both the first person shaking the second person's hand and the second person shaking the first person's hand), and then eliminate the duplicate handshakes. Neither student in the study solved the Flag or Handshake Problem precisely as I had anticipated, but in solving the problems they did create a system of notation that could account for ordered outcomes. I examine how one of the students did so in the data analysis.

Data Analysis

Michael's Notating Activity for the Outfits Problem

On 10/17/05, I presented Michael and his partner with the Outfits Problem. Initially, Michael solved this problem using tally marks (Figure 1a). His partner created a picture of three shirts and four pairs of pants, labeling the shirts "1", "2", and "3", and the pairs of pants "A", "B", "C", and "D". His partner then listed all of the outfits "A1", "A2", "A3", etc. When both students were done with their solutions, I asked them to explain to the other what they had done. After both did so, I requested that Michael use notation similar to his partner's notation to solve the problem. In response, Michael produced Figure 1b and 1c. The notation Michael produced in Figure 1a and 1b was representative of the type of notation he produced throughout the first two episodes for problems like the Outfits Problem. During these episodes, he did use tree diagrams and arrays, but only at my request, and he specifically stated that he did not fully understand arrays despite his partner's explanation of them.

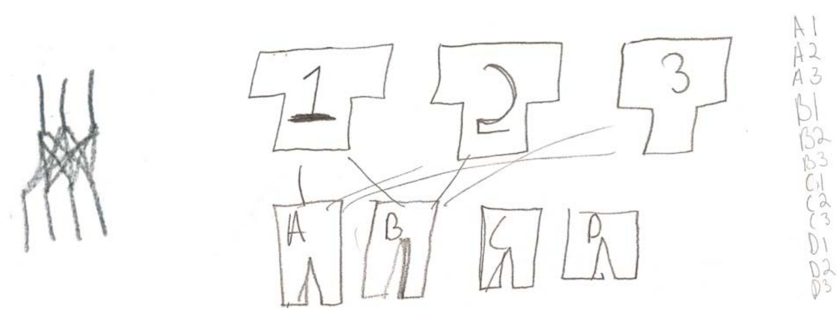

Figure 1a (left), Figure 1b (middle), & 1c (right) Michael's notation

An Analysis of Michael's Notating Activity

Michael's use of tally marks did not create a clear record of the result of putting a shirt with a pair of pants to make an outfit. Rather, the outfits were recorded each time he connected tally mark in the bottom row with a tally mark in the top row, leaving a jumble of lines between the two rows of tally marks. For this reason I requested that Michael use notation similar to his partner (Figure 1b & 1c). Using this notation entailed recording each outfit in a separate location, Figure 1c, and thus his notating activity in Figure 1c clearly recorded the outfits. It also entailed him in differentiating the two types of quantities (shirts and pants) in the problem by using a different type of character (letters and digits) for each. Using the letters and digits to differentiate between the two types of quantities was an essential first step in creating a system of notation that could account for order. That is, using letters and digits opened the possibility for him to use one type of character to denote the first element in an ordered pair and the other type of character to denote the second element in an ordered pair where the difference in character type might help him to distinguish between the meaning of the first and second character.

Michael's Notating Activity for the Flag Problem

During the third teaching episode, I presented Michael and his partner with the Flag Problem. In order to orient them to the Flag Problem, I asked that they each make two-striped flags in The Geometer's Sketchpad (GSP). During this time, the boys discussed whether to count a flag that had two stripes that were the same color (e.g., red in the top stripe and red in the bottom stripe), and the boys created flags in GSP where they used red in the top stripe (e.g., red in the top stripe and orange in the bottom stripe), and used red in the bottom stripe (e.g., black in the top stripe and red in the bottom stripe). However, we did not have an explicit discussion about the difference between, for example, a red-orange and an orange-red flag.

After the boys made six representative two-striped flags in GSP, I asked each boy to determine all of the possible flags they could make. Michael replied, "I'll do what I do", began by making a list of all of the colors. Then at my suggestion he associated a number with each color (e.g., writing "Red = 1", etc.), and recorded flags as shown in Figure 2. After he was done recording flags, I asked why he began his second column of notation with "2-2" instead of "2-1." He replied, "you have one to two, but you can't do two to one again because it's already used."

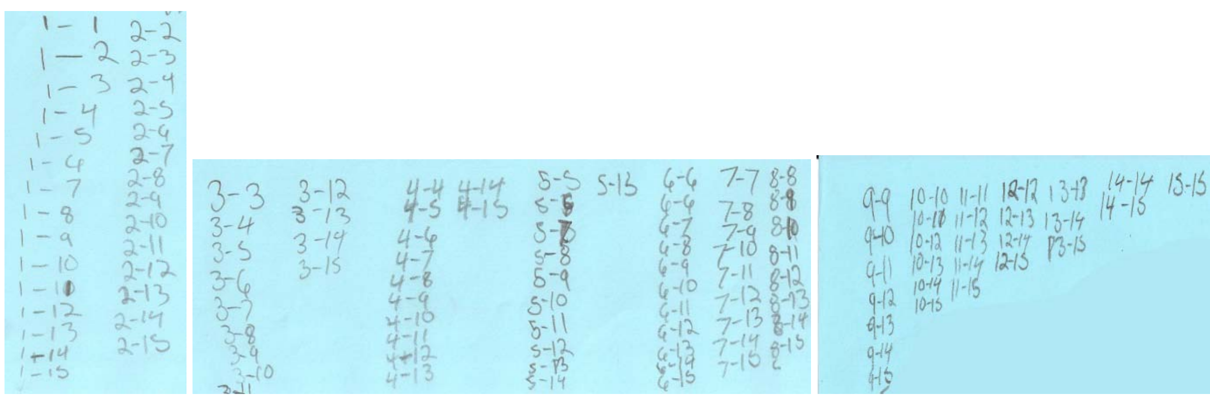

Figure 2. Michael's notation for the flags

An Analysis of Michael's Notating Activity

Michael's notation for this problem differed from his notation in the Outfits Problem—he listed the possible flags using two digits (e.g., "1-1") as opposed to a letter and a digit (e.g., "A1"). This modification in his system of notation indicates that he considered that the outcomes were produced from a single type of quantity (colors), as opposed to two different types of quantities (e.g., the shirts and pants). Considering the outcomes to be produced from the same type of quantity is an essential precursor to establishing a system of notation that includes ordered outcomes. That is, there is no impetus to order outcomes when there are two different types of quantities: Shirt A paired with pants 1 is the same as pants 1 paired with shirt A precisely because the two quantities being paired cannot be considered the same. In contrast, when a person creates outcomes from a single quantity, the question of whether color 1 paired with color 2 is different from color 2 paired with color 1 arises because the quantity used to create these pairings is the same, and so a person may use some other relevant feature of the problem to distinguish between these outcomes (e.g., by position, as in a top and bottom stripe).

In his solution, however, Michael did not seem to consider the difference between the flag notated as “1-2”, and the flag notated as “2-1”. Instead, his comment “you have one to two, but you can’t do two to one again because it’s already used” indicates that he seemed to consider the flag “2-1” and the flag “1-2” to be identical. So although Michael considered the flags to be produced by a single type of quantity, he had not yet differentiated between two outcomes based on the position of the stripe. Nonetheless, he had modified his system of notation for solving combinatorics problems—he used the same type of character (i.e., digits) to denote each stripe of the flag because he considered the creation of a flag to entail only a single type of quantity—an important step towards creating a system of notation that accounted for ordering.

Michael’s Notating Activity for the Handshake Problem

During the next teaching episode on 11/09/05, I presented Michael and his partner the Handshake Problem.

Excerpt 1: Michael’s notating activity during the Handshake Problem

M [writes “10 people” on his paper]: So its one A [Michael writes “1A”. He stops writing and looks up at the teacher.] A can’t shake his own hand, can he? [Michael clasps his left and right hand and moves them up and down as if to shake his own hand.]

T: No, A can’t shake his own hand.

M [Michael erases the “A” and replaces it with a “B”. He continues his notation writing “1C, 1D, 1E, 1F, 1G, 1H, 1I, 1J”. Michael begins a second column of notation writing “2A, 2” (Figure 3a). He stops writing and examines his notation for 10 seconds.]: No, wait. B is person 2. [Michael proceeds to write subscripts after each of the letters in his notation. After he writes “2A_1” (Figure 3b), he says to himself]: But didn’t they [person 2 and person A] already shake hands? Yes. [Michael erases “A_1”, replacing it with “B_2”.] Noooo. [Michael erases “B_2”, replacing it with “C_3”.] I could have just wrote two three, two four, two five [i.e., “2-3”, “2-4”, “2-5”]. Oh well. [Michael continues until he has produced Figure 3c].

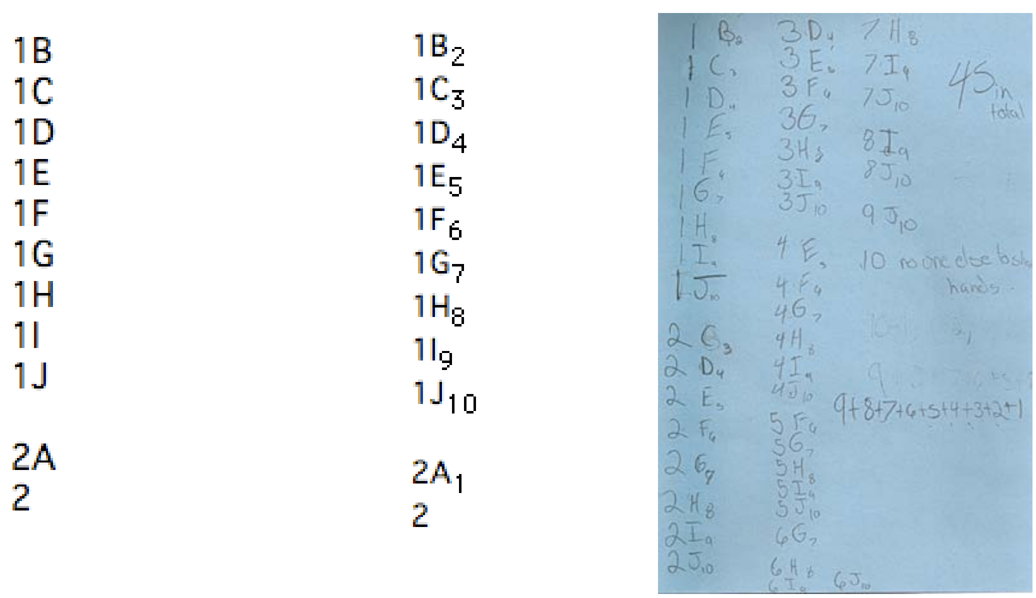

Figure 3a (left), 3b (middle), & 3c (right). Michael’s notation for the Handshake Problem

Analysis of Michael's Notating Activity

The statement of the Handshake Problem, like the Flag Problem, referred to only one type of quantity, ten people. However, Michael did not notate the Handshake Problem as he had notated the Flag Problem. Instead, he used one letter and one number to notate handshakes, which was similar to his notation for the Outfits Problem. Choosing to use one letter and one number provides indication that, at least initially, Michael was differentiating between what was symbolized by the letter and number—one symbolized a person who was doing the handshaking (the "handshaker"), and the other a person who was having his hand shaken (the "handshakee"). After Michael wrote down "1A" for the first handshake, he stopped and asked the teacher whether a person could shake his own hand—indicating that he considered person A and person 1 to be the same person, despite his initial differentiation between "handshakers" and "handshakees". After the teacher clarified that self-handshakes were not legitimate, Michael erased "1A" replaced it with "1B", and continued with his notation (Figure 3a).

Prior to listing the handshake "2B", which would have also symbolized a self-handshake, he stopped listing the handshakes, and said to himself, "No, wait. B is person 2." He, then, returned to the top of his list of handshakes, and used subscripts to identify which letters and digits symbolized the same person (i.e., he wrote "$1B_2$" to show that person B and person 2 were the same person). He continued to write subscripts after each letter, and after writing "$2A_1$", he stopped and said aloud, "But didn't they already shake hands? Yes." Michael's original intention of writing the subscripts was so that he could more easily identify and eliminate self-handshakes. However, by creating the subscripts he also identified handshakes that he had already counted, and eliminated these handshakes as well. Doing so meant that in the process of creating his notation for the Handshake Problem Michael had to consider the difference and similarity between the handshake he symbolized as 1B and the handshake he had symbolized as 2A, and with his addition of subscripts he determined that they symbolized the same handshake.

His use of letters and digits, and his subsequent introduction of subscripts enabled him to use features of his notation for the Outfits Problem and features of his notation for the Flag Problem. That is, he initially used letters and digits to differentiate between handshakers and handshakees, which was similar to what he had done in the Outfits Problem. Subsequently, he associated a number with each letter using subscripts, which enabled him to identify person B and person 2 as the same person. Making this association enabled him to conclude that he could have used two digits (rather than a digit and a letter) in order to notate this problem, stating, "I could have just wrote two three, two four, two five". Although he drew this conclusion, his more complex system of notation that included subscripts seemed essential to his progress towards developing a system of notation that accounted for ordering. It was essential to his progress because when he used only digits to notate the Flag Problem, he did not make a differentiation between a top and bottom stripe whereas in this problem he did differentiate between the first person shaking the second person's hand and the second person shaking the first person's hand in his system of notation.

Discussion

Michael's progress towards developing a system of notation that accounted for ordering outcomes was a dynamic process in which he used and modified his notation from earlier problems in his solution of later problems. For instance, in creating notation for the Flag Problem, he used two characters to notate each outcome as he had in the Outfits Problem, but he introduced using the same type of character (i.e., two digits) to denote each flag. By modifying his notation, he was able to capture progressively more complex aspects of the problem situations, and therefore his system of notation became more advanced over the course of the four teaching episodes. Moreover, the process of developing and refining a system of notation was both creative and time consuming (cf. Mason, 1996). It was creative in that it entailed Michael using notation in unconventional ways—he used subscripts to denote the equivalence of two people, an unconventional, but creative, usage of subscripts. It was time consuming in that it took approximately two hours of teaching time for him to develop a distinction that I probably could have suggested to him in a matter of moments (e.g., "you should use identical characters, and have the first character denote the top stripe and the second character denote the bottom stripe"). However, if students development of systems of notation is integral to their developing powerful mathematical reasoning, then it is critical for students to have experiences that offer the time and creative license to develop and refine these systems during their mathematics education. In turn, it is critical for researchers to map out trajectories for how students develop and refine systems of notation in a range of mathematical domains in order to design instructional sequences that can support this development.

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THE ROLE OF PRECALCULUS STUDENTS' QUANTITATIVE AND COVARIATIONAL REASONING IN SOLVING NOVEL MATHEMATICS PROBLEMS

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This paper reports results of an investigation of precalculus students' thinking as they responded to novel, precalculus-level applied problems. The subjects were six precalculus students and the data was gathered as the students were completing a one semester precalculus course. A primary goal of this study was to gain insights into the process by which these students made sense of the words in applied (word) problems. The study revealed the critical role of quantitative reasoning and the importance of students' building a mental structure of a problem's quantities and their relationships when orienting to novel applied problems.

Background

Mathematics educators have focused on the necessity of students having the ability to think about functions by attending to patterns of change (Kaput, 1985; Monk & Nemirovsky, 1994; National Council of Teachers of Mathematics, 2000; Vinner & Dreyfus, 1989). The National Council of Teachers of Mathematics (NCTM, 2000) Standards suggested that students should be able to analyze patterns of change in data and understand how to represent these patterns using various function representational contexts, such as formulas, graphs and data. It is well documented that high performing precalculus and calculus students have weak understandings of the function concept and have difficulty reasoning covariationally, or modeling function relationships involving the rate of change of one variable as it continuously varies with another variable(Carlson, 1998; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994). This ability to reason covariationally has been shown to be critical in using functions to model dynamic events (Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson, 1994). While the ability to reason covariationally and construct quantities has been shown to be foundational for understanding key ideas of calculus, there is little research that explains in what ways covariational and quantitative reasoning affect students' ability to respond to novel problem tasks. Thus, this study was designed to address this gap in the research literature.

Objectives and Research Question

In response to the widely reported difficulties students have using functions, we designed a study to investigate how a curriculum that focused on developing quantitative and covariational reasoning impacts students' development of these abilities. The purpose of our research was to understand what role quantitative and covariational reasoning played in a student's orientation to a novel problem that required students to use the function concept to represent how two quantities are related. Our research goals were to:

- Describe the ways of thinking students used to solve novel mathematics problems prior to receiving instruction in a redesigned precalculus course.
- Describe and explain how students' ways of thinking and problem solving abilities changed or remained the same over the course of one semester.
- Explain the role that construction of quantity and covariation of quantity played in students' orienting to novel applied precalculus problems.

Theoretical Framework

The study draws from Glasersfeld's (1995) view of radical constructivism that learning begins and ends with the learner. He claimed that each learner's mathematical reality is independent and unknowable to others in any absolute sense. Thus, we as researchers cannot be completely confident in our understanding of a student's mathematical reality, but we can describe ways of thinking so that, were a student to have them, would, from the student's perspective, give coherence and sense to their actions. The classroom is a place for students to develop ways of thinking about mathematics, although each student will use a personal way of thinking, since the reality he/she experiences is fundamentally tied to his/her prior experiences.

Quantitative and Covariational Reasoning

Quantitative and covariational reasoning are both explanatory and descriptive research frameworks. Each takes as central that the individual student constructs his/her reality. Specifically, both quantitative and covariational reasoning do not aim to describe reality as the student sees it, but instead the goal is to identify ways of thinking that explain students' responses and behavior in solving mathematical tasks.

In Thompson's (1990, 1993) characterization of quantitative reasoning, he described a quantity as a conceived attribute of something that one can imagine being measured. Quantity is a cognitive object specific to the individual student and because it is a cognitive object, a quantity is not the same for every learner. According to Thompson (1994), quantitative reasoning is the analysis of a problem-solving situation in terms of the measurable attributes of objects within a situation as conceived by an individual. A quantitative structure is a network of quantities and relationships between those quantities that comprise the student's conception of the situation. Quantitative reasoning focuses on the importance of developing students' ways of thinking to conceive of problem situations in terms of measurable attributes of objects within the described situation. As students engage in sense making in the orientation phase of problem solving, they conceptualize quantities; then form a basic notion of function by relating quantities in terms of invariant relationships among their values—as the quantities' values vary, the relationships remain the same (e.g., Bob's age is always 5 years greater than Jane's). We refer to this sense-making process as building a quantitative mental structure. Once an individual has built this mental structure, we conjecture, as does Thompson, that the structure that emerges from seeing how two quantities are related is what allows students to construct meaningful formulas and define functions relating measurable attributes of a situation.

Carlson, Jacobs, Coe, Larsen & Hsu (2002) indicated that covariational reasoning involves attending to the values of two quantities and how they change together. Specifically, covariational reasoning is a way of thinking that involves first understanding that if two quantities vary simultaneously, at any time that one quantity has a measure, the other quantity has on a measure too. Engaging in covariational reasoning involves the learner initially constructing a sustained mental image of two or more coupled quantities, and understanding that those quantities could take on different values—and those values are systematically related by the quantitative relationships as conceived in the situation. Carlson et al. (2002) delineated a framework of five ways of thinking about how two quantities' values change in tandem. Together, covariational and quantitative reasoning assume students' mathematical realities are distinct and independent from any other mathematical knower and allow one to describe how students' mental imagery affects their thinking. As a result, the frameworks are particularly powerful for documenting and explaining students' responses and actions in response to novel mathematical tasks.

Mathematical Problem-Solving

Based on the assumption that students create independent mathematical realities, this study assumed mathematical problem solving was an active, multi-tiered, internal process in which the students pose, use, reflect, and revise understandings in attempting to make those understandings fit the mathematical reality they experience. A great deal of literature has been written on mathematical problem solving (Blum & Niss, 2004; DeBellis, 1997; Goldin, 2000; Lesh, 1985; Lesh & Akerstrom, 1982; Romberg, 1994). Carlson, in (Carlson & Bloom, 2005) synthesized seminal studies in students' solving novel, applied mathematics problems and proposed a problem solving taxonomy focused on the processes and behaviors students exhibit and progress through in a common problem-solving approach. Their work, coupled with quantitative and covariational reasoning, was the basis for analyzing and describing students problem solving behaviors and utterances while working on applied mathematics problems. The mathematicians under observation in this study were efficient in constructing a picture or describing a mental image of the problem situation. Carlson & Bloom (2005) referred to this problem-solving phase as orientation. Making sense of the problem, organizing information, and constructing possible solution approaches were the key behaviors associated with the orientation phase. The orientation phase also includes the construction of a picture or mental image, which is a focal point of quanitative and covariational reasoning. The findings of this study suggested the need for further investigation of the role that quantitative reasoning plays in orienting to a novel applied problem, and the role that covariational reasoning plays in constructing meaningful representations of the functions that related the two quantities.

Method

Subjects and Setting

The authors tracked six students over the course of a one-semester, redesigned, precalculus course. All subjects participated voluntarily and received monetary compensation for their time. Two students were female and four were male. All were enrolled in a redesigned precalculus course that focused on developing students' ability to reason conceptually about functions and quantity as they solved complex mathematical problems. These students were identified as high-performing, middle-performing, and low-performing based on a pre-calculus assessment instrument designed to predict a student's success in a conceptually based precalculus course (Carlson, Oehrtman, & Engelke, 2009). Two students earned a C in the course, three earned a B, and one student earned an A. Three of the six students planned to enroll in calculus I the following semester.

Data Collection

All six students attended every class session over the course of the semester. The research team interviewed each student in an individual clinical interview at six different times during the course. The sequence of six interviews focused on gaining insights into students' understanding of function, exponential growth, and trigonometric relationships. The students attempted to solve three mathematics problems in each interview session that were not presented during class, In addition, at the beginning of each interview we asked each subject if she/he had remembered seeing the problem before. If the student remembered the problem, that problem was excluded from that subject's interview. This issue occurred only once, resulting in our eliminating the interviews on this problem from our study. Students were videotaped, with permission, during classroom sessions and clinical interviews. Students' written work was scanned as a backup if the camera was not able to capture what the student wrote during the interview or classroom session.

Analysis

We synthesized and coded using a combination of open and axial coding as described by Strauss & Corbin (1998). We initially employed open coding with the video data to identify episodes we believed fell into the orienting phase of problem solving. The next level of coding broke these episodes into specific categories of making sense of the problem, organizing information, and constructing possible modes of solution. The video data was then coded for episodes the authors believed the students used quantitative and covariational reasoning to orient to a problem. Other emergent categories included coordinating changes in number, anticipating the behavior of numbers, and anticipating the behavior of function thinking about quantity. We used axial coding to characterize how the coded categories were related. For example, we intended to explain how the initial sense making of the problem related to the student anticipating the behavior of numbers. In this way, we took over thirty emergent categories from open coding and abstract them into four major categories. The major explanatory categories that emerged from the data were identified as covariation of number, covariation of quantity, formation of quantity, and quantitative structure.

Results

Students oriented to problems by attending to the numbers in the situation, and students often explicitly described numbers as measuring an aspect of a situation. The results discussed here focus on students' work on the box problem that initially prompted students to develop a formula for the volume of a box formed by cutting equal sized square cutouts from a 14 inch by 17 inch piece of cardboard. We present two students' work that is representative of the larger group of six students we tracked during the semester.

Orienting with Numbers: Adam

Three of the six students approached the box problem by focusing explicitly on the numbers in the problem. This specific example displays the student orienting to the problem by focusing on given numbers, and attempting to find a formula given a specific cutout length. The student's focus on number is most apparent when he states that, "to be really sure, you would have to check as many numbers as you could until you knew the formula was right." Over the course of the semester, this student reconceptualized his approach to problem solving, even stating that, "I have to think about what the numbers represent before I try to use them", at the conclusion of the semester.

INT:	We talked earlier about volume, now I'd like to focus on that.
STU:	Okay.
INT:	The problem asks you to construct a formula for the volume of the box. What does that make you think might be useful?
STU:	Well, I need. Well, what I would do is in math I usually try something and try out a few things to see if it fits.

INT:	Could you say more about what you mean?
STU:	Well, basically in this case I just come up with a formula, and I am used to plugging in a few values to see if it works.
INT:	Okay, I think I see what you mean, but why don't you show me what you mean?
STU:	Alright, so I have this formula (14-2(1.5))(17-2(1.5))
INT:	Okay, so what led you to that formula?
STU:	Well, I took the numbers out of the problem, so I know the length and the width of the box, which are 14 and 17. Then I decided to try out if I cut a square corner that had sides 1.5 inches, so then I would multiply ((14-2(1.5))(17-2(1.5)), oh I would probably have to multiply by 1.5, so it's length and width and height.
INT:	Can you tell me why you think your formula is accurate?
STU:	Well, I am pretty sure, to be really sure, you would have to check as many numbers as you could until you knew the formula was right.

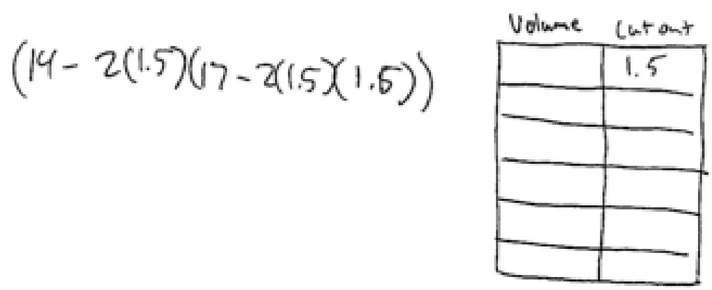

Figure 1. Orienting with number

Orienting with Quantity: Brad

Brad approached the box problem by attending to quantities. Brad focused on the formula as secondary to thinking about the relationship between the length of the side of the cutout and the volume of the box. Approaching the problem by thinking about the relationships between quantities allowed Brad to imagine the behavior of the box as the cutout changes, rather than needing to explicitly calculate the volume using numbers. As a result, the numbers for cutout length were simply a specific representation of a number of possible values measuring a quality of an object rather than a number in and of itself.

INT:	The problem asks you to construct a formula for the volume of the box. What does that make you think might be useful?
STU:	Well, the volume of the box is formed by the length, width and height of the box. The length, width, and height of the box all depend on how big you make the cutout.
INT:	Okay, I think I see what you mean, but why don't you show me what you mean?
STU:	Alright, so I have this formula x(17-2x)(14-2x).
STU:	Well, it's not hard to figure out what's happening to each piece of the formula. The length and width get smaller as you increase x, which is the cutout size. But the height, which is the same as your x, gets bigger. So you kind of have to figure out how the decrease in the length and width balance the increase in height.

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INT:	That’s really interesting, could you say more about what you mean by balance?
STU:	Well, you have to think about the volume, and the volume depends on umm… the length, width, and height of the box. So if the increase in the height outweighs the decreases in the length and width, the volume should increase. Umm… [long pause]. I’m just not sure how to determine the outweighing, I’d probablyplug some values in.

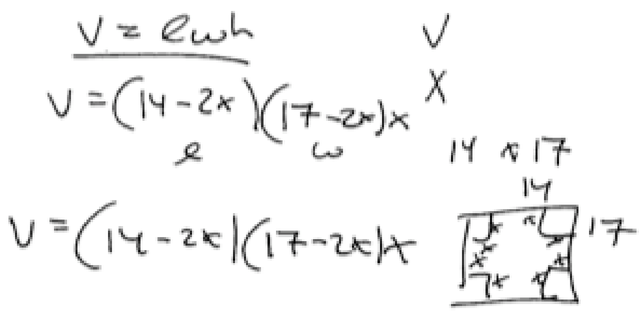

Figure 2. Orienting with quantity

Thinking Covariationally: Adam

Brad conceptualized measurable attributes of the box in the box problem was able to consider the issue of “balancing” the length, width and height of the box. Like Brad, Adam was able to imagine all possible values for the length and width of the box by constructing the qualities of the object and imagining those qualities having measures, and represented these values on the x axis. Adam’s image of a quantity’s measured attribute resulted in his image of a continuum of values represented by x, which allowed him to describe the volume as increasing to a certain point, then decreasing.

INT:	What can you say about how the volume of the box changes with respect to the cutout length?
STU:	Well, it goes up to a certain point, then it goes down, then it gets negative.
INT:	What are you thinking about as you tell me how the volume changes?
STU:	Well, my calculations show that the volume increases to a certain point,
then it	decreases.
INT:	Does your description match what you understand about the box?
STU:	Yeah, I guess so. I guess I got the formula from the parts of the box, and then I can imagine plugging in all sorts of numbers to each part of the formula and then multiply those numbers to get the volume. Like, once I have the box, I have that box, and the formula is all I need to find the volume, so I don’t need to think about the box.

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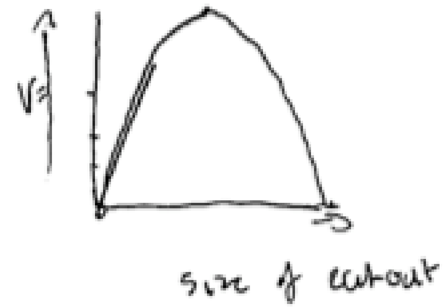

Figure 3. Covarying quantity

Discussion

The students' responses indicate that construction and covariation of quantity are central to student's problem solving approaches to novel tasks. Covariation of quantity is in contrast to covariation of number. The former involves the student's mental construction of a measurable attribute of an object and imagining that attribute having a range of measures, while the latter focuses on numerical representation without understanding the context that produced the number. In orienting to a problem solving task, if a student makes sense of the problem by thinking purely about plugging numbers into a formula he or she might be able to answer a question that asks about the volume of the box for a specific cutout value. However, that student might have great difficulty considering a problem that asks students to consider a dynamic situation like the box problem. If a student's mental image of a situation consists of measurable attributes of an object, he/she can imagine the attribute's measure varying. Imagining varying measures of an attribute supports a student developing a sustained image of two quantities changing in tandem. For students in this study, orienting to a problem by constructing quantitative relationships allowed them to think about the result of an action (changing the cutout length) on the length, width, and height of the box. Understanding how each attribute of the box changed allowed them to anticipate how the volume of the box varied while imagining varying values of the cutout length. The results suggest that it is critical for students to construct a mental image of the situation by reasoning quantitatively to solve novel problems with numerous related parts.

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THE ROLE OF THE RADIUS IN STUDENTS CONSTRUCTING TRIGONOMETRIC UNDERSTANDINGS

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Multiple studies have identified students and teachers having difficulty constructing meaningful and coherent understandings of trigonometric functions. Furthermore, research has revealed that students have difficulty reasoning about topics foundational to trigonometry, such as angle measure and the radius as a unit of measurement. This study conjectured that these foundational understandings are central to constructing coherent understandings of trigonometric functions. Stemming from this conjecture, an instructional sequence was designed to promote students constructing and leveraging understandings of the radius as a unit of measurement. This report discusses the results of an undergraduate precalculus student's interactions with these materials.

Background

Although trigonometry was once labeled "forgotten and abused" in mathematics education (Markel, 1982) due to the limited focus given to trigonometry by mathematics educators, multiple studies have recently shed light on the difficulties students encounter when reasoning about the sine and cosine functions. As one example, Weber (2005) found that students have difficulty constructing the geometric objects of the various trigonometry contexts as tools of reasoning (e.g., the unit circle and right triangles). Other studies have revealed that students have difficulty reasoning about topics that are foundational to trigonometry (Brown, 2005; Fi, 2003; Moore, 2009), such as angle measure and the radius as a unit of measurement.

Thompson (2008) recently called attention to the important role that ideas of angle measure and the radius play in conceptualizing the sine and cosine functions. He also outlined an instructional approach that he conjectured would promote students developing meaningful conceptions of the sine and cosine functions. This study builds on Thompson's ideas by investigating the implementation of instructional tasks whose design was based on the conjectures provided by Thompson.

Specifically, this study reports results of one student's interactions with instructional tasks designed to promote students constructing robust understandings of angle measure and the radius as a unit of measurement. The curriculum was also designed such that students could leverage these understandings when reasoning about the sine and cosine functions as the covariation of an angle measure and a second quantity. The design of the curriculum was also informed by research on reasoning abilities the researcher deemed critical for success in trigonometry, such as quantitative and covariational reasoning (M. Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Smith III & Thompson, 2008). The results presented in this report describe the student's thinking as he solved the instructional tasks. The student's actions reveal him constructing a coherent system of ideas of the radius as a unit of measurement and the sine and cosine functions.

Theoretical Perspective

Quantitative reasoning (Smith III & Thompson, 2008) provides a theory that emphasizes a learner centered approach to mathematics. Quantitative reasoning describes the processes of a learner conceiving of quantities and relationships between quantities such that these mental

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structures promote mathematical reasoning. More specifically, quantitative reasoning refers to the mental actions of an individual conceiving of a situation, conceptualizing measurable attributes of this situation (called quantities), and reasoning about relationships between these quantities. The mental structure generated from these processes offers a foundation upon which the student can reflect on and subsequently construct mathematical understandings.

As an example, a student might conceive of a ball traveling in a circle around some central point. As the student refines her or his image of this situation such that he or she constructs attributes admitting measurement processes (e.g., the quantities of a traversed arc length and a vertical distance above the central point), the student can subsequently engage in covariational reasoning (M. Carlson, et al., 2002). That is, the student can investigate the nature of the dynamic relationships between the relevant quantities. These investigations can lead to the production of mathematical representations (e.g., graphs and formulas) and understandings that emerge from the student's image of the situation. This process of a student engaging in quantitative reasoning and reflecting on this reasoning formed a central intent of the instructional sequence.

The main questions driving this study were: i) "What understandings of the radius as a unit of measurement do students construct during the designed instructional sequence?" and ii) "How do these understandings influence their conceptualization of the sine function in the contexts of the unit circle and right triangles?" Overall, the study investigated students' understandings of the cosine function in addition to the sine function, but for the purposes of this report only a student's conception of the sine function is discussed.

Methodology

The results of this study focus on one student, Zac, who was drawn from an undergraduate precalculus course at a large public university in the southwest United States. The precalculus classroom was part of a design research study where the initial classroom intervention (M. P. Carlson & Oehrtman, 2009) was informed by theory on the processes of covariational reasoning and select literature about mathematical discourse and problem-solving (M. Carlson, et al., 2002; M. P. Carlson & Bloom, 2005; Clark, Moore, & Carlson, 2008). The classroom instruction consisted of direct instruction, whole class discussion, and collaborative activity. The head of the precalculus curriculum design project was the professor of the course and the student discussed in this report was chosen on a volunteer basis. Additionally, the student was monetarily compensated for his time.

The study used a teaching experiment methodology (Steffe & Thompson, 2000) to gain insights into the student's thinking. The researcher (myself) conducted six ninety-minute teaching sessions with Zac and two of his peers, where the researcher acted as the students' instructor. Additionally, multiple interviews totaling four hours were conducted with Zac at various points during the study. The interview sessions offered additional insights into Zac's thinking as a result of the one-on-one setting of these interviews. The interview sessions followed a clinical interview protocol (Clement, 2000), with the added feature of the researcher presenting "on the fly" problems to Zac in response to the solutions he provided. A witness, who was a fellow researcher on the precalculus research team, attended all teaching and interview sessions and frequently met with the researcher to discuss their observations. Also, the researcher relied on video observations between teaching sessions to inform the design of the instructional and interview tasks.

All classroom and interview sessions were videotaped and digitized. Upon completion of the study, all video data was first transcribed (for both utterances and visual actions) and subsequently analyzed using an open coding approach (Strauss & Corbin, 1998). The researcher analyzed Zac's behaviors in an attempt to determine the mental actions driving Zac's solutions. Then, the researcher attempted to identify connections and consistencies in Zac's thinking in an effort to identify the relationship between Zac's conception of the radius as a unit of measurement and his reasoning.

A Brief Conceptual Analysis Of Trigonometry

In order to provide a background for the analysis of Zac's actions and the intentions of the instructional activities, this section briefly outlines the system of ideas that informed the instructional goals. A central goal of the study was to support the student in constructing a *process*, or derivation, of measuring an angle. As Thompson (2008) described, one coherent understanding of angle measure is the image of measuring the length of an arc subtended by the angle. In this approach, the degree is a unit of an angle's measure that corresponds to an angle subtending 1/360th of any circle's circumference, and the radian is a unit an angle's measure that corresponds to an angle subtending 1/(2π)th of any circle's circumference (provided the circles are centered at the vertex of the angle). In the case that the radius is used as the unit to measure an angle, the measure also conveys that the subtended arc length is so many times as large as the radius of the corresponding circle.

In order to leverage an image of measuring an angle that is reliant on a subtended arc length, as well as the radius as a unit of measurement, the sine function was introduced within the context of circular motion. The sine function was presented as the mathematical formalization of a covarying arc length and a vertical position. Also, the output of the sine function was defined as a measurement relative to the radius (Figure 1). This design decision was made to create a quantitative meaning of the output of the sine function that enabled the student to reason about the radius as a unit of measure for a circle of any size.

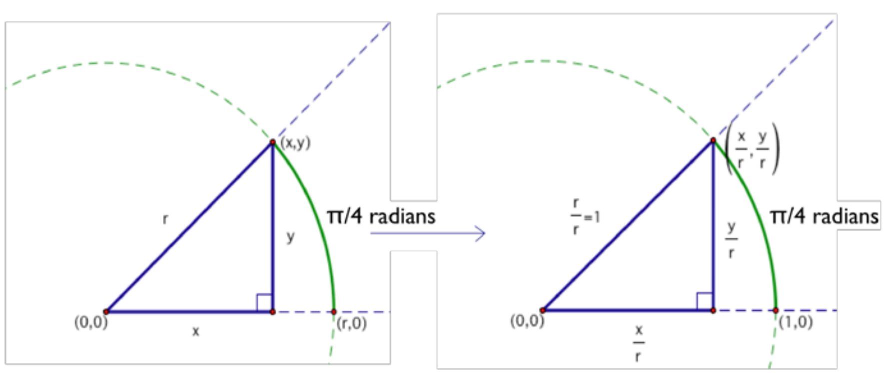


Figure 4. Measurement relative to the radius

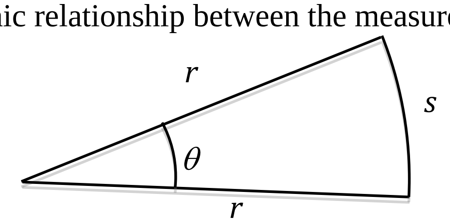
Results

Zac entered the teaching experiment with a self-admitted limited understanding of angle measure. When presented with an angle and the angle's measure during the pre-interview, Zac was unable to describe a clear meaning of the angle's measure. Zac claimed, "I never really thought about [angle measure]." Zac was then unable to measure an angle when given the supplies of a compass, a ruler, and a piece of string. After the researcher gave Zac the supplies, he immediately claimed that he was unable to accomplish the goal of measuring the angle. He also described geometric objects as having predefined angle measures (e.g., a straight line is 180 degrees), but he was unable to describe these measurements as the result of a process of coordinating quantities.

In response to the difficulties Zac encountered during the pre-interview, the instructional tasks were designed to engage Zac in creating tools (e.g., a protractor) to measure an angle. After Zac constructed an understanding of degree measurement that was consistent with the instructional goals (e.g., measuring a fraction of a circle's circumference subtended by the angle), Zac engaged in an activity prompting him to create a circle using a piece of string as the radius. He was then asked to measure the circumference of the circle in a number of string lengths. After Zac identified that 2π string lengths, or radii, rotate *along* a circle's circumference, the researcher asked Zac to explain using the radius as a unit of arc measure. Zac responded, "it simplifies a circle…the circumference of a circle is equal to two pi *r*, where the radius is the unit, not inches, or anything like that. So it simplifies it…using [the radius] as an actual unit… one radius, and then the circumference is six point two eight radius." Zac's actions reveal that he imagined measuring the circumference and radius of a circle in a number of radii, which implies that he conceptualized the radius "as an actual unit" of arc measure. This image enabled Zac to conceive of 2π radii rotating through any circle's circumference and a circle having a radius of "one radius" (e.g., the unit circle).

The Arc Problem

Using the following diagram, determine an algebraic relationship between the measurements r, θ, and s.



After Zac conceived of arcs as measurable in a number of radii, he flexibly solved problems requiring him to reason about radian angle measures. For instance, he had no difficulty converting between a linear measurement of an arc (e.g., a number of inches) and a number of radians. He justified his calculations by reasoning that an angle's measure in radians conveys the number of radii along the subtended arc for a circle (centered at the vertex of the angle) of any size. Also, he was able to generalize his conception of radian angle measures such that his reasoning did not rely on numerical values and executing calculations. As an example, consider his solution to The Arc Problem, and note that the symbolic relationship between the three quantities was not formalized previous to Zac's attempt to solve this task.

After establishing that the angle's measure was in radians, Zac constructed an algebraic relationship between the various measurements (Excerpt 1).

Zac: Alright. We'll say theta equals radians (*writing* $\theta = rad$), very very simple then. *r* theta is equal to *s* (*writing* $r\theta = s$). 'Cause theta is in radians, that means a percentage of the radius. Which would then be equal to this length (*tracing arc length*). So you multiply the percentage of the radius by the radius, you'll get the arc length.

Zac conceived of theta (a number of radians) as representing a fractional amount of the radius length (lines 2-4). Zac's image of a radian measure conveying a multiplicative comparison between an arc and the length of the radius led to his ability to *anticipate* multiplying the percentage of a radius length by the radius to obtain an arc length (lines 3-5). That is, Zac's algebraic representation stemmed from his ability to reason about measuring an *undetermined* arc length as a fraction of a radius length.

After Zac created his formula, the researcher prompted Zac to explain the formula $\theta = s / r$. As opposed to providing a procedural explanation (e.g., dividing both sides of his previous formula by *r*), Zac explained, "Well this is…a percentage of a radius length over a radius…a ratio, that'll give you a percentage of *r*." Thus, Zac's ability to conceive of a subtended arc length as a percentage of a radius enabled him to interpret a presented formula relative to a relationship between the quantities of the situation.

The Ferris Wheel Problem

Consider a Ferris wheel with a radius of 36 feet that takes 1.2 minutes to complete a full counter-clockwise rotation. April boards the Ferris wheel at the bottom and begins a continuous ride on the Ferris wheel. Construct a function that relates the total distance traveled by April and her vertical distance from the ground.

As Zac progressed through the study, his ability to reason about measuring a quantity as a fraction of a circle's radius enabled Zac to imagine measuring various attributes of circular motion in a number of radii. As an example, consider Zac's actions on The Ferris Wheel Problem. Zac first recognized that he needed to measure the distance along an arc from April to the standard position (e.g., 3 o'clock). He then constructed the expression of $\frac{Td - 56.5}{36}$ to represent this distance, while explaining that *Td* represented the number of feet April traveled along the arc swept out by her ride. Zac then explained that the entire expression represented the measure of the angle April swept out in a number of radians. He also explained that dividing her distance along the arc in feet by thirty-six represented a number of radius lengths from the standard position.

Next, Zac constructed the expression of $\sin\left(\frac{Td - 56.5}{36}\right)$ and stated that this expression represented "the vertical distance…a percentage of the radius length, which I then need to multiply by thirty six." In this case, Zac leveraged his understanding of the sine function outputting a quantity measured relative to the radius in order to convert the output unit. This resulted in Zac constructing the function $vd = f(Td) = 36\sin\left(\frac{Td - 56.5}{36}\right)$.

Zac continued modifying his solution and explained that the output of the sine function is a vertical distance above the center of the Ferris wheel, and that the correct function is $vd = f(Td) = 36\sin\left(\frac{Td - 56.5}{36}\right) + 36$ (e.g., a vertical distance from the bottom of the Ferris wheel). Thus, Zac's solution process included various refinements that reflected reasoning about the quantities of the situation and measuring these quantities in a fraction of the radius. Zac's ability to reason about measuring lengths as a fraction of the radius enabled Zac to convert between units of measurement as needed, while representing these conversions in his function.

The Empire State Building Problem

While site seeing in New York City, Bob stopped 1000 feet from the Empire State Building and looked up to see the top of the Building. Given that the angle of Bob's site from the ground was 56 degrees, determine the height of the Empire State Building.

Similar to his actions during The Ferris Wheel Problem, Zac frequently used the radius as a tool of reasoning within the various contexts of the instructional activities. For instance, during the first instructional activity that presented a right triangle context (The Empire State Building Problem), Zac used the hypotenuse of the right triangle to construct a circle and right triangle in the same orientation as Figure 1. Zac then correctly solved the problem using the sine and cosine functions while continually referencing the hypotenuse as "the radius." In response to Zac creating a circle and reasoning about a radius, the researcher asked Zac to further describe his actions (Excerpt 2).

Int: So what told you to put this in a circle like this? Why did you make that choice?

Zac: Um, to make it easier to understand. How I was originally taught was just with triangles. Now that we've started using circles it makes a whole lot more sense to me.

Int: So could you say a little bit about why it makes a little more sense now?

Zac: Uh, because I always just thought hypotenuse was, you know, just a side of a triangle. You know, you could use Pythagorean's Theorem to find out what it was very easily. And now that we've figured out, you know, now I'm looking at it and seeing it's the radius, it makes a lot more sense to be able to find, the horizontal and vertical distance according to the radius (*waving tip of pen across the radius*).

Zac's descriptions suggest that he found value in "using circles" to relate an angle measure and another quantity (lines 3-5). Also, Zac explained that he conceived of the hypotenuse as only a side of a triangle previous to participating in the teaching experiment (lines 7-8). He then explained that his understanding now consisted of conceptualizing the hypotenuse, or radius, as a unit of measurement (lines 9-12). Thus, Zac's image of the radius as a unit of measure created a tool of reasoning that enabled him to imagine using the hypotenuse of a right triangle as a unit length, as opposed to "just a side of a triangle."

Discussion and Conclusions

Zac's actions offered insights to his conceptualization of the radius as a unit of measurement and the reasoning abilities that followed from this conceptualization. Zac entered the study with a self-admitted limited understanding of angle measure, but his participation in the teaching experiment led to him constructing meaningful conceptions of angle measure. Specifically, Zac conceptualized measuring *along* a subtended arc in a number of radius lengths. This image enabled Zac to imagine the radius of any circle as a unit length, regardless of the size of the circle. Also, Zac conceptualized that measuring along the circumference of any circle in units of the radius yielded a measurement of 2π units. In other words, Zac's ability to imagine measuring quantities in a number of radii enabled him to conceive of any circle as the unit circle. Zac also explained that measuring quantities relative to the radius made circular contexts “easier” because this action simplified any circle to a circle with a radius length of “one radius.”

Zac’s conceptualization of the process of measuring a quantity in a number of radii also led to Zac conceiving of radian measures in terms of a quantitative relationship. Specifically, Zac fluently reasoned about radian measures as a multiplicative comparison between *indeterminate* measures of a length and the radius. For instance, Zac generated the formula of $s = r\theta$ without having to perform calculations between numerical values.

Zac engaged in similar reasoning in order to use the sine and cosine functions with circles of any size. On The Ferris Wheel Problem, Zac leveraged his quantitative understanding of the radius as a unit length to reason about measuring the distance traveled and the vertical position of a rider relative to the radius. His image of the quantitative relationship conveyed by a measurement relative to the radius enabled him to anticipate converting between measurement units without carrying out numerical calculations. This led to him defining a function that modeled the individual’s ride on a Ferris wheel.

The reasoning abilities exhibited by Zac also led to him creating coherence between right triangle trigonometry and unit circle trigonometry, which has been reported to be a difficult task for both students and teachers (Thompson, Carlson, & Silverman, 2007; Weber, 2005). Zac’s flexible conception of the radius as a unit of measure led to him imagining the hypotenuse of a right triangle as the radius of a circle. This enabled him to reason about measuring the legs of a right triangle relative to the hypotenuse. This image resulted in Zac using the sine and cosine functions in ways consistent with their use in a unit circle context (e.g., the outputs of the functions represented measurements relative to the hypotenuse, where the hypotenuse formed the radius of the circle).

Zac’s actions illustrate the importance of a student conceptualizing the radius in ways that it becomes a tool of reasoning. A student that develops an image of the radius such that they can imagine measuring quantities relative to the radius (by making a multiplicative comparison) is prepared to coherently use the radius and trigonometric functions in multiple contexts. The student can imagine measuring quantities relative to the radius (or hypotenuse) for a circle (or right triangle) of any size, while flexibly converting between the given measurement unit and a number of radii. That is, the student can conceive of any circle as a circle with a radius of one unit or any right triangle as a triangle with a hypotenuse of one unit. These reasoning processes can enable a student to fluently use the sine function (and cosine function) in novel situations that consist of circular motion or right triangles.

Endnote

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ANALYSIS AND INFERENCE TO STUDENTS' APPROACHES ABOUT DEVELOPMENT OF PROBLEM-SOLVING ABILITY

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It is widely agreed that the development of problem-solving ability should be viewed as the primary goal of school curriculum and instruction. While this goal has been a persistent part of mathematics education community for over a century, issues regarding how to develop problem solving skills among learners continues to be a major dilemma. There is little known about how students' problem-solving ability develops over time and ways in which these skills might be nurtured along a developmental trajectory. In our study we administered three non-routine problems to approximately 350 students in grades 5 and 8 to trace their mathematical problem solving skills at the end of elementary and middle grades. The content of problems concerned proportional reasoning, geometry and visualization and probability. These areas were selected since they have been identified as core content areas in K-8. The approaches from 5th and 8th graders to the same problems were studied, categorized, and using theoretical framework provided by Silver (1987). In particular, we sought evidence of *pattern recognition, representation*, *understanding*, *memory schemas* among children's work and ways in which the choices of children changed from 5th to 8th grade. Our results indicate an improvement in the area of "understanding of problems" from 5th to 8th grade, which could imply the existence of better content knowledge related to these problems. Preference to more "mathematical" approaches (i.e. equations and formulae) increased as the increased. According to Siegler (2005), students choose adaptively among the strategies they knew when they came across a problem. Thus the result indicated that more students were familiar with those strategies and were able to apply them based on the understanding of the problems. The memory schemata in the context of probability content were found to be poorly developed.

Standardized approaches and textbook representations were noted most frequently in higher grades. Variety and creativity, which are important features of mathematical thinking, were less distinguished. Various types of examples need to be used and the use of multiple representations when solving problems needs to be more encouraged in classroom instruction in order to expand memory schemata among children.

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IDENTIFYING THE REASONING STRATEGIES STUDENTS USED WHEN SOLVING RELATED RATES PROBLEMS

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Related rates problems serve as a source of difficulty for first semester calculus students. While many studies have investigated concepts related to solving related rates problems, there are few that focus on related rates problems themselves. Engelke (2007a, 2007b) found that students fixated on procedural steps, which prevented them from building a mental model of the situation. Without having a mental model of the situation, the students were less likely to engage in transformational and covariational reasoning about the problem situation, which led to procedural traps. We hypothesized that students who participated in a Supplemental Instruction (SI) workshop would engage in debates that would promote higher levels of reasoning while working on related rates problems. These higher levels of reasoning would foster the development of their mental models for the problem situations.

The students in our study were first year calculus students that are coming from two or more standard lecture courses and have the option to participate in an SI workshop. During five SI sessions the students worked in small groups to solve problems provided by the SI leaders. These sessions were videotaped and transcribed for analysis, and we were able to determine what types of reasoning structures the students were utilizing.

The coding process sought to locate, identify, and count the number of times the students used one of the four types of reasoning processes described by Lithner (2003, 2004): Global Plausible Reasoning (41%), Local Plausible Reasoning (24%), Identification of Similarities (15%), and Established Experiences (20%). While the students frequently began the problem solving process using the superficial reasoning strategies of IS and EE, they frequently progressed to plausible reasoning strategies as a result of engaging in debates about the problem solving process. It appears that the SI model created an environment that promotes LPR and GPR by having the students work on challenging problems, and subsequently strengthened the students’ problem solving abilities. The workshop setting moved students toward the LPR/GPR strategies; working on one’s own does not create the cognitive dissonance necessary to spark the debates.

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"I JUST DON'T GET IT": READING AND COMPREHENDING MATHEMATICS TEXT

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Our study examines the notion that many undergraduate students do not read their mathematics textbook due to weak reading comprehension skills. The National Council of Teachers of Mathematics (NCTM) *Standards* promotes mathematical discourse and writing yet fails to address the connection to reading comprehension skills. Sanacore and Palumbo (2009) describe the fourth grade slump, where students make a transition from *learning to read,* to *reading to learn*. They contend the transition requires that students possess strong comprehension skills and that students must acquire the capacity to transfer their knowledge of comprehension techniques for narrative text to informational context. Hall (2007) informs us that researchers "have typically focused on how to help content area teachers implement strategy instruction into their classrooms…. [perhaps] strategy instruction alone ... [are] not sufficient" (p. 307). There is limited research linking reading techniques to mathematics. Hyde (2006) informs us that many teachers focus on key words in a word problem as an instructional method to solve them, but this approach unintentionally tells students, "you don't have to *read;* you don't have to *think*. Just grab the numbers and compute" (p 3). In addition, Donahue (2003) indicate that many teachers in the middle grades and secondary grades "resist teaching reading because they think it is 'someone else's job'…" (p. 26).

We believe students' struggle in mathematics courses may be attributed to their weak reading comprehension skills. Our study examines three reading comprehension techniques that may improve mathematics understanding, knowledge and may promote students' learning of mathematics.

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THE EXPERIENCES OF A FOURTH-GRADE STUDENT WITH WORKING MEMORY DEFICITS AND A LEARNING DISABILITY IN MATHEMATICS WITHIN CONSTRUCTIVIST MATHEMATICS INSTRUCTION

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As constructivism becomes more prominent in special education and inclusion classrooms, general and special education teachers need teaching methods designed to help students with learning disabilities succeed within constructivist mathematics curriculum (Woodward & Montague, 2002). To develop these methods, special education researchers need to study the experiences of students with learning disabilities in constructivist mathematics to determine any special needs they may have. Certain characteristics of constructivism, such as multistep problems with heavy cognitive load and students presenting lengthy rationale for solutions, can potentially create difficult situations for students with learning disabilities who commonly have working memory deficits. While storage of information appears to be a primary cause of difficulty, processing efficiency and level of understanding of mathematical concepts are also potentially interrelated factors (Keeler & Swanson, 2001). The researchers in this study examined the experiences of a fourth grade student with a learning disability and working memory deficits within constructivist mathematics curriculum (Simon & Tzur, 2004). The participant was more motivated and successful with the curriculum when the instructor accommodated for her disabilities by helping her with storing information (sometimes by helping her develop her own strategies for storing and organizing information) and clarifying what the problem was asking.

Endnotes

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Chapter 11: Proof and Reasoning

Research Reports

Brief Research Reports

Posters

EXPLORING THE TEACHER'S ROLE IN A DISCOURSE-RICH ENVIRONMENT TO PROMOTE PROVING IN THE SECONDARY SCHOOL

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We investigated the secondary school teacher's role in engaging students in the process of proving. Classroom practices of two experienced mathematics teachers were examined. In particular, verbal exchanges between the teacher and student(s) that promoted students' need for and refinement of coherent arguments are discussed. Teachers used a variety of techniques to promote discourse about proof, including selection of rich tasks, use of strategic questioning, and peer critiquing of proofs. Students' recognition of and adoption of discursive rules of the mathematical community were evident in student critiques of peer proofs.

Introduction

There is no shortage of evidence that high school students struggle with reasoning and proof (Balacheff, 1988; Chazan, 1993; Harel & Sowder, 1998; Senk, 1985, 1989). Authors have speculated about the sources of student misconceptions by examining typical high school classroom interactions (Herbst, 2002a, 2002b). Researchers, educators, and policy makers have postulated that overemphasis on (or inappropriate use of) two-column proofs, focus on proving self-evident statements, and lack of attention to the purposes of and need for justification in mathematical settings contribute to the poor outcomes that have been documented (NCTM, 1989; Weiss, Herbst, & Chen, 2009).

In courses that emphasize mathematical proof and reasoning, teachers aim to help students learn to value, recognize, and produce valid mathematical justification. In this paper, we define mathematical justification as both a process and a product. The process of justifying requires that the justifier recognizes a need for a mathematical argument, analyzes the relationships and facts in a situation, and then constructs an argument that she or he finds convincing. It is also understood that in any social community, the argument or justification has limited value unless it also convinces others and helps them develop a deeper understanding of the underlying concepts (Hanna, 2000; Thurston, 1995). We use the phrases "mathematical argument" and "mathematical justification" interchangeably to mean a valid rationale for a mathematical claim. Such justifications may be presented verbally, pictorially, or in writing.

The goal of helping students learn to formulate clear, concise, and mathematically viable arguments is a lofty one. Not only must the students develop understanding of the facts and relationships in the subject matter, but, additionally, they must have sufficient facility with the language of mathematics to express their ideas in a way that is consistent with norms of the discipline. They must also possess sufficiently strong general communication skills to translate the ideas they form in their own minds into logically ordered statements that conform to mathematical syntax, as well as the syntax of the language in which they are communicating.

Again, for those who spend time in U.S. mathematics classrooms, it is clear that making meaningful progress in all of these areas is a tall order.

The goal of this study was to examine the classroom practices for teaching proof used by teachers who are recognized by colleagues, supervisors, and other informed observers as being particularly effective. We hope that by closely examining various aspects of the pedagogy used by these teachers, we might glean information about teaching practices that enable students to successfully learn to develop robust mathematical arguments.

Theoretical Perspectives

There are many factors that influence teaching and learning in a high school classroom. Among these factors are student motivation, school ethos, community values, access to educational resources, and family priorities. Although we acknowledge that such factors influence what happens in school, our investigation was focused on classroom events, decisions, and actions within the control of the teacher. In particular, we examined the verbal exchanges between teachers and students to characterize classroom events, teacher actions, and students' responses. Sfard (2000, 2001) characterized thinking as communication and learning as gaining access to a disciplinary discourse. She also described *object-level rules*, rules governing content of a discipline, and *meta-discursive rules*, rules governing the flow of information exchange within a discipline. Using her description, students' understandings may be assessed by determining the extent to which they are able to follow these discursive rules. Sfard's (2000) research also revealed that differences in classroom environments and expectations could lead to differences in students' perceptions of the meta-discursive rules. This suggests that to make inferences about students' justifications, researchers must gather evidence by means of verbal or written communication and should consider the meta-discursive rules of the classrooms in which the students participated. If the learning experience is successful, students' communications should reflect the rules and norms of the classroom, or, better yet, the rules and conventions of the discipline. If their communications are at odds with the meta-discursive rules of the learning environment or the discipline, we should reflect on why this has happened.

Maher (2009) asserted that proving can arise naturally from activities that teachers create for their classrooms and the questions that they ask. Elementary-school students in her study developed more sophisticated arguments when they were pushed by the researcher to elaborate on why they knew that their assertion was correct. Weber, Maher, Powell, and Lee (2008) showed that group discussion can promote improved reasoning by putting students in a position to have their warrants challenged. In this study, we sought to determine how the teacher's role in developing a discourse-rich environment, including the facilitation of group discussion, may influence students' developing awareness of meta-discursive rules for communicating mathematical justifications or object-level rules about the nature of proof in mathematics.

Research Questions

Three research questions guided our observations and data collection:

1. What practices were used by two experienced, highly recognized mathematics teachers to support a discourse-rich classroom environment?
2. What practices were used by the teachers to promote mathematical justification?
3. What evidence was there that students were able to communicate using the meta-discursive rules of the discipline?

Methodology

The first author spent several weeks observing and videotaping the geometry classes of two experienced, highly recognized high school mathematics teachers. One teacher's school was in a suburb of a large Midwestern city in the United States. His classes were proof-based, honors-level geometry courses. The other teacher's school was in a predominantly rural community in the Midwest. His classes were proof-based geometry courses and did not have an "honors" designation. Classes were videotaped and field notes were taken to record noteworthy events. Videos were reviewed in search of patterns of teacher behavior, particularly those that pertained to the teaching and learning of proof; more specifically, we looked for teacher practices that enhanced the quality and/or quantity of student participation in discourse related to justification and/or proof. We used an open-coding strategy (Glaser & Strauss, 1967) to look for patterns across the data for the two participants, resulting in such categories as directing [possibly modified] questions back to the student, directing students to question other students, probing for elaboration, posing discourse-promoting tasks, etc. Some videos were also transcribed as a way to analyze the discourse more carefully.

Results

Promoting Discourse

Both teachers used several common practices that may have contributed to the development of a discourse-rich environment. First, both teachers had chosen, independently, not to use the typical two-column proofs in their courses, but rather to have students write paragraphs to support their claims. The requirement that students speak and write arguments in their own words in both classrooms contributed to high levels of student engagement in lessons, as well as to a sense that there was not one correct way to formulate an argument.

Likewise, both teachers had their classrooms arranged in groups, with three or four students in a group. The teachers regularly incorporated group tasks into their lessons and asked students to consult one another about the viability of their arguments. Both teachers spent a majority of the class time circulating around the room, posing questions to groups and to individuals within the group. The teachers' questions were strategically constructed to encourage students to clarify or refine their mathematical ideas and arguments. These questions often included a restatement of the student's initial query or claim, followed by a request for greater clarity, a challenge about the student's claim, or a provocation to generalize and to connect with other ideas. Teacher questions were also used to draw group members into the conversation by directing specific students to comment on or to identify conflicts with prior assertions made by an individual in the group. Although such questions served to guide students' thinking by drawing attention to particular issues, the teacher questions did not typically offer additional information to the student about the correctness of the student's thinking. The suburban teacher was particularly adamant about not providing an evaluation of students' arguments, but rather, forcing the group or the entire class to assess the work of their peers.

The teachers' expertise at posing strategic questions was particularly evident when they were interacting with small groups. The following excerpts from classrooms discussions illustrate how teachers promoted discourse as well as an awareness of processes and norms of proving.

Teacher: I'm looking at your work and I'm looking at number 1 and you're saying that *x* has to be more than 12. Is that fair?

Students 1, 2 and 3: Yeah.

Teacher: Now, how did you get that?

Student 1: You're given that DF is greater than DE. And that means that angle E must be greater than angle F since they're opposites.

Teacher: OK.

Student 1: Since DF is opposite of E.

Teacher: OK. What can we use from yesterday's class to kind of put that into a more compact form? (Pause.) What rationale can we provide? What you said is fine, but what do we have from yesterday?

Student 3: The shortest side is opposite the shortest angle.

Teacher: Is that what's happening here? (Students agree.) So, I've had a look at what you have so far, but I'm wondering, I see x has to be more than 12 (what students had written), but it is possible for me to have x is, uh, 200? Is that ok? It's more than 12.

Student 1: Oh, wait.

Teacher: You just said, "Oh, wait." But why do you have to wait here?

Student 1: It can't be greater than, the angle measure can't be greater than 90.

Teacher (to Student 4): What were you about to say?

Student 4: That's what I was going to say.

Student 1: Both of these angles can't be greater than 90.

Teacher: So, is there an added level of restriction that you have to address here?

Student 1: Yeah.

Teacher: Figure it out. Call me back. (Leaves the table.)

The teacher moved to the next table, and looked at students' work. He called back to the previous table, "Student 1, come over here and have a conversation with Student 5 about number 1. Don't give it away; ask her the same question I asked you."

The following exchange happened at yet another table:

Teacher: So what's the minimum side?

Student 6: 12

Teacher: In this particular case, but what about the general case?

We see that the teacher used a combination of questions to advance students' reasoning about the problem and to promote the normative practice of justification through communication. He asked specific questions about the problem under investigation; for example, "You're saying that *x* has to be more than 12. Is that fair?" which probes to see whether the students are committed to their claimed answer. He also encouraged students to support or generalize their ideas; for example, in asking, "[True] in this particular case, but what about the general case?" the teacher directed students to broaden their thinking and to extend their ideas to assert mathematical generalizations. Also, this teacher used the tool of playing devil's advocate as a method for getting students to reexamine their work. For example, when one group had accepted a condition for a problem without thoroughly considering all possibilities, he asked "I see *x* has to be more than 12 (what the students had written), but it is possible for me to have *x* is, uh, 200?" This teacher also used students as questioners of other students, both as a method of more efficiently circulating around the room, as well as a way for different groups to gain insights from the mistakes or better strategies of the other groups. For example, "*Student 1*, come over here and have a conversation with *Student 5* about number 1. Don't give it away; ask her the same question I asked you." This further illustrates the teacher's effort to instill a norm of student-to-student questioning and justifying within the classroom community.

Promoting Justification

In order to help students see the need for mathematical justifications, the teachers chose rich, argument-eliciting tasks, used strategic questioning (as described in the previous section), and required students to formulate arguments in writing. Both teachers began their teaching about proofs with connections to real-life examples and with emphases on writing. The suburban teacher used the following task based on an actual court case (Abelson, 2006) to open the discussion about mathematical justification.

> *Is a burrito a sandwich?*
> Work with your group to WRITE a response to the above question. Be prepared to present to the class and defend your position with your peers.
> What are your underlying assumptions? What entails a convincing argument? Can you ***predict*** any counter-arguments from other groups? Can you defend your position?

Students engaged in extensive debate about the question and then read their justifications aloud to the class. The activity led to a need for clear definitions (e.g., “what is a sandwich?”), as well as the development of informal arguments about how to show that the criteria in a definition have been met. For example, one group stated that a burrito is not a sandwich because a sandwich requires two pieces of bread with stuff in the middle and the burrito only has one outer layer twisted and turned around the stuff in the middle. But another group claimed that a burrito is a sandwich, but a sandwich is not a burrito, just like a square is a rectangle, but a rectangle is not necessarily a square.

The teacher in the rural setting had students engage in the checkerboard problem (Fendel, Resek, Alper, & Fraser, 1998). The problem asked, “How many squares are there altogether on the [8-by-8] checkerboard?” A follow-up question asked students to generalize the solution to checkerboards of any size. Although the primary focus of the task was problem solving and communication, the emphasis on the “write-up” of the solution incorporated elements of generalization and reasoning. Students were given explicit detail about how to write complete solutions that included: a restatement of the problem; a description of the process used to solve the problem, including diagrams; a complete solution, including reasoning and a discussion of generalizations included; and possible extensions to the problem. After completing their own solutions, students were presented with a rubric for rating solutions and sample papers to illustrate performance levels. Although students had struggled with explaining their thinking before the activity, the use of rubrics to rate student writing helped reinforce the teacher’s expectation that writing should be clear and detailed enough so that the reader could follow the thinking of the solution writer. Later in the semester, the teacher was able to connect proof writing expectations to the criteria for high-quality mathematics writing articulated by students during this exercise.

Evidence of Adoption of Meta-Discursive Rules

There was evidence that students in the suburban class, in particular, became fluent in some of the meta-discursive rules of the discipline. In particular, students began to value efficient (i.e., concisely worded) arguments as well as fluid arguments in which ideas flowed logically in a manner that helped guide the reader through the proof. This evolution of students’ ideas was the result of intentional work done by their teacher to prompt students to articulate the rules or guidelines in making an effective argument. The teacher prompted this articulation in a variety of ways. Proofs were shared for public critique in small groups and in the whole-class setting. Typically, one proof was examined at a time and the teacher facilitated discussion about the validity of the proof. In those circumstances, students were left to draw inferences about the value of their own proofs based on their perceived connections to the proof under discussion. But in some instances, the teacher spent most or all of a class period having students critique each other's proofs in pairs and in small groups. Using this mechanism, students were given explicit feedback on their own writing, in a setting other than formal assessment by the teacher. It was during these discussions, that students heard evolving values of the class applied to their own work and had an opportunity to apply their developing standards for high-quality proofs to the work of others.

In the following excerpt, the teacher visited a student group in the midst of critiquing each other's proofs. The students illustrated their values about high-quality proofs in the critiques they offered each other. In particular, they demonstrated a preference for readability, conciseness, and efficiency.

Student 7: Move the angles in your proof. Maybe that would make it a little easier to understand. Do you get what I'm saying? Because you're doing ASA. You proved side first and then the two angles. So it would just be easier to understand if you did Angle, Side, Angle [referring to the order that the corresponding congruencies were presented in the proof].

Teacher: So are you making comments about readability or are you making comments about…?

Student 7: About flow.

Teacher: Ok. So does the fact that he wrote it out of order make it less powerful, or less accurate, or is it just…

Student 7: No.

Teacher: Better for readability?

Student 7: Yeah.

Teacher (to student whose proof was being discussed): Does that make sense?

Student 8: Yes.

Teacher: Student 8 had a comment here…

Student 9: Do you take points off for lengthwise, how big it is [the length of the proof]

Teacher: (shrugs)

Student 9: Do you have to make it even less [referring to the length of her proof]?

Teacher: Let's take it to your friends who are reading your work. What do you think of hers? I know hers tend to be a little bit longer, right? So what about it is longer? Is it extra stuff that should not be there or is it things that she has put in for extra clarity? Why is it longer?

Student 10: Well she puts in some stuff that doesn't make it any clearer. It's just there. And you could shorten it up a lot easier. Like, instead of saying, "CD is congruent to FQ because if two segments are congruent to the same segment, they are congruent," you could just say, "the transitive property." They both work.

Teacher: They both work. But one is a little bit more efficient than the other, right? Does that make sense?

Student 9: Yeah.

Teacher: So maybe what you want to look for as you're writing and as you're reading is: "If I were to shorten mine up, am I going to take away from the proof, or am I going to be more efficient in writing?" Good comments. Thank you.

The process of having students critique and refine each other's work is not typically observed in secondary schools, but was used very effectively in this setting. Each student took the responsibility to critique others' work, as well as offering his or her own work for examination. This activity puts students in the position of having to publicly demonstrate their understanding of the meta-discursive rules governing proof writing. Thus, we can see that students have embraced some of the discipline's standards for communicating mathematical ideas via proofs.

Discussion

The results of the study demonstrate that secondary mathematics teachers can develop the type of classroom environments that promote deep mathematical justification that lead to more effective student engagement in formal mathematical proof. Maher (2009) also described similar findings related to elementary children's experiences with proof and justification. Maher noted that proving can be an extension of children's natural desire to engage in sense making. The present work complements Maher's work and demonstrates how such sense making and proving might be promoted in high school classroom settings.

Students, in our study, began to demonstrate that they were thinking about proof in ways that are consistent with the values and practices of the discipline of mathematics, as Sfard advocated (2000). For example, in these classrooms, we observed students making conjectures, pushing other students to consider solutions and constraints more carefully, and constructively critiquing each other's work to make their arguments and solutions more efficient.

This study focused on the work of two teachers, recognized by peers and supervisors as particularly effective. Both teachers provided students with rich tasks to discuss in collaborative group settings. Beyond that basic structure, the strategic questioning used by the teachers prompted students to justify their thinking, their strategies, and their choices to their peers and to the teacher. The requirement that students publicly critique proofs made overt students' emerging values about what constitutes a convincing argument. Those values were consistent with those of the discipline, and provided evidence that meta-discursive rules related to proof had been adopted by students. Some of these effective practices are consistent with findings of Stylianides and Ball (2008), who concluded that knowledge of different kinds of proving tasks and knowledge of the relationship between proving tasks and proving activity were critical components of teacher's knowledge for teaching proof.

What needs further clarification is how the school environment or other factors contribute to developing teachers who are effective at helping students become adept at proving. Likewise, further exploration about links between teacher knowledge and effective teaching practices should be examined. If the techniques used by these teachers are so effective, why aren't all teachers using such techniques? We suspect that the techniques used by these teachers are not easy to acquire; they appear to require a significant understanding of the content, students' thinking, and the role of proof in the discipline of mathematics. We leave consideration of these possibilities open for further discussion and study.

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AN EXPLORATION OF FIFTH GRADERS' JUSTIFICATION SCHEMES WHEN ENGAGED IN PATTERN FINDING TASKS

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Pattern finding tasks were used to teach algebraic reasoning to a class of 25 fifth graders. The tasks involved students in generalizing rules and justifying their conjectures. Students' written work and data from the transcripts of the video and audio tapes of the class sessions and group discussions were sources of data. Results indicated that these fifth graders used analytic schemes, empirical schemes and also external authorities to justify conjectures. Implications for teaching and learning early algebra concepts are discussed.

Introduction

Increasing numbers of researchers and educators recommend that justifications and reasoning should be the core of any branch of mathematics education (For example, Mueller & Maher, 2009; National Council of Teachers of Mathematics (NCTM), 2000; Sowder & Harel, 1998). Justifications among other purposes support students' understanding of generalizations (Lannin, 2005), and are ways of verifying that a proposition is true, and why it is true (Bell, 1976; Healy & Hoyles, 2000). Additionally justifications are tools for understanding mathematical concepts (Hanna, 2000; Tsamir et al., 2009). Realizing the importance of justifications in mathematics education at all levels, NCTM (2000) includes justification proficiency as one of the requirements for elementary to high school students. Despite such importance, justifying conjectures proves to be a challenge to most middle grade students (Ellis, 2007; Lannin, 2005), high school students (Healy & Hoyles, 2000), preservice teachers (Richardson, Berenson & Staley, 2009) and inservice teachers (Knuth, 2002**).**

Since justifications are linked to understanding mathematical concepts (Hanna, 2000), and because analyzing students' reasoning as they justify conjectures is instrumental to identifying effective teaching strategies (Mueller & Maher, 2009; Sowder & Harel, 1998), the objective of our study is to analyze students' justifications for their algebraic conjectures. Thus, our research question is, how do fifth graders justify their algebraic conjectures when engaged in pattern finding tasks?

Research related to this question has tended to focus on justifications and proofs of students engaged in either different mathematical activities or different age groups and grade levels. In a study with sixth graders, Mueller and Maher (2009) reported students' reasoning and justification strategies while working on tasks involving fractions. The students used direct reasoning, reasoning by contradiction, reasoning by cases and noting the lower and upper bounds of a set to argue for what set an element belongs to. In another study with sixth graders working on pattern finding tasks, Lannin (2005) observed that students used deductive reasoning, giving empirical and generic examples, and also appeal to external authorities to justify their conjectures. Bergqvist (2005) found similar results in a study with students in their 11th year of schooling. Despite NCTM (2000) including pattern finding and reasoning behind conjectures in elementary grade levels curriculum, there seems to be a scholarly gap on how elementary school students justify their conjectures when involved in pattern finding tasks (Lannin, 2005). This study contributes to this area by analyzing fifth graders justification schemes so as to inform educators' choices of teaching strategies that will help develop students' mathematical reasoning.

Theoretical Framework

Proof schemes, according to Sowder and Harel (1998), are ways of ascertaining to oneself and persuading others about the truth or falsity of conjectures or any mathematical idea. These ways or arguments differ from person to person and an individual's schemes may differ from context to context. Sowder and Harel (1998) categorized the justification schemes into externally based schemes, empirical schemes and analytic schemes. These form the framework of our study.

Externally based schemes are arguments that are based on outside sources. They are divided into authoritarian scheme, ritual scheme and symbolic scheme. A student using authoritarian arguments refer to reference materials and people they assume to be more knowledgeable than themselves as basis for their justifications and not at the sense and correctness of the reasoning itself. Similarly, the ritual scheme is based on the form of the argument and not the reasoning. Symbolic schemes on the other hand, involve manipulation of symbols. The manipulation might be meaningless or meaningful but do not give reference to the context of the problem.

Empirical justification schemes are categorized as perceptual and example-based schemes. The example-based schemes involve inductive reasoning whereby one uses examples to verify a general case. Students using perceptual schemes would use images but disregard the generality of the context. Hence they may not be able to justify the same scenario when the images are transformed. In contrast, transformational analytic scheme uses deductive reasoning and takes into consideration the generality of the context. And lastly, an axiomatic scheme is a form of an analytic argument that is deductively organized and uses forms of axiomatic systems.

We adapted this framework to relate it to the context of pattern finding activities and elementary school students' cognitive abilities. Our interest was not on the formal organization of the justification but on the reasoning being evidenced (see Table 1). Therefore, the purpose of the study is to focus on fifth grade students' reasoning schemes and how schemes influence their abilities to justify algebraic ideas. Our specific research question is, what justification schemes do fifth graders use to verify and to convince others of the validity of mathematical conjectures when engaged in pattern finding tasks?

Methods

The participants in this study were 25 elementary school students in a fifth grade class at one of the science magnet schools located in a rural county in the South East of the United States. This class consisted of 14 girls and 11 boys, one African American, one Latino and 23 Caucasians. According to the class teacher, the ability of the students ranged from average to above average. One researcher taught the class for the three day teaching experiment, while others collected data through observations, audio and video recordings. The students' written work was also collected as a source of data. Pseudonyms were used during the teaching experiment and in this report too.

With an aim of teaching algebraic reasoning, the teacher/researcher created a constructivist classroom environment where students worked in small groups, shared their ideas with the whole class and established sociomathematical norms of justifying their conjectures (Store, Berenson & Carter, 2010). Each lesson or session lasted for 90 to 120 minutes. All the students were encouraged to write an input/output table to collect and organize their data. Students were asked to predict and justify how many people may sit around a train of 100 and n tables in pattern tasks (see Figures 1, 2 & 3) that were given during the teaching experiment.

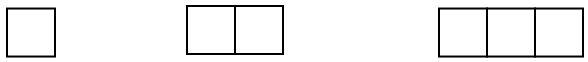


Figure 1. Day 1 pattern task (square tables).

Figure 2. Day 2 pattern task (triangle tables).

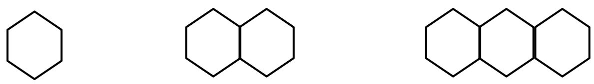


Figure 3. Day 3 pattern task (hexagon tables).

We used the categories in Table 1 developed from Sowder and Harel’s (1998) justification framework to analyze students’ justifications. The researchers coded and recorded the data until a consensus on the category of justification scheme was reached.

Externally based schemes	Authoritarian: Using ideas expressed by others without expressing ownership of the reasoning in those ideas.
	Ritual: Using the form of arguments that have been socially established by the class as acceptable without regarding the reasoning of the content in those arguments.
	Symbolic: Manipulating symbols without any reference to the context of the problem.
Empirical schemes	Perceptual: Using images without considering the generality of the context and manipulating the finite series of the pattern without any reference to the generality of the context.
	Example based: Verifying a few examples by using the values in the input/output tables show that the conjecture holds.
Analytic schemes	Transformational: Using images that show features of the general context and using operations that anticipate possible changes
	Axiomatic: Deductively organized justification. For example, using previously justified rules to argue for a new rule.

Table 1. Justification Analysis Codes (adapted from Sowder & Harel, 1998).

Results

Different justification schemes were evident on each day of the teaching experiment. During the first part of the session of the first day, a significant number (9/25) of students used non-contextual manipulation of symbols as a justification scheme. This scheme was not evident on the second and third days. In all the three days, analytic schemes and empirical schemes were evident. This section presents the results of the analysis for each day that includes examples and evidence of students’ reasoning schemes that were representative of all the schemes in this class.

Day 1 Task Justifications Schemes

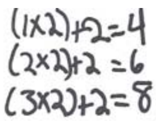
On this day, when students were asked to predict how many people could sit around a train of 100 tables, some conjectured 220 people and others 202 people. They all conjectured that 2n+2 people could sit around a train of 100 square tables. In justifying these conjectures, the following justification schemes were evident.

Externally based scheme. This scheme was evident when students were verifying their conjecture of 220 as number of people who can sit around 100 square tables. Students did not explain further than stating that, since 22 people can sit around 10 tables, and that 100 is 10 times 10, therefore 10 times 22 gives the correct number of people for 100 tables just as in episode 1. In this argument (See Episode 1), there is no reference to the context of the problem posed:

Janice: I got 220 people.
Teacher: Alright, we have 202 or 220. How are we going to be sure which is right?
Janice: Ten tables equals 22 people and so I did ten times 22 and got 220.

Episode 1

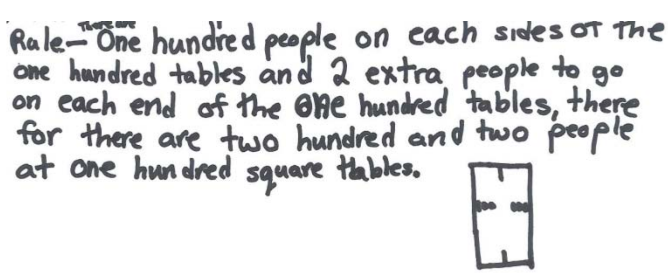
Empirical scheme. To verify that 202 people could sit around 100 tables, some students used their general rules to compute number of people that may sit around 4, 5, 6, …100 tables and used that to argue for the validity of their prediction. Example based scheme was evident when students were verifying that the rule 2n+2. Episode 2 contains an example of the computations used to verify the rule.



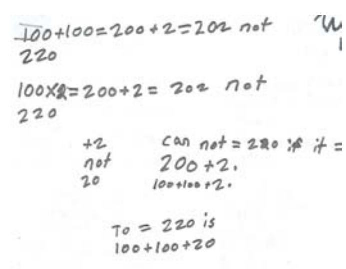
Episode 2

Analytic schemes. A perceptual scheme that recognized the general features of this problem was another common scheme on this day. Episode 3 illustrates an example of such a scheme.

In justifying that 202 was the correct number of people who could sit around 100 tables, the student in Episode 4 used his conviction of the model discussed in episode 3 and also his conviction that the rule 2n+2 was valid to support his conjecture by illustrating that the 220 conjecture contradicts these axioms.



Episode 3



Episode 4

Day 2 Task Justifications Schemes

On this task, to justify the rule n+2 for n triangle tables, the students were asked to explain the +2 in this rule. The following schemes were observed.

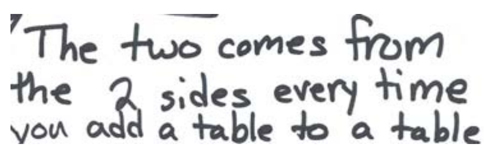
Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

Empirical schemes. Episode 5 illustrates such reasoning from a student's written work and an explanation during class discussions. He explained that, he tested his rule by plugging his input values from the t-table in his rule. Then he checked if the result from this operation was equivalent to the output values in the t table. When this worked for a few examples, he then concluded that his rule is valid.

Student: I did the number of tables plus two equals the number of people. And then I put each number in the T table (the t values) plus two equals (which gave me) the number in the people column right beside it.

Episode 5

Perceptual schemes were also evident on this task. To verify that n+2 was the rule for finding number who could sit around n triangle tables, some students used their visual observation that in building the train model, 2 seats were being added each time a table was added while others considered their perceptual observation that 2 seats were being removed from the sides that the tables were being joined. Episodes 6 from students' written work and episode 7 from transcripts of students' discussion illustrate this reasoning.

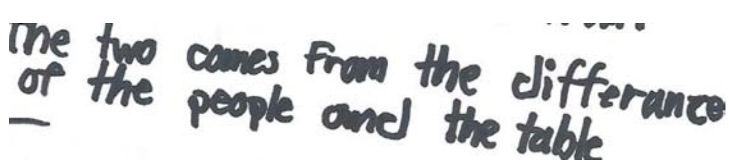


Episode 6

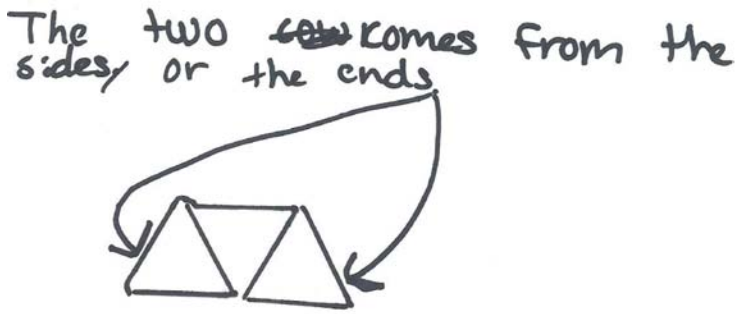
Brenda: Okay, this table (triangle).
Dan: Mm-hmm.
Brenda: Then you take away this (one side of the triangle) right here. They're (two triangles are) getting ready to be put together. So if you have this table and you put it together with this table you have to take this chair away (close one open side of the triangle) and on this table you have to take this chair away (close one open side of another triangle). And then you put them together.

Episode 7

Analytic Schemes. To argue for the +2 in the rule, some students used the general difference between the input and output values that they observed in t-table (episode 8) while others who used this scheme argued that the number the number of people will be the same as the number of tables on the train plus the 2 people who will sit at end sides of the train (episode 9).



Episode 8



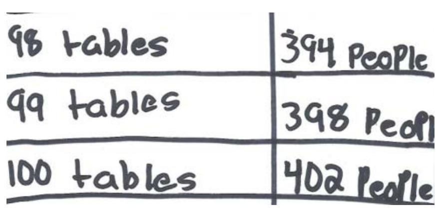
Episode 9

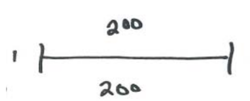
Day 3 Justifications Schemes

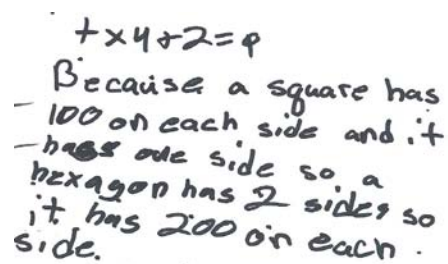
Empirical scheme. To test his rule, Wizard calculated number of people who can sit around 1 table, 2 tables, until 100 tables. He did this after finding the general rule for finding number of people who can sit around n hexagon tables as 4n + 2. Still, in his justification, he did not consider the generality of the context. Below are some of his written calculations (See Episode 10).

Analytic schemes. In justifying that 402 people could sit around a train of 100 hexagon tables, some students argued that each table was contributing 2 seats at the top and 2 seats at the bottom of the train, and that there were 2 seats at the ends of the train. The image in episode 11 was used to explain this thinking.

An axiomatic scheme to justify the rule for finding number of people who would sit around a train of hexagon tables was also used. In episode 12, the student used an already established fact about square train tables (2n+2 =P) and the characteristics of its model (one person sit at the top and another one person at the bottom of the model for each square table) to deduce that 4n + 2 = P for hexagon tables. Her written work shows this line of reasoning.







Episode 10 Episode 11 Episode 12

Conclusions

The purpose of this study was to identify justification schemes that the fifth graders used in this teaching experiment. The results of the analysis show that fifth graders are capable of justifying using both analytic schemes and empirical schemes and that they may also justify using externally based schemes. The type of questions that prompted students to justify was one of the factors that contributed to the range of justifications in the class. For example, justification by contradiction was only evident when students had to decide the correctness of one answer against the other. Similarly, using a general perceptual model was not common when working with triangle tables unlike with the other tasks because triangles did not present an obvious geometric model that related to the rule. To use the axiomatic scheme that was evident on day 3, this student employed the isomorphic characteristics of the tasks used during this teaching experiment to deduce that his conjecture on the 3rd day was correct. Since NCTM (2000) recommends that students should develop fluency with different forms of justifications, and because justifications are instrumental to mathematical understanding, we recommend that teachers provide contexts that support the development of both empirical and analytic schemes.

Endnotes

1. We wish to thank Dr. Kerri Richardson for her assistance in reporting this research.

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A TEXTUAL ANALYSIS OF REASONING AND PROOF IN ONE REFORM-ORIENTED HIGH SCHOOL MATHEMATICS TEXTBOOK

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This study examined the presentation and student development of reasoning and proof within one unit of a reform-oriented high school mathematics textbook. In the exposition sections, 24% of the sentences related to reasoning and proof while 12% of the problems asked students to develop reasoning and proof components. The most frequently occurring category within exposition sections was proof building blocks while supporting claims occurred most often in student problems. The text exposition rarely modeled pattern identification or conjecturing. Moreover, students were infrequently asked to identify a pattern, propose a conjecture, and create a proof for a single mathematical idea.

Introduction

Reform documents (National Council of Teachers of Mathematics [NCTM], 2000, 2009) in the United States have advocated the inclusion of opportunities for students in grades K-12 to engage in mathematical proof. Research has examined students' understanding of proof at a variety of different grade levels and found that they generally struggle in understanding this area (Harel & Sowder, 2007). Research by Healy and Hoyles (1998) suggests that the classroom environment can positively influence students' understanding of proof. One important component influencing what students learn within the classroom environment is the written curriculum (i.e. mathematics textbooks) that teachers use (Stein, Remillard, & Smith, 2007). However, less research has examined students' opportunities to engage in reasoning and proof within the written curriculum. An exception is a recent study conducted by Stylianides (2008) in which he examined reasoning and proof in the number, geometry, and algebra units of the grade 6, 7, and 8 middle school mathematics textbook series *Connected Mathematics Project* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998/2004). He found that 40% of the tasks asked students to engage in some kind of activity situated within his reasoning and proof framework. Of these tasks, 3% expected students to make conjectures, 12% expected more formal demonstrations of mathematical truth, and 62% of the tasks expected students to produce rationales or informal arguments. In addition, he found that reasoning and proof was not apportioned evenly across grades and content areas. For example, the grade 6 number unit contained the greatest frequency of proofs. This lack of attention to reasoning and proof within the written curriculum by the research community is a significant omission in our understanding of how students learn this important area as middle school and high school teachers in the United States frequently use mathematics textbooks in the course of their daily instruction (Grouws & Smith, 2000; Weiss, Banilower, McMahon, & Smith, 2001).

Research suggests that teachers interpret curricula in different ways resulting in an enacted curriculum that may differ in significant ways from the written curriculum (Stein et al., 2007). Nonetheless, while the written curriculum may not have a one-to-one correspondence with the enacted classroom lesson it is suggestive of students' opportunities to engage in a variety of different mathematical ideas. Consequently, this study used a framework adapted from

Stylianides (2008) to examine the exposition sections and student problems within one unit of a reform-oriented high school mathematics textbook for reasoning and proof elements.

Theoretical Framework

Stylianides' (2008) reasoning and proof framework consists of mathematical, psychological, and pedagogical components. The analysis reported in this paper draws upon the mathematical component of his framework, which consists of analyses of reasoning and proof segments by an individual or individuals who are mathematically proficient. However, a number of different adaptations were made to this framework. First, Stylianides conceived of his framework as a table, however, I found it more helpful to consider it as a model as seen in Figure 1, which depicts the relationships between different components. Mathematicians' accounts of their own work suggest that they begin by identifying patterns, making conjectures about those patterns, and developing arguments to show the validity of those conjectures (e.g., Polya, 1954).

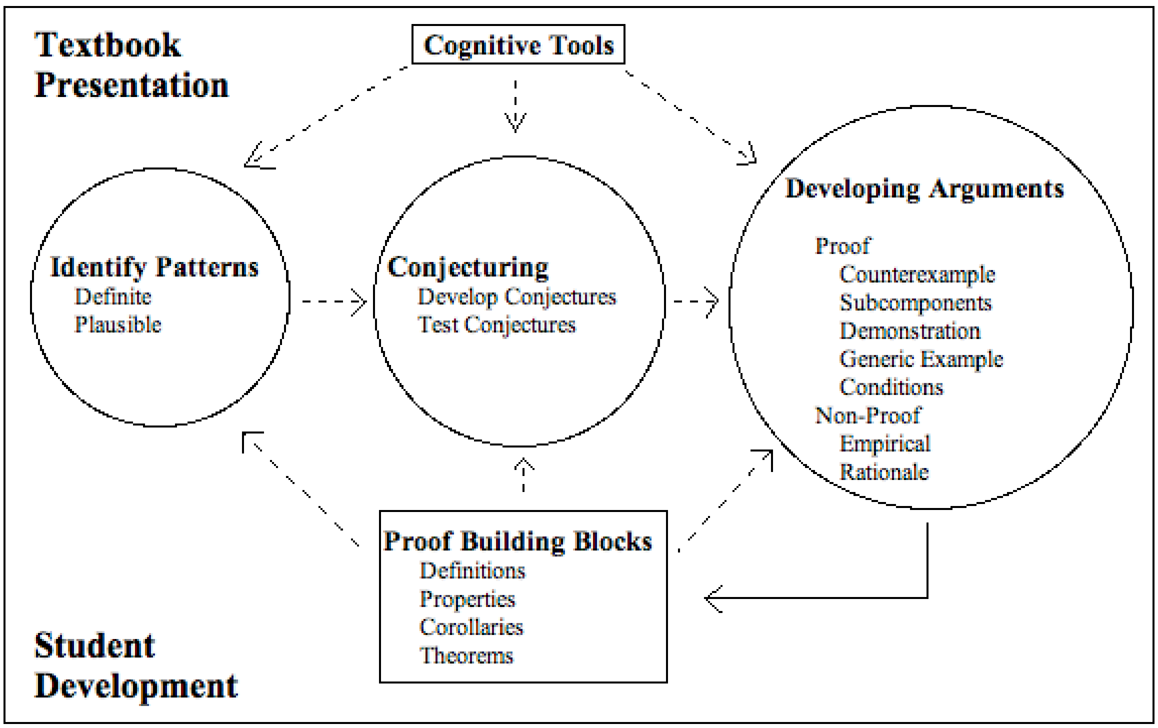

Figure 1. Reasoning and proof model describing the relationship between pattern identification, conjecturing, and developing arguments within textbook presentation and student development elements of curricular resources

Second, Stylianides' model focused solely on analyzing student work associated with reasoning and proof. However, an assumption of this study is that the textbook exposition sections can also influence students' understandings about reasoning and proof. Therefore, the framework was expanded to include textbook presentation. However, the dotted lines in the model suggest that student work or the textbook exposition could begin in one location and be confined to that area of the model or could progress to other components.

Third, the framework was adapted to include the potential role that cognitive tools (Zbiek, Heid, Blume, & Dick, 2007) can play in identifying patterns, making conjectures, and crafting arguments. For instance, students may use technology in the form of graphing calculators to examine a variety of representations of quadratic functions in order to make a conjecture about the number of times these functions may intersect the *x*-axis. Cognitive tools include microworlds, simulations, as well as technologies that enable the user to connect a variety of different representations (e.g., graphing calculators). As Zbiek et al. note, it is important to understand how learning is different within technological environments and this framework takes into account the role of technology in promoting students' understanding of reasoning and proof within curricular resources.

Fourth, although Stylianides' (2008) framework involves proofs that seek to concretize new knowledge, the role of this knowledge in the development of other proofs was not highlighted. Thus, there is an arrow leading from proofs to what I describe as *proof building blocks*. In addition, mathematicians develop other proof building blocks such as definitions (e.g., degree of a polynomial) that can be used in pattern identification or other components of the framework. This work is highlighted in the dashed arrows leading to the different reasoning and proof components.

Fifth, changes were made to the framework to account for the possibility that not only would students make conjectures but also be asked to test them. The student textbook can present a conjecture within the exposition sections and subsequently describe testing its validity or it may ask students to perform this action either with a conjecture that they themselves created or with a conjecture provided by the textbook authors. This conjecture testing may lead to justification in the form of proof or non-proof arguments.

Sixth, although mathematicians must develop complete proofs, a pedagogical device often used by school mathematics textbook developers is to ask students to provide specific components of the outline of a proof. Such an artifact is particularly prevalent in textbooks designed for school geometry where students are asked to fill in blanks for statements or explanations in two-column format. Consequently, the original framework was adapted to admit the coding of proof subcomponents.

Methods

Reasoning and proof has historically resided in the high school geometry classroom (Gonzalez & Herbst, 2006). However, NCTM has advocated the integration of reasoning and proof across different mathematics content strands (2000, 2009). An area of mathematics that many high school students in the United States complete is algebra, thus I was interested in examining how reasoning and proof were integrated within this common mathematics content area. There are a number of different areas within algebra that students have the opportunity to learn in the course of school mathematics. This study examined reasoning and proof within functions as this content area has been identified as a big idea in reform documents (NCTM, 2000) in mathematics and hence of importance in students' future mathematics studies. One unit on polynomial and rational functions was chosen from the student textbook in the third course of the reform-oriented high school mathematics program *Core-Plus Mathematics* (Fey et al., 2009). All analyses were conducted on the teacher's edition of this textbook as it contained the student textbook and expected answers to student problems on facing pages.

A *Core-Plus Mathematics* unit is broken down into lessons, each of which contains several investigations. Each lesson begins with a launch that is designed to introduce students to a specific context through which students learn about mathematical ideas in the investigation phase of the lesson. At the end of each investigation, students, with the assistance of the teacher, summarize the main mathematical ideas within the *Summarize the Mathematics* section. Afterwards, students are provided with additional practice in the *Check Your Understanding* and *On Your Own* sections of the text. The unit on polynomial and rational functions examined for this study involves properties of polynomials, operations on polynomials, completing the square, solving quadratic equations through the quadratic formula, vertex forms of quadratic functions, definitions and properties of rational functions, and operations on rational functions.

In order to determine the proportion of the exposition that involved presentation of reasoning and proof components the number of sentences appearing in exposition sections in the unit were counted. Exposition sections were those that included areas outside of numbered questions as well as sentences appearing in question sections that provided the reader with proof building blocks (e.g., definition). The total number of problems appearing in the unit was calculated in order to determine the percentage of problems devoted to reasoning and proof. While the *CPM* textbook indicates each question that students are to complete with a number or letter, each question may have several problems for students to complete. For example, the following appears as a single question in the student textbook. "5. Sketch a graph of the function $f(x) = x^2 - 6x + 13$. Explain how it shows that there are no real number solutions for the equation $x^2 - 6x + 13 = 0$" (Fey et al., 2009, p. 354). However, this was coded as including two problems since students must sketch a graph and develop an explanation. It was also possible to have rational numbers that were not integers as part of the problem count. A problem may have asked students to develop a conjecture and justify this conjecture within one sentence. Thus, this would have been coded as 0.5 for develop a conjecture and 0.5 for developing either a proof or non-proof argument.

Coding of reasoning and proof components within the *CPM* textbook began by proceeding through the written student textbook line by line. As sentences or questions were encountered in the textbook that were connected to the reasoning and proof framework either by using the definitions for each category described above or using the set of examples in Stylianides (2005) they were transcribed into a written electronic document. In addition, each example was coded for category within the reasoning and proof model (e.g. pattern), type (e.g., conjecture non-precursor), and technology use or not. A problem was considered to include technology if the answer in the teacher resources described the use of technology in the solution, the teacher resources showed work involving technology (e.g., a graphical representation on a calculator screen), or the student textbook asked students to use technology in answering the question in some way. In addition, a detailed justification was provided for each coding and the page number where each occurred in the student textbook was noted. Next, the instances of reasoning and proof in the electronic file were organized into tables by category (e.g., pattern) to facilitate analysis.

Conjectures were distinguished from patterns in two ways. First, the textbook sometimes specifically labeled student work as conjectures as in the following sentence: "Test your conjecture in Part c by graphing quadratic polynomial functions …" (Fey et al., 2009, p. 325). Second, if a sentence asking students to notice a relationship contained uncertainty then it was coded as a conjecture. This is seen in the example, "How do the results of your work on Problems 6 and 7 *suggest* a strategy for writing any quadratic expression …" (Fey et al., 2009, p. 351, emphasis added).

Rationales were distinguished from proofs using two conditions. First, an argument was considered a rationale if in addition to justifications involving assumed truths it also contained empirical justifications. Second, rationales contained one or more statements that were not assumed truths and were not justified. Assumed truths were determined by examining the algebra and function units from the third course as well as from previous courses. If these notions appeared within the *Summarize the Mathematics* section or were justified using proofs they became assumed truths for coding purposes.

The connection between the presentation of proof building blocks in the exposition sections and reasoning and proof components that students were asked to develop was also examined. That is, if a definition is presented in the textbook are students asked to develop conjectures involving this definition or are exposition sections isolated from student work? This analysis was conducted on electronic versions of the teacher's edition of the textbook, which contains the student textbook and the teacher answers together in one resource. As proof building blocks were encountered in the exposition sections of the student textbook, electronic searches were conducted for these words to determine if they appeared in student problems in later sections within the focus unit.

Results

Within the exposition sections of the focus unit there were 189 sentences, 44.5 or 24% of which were related to reasoning and proof. The breakdown of the different categories is shown in Table 1. As seen in the table the majority of the reasoning and proof presentations in the text were proof building blocks. These occurred twice as often as justifications of mathematical claims. The majority of these building blocks were definitions such as the following, "A **polynomial function** is any function with a rule that can be written in the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, where n is a whole number and the coefficients $a_n,\ a_{n-1},\ a_{n-2},\ ...,\ a_1,\ a_0$ are numbers" (Fey et al., 2009, p. 323, emphasis in original). Some of these building blocks came in the form of facts since they did not require a formal proof nor did they appear to be more recognized mathematical properties. One fact appeared on p. 342, "For higher-degree polynomials, graphs of two polynomial functions of the same degree can have different shapes." Only four of the 29.5 sentences associated with building blocks involved the presentation of statements regarded as theorems.

Interestingly, the exposition sections did not model identification of patterns, conjecture development, conjecture testing, or justifying mathematical ideas. Only one proof was provided in the exposition, a visual proof of completing the square involving the expression $x^2 + bx$. Proof subcomponents in the form of statements were presented in the text for the quadratic formula and the factor theorem. Students were asked to provide the explanations for the statements appearing in these proofs.

Table 1. Presentation of Reasoning and Proof Components

Category	Frequency	Percent
Supporting Claims	15	34%
Building Blocks	29.5	66%
Total	44.5	100%

Table 2 shows how the proof building blocks introduced in the exposition sections of the student textbook were connected to student problems involving reasoning and proof. Only five of the 29.5 instances of proof building blocks were isolated or not connected to student reasoning and proof problems. Some of the remaining 24.5 occurrences were coded multiple times thus there are a total of 28 connections to student problems involving reasoning and proof. Proof building blocks were most frequently connected to students' opportunities to identify patterns.

Table 2. Use of Building Blocks in Student Textbook

Category	Frequency	Percent
Isolated	5	17.9%
Pattern	9	32.1%
Conjecture	6	21.4%
Rationale	7	25.0%
Demonstration	1	3.6%
Total	28	100%

There were a total of 1,114 problems in the fifth unit of the third course of *CPM*. Of this total, 137 or 12% asked students to answer problems related to reasoning and proof. The breakdown of the different categories is shown in Table 3. Students were rarely asked to develop definitions or test conjectures while identifying patterns and supporting claims were the most frequently occurring reasoning and proof tasks. Students were asked to identify definite patterns only; no plausible patterns appeared in the text.

Table 3. Reasoning and Proof: Development

Category	Frequency	Percent
Creating Building Blocks	3	2.2%
Testing Conjectures	4.5	3.3%
Developing Conjectures	20.5	15.0%
Identifying Patterns	49	35.8%
Supporting Claims	60	43.8%
Total	137	100%

The *CPM* curriculum developers frequently expected students to use technology when identifying patterns, developing conjectures, and testing conjectures as seen in Table 4. For instance, within the second investigation of the first lesson the text asks, "How does the degree of a polynomial seem to be related to the number of zeroes of the related polynomial function? Test your conjecture by studying graphs of some polynomial functions of degree five and six" (p. 331). However, there were no demonstrated uses of technology within the exposition sections of the textbook.

Table 4. Technology Use Within Different Reasoning and Proof Components

	Technology Use	
Category	Yes	No
Identifying a Pattern	40	9
Developing a Conjecture	19.5	1
Testing Conjectures	3.5	1
Total	63	11

The greatest frequency of students' reasoning and proof development was within supporting claims as seen in Table 3. The breakdown of expected student answers between non-proof and proof based arguments is seen in Table 5. This table shows that there was an equal incidence of proof and non-proof arguments however; no generic examples appeared in the student textbook problem solutions.

Table 5. Breakdown of Supporting Claims by Non-Proofs and Proofs

Category	Frequency
Non-Proof	
Empirical	26
Rationale	4
Proof	
Demonstration	17
Generic Example	0
Subcomponents	13

Discussion

There were several noticeable shifts from Stylianides' (2008) analysis of reasoning and proof at the middle school level and this analysis of a high school mathematics textbook unit. First, there were fewer incidences of reasoning and proof found in the tasks from 40% at the middle school level to 12% at the high school level. Second, the majority of arguments that middle school students were asked to construct was in the area of rationales or informal arguments while high school students were expected to create informal and formal arguments with equal frequency. Third, the majority of the patterns at the middle school level were plausible while in this high school unit all of the patterns were definite.

Focusing exclusively on the high school unit examined here, there was evidence of a misalignment between the exposition sections and the problems that students were asked to complete. For example, the exposition sections presented proof building blocks such as definitions while students were rarely asked to perform this work in problems. Students frequently found patterns and developed conjectures using technology, but the authors rarely engaged in this action in the exposition sections. On the other hand, the definitions that were presented in the student textbook were frequently the source of students' work in other aspects of the reasoning and proof model.

As described earlier, mathematicians' work typically begins with noticing patterns, developing conjectures on the basis of these patterns, and determining and validity of conjectures through the construction of proofs. While students working on this unit from *CPM* experienced problems within these different areas they did not frequently travel through these three steps in succession. Students were asked to notice patterns, but rarely did they construct conjectures on the basis of these patterns. Students were asked to develop conjectures, but these did not follow from the patterns they had noted in other problems. When students were asked to explain their work this usually involved statements provided by the authors of the curriculum, not from conjectures that they had previously constructed in response to questions. Thus, there appeared to be a lack of alignment between how mathematics is practiced by mathematicians and the work that textbook authors asked students to perform. These findings suggest how curriculum developers could develop future secondary mathematics curricula. Providing students with more opportunities to develop proof building blocks and use them in all three areas of the framework.

Future steps in this research area include analyzing the reasoning and proof components within an entire textbook, across different courses of the same textbook series, or within different textbook series. Analyses of reasoning and proof elements within other textbooks could be used to compare textbooks and provide information to teachers who are considering textbook adoption. Additionally, analyses of the written curriculum could provide a springboard from which the enacted curriculum could be examined.

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ENCULTURATION TO PROOF: A PRAGMATIC AND THEORETICAL INVESTIGATION

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We report on an investigation in a transition-to-proof course of undergraduate students' epistemological shifts in mathematical argumentation and identify pedagogical factors that foster and/or constrain students' ability to create mathematical arguments. We view proof as a social process in which participants of a learning community move from peripheral participation to more full members of a community engaged in the mathematical activity of proving. This report analyzes the impact of a pedagogical intervention that modifies the classic format of a two-column proof and reifies the presentation of a proof and the questions asked by the reader of the proof. We summarize our analysis in a framework that characterizes enculturation to proof.

Introduction

Cobb (2000) describes three prongs that are characteristic of a profound shift in the field of mathematics education. The first shift is described as one from students as information processors to students acting purposefully in a mathematical reality they are constructing. A second shift is described as a view of students' mathematical activity embedded within the evolving classroom microculture and the larger cultural sphere. The third shift is described as one from theory informing practice to a view of theory and practice guiding each other.

Our work in teaching proof to prospective secondary mathematics teachers reflects all three emergent trends in the following ways. First, our perspective on learning proof is that it is not just a cognitive endeavor. We characterize learning as both a process of individual construction and a process of enculturation. Second, from our perspective, individual student activity is seen to be located within broader systems of activity and the norms constituted in class are reflexively related to shifts in beliefs about mathematics (Yackel & Rasmussen, 2002). Third, we engage in the development of a theoretical framing of students' enculturation into proof, grounded in classroom settings. In particular, our experience as teachers of a transition to higher mathematics class with an innovative pedagogical intervention evolved into a systematic study of the intervention.

Prior research literature offers a number of different frameworks for classifying students' justifications (e.g., Balacheff, 1988; Chazan, 1993; Harel & Sowder, 1998; Healy & Hoyles, 2000) and it points to a number of difficulties that students have in creating proofs (e.g., Tall, 1991; Selden & Selden, 2003, 2008; Weber, 2001). One of the difficulties that undergraduate students have is that they sometimes see proofs and proving as unrelated to their own ways of thinking (Seldon & Seldon, 2008). Part of the difficulty with proving involves the need for the learner's increased awareness of and sensitivity to disciplinary norms and what statements in an argument need to be justified.

In contrast to these cognitively-oriented frameworks, we view proof as a social process in which participants of a learning community move from peripheral participation to more full members of a community engaged in the mathematical activity of proving (Lave & Wenger, 1991). Proof is a central practice of the mathematical community. Furthermore, as prospective secondary mathematics teachers, they will likely teach proof. This report analyzes the impact on

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students of a pedagogical intervention to help answer the question: How can we characterize the enculturation process as students become more central members in the practice of proof?

Setting & Participants

The study reported here was conducted with undergraduates at a large, urban university in the United States. The participants were pre-service secondary teachers enrolled in an upper-division mathematics course intended to be an introduction to proof in upper-division classes. Of the 24 students (13 males and 11 females) in the class, 7 agreed to participate in individual interviews conducted at the end of the semester. The interviewees were 5 males and 2 females. They were all upper-division mathematics students, though four, as seniors, had participated in other upper-division mathematics classes; three were juniors and 'new' to proof.

The first author was the instructor of a transition-to-proof course intended for students who are interested in teaching secondary school mathematics. The second author conducted the interviews. This course is intended to precede their upper-division mathematics courses, though in practice this is not always the case. A primary goal of the course is for students to develop expertise writing proofs and solving problems. In this course, we take a view of proof as a convincing argument that answers the question 'Why?" Thus, the primary function of proof for the students in our undergraduate educational setting is verification and explanation (Hanna, 2000). Euclidean and non-Euclidean geometry is the content area in which we develop reasoning and communication.

The text, Henderson and Taimina's (2005) *Experiencing Geometry: Euclidean and Non-Euclidean with History* (3rd Edition) consists of a series of problems that ask students to make conjectures and then to justify the conjecture. A goal of this course is that students will develop personally meaningful solutions to problems by drawing on intuitive notions to understand geometric concepts. They are then asked to communicate their mathematical thinking and activity to others. Zandieh, Larsen, and Nunley (2008) describe the process of developing a proof starting from intuitive ideas and conjectures as consisting of two phases. The first phase is described as 'getting a feel for why the statement is true" and the second is described as "working out the details" (p. 135). Students are pushed to answer the "why", using this as an opportunity to develop a more formal argument. Our pedagogy focused on helping students understand why something is true and then, with the help of peers and the instructor, develop expertise in writing proofs more formally.

In this study, the structure of the class meetings typically involved working in small groups of four on challenging problems and presenting the group's progress on these problems in whole-class discussions. The class developed an expectation that students would also be expected to question and comment on each other's offerings and preliminary presentations. These questions and comments were usually in the form of questions posed to the presenter of the proof and his or her group.

The first author developed a pedagogical intervention intended to raise awareness of and sensitivity to the need to query support for statements made in mathematical arguments by drawing a line next to the presented proof and recording the questions asked of the proof creator. For example, Figure 1 illustrates one such scenario. A representative from a group presented their tentative argument. Other students in the class asked questions. Their questions were recorded to the right. (See Figure 1.) This intervention was a modification of the classic format of a two-column proof, which facilitates the creation and evaluation of a proof by making explicit statements and reasons (Herbst, 2002). This modified two-column format reifies the presentation of a proof and the questions asked by the writer or a reader of the proof. The pedagogical innovation emerged in the semester prior to the study and the instructor intentionally enacted the pedagogical innovation in a subsequent semester. We view the modified two-column proof format as a boundary object (Star & Greismeier, 1989), that is, an interface between the students of the class and the instructor as an experienced member of the community into which they were becoming more central participants. Just as mathematicians internally question their understanding and the validity of the proofs they read, so, too, students learned to ask these questions.

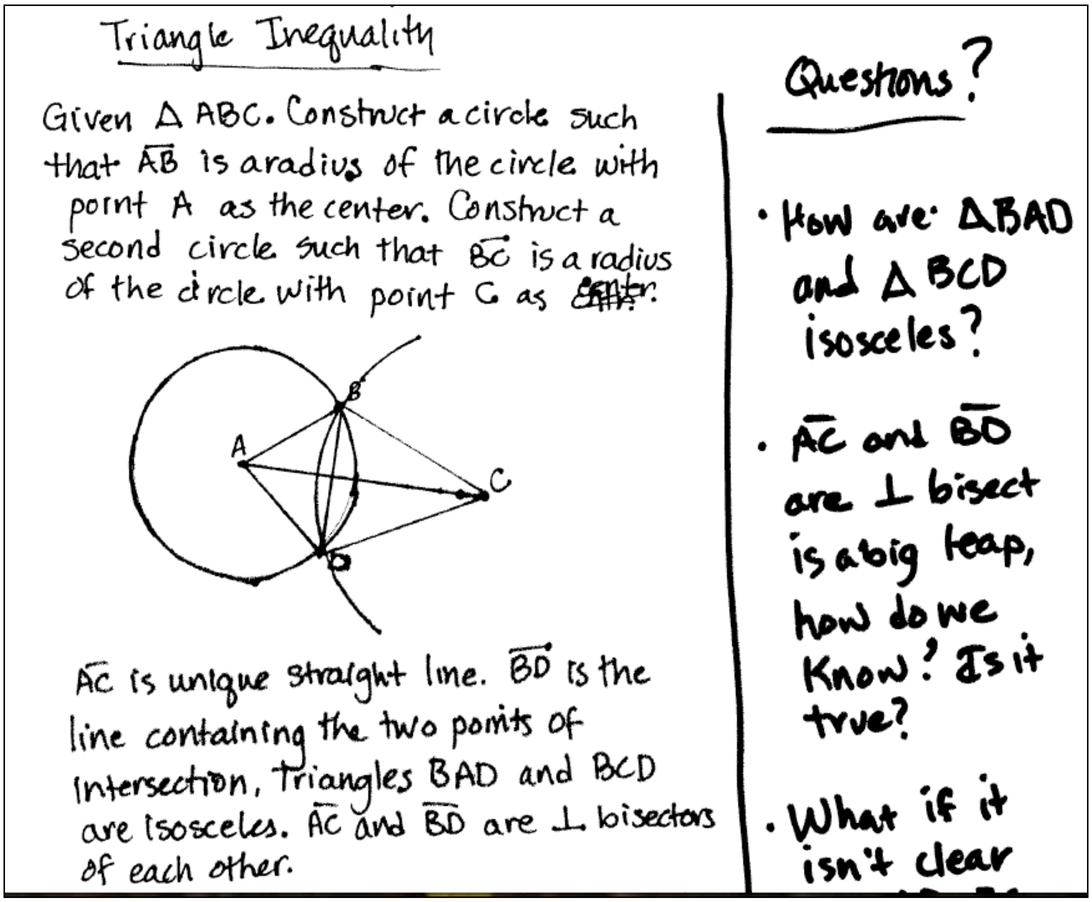

Figure 1. A portion of a proof along with recorded questions from students.

Methodology

Data

The data sources for the study drew from instructor's reflective journal, copies of artifacts collected during the semester, and video-recorded individual interviews. Specifically, the data corpus consisted of the following:

- Transcripts of post-semester clinical interviews with 7 students;
- The instructor's record of instructional decisions, which sometimes included accounts and interpretations of classroom events, as well as rationales for instructional design decisions;

- Captured collective work—for example, overhead transparencies, which include a group's convincing argument and a record of the class' questions in response;
- Students' relevant written work including responses to exam questions and selected homework that entails creating and critiquing proofs.

Seven subjects participated in post-semester semi-structured, task-based interviews (Goldin, 2000). The interviews were designed to reveal how students engaged in the practice of proof. Further, students were prompted to reflect on how they engaged in proof or rather how the activities and practices of the classroom community might have contributed to their engagement in practice of proof. Three tasks and reflection on the proof creation and proof critiquing process comprised the interview protocol.

The first task entailed asking an interviewee to construct a proof for a novel problem. In the second task, the interviewee was then presented with a student's convincing argument for the conjecture posed in the first task. Thus, the interviewee was asked to critique another's proof of the task that he or she had just constructed. For the third task, a problem the students had previously undertaken as homework was represented with another student's solution. In the second and third task, the task had the modified two-column format (that is, the pedagogical intervention students were familiar with from class) labeled Convincing Argument with an accompanying column for Questions.

After completion of the three tasks, the interviewer asked the students to reflect first on the process in which they had just engaged as they created convincing arguments and second on the process, as they had critiqued others' proofs, and finally, on the practice as situated in the activity of the classroom community.

Methods

Our initial goal for the analysis was to uncover the diversity of ways in which students were reasoning about purported proofs, their ability to construct proofs, and their perspective on the modified two-column format. We therefore engaged in what Strauss and Corbin (1990) refer to as open coding, which is the process of selecting and naming categories from the analysis of data. Specifically, we began the analysis for diversity of student reasoning by first examining the seven end-of-semester interviews. We then triangulated this analysis using the constant comparison method (Glaser & Strauss, 1967) by examining all documents collected during the semester that provided additional information on student use of the modified two-column format. These documents included copies of overheads that the teacher and her students produced in class, copies of student homework, and copies of student exams.

We then engaged in the process of making explicit connections between categories and sub-categories. This step is what Strauss and Corbin (1990) refer to as axial coding. The aim of this step was to put our analysis together in new ways. Specifically, we came to see our analysis as a paradigm case of the process of enculturation into mathematical proof. As Strauss and Corbin (1990) argue, a researcher's ability to see an analysis in new ways stems largely from his or her theoretical sensitivity. Sources of theoretical sensitivity include the research literature, professional experience, and personal experience.

Results

A major result from this analysis was the framework in Figure 2. This emergent framework portrays three dimensions along which we describe the transition of students from newcomer to more central participants. The central dimensions consist of the *Manner* in which a learner engages with proof, the *Criteria* that a learner brings to bear on proof and the *Positioning* of self with respect to proof. Our analysis revealed critical aspects of students' positioning in the transition along the continuum of newcomer to more central participant. Each of these dimensions contains several distinguishing features that differentiate newcomer from more central participant in the practice of proof. Moreover, these dimensions can be used to describe both activity as proof creator and proof critiquer. We illustrate briefly here using examples from the interviews of students we characterize as being more central participants or newcomers to the practice of proof.

A student who had some experience with proof prior to and concurrently while participating in this class carefully considered what was to be proven and took on the role of skeptic in proof creation. The initial pass through the proof tended to be wholistic, scoping out what needed to be done before beginning to write. A more central participant framed the proof: "So, now I am convinced that it works in all possible cases. So the next thing I would do is to break down into these cases and prove each case using arguments that I used, you know, and just assign everything variables and what not." As a proof critiquer, his *manner* was one in which he tended to review the proof first holistically, paying attention to the structure of the proof and playing the believing game initially. In contrast, a student who was a relative newcomer tended to read proofs to be personally convinced. After creating a proof, he was asked to critique another student's proof and tended to read the proof to be personally convinced. He said, "They pretty much followed everything I did. That's why I didn't question it….I don't see why I would question it."

The *criteria* a learner brings to bear vary in terms of the attention to and certainty about disciplinary norms. These can include, for example, whether in a proof critique, they can rely upon the geometric picture as evidence and the expectations for the efficiency and elegance. A relative newcomer doesn't provide or expect warrants unless they themselves are having difficulty following the argument. A more central participant is one who expects explicit warrants even in cases where they follow the argument. As one said, "I know it is true and he is not wrong. But if you are doing a proof you need support, you need a reason why and he didn't state it. It would have made a stronger argument." A newcomer tended to use variables and notation that encumbered his or her thinking about a problem, whereas a more central participant leveraged notation and tended to be explicit about what they were doing.

Finally, we frame a difference in the individual positioned as a tentative questioner versus one who takes ownership of the critique, who reads for the purpose of understanding someone else's point of view and speaks as if he or she was a member of a larger group. A tentative critiquer discussed the difficulty in taking on the role of critiquer, "The first time I saw this in class I had a problem with questioning...I feel like for me it is a little negative…like look at this person's proof. What's wrong with it?" But it's not you are just asking questions for clarity." A more central participant indicated a view of self as teacher/facilitator using language such as "Because it says P has to lie on the outside. It is maybe a special case. I want him to explore it more." A more central participant reads for the purpose of understanding another's perspective and sometimes critiques as if he or she is a member of a larger group. A more central participant was comfortable in the role of asking questions: "[As a teacher] I need to be able to ask questions because if I am not asking questions, then what I am doing is giving answers. I need to be able to leave the student to find their own responses or answers…"

We see these three dimensions of manner, criteria, and positioning as derived from the reflexive relationship between a individual and the community in which he or she participates. In the larger sense, the social norms are reflexively related to the beliefs and values of a proof creator and critiquer. The criteria one brings to bear on proof are related to the socio-mathematical norms negotiated within the classroom community. The manner in which they engage in the activity of creating and critiquing proofs is related to classroom math practices.

Conclusions

We propose that our analysis outlines indicators of the journey that students take as they become more central members of practice of creating and critiquing proofs. Our work contributes pragmatically to the pedagogy that fosters enculturation in ways that are commensurate with the discipline (Weber, 2002). In particular, we believe this intervention provides support for their future work as teachers. When asked to reflect on the usefulness of this pedagogical intervention, one student said,

> *It has definitely helped me…Just because you get the outside voices coming in. You get your partners telling you, 'I don't understand' and being a prospective teacher you can't be just like 'Here it is" …You need to be able to respond to questions and ..how you got there.*

Our analysis also showed that the pedagogical innovation played a role in promoting shifts in the positioning or identity as a mathematics teacher. We further propose that our framing of enculturation to proof as human and social activity offers an alternative lens which complements the extensive research on the cognitive aspects of understanding students' challenges with proof.

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Newcomer	More central
***Manner** in which a learner engages in proof*	
As proof creator • Uses concept images to structure thinking • Improvised, trail-blazing character • Labeling and notation not leveraged as a tool for thinking • Cannot take on the role of skeptic **As proof critiquer** • Pass tends to be local, line by line • Adds warrant and assumes writer had it in mind • Reads argument to be personally convinced	**As proof creator** • Uses concept definitions to structure thinking • Organized and reflective exploratory talk • Labeling and notation used as a useful tool • Can take on the role of skeptic **As proof critiquer** • First pass is wholistic, paying attention to structure • Plays the believing game and leaves markers to go back and confirm • Reads argument to ascertain if skeptics would be convinced
***Criteria** that a learner brings to bear on proof*	
As proof creator • Arguments sufficient to convince oneself • Attention to a small set of disciplinary norms • Appears unaware of proof templates **As proof critiquer** • Expects explicit warrants when personally in doubt • Attention to small set of disciplinary norms	**As proof creator** • Arguments must also convince skeptics • Attention to a larger set of disciplinary norms • Explicit awareness of proof templates **As proof critiquer** • Expects explicit warrants even in cases where they follow the argument • Attention to a larger set of disciplinary norms
***Positioning** of self with respect to proof*	
As proof creator • Tentative proof maker • Speaks as needing others to facilitate and assist • Speaks as if unaware of connection to a larger group **As proof critiquer** • Tentative identity • Tentative questioner • Speaks as if unaware of connection to larger group	**As proof creator** • Confident proof maker • Speaks as teacher/facilitator for others • Speaks as if s/he is a member of a larger group **As proof critiquer** • Takes ownership of the critique • Reads to give feedback to writer • Speaks as if s/he is a member of a larger group

Figure 2. Enculturation to Proof Framework (EPF)

JUSTIFICATION IN MIDDLE SCHOOL CLASSROOMS: HOW DO MIDDLE SCHOOL TEACHERS DEFINE JUSTIFICATION AND ITS ROLE IN THE CLASSROOM1?

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> *A good classroom proof would convince a skeptical mathematician, as well as explain to a naïve undergraduate [or middle school student]* (Hersh, 1993, p. 398).

Recent reform movements and standards documents call for all students in all grades to engage in mathematical justification. For this to be possible teachers need to understand mathematical justification themselves and know how they can use justification in their classrooms to promote learning. In this study we examined 12 middle school teachers' concepts of justification and the role they see it playing in their classrooms. Results indicate that 10 of the 12 teachers included 'explaining their students thinking' into their definition of justification. Six of the 10 coupled this with 'explaining why something works' while the other four did not. Eleven of the 12 teachers saw the role of justification in their classroom as helping students learn and 7 teachers used justification for assessment purposes. The teachers' concepts of justification and their vision of its role impact how they react to student responses.

Introduction

Consider the following two student solutions to the problem ½ + ¼ (see Figure 1).

Claire's Response	**Ahmed's Response**
I know that ½ is the same as two ¼ s. So, altogether I have three ¼ s, or ¾. 	I found a common denominator, which was 4, and then I added $\frac{2}{4}$ and $\frac{1}{4}$ to get $\frac{3}{4}$. $\frac{1}{2}=\frac{2}{4}$ $+\frac{1}{4}$ $\frac{3}{4}$

Figure 1. Two student solutions to ½ + ¼

Would you consider either (or both) of these solutions a justification? Why or why not? We encourage the readers to take a minute to decide their answer and reasoning before proceeding. Pondering the answer to this question raises the next question: A justification of what? There are two things that could be justified in this example, one is the answer (3/4) and one is the process

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of getting the answer. So we see that answering the question of whether either (or both) solutions are justifictions is not as straight forward as one may think. Whether one determines either Claire's response or Ahmed's response a justification will depend on two things: (a) one's own definition of justification and (b) how one sees the role of justification play out in a specific context (such as a middle school classroom). The literature provides several definitions for justification (or proof) that may also apply in the middle school classrooms. For example, "proof [or justification] is just a convincing argument, as judged by competent judges" (Hersh, 1993, p. 398) and in the middle school environment the middle school students would take on the role of justifying and judging (as competent judges). Another way of looking at justification may be "the process employed by an individual to remove or create doubts about the truth of an observation" (Harel & Sowder, 1998, p. 241). Both these definitions talk about proof (or justification) as convincing and/or removing doubt. In this paper we turn to middle school teachers and examine their concepts of justification and what they see as the purpose of using justification in their classrooms.

Purposes

In recent years several organizations have called for justification[2] to be a part of the mathematics education of students at all levels (i.e. NCTM, 2000). The NCTM *Principles and Standards* state explicitly that all students K-12 should *recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures; develop and evaluate mathematical arguments and proofs; select and use various types of reasoning and methods of proof* (p. 56).

In addition the NCTM *Standards* note, "reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts" (p. 56) and "should be a natural, ongoing part of classrooms discussions" (p. 342). With justification permeating all grades, many contexts, and classroom discussions, the question of how teachers think about justification arises. If teachers are expected to leverage and promote student justification, the teachers need to have an understanding of justification themselves as well as an idea of how to use it in their classrooms. As a field, we know relatively little about teachers' views and practice with respect to justification (Knuth, 2002a, 2002b).

In this paper we report on the first part of a three-year project designed to investigate how teachers' understanding of justification develops and how this understanding impacts, and is impacted by, their teaching. To begin this work, we documented, via interviews, teachers' conceptions of justification and its role in their classrooms, including how that manifests itself when they consider specific student responses.

Perspectives

Two major types of understanding have been identified in the literature, one focusing on "how" to do something the other on "why" it works. Hiebert & Lefevre (1986), for example, distinguish between conceptual and procedural knowledge. Conceptual knowledge "can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (pp. 3-4). By contrast, procedural knowledge focuses on "rules or procedures for solving mathematical problems" (p. 7). Teaching with justification honors children's thinking and centralizes their sense making through a focus on conceptual knowledge. However, in many classrooms the focus lies on developing procedural rather than conceptual knowledge.

> In many school mathematics classrooms, students often are encouraged to use results and methods based on the authority of the teacher, rather than on a mathematical argument that the method produces the desired result (Chazan & Lueke, 2009, p. 28).

Justification plays a crucial role in classrooms that do not follow what Chazan and Lueke describe as typical practice. In such classrooms, teachers expect their students to build their own understanding and explain why things work. For teachers to be able to enact classrooms that allow students to develop conceptual understanding and take on some of the mathematical authority through justification, the teachers need to explicate their own understanding of justification and need to be explicitly aware of how and why they use it. This will allow the teachers to work with their students to establish what counts as a justification and allows the teachers to give feedback to their students on the process of justification.

Few studies have sought to understand teachers' conceptions of justification and its role in the classroom. Knuth (2002a, 2002b) argued that teachers' conceptions of the nature and role of proof would shape their classroom practice with respect to proof. He found that High School teachers saw as the role of proof developing logical thinking skills, communicating, displaying student thinking, and explaining why, which seemed to be akin to showing a derivation (and not promoting insight, as in the case of an *explanatory* proof). Knuth (2002b) remarked that he was surprised that teachers did *not* talk about the role of proof in supporting student learning. Knuth noted that most teachers were also not committed to the idea that *all* students should encounter proof, and generally felt it was more appropriate for their higher-level classes. It is possible then that teachers see proof as more of a topic to be taught than as a core mathematical process.

In this paper, we examine how a group of middle school teachers who value communication and conceptual knowledge building in their classrooms define justification and describe its role in their classrooms.

Methods

Participants

Participants in the study are 12 middle school mathematics teachers. Half of them had previously participated in intense professional development geared towards promoting student discourse in mathematics classrooms. The participants were recruited for the study based on (a) previous involvement in professional development, (b) through connections to other teachers who had been involved in professional development, or (c) through connection to the research faculty. The participants were selected in this manner because this study aims to examine how teachers' conceptions of justification (and its role in the classroom) develop and thus it is essential that the teachers already expect their students to participate in classroom discourse; however, this discourse may not focus on justification yet. As part of the project the teachers participate in a two-year extended professional development that involves meeting with researchers for one week each summer and for three work sessions throughout the year. In addition the researchers collect data in their classrooms four times each year.

Data

The data examined in this study come from two 20 – 70 minute interviews conducted with the teachers at the beginning of this three-year project. The interviews were audio taped and transcribed for data analysis. Interview questions aimed to find out how teachers define justification and its role in their classrooms. Sample questions can be seen in Figure 2.

What is justification?	Why use justification?
- When you talk about justification [proof], what do you mean by that term? - Is either of Claire or Ahmed's response an example of what you consider to be justification? Why? - What makes a *good* justification? justification? Can you try to put this in words?	- Can you tell me a bit about justification in your classroom? - What are your goals with respect to justification in your classroom? - What do you think you can expect from your students at the beginning of the year with respect to justification?

Figure 2. Sample Interview Questions

Analysis

Data analysis began by reading all transcripts carefully and examining all answers teachers gave (across questions) for how they thought about justification and its role in the middle school classroom. Because little is known about teachers' conceptions of justification we followed the methodology of grounded theory with open coding (Strauss & Corbin, 1990), thus keeping an eye open for categories to emerge from the data and then refining those categories with further data analysis. Several members of the research team then composed summaries for each teacher focusing on what the teachers thought justification was and how they said they would use it in their classroom. These summaries were then used to develop initial categories for each of the two questions:

(1) What do the middle school teachers think justification is?
(2) Why do they use justification?

After developing the initial categories, each transcript was carefully coded using those categories, but keeping an eye out for new or more refined versions of the categories to emerge. Each coded statement was read by at least two members of the research team to ensure agreement on classifications and the development of new categories. In this paper we report on the initial categories that emerged from this data.

Results

What is justification?

Three themes emerged in the teachers' responses to the questions probing for a definition of justification (See Table 1 for the distribution of the teachers' answers as well as a summary):

1. A justification explains *why something works* (mathematically). This category focuses on the mathematical reasons for why something works. Sample answers were "I think that justification has more of the why behind it." [T5] and "justifying doesn't mean just being detailed … showing your procedures, but it meant being able to get to the crux of it." [T12] or "why does it work ... why does it work mathematically?" [T3]
2. A justification explains *student thinking* or *how the student knows*. This category is more personal than the first category in that the focus is on the student rather than on the mathematics. Sample answers were: "Justification… okay, like reasoning? As to why… your reasoning as to why you think your answer is correct. Looking at your process to justify why you think something is true. Not because, because 'I know it's true,' but your reasoning." [T2]. A typical question teachers want to ask in this category is: "how did you come up with that?" [T4]

3. A justification explains *what a student did*. This category focuses on what the students did in comparison to the first two that focused on student or mathematical reasoning. Sample answers to what justification is in this category were “being able to explain what you did” [T10] or “showing what they did to get there” [T5].

Table 1. Categorization of Teachers’ Responses to Questions Probing what Justification is

What is justification?	T5[3]	T4	T8	T12	T3	T7	T1	T6	T11	T2	T9	T10	# of teachers
1. explains *why* something works (mathematically)	1	1	1	1	1	1	1						7
2. explains *student* thinking or *how the student* knows	1	1	1	1	1	1		1	1	1	1		10
3. explains *what* a student did	1	1									1	1	4

As we categorized the teachers’ responses individually teachers could be categorized as holding any combination of the conceptions listed above. From the data in Table 1 we see that 7 of the 12, more than half the teachers, thought justification explains why something works (mathematically) and except for one case (T1) this was always coupled with justification is explaining student thinking or how the student knows. Another observation to make from the data is that 10 of the 12 teachers included the second category (a justification explains student thinking of how the student knows) in their definition of justification. A third observation is that there were only 4 instances of justification as explaining what a student did. The first three (T5, T4, and T9) coupled with other categories, thus only one sole occurrence. We are unsure whether the views of justification of the teachers selected for this study are representative of the larger population as many of them had sustained professional development regarding discourse in the classroom and most of them are teaching from reform curricula that include justification.

Why use justification?

Various themes emerged from our analysis of this question. One important dimension was the different constituents the teachers saw as benefitting from justification: the individual student, the group, or the teacher/assessor.

1. Justification is for individual students to help them learn the mathematics or gain a deeper understanding of the mathematics for themselves. This category emphasizes that the focus of the learning is on the individual student. Students are asked to justify so they understand the concept better. Sample answers in this category are “I think you further understand things when you can explain them to someone else.” [T5] or “[justification] makes for a deeper understanding of the mathematics” [T12]

2. Justification is for the other students in the classroom to learn from each other. This category emphasizes the shared authority in the classroom. Justification can be used as a way for students to learn from each other. Sample answers in this category include “I am just trying to get them to press each other a little bit more” [T1] and “make connections to other people’s thinking and learning, … listening to understand others and trying to make those connections.” [T1]

3. Justification is for the teacher or external examiners. Justification is used in the classroom to allow the teacher to see what the students are doing for both formative and summative assessment. This category also includes external examiners such as state tests. Sample answers in

this category include “you know, seeing, and being able to follow – as the facilitator – how they [the students] got there.” [T2] or “so I can see what you [the student] are talking about” [T11] and later this same teacher stated “and how I know they learned.”

Table 2 shows the distribution of the teachers’ answers among the various categories. As we categorized the teachers’ responses individually teachers could be categorized as holding any combination of the conceptions listed above. Table 2 indicates that 11 of the 12 teachers saw justification as a way to help students learn mathematics for themselves or from each other. The second category (helping students learn from each other) was always coupled with the first (helping students learn for themselves), however, not all teachers coupled the two.

Table 2. Categorization of Teachers’ Responses to Questions Probing the Role of Justification

Why use justification	T5	T4	T8	T12	T3	T7	T1	T6	T11	T2	T9	T10	# of teachers
Helping the students learn for themselves	1	1	1	1	1	1	1	1	1	1		1	11
Helping the students learn from each other		1		1	1		1	1	1				6
External: Formative/ summative assessment or standardized testing		1	1					1	1	1	1	1	7

Interaction between the definition and the role of justification

Comparing the data in Tables 1 and 2 reveals that almost all of the 11 teachers who viewed the role of justification as helping the students learn (for themselves or from each other) also thought justifications explain student thinking and/or how the student knows (the exceptions are T1 and T10). This connection makes sense: for a student to learn from justification, the student must explicate his or her current thinking/understanding to be able to build on it. This aligns with an emphasis on conceptual (rather than procedural) understanding. However, one teacher (T9) who defined justification as including an explanation of student thinking or how the student knows did not state that the role of justification was to help students learn (for themselves or from each other). This teacher (indicated the role of justification was for the teacher to assess the student. The conjunction of these two categories – justification explains student thinking and the role of justification is assessment– shows a link of assessment to student thinking (rather than a focus on the correct answers) thus connecting to the students’ conceptual understanding rather than accepting procedurally determined answers.

How do the teachers categorize Claire’s and Ahmed’s solutions?

The teachers as a group considered Claire’s solution a justification. Ten of the 12 teachers said so explicitly and 2 said probably (see Table 3). Sample reasons for a *yes* response were “Claire’s response is [a justification], um, mainly because of her drawing. I think that that representation helps.” [T1] or “Well, she's explaining... I mean, she's basically explaining what she's doing … and the picture really brings it together. I mean, this would work.” [T4] A sample response coded as *probably* was “This [Claire’s solution], I think from what I am seeing, this has more internal meaning to students [as opposed to Ahmed’s solution].” [T3]

Ahmed’s solution elicited a much more varied set of responses from the teachers. Four of the 12 teachers considered it a justification, 3 teachers did not consider it a justification, and 3 teachers stated that it depends on either what else the student knows or what the class had done before. The responses of 2 teachers were not detailed enough to make a definitive determination of their view. (See Table 3.) A sample response for *yes* (Ahmed's solution is a justification) was "But, yeah, this [Ahmed's] is kind of a little bit more sophisticated mathematically [than Claire's] … [but both solutions] are showing their steps. They're showing the process they went through and they're coming up with an answer" [T4]. A sample response for *no* [Ahmed's solution is not a justification] "because he is thinking of his rules and he is following it and unless he was thinking that, unless he said something that a common denominator means that, this, that I, right now there is no justification in it so he's just, so if I were to look at this without talking to him I would say, no, he is not justifying anything." [T11] Teachers whose responses were coded *depends* indicated that Ahmed's response might or might not be a justification depending on what they had done previously in the classroom, e.g., "it depends on where are we in the year in terms of fractions" [T10], or what level the students were at: "for an 8th grader that's pretty given, but for a 6th grader, I want to make sure he knows why it's working" [T8].

Table 3. Categorization of teachers' responses to Claire and Ahmed

Claire/Ahmed	T5	T4	T8	T12	T3	T7	T1	T6	T11	T2	T9	T10	# of teachers
Claire	Y	Y	Y	P	P	Y	Y	Y	Y	Y	Y	Y	Y=10; P=2
Ahmed	Y	Y	D	D	N	N	ND	ND	N	Y	Y	D	Y=4; N=3; D=3; ND=2

Note: (Y=yes a justification, N=not a justification, D=depends (on student background knowledge), ND=not determinable)

In general we notice that Ahmed's more algorithmic/procedural response elicited a much wider variety of response from the teachers than Claire. While all the teachers classified Claire's as a justification (or probably one) only 4 teachers did so for Ahmed. The variation seems to stem from the fact that teachers thought Ahmed's solution could represent a range of levels of understanding of the fraction concepts, from no understanding (a mere application of a learned procedure) to a thorough understanding of the underlying relationships, most notably why finding a common denominator of 4 is useful, and why 2/4 is the same as 1/2. For example, one teacher responded that is was not clear "whether he knows it because it's just a common denominator, or if he knows, like, 'cause I think it's the same rationale that Claire's using, one half equals two quarters" [T6]. Some teachers indicated that further information may lead to a clearer determination of what Ahmed knows, thus teachers in the "no" category may change their categorizations with additional information about the students' thinking. The stance of those indicating "depends" was that some light might be shed on Ahmed's understanding if one knew what had been established in his classroom community before he produced this solution. If this method was established in his class then it may stand as a justification.

Discussion

As evidenced across the teachers' conceptions of justification, its role in the classroom, and their particular analysis of Claire's and Ahmed's responses, this group of middle school teachers was strongly committed to justification as revealing student thinking and showing why something works and to the role of justification in both promoting and assessing student understanding. Their responses, and in particular, the teachers' discussions of Ahmed's justification, reveal high valuing of conceptual understanding – something that they, as teachers, could promote and assess by engaging students in the practice of justification. Indeed, this is at the heart of the variation in responses to Ahmed's response, as many teachers deliberated on whether the response offered evidence the student understood the fraction concept and/or the conceptual underpinnings of the algorithm to add fractions with unlike denominators.

These results contrast with prior results by Knuth (Knuth, 2002a). Unlike Knuth's secondary teachers, this group of teachers felt justification was appropriate for all their students and saw a close connection with justification and promoting student learning. A reason for these differences might be the teachers' extensive previous professional development (which may render them somewhat unrepresentative). Both studies, however, found that teachers thought a purpose of justification was displaying student thinking.

Conclusions

Teachers (and teacher educators) struggle to explicate what a justification is and why they use it. While teachers seem to agree that justification is a valuable practice to use in the classroom; they hold mostly implicit and intuitive notions of what a justification is and what makes a justification good. To be in a position to teach in a way that incorporates justification in the classroom, teachers need to move beyond intuitive notions of what makes a good justification and explicate what they are looking for and why. This would in turn support the development of students' conceptual understanding and teachers' ability to give feedback to students.

The results reported in this study are the first step in understanding how a group of middle-grades mathematics teachers think about justification and why they use it. This study provides a little insight into how teachers view justification. By understanding how teachers think about justification and why they use it, teachers can lay the foundation for discussions on how to capitalize from justification in their own classrooms. In our work with the teachers we have found that the teachers are beginning to move from simply having students share justifications in their classrooms to utilizing them more purposefully and are thinking more deeply about what to do with a justification once it has been shared.

Endnotes

1. The preparation of this paper was supported by a grant from the National Science Foundation (NSF). The views expressed are those of the authors and do not necessarily reflect the views of the NSF.

2. While we recognize that distinctions are made in the literature between the terms justification, reasoning, argumentation, and proof we are not examining the differences between those and will use them interchangeably in this paper.

3. The teacher pseudonyms were given for the initial ordering of the data. The data were then reordered to allow patterns to emerge. The ordering of the teachers stays constant throughout the paper.

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VIRTUAL OTHERS: ONE LEARNER'S MATHEAMTICAL ARGUMENTS IN RESPONSE TO AN ANIMATED EPISODE OF GEOMETRY INSTRUCTION

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This study looks at one mathematically inclined learner's interactions with an animated episode from a high school geometry classroom. The learner being observed worked on the same geometric task as the students in the animated episode and alternated working the mathematics and watching the video. We conceive the animated characters as virtual classmates for the learner to react to, thereby locating the learner's activity in their zone of proximal development (Vygotsky, 1978). The research questions are aimed at understanding the mathematical work done by the learner and how the positions of the animated characters supported the learner's mathematical work.

Introduction

Vygotsky (1978) described the zone of proximal development as the difference between what a child can do with the help of others and what they can do alone. In mathematics, this construct helps us understand that students may be able to solve problems and learn in groups in ways that might not be possible alone. When students work on mathematical tasks with others, teachers or fellow students, they have more resources available to them and their opportunities to learn increase by being exposed to others' thinking. The study described in this paper looks at one mathematically inclined learner's interactions with an animated episode from a high school geometry classroom. The learner being observed worked on the same geometric task as the students in the animated episode and she alternated working the mathematics and watching the video. We conceive the animated characters in the episode as virtual classmates for the learner to react to, thereby locating the learner's activity in their zone of proximal development. The research questions are aimed at understanding the resulting mathematical work done by the learner and how the positions put forth by the characters in the animation supported the learner's mathematical work.

This study (1) is primarily concerned with they ways that students learn how to do proofs. Students in schools learn how to do proofs inside the instructional situations of 'doing proofs' (Herbst & Brach, 2006). The instantiations of 'doing proofs' in the geometry classroom are not necessarily the same as the instantiation of proving outside of a classroom. A fundamental difference between doing proofs in the realm of mathematics and in the geometry classroom is that in the latter the claims to be proven by students are assigned by the teacher and often times are inconsequential; the work is done not so much for the sake of the claim proved but for the purpose of claiming that the class has learned how to prove. This is different than the realm of mathematics where proofs are done to justify claims and to expand the knowledge of the field (Hanna, 1983). In the realm of mathematics the claims to be proven might evolve over the course of the proof (Lakatos, 1976).

In the current study, learners were shown an animated classroom episode in which students are asked to make conjectures and prove claims about the angle bisectors of a quadrilateral. We recorded each learner's arguments and analyzed the structure and content of the arguments. By understanding how the learners interacted with the animation we can hypothesize how interactions with animated classroom episodes could expand students' opportunities to learn

We ask following research questions:

- How does the learner make sense of the discussion being presented in the animation?
- What role do the animated characters' arguments play in the arguments of the learner?

Data

The data to be used in this study was collected in individual meetings with a middle school student, Sonia, for ten, one-hour sessions each. In these sessions Sonia watched animations with the first author, and both participant and researcher would pause the animation to ask a question or give a reaction. Often the animation would be paused for many minutes while Sonia worked on mathematical questions inspired by the animation. At the time of this study, Sonia was about to enter the ninth grade. She would be taking geometry for the first time in ninth grade. These sessions were audio recorded and then transcribed and indexed for analysis.

These case study data could be framed in several ways, but we are looking at them as cases of a learner making sense of a classroom discussion about an open-ended task. In the case we can see the learner making sense of the discussion by interpreting the arguments in the discussion and creating her own arguments, perhaps drawing on the discussion in the animation.

The animation used in this study is an animated episode of geometry instruction entitled The Square (2). The Square is an eight-minute episode that shows a particular case of a teacher and her students interacting around the angle bisectors problem. After reminding students that the angle bisectors of a triangle meet at a point, the animated teacher gives her students several minutes to work on making conjectures about angle bisectors in a quadrilateral, which she says they will later try to prove. When time is up, the teacher calls Alpha to the board to share his conjecture. Alpha draws a square and its diagonals on the board and states the conjecture that "they bisect each other." The teacher dismisses Alpha's conjecture on the grounds that the problem is about angle bisectors, not about diagonals. Some students come to Alpha's defense, saying that angle bisectors and diagonals are the same thing. In response to this Gamma comes to the board to show that in rectangles angle bisectors and diagonals are not the same thing. Alpha then restates his conjecture as "In a square the angle bisectors meet at a point because they are the diagonals." Now that the conjecture has been refined, the teacher asks for volunteers to prove the statement. Lambda begins to prove the statement from his seat. In the following minutes the teacher and Lambda struggle to understand each other. Their communication is hindered by the fact that Lambda would like the teacher to erase one of the diagonals from the square and the teacher resists this action. Finally, the teacher erases one diagonal and Lambda completes his argument. The episode ends with the teacher calling for a two-column proof of Lambda's argument.

Method and Analysis

The conversations with the learner were analyzed to reveal the argument that she made in reaction to the animation. The arguments were modeled using Toulmin's arguments scheme (Toulmin, 1958). Toulmin's method of modeling arguments is a tool for describing the connections that an arguer could make between a set of data (hypothesis) and a conclusion. Importantly, connecting these two are a warrant, or reason to believe that the conclusion follows from the data, a backing, or further support for the warrant, a qualifier that conveys the arguer's confidence in the argument, and a rebuttal, or a counter argument that the arguer acknowledges (see Figure 1). Every argument may not make use of all these components, but Toulmin claims that all arguments can be fit into this form. Toulmin's scheme was designed as an alternative to formal logical argument models, so unlike standard mathematical proof, the warrant for an argument need not be logical connectives and the qualifier is not always "necessarily". This freedom from formal logic makes Toulmin's model useful for recording arguments that rely on inductive logic or intuition.

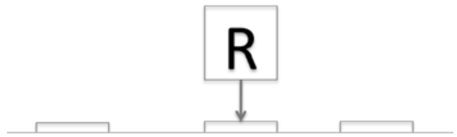

Figure 1. Toulmin's model of arguments

(Note: The model can be read as "D implies C with probability Q, on the basis of W, supported by B, unless R.")

Once Sonia's arguments had been mapped from the conversations they could be compared in terms of their structure. Below we showcase the argument made by animated characters in The Square and compare it with an argument made by Sonia.

Sonia

To begin I will present the map of the argument presented in the first few moments of the discussion in The Square. I will then present Sonia's argument that she produced in response to this argument, highlighting her interpretation of the animated characters' argument and describing the form and flow of her argument.

As stated above, the animation begins with the teacher posing the question, "what can one say about the angle bisectors of a quadrilateral?" The teacher calls Alpha to the board and he makes the claim that in a square the diagonals bisect each other. This is mapped below in the top three boxes of Figure 2, representing the data, "A square", and the conclusion, "In a square the diagonals bisect each other." This conclusion is warranted by the exploration that Alpha did independently after the teacher posed the problem.

After Alpha makes his initial argument Beta begins a second argument by asserting the conclusion that "Diagonals are angle bisectors." This is represented in the box at the far right of Figure 2. She bases this conclusion on Alpha's choice to look at a square and the warrant, provided by Alpha, that "The diagonals of a square cut it in half." This second argument, seen in the bottom half of Figure 2, and ending in Beta's conclusion, is the argument that Sonia uses to build her own argument. We now describe the argument that Sonia made in response to the arguments of the animated characters.

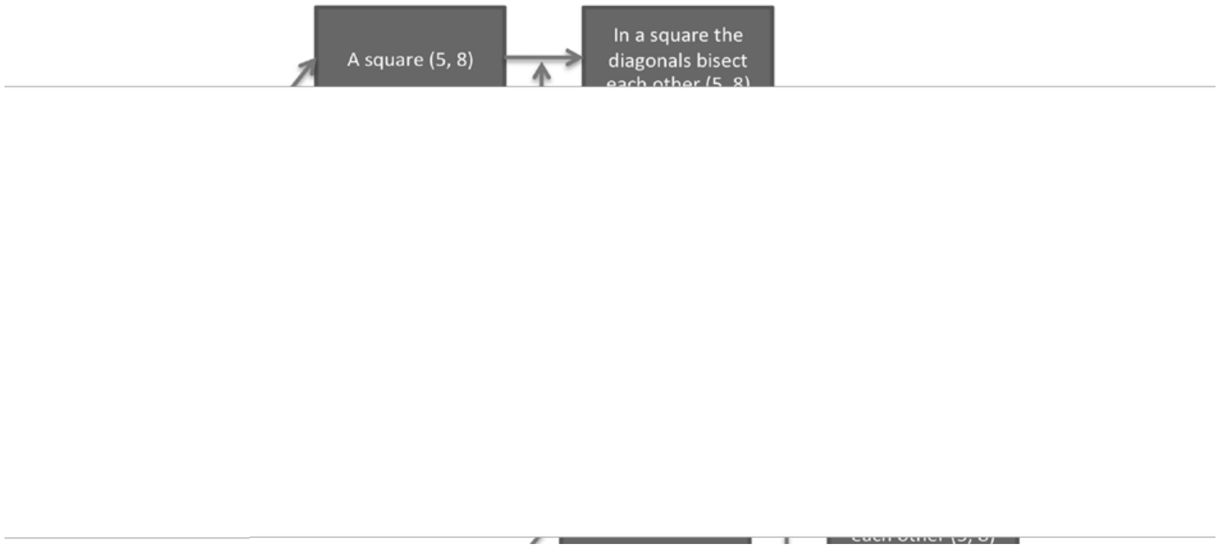

Figure 2. The initial problem and Alpha and Beta's contributions in The Square
(Note: Numbers in parenthesis correspond to the line in the script where assertion comes from)

From the transcript of the discussion we map Sonia's sense making of the animated students' comments. The result of this mapping is shown in Figure 3. In the figure, boxes that are filled and thin arrows represent comments made by characters in the animation. Boxes that are not filled and thick arrows represent comments made by Sonia. The goal of this mapping is to show the interaction between Sonia's thought process and the discussion in the animation as well as highlight the mathematical actions that Sonia employs while making sense of the discussion in the animation.

On the second day of the case study Sonia interprets Alpha's statement "Well, I just thought that the diagonals cut the square in half" (excerpt of the case study transcript can be seen in Appendix A). The conversation between Sonia and the researcher begins with the researcher asking Sonia what Alpha means by this and then asking her how this relates to the question the animated teacher asked, "what can one say about the angle bisectors of a quadrilateral?" Sonia goes on to interpret Alpha's claim that the diagonals of a square cut it in half and provide an argument in response to the teacher's question.

Sonia interprets Alpha's statement, "the diagonals cut the square in half" as meaning that the diagonal cuts the quadrilateral into two triangles of equal area. This is represented by the node at the bottom left of Figure 3 that reads, "The diagonal cuts the square into two triangles with equal area" that is set equal to filled node that reads, "The diagonals of a square cut it in half." Sonia says, the diagonal "cuts it in half triangle wise…like this side is the same as this side, the area." Sonia does not mention the area again after this and it is reasonable to think that she means that the two triangles are congruent, not that they have equal area. To support this, as examples she gives the rectangle and parallelogram as quadrilaterals that have diagonals that cut the shape in half and she gives the kite and its minor diagonal as a counter-example. These warrants are shown in the very bottom left of Figure 3 in the node that reads, "This is true in a quadrilateral where the opposite sides are the same length [like parallelograms and rectangles]." The counter-example of the kite is represented in the rebuttal node above which reads "Not in the case of a kite."

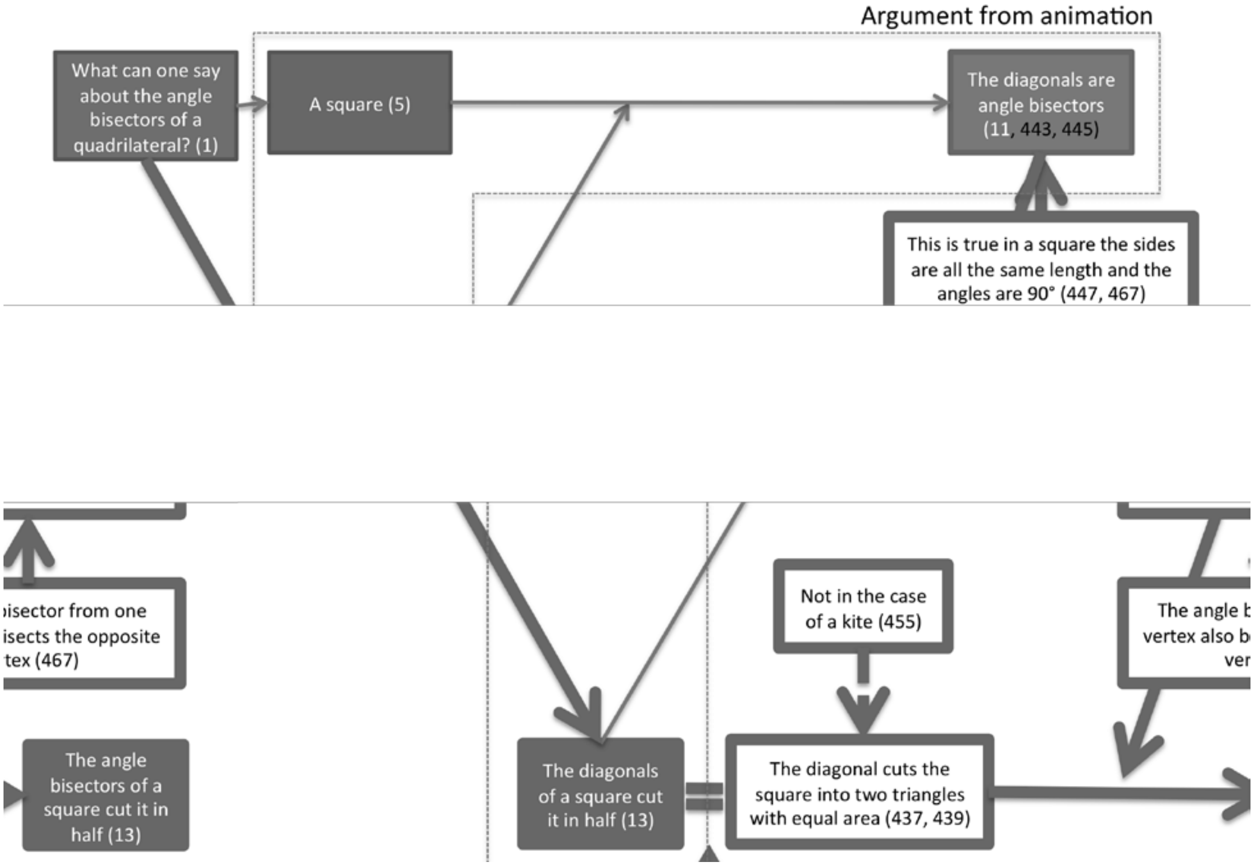

Figure 3. Mapping of Sonia's sense making

Sonia's use of "the diagonals cut the square in half" does not match with Alpha's. The different between the two uses can be seen when the phrase is extended to rectangles. For Sonia, the diagonals of a rectangle do cut it in half (by creating two congruent triangles), and for Alpha, it is likely that he means that the diagonals of a rectangle do not cut it in half (because the diagonals do not bisect their vertex angles).

One way to interpret Alpha's use of this phrase that is consistent with his actions in The Square is that in a square the diagonal is a line of symmetry. Using this interpretation, the diagonals of a rhombus also cut it in half, and the main diagonal of a kite cuts it in half. Sonia's interpretation encompasses Alpha's definition, but we see that Alpha's meaning accepts fewer quadrilaterals than Sonia's (specifically rectangles and parallelograms).

In The Square, Alpha uses the claim that the diagonals of a square cut it in half to support the inference that in a square the diagonals are the angle bisectors. This argument is represented in Figure 3 by the thin arrows connecting the filled nodes, "A square," "The diagonals are angle bisectors," and "The diagonals of a square cut it in half." Because Sonia interprets this claim differently than Alpha, she does not use this to support that inference. Rather, she uses the conclusion of Alpha's argument, that in a square the diagonals are the angle bisectors, to justify Alpha's response as an appropriate answer to the question, what can one say about the angle bisectors of a unfilled. This argument is represented in Figure 3 by the thick arrows connecting the unfilled nodes, "The diagonals of a square cut it in half," "The angle bisectors of a square cut it in half," and "The diagonals are angle bisectors." The first unfilled node is the data, followed by the conclusion and warrant of Sonia's argument. By using this claim she can transform the statement the diagonals of a square cut it in half into the claim that the angle bisectors of a square cut it in half. This statement now says something about the angle bisectors of a quadrilateral so it is a reasonable answer to the teacher's question.

Overall, from this analysis we can see the complicated nature of Sonia's sense making. Although the argument that she builds is not the same as the one presented in the animation it reflects flexible thinking and the ability to build and justify a mathematical argument. It also shows her ability to link the pieces of an argument to the question that the argument aims to answer. We hypothesize that Sonia's interaction with the animation was critical in her ability to involve in this type of argumentation despite the fact that she had not been exposed to deductive geometry.

Conclusion

In conclusion, this analysis shows how animated episodes of geometry instruction can be used as a basis on which learners can build mathematical arguments. The animations provide prompts for students to react to and claims for students to prove, refute, and expand upon. In dynamic and unpredictable classroom interactions animations can lay the foundation for productive mathematical discussions. The use of animations as prompts does not take away work from students but rather increases the amount of mathematical resources and operations that are available while they are working on a task and gives them virtual partners to think with.

Endnotes

1. The research presented here is made possible by NSF grant ESI-0353285 to Patricio Herbst. The opinions expressed here are those of the author and do not reflect the views of the National Science Foundation.

2. The animated story can be watched and annotated in http://grip.umich.edu/themat

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APPENDIX A: Transcript of Sonia's argument

Turn #	Speaker	Turn
437.		[Animation: Alpha: "Well, I just thought the diagonals cut the square in half. Teacher: "Alpha just said that the diagonals cut the square in half. Somebody elaborate on that."] Animation is paused.
438.	Interviewer	So what do you think that means? what do you think he might mean by that, "Diagonals cut the square in half."?
439.	Sonia	Like it cuts it in half triangle wise.
440.	Interviewer	So what…
441.	Sonia	Like this side [Interviewer: Uh-huh.] is the same as this side [Interviewer: Mm-hmm.] – the area.
442.	Interviewer	And how does that relate to the question about angle bisectors?
443.	Sonia	Well she asked for conjectures about what angle bisectors do and in the quadrilateral --
444.	Interviewer	Uh-huh, but when didn't she – but when he said, "The diagonals cut the square in half" --
445.	Sonia	Oh the diagonals are the same thing. Diagonals don't necessarily divide – they don't necessary bisect the angle.
446.	Interviewer	Yeah. So – so is…
447.	Sonia	But he's not wrong if it's a square.
448.	Interviewer	Why not?
449.	Sonia	Because they're at 90˚ angles at – on the side – on the shape that has the same length [Interviewer: Uh-huh.] on all sides, so the diagonals and the angle bisectors are the same thing – or they go through the same path [Interviewer: Yeah.] on the -- square.
450.	Interviewer	How do you think you would prove that?
451.	Sonia	Well diagonals divide the shape cleanly in half and so --
452.	Interviewer	That's what he just said though right?
453.	Sonia	Yeah [laughing] I was getting to the point – ok, so – and then --
454.	Interviewer	Do diagonals always cut the – cut the shape in half?
455.	Sonia	Oh I guess not.
456.	Interviewer	Well like what do [inaudible] where it does, and where it doesn't?
457.	Sonia	Well here you would have this small little kite things that doesn't divide the square into --
458.	Interviewer	Right that' not cut in half --
459.	Sonia	No. But…in a shape where the opposite sides are the same length, and then they have to cut the shape in half.
460.	Interviewer	So what's another example, besides the square?
461.	Sonia	A rectangle. 'Cause that cuts the shape in half.
462.	Interviewer	Oh yeah.
463.	Sonia	And this cuts the shape in half. [Interviewer: Yeah.] And on a parallelogram let's pretend – that's better – that's in half too.

464.	Interviewer	Mm-hmm, and so – and what are you drawing?
465.	Sonia	That was supposed to be a parallelogram.
466.	Interviewer	And what's the line inside?
467.	Sonia	The diagonal.
468.	Interviewer	The diagonal, yeah, ok.
469.	Sonia	On a square, the diagonal and the bisectors are the same thing because the length of the – of all the sides are the same since it's a square obviously and because all the angles are 90˚ angles [Interviewer: Mm-hmm.] then they have to – the bisectors bisects one and ends up bisecting the other even though there are four bisectors it just looks like two [Interviewer: Yeah.] 'cause it goes through [Interviewer: Yeah.]. So, it's kind of the same thing; diagonals and the square.

FACTORS INFLUENCING STUDENTS' PREFERENCES ON EMPIRICAL AND DEDUCTIVE PROOFS IN GEOMETRY

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The purpose of this study is to investigate what influences students' preferences on empirical and deductive proofs and find their relations. Although empirical and deductive proofs have been seen as a significant aspect of school mathematics, literatures have indicated that students tend to have a preference for empirical proof when they are convinced a mathematical statement (Porteous, 1991; Martin & Harel, 1989). Several studies highlighted students' views about empirical and deductive proof (for example, Chazan, 1993). However, there are few attempts to find the relations of their views about these two proofs. The study was conducted to 47 students in 7~9 grades in the transition from empirical proof to deductive proof according to their mathematics curriculum. The data was collected on the written questionnaire asking students to choose one between empirical and deductive proofs in verifying that the sum of angles in any triangle is 180°. Further, they were asked to provide explanations for their preferences. Students' responses were coded and these codes were categorized to find the relations.

As a result, students' responses could be categorized by 3 factors: accuracy of measurement, representativeness of triangles, and mathematics principles. First, the preferences on empirical proof were derived from considering the measurement as an accurate method, while conceiving the possibility of errors in measurement derived the preferences on deductive proof. Second, a number of students thought that verifying the statement for three different types of triangles - acute, right, obtuse triangles - in empirical proof was enough to convince the statement, while other students regarded these different types of triangles merely as partial examples of triangles and so they preferred deductive proof. Finally, students preferring empirical proof thought that using mathematical principles such as the properties of alternate or corresponding angles made proof more difficult to understand. Students preferring deductive proof, on the other hand, explained roles of these mathematical principles as verification, explanation, and application to other problems. The results indicated that students' preferences were due to their different perceptions of these common factors.

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JUSTIFICATION IN MIDDLE SCHOOL CLASSROOMS: AN ANALYSIS OF STUDENT RESPONSES TO TWO JUSTIFICATION TASKS

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Justification has been put forward as an important mathematical practice that should be present in K-12 classrooms (NCTM, 2000; Stylianou et al., 2009). Yet, research evidence shows that this practice is often absent from classrooms, notably those in the middle grades (Jacobs et al., 2006). For teachers to teach in a manner that centralizes justification, they must anticipate, elicit and build on students' thinking. To support this, it is helpful for teachers to have ways to think about potential student responses and important dimensions along which those responses might vary. The purpose of this study was to document the range of student responses on two justification tasks implemented in middle grades classes. In reporting the results, we use an analytic framework developed by our research team, which may be useful for categorizing student responses for justification tasks.

The data used for this poster is part of a larger NSF-funded project, JAGUAR (Justification and Argumentation: Growing Understanding of Algebraic Reasoning). Data has been collected from twelve participating (JAGUAR) teachers. For the purposes of this poster, we report findings from analyses of student work on two justification tasks, namely, the Hexagon task and the Number Trick task. The data reviewed included student work samples from both justification tasks; the objective here was to capture a wide range of student responses that could be categorized under our given framework of student *forms of reasoning*.

Our analysis revealed a broad range of student responses. We classified this range of student responses by its mathematical *form of reasoning*. For the Hexagon task, students provided justifications using verbal reasoning (describing the process), model-based reasoning- through the use of pictorial, algebraic and graphical representations, as well as empirical reasoning. For the case of the Number Trick task the responses showed elements of empirical, pattern-based, model-based as well as spatial-geometric reasoning.

These findings can support middle-school teachers in gaining a richer understanding of how students engage in justification-tasks while they develop their mathematical thinking and reasoning. The analytic framework may also be a valuable tool for both generating and analyzing student thinking.

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MAKING SENSE OF RATE OF CHANGE: EXAMINING STUDENTS' REASONING ABOUT CHANGING QUANTITIES

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In this qualitative study I examined how six high school students who have not taken calculus reasoned about changing quantities when interacting with mathematical tasks involving multiple representations of constant and varying rates of change. Employing a cognitive perspective on mathematical reasoning, I conducted a series of five individual, task-based interviews with each student. Using a three-phase data analysis process, I identified episodes where students attended to changing quantities, traced students' reasoning from their statements, written work and gestures to develop characterizations of their reasoning, and examined how students combined covariational, transformational, and proportional reasoning when reasoning about changing quantities. Students tended to reason in ways that remained consistent across the tasks, even though the tasks involved different contexts and different types of representations of mathematical quantities and relationships between quantities. Empirical evidence from this study supports that students without formal instruction in rate of change do combine covariational, transformational, and/or proportional reasoning when reasoning quantitatively about specific and non-specific amounts of change. I use a tool that I developed to summarize different combinations of reasoning and highlight affordances of combining different forms of reasoning.

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PROOF IS IN THE EYE OF THE BEHOLDER: MIDDLE SCHOOL STUDENTS' CONCEPTIONS OF CONVINCING JUSTIFICATIONS

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Some mathematics educators have defined proof as the process one undertakes to remove doubt, or convince oneself and others that a statement is true (Harel & Sowder, 1998). Thus, learning to do mathematical proof involves adopting the belief of what is mathematically convincing – namely, justifying by using arguments based on general premises in a logically deductive fashion. However, middle school students have yet to learn the norms of the proving in mathematics, norms that are often learned observing proofs as completed by their instructor and generalizing features of those proofs deemed to be "correct" by a mathematical authority. Understanding what features of justifications students find convincing, such as the use of examples or the use of general statements, can support instructional interventions that highlight important distinctions between proofs and non-proofs. Weber (2009) acknowledges: "The lack of research on how students *do* read mathematical arguments, as well as how they *should* read them, represents an important void (p.2)". This study addresses the need for research on students' reading and evaluating of mathematical proof by investigating: In what ways does the mathematical context of the statement influence students' choice as to which type of justification is more convincing?

Interviews were conducted with 21 seventh grade students. The students were asked to evaluate two kinds of justifications, an examples-based (EB) justification and a general, deductive (G) justification, for true statements about number theory and geometry. Of these 21 students, 16 chose the EB justification as more convincing than the G justification for the number theory task. In contrast, only 9 students found the EB justification more convincing for the geometry task. Analysis of students' explanations for their choice revealed that while students tended to value examples as a way to "show" what the statement was trying to prove, they mentioned features related to the justification's explanatory power only when evaluating justifications for the geometry task (n=7). However, despite context of the task, only two students mentioned choosing a general argument because examples do not prove the truth of a general statement. While the sample size limits the statistical significance of these findings, the results suggest that while middle school students tend to favor EB justifications as more convincing, they value the use of examples and explanatory statements differently across various mathematical contexts.

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THE CONTRIBUTIONS OF STORYTELLING TO MATH LEARNING, TEACHING AND MATH EDUCATION RESEARCH

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This poster presentation draws on personal experiences of completing an interdisciplinary degree in the area of education and mathematics and previous works dealing with story, learning and research (Reason & Hawkins, 1988). Story and storytelling is mentioned and used in various areas of math and math education. *The Alberta Outcomes with Assessment* (2002) talks about students being able to tell a story that could describe the shape of a function. Ian Stuart (2006) sums up proof writing as being able to tell a good convincing story. Some have used story to describe teacher practices (LoPresto & Drake, 2005). Others have used story to describe the field of math education research (Lerman & Tsatsaroni, 2003). Story seems to have a deep entangled role in math education research and the learning of math. The poster will explore the potential of storytelling and relationships between it and math learning, teaching and education research. I will share the story of completing a dissertation that is a joint expression of both mathematics and education. Storytelling was chosen as a mode to present the research because it allows me to express the "liveliness, the involvement and even the passion of [my] experiences" (Reason & Hawkins, 1988, p. 79). I am developing a dialogue between a fictional math graduate student and her education supervisor. The dialogic storytelling style is emerging in the math education literature (Nardi, 2008). In my poster I will explore how storytelling can be used to exhibit Lockhart's (2009) notion of mathematics being a creative journey of the imagination. Examples of storytelling are included to illuminate the role of intuition and struggle in math learning. Further research questions concerning math, math education, math learning and storytelling are included.

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VALID OR INVALID PROOFS AND COUNTEREXAMPLES: PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' PRODUCTIONS AND EVALUATIONS

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To implement current reforms regarding proof and counterexample in secondary school mathematics successfully, pre-service secondary mathematics teachers are expected to be able to produce proofs and counterexamples as well as to determine what arguments are acceptable in the mathematics community. Notably, what counts as a valid proof or counterexample really depends on an individual proof scheme (Harel & Sowder, 2007), and a way to better understand individuals' conceptions of proof and counterexample is by having individuals evaluate their own work (Stylianides & Stylianides, 2009). Drawing on research on proof and counterexample (e.g., Knuth, 1999), the aim of this study is to examine how pre-service secondary mathematics teachers evaluated their own proof and counterexample productions based on beliefs they and their mathematics professors held. The primary sources of data were the pre-service secondary mathematics teachers' semi-structured interviews. During the interview, eight pre-service secondary mathematics teachers met with the researcher individually and were asked (1) to rate the level of their confidence in terms of the validity of their production using a four-point scale, and (2) to indicate whether or not their mathematics professors would accept their productions as valid. Pre-service teachers' proof and counterexample justifications were categorized according to Harel and Sowder's (2007) and Knuth's (1999) taxonomies of proof. This qualitative analysis shows the distinction between what per-service teachers and their professors believed constituted proof and counterexample. The findings suggest that more attention should be paid to teaching and learning proof and counterexample, as participants showed difficulty writing proofs and counterexamples as well as held different perspectives on evaluating their own productions.

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Chapter 12: Teacher Beliefs

Research Reports

Brief Research Reports

Posters

CHALLENGING BELIEFS ABOUT CHILDREN'S MATHEMATICS LEARNING THROUGH VIDEO STUDY

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This study examines the extent to which pre-service teacher beliefs about learning mathematics are impacted by studying videos of children's learning. Elementary pre-service teachers from three universities participated and studied videos of children's learning stored at the Video Mosaic Repository of Rutgers University as an intervention unit in a required mathematics education course. A pre and post beliefs assessment was administered to participants as well as to comparison groups. A statistical analysis of beliefs data showed that across all sites there was a significantly greater change in experimental pre-service teacher beliefs about student learning as compared to the comparison groups.

Introduction

This study was premised on the notion that studying well-chosen videos of children engaged in learning mathematics can serve as a seminal resource for teachers to understand how students represent mathematical ideas and reason about them (Maher, 2008). Since videos are able to capture aspects of the emerging processes of learning, they can serve to engage pre-service and practicing teachers in new strategies for effectively teaching mathematical concepts to a range of students. Videos can be stopped and started again, as well as viewed multiple times, and thus provide a medium that supports reflection and consideration of beliefs and assumptions that underlie educational practice. Evidence in support of this notion comes from researchers who examined the effects of using video clips as an alternative to classroom-based field experience for prospective elementary teachers and found significant positive growth in beliefs scores among participants in the course sections that studied children's mathematical behavior on video (Philipp, et al., 2007).

Objectives of Study

Certain views about learning and teaching, and what it means to do mathematics, give rise to particular conditions that can be examined (Davis & Maher, 1990; Maher, 1988). The objective of this study was to investigate whether studying videos of children's learning under conditions that invited exploration, collaboration, and building meaning of mathematical ideas while working on strands of similar problems could affect pre-service teacher beliefs about how students learn. The study was part of a multifaceted research and development project aimed at preserving and making more accessible videos from the collection amassed by the Robert B. Davis Institute for Learning through more than two decades of research[1].

The collection contains unique and valuable video data and meta data (i.e., data about data, such as transcripts of video, coding schemes, students' work, observational and research notes, analytic commentaries) on how students build mathematical ideas and ways of reasoning over time in a variety of diverse school settings and across all grade levels, in several content domains (Agnew, Mills & Maher, 2010). Drawing from the longitudinal and parallel cross-sectional studies in urban and suburban contexts, a collection of videos has evolved that captures the development of children's reasoning, early proof making, and students' building of isomorphisms between and among problems of the same structure (Maher, 2005; Maher & Martino, 1996; Maher, Davis & Alston, 1992).

Videos from the collection were leveraged to conduct research in the context of pre-service teacher education through comparison of experimental instruction that incorporated the studying of video clips with instruction that did not. Instructors of courses for prospective elementary teachers selected videos from the collection that featured elementary and middle school aged children [Note 2]. The video clips from classroom research illustrated children doing mathematics, talking in pairs or small groups about their mathematical ideas, sharing those ideas with the classroom community, and offering justifications for their solutions. Interviews of children sharing their solutions strategies were also included. It should be noted that the broader context for research was attentive to the use of studying videos to raise teachers' awareness of the complexity of learning (Spiro, Collins, Ramchandran, 2007) by attending to the process through which mathematical reasoning unfolds rather than just the final outcome, and to the development of adaptive expertise (Bransford, Derry, Berliner, & Hammerness, 2006) to help teachers prepare to recognize and build upon instances of mathematical thinking in children's problem solving, including written work and problem-solving discourse. The focal point of the study presented here was video-based instructional invention for prospective teachers aimed at development of teacher beliefs consistent with current norms and standards of mathematics education.

Perspectives

Pre-service teachers form beliefs about learning from their own experiences as math learners and their observations from field experiences, and their assignment to field sites may be made on factors of convenience rather than course goals. While classroom observations and reflections on observations through keeping journals and/or engaging in discussion groups are often a component of teacher education programs, the process is not without its limitations. As other researchers have noted (e.g., Philipp et al., 2007), education faculty may find themselves unable to make much input on the assignments of pre-service teachers to classrooms for observation, such as to select sites more likely to exhibit reform-oriented mathematics instruction. Moreover, in classrooms where traditional, direct-instruction teaching practices are common in local schools, pre-service teachers who are exposed to theories and research evidence that support reform-oriented instruction as part of their teacher education programs may find themselves observing practices that are at odds with what they have been learning from their professors. Research suggests that, under those circumstances, pre-service teachers will not exhibit change toward reform-oriented beliefs about mathematics learning and teaching and their belief scores may actually decrease instead (Philipp, et al., 2007).

The opportunity to study videos of children doing and talking about mathematics can give pre-service teachers access to knowledge that might otherwise be unavailable. For example, children who are hesitant to engage in large group exchanges common in the traditional classrooms can be observed in videos of their learning in small group settings, where they may be more comfortable communicating and expressing their ideas with peers. Similarly, through video analysis pre-service teachers can study and carefully evaluate the contributions of students who are culturally and linguistically diverse and whose ideas when expressed may not be readily recognizable to teachers because of cultural and language differences. Another valuable feature in the use of videos for improving teacher practice is that videos make possible the study of teacher moves that play an important role in influencing learning outcomes. Teacher educators working with prospective teachers can use videos as tools for studying how students build new knowledge and how teacher actions influence student learning (Maher, 2008). As has been noted, for teachers to be able to teach mathematics as a conceptually oriented sense-making activity, teachers need to shift their beliefs about mathematics away from an idea of math as a set of facts and procedures that children find difficult, towards beliefs of children as sense-makers capable of developing sophisticated mathematical thinking (e.g., Philipp et al., 2007). A Beliefs Inventory was developed and implemented as an assessment for measuring the impact of studying videos featuring children's mathematical reasoning on pre-service teachers' beliefs about learning and teaching math. To this end, the following research question was posed:

What effect does studying well-chosen videos of children engaged in learning mathematics have on the beliefs held by prospective elementary and middle school teachers?

Methods of Inquiry

Participants in the study were undergraduate students enrolled in an elementary mathematics education course for pre-service teachers. One section consisted of students enrolled in a mathematics course who expressed interest in elementary school teaching. Participants came from three different universities in New Jersey. At each site, all participants from these courses were given the same Beliefs Inventory at the beginning and end of the course as pre and post-measurements. The instructional intervention for experimental groups included studying of selected video clips from the Rutgers Repository. Comparison groups of similar classes were given the same pre and post Belief Inventories; however, the comparison groups did not have access to the video-based instructional intervention.

The Beliefs Inventory included items that assessed beliefs about how mathematics is learned and how teachers influence (or not) children's learning. For this report, we discuss the subset of belief items directly related to the theme of children's learning mathematics that the intervention of studying videos might influence (See Table 1).

The pre and post-test Belief Inventory scores of the experimental groups and comparison groups were compared, first for all of the items and then for the subset of eleven of the thirty-three items that addressed children's mathematical learning. We report, here, on the eleven items in the Beliefs Inventory that align more closely to the themes contained in video episodes. These are given in Table 1 below.

The pre and post-test Beliefs Inventory averages were calculated based upon the following coding scheme: 1=strongly agree, 2=agree, 3=uncertain, 4=disagree, and 5=strongly disagree. Inverse Beliefs question items were considered negative beliefs and coded as: 1=strongly disagree, 2=disagree, 3=uncertain, 4=agree, and 5=strongly agree. The rationale for the reverse coding of negative beliefs question items is to achieve consistency for all eleven Beliefs Inventory items in that a student's post-test score being smaller than the student's pre-test score is interpreted as growth in Beliefs concerning children's learning of mathematics.

Results

Table 1 shows post-test Beliefs growth for both the experimental and comparison groups of students on each of the eleven items in the Beliefs Inventory; however, the Beliefs growth of the experimental students is consistently greater than that of the comparison students for each of the eleven Inventory questions. The probability of this occurring due to chance alone is (0.5).

Table 1. Post-Test Change in Beliefs Inventory – Experimental vs. Comparison Groups

Beliefs Inventory Item Descriptions	Experimental Group (EX) N=69		Comparison Group (CO) N=81		EX vs. CO Growth Difference
	Pre-Test Mean	Post-Test Mean	Pre-Test Mean	Post-Test Mean	
Math is primarily about communication.	2.76	2.01	2.71	2.75	0.79
Teachers should show students multiple ways of solving a problem.	1.91	1.54	2.09	2.21	0.49
All students are capable of working on complex math tasks.	2.70	1.97	2.96	2.69	0.46
Inverse of: Students will get confused if you show them more than one way to solve a problem.	2.58	2.06	2.17	2.09	0.44
If students learn math concepts before they learn the procedures, they are more likely to understand the concepts.	2.37	1.80	2.46	2.29	0.40
Inverse of: Only really smart students are capable of working on complex math tasks.	3.09	1.29	3.07	1.58	0.31
Inverse of: Learning a step-by-step approach is helpful for slow learners.	4.38	3.88	3.59	3.48	0.39
Mixed ability groups are effective organizations for stronger students to help slower learners.	2.13	1.77	2.49	2.43	0.30
It's helpful to encourage student-to-student talking during math activities.	1.96	1.29	2.15	1.75	0.27
Learners can solve problems in novel ways before being taught to solve such problems.	2.40	1.77	2.64	2.30	0.29
Inverse of: Conflicts in learning arise if teachers facilitate multiple solutions.	2.58	2.15	2.49	2.32	0.26

The experimental versus comparison growth difference in Table 1 was calculated as experimental (pretest – posttest) mean – comparison (pretest – posttest) mean.

Table 2 reports the mean pre and post averages across the eleven Inventory items. A student t-test on the pre and post test means provides evidence of a significant post-test growth in Beliefs for both the experimental and the comparison students; however, a student t-test on the growth means of the experimental versus comparison groups supports the claim that the growth of the experimental group, on the average, is significantly higher than that of the comparison group (t = 5.6 with a significance level of 0.0001).

Table 2. Statistical Test of Significance of Mean Posttest Beliefs Growth of Experimental (EX) and Comparison (CO) SS and the Significance of EX vs. CO Mean Differences

	EX Group Means		CO Group Means		EX vs. CO Growth Difference
	Pre-Test	Post-Test	Pre-Test	Post-Test	
Overall Mean	2.49	1.95	2.48	2.36	0.42
Student t-test: significance of post-test change	t = 7.87 (Sign = 0.0001)		t = 2.95 (Sign =0.002)		t = 5.46 (Sign = 0.0001)

Table 3 summarizes the linear regression parameter estimates and tests of significance (R^2 = 42.0%) for post-test growth as a function of the student's pretest score for each institution for experimental and comparison groups.

Table 3. Linear Regression Parameter Estimates and Tests of Significance

Term	Estimate	Std Error	t Ratio	Prob>|t|
Intercept	-0.812454	0.206413	-3.94	0.0001
Student Pre-Test Average	0.4341682	0.076685	5.66	<.0001
Exp or Comparison [CO]	-0.193902	0.036937	-5.25	<.0001
Exp or Comparison [EX]	0.1939018	0.036937	5.25	<.0001
Exp or Comparison [CO](Student Pre-Test Average-2.48252)	-0.310746	0.075823	-4.10	<.0001
Exp or Comparison [EX](Student Pre-Test Average-2.48252)	0.3107464	0.075823	4.10	<.0001
Site 1	-0.266831	0.098461	-2.71	0.0076
Site 2	0.0289424	0.065397	0.44	0.6588
Site 3	0.237889	0.065801	3.62	0.0004

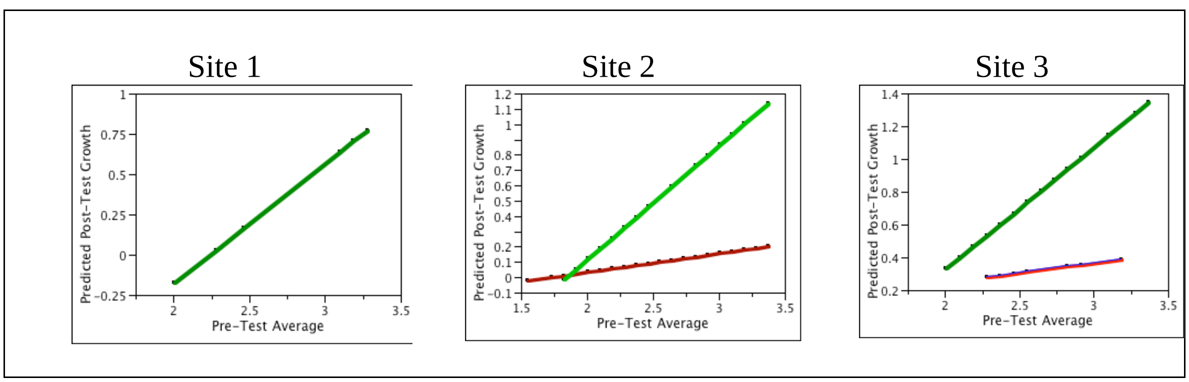

Figure 1. Predicted Beliefs Inventory post-test growth for the average of eleven Beliefs

The results of a regression model for prediction of a student's post-test Beliefs growth mean as a function of the student's mean pre-test score, experimental or comparison group participation, and educational institution are given. The analysis provides evidence that all these factors are statistically significant; however, the trend is consistent: (1) students with higher pretest Beliefs mean scores (i.e. significant number of coded undecided and disagree item scores) tend to have higher post-test Beliefs growth and (2) student's with a significant number of coded undecided and disagree beliefs tend to have higher expected levels of growth in the experimental as compared to the comparison groups.

Figure 1 is a graph of the student's predicted post-test growth versus the student's pre-test mean score for the experimental (EX -----) and comparison groups (CO -----) of each of the three participating education institutions.

Inventory items versus a student's corresponding pretest inventory score

Table 4 below examines the individual questions in the Beliefs Inventory and tabulates the nature of the student's post-test score as an improvement in beliefs, a lowering in beliefs or no change. Table 4 shows that of all the student beliefs responses in the experimental group that were coded as disagree, a total of 60.6% of the post-test responses indicated a post-test gain for the experimental students versus 37.6% for the comparison group students. Similarly, Table 4 reports that of all the question items that were scored as undecided in the pre-test, the post-test scoring indicated a 69.7% beliefs gain for the experimental students versus a 49.7% gain for the control group students. It is useful to note that a student pre-test score of agree to an inverse belief question item is a negative belief and is tabulated in Table 4 in the disagree category of the positive beliefs question items.

Table 4. Analysis of Student Post-Test Response to Belief Inventory Questions

Pre-Test Item Response	% With Improved Post-Test Responses			% With Lower Post-Test Response		
	EX	CO	EX - CO	EX	CO	EX - CO
Agree	0.0	0.0	0.0	10.4	14.5	- 4.1
Undecided	69.7	49.7	20.0	5.9	21.5	-15.6
Disagree	60.6	37.6	23.0	0.0	0.0	0.0

Discussion

The significant changes in pre-service teacher beliefs in this study supports previous research that suggests that studying videotape data of students engaged in mathematics leads to positive growth in beliefs scores of pre-service teachers (Philipp, et al., 2007). It also extends the findings to include a larger elementary teacher population. In addition, the videos from previous studies featured early grade students working with number ideas while the videos for this study involved elementary to middle school students attending to open-ended fraction and counting problems.

It is important to note that of the thirty-three belief questions, only the eleven that specifically focused on student learning emerged as different between the experimental and comparison groups. The eleven belief questions included in the growth analysis of this report are those that were aligned with the content of the intervention videos and associated student activities. These included questions dealing with the ability of students to solve complex problems in novel ways in a group problem-solving setting, the value of student interaction and communication, the benefits of students offering multiple representations of ideas and approaches to solutions, as well as the importance of students learning concepts through problem solving prior to being introduced to procedures. The twenty-two belief questions that were excluded from the growth analysis presented in this report were those that were not aligned with the videos of children doing mathematics. An analysis of these non-aligned belief items revealed that there was significant pre-service teacher growth in these areas; however, the growth was comparable for both the experimental and comparison groups.

The findings, while encouraging, raise important questions for further study. For example, it would be useful to know in more detail how instructors made use of the videos at the various sites while working with the pre-service teachers. Were videos studied individually by pre-service teachers as homework assignments? Were they discussed in small groups, or viewed as a whole class activity led by the instructor? Did instructors focus on the pre-service teachers' reflections about how children learn? Additional considerations might be whether (or not) the treatment provided opportunities for multiple viewings and engaged participants in discussion of children's mathematical behavior. Analyses of workflow data, which document the specifics of which videos were used and how they were used, might provide insight into the *process* of teacher belief change. In addition, for those pre-service teachers who were asked to respond to reflection prompts after viewing the videos, an examination of their responses can give insight into how the changes in beliefs evolved.

Other research questions also arise from the study of teacher beliefs and changes in those beliefs. For example, accompanying a change in beliefs, is there a change in pre-service teacher content knowledge? Does a change in beliefs influence what a teacher notices on the videos that are being studied? Does what a teacher notice change over time? Perhaps *how* an instructor decides to use the videos is not as critical in changing beliefs as is having *access* to the video collection so that teachers can become more aware of the details in children's learning

Endnote

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2. The Video Mosaic website is located at: http://www.video-mosaic.org/

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PLACE-BASED PRACTICES IN RURAL MATHEMATICS INSTRUCTION: A SELECT CROSS-CASE COMPARISON STUDY

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This project studied efforts to connect mathematics instruction to community and place. Data came from interviews, observations, document analysis, and student work samples from seven sites across the United States. Analysis of the data suggested that such efforts usually result from the initiative of a particular teacher. These teachers build connections in order to sustain local practices, values, and in some cases, the community itself. A number of common affordances and constraints to place-based education emerged from the data. These reflected place-specific conditions as well as the extensiveness of various forms of support (e.g., personnel, money).

Introduction

The study from which this paper arises sought to understand the struggle to make connections between school mathematics and rural communities. The two primary research questions were: (R1) How do rural schools connect mathematics education to local communities and places? and (R2) What conditions enable and constrain such efforts? Rather than identify a set of “best practices,” a term that usually focuses too narrowly on curriculum, this study adopted a more sociological perspective by striving to understand the complex set of human exchanges and practices involved in attempts to connect mathematics instruction of rural schoolchildren to the people, land, and cultures of their communities. Many times, such practices reflected the deeply personal passions of an individual or group of educators or local community members. They always required intense creative efforts on the part of the educators—far beyond what is required to follow the inherited curriculum. As such, cases meriting study are not common, and studies of these phenomena even less so, though some examples of such work demonstrate the potential of place-based approaches (Green, 2008; Howley, Howley, & Perko, 2009).

Educational interventions that connect instruction to local people, land, and cultures are one route to authentic assessment and instruction (Sarkar & Frazier, 2008). They draw on existing knowledge at individual, school, and community levels, reflecting consistency with (social) constructivist models of learning (Smith, 2002). While forging connections between instruction and community is no less applicable to non-rural schooling, place-based practices are seen as an especially promising approach for rural schools given the primacy of community, family, and the land in rural areas (Theobald, 1997). This perspective motivated the focus of this study on rural places. One measure of the fit of “place” to “rural” is suggested by a count of the number of ERIC documents tagged with the descriptor “Place Based Education.” As of this writing, 191 documents bear the descriptor and nearly all also carry a rural descriptor or display strong rural connections. The study is therefore simultaneously an interrogation of the compatibility of place-based approaches and rural education.

A focus on mathematics education stems not only from an interest in improving mathematics instruction in rural schools, but also from the recognition that quality mathematics instruction is essential to the success of both rural schoolchildren and rural communities. In the emerging green economies, the place of rural communities figures significantly. Alternative energy

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production seems tied to corn, soy, and other rural products. The skilled implementation of new technologies will necessitate all levels of expertise in math, agriculture, science, and engineering.

Focusing on the *struggle* to make connections as described above is not to ignore issues of pedagogy and psychology, but rather to see them within a broader set of affordances and constraints that structure educational practice. Pedagogy generally results from the struggle and stands as a reflection of an educator's prior negotiations—of resources, of values, and of risks. If Otto Von Bismarck's saying about politics being the art of the possible is to be believed, then the classroom may be its premier gallery. Inherited curricula come packaged in the prior approval of some authority (editors, departments, parents, administrators). Place-based approaches studied here were not off-the-shelf curricula, so bringing these approaches to the classroom involved negotiating issues of time, curriculum coverage, accountability, and support. The cases profiled in this study are examples of creativity, risk, and craftsmanship at work in classrooms across rural America.

Background and Relevant Literature

Mathematics education in rural schools is often considered from a deficiency perspective (Howley, Howley, & Huber, 2005). Alternative perspectives, however, recognize the affordances (Gibson, 1977) that structure opportunities within a given social setting, or the "funds of knowledge" (González, Moll, & Amanti, 2005) that similarly reside within the community and reflect the resources, skills, and knowledge available for action toward some end. This study shared the belief that everyday rural life contains important resources that can improve mathematics instruction. These alternative perspectives both structured and motivated the investigation. Further examinations of the obstacles and affordances may be found in Klein (2007, 2008, 2010).

In defining the central substantive interest as "connections between school mathematics and rural communities," the study draws from literature on "place-based education." Though reported instances of place-based education are comparatively common in social studies, literature, and science classes, they are almost non-existent in mathematics classes and even less common than empirical research about mathematics education in rural schools (Silver, 2003). Place-based education is itself a work in progress (Gruenewald, 2003), and is simply defined as learning wherein local "place" is an essential feature. A number of writers in rural education conceive place as the interaction of community and land—land that influences community and vice-versa; in other words, a "land ethic" (Haas & Nachtigal, 1998). To focus curriculum on these interactions, particularly in face of efforts to standardize curricula around so-called national interests—reflects what Kliebard (2004) described as "curriculum struggle."

Earlier studies of such curriculum dynamics are comparatively rare. One of the first relevant empirical studies examined the work of teachers and community members in five districts to design place-based activities and infuse them into their schools' curricula (Wither, 2001). The study examined the development of the activities as well as the challenges faced during implementation. For example, teachers not involved with the planning valued place-based education less highly than the need to address the more self-evidently pressing school problems (e.g., test scores, AYP). Principals also offered only limited support, according to Wither. Green (2008) further showed how difficult place-based education is to sustain absent continued funding and third-party assistance. These difficulties were acknowledged as possible constraints to be aware of in the present study.

Methodology

This paper is grounded in the broader work of an NSF-supported center for teaching and learning focused on improving the teaching, learning, and understanding of mathematics education in rural America. The researchers on this project are affiliated with this center and have inherited from it an extensive collection of research that directly supports all aspects of this study. From this center's network of rural education contacts, members of the research team contacted 58 nominators who were familiar with local practices. They were asked to nominate sites that were engaged in connecting community and place to mathematics instruction. The nomination process resulted in 61 sites, from which the team used purposeful sampling techniques to select seven representing variety in geographic location, grade level, and extensiveness of community and place engagement.

The team developed data collection protocols for interviews, focus groups, observations, and document acquisition. A member from the team spent one week at each site conducting classroom and community observations as well as focus group and individual interviews with teachers, parents, students, administrators, and community members. Data for the seven sites included 85 interviews, 27 classroom observations, and 30 sets of field notes. Individual and focus-group interviews were audio recorded, transcribed, and coded initially using qualitative software to produce a draft coding scheme. A subsequent review generated codes that were best suited to the cross-case comparison and organized into conceptually distinct categories (Strauss & Corbin, 1998).

Next, a set of cross-case themes emerged based on clusters of coded data and draft case analyses. A second researcher reviewed data and determined the extent to which they confirmed or disconfirmed the proposed themes. Secondary analysis revealed that the challenges of integrating place and community into mathematics instruction were more pronounced than initial analysis suggested. The final case studies reflect a cross-case "conversation" that depicts, in broad terms, the character of the work undertaken at each site to integrate place and community into mathematics instruction. These are organized into a forthcoming monograph.

Analysis

South Valley Elementary School (SV) (all names given are pseudonyms). South Valley Local School District is a small district located in rural Ohio. Ms. Ball, graduated from South Valley High School and now teaches six grade mathematics at South Valley Elementary. She engages students in a range of community-inspired mathematics projects throughout the year, including St. Jude's math-athon, Relay for Life, Pi Day, and a stained-glass project that focuses on geometry and measurement in creating artwork. For the Relay for Life project, students collect and track money, time laps, and graph results.

Magnolia City Schools (MC). Magnolia City School District is located in a rural southern Alabama county where 55.65% of students receive free or reduced lunches. Magnolia City students participate in an aquaculture program and actively maintain the fish environments (monitoring pH, population size, health), track and foster fish growth, and eventually sell the fish at a community fish fry that generates funds to sustain the program.

Meriwether Lewis Junior-Senior High School (ML). Located in the a mountainous region of Washington, the school had just over 250 students in grades 7-12 in 2007-2008. Ms. Jay is the only middle school mathematics teacher and invites community members to her classes to describe the mathematics they use in their daily work. This has included a local fiber artist, a bicycle shop owner, and a video game designer. For "Math Communities," parents lead small groups of students through multi-step word problems every 2-3 weeks.

Eastcove Community School (EC). The school has a student body (pre-K to 12) of just 71 students and is located on a small island off of the coast of Maine (so small that the town's website lists the name of every baby that lives in the community). The principal and some teachers have instituted several place-based initiatives including the design and construction of pea-pod boats and an all-electric vehicle that was demonstrated in Washington, DC.

Green Mountain School (GM). This is the only non-profit private school in the study. The school enrolled 127 students in grades preK-8 in 2007-2008. The school is located on a wooded campus in rural Vermont. Students engage in "tree plot math:" a six-week project related to the local industry of timbering. They are given tracts of land, shaped sometimes as triangles, circles, or other shapes and they gather data about the trees in their plot, graph the results, calculate the worth of their trees, visit the mills, and occasionally present their findings to the schools' board of trustees (to decide whether or not to log the tract).

Confluence District Collaborative (CD). Located in a Great Plains state, the collaborative brings four school districts into one administrative unit with one superintendent and central office staff. The schools in the district have poverty rates at or below the national average of 17.4% and educate students from a wide geographic region. Moreover, survival of the individual schools and districts involves significant sharing of resources, including teachers travelling between schools to teach classes. Unlike other sites where place-based efforts were establishing themselves, CD exemplified the decline of a nationally-renowned site for place-based work. The research team found that, at CD, most informants there viewed place-based approaches as appropriate only for lower-level students or not as possible given the distance-learning technologies being used to solve the teacher travel problems. Studying this decline is important in understanding what sustains or inhibits such work.

Lafayette County High School (LC). This school is a comparatively large school (over 1,000 students in grades 9-12) in rural Kentucky. Because of the initiative of a math teacher and an ag teacher, the school operates a lutherie class in which students craft ukuleles from raw lumber over the course of the year. Each student makes two ukuleles: one to keep and one to sell at a community show for funds to sustain the program. Measurement, scale geometry, and trigonometry are emphasized connections.

Results

The authors reported first findings of the study in 2009. At that time, the first round of case analyses did not include Lafayette (LC) even though the data had been collected. In addressing RQ1, how connections were made, and RQ2, what supported and inhibited these practices, the following pre-LC themes emerged: (1) the influence of place-specific conditions, (2) initiative and support, and (3) "the praxis of the real." These themes structured our answers to both research questions and inform a deep, cross-case analysis currently underway.

The influence of place-specific conditions characterizes significant evidence of how the place-based work is both supported and limited according to the particularities of the place in which that work is being done. This theme was further subdivided based on review of the data. *Economic priorities and strengths of the area* was an oft-cited theme, such as at Eastcove where the importance of fishing to the island led to students' use of scaling, proportional reasoning, and representation in the design of "pea-pod boats." At Magnolia City, aquaculture and a school program involving turf production and maintenance reflected community dependence on tourists who frequent area resorts. *Isolation and population density* limited teacher-to-teacher collaboration and student-parent access to school grounds to support these initiatives, and members of the research team perceived this theme inconsistently. Though it arose frequently at ML, a secondary analysis of the data didn't offer significant cross-case support for "isolation" as a substantial sub-theme. Finally, *community networks* were often mobilized to support instruction, such as at ML where the teacher depended on parents and community members to share with her classes the ways that they use mathematics at work or as part of a hobby.

"Initiative and support" recognized how dependent many of the cases were on a particular teacher or educator (e.g., people, time, money). At ML, Ms. Jay was a wellspring of initiative, but indicated that collegial or administrative support was largely rhetorical. She told the research team that motivating parents to volunteer was a time-intensive and frustrating process that usually engaged the same set of parents repeatedly. That said, she continued to fight to keep the program going, reflecting her initiative and considerable skill at recruiting support. Moreover, she received grants from the local educational foundation (made up of area parents and businesses), which allowed her to compensate classroom guests for the time they took off work. This creative solution grew from her knowledge of the particular local economic realities that might prevent community participation.

The politics behind the Confluence District Collaborative left many teachers unwilling or unable to support place-based instruction, either as an act of resistance, or for want of will or means. Programs at several sites were driven by the commitment and passions of one or two educators with other teachers' and administrators' non-participatory approval.

Finally, the "praxis of the real" refers to the frequent comment in interviews at all sites studied, that "connecting mathematics to community and place" meant identically, "connecting mathematics to the 'real world'." Moreover, study participants frequently commented how challenging it was to forge these connections, and how some mathematics content and some mathematics students were not fit for the challenge. This paper focuses on this theme and how Lafayette's case analysis has nuanced the research team's understanding of this theme.

Inherent to the challenge of making connections to "the real" is deciding "*whose* real?" A seventh grade student's reality may have to be expanded to include the considerations involved in logging a tract of land if Tree Plot Math at GM is going to be considered meaningfully "real." At LC, the vocational agriculture teacher confessed that his greatest surprise in teaching the lutherie class was discovering that most students couldn't read a tape measure—it simply wasn't part of their reality to that point. Evidence from the interviews reveals a tension in this kind of work between *enforcing* and *evoking* reality. When an activity becomes curriculum, it becomes part of an *enforced* school reality rather than a student's reality. Math Communities at ML engaged students in problems where their behavior was monitored as part of a prescribed path to a closed-ended solution. It should be no surprise that, since the introduction of Math Communities at ML, student scores on the problem-solving portion of the state WASL test have increased. High-stakes assessments measure connections between school realities. *Evocation* of a reality, on the other hand, may lead to more *memorable* experiences, but its open-endedness can make it difficult to prescribe outcomes as is currently the practice of modern lesson planning. As LC shows, the crafting of a stringed instrument takes two hours per day, five days a week for the entire academic year. "Evocational training" apparently takes time—a precious commodity given the pressures of accountability. This tension suggests that the *praxis* is as difficult to nail down as the meaningful *real*ity. In the words of a Green Mountain teacher:

> *Some specific skills they may not retain better from working, or learning, in the woods or learning out of working in a textbook. However, their understanding of how they can use those skills to be in their environment and have it make sense, and actually get something out of it.. and the other things they learn about…their environment, affecting the economy…those things they all remember… the woods aren't just there but there is actually something in them* [emphasis added].

This passage also highlights the ties between "the memorable" and "the real" for students of a given age. Mathematical objects, it would seem, are not as memorable as a tract of woods. One must remember, though, that Tree Plot Math, like many of the projects studied here, are episodic, and that traditional classroom learning likely predominates.

A second question that arose from more careful consideration of this theme is analogous to "whose reality?" It asks, "whose praxis?" Many teachers at CD felt like the calculus teacher at one of the high schools:

> *I teach upper math, so I don't know. I teach trig and calculus. So I doubt if we do any calculus in local places around here. I don't know how we have time to do that. I am not sure that the upper math classes—maybe a lower income average are going to need that—but not my upper math class…Because my kids that I teach now are interested 'in the math,' I don't have to make it flowery, I don't have to make it enjoyable, I don't have to make it fun, they just want to know, what's the math, the theory behind it.*

In the case of Eastcove, the fishing life of the island and the seasonal tourism was an ever-present reality that the principal of the school connected directly to the survival of the community: "Of all the islands in [the state], there are only three left...and, so, we feel like we've got to work hard to keep this community going, and part of it is connecting the kids with the community all the time in whatever ways we can." The question of "whose praxis?" therefore is one that questions ownership of (relatedly, "buy-in to") the purposes and practices of learning.

The Real, A Place Revisited

When the Lafayette (LC) case study was complete, the team realized that the "praxis of the real" failed to capture important nuance present in the data. What emerged from LC was evidence of the recognition that considerable fear surrounds getting local, everyday mathematics into the open. Whereas math indeed may *be* everywhere, it remains largely unrecognized—what we consider "locally *immanent"* (indwelling). Few seem willing to take on the project of "revealing" the everyday-ness of mathematics. This irony is captured well by "immanence" which is defined both as "remaining or operating within a domain of reality or realm of discourse" and "inherent…having existence or effect only within the mind" (both from Mish, 1984, p. 601). The first meaning captures the confinement of everyday mathematics to the unrecognized obscurities of daily life, while the second captures the Platonic sense of ideal mathematical objects existing only in the mind. Yet while *immanent* in both senses, it is also *imminent*, or "ready to take place; esp: hanging threateningly over one's head" (Mish, 1984, p. 602). Despite math's utility, it nonetheless seemed to "hang threateningly over study participant's heads" including adult informants in schools and in the community. The reality to which school mathematics might connect is one in which mathematics is both immanent and imminent. Examples of this were found in abundance on re-inspection of data from other sites.

While the need to make mathematics fun was not a universal feature of the sites studied here, the place-based approaches were described as “memorable” and “relevant” by students, teachers, parents, and community members. Adults at all sites made familiar confessions. At Lafayette:

> *Cabinetmaker (father of former student): I wish I could say…I loved school, but math?... Geometry I was into, but when it came to algebra, it was over my head, you know?*
> *Teacher: Around the country [nation] students really have a hard time with mathematics and just seeing the relevance of mathematics.*
> *Superintendent: And for some reason in this area is a very feared subject by—by many kids. They’re—they have a—a fear of mathematics and, you know, they—they’re kind of turned off to math.*

These same adults who confess a fear (imminence) of mathematics also report using it in their work (immanence). This sentiment was echoed by adult participants at nearly every site. As such, it seems an appropriate and more encompassing thematic frame for understanding the praxis of the real. This insight is confirmed repeatedly in the transcripts. Perhaps its best formulation was by a former student of the lutherie program at Lafayette: “The side of math you don’t see every day is the everyday math.” Her statement is rich with irony. Nevertheless, it suggests that the real challenge is in the “seeing,” rather than the “connecting.” Ongoing cross-case analysis will further bear out the appropriateness of this new thematic frame for each case.

Three cautions seem warranted in understanding the portion of results presented above. First, examples of rural schools connecting mathematics to education are promising but rare, so generalizability is not appropriate. Second, math-place or math-community connections happen (R1) because of and despite the affordances and constraints (R2) found there, so these questions can’t be considered independently. Conditions that enable and constrain place-based approaches, as evidenced by the themes of *initiative and support* and *place-specific conditions*, were far easier to see than the intricacies of *how* such connections are made. Nevertheless, evidence of how they were made yet bore the scars of their negotiated creation. Third, “connection” was a difficult concept to probe as it begged so many other questions: Of what? To what? In what way? A deeper level of cross-case comparison is underway to address these issues. It suffers from the same problems of defining context as do studies of transfer of knowledge (from what to what?).

Ongoing analysis is motivated by acknowledged limitations in the first and second passes of data. These limitations reflect the difficulties inherent to understanding community practice and are well noted in methodological literature (e.g., Geertz, 1973). For instance, members of the study team kept asking one another for each site, what the “case” was. At four sites (GM, SV, LC, ML), the presenting “case” seemed defined by the practices of a particular teacher. At CD, the collaboration itself seemed to be the case. Eastcove (EC) had the broadest scope and invoked place-based education explicitly, though even there the project was sustained by a group of teachers trained in place-based pedagogy. Making connections rested in nearly every case on the insights and passions of particular educators. Moreover, these passions seem to result from an inspirational antecedent experience or contact. The founder of the lutherie program at LC was inspired by a neighboring school’s raptor rescue program. Similar stories surfaced at GM and ML. Further analysis and team discussion will involve clearly defining the research case.

Regarding the purposes for engaging in place-based projects, place-based and rural education literature suggests “sustaining rural communities” is the primary purpose. Eastcove demonstrated this most strongly, as noted in the principal’s comments. The Confluence Collaborative, despite mathematics not being implicated to the same degree there, also reflected this to some degree in that the purpose of the collaborative is to provide an administrative unit that allows schools in these areas to continue operation, thus sustaining one of the anchors of rural communities (DeYoung, 1995). Projects at LC, GM, MC, and ML connected mathematics instruction to local values and practices, thus promoting sustainability of the lifeways that predominate there. The purpose of these projects is an important factor that seems connected to both R1 and R2. Ongoing analysis and further review of the extant literature will help to clarify this connection.

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QUANTITATIVE LITERACY: PRESERVICE TEACHERS' MATHEMATICAL KNOWLEDGE, BELIEFS, AND DISPOSITIONS

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Quantitative literacy is a way of thinking and understanding that combines an individuals' mathematical content knowledge, beliefs about the nature and utility of mathematics, and mathematical disposition. Since quantitative literacy skills are usually developed within mathematics courses, this study seeks to discover the quantitative literacies of future mathematics teachers. Seven preservice teachers, both elementary and secondary, were interviewed about their mathematical knowledge, beliefs, and dispositions. Participants were also asked their opinions on the notion of quantitative literacy, and how they will approach these topics in their future classrooms. The preservice teachers possess strong content knowledge of and positive attitudes and beliefs toward mathematics.

Introduction

Wilkins (2010) describes quantitative literacy as "a habit of mind that is characterized by the interrelationship among a person's everyday understanding of mathematics, his or her beliefs about mathematics, and his or her disposition toward mathematics." We use this conception of quantitative literacy to frame our study of the level of quantitative literacy of preservice teachers.

Cognition. Cognition, or knowledge, describes more than the mastery of school mathematics. Although skills needed for quantitative literacy are appropriate topics for elementary and middle school coursework, quantitative literacy topics are often deemphasized or skipped entirely in favor of other content (Steen, 1990). A quantitatively literate person has the skills required for a productive life, including the ability to count, tell time, use money, measure, read tables and graphs, estimate, and understand statistics as they are presented in society and media (Cockcroft, 1982; Steen, 1990).

Beliefs. Beliefs play an important role in a person's quantitative literacy (Author, in press, 2000). De Corte, Op't Eynde, & Verschaffel (2002) consider three categories of beliefs: beliefs about mathematics, beliefs about mathematical learning and problem solving, and beliefs about mathematics teaching. For example, one belief consistent with the notion of quantitative literacy is that mathematics is a dynamic field in which new ideas emerge, rather than a collection of absolute facts and formulas. Other beliefs about mathematics learning and problem solving that are not consistent with quantitative literacy include the idea that memorization is necessary for success in mathematics, or that the teacher is the final authority on mathematics and will always have the correct answer to a problem.

Disposition. Disposition refers to an "at-homeness with numbers" (Cockcroft, 1982, p. 11), in which people are comfortable with using mathematics. "Being quantitatively literate does not necessarily imply that one must enjoy the study of mathematics, but instead implies that one possess a willingness to engage in situations that require a functional level of quantitative reasoning" (Wilkins, 2000, p. 408). Further, disposition also refers to a person's understanding of the value of mathematics (Wilkins, 2010).

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Quantitative Literacy and Teaching

In order for teachers to effectively promote quantitative literacy among students, they themselves must also be quantitatively literate. While there is little research that analyzes all three components of teachers' quantitative literacy, many studies singularly or dually focus on cognition, beliefs, and/or disposition.

Various studies have examined preservice teachers' cognition, or content knowledge (e.g., Ball, 1990). While there are significant differences in content knowledge between elementary and secondary preservice teachers (Ball, 1990), both groups lack conceptual knowledge in many areas (Ball, 1990; Thanheiser, 2009). Ball (1990) found that despite the procedural adeptness of both elementary and secondary preservice teachers, their understanding of mathematical ideas was rule based, and they did not connect procedures with mathematical principles. Even though secondary education students had majored in mathematics, and scored higher on content knowledge, they showed no difference from elementary students in explaining and connecting mathematical meanings and ideas.

There are also many studies that have focused on teacher or preservice teacher beliefs and dispositions (Ball, 1990; Thompson, 1984; Wilkins, 2008). In general, secondary preservice teachers enjoy mathematics and have positive attitudes towards teaching mathematics (Latterell, 2008; Ball, 1990). Elementary teachers have more varied opinions; some greatly enjoy mathematics while others dislike it, or even fear mathematics (Ball, 1990; Liu, 2008).

Another belief shared by many preservice teachers is that mathematics is a collection of rules, and memorization is necessary for understanding mathematics (Ball, 1990; Liu, 2008). This limited view of the nature of mathematics held by some preservice teachers is unfortunate, because it has been found that teachers' beliefs about mathematics and teaching shape instruction (Thompson, 1984). For example, Thompson found that teachers who believed that mathematics was a set of facts and procedures tended to teach in a manner that projected that belief. Another teacher, who viewed mathematics as a set of interrelated topics, used classroom activities that allowed students to explore and make connections among mathematical ideas (Thompson, 1984). Author (2008) also found a relationship between teachers' beliefs and instruction. In this study, teachers with positive attitudes towards mathematics were more likely to use inquiry-based instruction in their classes. These findings are encouraging in that other studies have found that mathematics methods courses can positively influence preservice teachers' beliefs and attitudes (e.g., Wilkins, 2004). The purpose of our study was to explore the quantitative literacy of preservice teachers by considering two specific questions:

1. What are elementary and secondary preservice mathematics teachers' opinions of quantitative literacy?
2. In what ways are preservice teachers quantitatively literate (as related to their mathematical knowledge, beliefs and disposition)?

Methods

A phenomenological design was implemented to explore and describe the beliefs and knowledge of preservice teachers related to quantitative literacy. A phenomenological design was implemented because it suited the purpose of the research study; to examine the experiences and perspectives of preservice teachers, and how those experiences and perspectives were articulated by the participants (Rossman & Rallis, 2003). Seven preservice teachers in their fifth year of a teacher preparation program were interviewed. All of the preservice teachers were enrolled in a university degree program that leads to teacher licensure and a master's degree in education during five years of study. Four of the preservice teachers (Alicia, Bess, Julie, and Meghan) were students in an elementary program. Two of the elementary preservice teachers (Alicia and Meghan) had undergraduate degrees in English while the other two (Bess and Julie) had undergraduate degrees in Interdisciplinary Studies. The remaining three preservice teachers (Briana, Mike, and Victoria) were students in a secondary mathematics program. All three of the secondary mathematics preservice teachers had undergraduate degrees in mathematics. At the time of the interviews, all of the preservice teachers were enrolled in a methods course for elementary and middle school mathematics.

One limitation of the study is the dual relationship that the researchers had with the participants in the study. The researchers personally knew all but one of the participants due to the fact that the researchers had taken a class with the participants in a previous semester. However, the researchers and the participants had never before discussed the matter of quantitative literacy. Therefore, it was believed that the dual relationship would not adversely hinder the current study.

Data were collected through standardized open-format interviews. The researchers developed preliminary interview questions and then sought advice from other scholars and researchers about the effectiveness and clarity of the research questions. Once the interview questions were finalized, the researchers matched up each interview question with a research question to make sure that the interview suited the purpose of the study. Table 1 synthesizes the link between research questions and interview questions.

Table 1. Research Questions in Relation to Interview Questions

Research question	Interview question
1) What are preservice teachers' opinions of quantitative literacy?	M2, M3, M4, O1, O2, O3, O4, O5, T6, T7
2) In what ways are preservice teachers quantitatively literate?	
a) Knowledge	P1, P2, M3, T1, T2, T3, T4
b) Beliefs	P1, P2, M1, M3, O3
c) Disposition	P1, P2, T1, T5

The standardized interviews consisted of questions on the subject's mathematical background, beliefs, and attitudes. The preservice teachers were asked to describe their experiences as a mathematics student in elementary, middle, and high school as well as college (P1, P2)? To determine some of the preservice teacher's beliefs and views about mathematics, we asked: Is math unchanging (M1)? What math skills are important for everyone to have (M2)? Why should people learn math (M3)? How does your view about why people should learn math affect what you will do as a math teacher (M4)? After the researchers presented the preservice teachers with a definition of quantitative literacy (Wilkins, 2010), the participants were asked to comment on the following questions: What do you think of this definition (O1)? Based on this definition, what level of school mathematics do you think is essential to be quantitatively literate (O2)? What is the importance of being quantitatively literate (O3)? Assuming we need to develop quantitative literacy in students, what is the role of a math teacher (O4)? Do you think people besides math teachers have a role in helping people become quantitatively literate (O5)?

The participants were also asked to perform two quantitative tasks, which highlighted both the cognitive and dispositional aspects of each participant's quantitative literacy. One task involved calculating a tip at a restaurant, and the other task involved understanding averages based on a story from a fictional news report (T3, T4). After each task, participants described how the problem made them feel (T1, T5), and whether they thought the task required skills important for all citizens to have (T2, T6, T7).

The interviews were videotaped and transcribed. The transcriptions were then coded in order to organize, analyze, and interpret the interview data. Themes and categories of responses were organized using constant comparative analysis and code mapping (Anfara, Brown, & Mangione, 2002). First, the first two authors individually coded interviews. Next, the researchers discussed emerging themes and exchanged and recoded interviews in order to help ensure the credibility and dependability of the study. Initial codes and memos were then further categorized into themes and patterns that were relevant to the purpose of the study and that would help address the research questions.

Results

Participants' Cognition

Certainly, using only two tasks made it difficult to assess the participants' cognition during the interviews. However, all preservice teachers seemed to possess strong mathematical content knowledge relevant to tasks used in the study. The skills they demonstrated included performing basic computations, understanding averages, analyzing data given in paragraph and table formats, and discussing mathematical ideas as they apply to real-life situations.

When asked what mathematics skills were important for everyone to have, the preservice secondary teachers mentioned high school math courses as well as everyday quantitative skills. Elementary preservice teachers also cited *rounding* as an important skill. All participants felt that problem solving (or critical thinking) were essential mathematical skills for all citizens. Below are representative quotes from two of the preservice teachers.

> Definitely algebra skills, into Algebra II, just knowing how to manipulate equations…People have to have a good concept of fractions and percentages and decimals… [in order] to be a responsible citizen, which includes things like going to the grocery store or reading facts from a newspaper…People need to know how to interpret what's going on, and they need to know how to take information they receive, and sort of understand what's being said. (Briana, secondary)

> Adding, subtracting, multiplying, dividing. Being able to find out percentages without having to use a calculator, for things like going to restaurants and tipping or figuring out sales. (Alicia, elementary)

Participants' Beliefs

Of the three components of quantitative literacy, the preservice teachers elaborated the most on their beliefs about the nature of mathematics and about the teaching and learning of mathematics. Although the preservice teachers were not directly asked about their perceived ability in mathematics, many of them commented on their beliefs about mathematics ability. They believed individuals have an innate ability or disability for mathematics. Bess (elementary) described this belief well when she said, "I think that's just the way my brain is. You know, some people are math people, some people are not." Coupled with this belief, the preservice teachers did not demonstrate that success (as measured by grades) affected their belief about whether or not they had natural mathematics ability. Meghan (elementary) explained that her grades in mathematics classes were typically A's or B's, but that she had to work incredibly hard in order to earn those grades. Even though she was successful (as measured by grades) in mathematics classes, she still considered herself terrible at mathematics. Both high and low-achieving preservice teachers related being 'good at math' with not having to work hard at it.

To further understand their mathematical beliefs, subjects were asked questions dealing with the certainty, simplicity, and sources of mathematical knowledge. To discover if the participants viewed mathematics in a static or dynamic manner, we asked if they thought mathematics was unchanging. One preservice teacher, Mike (secondary), explained that mathematics was constant, that the principles and concepts of mathematics were unchanging. However, the discovery of new principles and concepts also changes mathematics. By using the word *discover*, he seems to imply that mathematics itself does not change. He gave the example of discrete mathematics not existing twenty years ago, but the mathematics underlying the field has always been true, whether or not humans had 'discovered' it. Briana (secondary), Bess (elementary), and Alicia (elementary) felt similarly in that mathematics is always changing because of new developments, but that properties were always true, and thereby did not change. Bess (elementary) said that "two plus two will always equal four, so…basic things are unchanging." Alicia (elementary), Julie (elementary), and Victoria (secondary) commented on how mathematics changes depending on how it is used.

An important belief about the nature of mathematics that was conveyed by participants was the presence of more than one valid method to solve a problem. The preservice teachers were asked what their response would be to two students in their classroom who were arguing because they used different methods to solve a problem. All of the preservice teachers, either implicitly or explicitly, responded that they believed more than one method to solve a problem. All seven participants believed that students should use the most meaningful and sensible to them, even if the chosen method differed from another students'. In addition, some of the preservice teachers believed that students could even use a method that they developed on their own. Meghan's (elementary) response to the interview question was "I always say 'use the method that works best for you'…as long as it works consistently and as long as I can understand it…and it's not just a fluke, they are more than welcome to use it." Alicia (elementary) mentioned multiplication as an example. She said that when she learned how to multiply two-digit numbers, there was only one way to do it. However, since taking her mathematics methods course, she realized that there were many different methods and was able to list at least five different techniques for teaching multiplication during the interview.

Closely related to the belief that there is more than one valid method to solve a mathematics problem is the belief that the reason or justification for a problem is more important than the actual answer. According to the participants, students could use different methods to solve a problem as long as they understood the reasons for using those methods. Many of the preservice teachers stressed the significance of *process* over *solutions*. Below are quotes from two of the preservice teachers expressing this idea.

> *You got the answer, but that's not the important part…The important part in math is how you came to understand it and the method you used.* (Julie, elementary)

> *The processes that are involved in stepping logically through an equation are just as important as the right answer- no, more important than the right answer. Because all throughout life, people are making decisions. And good decisions are made when people can step logically through what's going on.* (Briana, secondary)

An encouraging belief that emerged from the interviews was that the preservice teachers valued conceptual understanding over memorization. Instead of placing importance on procedures and skills, the preservice teachers thought that it was important for students to develop conceptual understanding of mathematical ideas.

Another belief about the nature of mathematics, especially related to the utility of the subject in real life, was that mathematics is useful and important. All seven of the participants expressed the belief that mathematics is used every day life and essential for thriving in society. Julie (elementary) commented: "It [math] is everywhere! You can't do anything in the world, you can't really do anything without math!" Because math is so useful in everyday life, the preservice teachers believed that it was important to show students just how useful and relevant mathematics really is. Briana (secondary) thought that this involved more than just telling students that you need mathematics in any job. She thought students had heard that argument plenty of times and that it didn't really mean much to students; instead, students needed to be shown examples.

Disposition

When asked if they enjoyed math, all of the secondary preservice teachers, and two of the elementary preservice teachers responded enthusiastically. However, the two elementary preservice teachers who expressed a dislike for mathematics, Alicia and Meghan, both said they appreciated mathematics more since taking their mathematics methods course. Both positive and negative views towards mathematics were attributed to their past successes or failures in mathematics coursework.

Another important finding related to disposition is that these elementary preservice teachers were not apprehensive about teaching mathematics. Bess and Julie were both extremely excited about the opportunity to teach mathematics, a subject that they both greatly enjoy. Despite their dislike of mathematics, both Alicia and Meghan were confident that they could teach mathematics in an elementary classroom due to *low* levels of mathematics addressed in elementary school curriculum. Alicia was excited to teach using strategies she learned in her mathematics methods course. Meghan was assured because she felt elementary mathematics was easy to understand. "I feel it's a little easier so I'm not afraid of it. The hardest thing I'll probably have to teach would be basic algebra, multiplication, or long division and I can do all those things well."

Conclusions

The preservice teacher participants in this study were generally quantitatively literate and possessed strong content knowledge and positive beliefs and dispositions. Participants considered quantitative literacy to be of importance, and stated that they planned to promote quantitative literacy in their classrooms. All participants showed some awareness of quantitative literacy, whether or not they recognized it by that name. They held several beliefs consistent with the goals of quantitative literacy, in particular, those about the nature of mathematics and its value and utility in society. In addition, the preservice teachers believed that mathematics should be taught conceptually and not just procedurally. Further, teachers seemed to possess a positive disposition toward mathematics and a willingness to engage in mathematics teaching (cf. "Curriculum", 1989; Wilkins, 2010).

In contrast with the findings of previous research (e.g., Ball, 1990; Liu, 2008), in our study we found that preservice teachers viewed mathematics as more than a set of procedures and facts. In accordance with that belief, all participants stated that they planned to stress conceptual understanding over memorization in their instruction. They also expressed that they would try to motivate their students by showing them the usefulness of mathematics.

The participating preservice teachers were excited about teaching mathematics. For a majority of the participants, this excitement seemed to be due to their positive dispositions; even those who seemingly did not enjoy the subject. In this case, this change in attitude and beliefs was attributed to their experiences in a mathematics methods course.

Future research on the significance of quantitative literacy in teacher education programs would be valuable to better understand how to promote preservice teachers' skills in the area of quantitative literacy and their potential to promote quantitative literacy in their future classrooms.

There is little we do in America that is more important than teaching. Effective teaching of mathematics requires appropriate pedagogical and mathematical foundations, but thrives only in an environment of trust which encourages leadership and innovation (National Research Council, 1989, p. 57). Such an environment would most certainly also develop students' quantitative literacy – their knowledge, beliefs and disposition in relation to mathematics.

Endnotes

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UNDERSTANDING OBSTACLES TO TEACHER EDUCATION FOR POST-SECONDARY TEACHERS OF MATHEMATICS

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In an effort to inform teacher education for future post-secondary teachers of mathematics, a dialogue with mathematics graduate students was established in order to explore their experiences and perspectives of mathematics teaching. Using hermeneutic inquiry and thematic analysis, a series of conversations was analyzed and interpreted with attention to themes and experiences that had the potential to influence the mathematics graduate students' ideas about and approaches to teaching. The structure of their graduate work, their views of mathematics and themselves as mathematicians, along with expectations of behavior in their department, had implications for how the graduate students formed their identities as teachers. Themes that are explored in this report are teacher versus professor, replication, and resignation in the participants' views of who and how they could be as post-secondary teachers of mathematics.

Introduction

Mathematics departments constitute one of the largest service departments within institutions of higher education, providing prerequisite courses for students in diverse disciplines such as engineering, physics, chemistry, business, medicine, psychology, and education. As a result, the teaching of mathematics at the university level is quite important in undergraduate education and professors, instructors, and graduate teaching assistants in mathematics have a wide-reaching influence on the education of future researchers, teachers, and mathematicians (Golde & Walker, 2006). While there has been an increased focus on undergraduate mathematics education, the format and style of post-secondary mathematics teaching has remained problematic for undergraduates (Alsina, 2005; Kyle, 1997; NSF, 1996).

In the effort to improve university mathematics teaching, mathematics graduate students have recently become subjects of investigation. The most recent research into mathematics graduate students' teaching has examined their classroom practices and possible connections between their practices and beliefs about teaching and learning. Researchers concluded that newly acquired positive attitudes and beliefs about teaching mathematics did not produce hoped for changes to graduate students' teaching practices (Belnap, 2005; Speer, 2001). When graduate students in mathematics could speak of teaching using reform-oriented terminology, these students also reported that they maintained a traditional, lecture-style form of instruction (Belnap, 2005). Other research has shown that enrollment in a course in pedagogy did not produce expected changes to mathematics graduate students' teaching practices.

In light of these conclusions, it appears that the experiences of mathematics graduate students and the development of their teaching practices are not yet well understood. Such an understanding is important in the field of mathematics teacher education as almost seventy-five percent of mathematics PhDs will become professors at post-secondary institutions dedicated to undergraduate education rather than research (Kirkam et al., 2006). Moreover, because professors of mathematics and mathematics graduate students often represent the last models of mathematics instruction for future elementary, secondary, and post-secondary mathematics teachers, university mathematics teaching has a far-reaching influence on teaching at all levels

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(Golde & Walker, 2006; Shulman, 2004). As previous studies have found that informing mathematics graduate students of different approaches to teaching, student learning, and curriculum reform did not alter classroom practices, something remains to be explored.

Orientation of the Research

Within the graduate students' lives in mathematics exists a complex and intricate interplay among the structures that mathematics graduate students encounter, their feelings about mathematics and themselves, their interpretations of the nature of mathematics, and their sense of their new role as teachers. Through their involvement in their graduate programs and the routines of a department of mathematics, graduate students' views of the discipline and its teaching are shaped. Exposure to new contexts of mathematics teaching, learning, and research have the potential to be interpreted as having implications for how they should be and convey their work as mathematicians (Austin, 2002). Further, their own ideas and beliefs also appear to have an influence on mathematics graduate students' teaching (Speer at al., 2005).

In thinking about the idea of mathematics graduate students becoming future academics, professors, researchers, or mathematicians, where do they look for who they are supposed to become? For indications of how they should be, of how they attend to their work, their students, their discipline, and themselves? How might the graduate school experience shape the identity of these future mathematicians as teachers of mathematics? What has meaning for them in how they present themselves within their disciplines and in the classroom? What is it that mathematics graduate students interpret or understand their lives to be like in mathematics? What might they interpret as having significance for who and how they should be as mathematicians and as post-secondary teachers of mathematics?

In raising such questions, this research project is an exploration of the lives of mathematics graduate students and is concerned with how mathematics graduate students become mathematicians, the experiences they go through, and what has meaning for them as they become mathematicians and post-secondary teachers of mathematics. In order to explore the answers to these questions, hermeneutic inquiry was chosen as a way to seek an understanding of these phenomena. Hermeneutic inquiry is important for this project as "it is the interpretive study of the expressions and objectifications of lived experience in the attempt to determine the meaning embodied in them" (van Manen, 1997, p. 38). Hermeneutic inquiry helps one to unearth the ways and the whys in which we understand life and lived existence, and how we can create and find meaning through experience, language, and social engagement (Brown, 2001; Gallagher, 1992; Smith, 2006).

Mathematics graduate students' experiences, their professors' teaching, departmental expectations, and their teaching assistantships all have interpretive implications for what mathematics graduate students make important in their lives. As graduate students' knowledge of their future worlds develop "as a consequence of their encounter with the department: semi-automatic, barely conscious interpretations of what teachers say and do" (Gerholm, 1990, p. 264), hermeneutic inquiry opens up a space for understanding interpretations within these different encounters. For this research project, hermeneutic inquiry affords attentiveness to the questions: What variety of experiences do mathematics graduate students encounter as they progress through their graduate programs? Among their experiences, what in particular is taken as having meaning for who and how they should be as mathematicians and as teachers of mathematics? How are these experiences interpreted to have meaning for how one lives a life in mathematics? Finding answers to these questions will help to deepen the understanding of teaching and learning in post-secondary mathematics and provide guidance in structuring post-secondary teacher education in mathematics.

The Research Study

Graduate students in mathematics from an urban, doctorate-granting university were approached to be participants in this study. Six students agreed to participate. The group was fairly diverse in their backgrounds: three were master's students, three were doctoral students, and they ranged from a first semester master's student through a final year doctoral student; four were men, two were women; their ages ranged from 22 to 33 years; and there were four nationalities among them. While each of their paths to graduate study in mathematics was distinct, all but one of the participants expected to work in academia once they completed their degrees. During their graduate programs in mathematics, each of the participants had been assigned to teaching assistantship duties such as tutoring workshops where they helped students one-on-one with homework exercises, grading homework and exam papers, or leading one-hour tutorial sessions during which they presented mathematical topics similar to those in the affiliated lecture section of the course.

Carson (1986) and Van Manen (1997) propose conversation as a mode of doing research within hermeneutic inquiry to explore and uncover one's own and others' interpretations and understandings of experience. In consideration of this, over a period of six months, a series of five audio-recorded conversations were conducted with the research participants. The first two meetings and the final meeting were conducted with each participant individually, where each meeting lasted approximately one hour. The third and fourth meetings were conducted with all participants present, each lasting just under three hours. A recursive process was used in which the topic of subsequent conversations was based upon themes from previous conversations. Throughout the project, the research participants had the opportunity to review the analyses in a collaborative effort to refine, augment, and improve the reporting of their experiences.

Each conversation was transcribed by the researcher, who listened for the topics of conversation and the language used by each of the research participants. Notes were made of the congruence among the research participants. These similarities were not limited to broad categories of their lives, such as how they each had to attend to their teaching assistantship duties or their graduate level course work. Rather, it was opinions and perspectives about various aspects of their experiences that appeared to be in common. These similarities were grouped into themes using the guidelines of thematic analysis described by Braun and Clarke (2006). The themes and the participants' comments within each theme were then assembled and analyzed using a hermeneutic, interpretive lens to understand what facets of their lives in graduate school were taken as having meaning for their identities as mathematicians and post-secondary teachers of mathematics.

Findings

For the mathematics graduate students who participated in this study, what did it mean to become a mathematician and a professor of mathematics? How did they understand their roles as mathematics teaching assistants and possible future professors of mathematics? How did they experience and make meaning of the various suggestions they encountered about teaching? What was important for success in their programs? What did the participants interpret as having meaning for who and how they should be as mathematicians and as professors of mathematics?

The concept of legitimate peripheral participation offers an interesting lens through which to interpret, understand, and describe what was happening for the mathematics graduate students. Lave and Wenger (1991) wrote “Communities of practice have histories and developmental cycles, and reproduce themselves in such a way that the transformation of newcomers into old-timers becomes remarkably integral to the practice” (p. 122). Further, Lave and Wenger (1991) claimed that “even in cases where a fixed doctrine is transmitted, the ability of the community of practice to reproduce itself through the training process derives not from the doctrine, but from the maintenance of certain modes of co-participation in which it is embedded” (p. 16, emphasis added). The “reproduction of the community” and “maintenance of certain modes of co-participation” are seen in the following themes and quotes from different participants: teacher versus professor, replication, and resignation, which were heard throughout the conversations with the participants.

Teacher Versus Professor

> *“It is difficult to ask a professor to teach”*
> *“The first thing we need to get across is that professors and teachers are two completely different things”*
> *“Professors are demanded to teach”*
> *“This idea that they [the professors] need to explain it enough or well enough in order for us to understand it is absurd”*

What became clear in this project is that the participants were on a path to becoming mathematicians, not post-secondary teachers of mathematics. They learned about the discipline of mathematics through their coursework and about mathematical research through undertaking their theses and dissertations. To be successful in their programs, they were compelled to become skilled in mathematics by mastering their coursework through earning high marks and undertaking a research project. In contrast to the mathematics they were required to become skilled in, the mathematics graduate students were not required to demonstrate competence in teaching, in how they interacted with students, how they presented material to a class, or how they assessed students’ learning. There was no point in their programs where they were evaluated on their teaching, even though teaching had the potential be a significant part of their future profession. The research work that they had to focus on, in a way, became the sole indicator of what would be expected of them in their careers.

As such, their identities became tied to becoming a professor, rather than a teacher, where the research work of a professor was more important and more esteemed than that of a teacher. The research participants marked out what they observed as the separate territory of professors and teachers. They insisted that teaching was not a part of a professor’s role. They described a professor’s role in the classroom as solely presenting mathematical material, with absolutely no requirements to assist students in their understanding of the mathematics. Gaining such an understanding was the sole responsibility of the students. It is interesting to note, however, that the participants had expressed enthusiasm for and interest in helping undergraduates to learn and understand mathematics. Yet, while observing what occurred in their department, in an act of “maintaining certain modes of coparticipation,” they declared that, as professors, they would not be required to teach, nor should they be expected to, as seen in the final quote above.

Replication

> *"It's easy to keep teaching calculus like this. We've been doing it forever"*
> *"You just do examples. In 50 minutes, it's over"*
> *"How many ways can you skin a calculus class?"*
> *"They are pretty different sections, but you pretty much do the same thing"*
> *"You could teach a little bit better, but I don't know how much variety you can actually put in. How much different is professor A from professor B?"*

The reproduction of the community resonates in above quotes from the research participants, which illustrate the theme of replication of both mathematics teaching and of mathematicians. Similar to Lave and Wenger's (1991) idea that communities "reproduce themselves" (p. 121), the post-secondary teaching of mathematics, as viewed by the participants, appeared to be a practice of replication, a reproducing of others' teaching and the material in mathematics textbooks. Specifically, one participant spoke of the structure of all mathematics courses as "definition, theory, example," where replicating the fixed structures of mathematics texts and courses was a sufficient, even sanctioned, way of being in the classroom. Further, Jardine (2006) has written that in mathematics there exists a "mood of detached inevitability: anyone could be here in my place and things would proceed identically" (p. 187), signaling the replication of identity amongst mathematics teachers. This view echoes in the language of "professor A and professor B" used by one of the participants, which spoke to an interchangeability between professors, as though their identities might be so alike or the differences so insignificant that it would not matter who was in the classroom.

Another participant's experience of being told that she was prohibited from saying "I don't know" in response to an undergraduate's question represented a disclosure that certain forms of behavior were considered illegitimate within the department of mathematics. In her peripheral role as a teaching assistant, she had to abide by this prohibition in order to become similar to those in the department on her way to becoming a mathematician. With regard to his own teaching, one participant spoke of how he could not work "outside of a certain box" in the department. As a result, he no longer appeared to have a concern for his teaching, saying, "I would not be able to change things even if I wanted to." When this participant spoke of his hopes for his future career as an academic, teaching was no longer of consequence to his success as a mathematician and future professor. In the final year of his doctoral program, this participant was an illustration of what Lave and Wenger refer to as the "transformation of newcomers into old-timers" (p. 121) and how "an extended period of legitimate peripherality provides learners with opportunities to make the culture of practice theirs" (p. 95).

The structures of the participants' programs, their teaching assistant work, and the suggestions that were put forth by the department either through direct communication or the lack of it seemed to point to a particular, sanctioned way of being and becoming a mathematician, a way of being which implied not only that teaching was unimportant, but also that it would be determined solely by what had been observed in texts and other professors' classrooms. Throughout the analysis of the recordings and transcripts, it became apparent that the participants began to make certain tasks more important than others through what they were and were not allowed to do as newcomers in the department. In this regard, it seemed that the participants were being primed for a particular way of being.

Resignation

> *"That's never going to happen in math"*
> *"There's nothing I can do about it"*
> *"No one is going to listen to me."*
> *"I hope I'll be different, but I don't think I will be"*
> *"I guess the system just drew the love out of it"*

Unfortunately, the act of replication of mathematics teaching and the thought of taking on a particular identity in mathematics evolved into feelings of resignation among the participants. With regard to his current role as a graduate student, one participant said, "You can't have an opinion, you can't have anything except the fact that 'yeah, this is true.'" Here it seemed that this participant was resigned to a passive position with respect to his own learning, and that he must accept the facts not only of mathematics, but also of how he could engage in the department. Further, when speaking about the possibilities for his future teaching practice and, in particular, about the use of discussion in a mathematics classroom, he said, "that's never going to happen in math," a statement that expressed a resigned view that there are no alternative possibilities for what can occur in mathematics classrooms. Concerning his own observations of the ways in which the undergraduates were being taught by professors in the department, another participant remarked "I might have the same complaints, but there's nothing I can do about it," signaling a resignation to being unable to change the way mathematics courses are taught or structured.

Many of the participants described how they had entered their graduate programs interested in and excited about teaching. However, each of them expressed uncertainty and even frustration about this part of their current and future roles in mathematics. As a teaching assistant, one participant spoke about losing his optimism as a teacher, saying, "I've seen it [his optimism] just completely smothered here" and that his experience in helping students had been "completely disappointing" in the context of his teaching assistant work, in wanting to provide more meaningful experiences for undergraduates. When one participant spoke of his future role as a mathematics professor, he expressed an interest in being different from his own professors, but then followed this interest with "I don't think I will be," resigning himself to a particular way of being.

As a graduate student still enrolled in courses, another participant spoke about how "we're both [professor and students] going through the motions. And maybe that's the point, then, of why be enthusiastic [about teaching] because we're just going through the motions." Finally, one participant, in speaking of her love for mathematics and love for teaching, said, "the system [graduate program in mathematics] just drew the love out of it." Without guidance or encouragement, the participants' hopes for who they could be and what they could do as teachers of mathematics were diminished. Without a forum to discuss their views and explore different ideas for teaching, they were left to find meaning in their experiences and each participant had become resigned to a notion that there was only one way to teach mathematics and one way to be as a professor of mathematics.

Conclusion

Through their experiences as graduate students, the participants had observed that post-secondary mathematics teaching takes on a particular form and that mathematicians seem to take on particular identities. As graduate teaching assistants, they encountered rules and structures that did not allow or support them to diverge from a particular form of interaction with undergraduates. Their interpretations of mathematics curricula, such as calculus, as emblematic of mathematics also seemed to bind them to a specific way of presenting mathematics. It appeared that the participants were not just learning mathematics, but also how to be in mathematics.

As the graduate students moved to the next step on the ladder toward the completion of their programs, none of the steps seemed to address how they interacted with students or what their teaching practices might be like. In other words, the itinerary of progress through the department did not explicitly address their teaching, other than to suggest that it take on a particular form. It is interesting to note that as the three doctoral students got closer to earning their degrees and formally being mathematicians, teaching became less important as their supervisors and the department worked to find them other sources funding so that they would not have to teach, as though, as they became more central to the community of mathematicians, teaching itself was no longer seen as a legitimate task. The resignation that is described here seemed to develop from the participants having to give up their hopes and expectations in order to be considered legitimate within the department. But, in attending to what was legitimate in the department, there seemed to be a considerable cost. What they had to produce as mathematicians was, in a way, how they seemed to present themselves as mathematicians and as mathematics teachers – restricted and disconnected from the creative and active processes that inspired mathematics, including the mathematics they were most interested in and passionate about.

Framed by the idea of legitimate peripheral participation, through their process of becoming mathematicians, the participants in this study seemed to move from a peripheral position to a slightly more central standing in the community as their identities became closer to that of the mathematicians in their department. As they learned of the relative importance or unimportance of different aspects in the life of a mathematician, they grew into the community of mathematicians and reproduced the community. This transition was not overt, nor was it explicitly stated anywhere. The participants did not report a public statement or even an acknowledgement that they had to abandon their own ideas about teaching, that they should no longer consider teaching important and, by maintaining “certain modes of coparticipation,” they would move toward a more central position in the department. Rather, it seemed that the set-up, the structure of the department, the behaviors that deemed were legitimate, and the progression to becoming a mathematician rendered it so.

The goal of this project was to understand what the obstacles might be for post-secondary mathematics teacher education. What was true for the participants in this study is that mathematics graduate students seldom learn explicitly about what it means to be an instructor or professor of mathematics, and they are left to create meaning and develop proficiency amongst themselves. Indeed, in order to earn a graduate degree in mathematics, they must become proficient in high-level mathematics and little attention is paid to their development as teachers, to who they are, who they want to be, or will be as teachers of mathematics. The participants in this study interpreted their success and their lives in mathematics to be restricted to a particular way of being and of presenting mathematics. As a final point, then, if the current structures and suggestions of what is important to graduate study in mathematics remain in place, it is unlikely that new teacher education programs that are established for mathematics graduate students will produce hoped for changes to teaching in post-secondary mathematics.

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INFLUENCING PROSPECTIVE SECONDARY TEACHERS' BELIEFS AND PRACTICE

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Determining and changing teacher beliefs and exploring their influence on instruction have been a focus for educational researchers over the past few decades. This study investigates a group of prospective secondary mathematics teachers following attempts to facilitate change through specific pedagogical tasks in field experiences. The participants' responses to these tasks formed three categories: affirmation of beliefs, superficial belief and practice change, *and* substantive belief and practice change. *This paper illustrates participants' development through accounts of three individuals. Results suggest possibilities for teacher preparation for reform-oriented instruction; yet also reveal the complexity of changing beliefs and instructional practices.*

Introduction

Current mathematics education reform is built around the idea that the study of mathematics should focus on five mathematical processes: problem solving, reasoning and proof, communication, making connections, and representational flexibility (NCTM, 2000). This view differs greatly from the traditional study of mathematics as a set list of rules and procedures to be memorized and routinely manipulated (Romberg, 1992). Since the perspective of the study of mathematics has changed, this challenges how mathematics is taught in classrooms (NCTM, 1991). If mathematics should be explored and investigated in the classroom, the traditional view of the mathematics teacher as an imparter of all knowledge no longer holds up. To meet the demands of mathematics education reform, mathematics teachers must take a fundamentally different approach to the teaching of mathematics. According to Thompson (1992), reform-oriented mathematics instruction should be characterized by students engaging in "purposeful activities that grow out of problem situations, requiring reasoning and creative thinking, gathering and applying information, discovering, inventing, and communicating ideas, and testing those ideas through critical reflection and argumentation" (p. 128).

After twenty years of reform efforts, widespread changes in the teaching and learning of mathematics have not been observed (Hiebert et al., 2005; Stigler & Hiebert, 1999). One reason for this perpetuation of traditional teaching methods is the close relationship between beliefs about mathematics and practice (Raymond, 1997). These beliefs are built from prospective teachers' *apprenticeship of observation* as students in classrooms where the traditional view of mathematics persists (Lortie, 1975). Mathematics education research has found that beliefs are difficult to change (Grootenboer, 2008), do not necessarily match practice (Grant, Hiebert, & Wearne, 1998), and are resistant to change (Philipp, 2007; Thompson, 1992).

Theoretical Framework

Differing models for initiating change in teachers' beliefs have been suggested and investigated to varying degrees. Guskey (1986) cited the ineffectiveness of presuming that changing teacher beliefs, attitudes, and perceptions would have to precede any change in classroom practice. He suggested a model (Figure 1) proposing the necessity of change in

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classroom practice leading to change in teacher beliefs once teachers observed positive change in student learning outcomes (Guskey, 1986, p. 7). This model was supported by von Glasersfeld (1993) who suggested, “If one succeeds in getting teachers to make a serious effort to apply some of the constructivist methodology, even if they don’t believe in it, they become enthralled after five or six weeks” (p. 37).

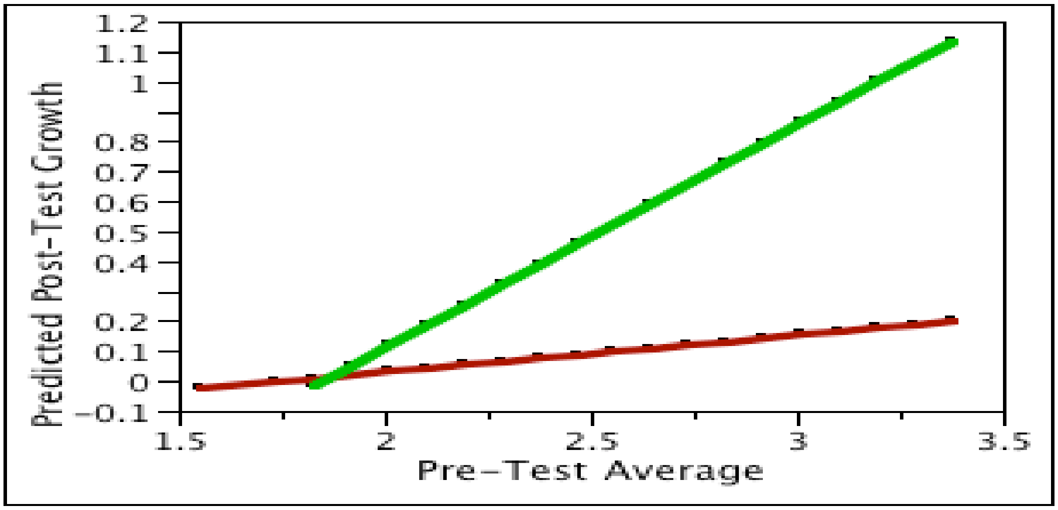

Figure 1. A model of the process of teacher change (Guskey, 1986, p. 7)

Implicit in a discussion of teacher beliefs and practice is a teacher’s sense of efficacy. Teacher efficacy is characterized by Tschannen-Moran, Woolfolk Hoy, and Hoy (1998) as “the teacher's belief in his or her capability to organize and execute courses of action required to successfully accomplish a specific teaching task in a particular context” (p. 233). Labone (2004) suggested prospective teachers’ efficacy beliefs had not yet developed and were more malleable than in-service teachers’ efficacy beliefs. However, prospective teachers enter teacher preparation programs with well-established beliefs about teaching (Weinstein, 1989) and efficacy beliefs related to traditional approaches to teaching built during their apprenticeship of observation could have already been formed. Therefore, it is important to consider the possibility of initiating changes in prospective teachers’ efficacy beliefs. Wheatley (2002) found that eliciting such a change might require challenging existing beliefs in a way that would cause teachers to doubt their beliefs.

Using Guskey’s model for the process of teacher change as well as the influence of prospective teachers’ sense of efficacy, the purpose of this research project is to explore the influence of specific, required, research-based strategies on prospective secondary mathematics teachers’ existing beliefs and practices regarding the teaching and learning of mathematics. The following research question guided our investigation: How does emphasis on specific pedagogical strategies to elicit student thinking in field experiences impact prospective secondary mathematics teacher beliefs and practice?

Methodology

Participants in this study were four graduate students in a Master of Education program working to obtain a 7-12 integrated mathematics teaching license. The prospective secondary mathematics teachers in this program achieve the equivalent of a Bachelor of Science degree in mathematics in order to enroll. The program is a five-quarter program, starting in the summer with courses focused on mathematics methods and learning theory. Throughout the fall, winter, and spring quarters, the prospective teachers are in progressively more teaching intensive field experiences while taking methods, multicultural education, and research courses in the evening. During each 10-week quarter the prospective teachers are in the field for 12 hours per week in the fall, 15 hours per week in the winter, and full-time student teaching in the spring. Field experiences consist of a combination of middle or high school in an urban or suburban setting. The final summer quarter is devoted to finishing an individual action research report. The investigators in this study gained access to the participants as university field supervisors. The university program manager randomly assigned the participants to each supervisor. This research project focuses on data collected in relation to field experiences during the fall, winter, and spring quarter of the 2009-2010 school year.

Pedagogical Tasks

The participants were required to implement specific, research-based, pedagogical tasks in their field experience sites. The design and motivation for each pedagogical task was to provide participants with opportunities to elicit student thinking in their classrooms. One task was called *the lesson in teacher silence* and was designed to prevent participants from the traditional approach to teaching typically characterized by a teacher explaining his/her thinking for the majority of the lesson. The participants were supported in the development of a lesson focused on posing an engaging, open-ended problem for students to work on collaboratively followed by a student-led discussion of their thinking about the problem. In this lesson, participants were required to hold back their desire to interject their thoughts into the students' solution strategies and record evidence of student thinking on a clipboard as they circulated the classroom. Other examples of pedagogical tasks included: questioning strategies, one-on-one interviews with students, and problem posing.

Data Sources

This paper presents preliminary results from the first 5 months (September – January) of a 9-month research project. Data collected and considered for this research include: 1) beliefs survey, 2) field experience artifacts, 3) classroom observations, and 3) semi-structured interviews.

The beliefs instrument used was the Mathematics Beliefs Instrument (MBI) developed by Hart (2002). It is a 30-item Likert scale instrument designed to determine respondents' beliefs consistent with NCTM standards, beliefs about teaching and learning mathematics, and teacher efficacy. Participants completed the survey before the beginning of their field experience (September) and at the conclusion of their field experience (December). They will also complete the surveys at the end of their winter field experience (March) and student teaching (June).

Field experience artifacts consist of any documents created by the participants in relation to field experience. This includes lesson plans developed and implemented by the interns, reflections on participant observations and teaching experiences, and electronic correspondences from participants in relation to field experiences.

Classroom observations occur throughout each of the three 10-week field experiences. Participants were observed for three of eight instructional periods during their initial field experience in fall quarter. At least three additional classroom observations will take place for each remaining quarter in the project. During observations, extensive field notes are taken.

Concurrent to each classroom observation, participants participate in a semi-structured interview before and after the observation for approximately 30 minutes each time. Each interview is audio taped and subsequently transcribed for analysis. Participants are also involved in weekly group meetings with their supervisor to discuss field experiences. These meetings are approximately 60 minutes and are also audio taped and transcribed.

Analysis

From an interpretive case study approach, analysis is ongoing, recursive, dynamic, and emergent (Merriam, 1988). All data sources collected by this point in the research project were read, re-read, and independently coded by the researchers. Emerging themes were reviewed and incorporated into a preliminary adapted model for the process of prospective teacher change. Validity and trustworthiness issues suggested by Lincoln and Guba (1985) are addressed through triangulation of data across multiple sources and independent coding by two researchers, which allows for cross-validation of results.

Results

Through data analysis three broad categories of belief and practice change were identified: affirmation of beliefs, superficial belief and practice change, and substantive belief and practice change. Although there were three broad categories that emerged in the analysis, participants were not bound to a single category but often exhibited behaviors consistent with a single category. Brief, preliminary case studies for three participants exhibiting behaviors in each category are shared. The fourth participant is not explicitly discussed in this paper because her story is similar to others.

Participants in the affirmation of beliefs category did not attempt to engage in required instructional change or reflect upon the new learning opportunities and consequently demonstrated no change in practice or beliefs about the roles of teacher and students. This category is largely defined by a discomfort with operating in novel mathematical discourse patterns or a disinterest in instructional change.

Episode 1: Sara's Story

Sara initially measured slightly above average relative to her peers on the MBI, but planned traditional teacher-centered lessons that often focused on procedural competency. Sara was very reluctant to implement a student-centered, activity driven lesson and viewed the planning of such a lesson as a burden. In spite of her reluctance, she planned an activity requiring group problem solving on a geometric proof task involving angle measures. Although student engagement was not problematic and groups actively worked towards understanding the problem, Sara disengaged during group work time and chose to sit at her desk and grade papers. Following the lesson, Sara was very concerned about whether or not she had successfully implemented a novel teaching strategy, but made limited mention of what she perceived as different about the student thinking and learning opportunities that occurred during the lesson. Reflecting on the lesson, she voiced a discomfort and lack of confidence in facilitating small group classroom interactions. Sara showed almost no change on the MBI following her initial field experience, and approached lesson planning and instruction in a traditional, teacher-centered way throughout her field experiences.

Those demonstrating superficial changes in beliefs and practice struggled with the implementation of the required tasks and reflective discussions, but showed an awareness of a need for change. Participants in this category showed the greatest questioning of beliefs, but beliefs change did not coincide with a change in instructional planning or practice.

Episode 2: Jim's Story

Jim initially measured well below average relative to his peers on the MBI, and planned very teacher-centered lessons. Jim initially had low efficacy for teaching, focused almost exclusively on classroom discipline, and viewed mathematics as a field of study that was best learned through practice and mental discipline. Jim was uneasy changing his instructional approaches, but planned lessons built around implementing questioning strategies to elicit student thinking. Implementing these lessons was a challenge for Jim as he struggled to recognize opportunities to expose student thinking. Following his initial field placement, Jim's MBI scores demonstrated substantial variation from the first to the second administration and he exhibited an increased desire to try new instructional strategies (even though his instruction remained static). In his second field experience, Jim was placed with a mentor who implemented more student-centered instructional strategies, and has continued to voice an interest in trying similar strategies.

Prospective teachers participating in the study demonstrating substantive belief and practice change successfully implemented the required pedagogical tasks and engaged in deep reflective discussion following the tasks. These participants expressed substantive change in beliefs related to the roles of teacher and students, which coincided with substantive change in instructional practice.

Episode 3: Michelle's Story

Michelle initially measured above average relative to her peers on the MBI, and voiced a desire to teach towards conceptual understanding for her students. Michelle planned and executed several lessons emphasizing student-centered activities. She demonstrated a confidence in facilitating student-centered discussions. Following these lessons, Michelle often reflected on how she viewed learning as complex and that without challenging activities and assessments it is difficult to know if students truly understand the material. After planning and implementing a lesson emphasizing what she feared would be learned as a "rote procedure", Michelle interviewed one of her students to better understand what was learned. Following this interview, Michelle voiced a concern with her teaching approach and expressed limitations to the quality of learning that took place. Michelle showed an increase on the MBI and also has been very consistent in attempting to implement student-centered instructional approaches.

Discussion

Although the results are preliminary, they appear to support Guskey's model for changing teacher beliefs. All participants were required to enact some form of change in classroom practices, placing emphasis on student-centered activities. Those who engaged with new modes of student learning and thinking showed different degrees of belief and instructional change, while those who did not engage with the new modes of student learning and thinking showed no evidence of change. Results suggest possibilities for effective teacher preparation reinforcing reform-oriented instruction; yet also reveal the complex intricacies of changing beliefs and instructional practices.

Based on the preliminary results of this study, an adapted model (Figure 2) of Guskey's process of teacher change is proposed for prospective teacher education. Replacing the component of staff development is university-based experiences. These experiences include teacher preparation coursework and interactions with university field supervisors where prospective teachers are exposed to issues related to the teaching and learning of mathematics. The next component of the model that was adapted was the observed change in student learning outcomes. Due to the structure of field experiences, prospective teachers often have a limited frame of reference with respect to analyzing student learning outcomes and have difficulty observing substantive changes in this area. Considering prospective teachers' limited conceptions of student thinking, the purpose of instructional changes was to increase opportunities to elicit student thinking. Preliminary data analysis suggests that teacher efficacy in dealing with novel mathematical discourse as well as teacher engagement with novel mathematical learning opportunities plays a role in the type of belief and instructional practice change. The variety of prospective teachers' efficacy and attentiveness to student thinking influences the nature of prospective teacher belief change. Unlike Guskey's proposed model that culminates with teacher belief change, the adapted model suggests a more complex range of prospective teacher belief change.

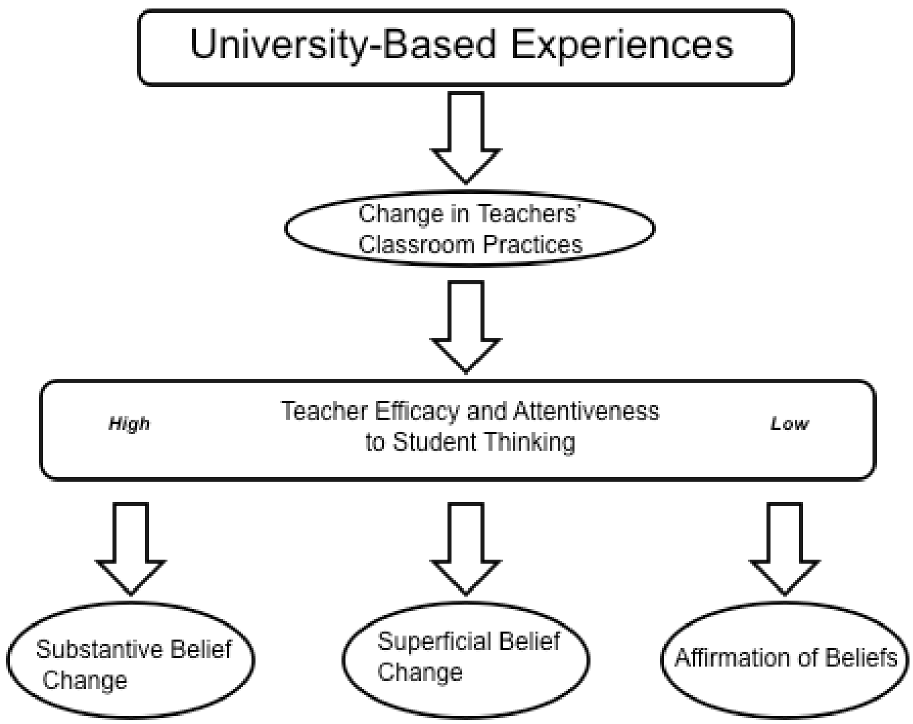

Figure 2. A model of the process of prospective teacher change

Conclusion and Implications

Prospective teachers often have a limited experience set to draw upon when planning, and rely heavily upon their experiences as students to inform their instructional practices (Lortie, 1975). Therefore, part of the role of a prospective education program is to encourage prospective teachers to engage in a variety of pedagogical strategies. Findings from this study indicate that requiring prospective teachers to plan and execute a lesson or learning experience that differs from their traditional teaching practices leads to a variety of types of teacher belief and practice change. Participants who seemed to demonstrate high levels of efficacy and engagement with student thinking showed significant and substantive changes in beliefs and instructional practices, while participants who did not seem to have high teacher efficacy and engagement with student thinking showed no change in beliefs. Participants who seemed to demonstrate moderate levels of teacher efficacy and engagement with student thinking showed high levels of questioning of beliefs, but struggled to enact change in instructional approaches.

Guskey's proposed model for teacher change emphasizes a change in instruction and student outcomes prior to a change in teacher beliefs. An adapted model for prospective teacher education has been proposed; emphasizing the fact that university based experiences can play a significant role in changing instruction. This change in instruction opens up new opportunities for teacher learning and exposure to new modes of mathematical discourse. Instructional change, filtered by teacher efficacy and engagement with these new student-learning opportunities, leads to a range of belief reactions. In light of the findings of this study suggesting that prospective teachers with moderate efficacy and engagement with new student learning opportunities encounter substantial questioning of beliefs and Wheatley's (2002) finding that belief change may require teachers to doubt their beliefs, this adapted model is potentially fruitful for prospective teacher education.

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MEASURING BELIEFS ABOUT CONCEPTUAL PROGRESSION AMONG PRE-SERVICE SPECIAL EDUCATION TEACHERS

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In this study, we attempt to measure whether an elementary mathematics teaching methods course can affect the beliefs of a group of 14 pre-service special education teachers, and how those changes correlate with what the course instructor notices. Using various metrics, including the IMAP instrument and classroom observations, we find that a reform-oriented course can indeed move teachers to believe that mathematics teaching should start with conceptual understanding before procedural fluency.

Introduction

Principles and Standards for School Mathematics states that, "Equity requires high expectation and worthwhile opportunities for all. . . . Expectations must be raised–mathematics can and must be learned by *all* students." (2000, pp. 12-13). Following the publication of PSSM, a year later , the National Research Council published *Adding It Up*, reinforcing this same idea of equity through research, and specifically addressing the need to teach students with special needs, "It has long been assumed that children with moderate, mild, and borderline mental retardation or learning disabilities are not capable of meaningful or conceptual mathematical learning and, thus, unlike other children have to be taught by rote" (2001, p. 341). This dismissal of special education practices as rote and procedural added friction between the mathematics education and special education communities, stoking the fire of the ongoing 'math wars' and drawing lines in the sand between traditional and reform-oriented mathematics educators (Schoenfeld, 2004). Five years later, NCTM published the *Curriculum Focal Points* (2006) to pinpoint specific mathematical topics, skills, and curricular strands for each grade level (Schoenfeld, 2007), shedding more light on the idea that perhaps the two feuding sides of the 'math wars' were not as far apart as previously imagined.

One area in which these theoretical battles between constructivists and embedded skills proponents are especially fierce occurs in our preparation of special education teachers (Woodward & Montague, 2002). According to Allsopp and colleagues (2003), current special education pre-service programs do a minimal job preparing teachers to teach mathematics to children with special needs. And while Brownell, Ross, Colón, and McCallum write, that "research in special education teacher education is almost nonexistent" (2005, p. 248), one thing we do know is that particularly effective instruction strategies for teachers of students with special needs should "ensure that the sequence of instruction moves from the concrete, to the representational, to the abstract" (Allsopp, et al., 2003, p. 4). These statements can seem like attacks from the mathematics education community dismissing the ways in which special education teacher educators prepare their teachers.

But fortunately, some special education researchers such as Woodward & Montague have found specific teaching techniques supported by both research in mathematics education as well as special education. One particular teacher belief that pre-service teachers are expected to embrace involves a conceptually guided approach to mathematics understanding; conceptual understanding must come before learning procedural algorithms and "problem solving and

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analysis [must] replace drill and practice" (2002, p. 96). Research into how pre-service special education teachers embrace this belief is also minimal (Allsopp, et al., 2003; Brownell, et al., 2005).

In this paper, we will describe how a pre-service elementary mathematics methods course might help pre-service teachers build beliefs about mathematics teaching that align both with research from the fields of mathematics education and special education–specifically looking at the belief that students must build conceptual understanding of a mathematic computation before learning the procedural algorithm. Additionally, we will explore how pre-service teachers' actual beliefs about this conceptual progression, when measured using validated instruments, might align with issues the course instructor notices.

Perspectives

Pre-service Teacher Development

The Cognitively Guided Instruction (CGI) body of work is a professional development elementary mathematics teacher framework that comes directly from the mathematics education research community (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema, et al., 1996; Franke, Carpenter, Levi, & Fennema, 2001). A solid companion to a CGI-based pre-service program includes Ginsburg's clinical interview framework (1997) along with textbooks that support this approach to teaching (e.g. *Elementary and Middle School Mathematics: Teaching Developmentally*, 2010).

Understanding how these perspectives and teaching ideas might be taken up by special education teachers is exactly the type of research in need of undertaking in order to better understand how to prepare teachers as they try to navigate between traditional and reform-oriented strategies (Bottge, Rueda, LaRoque, Serlin, & Kwon, 2007; Brownell, et al., 2005; National Council of Supervisors of Mathematics, 2008; Woodward & Montague, 2002).

Specifically, one teaching belief that bridges both the special education and the math education areas concerns the idea that students should build conceptual understanding of a mathematical idea before learning the procedural algorithm (Woodward & Montague, 2002). For instance, this means that students understand conceptually what it means to add triple-digit numbers to each other through exploration of pictures, manipulatives, or other models before learning the traditional algorithm. This might play out as a student using base-ten manipulative blocks to model a problem like "149 + 286" before learning the traditional algorithm that involves lining up each number according to place-value and then moving right-to-left through each digit, making sure to carry over extra digits into the next place value.

Measuring pre-service teacher beliefs

Measuring and understanding specific pre-service math teacher beliefs is somewhat problematic (Philipp, 2007; Thompson, 1992). Often, a Likert-scale survey question is used to measure beliefs by asking teachers to rate their level of agreement to statements like, "Students need to build an understanding of a mathematical concept before learning the algorithm". This is tricky because capturing how a respondent interprets the statement is difficult, respondents are not given an opportunity to explain the importance of how they respond, and items on a Likert-scale are not situated within a specific context (Ambrose, Clement, Philipp, & Chauvot, 2004).

The Integrating Mathematics and Pedagogy (IMAP) instrument moves beyond traditional Likert-scale questions to elicit authentic understandings of pre-service teacher beliefs through the use of respondents impressions to student work samples and videos of teacher-student interactions (Ambrose, et al., 2004; Philipp, 2007). Not only is the IMAP instrument a validated belief measurement tool specifically targeted to pre-service Elementary teachers, it also reflects much of the CGI conceptual framework for helping pre-service teachers listen closely to children's thinking (Philipp, 2007).

Further study of teacher beliefs requires structured classroom observation as well. Simon and Tzur's (1999) suggest the use of a *set* as the unit of analysis for studying an account of practice for a teacher. The use of a set becomes a robust analytical tool for comparing and analyzing specific pieces of a teacher's classroom practice. According to auhtors, a set should consist of "at least two consecutive related mathematics lessons and interviews before, between, and after the lessons" (1999, p. 258).

In this study, we attempt to respond to the call voiced within the special education research community for the need to conceptualize how to build understanding among preservice teachers through "carefully scaffolded activities using authentic tasks" (Woodward & Montague, 2002, p. 96). Simultaniously, we aimed to respond to the goal of moving pre-service teachers away from the idea that good math teaching involves a lot of 'telling' (Philipp, 2007), as proposed withing the mathematics education community. Using the IMAP instrument to measure teacher beliefs and Simon and Tzur's notion of a classroom analysis *set*, we explore the belief that students must learn mathematical concepts before they learn procedures, or else risk never building full conceptual understanding.

Methods

Data

The participants in this study were 14 elementary pre-service teachers within a special education cohort at a large, public university in the Southwestern United States. They were in their second-to-last semester of their professional development sequence and just about to enter their student teaching semester. The course instructor was the first author of this study. The second author also taught a similar general education elementary math methods course that is separate from this study.

In the same semester that the pre-service teachers were enrolled in this course, they were also taking another course that heavily emphasized a direct-teach model of math lesson planning. So during this study, the pre-service teachers were in classrooms using direct-teach style lesson plans, while exploring reform-based mathematics learning in their methods course. One interesting note about this cohort of teachers is that they were the first class mandated to pass the state-wide high-stakes exam in order to advance to the next grade within their K-12 careers–the first students in this particular state to experience a full K-12 of high-stakes testing culture.

Data for this study consists of four specific pieces. First, baseline teacher beliefs were taken from teacher journal entries at the beginning of the semester asking teachers to reflect upon this question, which relates to Chapters 2 and 3 of the Van de Walle, et al. textbook (2010):

> *Not everyone believes in the constructivist-oriented approach to teaching mathematics. Some of their reasons include the following: There is not enough time to let kids discover everything. Basic facts and ideas are better taught through quality explanations. Students should not have to "reinvent the wheel." How would you respond to these arguments?*

The second unit of data was the course instructor's comments and analysis on each teacher's final project, which reflected the course instructor's understanding of each teacher's belief that students need to learn mathematics concepts before being introduced to the traditional algorithms.

The third unit of data involved teacher responses to the IMAP instrument, which teachers completed during the final class period of the semester. Teachers were informed that their responses to the IMAP instrument would have no bearing onto their final grade for the course. For the purpose of this study, we concentrate only on two sections of the IMAP instrument.

In the first IMAP section, teachers looked at five student responses to the problem "149+286" and responded to the prompt "Consider just the strategies on which you would focus in a unit on multi-digit addition. Over a several-weeks unit, in which order would you focus on these strategies?" One of the student responses is the traditional algorithm, another uses base-10 block manipulatives, and the other three represent various created strategies.

In the second IMAP section, teachers watch two video clips of a teacher working with a student on dividing whole numbers by fractions. In the first clip, the teacher guides the student through the traditional algorithm of changing the division sign to a multiplication sign and flipping the second fraction, without explaining why this is necessary. Using this algorithm, the student is able to solve several additional problems involving fraction division. Teachers are asked to respond to this clip as well as comment on the prompt, "Would you expect this child to be able to solve a similar problem on her own 3 days after this session took place?" Teachers then watch the second clip, which takes place three days later, in which the student is asked to solve, "6 ÷ 1/3". The student is unable to remember the algorithm, incorrectly changing the division sign into an addition sign, and struggling to figure out what to do. Teachers were again asked to respond to the clip, as well as answer the prompt, "Provide suggestions about what the teacher might do so that more children would be able to solve a similar problem in the future.

The last unit of analysis involved a case study of a student, Christin (a pseudonym), in an attempt to utilize Simon and Tzur's (1999) concept of a *set* for analyzing classroom practice. Christin was selected because she happened to be sitting near the front of the classroom on the first day of class. The course instructor observed three specific *sets*: at the beginning, at the midpoint, and at the end of the semester. Each *set* consisted of an analysis of Christin's lesson plan for the day, a pre-observation interview, a classroom observation, and a post-observation interview. Christin's post-observation interview was conducted the day after the lesson rather than immediately after she taught, in an attempt to allow time to self-reflect. Christin's classroom was a special education resource room consisting of eight 3rd to 5th grade students with varying special needs. Unfortunately, the logistics of this particular student's classroom placement did not allow us to observe consecutive math lessons to fully follow the *set* framework.

Analysis

Analysis of this data took four steps. First, we analyzed the student reflection assignment from the first day of class to look at the belief system that teachers were starting at before they entered the course. Second, we coded the IMAP instrument results for Belief 4, sections 3.3 and 9 using the IMAP rubric. We started by completing the IMAP coding training exercises until we both achieved a 100% match with the rubric codes. Then, each author of this paper coded anonymous teacher responses individually before convening to compare our scores. The inter-rater agreement between both authors was 92.9%, which represents a strong agreement. Third, the classroom instructor looked through the graded final reflections for each teacher, categorizing the evidence to support each teacher's beliefs using a similar 4-point scale as the IMAP rubric for comparability. Finally, we looked deeper into *sets*, including classroom observation notes, interview notes, and classroom video of Christin's teacher practice to gain a deeper understanding and connection with how our results might correlate with actual classroom practice.

Results

The results of our analysis are presented in Table 1, in which pre-service teachers are categorized as showing 'No evidence', 'Weak Evidence', 'Evidence', or 'Strong Evidence' of the belief, "If students learn mathematical concepts before they learn procedures, they are more likely to understand the procedures when they learn them. If they learn the procedures first, they are less likely ever to learn the concepts." (Ambrose, et al., 2004)

Table 1. Number of teachers showing evidence of the belief using three measures at two time points

	Beginning of Semester	End of Semester	
Evidence Level	First assignment	Final Project	IMAP instrument
No Evidence	**1 (7.1%)**	0 (0.0%)	0 (0.0%)
Weak Evidence	**7 (50.0%)**	3 (21.4%)	3 (21.4%)
Evidence	5 (35.7%)	**9 (64.3%)**	5 (35.7%)
Strong Evidence	1 (7.1%)	2 (14.3%)	**6 (42.9%)**

At the beginning of the course, eight of the fourteen teachers showed no evidence or weak evidence of the belief that teachers should learn mathematical concepts before they learn procedures. At the end of the course, according to the course instructors' impression of the teachers on the final project, eleven of the fourteen teachers now showed either evidence or strong evidence of this belief. This shows substantial change in the beliefs of the teachers through their experiences in this semester. However, from the course instructor's impression, only two teachers displayed strong evidence of this belief.

Data from the IMAP instrument shows almost the same results as the course instructor's impressions. Again, zero teachers showed no evidence of the belief and three teachers showed weak evidence of the belief. However, the results from the IMAP instrument show substantial change in the number of teachers who showed strong evidence of this belief rather than just evidence. While the instructor's impression was that only two teachers showed strong evidence of this belief, the IMAP instrument shows that six teachers showed strong evidence of this belief. Since the IMAP instrument is a validated tool for measuring teacher beliefs, we conclude that the number of teachers displaying strong evidence of this belief is actually much more than the course instructor originally thought. Furthermore, since the number of participants in this dataset was so small (N=14), we did not perform further statistical analysis.

The specific results for our case-study teacher, Christin, found that there was no change in the amount of evidence she displayed for this specific belief. Only one other teacher showed no change in the three measures, displaying evidence of the belief at all three metrics. However, Christin showed weak evidence of the belief on all three metrics. We can conclude that Christin did not change much in her beliefs through the course.

This result is consistent with the classroom analysis *sets* we observed from Christin's classroom. In the three *sets* collected throughout the semester, she continually emphasized a belief that teaching students procedural steps was just as, if not more, important than conceptual understanding.

During the pre-observation interview of her last set at the end of the semester, Christin explained that she was still struggling with letting go of control for students to struggle and come up with the conceptual understanding on their own. She started a lesson involving multi-digit subtraction word problems with an overhead transparency of the six steps for problem solving and had students in her classroom read through each step out loud as a whole group. At one point during the lesson, when most of the students seemed to be struggling with one particular problem, Christin simplified the problem situation into a numerical subtraction problem, placing one digit on top of another to show students that they should use the traditional algorithm for subtraction, and instructed students to, “solve it.” Our observation notes for this lesson expressed frustration at Christin’s continued belief in emphasizing procedural step-by-step thinking when working with her students and wondered why she continued to allow this to permeate her classroom practice. During the post-observation interview, we used non-evaluative language to tell Christin we noticed that she started breaking the problems down into specific steps. She acknowledged that this was something she is working on moving away from, but also responded, “I want them to succeed. I want them to feel good at doing what they’re doing and not feel like, “I’m so frustrated I’m going to shut down”.

Collectively, these three *sets* of Christin’s classroom practice reveal a continued belief that procedural steps should come before conceptual understanding, a deep belief that mathematics involves following steps. This shows an alignment with her classroom practice to the beliefs that she evidenced through our three metrics. While Christin’s *set* was not an ideal case to test if our metrics from the IMAP instrument and the final project would correlate with classroom practice, we can at least conclude that these metrics are accurate when measuring no change and weak evidence of the belief that conceptual understanding must come before learning a procedural solution.

Discussion and Conclusion

This study reveals that the IMAP instrument seems more finely tuned to elicit the precise differences between just showing evidence and showing strong evidence of specific teaching beliefs, ones that the course instructor was not able to see. Perhaps our view of our own teachers through the lens of instructor does not allow us to see the fine-grain belief changes in our pre-service teachers. We were able to pinpoint a specific teacher’s beliefs when observing their classroom practice through the framework of a *set* and accurately predict their beliefs. But perhaps, just reading teachers reflections, lessons plans, and other assignments does not reveal to us deep level understandings of our pre-service teachers’ beliefs.

In the end, we found that our reform-oriented methods course did indeed create substantial change in special education pre-service teachers thinking about math teaching. This answers a need in the literature to show that special education teachers can indeed embrace reform mathematical ideas as their own through a pre-service math methods course. We can conclude that it is possible to help empower special education pre-service teachers to embrace reform styles of mathematics teaching that emphasize conceptual understanding before procedural fluency, something that both special educators and math educators agree on (Woodward & Montague, 2002). And even in the midst of continually writing and implementing direct-teach style math lessons in their field placements, the pre-service teachers moved towards a belief in mathematics teaching that emphasizes conceptual progression.

However, three potential issues for drawing conclusions from these results should be taken into account. First, the primary author was also the instructor for the course. And, the IMAP instrument requires teachers to write their first and last names, so it was *not* taken anonymously. So even though the teachers were told that their responses would have no bearing on their final grade, teachers still knew that the course instructor would be reading their responses and therefore they could have responded thinking that their responses might affect their final course grade. The responses were not read until after the final grades for the course had been submitted. Second, while we measured teacher beliefs at the beginning and the end of the semester, we did not use the same measures. We did not have a formal pre/post measure in place to properly measure changes in belief. Third, we find it hard to draw an accurate conclusion from Christin's classroom observation *sets*. Since she did not evidence any change in her beliefs, we cannot formally conclude that our metrics will always match a teacher's classroom practice. Perhaps our metrics only work when a teacher evidences no change. We further wonder if Christin might not be an anomaly at all. If we had the time and resources to deeply observe every one of our special education pre-service teachers, would we have seen the same reliance on procedure and emphasis on a 'follow-the-steps' approach to mathematical learning?

Through this analysis, we learned that using a trusted mechanism to measure teacher beliefs, such as the IMAP instrument, took an incredibly long time to properly code and analyze. It took two researchers two weeks to code, discuss, and ultimately agree on the codes for just one of the seven beliefs that the IMAP instrument measures. Our original intention was to use the IMAP instrument as a formative assessment tool to elicit the shifting beliefs of our pre-service teachers. However, the lengthy coding and analysis mechanism of the IMAP instrument make that an incredibly time-intensive task.

We have a hard time concluding that these beliefs will directly affect a teacher's classroom practice. Perhaps this link from special education teacher beliefs to actual practice is another venue for continued research into special education mathematics education. We were able to observe what happens when a teacher shows weak evidence of a belief, but what about the classroom practice of a teacher who shows strong evidence?

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ON RESPONSIBILITY: THE CORRELATION BETWEEN EFFICACY AND SENSE OF RESPONSBILITY TOWARDS TEACHING ALL CHILDREN IN MIDDLE GRADES MATHEMATICS CLASSROOMS

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Based on the novel framework of Academic Agency, this study explored how teacher's sense of efficacy influenced overall sense of responsibility in instructing all children in the nature of mathematics. Survey instruments utilized to collect individual data included a modified version of the Teaching Efficacy survey, Mathematics Teaching Efficacy Belief Instrument and Teachers' Belief Form I. Pearson Correlation was employed as an analysis tool to identify relationships existing between variables. The noteworthy finding signified that a positive relationship exists between a teacher's sense of efficacy and overall sense of responsibility in assuring that all children have the opportunity to advance.

Introduction

Historically, educational practices have failed to provide access to quality educational experiences for children who fall outside of what may be considered the majority. National and international assessments of student progress indicate that there are disproportionate numbers of underrepresented students based upon multiple subgroups who consistently fall below the level of achievement of majority students. The No Child Left Behind (NCLB) Act was developed to illuminate these discrepancies and mandate that something be done to correct this inequitable situation. In response to this Federal mandate, multiple initiatives have been funded with, as of yet, little or no impact on the mathematics achievement of underrepresented students. Knowing that the greatest factor determining success in the classroom is the teacher, taking a careful look at what level of responsibility teachers have toward the goal of teaching all children becomes not only reasonable but also essential for the success of all children.

To address what is meant by a teachers' sense of responsibility the micro-political context in which it exists must be acknowledged. Teachers' sense of students' ability (Roscigno & Ainsworth-Darnell, 1999) and teachers' sense of responsibility for students' learning both individually and collectively (Lee & Smith, 2001; Lee & Loeb, 2000; Diamond, Randolph & Spillane, 2004) define the mirco-political context. Daily interactions provide the individual sense making mechanisms for how educators value and address differences that influence student outcomes (Diamond, Randolph & Spillane, 2004; Lareau & Horwat, 1999; Roscigno & Ainsworth-Darnell, 1999). This is what is frequently referred to as collective sense of responsibility in the classroom.

A collective sense of responsibility is established by organizationally defined indicators for student outcomes (Lee & Smith, 2001). Lee and Smith (2001) identified three components to measure a sense of collective responsibility: 1) teachers' innate sense of responsibility for student learning, 2) a teacher's willingness to modify teaching strategies to address students' needs and 3) a teacher's sense of efficacy in their teaching practices. Therefore, collective responsibility exists on a continuum; on one end are schools where teachers acknowledge their success and failures in the classroom and accept responsibility for student success and at the other extreme, teachers who take little responsibility for student success and blame the failure of students advancement on conditions such as student ability, socio-economic level, or lack of motivation of the student (Lee & Loeb, 2000; Lee & Smith, 1996).

Focusing attention on responsibility, two conjectures exist concerning the attribute of responsibility. Responsibility can be defined as either virtuous or accountable. Silverman (2009) posits that a responsible and virtuous person will engage in those areas that are driven by morality and how the choices that they make regulate their behavior. A responsible and accountable person on the other hand is driven by rules, outcomes, and punishment such as high stakes testing. For an individual that adheres to accountability, state mandates and political forces influence the manner in which teachers base educational decisions. Therefore, constructs that are imposed from external forces may diminish a teacher's sense of responsibility in a classroom setting.

Previous research studies have shown that the avoidance of responsibility may serve as a means of dissonance reduction (Brock & Buss, 1962; Cooper, 1971; Lerner & Mathews, 1967). Dissonance reduction is the cognitive state where an individual fails to accept responsibility for behavior or the consequence of behavior based upon their level of investment (Cooper, 1971). Cooper (1971) identifies two processes by which an individual may avert responsibility: 1) the environment, individuals' claim they were forced to behave in an irresponsible manner, 2) consequences of their attitude-discrepant behavior was unforeseeable.

Teachers act and react to daily occurrences based upon their own previous life experiences. According to Thompson (1992), a "teacher's beliefs appear to act as filters through which teachers interpret and ascribe meanings to their experiences as they interact with children and the subject matter" (p.139). Good and Brophy (1990) state, a teacher who believes all students can excel in school will change their behavior to make sure that success is attainable for all learners. In many ways a teacher's expectations become a self-fulfilling prophecy in the classroom. If a teacher expects more, they will receive more from the students in the classroom.

Ross (1998) proposed that reflection on the results of one's teaching is the most important contributor to teaching efficacy. Self-reflection is difficult. Individuals do not want to be faced with their weaknesses, but to improve classroom practices reflection upon action becomes essential. When comparing the level of efficacy and reflection, a person with a high sense of efficacy would tend to reflect on classroom practices, identify areas for improvement, change their behaviors, and adapt a new plan to address their goals, whereas a person with lower self-efficacy may acknowledge that their practices could be improved but fail to implement any type of significant change because they feel the goal is unattainable.

In a study conducted by Ashton and Webb (1986) they found that teacher expectations and commitment to responsibility were altered by student characteristics such as socioeconomic class, race, or classroom behavior. Thus, a teacher that demonstrated a low sense of efficacy failed to accept any responsibility for student achievement. Teachers with high efficacious feelings cited more positive relationships with students and took a greater responsibility in reaching children. Although personal influences such as efficacy are not seen as a direct link to a sense of responsibility, the importance of individual belief systems are central to a learning environment.

For the purpose of this study, efficacy was identified as what individuals believe they can accomplish in a specific situation. Therefore, taking control of situations and knowing how to proceed becomes paramount to an individual's sense of efficacy. The notion of efficacy can be seen through two aspects of control. First, the strength of the efficacious beliefs becomes a determinant of how the individual chooses to use their personal capabilities and resources to achieve a goal. The second factor of control is how individuals see themselves in relation to the environment and how are they able to change the situation to aid in addressing the challenge at hand (Bandura, 1993).

Research conducted by Hoy and Woolfolk (1990) established that teachers with a low sense of instructional efficacy tend to possess a more custodial orientation. Their study looked at individuals either entering student teaching, taking teaching methods courses, or students taking developmental psychology classes. Through various measurement instruments, three of the four major hypotheses stated for the study were confirmed: pupil-control ideology of student teachers became more custodial, the social problem-solving orientation of the student teacher became more controlling, and the general sense of teaching efficacy for the student teachers also declined. Custodial orientation, because of limited student involvement, does not allow for motivation from within the student to drive instruction. Teachers that identify with a custodial orientation tend to believe that education is incapable of overcoming the students' limitations. These limitations may be both internal and external. Student ability would be identified as an internal limitation, whereas living environment would be considered an external limitation.

In an educational setting where the objective is procedural, students fail to take ownership of the learning process and struggle to move forward. Ross, Hogaboam-Gray, and Gray (2004) reported that teachers with a high sense of efficacy try harder, encourage student autonomy, focus on needs of low-ability students, and help to transform students' perception of their ability.

Purpose

The intention of this study was to identify how efficacy may influence a teachers' sense of responsibility to improve the outcome of students who fail to flourish in mathematics. The question guiding this inquiry was, Do relationships exist between a teacher's senses of efficacy, both general teaching efficacy and mathematics teaching efficacy, and their overall responsibility that will assist us in learning more about how to teach all children in middle grades mathematics classrooms? Annual educational statistics illustrate that each year millions of children fail to make adequate progress in personal mathematical understanding so the focus on teaching becomes: who is responsible and how is that responsibility defined so that underrepresented children who fall into a myriad of demographic categories such as socio-economic status or gender, receive adequate mathematics instruction? One would assume that a classroom teacher should hold a significant level of responsibility but, this assumption is neither straightforward nor clearly defined.

Theoretical Framework

For this study, only a small portion of the Academic Agency framework was studied. The complete framework of Academic Agency consists of three categories: efficacy, commitment, and knowledge. Each individual construct addresses components of the educational setting that tend to influence student learning. The classification of efficacy encompasses three forms of efficacy: self-efficacy, teaching efficacy, and mathematics teaching efficacy. The second division, commitment focuses on the instructional and academic emphasis of an academic setting. Knowledge, the final piece of the triad, brings attention to not only content specific knowledge but also pedagogical content knowledge (see Shulman, 1987 for additional information regarding PCK) and mathematics knowledge for teaching (see Ball & Bass, 2003; Ball, Hill, & Bass, 2005; Ball, Thames, and Phelps, 2008; Bass & Lewis, 2005 for additional information regarding MKT).

As depicted in Figure 1, the premise of Academic Agency states that only when an educator has taken the responsibility to act and react, based upon efficacy, commitment, and knowledge, to the needs of the student they become agents of change. The purpose of this study was to identify the influence that efficacy has on an individual's sense of responsibility within the classroom. In future publications, data will be analyzed to attend to the remaining two sub-sections of commitment and knowledge.

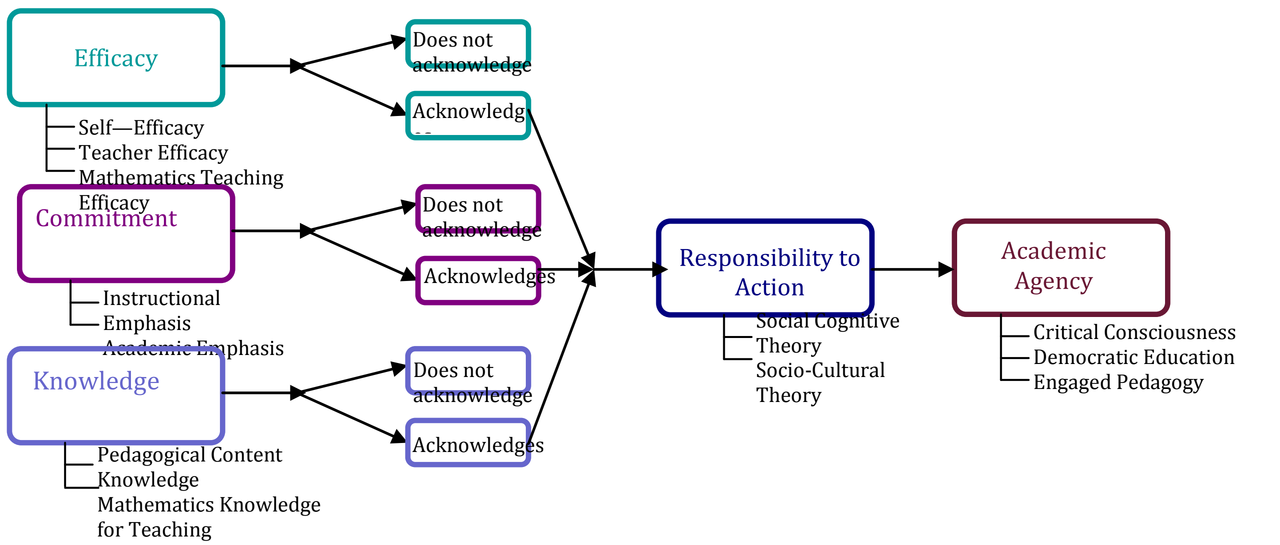

Figure 1. Theoretical Framework of Academic Agency

Methods of Inquiry

To explore whether a relationship exists between efficacy and responsibility three quantitative survey instruments were used. Survey instruments were presented to consenting middle grades (grades five through eight) mathematics teachers in three school districts: one rural, one suburban, and one urban, in central Ohio. Participants in the study included individuals who have taught one to thirty years, with the majority of participants being female. A total of forty-nine usable survey instruments were analyzed for this study.

Participant survey included, a modified version of Gibson and Dembo's (1984) Teacher Efficacy survey measuring overall teaching efficacy, The Mathematics Teaching Efficacy Beliefs Instrument (MTEBI-A) to look at content specific efficacy, and a modified version of Silverman's (2009) Teachers' Beliefs Form I to address responsibility. A Likert scale was employed to record individual responses. The range of the Likert scale was from 1 to 6 with 1 representing strongly disagree and 6 corresponding to strongly agree.

Teacher Efficacy

Woolfolk and Hoy (1990), using Gibson and Dembo's original scale, constructed a 22-item Teacher Efficacy Scale to examine the meaning of efficacy for prospective teachers. This modified version is the instrument I chose for this study. Woolfolk and Hoy's efficacy instrument encompasses 16 items from the Gibson and Dembo (1984) Teacher Efficacy Scale that produced adequate reliability and 4 others that address the adequacy of the teacher's preservice preparation. The two additional survey questions include two original RAND items, item one: "When it comes right down to it, a teacher really can't do much because most of a student's motivation and performance depends on his or her home environment" and item two:

"If I try really hard, I can get through to even the most difficult or unmotivated students." Item one of the RAND items focuses attention on teaching efficacy while item two looks at personal teaching efficacy. Through previous studies using the RAND items, results have indicated that teaching efficacy and personal teaching efficacy are independent. Hoy and Woolfolk (1990) reported a Cronbach's alpha coefficient of α=.74 for the teaching efficacy (TE) scale and α=.82 for the personal efficacy (PE) scale.

Mathematics teaching efficacy.

The Mathematics Teaching Efficacy Beliefs Instrument (MTEBI-A) (Enochs, Smith & Huinker, 2000) for in-service teachers, was used to measure mathematics teaching efficacy. The MTEBI-A is comprised of 13 personal mathematics teaching efficacy (PMTE) items and 8 mathematics teaching outcome expectancy items (MTOE). The MTEBI-A was developed based on the STEBI-A, a science efficacy belief instrument created by Riggs and Enochs (1990). Enochs, Smith and Huinker (2000) reported a Cronbach's alpha coefficient of α=.88 for the PMTE scale and α=.77 for the MTOE scale. A sample question is: "Even if I try very hard, I will not teach mathematics as well as I teach most subjects.

Teacher responsibility.

To explore teachers' sense of responsibility, the Teachers' Beliefs Form I, developed by Silverman (2009) was utilized. An abbreviated 23-item version of Silverman (2009) Teachers' Beliefs measured specific aspects of teacher beliefs regarding specific aspects of responsibility in the classroom setting. Silverman in her original study reported a Cronbach's alpha of α=.505 for items that addressed overall responsibility. A sample responsibility question is: "Teachers are professionally responsible for reaching all of their students".

For this study, the following reliability statistics were reported; Teacher Efficacy α=.807, Teaching Efficacy-Teaching α=.633, Teaching Efficacy-Personal α=.791, MTEBI α=.792, MTEBI-MTOE α=.711, MTEBI-PMTE α=.737, and Overall Responsibility α=.604.

To recruit participants, site visits were conducted. At each setting participants were presented with a detailed description of the study, a consent form which they were asked to complete at that time, and the survey instrument. The consenting document allowed individuals to indicate either they consented or did not consent to participate in the study. Teachers at the rural setting completed the survey the same day as the site visit. At the remaining research sites, participants completed the questionnaire individually.

As data was received individuals were given unique identifying numbers and all data was de-identified. Individual responses were then recorded in SPSS. To prepare for the initial analysis, the mean of each main variable (teaching efficacy, mathematics teaching efficacy, and overall responsibility) was computed. To prepare for the second correlation study, the summated score of each subscale within teaching efficacy and mathematics teaching efficacy were calculated.

Analysis of the data consisted of identifying any significant correlations that existed between variables and the level and direction of their relationship. The statistical measure Pearson Correlation was utilized to identify the degree and direction of existing relationships. The first level of analysis looked at the interactions of overall responsibility, teacher efficacy and mathematics teaching efficacy. The second correlation study addressed how the sub divisions of each instrument influenced one another as well as overall responsibility and assessed the association between variables.

Results

The results from the study, shown in Table 1, show a connection between a teacher's sense of overall responsibility and their sense of efficacy, as well as a corresponding relationship between teaching efficacy and mathematics teaching efficacy. In the initial correlation analysis, as overall sense of responsibility increased, a teacher's sense of teaching efficacy also increased and showed significance at the p-value of .05, N=43.

Table 1. Pearson Correlation for Overall Responsibility, Teaching Efficacy, & Mathematics Teaching Efficacy

	MEAN TEACHING RESPONSIBILITY	MEAN TEACHING EFFICACY	MEAN MATH TEACHING EFFICACY
MEAN TEACHING RESPONSIBILITY	1		
MEAN TEACHING EFFICACY	.393*	1	
MEAN MATH TEACHING EFFICACY	.212	.630*	1

*Note. * Correlation is significant at the 0.05 level (2-tailed).*

For the second phase of this study, the correlation between the subscales of teacher efficacy, mathematics teaching efficacy and the overall sense of responsibility once again showed evidence of a positive relationship between variables (See Table 2). As the subscales for teacher efficacy increased the overall sense of responsibility increased and was significant at a p-value of .05. Teacher efficacy also responded positively to the measure of mathematics teaching efficacy and outcome expectancy measured by the MTEBI-A.

Table 2. Pearson Correlation for Overall Responsibility, Mean Teaching Efficacy, Mean Personal Efficacy, Mathematics Teaching Efficacy-Outcome Expectancy and Mathematics Teaching Efficacy-Personal Efficacy

	MEAN TEACHING RESPONSIBILITY	MEAN TEACHING EFFICACY	MEAN PERSONAL EFFICACY	MTEBI-OUTCOME EXPECTANCY	MTEBI-PERSONAL EFFICACY
MEAN TEACHING RESPONSIBILITY	1				
MEAN TEACHING EFFICACY	.345*	1			
MEAN PERSONAL EFFICACY	.311*	.474*	1		
MTEBI-OUTCOME EXPECTANCY	.311*	.596*	.660*	1	
MTEBI PERSONAL EFFICACY	.242	.212	.419*	.345*	1

*Note. * Correlation is significant at the 0.05 level (2-tailed)*

The results of this study provide an entry point for further research. Analysis of the survey responses identified compelling evidence that as teachers' sense of teaching efficacy increases an individual embraces more responsibility in their professional role. Further research is needed to delve into the more complex and challenging discovery regarding why this phenomenon exists.

Conclusion

The findings from this study begin to identify a gap that exists in current research. Ongoing research on teaching can contribute to our understanding of the role of teaching efficacy and its extended influence on overall responsibility. To promote the advancement of mathematics education, educators must focus attention on how they define their responsibility in the classroom and the steps that are taken to enhance the classroom experience so that all children begin to find success in the middle grades mathematics classroom.

Efficacy is only a piece of the entire Academic Agency model; therefore, a more comprehensive view of responsibility will be acknowledged once all components (efficacy, commitment, and knowledge) are analyzed and their influence is established in future publications.

The analysis described in this study involves clear limitations, including a small sample size and limited research sites. Because the sample size is limited, I am hesitant to make any type of generalization based upon the evidence collected. To more completely address the influence of efficacy on an individual's overall sense of responsibility, a larger sample will need to be collected and multiple research sites will need to be included. As previously stated, this preliminary study provides an entry point for a much larger and necessary study.

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QUANT PARTICIPANTS' PERSPECTIVES REGARDING THE IMPLEMENTATION OF HIGH-LEVEL TASKS

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Quantifying Uncertainty and Analyzing Numerical Trends (QUANT) is a yearlong professional development program to expand secondary school mathematics teachers' statistical proficiency for teaching*. This study investigated the perspectives of the 2009–2010 participants regarding their implementation of high-level tasks and the obstacles they encountered when attempting to apply them in their classrooms. The findings indicate that the program helped the participants gain confidence in their teaching and that the participants found the program to be valuable for exchanging ideas and experiences. However, the participants indicated that instructional time and class duration critically support or inhibit implementation of high-level tasks.*

Introduction

According to Boston and Smith (2009), teachers are more likely to implement high-level tasks if they are exposed to professional development (PD) programs that support the implementation of such tasks. The goal of PD programs is to improve student learning, and student performance will not improve unless teacher performance improves (Loucks-Horsley et al., 2003). To be effective, technology-intensive PD programs need a clear content focus that is relevant to the participants (Foley, 2002). In a research study conducted at three Boston schools, students whose teachers received need-based PD performed significantly higher on a normed achievement test than students whose teachers did not receive PD (National Research Council, 2004). PD programs are especially crucial when teachers are required to apply new technologies and approaches in their classrooms (NRC, 2004). However, applying high-level cognitively demanding tasks inside the classroom is not straightforward for teachers, even with PD experience and support.

Mishra and Koehler (2006) argue that developing proficiency for teaching requires a thoughtful interweaving of technology, pedagogy, and content knowledge: A teacher who has deep pedagogical knowledge understands how students construct knowledge and acquire skills, and develop a positive disposition toward learning mathematics and statistics using technological support. Kynigos and Argyris (2004) show that teachers' beliefs and practices are established over time while they are "engaged in an innovative 'mathematical investigations' school program, based on the use of exploratory software" (p. 247). Their research results show that "teachers refer to a variety of aspects of the learning situation in which they intervene rather than just the mathematical concepts and ideas" (p. 247).

The QUANT PD program is intended to improve teachers' technological pedagogical content knowledge (TPACK) in the areas of data analysis, probability and statistics (Reed, 2009). The first QUANT institute took place at Ohio University in Athens in June 2008. The second and third QUANT institutes took place at Ohio Resource Center in Columbus in August 2008 and in

June 2009, respectively. The key aims of the QUANT program are (a) to deepen participants' understanding of content and pedagogy, (b) to develop participants' knowledge of the *mathematical tasks framework* (Stein, Smith, Henningsen, & Silver, 2009), (c) to deepen participants' working knowledge of instructional technology, and (d) to encourage participants' use of high-level tasks during instruction of data analysis, probability, and statistics in their classrooms.

In keeping with the National Research Council (NRC, 2001), the QUANT program promotes the five major components for developing mathematical proficiency—*conceptual understanding, procedural fluency, strategic competence, adaptive reasoning,* and *productive disposition.* These components apply to the teaching and learning of statistics just as they do to the teaching and learning of mathematics. The QUANT materials development and program implementation are based on this premise (Foley, et al. 2010). The QUANT development team defines statistical proficiency for teaching as "teachers' knowledge that is useful in promoting the statistical proficiency of their students" (Foley, Strayer, & Regan, 2010, p. 4).

The Intervention

Eight high school mathematics teachers participated in the June 2009 QUANT institute held at Ohio Resource Center in Columbus. The program explored rich and inquiry-based activities in mathematics and statistics using manipulatives and enhanced technology such as the TI-*n*spire CAS software and TI-*n*spire CAS handheld computer (Wagner, 2009). Pedagogically, the participants used the *mathematical tasks framework* (MTF) and the *task analysis guide* (TAG) as presented in Stein et al. (2009). The purpose of introducing the participants to MTF and TAG was to enhance their ability to identify the two categories of mathematical tasks associated with high-level cognitive demands—*doing mathematics and procedures with connections*—as well as with two categories of mathematical tasks associated with low-level cognitive demands—*memorization and procedures without connections.* The participating teachers were then expected to select or create tasks based on their overall goals for student learning. A key aim of the QUANT program was to help teachers set up, enact, and maintain high-level tasks.

A follow-through workshop was held four months later at the Ohio Resource Center. To address the importance of Stein et al.'s (2009) TAG and to emphasize the productive disposition component of statistical proficiency, participants were asked to prepare and present lessons that reflected their capacity to teach statistical concepts using high-level tasks. These lessons gave the participants the opportunity to practice the new skills they had learned.

The Study

This study explored participants' perceptions of the effectiveness of the 2009–2010 QUANT program and identified perceived obstacles that prevent teachers from enacting high-level cognitively demanding tasks when teaching mathematics. Because the QUANT program specified that the participants would be able to present high-level tasks in their classroom following the program, the research team investigated the degree to which participants actually applied, or were intentionally willing to apply, what they had experienced in the QUANT program. The study examined whether the QUANT program's activities aligned with the participants' teaching goals. Specifically, the investigation evaluated the extent to which the participants, subsequent to the intervention, used high-level tasks when teaching mathematics.

Research Questions

- How and to what extent did the QUANT participants intend to apply high-level tasks in teaching mathematics?
- What, if any, changes did the QUANT participants make to their teaching methods as a result of following the TAG and using the TI-nspire handheld?
- What did the QUANT participants identify as obstacles that prevented their implementation of the TAG?

Methodology

This study collected data through face-to-face interviews with 5 of 8 participants, using a question protocol designed by the researchers, consisting of 11 specific, open-ended questions Follow-up questions were used when appropriate.

Each interview lasted 60 min. The participants were asked to reflect on negative or positive experiences when they applied what they had learned in the QUANT program and ways in which this affected their daily practice. Additionally, they were encouraged to comment on the classroom environment that could support such implementation. Other specific questions addressed the usefulness of the QUANT materials and resources that were given to participants to help them in applying the updated teaching strategies.

Meetings with 2 of the interviewees took place in their classrooms, which provided insight into their teaching environment. In order to gain the most accurate self-reported answers, researchers assured the interviewees that the purpose of the study was not to evaluate their teaching ability or their mathematical knowledge, but to further understand the effectiveness of QUANT program, based on practitioners' own input.

All interviews were recorded in their entirety, transcribed, and analyzed by the team of researchers. Common themes and differences among the interviewees' responses were identified. Initially 3 core codes teachers' perceptions regarding participating in PD program, implementing high-level tasks, and obstacles that may prevent any change were used in analysis.

Findings

Teachers' perceptions regarding participating in PD program

The analysis of the interview transcripts indicated that 4 of the 5 participants reported improved attitudes about their teaching as a result of exposure to the QUANT program. The participants were mathematics teachers who also taught statistics. Some of them had begun to teach statistics recently. They all reported that they felt more comfortable when they taught statistics and tried to apply higher-level tasks after participating in the QUANT program. Participating teachers indicated that they were doing a better job compared to previous years. They claimed they were trying to change their teaching methods as a result of attending the QUANT program. The QUANT program helped them gain confidence in their job. Additionally, some of the teachers reported that it was praiseworthy to participate in QUANT program because they had been feeling isolated due to absence of other statistics teachers in their school. QUANT program was the opportunity for them to meet other mathematics teachers who taught statistics and it gave them opportunity to connect with each other, discuss some activities or share ideas. Following is typical of comments shared by this group.

> *I really enjoyed meeting other teachers who are teaching the same subject and are interested in the same topics … I thought lots of the activities we did were interesting and valuable.* (Participant A, interview transcript, October 2009)

On the other hand, one widely mentioned issue was that attendance of a PD program was entirely self-motivated. Several participants reported that mathematics teachers felt so comfortable and secure in their jobs that they did not appreciate PD programs and did not see the value in the support provided by such programs. They were unwilling to sacrifice their time to attend a PD program.

> *You know the teachers who have been there for a long time are not interested in doing your thing. A lot of them will become comfortable, and asking them to do something new is just increasing their work and there is some resistance to that. I do not think that it is a coincidence that attendance of the QUANT was not good. Even though they pay for the hotel, breakfast, and dinner, and lots of giveaway materials, [they] still could not get people to give up two weeks of their summer because people are just not interested in having to do more work.* (Participant B, interview transcript, October 2009)

Thus, the participants in the study considered themselves highly self-motivated.

Teachers' perceptions regarding implementing high-level tasks

Four participants indicated that their level of comfort and confidence had improved as a result of realizing the impact of implementing high-level tasks on students' learning of mathematics. One such individual was Participant C a high school mathematics teacher who had been teaching statistics for more than 4 years. Participant C discussed how students had become more actively engaged in meaningful discussions about learning in the classroom because of the implementation of high-level tasks. Participant C stated that her students were showing improvement in their ways of solving mathematical problems and they were asking deeper questions as a result of shifting her teaching style towards the implementation of high-level tasks, as depicted below:

> *My students are asking questions more deeply than I am used to, and they begin to ask about the special cases that we never talked about it in the class.* (Participant C, interview transcript, October, 2009)

Because participants spent a great deal of time engaging in solving rich problems during the QUANT program, Participant C was the one of the participants who spent time to think about questions, restate questions in a test and ask challenging questions during the discussion so as to lead the students to think.

During the interview, participants' comments demonstrate an improved level of implementation of high-level tasks in their classrooms as a result of attending the workshop. They claimed thing real-life data from sport or business to capture students' interest and to help them model their mathematical knowledge. Specifically, 4 out of 5 participants referred to particular tasks from the QUANT materials that they had implemented to enrich students' experience and to increase their mathematical thinking.

All of the participants also expressed that the QUANT program helped them break from traditional methods of teaching mathematics, resulting in greater student learning. Participant D stated he had changed his teaching methods as a direct result of the QUANT program. He had changed the sequence of tasks in his lessons. Instead of teaching all the traditional techniques and procedures from the beginning of the lesson, he opened with a higher-level task first, leading to the main topic of the lesson plan. If necessary, Participant D would reteach or apply scaffolding.

Teachers' perspectives regarding the obstacles that may prevent change

One of the main goals of this study was to understand why applying high-level tasks and rich activities using technology in the classroom may be difficult to implement. The transcripts from the interviews showed that all participants identified time as the main constraint. With specific content to be covered in a limited time frame, the participants expressed concern that they had to sacrifice quality teaching to achieve the meet of all objectives. The 5 participants in this study were currently teaching in various school districts, with different teaching experiences, and were engaged with students from a wide range of academic proficiency. Even though there was no direct question that examined the effect of time on applying high-level tasks in the classrooms, 4 of the participants commented that time alone could influence their decision of applying high-level tasks in the classroom. Interestingly, the only teacher who did not count time as the main obstacle when enacting rich activity was the only one who taught in a 90-min class period. The results indicated that class-period duration could support or inhibit implementing high-level activity while teaching mathematics.

Another critical point raised by one participant was that time in outlines was limiting factors to a shift from traditional teaching to using high-level tasks was the level of the cognitive abilities of students accordingly. Students with high cognitive abilities and more motivation to learn do not need to experience high-level tasks because they could find them on their own.

> *The upper level students have lots of materials, so they have to go wide, but I think with students with lower level ability, deeper is better for them. They are expected to know lots of materials and if you go wide, but not deep, then they will not understand. I think different approach depend on the ability of the students.* (Participant B, interview transcript, October 2009)

A more significant point reflected in the same participant's response was that she did not see the importance of all the mathematical concepts in students' lives; she saw mathematics as a preparation for college only.

According to the participants, students' attitude and their willingness was another obstacle regarding implementation high-level tasks. Three participants stressed that students did not want to think on questions or problems deeply. One of the most experienced participants mentioned that forcing the students on high-level thinking caused frustrations for many students because the students had been conditioned to solve low-level tasks. According to one of the participants, it was more enjoyable to teach students who are curious than students who are taking the course for credit only. Because it is more pleasant to teach AP classes, they deserve high-level activities; for other student, their goal is to pass the course with a grade that is shown in their transcript, and her job is to help them achieve this goal. She indicated that subtle changes can occur only with motivated, high-ability students.

The QUANT program emphasized the development of mathematics teachers' TPACK. Nevertheless, some participants indicated that the unavailability of the TI-*n*spire in their schools would inhibit their ability to apply what they had been experiencing in the workshop. Further, because such technology requires continuous practice, the fact that teachers were not currently able to use it in the classrooms affect their ability to use later even when it became available.

Conclusion

Most of the time, teachers do not use high-level tasks or technology when teaching mathematics (Stigler & Hiebert, 1999; Tarr, Grouws, McNaught, & Sutter, 2008). On the other hand, according to Boston and Smith (2009) and NRC (2004), attending PD programs can support teachers' implementation of new approaches and technologies such as high-level cognitively demanding tasks and the TI-*n*spire. The 2009–2010 QUANT program was developed to meet these criteria.

This article reported on the ways that the QUANT program potentially impacted teachers' practices. Through face-to-face interviews, the research team was able to gain insight into the participants' perspectives to see how the teachers themselves evaluated their experience. These data presented the participants' point of view regarding the most valuable benefits from attending the workshop.

While the results show that the QUANT program convinced the 5 interviewees about the importance of engaging students in high-level tasks, most of them expressed concern that they would not be able to do so because of the pressure to cover specific content in limited time, because of the attitudes and willingness of their students, and because of the limited availability of the technology.

The results suggest that when designing PD programs to improve teaching, time management should be addressed as a critical issue that teachers need to consider. Class duration and the syllabus to be covered are two factors that cause an unspoken pressure on teachers. As a result, teachers face the dilemma of privileging the inclusion of high-level cognitively demanding tasks over *covering* the required materials.

Regarding the technology implementation, teachers may not use technology in their classroom after attending PD programs because of the lack of its availability. As a result, PD developers should consider means for providing participants with technology for their classrooms.

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TEACHERS' DISCUSSION OF MATHEMATICS AND PERCEPTION OF LEARNING

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In semi-structured interviews conducted with Scaling-Up SimCalc participants, treatment teachers discussed more mathematic topics they described as "newly introduced" than "basic", while control teachers discussed more "basic" than "newly introduced." Despite their presence in classrooms where students achieved significant learning gains on both "basic" and "newly introduced" topics, and their own testament on the conceptually difficult mathematics the SimCalc unit presented, treatment teachers still expressed that the SimCalc materials may be too advanced for their students. Teacher perceptions of student ability need to be addressed if we hope to encourage the exploration of more conceptually difficult mathematics in K-12 education.

Introduction

SimCalc MathWorlds® is a piece of educational software that facilitates students' exploration of mathematical ideas related to rate and proportionality by supporting dynamic graph creation and manipulation, in which graphical representations are tightly coupled with representations in a simulation "world". By linking measurable experiences and events to formalized mathematic expressions, SimCalc MathWorlds enables more students to learn the mathematics of change and variation (Kaput, 1994). Over the past 15 years, SimCalc MathWorlds has been utilized in a number of small-scale studies with great success. More recently the Scaling-Up SimCalc study has shown that the use of SimCalc MathWorlds in combination with corresponding teacher professional development and curriculum can be used with success in a wide-range of classrooms with a wide-range of students (Roschelle et al., in press; Tatar, 2008).

But finding significant outcomes for learning with SimCalc is just the beginning of the discussion. How can we ensure that SimCalc MathWorlds will not just work for a variety of teachers participating in the study, but that the teachers perceive the benefits of using the SimCalc resources and will actually choose to use it in the future? To answer this question, we turn to the corpus of phone interviews conducted with participating teachers, in which the teachers discussed their assessment of the conceptual difficulty of the SimCalc unit and articulated a wide-range of beliefs on students' mathematical abilities. We argue that teachers are the gatekeepers of resources used in their classrooms, and that if we hope to influence their choice of resources to use in the future (such as SimCalc MathWorlds) then we must first understand teacher perceptions and goals.

Our phone interviews with teachers in the SimCalc study reveal that the positive learning outcomes may not always be apparent to teachers, even when they articulate the benefits of the curriculum themselves. In this paper, we illustrate how treatment teachers participating in the study discussed the mathematics content of the Scaling-Up SimCalc unit as compared to similar discussion on curriculum content with control teachers. Also, we demonstrate how the treatment teachers' felt about the SimCalc unit's conceptual complexity in relation to their beliefs on 7th grade students' abilities and their state-mandated standards and assessment. We demonstrate by example that when we consider we empirical success of interventions such as Scaling-Up

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

SimCalc, we must also consider teacher perception on conceptually difficult topics and its relation to their teaching objectives and perception on student abilities.

Context Of The Study

SimCalc MathWorlds® software supports dynamic graph creation and manipulation, in which graphical representations are tightly coupled with representations in a simulation "world". Students can step through the motion graph, observe characters in the world, and examine tables with corresponding values. The graphs can be directly edited with the mouse, and the simulation world and tables will immediately reflect these edits. The image above (Figure 3) is a screen-shot from a SimCalc MathWorlds activity used by the participants in Scaling-Up SimCalc study. In the image, the world shows two vehicles going on a road trip where the lower line (usually displayed in red) represents a van and the upper line (usually displayed in yellow) represents a bus. With this particular MathWorlds graph and simulation, the two vehicles would start at position 0 at time 0, and when the student presses play the bus and van will move in correspondence with the graph, arriving at their final destination 180 miles away after 2 hours for the bus and 3 hours for the van. There are several worlds the students can interact with, such as elevators in motion, ducks swimming across a pond, and clowns marching across the screen. Definition and direct manipulation of graphically editable functions, hot links between graphically editable functions and their derivatives or integrals, and connections between representations and simulations are thought to be crucial components of making the mathematics of change and variation accessible to a wide range of students (Hegedus & Kaput, 2004; Roschelle & Kaput, 1996).

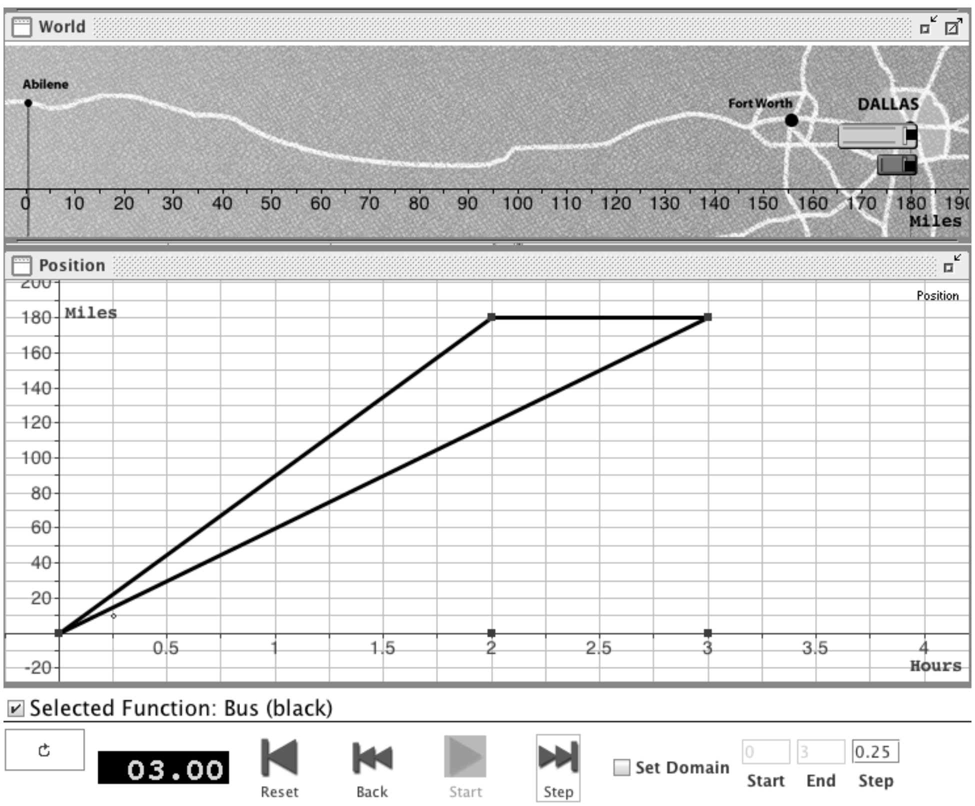

Figure 1. SimCalc MathWorlds® Screen-shot

The SimCalc MathWorlds approach presents mathematical ideas graphically rather than algebraically, which allows for the student to first understand the concepts dynamically and then move to numeric and algebraic functions later . Students can play the simulation to watch the characters move in correspondence to the position graph they created, therefore experiencing the mathematical constructs of algebra and calculus as dynamic, motion based events (Kaput, 1994).

Over the past 15 years, SimCalc MathWorlds has been evaluated in numerous small-scale design-research and quasi-experimental studies showing positive and promising results (Hegedus & Kaput, 2004; Nickerson et al., 2000; Roschelle & Kaput, 1996; Stroup, 2005; Tatar, 2008). However, policy makers at the local, state and federal levels, school administrators and legislators wanted what they would call a systematic demonstration of value. They wanted “proof” that SimCalc worked in wide variety of classrooms. To do this, the researchers conducted a series of randomized, controlled experiments replicating and extending the hypothesis that a wide variety of students from a wide variety of settings could benefit from the use of SimCalc MathWorlds (Roschelle et al., in press; Tatar, 2008).

Scaling-Up SimCalc Study Setting and Procedures

Since rate and proportionality is a central topic of 7th grade, 7th grade math classes and teachers were chosen to be participants in the first and most extensive study (Roschelle et al., in press; Tatar, 2008). The researchers implemented a delayed treatment design with two conditions to test their hypothesis. Teachers were assigned to condition randomly by school (that is, teachers in the same school were in the same condition). The experimental, or treatment group, was assigned to use SimCalc during year one, while the delayed treatment, or control group, was assigned to use SimCalc during year two. In year one, the control teachers were asked to teach their normal rate and proportionality unit. Students from both conditions were given a pre-test before their unit on rate and proportionality, as well as an identical post-test once the unit was completed. Full presentations of the experiments and their results are published in (Roschelle et al., in press; Tatar, 2008).

From an experimental point of view, study location has implications for how the study is conducted and for the generalizability of results. Diversity in teachers, students, and settings was necessary, but availability was also a consideration. The researchers chose to base the Scaling Up SimCalc study in Texas for three reasons:

1) The Charles A. Dana Center, which is a leader in math and science teacher professional development in Texas, and has an on-going relationship with teachers and schools, was willing to participate.
2) The State of Texas gathers comprehensive yearly data about schools and teachers that helped characterize the sample.
3) State standards and testing had been in place in Texas for longer than other states, and were more stable.

The exact pedagogical goals of the experiment were influenced by this choice. 7th grade math classrooms in Texas typically focus on a formula-based approach to rate and proportionality (a/b = c/d) that has its foundations in elementary school mathematical, fractions. A typical problem requires the student to solve for a single unknown value when given three numbers in a proportional relationship. An alternative approach to rate and proportionality is to emphasize its relationship to algebra by formulating problems as function based (y=kx). A function-based approach requires students to find a multiplicative constant that maps a set of inputs to a set of outputs. To show that the intervention was successful, students in the treatment condition needed to (1) learn standard mathematics to the same degree or better than their peers and (2) learn mathematics beyond what is normally taught (Tatar, 2008). Pre and post-tests were developed to evaluate the students on standards for their grade level, as well as more advanced topics. To develop the pre and post-tests, the researchers used questions from the “TAKS”, or Texas standards exam, to evaluate requirement number one. The TAKS (Texas Assessment of Knowledge and Skills) mathematics exam for 7th grade focuses on formula-based questions. To test requirement number two, the researchers developed additional function-based questions on rate and proportionality.

Scaling-Up SimCalc Study Results

Year one of the study was completed with 95 teachers and 1621 students in 7th grade math classes throughout Texas. Students in the treatment condition had a higher mean difference score, or gain score, compared to their peers in the control condition, $t(93)=9.1$, $p < .0001$, e.s. 0.84, using a two-level hierarchical linear model with students nested within teachers. This indicates that students from the treatment group learned more, as measured by the test, than students in the control group (Tatar, 2008). Furthermore, students in the treatment condition did even better than students in the control condition on the complex, or function-based, portion of the test as opposed to the simple, standards-focused, formula-oriented portion ($t(93)=10$, $p < .0001$, e.s. 1.22) (Tatar, 2008).

The Meaning of the Experimental Findings

Based on these experimental findings, the Scaling Up SimCalc study was a great success. Especially when combined with replicated experimental findings, it suggests a clear causal association between SimCalc and student mathematics learning. However, data from the year one teacher phone interviews suggested that the relationships between the teachers, students, and SimCalc content were complex. The teachers’ did not seem to perceive the substantial learning gains taking place in their own classrooms; however, they expressed opinions on the SimCalc unit’s difficulty and their students’ abilities that may have serious implications for their future use of SimCalc resources.

Post-Unit Teacher Phone Interviews

Each teacher phone interview was scheduled within 10 days of his or her administration of the post-test. We used a semi-structured interview protocol, and the interviews always started with the questions “what went well?” and “what went poorly?” From that point the interview continued with questions divided into the following six topics: Teaching Experience, Mathematical Content, Technology, Students, Collaboration, Support, and Research. After each teacher response the interviewer would ask follow-up questions to gain further description from the teachers. There were also additional “wrap-up” questions at the end of the interview. In both years of the Scaling-Up SimCalc study we conducted phone interviews with every participating teacher, however, for the purpose of this report we will focus on interviews conducted during the first year of the study.

During the first year of the Scaling-Up SimCalc study we conducted 95 interviews in total, which amounted to 5163 minutes of audio data. 48 of those interviews were with delayed treatment (control) teachers, while the remaining 47 were with immediate treatment teachers. The corpus of interviews were transcribed and analyzed using a Grounded Theory perspective through an iterative coding process. The analysis resulted in codes such as teaching philosophy, school environment, technology concerns, administration support, collegial support, and mathematic topics discussed. For the purpose of this report we are reflecting solely on the “mathematic topics discussed” code (or “Math Topics” for short), but a complete description of the codes and analysis can be found in Kurdziolek, 2007.

Teacher Discussion of Math Topics

During the interview, the teachers were asked to describe math topics that went well or were difficult for their students either while using the SimCalc materials (immediate treatment teachers) or using their usual curriculum on rate and proportionality (delayed treatment/control teachers). The responses to these questions were coded and categorized as “difficult” or “went well”. Also, while responding to these questions, the teachers would often characterize the math topic as either a “basic” skills that the teacher felt students should have mastered before this point in time, or as topics that were being “newly introduced” in the SimCalc unit or their usual curriculum on rate and proportionality. The following two quotes are examples of typical teacher responses to these questions:

- *“Well, the kids got really confused when you had - when there was a graph where the scale wasn't by 1's. And so they are thinking, 'well, I'm moving up one' but really they are moving up 20 spaces because it is a scale of 20 and so that was really confusing. And then, trying to get them when they are drawing the triangles to, you know, go all the way on the line and where it's an intersection point - an easy to read point - because sometimes they would go over to the line on the graph and it wouldn't be an easy to read point and so their estimating and dividing the slope wouldn't, you know, come out right.” – Treatment teacher description of mathematical topics that were difficult*
- *“The graphing was great and it really - what's the word here - they've gone over it in science and stuff before but "slope?" They had never heard of it. They understood it. The constant - the constant speed - just that that was a straight line. That went well. They understood that no slope was an object standing at rest or, you know, not moving. So, that all went well. The graphing was great. The technology, you know - they wouldn't have gotten it as well as they did without the software and that went really well.” – Treatment teacher description of mathematical topics that went well*

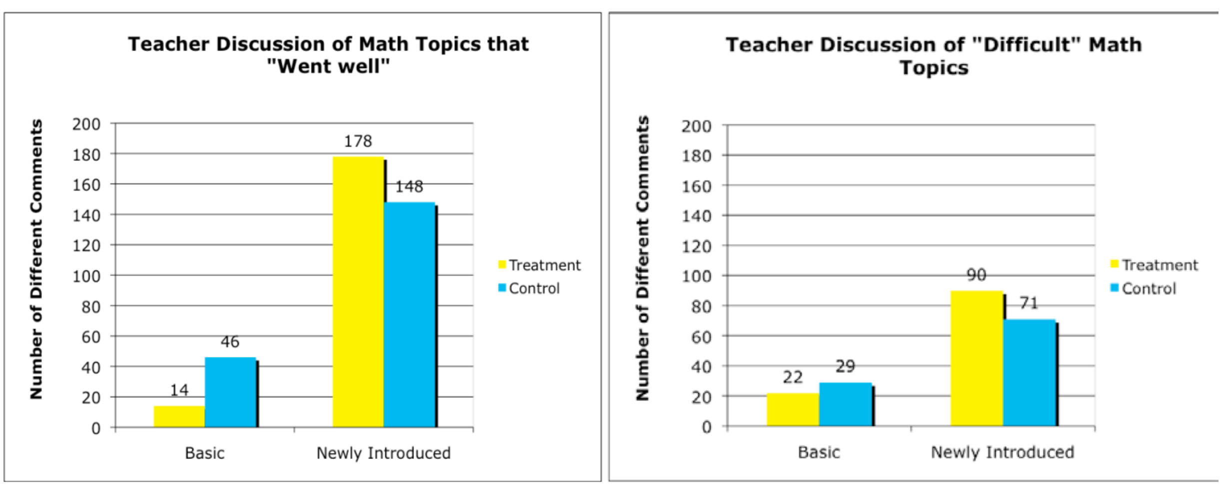

Figure 2. Number of teacher (treatment and control) comments on “basic” or “newly introduced” topics going well for their students (left) and being difficult for their students (right)

After all the year one, teacher phone interviews had undergone the coding process, we counted the number of teacher responses for the math topics that “went well” or “were difficult” and characterized by the teachers as “basic” or “newly introduced”. Teachers could have discussed multiple math topics throughout the course of the interviews, and each separate math topic discussed was included in the count. The maximum number of separate math topics discussed in any one interview was 12, and the minimum 2. The graphs above (Figure 4) show the results of this simple count between the treatment and control conditions. Overall, the teachers talked about more topics that went well than topics that were difficult. For both math topics that “went well” or “went difficult”, control teachers talked about more topics they described as “basic” than treatment teachers, and treatment teachers talked about more “newly introduced” topics than the control. This can be seen in Table 1.

Table 1. Number of treatment and control teacher comments on basic or newly introduced math topics

	Basic Topics	Newly Introduced topics
Treatment	36	268
Control	75	219

By running a chi-square analysis on this table we get the value of $\chi 2(1) = 18.47$ ($p < 0.001$), which means that treatment teachers routinely discussed more newly introduced topics than control teachers during the interviews, and that control teachers discussed more basic topics than treatment teachers. This result suggests that the SimCalc materials influenced the type of math topics treatment teachers discussed with us in the interviews, and presumably presented to their students as suggested by our experimental results. Also, since the treatment teachers did not discuss “basic” topics as much as control teachers, they presumably spent less time with their students on math topics that they would describe as basic while teaching the SimCalc unit.

Teacher Discussion on SimCalc Unit Difficulty

While the teachers were discussing the mathematical topics in their usual rate and proportionality unit or the SimCalc unit, they would often provide characterizations of the students’ capacities in their school/class, characterizations of the rate and proportionality unit’s difficulty level, and characterizations of their students’ ability in relation to the unit’s difficulty. In the case of the immediate treatment teachers, several (31 out of 47 – need to change these numbers) commented on how they and their students coped with using the SimCalc unit for the first time, and (24 out of 47) on how the SimCalc unit compared to their state-wide mathematics standards and assessment (TEKS/TAKS). Below are comments from four immediate treatment teachers on the SimCalc unit’s difficulty and their students’ abilities.

- *“For my students, that's just my opinion. I felt like it was a little difficult for them. Especially, not knowing - they've been so conditioned to choose these multiple choice questions that when they're faced with something that shows them to think and to give an answer themselves, it was a little complicated. I would venture to say, if they were to have math maybe like this for maybe two years consecutively, you'd see a big change.”*
- *“I saw the whole unit as being pretty pre-algebra unit, not a pre pre-algebra unit. It struck me as a regular 8th grade, advanced 7th grade type curriculum, even towards the beginning part. And, I feel pretty good that we're teaching aligned with what the Essential Knowledge and Skills are for the 7th grade, as far as 7th graders. But, I love*

the stretch. I don't want to say I don't want to do that. It's just that I might do the more required earlier because we've got to make sure their solid in all their standard skills first."

- *"Yeah. Neither I nor my teammate, we didn't try this with any of our regular classes. And she said, I wouldn't even try it with a regular class. Now, next year, I'm not going to have 7th pre-AP, I'm going to have 6th pre-AP and 7th regular, so I'll be doing it with a regular class but I will be doing it differently. To the extent that resource doesn't exist, I'll be trying to think of a way - because if a regular kid misses a day of this - the most likely ones to miss are the ones that are most likely to get a huge hole that they won't be engaged and then they'll be totally off track."*
- *"I would have liked to have been able to go more in depth with the graph as in slope and things like that but I just did not feel like my students were ready for that yet. But, I wish - my goal would be that they would be ready for that so then when they go into algebra, or pre-algebra, they will have heard those terms before and understand what it means so they can connect it all."*

The four teachers quoted above expressed that the SimCalc unit was a stretch for their regular 7th grade students. In the first two quotes, we see that the teachers assessed the SimCalc unit difficulty in relation to their students' abilities and background, as well as their state standards and assessments. In the case of the last two quotes, both teachers felt that the SimCalc unit was more appropriate for advanced 7th grade or 8th grade students, and they may not use SimCalc with regular 7th grade classrooms in the future.

Out of all the immediate treatment interviews, only one teacher described the SimCalc unit as being more conceptually difficult than their usual unit on rate and proportionality *and* within the realm of regular 7th grade students' abilities.

> *"In 7th grade. Because they are doing those linear equations and you might as well. I mean, you are doing unit rate - you might as well. So, it gave me a new experience as to whether they had the ability to do slope or not and, although we didn't go over it a lot, you know, I had to, you know, do one or two of the problems and then go on to the next thing. It did let me see that they could/would be able to slope in 7th grade."*

While the teacher quoted above was teaching the SimCalc unit, (s)he realized that 7th grade students had the ability to understand the concept of slope. Perhaps in the future this teacher will present slope, a conceptually more difficult math topic, to his/her students even though slope is not a topic on the annual 7th grade TAKS. Unfortunately, this teacher is an exception to the general trend of comments on the SimCalc unit difficulty in the post-unit teacher interviews. Overwhelmingly, when treatment teachers discussed the conceptual difficulty of the SimCalc unit they also describe topics or parts of the unit they would not cover in the future.

Conclusions

The Scaling-Up SimCalc study provides a compelling example of an intervention being used in a wide variety of classrooms with a high degree of success. After the first year of the Scaling-Up SimCalc study, researchers obtained statistically significant results indicating that students in the treatment condition had higher learning gains than their peers in control condition classrooms. This could be seen as a treatment (SimCalc) having a direct effect on outcome measures (student gains). However, this description of the intervention is deceptively simple. Students' relationship to much of the material and content SimCalc provided is filtered through teachers, in the sense that teachers received the SimCalc materials and were ultimately the ones who decided how and when the students accessed those resources. As Eugene Judson (2006, p. 583) put it, "When establishing any classroom innovation, it is the teacher who is the key determinant of implementation."

The SimCalc researchers who developed the replacement unit intended for the curriculum to be more advanced than the usual 7th grade curriculum and focus on function-based proportionality, yet still on par with an average 7th graders ability. While treatment teachers recognized and described the SimCalc unit as being more conceptually difficult than their usual curriculum, they did not perceive their students' learning gains on both the "simple" and "complex" mathematics. This perception of the SimCalc unit certainly affects the potential use of SimCalc resources in the future, but more importantly, represents a challenge to be addressed in the future. How can we increase teacher awareness of student learning gains? This is an issue to be addressed if we hope to encourage the exploration of more conceptually difficult mathematics in K-12 education.

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TEACHERS' UNDERSTANDING OF THE PHRASE "TRANSLATING FROM ENGLISH TO MATH"

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Teachers' interactions with algebra content are shaped by varied and complex factors such as her/his knowledge and beliefs as well as by the representation of the algebra content. This study examines two separate groups of teachers who explained multiple algebra problems as requiring students to "translate from English to math". Interviews revealed conflicting beliefs about whether the act of "translation" is mathematical. The teachers also expressed concerns that the wording of the problems was an obstacle to the mathematics. The findings broaden the discussion of how teachers' beliefs about mathematics may impact their interactions with curricular materials.

Introduction

In its communication standard, the National Council of Teachers of Mathematics (NCTM, 2000) notes that ``because mathematics is so often conveyed in symbols, oral and written communication about mathematical ideas is not always recognized as an important part of mathematics education" (p. 59). The preceding quote acknowledges a relationship between symbol use in mathematics and perceptions of what constitutes mathematics education. In the following pages, I will present research and perspectives which complexify teachers' perceptions of the relationship between symbolic and verbal representations in mathematics. Specifically, I investigate groups of teachers' categorizations of algebra problems as about "translating from English to math" and how their interpretations of this phrase may reflect particular views of mathematics.

The sites for the present study were two 3-day workshops (2006 and 2008) at a large university in the southwestern United States. The participants were teams of mathematicians, mathematics educators, and teachers from geographically diverse locations. The teachers were middle and high school teachers of algebra; the mathematicians and mathematics educators were university professors (See Table 1). A focal point of the workshop was a set of 12 algebra problems spanning content from pre-Algebra to Algebra II. The problems were compiled by the workshop organizer Chris Norris, a mathematician, and administered in the participating teachers' classrooms. The problem set was the same for both workshops. Most of each workshop was devoted to discussing the workshop problems and analyzing student work. My data is primarily drawn from a particular task in which the workshop participants, within their professional groups, analyzed and categorized the 12 algebra problems. This categorization task required each professional group to negotiate and justify categories for the set of 12 problems; the instructions for the task were summed up by Norris in 2006 as "So what do you think is the sort of mathematical content of the problems and do they fall naturally into different groups?"

Table 1. Number Workshop Participants and their Professional Groups

Year	# of Teachers	# of Mathematicians	# of Math Educators	Total
2006	4	6	4	14
2008	8	5	0	13

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

The results reported herein are from a larger study of these two workshops which investigated the ways in which the participants made sense of the workshop algebra problems. Herein, I focus on a surprising theme which emerged in these workshops; in both 2006 and 2008, the groups of teachers (which shared no common members) categorized a large portion of the workshop problems as being about translating from English to mathematics. In 2006, five out of the 12 problems were categorized as such and in 2008 six problems were categorized at such. For example, number eight (Figure 1) was categorized as being about "translating" by the teachers in both workshops. The questions which I investigate are: (1) How did the workshop participants interpret the phrase "translating from English to math"? and (2) What insight does this provide about participants' interactions with the algebra problems?

To convert from miles to kilometers, Abby takes the number of miles, *m*, doubles it, then subtracts 20% of the result. Renato first divides the number of miles by 5, and then multiplies the result by 8.

a. Write an algebraic expression for each method.

b. Use your answer from part (a) to decide if the two methods give the same answer.

Figure 1. Problem Number 8

Perspective

My primary focus was on the problem categorization tasks at the workshops: I conceptualized this task as mathematicians, mathematics educators, and mathematics teachers *interacting with* the workshop algebra problems. As such, there are three overlapping categories of research which make up the theoretical framework: (1) research on the learning and teaching of school algebra, (2) research on interactions with curricula, and (3) research on the beliefs, knowledge, and dispositions of the three professional groups. The inclusion of research on school algebra in the theoretical framework provides structure in which to situate not just Norris' algebra problems but also the interpretations of and interactions with these problems by all participants in the workshop. Indeed, a view of school algebra was influencing the workshop participants' interactions with these problems.

Kieran (2007) describes the content of school algebra as ranging between reform-oriented and traditional. Reform-minded algebra often is characterized by an emphasis on functions, real-world problems, and a value on multiplicity of techniques (such as the use of technology). Included in this the view of algebra are the multiple ways of representing functions including graphical, tabular, and symbolic representations. A traditional view conceptualizes algebra as generalized arithmetic and often focuses on symbolic manipulations, recognition of forms, simplification of expressions, solving equations, and factoring polynomials. The traditional view puts an emphasis on letter-symbolic aspects of algebra, reformed algebra emphasizes a combination of mathematical representations.

Analysis of these interactions with the algebra problems is also framed by research on interactions with curricular materials in general (the workshop problems being considered curricular materials). The present study is, in part, framed by Remillard's (2005) description of curriculum use as a participatory relationship between teacher and curriculum. Her framework acknowledges that teachers construct their understanding of curricular materials yet are also influenced by these materials. Of particular interest are the characteristics of curriculum, including the more ``subtle and often unintended" components, which affect the teacher-curriculum relationship.

There are surely many factors which influenced the interactions of the workshop participants with the problem set. These factors include the presentation and content of the problems. They also include less tangible characteristics of the workshop participants who are interpreting these problems. For this reason, the theoretical framework also includes research on these disciplinary groups' beliefs about and dispositions toward mathematics in general and algebra in particular. The few studies about mathematicians’ and mathematics educators’ beliefs about mathematics indicate a diversity of views (Mura,1993, 1995; Burton, 1999).

Of the research on teachers’ beliefs, the work of Nathan and Koedinger (2000a, 2000b) is particularly relevant. They investigated how 67 secondary school mathematics teachers and 35 mathematics education researchers predicted student difficulties with a set of arithmetic and algebra problems. Most of the researchers and teachers predicted greater difficulty with word-problems and algebra than with symbolic problems and arithmetic thus endorsing a view which Nathan and Koedinger referred to as the *Symbolic-Precedence Model*; students' symbolic reasoning strictly precedes verbal problem solving and arithmetic skills precede algebraic skills. This view is similar to that commonly portrayed in textbooks. However, analysis of student problem-solving strategies on these problems indicated that a *Verbal-Precedence Model* prevailed; that is, “contrary to teachers' expectations, students experienced greater difficulties when solving symbolic-equation problems than when solving verbally presented problems” (p. 179). In the follow-up study, 107 K-12 teachers were asked to rank-order (by difficulty) the mathematics problems from the previous study and to complete a Likert-scale assessment about the teachers' beliefs. They found that, across grade levels, teachers’ responses tended to reflect reform-oriented views of math. Elementary teachers were the most likely to agree with reform-oriented views and high-school teachers were the least likely. Middle school teachers did best at ranking the difficulty of the problems (as measured by comparison to student responses) whereas teachers from other levels displayed a Symbolic-Precedence Model.

Methods

Data included audio and video recordings from the 2006 and 2008 workshops and observational data/field notes from the 2008 workshop. The recordings from the problem categorization tasks were transcribed and coded, as were other relevant parts of the workshop (e.g., Norris’ description of the categorization task). There were three rounds of coding based both on the theoretical framework and emergent themes.

Interviews lasting from 24 to 60 minutes were conducted with all but one participant in 2008 (other than Norris, no 2006 participants were interviewed). Each interview was transcribed and coded. The open-ended interviews included questions about participants’ understanding and use of the phrase “translating from English to math” within the workshop. In general, I collected multiple data sources and subjected them to methodical, iterative analyses while simultaneously documenting my reflections and decisions. After analysis was complete, I conducted member checks with three of the participants (one from each professional group).

Results

On one level, ``translating" refers to converting from a verbal representation to a symbolic representation. However, I was intrigued by the communication of this idea through the phrase ``translating between English and math". Is there an idea of algebra or mathematics which might be underlying the use of this phrase? What idea of algebra might be inspired in a student who hears this phrase? Is the act of translating, itself, a part of mathematics?

In (a)-(c), (i) Write an algebraic expression representing each of the given operations on a number b. (ii) Are the expressions equivalent? Explain what this tells you. (a) "Multiply by one fifth" "Divide by one third" (b) "Multiply by one fifth" "Divide by five" (c) "Multiply by 0.4" "Divide by five halves"

Figure 2. Problem Number 5

The following dialog from the teachers' 2008 categorization activity is exemplary of the way in which they spoke about "translating." Judith Hardy (a middle school teacher) and Nadya King (a high school teacher) are referencing problem number five (Figure 2) which both the 2006 and the 2008 groups of teachers categorized as "translating."

Judith: And I think it goes back to what you were saying about converting the language of English to the language of math too. That you have to have, I think, a pretty decent grasp of English and then be able to convert that.

Nadya: Yeah, because we were portraying equivalent expressions again. But before we can even get to that comparison we have to dig through all those words.

In the group discussions in both 2006 and 2008, the phrase "translating from English to math" was picked up upon and used by members of all disciplinary groups. During the problem categorization tasks in 2006, the mathematicians brought up translating twice (numbers 5 and 8) and the mathematics educators mentioned this sort of "translating" once (number 7). The phrase itself was never explicitly discussed during the workshop. However, in 2008, I asked about "translating from English to mathematics" in the interviews. The mathematicians all agreed that the phrase is potentially misleading since they believed that this sort of translation is *in itself* a mathematical activity. The responses amongst the teachers were a bit more mixed (no mathematics educators were interviewed).

Keith Young (high school teacher, 2008) spoke about translation in a way that perhaps evoked an image of a boundary between mathematics and English where the mathematical activity happens on just one side of the boundary.

Interviewer: But you could argue that the math *is* doing that translation. It's not like it becomes [math] as a process of it, but the math is the process.

Keith: Mmmm, you see I understand what you're saying there but the main reason for doing that translation is so you can easily do the mathematical manipulations that you need to do. But then once you've manipulated it, whatever form you've manipulated it in, you've got to be able to take that and put it back… into English terms.

Greg Davis (high school teacher, 2008), after a similar prompt, more explicitly expressed a sentiment that the mathematics begins after the verbal representation has been translated: “I see what you're saying but I don't think necessarily mathematics is being able to translate, I think a skill you need in order to do mathematics well is to be able to translate.”

The phrase “translating between English and math” may indicate a range of underlying views of mathematics. Users of this phrase may think that the mathematical activities begin after the translating has been completed or may feel that the translation *is* a mathematical activity. The mathematicians in 2008 all agreed with the latter analysis though the 2008 group of teachers expressed more varied opinions.

The Wording of the Problems

In Remillard's (2005) conceptualization of the participatory relationship between teacher and curricular materials she highlights the roles of both the teacher and the curricular materials. The preceding discussion suggests that teachers’ beliefs about “where the mathematics begins” may have influenced their interpretations of the problems. But, conversely, the interactions between the workshop participants and the set of 12 algebra problems were also influenced by the problems themselves.

During the 2006 problem categorization task, the teachers described the wording as problematic for their students on seven out of the 11 problems which they discussed for more than one minute. The group of teachers in 2008 discussed the wording on 10 of the 12 problems as a source of difficulty for students. There were a variety of reasons cited when the wording was criticized. For example, Ursa Harper (MS teacher, 2006) spoke about unfamiliar wording as an obstacle on number 9 (Figure 3):

> And I didn't like exactly the way D [the correct answer] was worded: “The number people of people who decide not to go if the price is raised by a dollar.” And I teach them it's the “change per” and the word “per” wasn't in there.

The wording of problems with regard to symbol use was also seen as an obstacle at times. For example, several teachers perceived the symbols “\$*p*” (*p* dollars) in number 9 to be confusing for their students. Other problems presented difficulty because of their structure. For example, on number 5 (Figure 2), teachers from both workshops referenced students interpreting “In [parts] (a)-(c)” as “variable a minus variable c”.

> If the tickets for a concert cost \$*p* each, the number of people who will attend is *2500 - 80p*. Which of the following best describes the meaning of the 80 in this expression?
>
> A. The price of an individual ticket.
> B. The slope of the graph of attendance against ticket price.
> C. The price at which no-one will go to the concert.
> D. The number of people who will decide not to go if the price is raised by one dollar.
>
> Explain how you chose your answer.

Figure 3. Problem Number 9

During the teachers' categorization task in 2006, Tom Luft (middle school teacher) summed up some of the teachers’ concerns:

> I think that could be a general comment overall. I thought the directions on several problems the kids are like, “What am I supposed to do on this thing?”. And, you know, and I would clarify the directions because we wanted to get, we wanted to get data on the problems not on their ability to interpret the directions on the problems.

But to what extent does ”clarifying” the problem change the problem? Luft's quote seems to imply that changing the wording of a problem does not change the problem.

Discussion

Throughout the workshops I observed interactions which served to complexify the relationship between the written and the understood. It was the wording of the problems which was often cited, especially amongst the teachers, as a source of confusion and difficulty for the students. In parallel to this, the teachers from both workshops categorized about half of the problems as being about “translating from English to Math”. At times, the remaining computational tasks (after the “translation”) were described as “doing the math”.

As discovered in the 2008 interviews, there were two prominent ways to conceptualize the act of “translating”: (1) it is what a student needs to do in order to get to the mathematics (i.e., the translation is not mathematics), or (2) it is, in itself, a mathematical activity. The second of these perspectives was endorsed by all of the mathematicians yet there was variation within the group of teachers. Cutting across these two perspectives is the student activity of converting from a verbal representation to a symbolic representation; the difference is in how this activity relates to and fits in with ideas about mathematics. As Kieran (2007) reported, a focus on multiple mathematical representations is associated with a reformed view of algebra and the representation itself can serve as a source of meaning for an algebra problem.

The wording of the workshop problems was, at times, described as an obstacle to students' success. Indeed, this perception is consistent with Nathan and Koedinger's (2000b) finding that (7th through 12th grade) teachers and mathematics educators perceived verbally presented problems to be more difficult than symbolically presented problems. Perhaps their description of this “symbolic-precedence” is related to a particular view of mathematics or of algebra. Within the workshops, one manifestation of the criticism of the problems' wording was Tom Luft's (Middle School teacher 2006) clarification of the problems for his students. Luft was perhaps communicating an opinion that the wording of a problem is largely separate from the mathematics of the problem. This may correspond to the view that mathematics begins after the translation from English.

Certainly, a student who solves the “clarified” problem has engaged in a different task than a student who solves the originally-worded problem. The question, however, is whether or not the student has engaged in a different *mathematical* task. The answer is perhaps, among other things, dependent on an individual's view of mathematics. If the wording of a problem is an obstacle then, for some, the problem may be *about* the wording; that is, the problem may be about “translating from English to math”.

It is possible that the teachers' concerns about the wording of the workshop problems were, at least in part, byproducts of the unusual circumstances through which the workshop problems were introduced to their classrooms. For example, the problem set was distributed to teachers with no specific instructions for implementation; they were free to do what they pleased with the problems. So Tom Luft's "clarification" of the problems may not have reflected a particular view of algebra. Rather, it may have been nothing more than a practical means through which to produce materials for the workshop; he may or may not have believed that he was changing the mathematics of the problems by clarifying them. Nonetheless, the interviews with teachers from 2008 (Luft was in the 2006 group which was not interviewed) did reveal varying opinions about the relationship of "translating" to mathematics.

But if we accept the view that "translating" *is* a part of doing mathematics, then any sort of clarification (regardless of the motivation) needs to be done with care. Hiebert and Grouws (2007), in their survey of literature on teaching for conceptual understanding, cited as a key feature "the engagement of students in struggling or wrestling with important mathematical concepts" (p. 387). Is "translating from English to math" an "important mathematical concept" with which students should be allowed to struggle?

The phrase "translating from English to math" may, in itself, communicate a viewpoint that mathematics begins after the translation is complete. During the workshop teachers' presentations about their students' work on the problems, there were instances where the workshop participants would look at a student's work and say something like, "This student understood the mathematics but just couldn't put it in writing." For example, during the teachers' presentations in 2006, Tom Luft commented: "I have those kids that know how to do stuff but they can't explain it. They're not verbal and they're not linguistic but they are so mathematical, they can just do stuff." Is this a description of students' difficulties "translating from *math to English*"? Regardless of the view of mathematics held by a teacher, mathematician, or mathematics educator who says phrases such as "translating from English to math" or "the student understood the mathematics but couldn't write it", these phrases may imply a divide between the written and the understood in mathematics, between the wording of a problem (as not being mathematics) and the computations required by the problem (as mathematics).

But what is the appropriate way to discuss "translating"? Is it a skill or ability which is subsumed by the concept of *transformational fluency*? Is it a component of mathematical knowledge that needs to be labeled and made explicit? If we accept the view that "translation" is a part of algebra and mathematics, then what is an appropriate way to describe a students' ability to take a mathematical approach to a verbally described situation? If a teacher holds a view that mathematics does not include "translating" then how does that affect her/his pedagogical decisions? For instance, how do teachers working with English language learners make use of English word problems? And do the decisions they make reflect or endorse the particular views of mathematics just described? As Drouhard (2004) noted, "the way you teach algebra depends dramatically on what you believe algebra is; therefore this question is worth addressing" (p. 44).

At the very least, the present research has shown that the use of the phrase "translating between English and math" can correspond to varying views about which parts of a word problem are mathematical. Furthermore, the wording of a problem may be perceived as an obstacle to student engagement with mathematics rather than as a part of the mathematics. However, it would be useful to unpack and to characterize the ways in which wording may present obstacles. For instance, if a problem is difficult because the wording is unfamiliar then that has different consequences than if the wording is plainly inaccurate. Indeed, interpreting problems which use unfamiliar wording or symbols is an activity which many students are often required to do on standardized assessments. The ability to flexibly translate from verbal to symbolic representations may be undermined if students are not given opportunity to constructively struggle with unfamiliar wording or if they have a view of mathematics which does not include "translation" as mathematical skill or knowledge.

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ASSOCIATION BETWEEN SECONDARY MATHEMATICS TEACHERS' BELIEFS, BACKGROUND CHARACTERISTICS, AND DIMENSIONS OF CURRICULUM IMPLEMENTATION

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Beliefs represent a determinant of behavior and a cluster of relationships among systems of beliefs (Green, 1971). Teachers' beliefs about the teaching and learning of mathematics significantly influence not only how the subject is taught but also how students are expected to learn (Leder, Pehkonen, & Törner, 2002). As a result, it is imperative that researchers examine the extent to which beliefs influence the *process* of teaching and the *product* of learning within classrooms (Philipp, 2007). Therefore, we present correlations between teachers' beliefs, background characteristics, and dimensions of curriculum implementation.

We collected data from 151 mathematics teachers who participated in the Comparing Options in Secondary Mathematics: Investigating Curriculum (COSMIC) project, a 3-year quasi-experimental longitudinal study of the impact of two types of curriculum on student learning, a *subject-specific* approach (Algebra 1, Geometry, Algebra 2, Pre-Calculus) and an *integrated* approach (Integrated 1, Integrated 2, Integrated 3). The participating teachers completed an Initial Teacher Survey (ITS) to provide information on background characteristics and beliefs about the teaching and learning of mathematics. Additionally, we conducted 326 classroom observations, generated opportunity-to-learn indices, and documented teachers' fidelity of implementation. We analyzed data using factor analysis, a comparison of group means, and bivariate correlations. Findings suggest that teachers' beliefs clustered around three factors: *reform practices*, *didactic approaches*, and *self-efficacy* (Tarr et. al, 2010). Furthermore, *reform practices* correlated positively with the classroom learning environment, technology use, and teachers' alignment with NCTM Standards. Alternatively, the belief factor, *didactic approaches,* was negatively associated with attributes that correlated positively with *reform practices. Self-efficacy* correlated negatively with teaching experience. None of the beliefs were significantly correlated with fidelity of implementation and opportunity-to-learn indices.

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Chapter 13: Teacher Knowledge

Research Reports

Brief Research Reports

Posters

A PROBLEM WITH THE PROBLEM OF POINTS

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In this article, we claim that reducing certain problems to consideration of the sample space, which consists of equiprobable outcomes, may not be in accord with learners' initial ways of reasoning. We suggest a "desirable pedagogical approach" in which solving the problem builds on the set of outcomes identified by learners and acts as a bridge towards mathematical convention. Through detailing prospective high school mathematics teachers' engagement with a famous probabilistic problem, which they had not encountered previously, we discuss the participants' mathematical solutions and demonstrate the convincing power of the suggested pedagogical approach.

Introduction

The purpose of this article, in general, is to contribute to research on the probabilistic knowledge of teachers. According to Jones, Langrall, and Mooney (2007), with respect to probability, "research on teachers' mathematical content knowledge, pedagogical content knowledge, and knowledge of student learning [i.e., probabilistic knowledge] is limited" (p. 933). Specifically, our intent is to inform future developments associated with teaching and successful student learning about the sample space.

In order to start our discussion, consider the following problem: In a family of three children, what is the probability that there are two boys and a girl? Individuals well versed in probability, or experienced readers, will determine that the probability is 3/8. However, many novice learners, including prospective elementary school teachers, claim initially that this probability is 1/4, considering that 'two boys and a girl' is one out of four possible options (the other three being all boys, all girls, two girls and a boy). Claiming that the sought probability is 1/4 is quite common and similar examples (e.g., Speiser & Walter, 1998) are found throughout probability education literature.

Literature Review

Although there is an abundance of sample space literature found in mathematics education (e.g., Fischbein, 1975; Green, 1983; Nilsson, 2009), there are "conflicting results in this research" (Jones et al., 2007, p. 920). Despite the conflicting results, sample space research is categorized into different domains, which, in a broad sense, include sample space generation, incomplete generation leading to multiple sample spaces, and the assignment of probabilities based on sample space.

The majority of existing sample space research is focused on the generation of outcomes of the sample space and these studies indicate that students have difficulty in determining all possible outcomes that could occur in a variety of tasks and situations. Researchers use combinatorics (e.g., English, 2005), symmetry (e.g., Borovcnik & Bentz, 1991), and levels (e.g., Jones, Langrall, Thornton & Mogill 1997) to account for students' abilities or inabilities to list the set of all possible outcomes, i.e., the sample space. Building upon the notion of incomplete generation of the sample space, certain researchers discuss the existence of more than one sample space for particular students. For example, "to justify the probabilities for the outcomes

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of dice games, learners construct informal sample spaces" (Speiser & Walter, 1998, p. 61). The inferred structure of these informal or personal sample spaces has been used to demonstrate particular anomalies found in probability education research. Speiser & Walter demonstrated that certain individuals found, for example, the outcome (4,5) for the experiment of rolling two identical dice to be indistinguishable from the outcome (5,4). The researchers extrapolated the lack of discernment between pairs to all of outcomes (e.g., (3,4) and (4,3)). Consequently, the researchers hypothesized and subsequently concluded that the sample space employed by certain individuals answering their question consisted of 21 possible outcomes and not 36 outcomes, because individuals treated the outcomes (4,5) and (5,4) as one outcome. Alternatively stated, the inability to generate all possible outcomes of the sample space led to multiple sample spaces and, consequently, to various probabilities.

"Although the concept of sample space appears to be a relatively straightforward aspect of the mathematics of random phenomena, it is more subtle and elusive than it appears" (Jones, Langrall, & Mooney, 2007, p. 920). In line with the aforementioned statement, and recognizing the lack of research on teachers' probabilistic knowledge related to the sample space presented earlier, we draw upon a famous probabilistic task as our medium for investigation. To aid our investigation and in an attempt to synthesize and rectify conflicts in sample space research, we first introduce our theoretical constructs.

Theoretical Constructs

In this section we introduce a working definition for two constructs: the sample set and a desirable pedagogical approach. We start by providing a rationale for our notion of 'sample set', which is based, in part, upon lack of common terminology used in sample space research and our analysis of the possible ways for listing outcomes.

The Sample Set

While the notion of sample space plays a critical role in many probabilistic considerations, it has been described in literature (e.g., Konold, 1989) that individuals' ideas of sample space may be inconsistent with normative mathematical understanding. As presented in the previous section, researchers use different expressions to refer to unconventional sample space descriptions identified by students; for example, they refer to informal sample spaces, or multiple sample spaces. The lack of common terminology to describe student-generated lists of outcomes – that are not in accord with the conventional sample space – can be confusing. Further, research does little to distinguish between an erroneous or incomplete listing of outcomes from an unconventional listing. To overcome this confusion we introduce the notion of a sample set (to be defined below), and in what follows we provide further motivation for this construct.

Determining the probability that two heads and a tail result from three flips of a fair coin – a task isomorphic to the one presented in our prelude – requires listing the set of all the possible outcomes, i.e., the sample space. However, as Chernoff (2009) demonstrated, subtleties associated with what is meant by the term "outcome" (i.e., an element of the sample space) can lead to more than one sample space listing for a particular experiment. For example, one potential listing of the set of all possible outcomes is: *3 heads and 0 tails* (i.e., HHH), *2 heads and 1 tail* (i.e., HHT, HTH, THH), *1 head and 2 tails* (i.e., TTH, THT, HTT), and *0 heads and 3 tails* (i.e., TTT). As such, we define a *sample set* as any set of all possible outcomes, where the elements of this set do not need to be equiprobable. In other words, {3 heads and 0 tails, 2 heads and 1 tail, 1 head and 2 tails, 0 heads and 3 tails} is a sample set for three flips of a coin because all possible outcomes are listed as a set, but the outcomes are not equiprobable. For this example, the outcome *2 heads and 1 tail* (i.e., HHT, HTH, THH) is three times as likely as the outcome *3 heads and 0 tails* (i.e., HHH). In general, a sample set is a set of all possible outcomes, that is to say the sample space, which is not listed in conventional form.

Desirable Pedagogical Approach

We consider a *desirable pedagogical approach* (for the purposes of this discussion) as one that uses the learner's ideas as a starting point. From this point, where possible, building on initial ideas can develop conventional mathematical solutions. We consider this approach more desirable than attempting to impose mathematical convention, while ignoring the learner's (initial) interpretation. This approach is in accord with ample research in mathematics education that advocates for building on students' knowledge (e.g., Mack, 1990).

Our claim, in this instance, is that consideration of equiprobable outcomes, while correct mathematically and foundational to the theory of probability, may not be a desirable pedagogical choice in certain situations. We contend, consideration of the alternative descriptions of the sample space, that is the sample set, made by students is a desirable initial idea to embrace. Our research is seeking pedagogy that, without compromising mathematical rigor, acknowledges the learner and serves as a bridge between personal (sometimes naïve) and conventional knowledge.

Task & Participants

Participants in our study were 30 prospective high school mathematics teachers enrolled in a problem-solving course. This course is a "companion" to the traditional methods course for teaching high school mathematics, and is usually taken in the last semester of studies before teaching certification. In this course the students are expected to enhance their personal problem solving skills through a variety of problem solving experiences and to examine pedagogical approaches for making problem solving an integral part of their future classrooms. The research we present here is based upon responses to a probability task and the classroom discussions that followed the task.

The "classical" problem described in this section, known as the Problem of Points, has developed into many variations and is presented in a number of sources (e.g., Davis & Hersh, 1986; Devlin, 2008) as the problem that started the development of the field of probability. In several particular instantiations of this problem, the two players are named Pascal and Fermat. While it is unlikely that these prominent mathematicians entertained themselves in this way, we use their characters to exemplify and honour the problem.

> *Pascal and Fermat play a game that involves flipping a coin. If the coin comes up Heads, Fermat gets a point. If it comes up Tails, Pascal gets a point. A stake of 100 francs is put up and the first player to get 10 points wins the whole stake. However, the game is interrupted (with no possibility to resume and finish) when Fermat has 8 points and Pascal has 7. How should the 100 francs be divided?*

Results & Analysis

Responses from the 30 individuals who completed the Problem of Points fell into five different categories, which are presented below according to how participants divvied up the 100 francs between Pascal and Fermat.

Solution 1: 50 to 50, or 100 to 0

Three participants chose to either split the stakes between Pascal and Fermat each getting 50 francs or give the leading player all 100 of the francs.

> Olivia: Fermat is in the lead and the game is over so according to most sporting events, if the game is 50% complete then the winner is whoever is in the lead.
>
> Peter: Although it would seem that probability should influence the answer, given the limited number of trials, really anything can happen. This means that there is really no sure winner. Hence it should be all or nothing to the winner. In this case, there is no winner, hence the money should be divided equally.

Two individuals (one of which was Olivia) decided to give all the francs to the leading player because, as they mentioned, one cannot predict the future and, as such, the game is, in essence, over once it is interrupted. In a similar fashion Peter, who decided to split the francs evenly, also made reference to the unpredictability of the future. However, given this unpredictability, and because of the slim lead in the game, the francs according to Peter should split evenly between the two players, as there is no certain winner.

Solution 2: $46\frac{2}{3}$ to $53\frac{1}{3}$

Unlike the focus on the future in the previous section, a large number of individuals, 9 in total, decided to split the pot between Pascal and Fermat based on events that had occurred, i.e., the past.

> Quince: So far there have been 15 coin flips and Pascal has won 7 of 15 and Fermat won 8 of 15...The remaining flips are chance. Therefore, their outcomes are unknown. The division is based on flips already won.

In a simple and straightforward arithmetic calculation presented for this solution, Fermat has obtained 8 wins and Pascal 7, bringing the total number of wins to 15. For Fermat, he has achieved 8 of the 15 possible wins, which is $53\frac{1}{3}$%; therefore, he deserves $53\frac{1}{3}$ of the 100 total francs. Similarly, Pascal has achieved 7 out of a possible 15 wins, which is $46\frac{2}{3}$%; therefore, he deserved $46\frac{2}{3}$ of the 100 total francs.

Solution 3: 47.50 to 52.50

Three individuals, in their responses, combined elements of the first two responses presented above.

> Robert: Fermat had 80% of the required wins (goal was 10) and Pascal had 70% of the required wins (goal was 10). So, Fermat gets 80% of 1/2 the pot, Pascal gets 70% of 1/2 the pot and of the 25 left, they each get half.

Those who decided to split the pot 47.50 to 52.50 argued that Fermat had achieved 8 out of the 10 required wins and, thus, deserved 80% of half of the pot, or 40 francs. Similarly for Pascal, who had achieved 7 out of the 10 required and, thus, deserved 70% of half of the pot, or 35 francs. This approach is similar to the $46\frac{2}{3}$ to $53\frac{1}{3}$ solution above; however, when adding the numbers together, the total only comes to 75 francs. Given there are 25 francs remaining, it was subsequently determined that the remaining 25 francs should be split evenly between the two individuals – which has similar elements to the 50-50 solution presented above – so that they each receive an extra 12.50 francs, bringing to the totals to 47.50 and 52.50 francs each.

Solution 4: 60 to 40

In an attempt to cope with the unpredictability of the future, the majority of individuals, 11 in total, decided to divide the francs in a 60 to 40 split. Sonya's answer, presented below, accurately represents the approach for 9 of the 11 individuals who answered with a 60 to 40 split.

> Sonya: Because Fermat has 6 possibilities and Pascal only has 4 possibilities to win so it should be divided 60% to 40% and therefore Pascal should get 40 francs and Fermat should get 60 francs.

Given Fermat had eight wins, he only required two more wins in order to achieve his goal of ten wins. Similarly for Pascal: with seven wins under his belt, Pascal only required three wins to achieve his goal of ten wins. At this point, many individuals demonstrated the same approach; they listed all of the possible ways that Pascal could obtain three heads and all of the possible ways that Fermat could achieved two heads. Each participant found four possible ways that Pascal could win (i.e., TTT, THTT, TTHT, and HTTT) and six possible ways that Fermat could win (i.e., HH, HTH, THH, TTHH, THTH, and HTTH). As such, Fermat would have 6 possibilities to win, Pascal would have 4 possibilities to win and, therefore, the pot should be split 60 francs to 40 francs. As mentioned above, this was the most frequent solution.

Interestingly, two of the 11 individuals who responded with a 60 to 40 split arrived at their answer in a different fashion than discussed above.

> Ursula: Pascal has two ways to win: 10-8 or 10-9. Fermat has three ways to win: 7-10 or 8-10 or 9-10. Since there are 5 ways to win Pascal should get 2/5 of the pot and Fermat should get 3/5 of the pot.
>
> Timothy: Because Fermat had one more point than Pascal, Fermat should take 60 francs and Pascal should take 40.

While Ursula's division of the francs is the same as the other 9 individuals who responded with a 60 to 40 split, her reasons for the 60 to 40 split are quite different. Ursula's 3 to 2 ratio happens to be an equivalent ratio for the possibilities of Pascal and Fermat winning in the other nine individuals' responses. Timothy's reasoning was not elaborated upon further and remains unclear to us; however, we suspect that the "one more point than Pascal" changes the split in francs from 50:50 to 60:40.

Solution 5: 31.25 to 68.75

Only responses from 4 out of 30 participants (approximately 13%) were in accord with the conventionally correct probabilistic solution. Amongst the correct responses, all four individuals utilized a similar tree diagram to answer the task. In utilizing a tree diagram, all four individuals' calculations were similar in fashion, as represented with Vincent's response below:

> Vincent: Fermat: 1/4 + 2(1/8) + 3(1/16) = 11/16. Pascal: 1/8 + 3 (1/16) = 5/16. As such, 11/16 goes to Fermat and 5/16 goes to Pascal.

As shown with the recreation of the tree diagram in Figure 1, the solution considered the game in progress and the branch terminates when there is a winner. The 10 endpoints correspond to the 10 options identified below. However, the probabilities assigned to each option are different. The total chances of Fermat (Heads) winning is given by 1/4 + 1/8 + 1/16 + 1/8 + 1/16 + 1/16) = 11/16, attending to the black endpoints (seen in the tree diagram). Similarly, the chances of Pascal (Tails) winning are given by 1/16 + 1/16 + 1/16 + 1/8 = 5/16, attending to the white endpoints (also seen in the tree diagram).

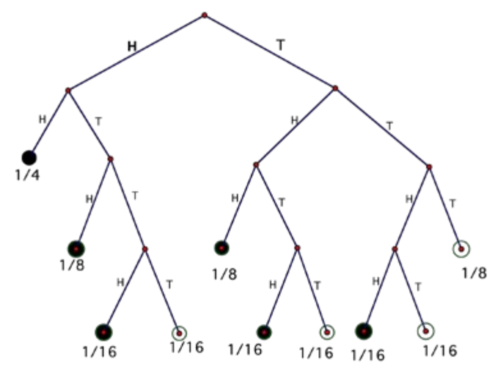

Figure 1. Tree diagram utilized in correct responses

We note that this solution is similar to the one presented in previous section (see, for example, Sonya's excerpt) in identifying 10 possible outcomes. However, the crucial difference is that that Sonya and others mistakenly considered the 10 outcomes as equiprobable, while Vincent and others assigned appropriate probabilities to each of the outcomes. It may be the case that mathematical symmetry (and along with it the solution's simplicity) is lost by considering outcomes that are not equiprobable, i.e., considering the sample set. However, what is gained, and what we intended to discuss with the group, is the principle of starting with learners' ideas and developing from those ideas, rather than rejecting them and imposing a different frame of thought.

Discussion

We started our debrief the next class by presenting all 5 possible solutions and asked participants whether they wanted to abandon their solution and accept a different one. Most participants who initially presented Solutions 1, 2 or 3 switched their preference towards Solution 4, finding it "more reasonable" and "more probabilistic". Participants who initially suggested Solutions 4 or 5 appeared to keep their initial preferences.

Ignoring the notion that the players would normally stop once the game is finished is, arguably, counterintuitive, yet concurrently a key component to detailing the conventional sample space for the Problem of Points. Presenting, or trying to elicit from the participants, the approach of listing all (2^4=) 16 outcomes resulted in considerable resentment. Participants' solutions were based upon the game stopping once somebody had won. Even though four flips would assure that the game ends, in most cases four flips were not necessary; the game can end within two or three steps. Consequently, our listing all approach was deemed not only as inconsistent with "common sense," but also inconsistent with the rules of the game. The students insisted that the game must stop once there is a winner and suggested a different set of possible scenarios. They insisted that the listing of possible outcomes should stop once there are two heads or three tails on the list. This consideration results in the ten options, where Fermat (Heads) is the winner for six options and Pascal (Tails) is the winner for four options as seen in Solution 4, and not the 16 outcomes we had presented.

With us using 16 outcomes for the Problem of Points and the participants using 10, we implemented what we consider a desirable pedagogical approach. To begin, we focused our attention on the 60 to 40 responses to the Problem of Points. The 10 possible ways that the game could finish – the four possible ways that Pascal could win (i.e., TTT, THTT, TTHT, and HTTT) and six possible ways that Fermat could win (i.e., HH, HTH, THH, TTHH, THTH, and HTTH) – were presented on the board. Next we employed an alternative representation of the tree diagram that certain individuals had used in their correct responses to the Problem of Points. We connected the diagram, similar to that on Figure 1, to the 10 outcomes listed by participants, as shown in Figure 2.

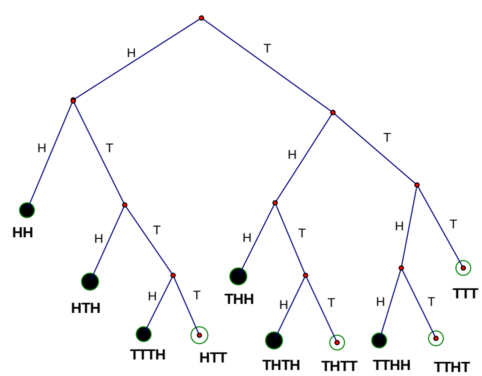

Figure 2. Placing 10 outcomes on a tree diagram

At first, drawing the tree diagram did not help. After all, the 10 outcomes featured in the 60 to 40 decision remained listed, but in a different visual representation. The time came for us as a group to focus on whether or not each of the 10 outcomes was equally likely. As such, participants' attention was drawn to the "length" of each of the branches, which indicated that all 10 of the outcomes presented were not equally likely. Resultantly, the appropriate probabilities of each branch were easily calculated, which led to the conventional solution. Having dealt with the correct mathematical solution to the Problem of Points, the conversation shifted back to the 10 outcomes that were still listed on the board. Having recognized that the 10 outcomes were not equally likely, it soon became apparent what they represented: the sample set. As such, the student-generated sample set was employed to bridge the participants' answers from the 60 to 40, where equiprobability was incorrectly assumed, to the 31.25 to 68.75 response in which different correct probabilities of each scenario are acknowledged. Accompanying our discussion with the participants of, what we now call, the sample set were conversations about the treatment of the sample set, from both a pedagogical and mathematical perspective, as either correct or incorrect.

Conclusion

There is no denying the probabilisitic and historical significance of the Problem of Points, in paving the path of the development of theoretical probability; however, we have also demonstrated its pedagogical significance, in having a considerable number of prospective teachers to "fall into the pit" of popular mistakes and assisting them on their way out. It is a common experience of those working in teacher education that guiding prospective teachers through experiences similar to those of their future students enhances both their mathematical understanding and their pedagogical sensitivity. However, in assuming that teachers' knowledge of the subject matter significantly extends beyond that of their students, such experiences may be not easily designed. We attempted to provide such an experience with the probabilistic task and the notion of sample space, being aware that "teachers may make pedagogical decisions based on their own understanding of the task rather than their students' current understandings" (Stohl, 2005, p. 358). Further, we have presented and detailed particular constructs to aid in our contention that probabilistic reasoning is not detached from subjective considerations. Instruction often asks students to abandon initial/naïve ideas and buy into conventional ones. We suggest that, where possible, learner's naïve incomplete or erroneous ideas can be adjusted and bridged towards the correct conventional ones, rather than rejected and abandoned. Our assessment as well as reaction of prospective teachers experiencing this approach is very positive. As Stohl (2005) contends, "teachers must develop their own understanding of the complexitites of probability concepts" (p. 362). Alternatively stated, "what is good for the geese is good for the gander." However, further research will determine to what degree our constructs will be beneficial for the learning and understanding of probability.

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EXAMINING CONTENT KNOWLEDGE FOR TEACHING IN A FORMATIVE ASSESSMENT PROJECT IN NETWORKED MIDDLE GRADES CLASSROOMS

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Twenty-four seventh-grade teachers participating in a research project focused on formative assessment in a networked classroom were given pre-and post- assessments of content knowledge for teaching. This paper examines several interesting differences in content knowledge for teaching between and among the two groups and suggests possible links between the differences and the content of the two models of professional development in which the participants were engaged.

Background

This paper reports on a three-year research project funded by the National Science Foundation[1] that is investigating feasible models of implementing formative assessment in mathematics classrooms using networked technology. Although proven to be effective, formative assessment has been challenging for teachers to implement in their classrooms (Ruiz-Primo & Furtak, 2006; Shavelson, R. J., Yin, Y., Furtak, E. M., Ruiz-Primo, M. A., Ayala, C. C., Young, D. B., et al., 2006; Yin, 2005). Similarly, using technology to implement formative assessment practices has also been a challenge for teachers (Owens, Pape, Irving, Sanalan, Boscardin, & Abrahamson, 2008). There are different points of view regarding how to assist teachers in developing facility in implementing formative assessment strategies using networked technology, and our group's research was designed to compare the efficacy of training in both formative assessment and technology simultaneously with training in formative assessment strategies prior to implementing networked technologies. The project was designed to examine differences resulting from two different professional development models over a two-year period of time. Input from the eight Advisory Board members along with numerous articles on professional development, formative assessment, technology and related topics (Ayala & Brandon, 2008; Black & Wiliam, 1998a; Black & Wiliam, 1998b; Black & Wiliam, 2005a; Black & Wiliam, 2005b; Gearhart, & Saxe, 2004; Guskey, 2007/2008; Stiggins, 2002, 1997; Stiggins, Arter, Cahappuis and Chappuis, 2004; Wiliam, Lee, Harrison, & Black. 2004; Wiliam. 1999) were taken into consideration. A thorough discussion of the design considerations for formative assessment and technology in a networked classroom is discussed elsewhere (Olson, Im, Slovin, Olson, Gilbert, Brandon, Yin, 2010). The research reported in this paper reflects data on teacher content knowledge for teaching collected during the first year of participant involvement.

Methodology

Thirty-two teachers were assigned to two groups, which for purposes of this paper will be referred to as FA and NAV. During the first summer of the project, teachers in FA participated in five days of professional development on the use of classroom formative assessment. During the same time, teachers assigned to NAV participated in professional development on the use of classroom formative assessment using the TI-Navigator™ networked classroom system[2]. Project facilitators planned the agendas together with the goal of providing as similar an experience with strategies for formative assessment as possible. In all cases where practicable, the same or very similar mathematics tasks were included. The formative assessment professional development in both groups was based on the same framework. Targeted content (i.e., mathematics content addressing several Hawaii content standards) that teachers were expected to teach during the third quarter of the school year was selected as a focus.

Formative Assessment

The formative assessment portion of the professional development was based on the Stiggins (1997) model and also drew from the Ayala & Brandon (2008) emphasis on the ongoing communication aimed at understanding student thinking (and misunderstandings) within the process. Throughout, the project staff emphasized the interplay between teacher and student as the centerpiece of the construct. Figure 1 lists the seven strategies of the Stiggins model for formative assessment. A teacher's ability to effectively implement formative assessment requires facilitating the shift in student's initial understanding towards where she needs to be. The aspect of formative assessment where teachers focus on how students are understanding (or misunderstanding) relies on the teacher's knowledge of the mathematics used in teaching, and led us to the examination of teacher content knowledge reported in this paper.

Where am I going?

1. Provide a clear and understandable vision of the learning target.
2. Use examples and models of strong and weak work.

Where am I now?

3. Offer regular descriptive feedback.
4. Teach students to self-assess and set goals.

How can I close the gap?

5. Design lessons to focus on one aspect of quality at a time.
6. Teach students focused revision.
7. Engage students in self-reflection, and let them keep track of and share their learning.

Figure 1. Seven Strategies of Assessment for Learning

Teacher content knowledge

It is widely established that teachers need to possess a deep and fundamental understanding of the mathematics they teach. For example, when teachers differentiate problems (responding to individual variance) to challenge and/or provide additional scaffolds for students, they use their mathematical understanding to: (1) listen to students' explanations of unconventional solution strategies to determine whether or not they are likely to lead to generalizable approaches, (2) press student thinking through appropriate questioning, and (3) create or select formative and summative assessment problems that are mathematically similar to the work done in class.

Studies in classrooms have found that improving teachers' mathematical knowledge for teaching significantly affects students' learning of mathematics (e.g. Hill, Rowan, & Ball, 2005). While teachers' undergraduate mathematics courses support their learning up to a point, Monk (1994) found that beyond five mathematics courses, the number of mathematics courses teachers had taken less significantly affected students' learning. In fact, Adler and Davis (2006) suggest that advanced courses may encourage teachers' compression and abbreviation of mathematical knowledge. This is problematic, since unpacking mathematical knowledge can provide entry points for students to understand, and therefore is *necessary* for teaching. This *unpacking* has begun to focus research into teachers' mathematical knowledge as it concerns the depth, connectedness, and explicit articulation of the mathematics of teaching (Ball, 2003; Ma, 1999). Knowing how to respond appropriately to students' questions and developing the ability to choose or create questions and problems targeting specific mathematical concepts is at the center of the content knowledge needed for teaching (Ball, 2003). Studies involving teachers of elementary students have found that improving their mathematical knowledge for teaching significantly affects students' learning of mathematics (e.g. Hill, Rowan, & Ball, 2005).

To measure the growth of participants' content knowledge for teaching (CKT), the University of Michigan's *Learning Mathematics for Teaching* (LMT) instrument was administered at the beginning of the summer institute in year one, and then again after one year of participation. The LMT measures have been shown to be a significant predictor of student achievement (Hill, Rowan, & Ball, 2005). The results of these test administrations are reported in the next section of this paper, with our purpose being to examine the aggregated teacher scores as well as the results between the two groups. We will examine the aggregate scores, and will also take a closer look into the differences in pre-and post-test responses for both groups and for the individual teachers.

The items on the LMT test are not released, and so we will describe the mathematical content of the tasks only in general terms, and will not directly refer to any specific calculations or tasks. A similar task could be the *Staircase Problem* shown in Figure 2. There are multiple ways that this problem could be represented using diagrams. Two representations are shown in Figure 3. Suppose that the task was to identify the correct expression that should accompany the diagram in Method *f*. Three possible choices are seen in Figure 4.

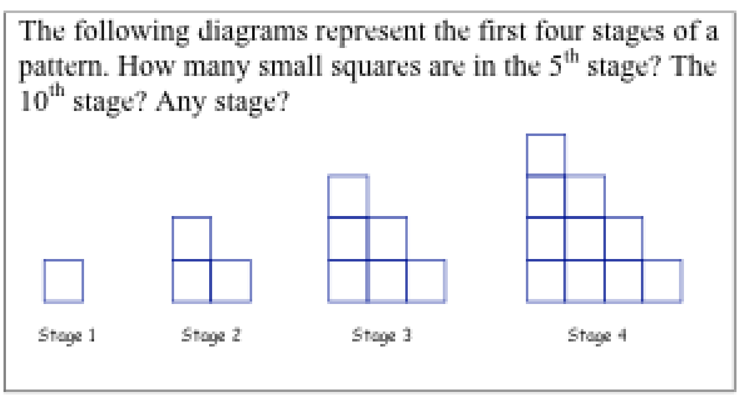

Figure 2. The Staircase Problem

In Method *f*, the area is the sum of the area of the large grey triangle plus the area of the smaller triangles. This can be seen as (1/2) times n^2 plus n times (1/2), since the area of the smaller triangles is 1/2 the area of the n smaller squares. In this case, the correct response would be A. It needs to be emphasized that even though all of the expressions are mathematically equivalent, response B corresponds to Method *g*, and neither response C or B clearly demonstrate the effect that dividing by 2 as shown by the diagonal would have on the diagram.

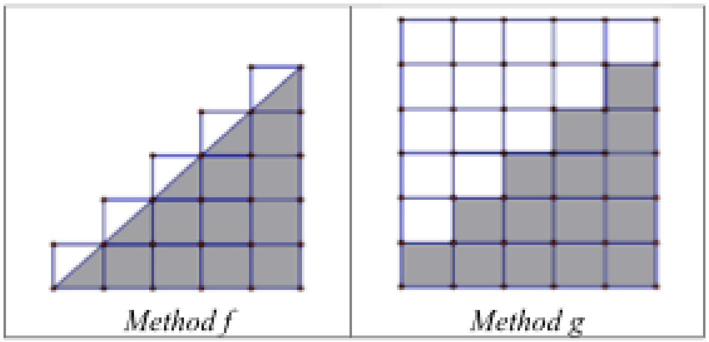

Figure 3. Two possible representations

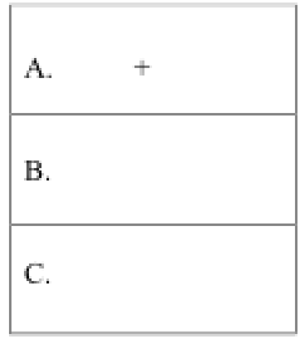

Figure 4

Findings

The results reported here are for the twenty-four teachers (twelve each from the FA and NAV groups) who fully completed the pre-assessment and post-assessment of the LMT instrument. The specific LMT test chosen was *Middle School Patterns Function and Algebra – Content Knowledge*. All items on the assessment are constructed to highlight the content knowledge that is involved in teaching. The content strands of this test include items intended to assess teacher's fluency with determining and interpreting patterns, functions, expressions, equations, and representations. The instrument consisted of 29 responses in the form of multiple-choice questions. The pre-test was given at the beginning of June 2008, prior to any professional development activity, and the post-test was given at the beginning of May 2009 at the final follow-up session for the school year. At this point in the project teachers had participated in five days of full-cohort professional development in June 2008, five half-day follow up sessions during the school year, and at least three coaching visits from project staff. It needs to be noted that not all participants were able to attend every professional development session, although a large majority of participants were in attendance at every session.

Descriptive data for the pre-test and post-test is given in Table 1. The overall score for the aggregated group on the post-test was a respectable 73%, but the overall scores mask some significant differences. For example, on several items, virtually all of the teachers answered correctly.

Table 1. Pre- and Post Assessment Data

Group	Statistic	Pre-test	Post-test
FA	M	19.167	19.5
	SD	4.303	4.275
NAV	M	20.25	22.333
	SD	5.228	4.008

Across both groups, we found that the participants experienced the greatest difficulty on items involving functions, with a particular challenge on characterizing and identifying the different types of functions. This paper will first report on the overall data findings, and then examine the results on the tasks by content strand.

A student pre- and post-test in algebra was also administered, and the teacher's CKT scores proved to be a significant predictor. For each one point gain on a teacher's post-test score, their students achieved 0.448 higher points on the student post-test after accounting for the influence from the other teacher variables (Olson, Im, Slovin, Olson, Gilbert, Brandon, & Yin, 2010).

The mean number of correct responses for the NAV group increased by over 2 points, about 7%, and the mean for the FA group increased by one-third of a point, about 2%. No individual successfully answered all questions correctly on either test. Three FA and two NAV teachers had fewer correct answers on the post-test than on the pre-test. The largest individual gain from pre-test to post-test was 8 and the greatest decline was 6. The mean scores on the tests give an indication that the questions were of sufficient complexity for the teachers.

A look inside the responses by task

It is interesting to portray the differences between the two groups by considering the variation of responses on the individual items. For each group, the sum of the total number of correct responses by item on the pretest was subtracted from the sum of post-test correct answers. Table 2 shows that the sum of the differences of correct answers by item from the pre-test to the post-test was 4 for the FA group and 23 for the NAV group. These large discrepancies can be explained by further disaggregating the data by task.

Table 2. Comparisons of Gain by Task

Group	Statistic	Overall gain of correct responses from pre- to post-test
FA	M	4
NAV	M	23

For the FA group, there were 12 items where the number of correct answers increased on the post-test and 11 items where there were more incorrect responses. No change occurred on 6 items. On the individual test items, the largest gain on the post-test was 5 more correct answers, and the largest loss on an individual item was 4 more incorrect answers. This data is shown in Table 3.

Table 3. Comparison of number of changed answers

Group	Correct answers *increased*	Correct answers *decreased*	No change occurred
FA	12	11	6
NAV	14	7	8

For the NAV group, there were 14 items where the number of correct answers increased on the post-test, and 7 items where there were more incorrect items. No change occurred on 8 items. The largest loss on the individual test items for the NAV group post-test was 2, which occurred on two items, and on one of the items this group had 5 more correct answers.

It is interesting to note the items yielding the largest changes in correct/incorrect answers from pre- to post-test occurred on different items for each of the groups. The largest gains for the FA group were seen on two items that asked for the teachers to select the linear relationship from a list of expressions. There were nine more correct responses, which increased the percentage on the two items from 46 to 84%. For comparison, on the same post-test, the NAV group's scores on these same two items increased by four percent, from 84 to 88%.

The largest post-test gain for the NAV group came on a task that asked if there were no solutions, exactly one, or more than one solution to a compound inequality (there were no solutions). Five more teachers answered the question correctly on the post-test, with the result that nine of the twelve NAV group answered successfully. For this item, the most popular incorrect answer for the NAV group on both tests was to say that there were multiple solutions. Only on this inequality task were there more than three additional correct answers from the pre- to the post-test for the NAV group. The result on this item for the FA group was one correct answer on the pre-test, and three correct responses on the post-test. Of the incorrect answers for the FA group on both tests, all responses were "more than one solution" with the exception of one "not sure" response on both the pre-and post-test.

Analysis of changed responses

Also of interest was the number of answers that were changed from the pre-test to the post-test. These results are reported in Table 4 below. In the FA group, 8 teachers changed over 10 answers from pre-test to post-test while only 3 teachers did so in the NAV group. It also should be noted that not all of those answers were changed from incorrect to correct. In fact, in the FA group, on average, less than half of the answers changed resulted in correct solutions. While the NAV group scored higher, still less than 60% of the changes resulted in correct solutions. It should be noted that both administrations of the test involved the same items.

Table 4. Number of answers changed from pre-assessment to post-assessment

Group	Statistic	Total Number of Answers Changed	Number of Answers Changed from Incorrect to Correct	Number of Answers Changed from Correct to Incorrect
FA	M	10.333	3.923	3.917
	SD	3.750	2.094	2.193
NAV	M	7.833	4.500	2.500
	SD	5.3054	3.205	2.0226

Table 5

Term	1	2	3	4	5
Value	1	3	5	10	15

Item Analysis

There were several individual items of particular interest, which serve to highlight the differences on the pre-test and post-test data between the two groups. Two of the tasks required teachers to differentiate between linear, quadratic, and exponential functions. The first of the tasks asked teachers to identify a growing sequence depicted with square tiles. Table 5 can demonstrate this pattern, although the table representation does not show the pictorial pattern of growth of the function. On the pre-test, the FA group had four correct responses, and three on the post-test, a lower overall score. All four of the correct responses to this item on the pre-test were changed to incorrect responses on the post-test. The three correct responses for this item on the post-test had been incorrect responses on the pre-test. The NAV group had seven correct responses on both the pre-and post-tests. Of the seven correct responses, one teacher changed from incorrect to correct on the post-test, and one switched from correct to incorrect.

The second task asked teachers to identify the growth of a function given a contextual example. In this two-part task, one section featured linear growth and the other was exponential. On the linear part, the FA group had seven correct answers on both the pre- and post-test, and the NAV group had seven correct answers on the pre- and six on the post-test. On the second task section, the FA group had six correct answers on the pre-test, but only 2 on the post. The NAV group had five correct responses on the pre-test and six on the post-test.

On another task, teachers were asked to identify an approach to solving an equation that involved dividing both sides of the equation by the variable *x* and thus division by zero must be considered. On the post-test, the number of correct responses from the FA group decreased from four to one. The NAV group's scores improved from four to six.

The lowest overall number of correct responses for both groups on both tests came on an item similar to *The Staircase Task* in Figure 2 above that asked teachers to identify which expression on a list that matches a visual representation such as what is seen in Figure 3. This item asked teachers to identify expressions that matched an area diagram where one of the dimensions was the variable *x*. An incorrect response on such an item is a nice way to distinguish between simple mathematical content knowledge and CKT. The content knowledge used in teaching is about trying to understand student thinking and having thought about the differences among approaches to solving the problem (all components of CKT). In teaching, this knowledge highlights the differences between the mathematical knowledge used in teaching and common content knowledge, the mathematical knowledge and skills used in settings other than teaching (Ball, Thames, & Phelps, 2008).

The FA group's scores on the pre- and post-test were five and three correct responses respectively. Only one member of the NAV group correctly answered this item on the pre-test, although four correctly answered on the post-test. The salient feature of this task is that the expressions shown were *mathematically equivalent*, but did not represent the situation shown in the diagram.

Discussion

The findings are interesting in light of the content of the professional development that included the summer professional development, follow up, and coaching. While we attempted to maintain consistency in the formative assessment strategies and most of the mathematical tasks provided for participants, these results give cause to consider how the two models of professional development affected the resulting differences in the results on the teacher content knowledge assessment. Certainly, the use of the graphing features using the TI-Navigator system highlighted the tabular and graphical representations of functions and easily allowed for an examination of a variety of linear functions and expressions. The teachers in the FA group did not have any experiences with TI-Navigator, and perhaps that is why there was a decrease of 4 correct answers when asked to identify functions for a growing sequence and a decrease of 4 for identifying an exponential function, yet this did not happen for identifying a linear sequence.

As the professional development spent significant effort to discuss equivalent expressions, it was surprising when the teachers in both the FA and NAV group were not able to successfully answer that question.

Teachers' overall averages on the post-test were respectable. While there were a large number of answers changed from pre-test to post-test, less than 60% of those changes resulted in changing an answer from incorrect to correct. In the FA group, on 12 of the 29 items there was an increase in average correct, while on 11 of the 29 items the average declined from pre-to post-test. For the NAV group, these numbers were 14 and 7 respectively. While teacher content knowledge has an impact on student achievement, less is known about organizing professional development to expand teacher knowledge. It appears that the NAV model was more useful in improving overall teacher knowledge and also resulted in fewer teachers changing answers from correct to incorrect. Further work is needed to determine what parts of the professional development were helpful so that teachers made changes from incorrect to correct as well as examining what might have impacted those teachers who changed from correct to incorrect. Perhaps the emphasis on different reasoning strategies and solutions might have hindered the FA group while multiple representations might have helped the NAV group.

Endnotes

1. The research reported in this paper was funded by National Science Research and Evaluation on Education in Science and Engineering (REESE) program. The views expressed in this article are the views of the authors and do not necessarily represent the views of the National Science Foundation.

2. TI-Navigator™ is a networking system developed by Texas Instruments that wirelessly connects each student's graphing calculator to a classroom computer.

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EXPLORING TEACHERS' MEASUREMENT DIVISION KNOWLEDGE

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This study provides fine-grained analysis of teachers' reasoning as they engage with quotative division tasks. We consider teacher knowledge at a fine grain in an effort to examine the content knowledge teachers have that may be important for student learning. This qualitative analysis considers the mathematical knowledge that six middle-grade teachers used in a professional development program and during interviews. In all of the items, fraction division situations were approaches through linear or area models. Our analysis focused on three knowledge components critical for reasoning about measurement fraction division situations: partitioning operations, flexibility with referent units, and interpretations of the measurement division model.

Introduction

In this study, we examined the knowledge components that teachers use as they engage in measurement division tasks that involve fractions. Our interest is in examining teacher knowledge and the ways teacher understanding develops over time. Specifically, we consider three knowledge components teachers use in approaching measurement division situations.

Over the past few years, researchers have built from Shulman's (1986) proposition about different kinds of knowledge teachers need to be successful in their practice. Shulman's concept of pedagogical content knowledge (PCK), specifically, has led to ongoing discussion about the unique knowledge teachers need in order to support student learning. In last two decades, researchers have expanded the notion of PCK through the development of fine-grained conceptualizations of this knowledge for teaching mathematics. Ball and her colleagues (e.g., Ball, Thames, & Phelps, 2008) have sought to define the specialized knowledge teachers need that the 'man on the street' does not. They have named this body of knowledge *mathematical knowledge for teaching* (MKT). While we recognize there are multiple ways to frame this specific body of knowledge, our interest in this paper is confined only to a sliver of the work. We are specifically interested in the specialized content knowledge teachers have related to quotative fraction division and the affordances and limitations of that knowledge in terms of its potential for supporting student learning.

Previous research on teacher understanding of fractions has been somewhat limited. We know from research focused on preservice and inservice teachers, that their conceptions of fraction division are lacking (e.g. Ball, 1990; Simon, 1993; Ma, 1999; Tirosh, 2000; Li & Kulm, 2008). Specifically, we know that teachers, both preservice and inservice, struggle to provide appropriate word problems for division situations, instead they provide word problems that focus on multiplication by a fraction. For example, in an example such as 2 ÷ ½, teachers suggest examples involving making half a batch of a recipe rather than focusing on questions about how many groups of one-half are in two (Ball, 1990; Simon, 1993; Ma, 1999). These shortcomings are important as demonstrated in one case study by Borko et al. (1992). In that case, a preservice teacher was unable to answer basic questions from students about why the invert-and-multiply algorithm works resulting in an explanation that actually focused on fraction multiplication. Ma (1999) argued that teachers need a profound understanding of fraction division and found that one or more pieces of knowledge necessary to teach fraction divisions were missing in her sample of US teachers. In short, prior research has uncovered that teachers have significant gaps in their understanding of fraction division, however, fine-grained work has not yet been undertaken to understand how teachers reason about division with fractions. Since teachers' MKT enables or constrains their ability to orchestrate mathematical discussions that provide students with opportunities to develop meaningful understanding it is important for teacher educators and professional developers to understand how teachers know mathematics (Silverman & Thompson, 2008). With this regard, we tried not only to understand the teachers' constraints but also to determine the basis for their constraints.

For our analysis, we drew from the ideas that emerged from the Fractions project (see below), which focused on children's' knowledge of fractions and their operations. The analysis framework from the fractions project allowed us to consider the kinds of compartmentalized knowledge focused on various stable operations that had been seen in children. In our analysis, we also saw some aspects of this knowledge in our teachers.

Using this model, we previously considered teachers' reorganization of their measurement division (MD) knowledge by refining their partitioning operations and units through the sequence of problem situations in which the relationship between the dividend and the divisor becomes more complex (Lee & Orrill, 2009). This study extends that work by focusing on six teachers participating in a professional development course (PD). While it is beyond the scope of the article to describe Steffe's fraction schemes in detail, we offer a brief description of a few schemes that we observed from our teachers under the following framework.

Theoretical Framework

The perspective of knowledge we use in this research draws on literature on cognition in mathematical and scientific domains (e.g., DiSessa, 1993; Izsák, 2008). The perspective of knowledge suggests that developing accounts of knowledge used by learners requires inferring diverse, fine-grained knowledge elements that they activate and how those elements function together as the learner tries to accomplish particular goals. In this study, the analysis led to three categories that serve to highlight the diversity of knowledge elements that our participants used – partitioning operations, flexibility with referent units, and interpretation of measurement division situations (how many groups of divisors in dividend versus using repeated subtraction).

The ideas of iterating and partitioning have been determined to be fundamental for children's fraction knowledge development (Piaget, Inhelder, & Szeminska, 1960), and Kieren (1976) provided a logical analysis of the foundations of rational numbers that considered the notions of unit and partitioning as mechanisms for developing knowledge of fractions. Building on the frameworks of Piaget et al. and Kieren, scholars in mathematics education have explored the relationship between partitioning and units with fractions (e.g., Behr, Harel, Post, & Lesh, 1992, 1994; Confrey, 1994; Mack, 1990, 2001; Steffe & Olive, 2010). They have provided insights into how students draw on their knowledge of partitioning to reconceptualize and partition different types of units to solve problems involving multiplication of fractions. In particular, students' use of partitioning was influenced by their ability to view fractional parts in relation to the referent, or unit, whole. We consider this ability keeping track of *referent unit (or whole)* in this study. In fraction multiplication, viewing fractional parts in such a manner helped students to consider a referent unit and to partition that unit by forming iterable units or splitting quantities in ways that reflected the multiplicative nature of the problem (Steffe & Olive, 2010).

By adopting a view of fractions as *measurement units*, teachers are able to reason about fraction division situations in meaningful ways. For example, teachers can think of a problem like 2/9 ÷ 1/3 as asking, "How many thirds are in two-ninths?" In our previous work, we found teachers needed to attend to both the correct referent unit and they needed to develop flexibility with it as the relationship between the dividend and the divisor got complex (Lee & Orrill, 2009). One focus of the PD was on developing this kind of flexibility with referent units. For example, focusing on a teacher's ability to keep track of the unit to which a fraction refers (e.g., in a problem such as 1/5*1/4=1/20, the 1/4 refers to a whole, but the 1/5 refers to a portion of the 1/4) and to shift their relative understanding of the quantities as the referent unit changes. We observed that when teachers did not explicitly attend to the referent units, they were unable to identify diagrams that best modeled the problem. Based on our analysis, however, we assert that having this flexibility is necessary but insufficient for quotative division. Rather, it needs to be paired with refined partitioning operations.

Common partitioning is defined as an operation used to re-partition a partitioned bar until a common number is found (Steffe & Olive, 2010). It accompanies children's explicit attention to the *co-measurement unit,* which is a measurement unit for commensurable segments, which are segments that can be divided by a common unit without a remainder. In our study, teachers used this operation when the divisor could not clearly measure out the dividend (i.e. denominators are relatively prime). For instance, they could coordinate and compare two composite units, such as 3s and 5s for 1/3 ÷ 1/5, to determine a common multiple, 15. Steffe and Olive described *cross partitioning* when they observed children solving the problem $\frac{1}{3} \times \frac{1}{5}$ by partitioning a bar vertically into three parts and horizontally into five parts and get 15 of one-fifteenth automatically. In order for children to calculate the correct answer to the problem, they needed to know that the quantities of the resulting fractional part was in reference to one whole not one fifth. Steffe (2004) stated that the use of a cross partitioning operation in a fraction multiplication context differed from the common partitioning operation in that the former provides a simultaneous repartitioning of each part of an existing partition without having to insert a partition into each of the individual parts. Even though cross partitioning is the fundamental operation for producing *fraction composition scheme*, which is a multiplying scheme for fractions, we observed some interesting consequences in applying it to our teachers' reasoning about division. Two mediating factors seemed to shape their approach. First, we observed that cross partitioning in division is inherently different from in multiplication because of the different roles referent units play in the two operations. Second, our teachers had already mastered 'invert and multiply' algorithm, which shaped their relationship to the operations.

Methods

This study is part of a larger project, Does it Work (DiW), which is concerned with what teachers learn in a professional development experience and how that learning translates into classroom practice.

In this analysis, we focused on six teachers, three sixth grade and three seventh grade, who participated in PD that met three hours one evening per week for 14 weeks . Of the fourteen teachers in the courses, eight were selected to participate in in-depth interviews about three assessments given as part of the research effort. Of those teachers, the six who demonstrated sophisticated reasoning about fraction division were considered for this study. The other two teachers relied strategies such as finding the drawing that matches to numerical calculation or identifying requisite features that were not grounded in quantitative reasoning.

The PD focused on fraction and decimal operations as well as proportional relationships. Each class meeting was videotaped using two cameras and the resulting video was combined into a restored view (Hall, 2000) using picture in picture technology. For this analysis, we focused on the two class meetings concerned with measurement (quotative) fraction division. In these two sessions, the teachers engaged with several tasks that relied on number line and area model representations and participated in discussions about using representations for fraction division.

Our analysis also considered participants' responses to two division items on the written assessments that were administered as part of the larger project. One of the items focused on using an area model to illustrate quotative division of fractions while the other was a number line item that implied partitive division (e.g., answering the question, 'what number is ½ one-fifth of'), but that our six participants approached quotatively (e.g., answering the question, 'how many 1/2s are in 1/5'). Because our assessment items are secure, all descriptions depict similar items to those actually used. In the interviews about these assessments, we asked teachers to discuss their reasoning for the choices they made on each item. In this way, we were able to gain insight into the mathematics that teachers used to analyze the tasks. Each interview was videotaped using two cameras as with the professional development sessions. Additionally, the interviews were all transcribed verbatim. As part of the data analysis, lesson graphs of the interviews were also created to allow easy reconstruction of the interviews.

Data analysis occurred in stages. The first stage was ongoing analysis throughout the implementation of the professional development course. Both researchers and the PD facilitator debriefed at the end of each session. Our discussion focused on how the participating teachers were making sense of the content and planning for future sessions. Immediately after each session, the facilitator (who was also a member of the research team) created annotated timelines of each session. These summaries provided written description of teachers' mathematical activities and interactions with the instructor as well as emerging key points in teachers' reasoning that were taken into account for the next session. The second stage of analysis was retrospective analysis. We worked from our initial lesson graphs that outlined the key incidents of the class meeting to identify key episodes. Then, we memoed (Strauss & Corbin, 1998) teachers' mathematical reasoning to determine which categories of fraction reasoning were important for teachers working in measurement division contexts. The three emergent categories that were at play in the teachers' reasoning were: (1) partitioning operations, (2) flexibility with referent units, and (3) the interpretations of measurement division model. In the third stage of analysis, we categorized our six teachers' reasoning in terms of the categories, and generated hypotheses that were then united into comprehensive accounts. We then compared the generated accounts of our hypotheses of teachers' reasoning with each of the interview lesson graphs. Specifically, we noted every time a teacher showed indications of each of the three categories, and took notes on lesson graphs of our inferences. Finally, we generated inferences about teachers' reasoning under the three categories of knowledge components.

Results

To summarize our main findings, while only two teachers used measurement models for division at the outset (Claire and Donna), all six used quotative interpretations by the end of the professional development. The six teachers' approaches to measurement division differed in terms of partitioning operations, ability to keep track of the referent unit in a flexible manner, and ways of interpreting quotative division problems. Also while children used cross partitioning operations to find a common multiple of two fractions, our teachers modified the approach. They used it to get the same denominator as using the 'invert and multiply' algorithm.

To highlight teachers' thinking in relation to cross partitioning and keeping track of the referent unit, we consider Walt and Claire. Both were able to interpret the area model item on the assessment, which showed the case where they could measure the dividend by the divisor such as 1/3 ÷ 1/9. However, when asked to determine a number line model that relied on a partitive interpretation to show 3/4 ÷ 1/3, both experienced difficulties and changed to a quotative division model. Claire drew an area model and cross partitioned the bar into four columns by three row pieces. Then she shaded three columns to represent 3/4, and shaded in four pieces of 1/12 much darker than the three fourths (Figure 1a). After she reminded herself to find the number of one thirds in three fourths, she measured out the number of four twelfths (=1/3 of 1) from three fourths, and told us that 2 and 1/4 groups of 1/3 could fit into 3/4. She correctly chose the referent unit of one third as one and she recognized that the 1/12 remaining after she counted two sets of 4/12, was 1/4 of 1/3. When teachers confound the referent units in this case, often they would consider one third in reference to three fourths (Figure 1b & 2b), as in multiplication where the goal is to identify parts of parts (e.g., 1/3 of 3/4), and then count the number of three twelfths (=1/3 of 3/4) inside three fourths, and get 3 as a quotient for 3/4 ÷ 1/3 (Figure 1b).

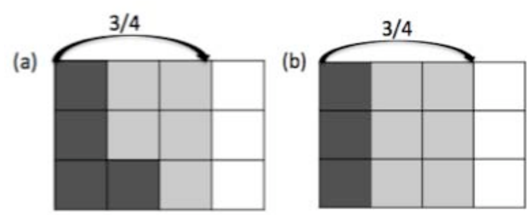

Figure 1. (a) Reconstruction of Claire's area model for solving 3/4 ÷ 1/3, (b) Teachers' Common errors for showing 3/4 ÷ 1/3

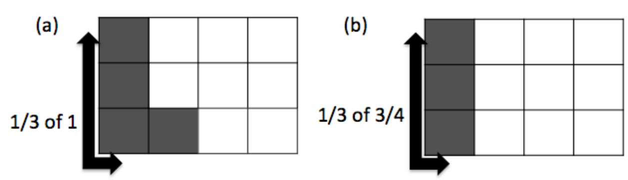

Figure 2. (a) Correct choice of 1 as a referent unit to 1/3, (b) Incorrect choice of 3/4 as a referent unit to 1/3

As described above, Claire was able to appropriately reason with an area model. However, in the post-post assessment interview, she demonstrated constraints in her common partitioning operation. She was able to break the number line into four parts and identify the length of three fourths, and then she repartitioned the same number line into three parts without considering the four parts that were already divided (Figure 3). We suggest that this indicates that she was unable to coordinate three-fourths and one-third simultaneously on the number line using common partitioning. She was not able to replicate the common partition from the area model in the number line situation. This was particularly interesting given that she had been able to do this in the first post-assessment the post-assessment interview.

In contrast to Claire, Walt was able to generate a number line using a common partitioning operation but failed to apply referent unit reasoning to his interpretation of the quotient. Actually, he first showed us the drawing similar to Claire's, but when we asked him how he would determine the answer to 3/4 ÷ 1/3 with the model, he told us that he needed to find the common denominator, 12, of 3/4 and 1/3. He then drew a second number line (Figure 4), divided it into twelve parts, and marked every fourth and every third using a curved line. Then, he counted the number of one-thirds that fit into 3/4. Once he counted the two 1/3s, he struggled with the remaining part, which was 1/3 of 1/4 of the whole. The remaining part would be referred to as 1/4 in the quotient because it is 1/4 of another 1/3 piece that would fit into the original quantity 3/4. When asked about the common partitioning, which was not adequately supporting his reasoning, Walt explained that he had calculated the common denominator to determine how to partition the line. Walt was able to generate a correct response to this item, however, his reasoning was not sophisticated compared to an approach that reasoned through the common partitioning and demonstrated the ability to keep track of the referent unit.

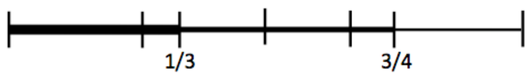

Figure 3. Reconstruction of Clair's number line model for solving 3/4 ÷ 1/3

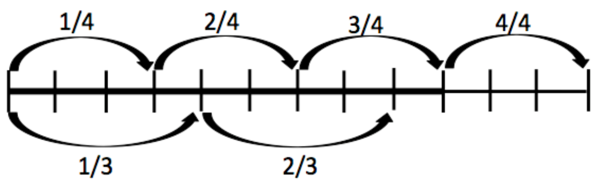

Figure 4. Reconstruction of Walt's model for solving 3/4 ÷ 1/3

The cases of Walt and Claire highlight the necessity for common partitions and their reliance on drawing from what they already know. Specifically, we Claire used area models and Walt relied on common denominators. These pre-existing understandings were insufficient for generating appropriate answers. In Claire's case, the area model did not allow her to apply an understanding of common partitions to a number line model, whereas Walt's common denominators did not support him in readily making sense of the number line.

Carrie and Donna demonstrated models other than measurement division in their work. Both Carrie and Donna relied on a repeated subtraction model of division in the posttest interview and Donna demonstrated that understanding in the course and in other interviews as well. This led to limitations in interpreting situations where the divisor was smaller than the dividend. Carrie's understanding was further limited by her inflexibility in interpreting referent units. In fact, Carrie often discarded referent units to simplify the problem. She had no sophisticated way to reason about the quotient in a division problem because she would not consider the whole, only the parts of the problem.

In summary, we saw that simply developing a measurement model for division was insufficient for this professional development work. The teachers were all able to move toward measurement division models, yet most experienced significant limitations in their abilities to reason about the division situations. Particularly important in this work is the realization that, unlike children, the teachers relied on the algorithm to identify a common denominator to re-partition the number line before they attempted to reason with the drawn quantities. In fact, some teachers were unable to identify the common partition leading them to fail at operating on the number line. This pre-existing procedural knowledge suggests that teachers' knowledge development trajectory might be different from children's.

Conclusions and Implications

Conceptual analysis of teachers' knowledge at this grain size allows us to develop an understanding of teachers' capacities to reason about fraction division. While this report offers only a glimpse into teachers' reasoning, it raises important issues. We note that the categories from research on students are valid lenses for considering teacher knowledge. However, there are limitations in relying on research about children's' reasoning to make sense of teacher knowledge. In this case, the teachers had strong procedural knowledge that helped them work around some operations, such as common partitioning. This is an issue because, as noted by Steffe and Olive (2010), common partitioning is a key operation for constructing more advanced operations. Thus, students need opportunities to engage in activities that support the development of this operation, however, teachers who cannot reason with it are unlikely to incorporate common partitioning into their students' classroom experiences.

Most studies have not looked at teachers' knowledge any closer than simply saying that teachers' reasoning about fraction measurement division is insufficient. While the studies have provided the pioneering work in identifying knowledge teachers lacked (in this case, fraction division), more studies are needed to uncover how teachers can reason about fractions and how we support them in developing their knowledge for teaching. That is, teachers need to have opportunities to learn mathematics, to understand how key understandings can empower their students' learning, and the actions they, as teachers, need to take to support students' development of understanding (Silverman & Thompson, 2008).

By building on understanding of teachers' mathematical concepts and operations in the ways illustrated here, we propose that a richer understanding of teachers' understandings was identified, which could be used as the foundation for developing stronger professional learning opportunities for teachers.

Endnotes

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INFORMING INSTRUCTIONAL DESIGN: EXAMINING ELEMENTARY PRESERVICE TEACHERS' STRATEGIES IN A PARTITIVE QUOTIENT PROBLEM

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In order to demonstrate a thorough understanding of rational number, learners need to understand each of the subconstructs of fractions, including the partitive quotient. This paper explores elementary preservice teachers' understanding of the partitive quotient subconstruct of fractions by examining their strategies presented in whole group and the discussions that ensued. Based on the analysis, we reveal some blind spots in terms of PSTs' understanding of the partitive quotient subconstruct, and propose implications for the design of instruction.

Introduction

Researchers generally agree that teachers need a robust understanding of the mathematics content they teach (RAND Mathematics Study Panel, 2003). Although there is less agreement about the precise nature of the mathematics knowledge teachers need to possess (e.g., Ponte & Chapman, 2002), researchers have increasingly emphasized that teachers need to know how and when to use procedures, the concepts underlying different procedures, and how various concepts are related to each other (Kilpatrick, Swafford, & Findell, 2001; Ma, 1999). Specifically in the domain of rational numbers, however, much of the extant research highlights limitations in preservice teachers' (PSTs) understanding of rational number concepts and procedures (e.g., Newton, 2008; Tirosh, 2000; Ball, 1990; Li & Kulm, 2008; Behr, Khoury, Harel, Post, & Lesh, 1997). Despite this growing body of research, little is known about PSTs' understanding of what some researchers argue is the foundational rational number subconstruct - namely the partitive quotient subconstruct.

Defined generally as fraction division, Kieren (1976) and Freudenthal (1973) argue that the partitive quotient subconstruct is foundational to a more holistic understanding of rational number. As fraction division is an important topic in elementary mathematics, albeit conceptually challenging to understand (Ma, 1999), it is important to determine the extent to which PSTs are mathematically prepared for some of the instructional tasks they will encounter as classroom teachers (Eisenhart et al., 1993). Teaching mathematics for understanding requires considerable knowledge and skill on the part of the teacher (Even & Ball, 2009). Building on previous work focused on young children's partitioning strategies used in partitive quotient (PQ) problem situations (Charles & Nason, 2001; Lamon, 1996), the authors examine four typical PSTs' PQ problem strategies presented in class and the follow-up discussion that ensued around those strategies, and discuss how the ways in which they partitioned, quantified, and ultimately discussed the strategies reveals the extent of their ability to build and understand the PQ subconstruct from the problem situation. Our aim in this paper is to inform the design of instructional tasks related to the PQ subconstruct to be used with PSTs, thus illustrating the process by which a math teacher educator may go through as they enact a task and examine PSTs' work on the task in order to inform subsequent instructional decisions.

Framework

Building the PQ Subconstruct

Within the domain of rational numbers, researchers have identified five subconstructs that constitute the rational number construct: part-whole, operator, decimal fraction, PQ, and measure (Kieren, 1976; Post, Behr, & Lesh, 1982). See Behr, Lesh, Post, & Silver (1983) for a discussion of the relationship between these different subconstructs. The PQ subconstruct of rational number is defined as the quotient resulting from partitive division such that given a quantity x (the dividend) and a quantity y (the divisor), when x is divided into y shares, the resulting amounts to be shared are of size x/y (Kieren, 1976). In other words, the idea underlying this procedure is to consider a PQ problem situation in terms of partitive division and to then consider the results of the division as a fraction. The PQ subconstruct also allows for quantification of the amount resulting from dividing a quantity into a given number of parts (Kieren, 1980).

Indeed, researchers argue that, for children, the major cognitive structure underlying the PQ subconstruct is partitioning, which is the ability to divide an object(s) into a given number of equal parts (Kieren, 1976). Charles & Nason (2001) argue that partitioning strategies have to generate equal shares and the equal shares need to be quantifiable. That is, partitioning should facilitate two conceptual mappings: (1) mapping the number of people (y) to the fraction name (yths), and (2) mapping between the number of objects to be shared (x) and the fraction name (yths). If these conditions are not met as part of the partitioning component, then learners do not abstract the PQ subconstruct even if they are able to complete the task, thus solving the task procedurally without demonstrating a more conceptual understanding of the underlying subconstruct.

Beyond partitioning, quantification is a centrally important component that helps build up the PQ subconstruct. Quantification generally involves the aforementioned conceptual mappings that are often implicit in partitioning. Thus, a strategy that facilitates building of the PQ subconstruct must fulfill the following requirements: the outcome of the partitioning component needs to facilitate the aforementioned conceptual mappings (i.e., mapping the number of people (y) to the fraction name (yths), and mapping between the number of objects to be shared (x) and the fraction name (yths)); the outcome of the quantification component should reflect these two mappings.

However, as compared to children's dependence on partitioning in order to solve PQ tasks, (e.g., Charles & Nason, 2001; Lamon, 1996) we argue that PSTs employ quite diverse strategies, not necessarily partitioning, and rely on their prior knowledge of other fraction subconstructs and fraction operations in order to solve a PQ task. For example, PSTs may be able to employ a partitioning strategy that facilitates the two aforementioned conditions, but are unable to generate the correct fractional answer because of their limited understanding of fraction operations (Li & Kulm, 2008; Ball, 1990). Given PSTs' prior knowledge of fraction concepts and operations, we hypothesize that PSTs will not rely solely on partitioning strategies or their understanding of fractions, but also on their prior knowledge of fraction operations.

Given the centrality of rational numbers and operations to the elementary school curriculum, it is important for teachers to have a thorough understanding of the concepts and operations within this domain of mathematics. Although much of the extant research explicitly involving the PQ subconstruct focuses on children's strategies employed in such problem situations, this body of research provides a foundation for understanding the nature of PSTs' knowledge in this domain given their often limited knowledge of such concepts as documented in past research (e.g., Ball, 1990; Newton, 2008; Tirosh, 2000). The authors hypothesize that while some of strategies employed by PSTs will be similar to those employed by young children, other strategies will also be employed given PSTs' prior knowledge of rational number concepts and operations.

Methododology

This study took place in the context of a required mathematics content course for elementary PSTs at a large public university in the Midwestern United States. The content course is designed around developing PSTs' mathematics knowledge needed for teaching (see Ball, Thames, & Phelps, 2008). The content of the course includes place value and number operations, rational number concepts, and aspects of number theory, and provides PSTs with opportunities to develop their abilities to engage in mathematical tasks central to teaching. The learning environment in the content course is largely collaborative in nature, where PSTs are encouraged to pose questions, share their ideas, as well as understand and discuss other people's strategies. The classroom excerpts used in this analysis were selected from two sections of the content course that were offered in the Fall 2008 semester. The task used as part of this study was specifically designed to deepen PSTs' understanding of the PQ subconstruct in a way that PSTs could engage with the more conceptual aspects of this subconstruct as opposed to simply solving naked fraction division problems (e.g., ½ ÷ ¾). Called the Cake Problem, the task reads as follows:

> *Roger has 3 cakes that he wants to share equally among 5 friends.*
> *How much cake can he give to each friend?*

An important aspect of the task design was to create a potentially unfamiliar context for PSTs, one that would highlight for the instructor where PSTs were in their ability to build the PQ subconstruct. Thus, the task was worded in such a way so as to not trigger for PSTs that a fractional answer was required. PSTs were asked to solve this problem first individually, then discuss their work on the task in small groups, and finally discuss their strategies with the whole group. As they worked on the task, instructor feedback was limited to probing PSTs to explain their thinking for purposes of making their thinking about the task explicit.

Results

PSTs' work on the task is taken from transcripts of videotapes of small and whole group discussions, chalkboard pictures of their strategies, as well as from their written solutions. The aim of this analysis is not to draw conclusions about individual PSTs' understanding, but rather to examine the strategies employed and discussions around these strategies for purposes of informing the design of instructional tasks involving the PQ subconstruct.

Various strategies were identified in the analysis. For example, distribute partitioning strategy (Lamon, 1996), chunking partitioning strategy, preservation of pieces partitioning strategy (Lamon, 1996), and proceduralized partitive quotient strategy (Charles & Nason, 2001). Based on the data, the authors found that some PSTs were able to solve the problem using their own strategies, but became easily confused by other strategies, were unable to make connections between different strategies, and were unsure why other strategies were valid. The four most common strategies and PSTs' discussions around these strategies are analyzed in the following excerpts:

Excerpt 1: 1/5 +1/5 +1/5 =3/15?

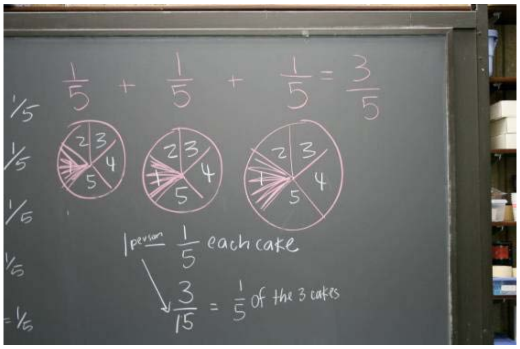

Figure 1. Distribute Partitioning Strategy

In her explanation of the distribute partitioning strategy (Figure 1) during the whole group discussion, one PST drew three circular objects, then divided each object into five pieces, "…because there are 5 people." She then shaded one part from the first object, another part from the second object, and another part from the third object, totaling three parts. She stated, "Okay, so that's one person [share]." She proceeded to number each part in each whole object in order that each person received three parts (i.e., one part from each object). This is illustrative of the distribution partitioning strategy whereby PSTs share out one part from each object to each person, and is similar to Lamon's (1996) distribution strategy. The PST stated, "Each person gets 1/5 of each cake." Another PST interprets the same picture differently. She stated that this is the same thing that she did. What is different is that she counts how much cake each person received by considering 3 pieces out of 15 pieces.

Following this discussion, one PST (Student 1) proposed a connection between these two explanations. However, her classmates largely reject it:

Student 1: Well if each cake is broken up into fifths...like so, like each person gets one fifth. It's just one fifth plus one fifth plus one fifth, which you know would be like three fifths. So if you think about...

Students: It should be three fifths.

Student 2: You don't add the denominator because they have common denominators.

…

The two explanations presented here are evidence of building of the PQ construct, as it facilitates the two mappings. However, Student 1's failure to connect these two strategies shows limited understanding of the PQ subconstruct. She did not see the two different units that are used to represent the fractions, which is the critical point underlying the connection that is implicit in Student 1's strategy. That is, to understand the relationship between these two explanations, PSTs need to realize a reunitizing process that was implicit for most of them. Additional evidence to support this claim can be found from other students and Student 2, who refuted Student 1 by pointing out the answer is wrong: 1/5 + 1/5 + 1/5 is 3/5, not 1/5, without paying attention to the units involved in these two different strategies and the reunitizing process that connected them.

Excerpt 2: Some people got 3/5 of a cake, and some people got 3/5 of two cakes?

PSTs who utilized the chunking partitioning strategy (Figure 2) partitioned each object into five parts first. However, instead of distributing a single part from each whole object to each person, PSTs who utilized this strategy denoted three consecutive parts from each object as each person's share. Then, they determined that each person's share consists of 3/5 of a cake.

The partitioning of the cake in the chunking strategy is different from the distribution strategy. The difference resulted in a confusion with the quantification of these two different partitionings. After the presentation, one PST showed her confusion when she asked how each person's share from different cakes is the same as taking each person's share from the same cake.

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

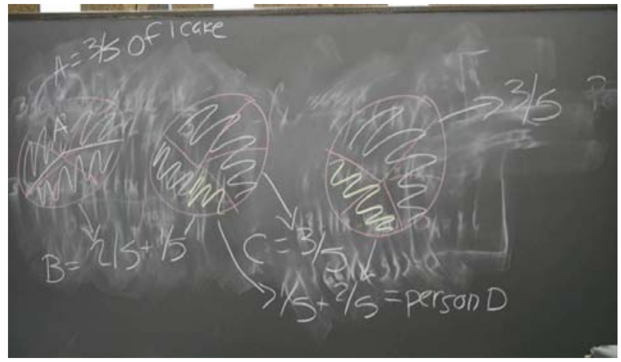

Figure 2. Chunking Partitioning Strategy

Student 3: Um, I don't know what we were talking about over here we can't have three fifths of a cake because we would have some left over like.

Student 3: Like you can't get three fifths of a pie (cake) because there's going to be two pieces left over and you have to borrow a piece from the other pie (cake).

Teacher: So you're saying some people are going to have to get cake from two different pieces from two different cakes.

Student 3: Yeah everyone can't get three fifths of a pie (cake).

Student 4: Unless you do it that way. (Referring to the distribute partitioning strategy.)

Student 3: Yeah like that. ((3 sec pause)) I don't know cause they can say that three fifths is three fifths of a pie (cake) and that's not of a pie (cake).

Student 3: Because everyone can't get three fifths of each of the cakes. I mean they can because there's like two they are going to have to borrow one from the other cake so it is like two cakes.

There are two ways to quantify the chunking partitioning strategy. The strategy represented on the board quantified by dividing a unit into fifths and designating 3 (3/5), which is the same as the quantification involved in the distributing partitioning strategy. Another way is dividing 3 units into 5 parts (5 different chunks). These two quantifications result in the same quantity. However, according to Kiren (1980), this presents different problems for the learner, which is also illustrated by Student 3’s confusion.

Excerpt 3: I don’t know, but I feel like visually this is cool.

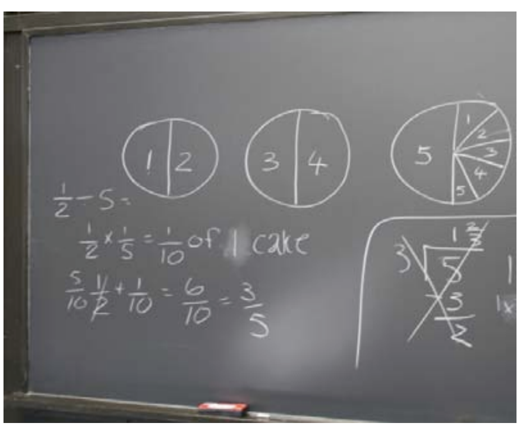

Figure 3. Preservation of Pieces Partitioning Strategy

The PST (Student 5) who presented this preservation of pieces partitioning strategy (Figure 3) drew three whole objects, divided each object in half, which resulted in six half objects. She then distributed the six halves to each person with one half-object remaining. This partitioning shares many common features with Lamon’s (1996) preservation of pieces partitioning strategy in that PSTs attempted to make the fewest number of cuts possible. She continued the partitioning by subdividing the remaining half object into five pieces, and then sharing each fifth to each person, as indicated by their numbering or shading of each fifth. When she quantified the remaining half object that was sub-divided into five parts. For example, in order to determine the size of the five small parts in relation to the whole object, the PST multiplied 1/5 by 1/2, which resulted in 1/10. Finally, in order to quantify each person's share, she added the 1/2 cake and the 1/10 together using fraction addition, which resulted in an interpretation of the answer as 3/5 of a cake in each person's share.

Although the PST who utilized this strategy came up with the correct answer in the context of the problem, she did not complete the necessary conceptual mappings as part of their partitioning and quantification components.

> Student 5: Okay so everyone got half a cake and then we had one half of a cake left over so we divided that by five people also. So everyone's going to get half a cake and then whatever they get from here, which is one piece. So we want to know how much is each piece so we did half divided by five people and which is like half times one over fifth. So each one gets one tenth of that (remaining) half.

As illustrated above, when she was representing, she explained that each one gets one tenth of that (remaining) half. Then she added 1/2 of a cake and 1/10 of a half together (Figure 3 was the final manifestation of her work and different from what she originally wrote down). Obviously, the quantification does not reflect the partitioning.

Moreover, due to the complicated partitioning, PSTs were not able to explain whether this strategy is valid. As Student 6 commented below:

> Student 6: I feel like visually this is cool. This is kind of interesting but with numbers it's hard to explain I feel.

Again, as with the chunking partitioning strategy, the essential quantification that dividing 3 units into 5 parts is implicit. Each of the 5 parts are made of 1/2 of a cake and 1/10 of a cake. PSTs were able to do the addition without recognizing what the addition really means in the context of the strategy.

Excerpt 4: I don't know which to divide first.

In this strategy, PSTs obtained the fraction by dividing 3 by 5. This strategy is the same as Charles & Nason's (2001) proceduralized PQ strategy, which is a high-level strategy that indicates the abstraction of the PQ subconstruct. However, we found that the proceduralized strategy might lead to building of the PQ subconstruct, but also might leave the understanding of PQ construct at procedural understanding level. As Student 7 stated in her presentation:

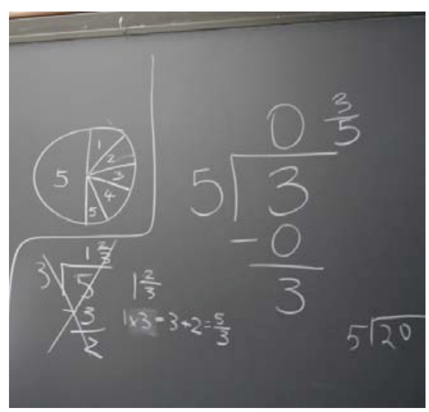

Figure 4. Proceduralized PQ

> Student 7: Um. I did, well at first I did five divided by three and I realized this was wrong. So, um, I finally figured it out that it's three divided by five.

> Student 7: Well at first I didn't. I did five divided by three and then I… So this is what I did before I did this. So then I got. I realized that three does go into five once and then I did minus three and then have two so two

thirds and then that one is wrong. So then I did it this way and then I got zero minus zero equals three and I wanted it in a fraction so then I got three fifths.
Teacher: Everybody convinced? Okay, just cause the other one was wrong can this one be wrong too?
Students: Yeah.
Teacher: Can they both be wrong?
Teacher: Why is this really the right thing to do?
No response.

Students are able to figure out the answers to the PQ problem, but are unable to explain why the different strategies are valid. Moreover, they rely on the long division algorithm to obtain the answer rather than producing the fraction directly. This might be due to the disconnection between fraction as division and division as fractions. PSTs may not see the PQ problem as a partitive division problem.

Discussion

The authors examined the strategies presented by PSTs with a PQ task and the follow-up discussion involving those strategies. As discussed previously, we posited that PSTs would employ diverse strategies, not necessarily partitioning, and rely on their prior knowledge of fractions and fraction operations in order to solve the PQ task. Based on our strategy analysis, we argue that, given PSTs' prior knowledge of fraction concepts and operations, they are able to build the PQ subconstruct. As evidenced in the class discussion, however, the building of the PQ subconstruct resulted from a largely procedural understanding of partitioning and quantification processes, with limited conceptual understanding of those processes. For example, PSTs were unable to make connections between different strategies (Excerpt 1), became easily confused by some strategies (Excerpt 2), and were unsure why some strategies were valid (Excerpt 3 & 4). Though not part of this analysis, we have designed additional follow-up tasks to help facilitate PSTs' conceptual understanding of the PQ subconstruct.

Based on our analysis, the knowledge PSTs needed to develop a conceptual understanding of the PQ construct are reunitizing, two fundamental ways of quantifying a partitioning, as well as partitive division in the PQ context. For PSTs, these three features of fractions are dramatically different from whole number contexts. For example, when using whole numbers to represent quantities, they do not consider the unit and reunitizing because the unit is always "1." However, whole numbers, a subset of rational numbers, cannot be considered as a different and isolated number system from rational numbers. Thus, we propose that in order to facilitate a thorough understanding of the PQ construct, teacher educators need to reveal the implicit connections between whole numbers and fractions as the following:

- For whole number problems, no reunitizing is involved due to the consistency of the unit; for fraction problems, reunitizing is involved because the referent unit varies, as in the Cake Problem where the unit could be 1 cake, 3 cakes or half of a cake.
- For whole number problems, there is only one way to quantify: divide all the units into a certain number of parts because it does not involve dividing a unit. Using this way to think about fraction problems is valid because all of the units are divided into a certain number of parts in both situations. However, the answer usually cannot be

obtained directly. For example, in excerpt 3, to quantify each part (i.e., 1/2 of a cake and 1/5 of half a cake), fraction addition and multiplication are employed.

- For whole number problems, the quotient of the division can be represented using fractions, which is similar to the PQ problem used in this study. For example, 6 cakes shared with 3 people can be represented as 6÷3= 6/3 =2, while 3 cakes shared with 5 people could be represented 3÷5=3/5.

Following the discussion of the uniformity of whole numbers and fractions in terms of these three features, the variance of the features of fractions should be emphasized. The following is the work that mathematics teacher educators can do to emphasize a more conceptual understanding of the PQ subconstruct: as to reunitizing, we propose to emphasize the importance of unit in representing a quantity using fractions. In their previous learning, only two elements of fraction are emphasized, denominator and numerator. Unit as an essential component of fractions is usually ignored. As to two fundamental ways of quantifying a partitioning, first, the properties of partitioning as Lamon (2000) discussed should be made explicit to PSTs. Second, PSTs should be encouraged to explain different partitioning using the two quantifying processes.

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MATHEMATICAL KNOWLEDGE FOR TEACHING IN ACTION: SUPPORTING ARGUMENTATION IN THE CLASSROOM

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This study examines how middle school math teachers' mathematical knowledge for teaching (MKT) can support their teaching. This post-hoc analysis examines data collected around a teacher professional development program designed to help teachers learn to facilitate mathematical argumentation in their classrooms. Findings show a range of MKT across the teacher sample and that the amount of argumentative talk in teachers' classrooms is strongly related to teachers' MKT. Contrasting illustrations of classroom discourse of a high-MKT/high-talk teacher and a low-MKT/low-talk teacher illustrate important ways MKT can be used in the actions of teaching. Implications for research and professional development are discussed.

Introduction

The purpose of this study is to examine the claim from commonsense wisdom, recent education policy discussions, and emerging research that teachers' mathematical knowledge for teaching (MKT) plays an important role in their teaching of mathematics. In particular, we investigate how teachers use MKT moment-by-moment in the teaching moves that they enact to support classroom discussion, specifically mathematical argumentation.

Hill et al. (2008) began an important line of inquiry examining the relationships between teachers' MKT and their instructional practices. They reported a series of case studies investigating elementary mathematics instruction. These studies traced connections between teachers' MKT and what they call "Mathematical Quality of Instruction" (MQI) observed in classroom teaching. Mix's categories characterize the rigor and richness of the mathematics of the lesson, including teachers' mathematical errors, the appropriateness of teachers' responses to their students' ideas, and precision of mathematical language. Across the case studies, they found positive and strong associations between MKT and MQI, as well as a number of mediating factors that support or hinder the teachers' use of their knowledge in practice.

The present study builds on this literature with a post-hoc analysis of data collected in the context of the Bridging Professional Development Project, a teacher professional development program designed to both improve teachers' MKT and develop their skills to facilitate mathematical argumentation in the classroom. The program targets middle school mathematics teachers teaching in underserved urban districts. We examine how teachers' MKT relates to the argumentative discourse that occurs in their classrooms when they teach a 2- to 4-day replacement unit designed to foster argumentation around important middle school mathematics. Although the Bridging Project was not planned specifically to investigate links from MKT to classroom discourse, the research design entailed a rich and relatively controlled context for systematic exploration of this relationship. The design included both direct assessment of specific MKT (using a test modeled on Hill and colleagues' work) and a systematic analysis of transcripts of classroom discourse. The design provides several methodological affordances that have not been present in prior research: (1) the content of the classroom instruction is held relatively constant, as all participating teachers (within a year) teach with the same instructional unit; (2) the content of the MKT assessed is highly aligned with the content of the mathematics lessons taught and observed; and (3) the sample size is large enough to afford statistical inferences. Furthermore, while most prior research on MKT has focused on elementary school, this study examines middle school teaching and learning.

In this post-hoc analysis, we characterize the distribution of MKT across the middle school math teacher sample and analyze possible relationships between teachers' level of MKT and characteristics of their classroom discourse. We look for MKT in action—how teacher's knowledge of mathematics is used—or not—in the moment of instruction.

Theoretical Framework

Mathematical Argumentation

Mathematics educators, mathematicians, and philosophers agree that mathematical argumentation is essential for learning mathematics. It can provide students with opportunities that enable them to construct conceptually rich understandings and develop a sense of ownership in the construction of knowledge. It can also be empowering to students, enabling them to develop intellectual autonomy and confidence as they become active co-constructors of mathematical arguments, formulating and determining the validity of their own justifications. Indeed, mathematical argumentation is highlighted in national math education policy documents as part of the development of essential mathematical proficiencies.

In deciding how to frame to teachers the mathematical practice of argumentation, we began by examining Toulmin's oft-cited structure of argumentation (2003). In this scheme, an argument in any discipline consists of core elements, such as a claim about something in the world, data used as evidence to support the claim, and warrants that explain the relationship of the data to the claim. To that make this scheme practical for teachers' use in the classroom, we define conjecturing as a process of "conscious guessing" to create mathematical statements of as-of-yet undetermined mathematical validity and justifying as a process of explicating one's reasoning to establish the mathematical validity of a conjecture. Finally, we also emphasize concluding as coming to agreement about the validity of an argument.

Conjecturing, justifying, and concluding, then, are the elements of our framework for argumentation, in our work with teachers. Teachers learn to make language-based mathematical arguments to justify solutions, compare and contrast alternative solutions, and establish generalized claims. Specialized forms of these categories of knowledge are needed in different aspects of teaching—planning, responding flexibly to students, connecting content to students' future studies, exploiting connections among mathematics topics, and more.

Mathematical Knowledge for Teaching

Foundational to teaching with argumentation is deep knowledge of both the mathematical content that is the object of the argument and the mathematical practice of argumentation. We conceptualize these understandings in terms of the construct of MKT, which Ball, Hill, and others use to characterize the specialized mathematical knowledge, skills, and practices that are grounded in, but not confined to, use in the classroom, at an appropriately higher level than what students are expected to learn (e.g., Ball, Lubiensky, & Mewborn, 2001; Hill et al., 2005) Our project focused on the mathematics of proportionality in geometry as well as properties of geometric shapes in coordinate geometry.

The Bridging Teacher Professional Development Model

The Bridging PD has been implemented as a 2-week summer workshop designed to foster intensive teacher learning. The training in week 1 focuses on deepening MKT, while the training in week 2 builds on this knowledge to develop the pedagogical aspects of facilitating mathematical argumentation. The pedagogical approach, which is described in detail elsewhere (see Shechtman & Knudsen, in press), draws on methods from improvisational theater and emphasizes the development of a repertoire of teaching moves—discrete units of instructional practice that teachers use to scaffold classroom mathematical argumentation. The workshop is designed to help teachers, within a two week timeframe, develop a repertoire of teaching moves that they can use flexibly, responsively, and appropriately given the moment-by-moment needs of the students as they engage in mathematical argumentation. Summer workshops activities include:

- ***Building mathematical knowledge for teaching.*** Using adult versions of a classroom unit for teaching proportionality, teachers learn to make mathematical arguments for themselves, while deepening their proportionality MKT. They analyze brief classroom scenarios focusing on students' mathematical thinking. Teachers learn to make their whole-group arguments in terms of conjecturing, justifying and concluding.
- ***Script read-throughs.*** Teachers take on parts in fictionalized transcripts of classroom discourse. Images of argumentation take on a first version of reality through these read-alouds. This builds teachers' awareness of new tools and practices particularly as compared to their own current practices, without requiring independent use of them.
- ***Teacher improv games.*** These activities provide an experience of teaching that shares the improvisational nature of "the real thing"—classroom teaching. The games have rules that constrain teachers' actions, enabling them to consider the use and purposes of teaching moves. The games advance teachers' understanding and use of new practices, including the affective component of doing something new.
- ***Writing the script: lesson planning.*** Teachers use "think alouds" with the prompt, "what are you actually going to do for this lesson?" to compose detailed lesson plans with step-by-step anticipation of classroom action when they teach with the Bridging curriculum unit.
- ***Rehearsal.*** Teachers choose a portion of their lesson to rehearse improvisationally, with the whole group acting as the class. Lesson segments are chosen to highlight potentially difficult places in the flow of the lesson: for example, orchestrating getting agreement about the validity of a justification. In the rehearsals, teachers apply new moves and tools to potential classroom interactions.

Methods

Research Design

In this early design phase of the Bridging Project, we implemented the PD within the structure of a small-scale 2-year randomized experiment impact study with complementary case studies. Teachers were recruited from high-poverty urban districts in the San Francisco Bay Area. A total of 35 teachers attended the workshops across the two years. Teachers were randomly assigned to either a Treatment or Control group. Each year, teachers in the Treatment group received the full 2-week Bridging intervention. Teachers in the Control group received only the week 1 MKT training but not the week 2 training addressing pedagogy (during that week, they received a different but equally valuable workshop focused on vertical coordination across grade levels). We implemented two iterations of this workshop in the summers of 2006 and 2007. During the school year, teachers were asked to teach a 2- to 4-day instructional unit using curriculum materials from week 1 of the summer workshop. To examine classroom discourse during these lessons, classes were videotaped using a set of microphones with the capacity to capture all student and teacher talk in the classroom. All whole-class discussion was transcribed verbatim for analysis.

The main hypothesis of the experiment was that the classroom discourse of Treatment group teachers (compared to that of Control group teachers) will have more and longer episodes of mathematical argumentation. In our first-year iteration, Treatment classrooms had significantly more episodes of argumentation (effect size 1.22), while in the second-year iteration there was no significant difference between groups. In both years, there were trends, though not significant, that argumentation episodes were longer in the Treatment group. In the present post-hoc analyses of the data from this experiment, we focus on examining relationships between teachers' MKT and classroom discourse.

Measures

Each year, teachers were given an assessment of their MKT before and after their 2-week PD program. The assessment was developed in alignment with the content of the PD. The research team followed a sequence of steps to rigorously develop and validate assessment: (1) establish a mathematically-precise conceptual framework to underlie both the curriculum and assessment, (2) work with a panel of experts to develop a pool of potential items, (3) conduct cognitive think-alouds, and (4) field test the items with about 20 teachers to establish item and test characteristics. The year 1 assessment focused on similarity across the strands of geometry, algebra, and number; it had 23 items and a Cronbach's alpha of .84. The year 2 assessment focused on coordinate geometry; it had 30 items and a Cronbach's alpha of .94. Both assessments contained a mixed of multiple choice questions assessing conceptual knowledge and items requiring the construction of mathematical definitions and justifications.

To examine argumentation in classroom discourse, we developed the Mathematical Argumentation as Joint Activity in the Classroom (MAJAC) coding protocol to locate and analyze episodes of mathematical argumentation in transcripts of whole-class discussion (*Shechtman, Knudsen & Kim, 2008*). It is implemented by coders who have extensive experience in the mathematics classroom, either through teaching or research. Two independent coders read through the transcript and record the line numbers on which they observe entry and exit points for episodes of classroom mathematical argumentation. Consensus is then sought through discussion.

Results

Distribution of MKT in the Teacher Sample

Teachers came into the workshop with a wide range of MKT as measured by the Bridging assessments. In both years, teachers' pretest scores were distributed across the range from 10-20% correct to about 90% correct. This affords an analysis of how varying degrees of MKT impact practice. Also, teachers' mathematics learning in the MKT workshops was significant, though modest. From pre- to post-workshop, MKT grew a mean of 0.5 SD [$t(23) = 4.2$, $p < .001$] and 0.3 SD [$t(20) = 4.5$, $p < .0001$] in years 1 and 2, respectively.

Discourse Patterns Across the Sample

Figure 1 shows an important pattern we observed in examining the relationship between MKT and the amount of talk teachers' classrooms spent in argumentation. Overall, there was a robust relationship across years that the higher the teacher's MKT the more argumentative talk occurred in their classroom. Note that the correlations are almost identical for pretest scores.

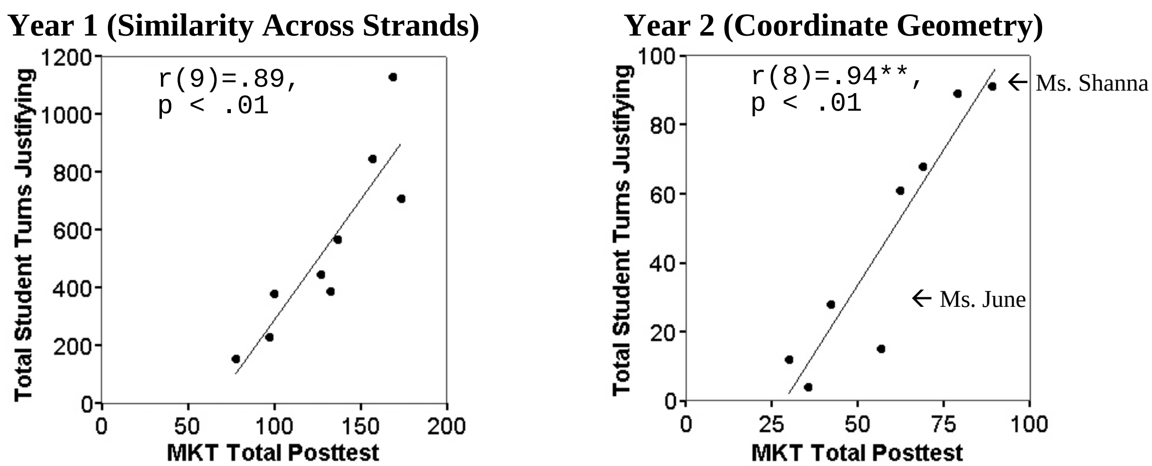

Figure 1. In the treatment groups, correlations between teacher MKT and total student turns justifying in whole-class discussion during 2- to 4-day instructional units

Illustrations of MKT in Action

To begin to unpack how teachers use MKT in action, here we examine excerpts from the classroom discourse of two year 2 teachers. Ms. Shanna was the highest-scoring teacher on the MKT post-test in the year 2 workshop and had the highest amount of classroom argumentative talk, while Ms. June was one of the lowest-scoring teachers and had relatively little classroom argumentative talk. In this limited space, rather than focus directly on the amount of argumentative talk, we have chosen to focus on an activity toward the beginning of the Bridging curriculum that lays the conceptual foundation for the argumentation to follow. In this activity, teachers facilitate a whole-class discussion in which the class co-constructs the mathematical definition of a rectangle. This in itself is argumentative, as posing a definition requires both making conjectures that particular statements capture all the appropriate cases of a rectangle and justifying these conjectures. At the heart of these illustrations, drilling down into the teachers' actual assessments reveals significantly different conceptual understandings (or misunderstandings or confusions) that the two teachers held. When asked to write a precise definition of "rectangle" in terms of what a rectangle would be on a coordinate plane, the two teachers wrote the following:

Ms. Shanna: A rectangle consists of 4 line segments intersecting at 4 vertices. Each vertex consists of 2 lines whose slopes are negative reciprocals of each other.
Ms. June: A rectangle's 2 vertical line segments share the same x value and the 2 horizontal segments share the same y value. The 2 line segments are parallel to the x and y axes, and the points of intersection of these line segments are perpendicular. 4 right angles are formed at the vertices.

Ms. Shanna. An excerpt of classroom dialogue from Ms. Shanna's class is as follows:

1	Ms. S	What's, what's your definition of a rectangle?
2	Students	Sides. Squares. Two sets of parallel lines.
3	Ms. S	I will take a quiet hand… Okay, John?
4	John	A shape with four sides. Four pairs of parallel sides… I mean, not four parallel sides. Two parallels.
5	Ms. S	So, it's a shape. [Writes "A shape…" on the board.]
6	Students	Two parallel sides. The length is uneven of the width. A stretched square.
7	Ms. S	It's a shape with two parallel sides. [Writes this on the board] What else?
8	Student	A longer square.
9	Ms. S	A longer square?
10	Student	A stretched--
11	Ms. S	Stretched? Is that what you mean?
12	Student	The width is stretched out.
13	Ms. S	The stretched
14	Student	A stretched square, huh? [laughter]
15	Ms. S	Okay. Larry.
16	Larry	The width is longer than the length.
17	Ms. S	Okay. I like that…the width is longer than the length. [Writes this on the board] And how do you know FOR SURE, how do you know for sure that what you drew is a rectangle? I have a question for you. Here's two short sides. And here's two long sides. Is that a rectangle? [Draws a trapezoid on the board]
18	Students	[simultaneously] No. That's a <inaudible>. Is that a heptagon?
19	Ms. S	Well, we know it's not a rectangle.
20	Students	<inaudible>
21	Ms. S	It's a trapezoid, ok? Right?
22	Student	Four 90-degree angles with two sets of parallel lines.
23	Ms. S	Oh okay, I like that. [Writes this on the board] Okay, four 90-degree angles with-
24	Student	Two sets of parallel lines.
25	Ms. S	Two sets of parallel lines.

Ms Shanna's MKT seems to have supported many of the moves that she makes here. She starts by collecting conjectures from students about the key aspects of the definition of a rectangle. In general, she is able to pick and choose from among the many statements that students make in order to highlight important mathematics. She keeps the focus on the mathematical properties of rectangles, such as lengths of sides and angles, which could help move students to a more mature view of geometric shapes as more than the colloquial understanding as holistic images. By line 17, she is poised to make an important argumentative move: she provides students with a counterexample. In the context of defining a rectangle, a counterexample is a shape that should fit the definitions students have given, but that is clearly not a rectangle. This counterexample "breaks" the incorrect definition students had made and helps them to see what aspects of a shape are necessary and sufficient to define a rectangle. The definition also, of course, includes squares, but Ms Shanna had, perhaps wisely, not taken that on. A discussion of whether or not squares are rectangles can become more of a debate than an argument, unless students are specifically equipped with the ideas to take it on.

Ms. June. In Ms June's class, defining a rectangle was spread over two days. The discussion was a lengthy one, summarized through these two excerpts.

1	Bruce	A rectangle…its vertical lines are longer than its horizontal lines.
2	Ms. J	Uh, okay. Write that down… He said the vertical line is longer than the horizontal. So this is an assumption he is putting out there. He's making a conjecture that, to be a rectangle, the vertical line must be longer than the horizontal… What's another property of a rectangle that you can know? And we're just putting them up here. Some of these may, we may decide are not good definitions, and we might want to, um, modify.
	<several minutes of talk>	
3	James	Uh, I was going to say two lines are…horizontal and two lines are vertical…The two, uh, the two lines that are horizontal are, um, equal, and the two lines that are vertical are equal, too.
4	Jason	What about, about parallelogram?
5	Ms. J	It's a parallelogram? Jacob, Jacob.
6	Jacob	Uh, there are four right angles.
7	Ms. J	There are four right angles, okay? Is there anything else you want to add to the definition?
8	Student	Yes, please.
9	Joan	Oh! <raises hand>
10	Ms. J	Yes.
11	Joan	It's like a square, but longer.

As in Ms. Shanna's class, students start from their basic colloquial understanding of rectangles as holistic images that are qualitatively different than squares. Ms June responds by naming Bruce's statement as conjecture, by asking for another property of rectangles, and warning that some of the statements that were on public display would not be true in the end. By asking for a definition, then calling a proposed definition a conjecture and then asking for properties, Ms June leaves the difference between these terms unclear. Note also the parallel to her written definition on the assessment that lacks mathematical precision with the omission of the basic premise that the object is a quadrilateral. It is important to note in this classroom, it is a *student* who offers a counterexample, but while Ms. Shanna was able to leverage her counterexample to lead the students to clarify their thinking, Ms. June does not explicitly disambiguate the counterexample from other properties. In the end, it is not clear what was left "standing" as a definition of rectangle.

On the second day of our observations, Ms June begins with a review:

12	Ms. J	Who knows a definition for a rectangle? Who has a definition for a rectangle? Jill?
13	Jill	It's like a square but the base...
14	Ms. J	It's like a square.
15	Jill	The width and the length are the same and the height is the same too but so if you add them all up.
16	Ms. J	Okay, you're saying the length and the width are the same?
17	Jill	Which one is the length and which one is the width?

18	Ms. J	Um, Jill I appreciate what you've done. Hold on. Andy, can you add on?
19	Andy	Yeah. She's right. It's like a square. But the width of it is a little bit longer and so that's what makes it a rectangle.
20	Ms. J	Okay. The length and the width are two different dimensions. And traditionally we would say the length is longer, but using what property does it, that we can change the length and width around. What property allows us to change them?
21	Myra	Commutative.
22	Ms. J	Commutative property of addition. Okay. Now, what is the quadrant plane?

Ms June's moves still do not lead students to a mathematically precise definition of a rectangle. Instead, she gets sidetracked into talking about labeling conventions. Furthermore, she goes on to incorrectly invoke the commutative property of addition as the reason "we can change the length and width around." This is much in line with Hill and colleagues' finding that low MKT was associated with teacher errors.

Discussion and Conclusion

This post-hoc analysis of data from a teacher professional program examined MKT in the action of teachers' facilitation of mathematical argumentation in their middle school classrooms. We found that teachers in the program represented a broad range of MKT backgrounds and that teachers' MKT strongly predicted the amount of argumentative talk in their classrooms. We examined the discussions of a high-MKT/high-talk classroom and a low-MKT/low-talk classroom, which reveal how teachers may leverage their knowledge to focus classroom discussion, select appropriate mathematical threads to follow in the discussion, provide mathematical precision, and avoid confusing errors. Future analyses will examine other ways that MKT is used in action to explain the finding that MKT supports greater argumentative talk.

These findings have important implications for research and practice in teacher professional development. While it is an intuitively appealing notion that teachers' MKT is important for classroom practice, several studies have failed to find strong direct correlations between teacher knowledge and student achievement; it is therefore important for the field to examine the ways in which MKT helps, is irrelevant to, or even hinders student learning in the context of the full system of classroom instructional resources (Shechtman, Roschelle, Haertel & Knudsen, in press). In the MQI tradition, this study helps to illuminate where and how MKT is important specifically in moment-by-moment practice. Future research will need to continue to unpack how MKT is used in the action of teaching, the impacts this may or may not actually have on student achievement, and how these findings can be used to improve teacher training.

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THE INTERPLAY OF BELIEFS AND PROCESSES IN THE CLASSROOM MATHEMATICS OF A NOVICE SECONDARY MATHEMATICS TEACHER

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As part of a larger study, which defined mathematical knowledge as participation in mathematical processes, we investigated the mathematical knowledge of a prospective secondary mathematics teacher and how he used mathematical processes in his teaching, both as a preservice teacher and later in his first year of teaching. His personal mathematics was characterized by a tendency not to place primary emphasis on mathematical processes. His stated core beliefs about mathematics teaching might have been expected to spur a more intensive use of mathematical processes but, as he envisioned these beliefs, did not conflict with the role of mathematical processes in his mathematics. The result was classroom mathematics that echoed personal mathematics and made little use of mathematical processes.

Introduction

Of growing importance in the area of mathematics teacher education are the connections among what teachers know, what they do in the classroom, and what students learn. Current research has made progress in assessing relationships between what elementary teachers know and their students' mathematics achievement (Hill, Rowan, & Ball, 2005). Other studies (e.g., Eisenberg, 1977; Monk, 1994) examined the relationship of secondary mathematics teachers' knowledge to students' achievement through proxies of courses, GPAs, years of experience, and degrees. Researchers have also attempted to measure secondary mathematics teacher knowledge through tests of content knowledge (e.g., Eisenberg, 1977; KAT, 2005). This study investigated the relationship between the mathematical knowledge of teachers and how they draw on that knowledge in their teaching.

Researchers have conceived mathematical knowledge in various ways including knowledge of school mathematics, knowledge of mathematics for teaching, and knowledge of collegiate mathematics. Often mathematical knowledge is represented as sets of procedural skills and conceptual understandings, but for both theoretical and pragmatic reasons, we follow Zbiek, Peters, and Conner (2008) and cast mathematical knowledge as the ability to participate in mathematical processes. Doing so allowed us to look at mathematical knowledge across different subject matter in a generalized form. We selected processes that are essential to mathematical thinking: defining, justifying, generalizing, and representing. We distinguish these processes from their products: definition, justification, generalization, and representation. The goal of our research was to characterize novice secondary mathematics teachers' use of these processes and products in their personal problem solving and in their classrooms and to characterize the relationship between them. We chose to focus on novice secondary mathematics teachers because it allowed us to observe the challenges that novice teachers face in using their own mathematical knowledge in their teaching practice. We observed their use of mathematical

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processes over an extended period of time, affording us the opportunity to characterize the enduring features of their mathematical knowledge.

The influence of mathematical knowledge on what happens in the novice teacher's classroom may be overshadowed by their focus on survival. Research and theory have suggested that novice teachers usually focus first on classroom management, then on teaching concerns and student concerns (Fuller & Brown, 1975; Fuller, 1969; Berliner, 1989, as cited in Berliner, 1991; Kremer-Hayon & Ben-Peretz, 1986). Studies of novice teachers show that issues of student behavior and classroom management are in the top three concerns of novice teachers (Adams & Krackover, 1997; Pigge & Marso, 1992; Veenman, 1984; Westerman, 1990). The need to attend to survival issues may account for research findings that novice teachers are often less attentive to monitoring student learning and adjusting instruction to meet student needs (Artzt & Armour-Thomas, 1997; Westerman, 1990) than to concerns about their actions in their role as teachers. However, Levin, Hammer, and Coffey (2009) observed that some novice teachers do attend to student thinking even before resolving classroom management issues. They suggest that attention to how a novice teacher is framing what is occurring in the classroom may account for why some novice teachers seem to attend to student thinking and some do not.

One factor that would seem to impact the way teachers frame what is occurring in the classroom is their set of beliefs about mathematics and its teaching. The relationship between teachers' beliefs about the nature of mathematics and the nature of teaching mathematics and their instructional practice is complex (Thompson, 1984, 1992). Other research implies that teachers' beliefs influence their instructional practice (Pajares, 1992; Thompson, 1984, 1992). For instance, Thompson (1984) found a consistency between the practices of three junior high mathematics teachers and their beliefs about the nature of mathematics and their instructional practice. Thompson concluded that "[m]any factors appear to interact with the teachers' conceptions of mathematics and its teaching in affecting their decisions and behavior, including beliefs about teaching that are not specific to mathematics" (p. 124). Likewise, in a study of six elementary teachers, Raymond (1997) found some inconsistencies between teachers' professed beliefs and their instructional practice, and she identified several categories of factors that influenced these teachers' practices. These studies are illustrative of research suggesting that teachers' beliefs play an important role in their instructional practice but that these beliefs are just one of many factors that influence practice.

Green's (1971) metaphor for belief systems provides a lens for examining the relationship between beliefs and instructional practice. Green identified three dimensions of belief systems that deal with how beliefs are held including the psychological strength of beliefs. Central or core beliefs are strongly held beliefs that are resistant to change. The centrality of beliefs can be used to examine the relationship between teachers' beliefs and their instructional practice (see Beswick, 2007).

Methodology

This study is one of several case studies we conducted with prospective secondary mathematics teachers documenting how prospective/novice teachers use mathematical processes and act on the products of those processes in their personal mathematics and in their teaching. Each teacher who participated in the study engaged in task-based interviews and several series of teaching observations spread over a span of four to five semesters. Each observation included pre-observation and post-observation interviews focused on clarifying participants' use of processes. This paper reports on the case of Russ (a pseudonym). At the close of data collection, Russ was a full-time first-year teacher of mathematics at a middle school in an Eastern state. Over the 2.5 years of involvement in the study, Russ participated in five 1.5 to 2 hour task-based interviews, and 13 observation cycles. These cycles included asking Russ about the extent of his agreement with several statements about the nature of mathematics and mathematics teaching.

Table 1. Schedule of data sources

Research Data Source	Setting	Timing
Interview 1. Count	Math methods I	Spring 2007
Interview 2. Area	Math methods I	Spring 2007
Interview 3. Wrap	Math methods II	Fall 2007
Interview 4. Cube	Math methods II	Fall 2007
Observation cycle[1]	Pre-student teaching field experience	Fall 2007
6 consecutive observation cycles[1]	Student teaching	Spring 2008
Interview 5. Defining	Student teaching	Spring 2008
5 consecutive observation cycles[1]	First year teaching	Fall 2009

[1] An observation cycle consists of a pre-observation interview, an observation, and a post-observation interview. For observation cycles on consecutive days, the post-observation interview for day n occurred at the same time as the pre-observation interview for day n + 1.

Interviews were focused on clarifying Russ's use of processes in his personal mathematics and his teaching. All task-based interviews were videorecorded, transcribed, and annotated. The pre- and post-interviews and the classroom observations were audiorecorded, transcribed, and annotated with two observers' notes and still photos. Transcriptions were coded for instances of Russ's use of the processes and/or products. After initial coding, the coded instances across all transcripts were elaborated and then categorized into the four process categories. These categories were then analyzed for emerging patterns of use of the processes and products, for Russ's personal mathematics and separately for his teaching. The patterns were then checked for consistency across his personal mathematics and teaching.

Results

We observed Russ in two preservice settings in the context of second-year algebra courses and during his first year of teaching. At the time we observed his first year of teaching, Russ was teaching an introductory algebra course in a full-year block schedule (double period) to middle school students. Throughout our observations, Russ displayed refined skills in classroom management that are atypical for novice teachers. Russ was remarkably comfortable in front of his class, commanding the attention and respect of his students naturally. In the rare instances in which students were not behaving as Russ would have preferred, he was quick and effective in addressing the issue and proceeding with his lesson. Russ was reflective of his own role as a teacher and revealed strong stances regarding this role.

Russ's use of mathematical processes

Our analysis of data revealed consistencies in Russ's mathematical processes across the 2.5 years of data gathering. Following are illustrative examples of our observations.

Russ's use of processes in personal mathematics. Russ's generalizing and justifying are illustrative of his use of processes in his personal mathematics. Russ's personal mathematics is often informal. When Russ justifies his claims, he typically does so by reasoning from a limited number of cases: checking algebraic results by substituting a few numerical values and verifying the generality of a geometric observation by checking a few instances. Once he has checked a few instances, he makes no attempt to reason formally to a general result. For example, when Russ is asked to relate a given formula for the area of a given equilateral triangle to a geometric problem on which he is working, he justifies that $\frac{(x \cdot d)^2\sqrt{3}}{4}$ represents the area of a particular triangle by substituting 1 for x and 2 for d and verifying that the result makes sense for the related diagram. He engages in no further verification of the formula.

Russ's pattern in generalizing is that he generalizes to a set of mathematical cases based on examining a small subset of those cases. Although Russ is confident in generalizing and stating generalizations, he makes no attempts to justify those generalizations unless prompted to do so, and even though his explanations are usually in the ballpark, they are typically not sufficient mathematically. For Russ, generalizing seems to be about pattern recognition, both when he is generalizing (looking for patterns) and when he applies generalizations (extending the form of a generalization to a new situation without examining the applicability of the generalization). Russ does not prove his generalizations for a global applicability; rather he tests a few cases for consistency. In the context of a task-based interview (see Figure 2), upon observing the graph associated with the square diagram in Figure 2a, Russ interprets the shape of the graphs as reflecting the fact that the rates are increasing faster in some places, but he does not seem to recognize that this shape is related to the positioning of sides of the square in relation to line l. He immediately generalizes the curved graph for the square example (see Figure 2a and Figure 2c) to the triangular example (see Figure 2b and Figure 2d) without accounting for the effect on rates of change of differences in orientation of the sides.

Russ's use of processes in classroom mathematics. In class, it is not uncommon for Russ to ask students to make generalizations based on an insufficient set of cases. For example, one generalization that Russ aims to teach is that the vertex of a function of the form $f(x) = |x - b| + c$ is the point (b, c). Having identified the vertex of only one example, $y = |x|$, Russ asks students to guess the identity of the vertex of $y = |x\text{-}1|+3$ without previously exposing them to functions that have a vertex that is not at the origin.

Although Russ is confident in stating generalizations in his classes, he does little generalizing and makes no attempts to justify generalizations unless specifically prompted by students to do so, and even though his explanations are usually in the ballpark, they are typically not sufficient mathematically. For example, following up on Russ's generalizations, the interviewer asks, "how do you know that the absolute value function is V-shaped and how do you know that a parabola is U-shaped?" Russ explains that the graph of the absolute value function consists of two lines with the same constant rate of change, and that the parabola is not V-shaped because "the rate of change from point to point is different." Later he adds to his argument that a parabola is U-shaped stating that "I know they're not straight and so if they're somewhat curved then they have to be that U." Russ presents the pieces of an argument, relying (not unexpectedly) on what is visible, but he does not structure his argument into a coherent whole.

Consistency in Russ's processes. In his personal mathematics, Russ's informal reasoning can be strong but is not infrequently less than accurate, and it seldom relies on formal mathematics. Without verifying his claims with formal mathematics, these inaccuracies at times go unrecognized. These same features also seem to characterize his classroom mathematics.

Russ's Beliefs

Our observations of Russ's problem solving showed that most of the time he did not engage in mathematical processes as we defined them but at other times he showed the capacity to do so. We viewed his lack of engagement in processes as a preference or tendency and thought it possible that his beliefs about processes or about teaching were dominant in his problem- solving strategies.

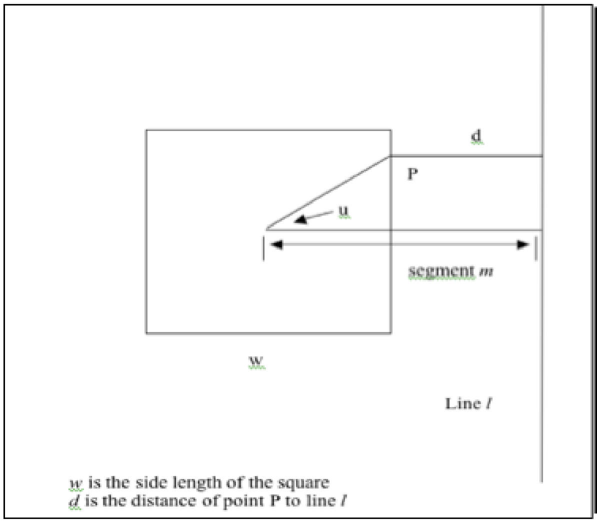

Figure 2aa

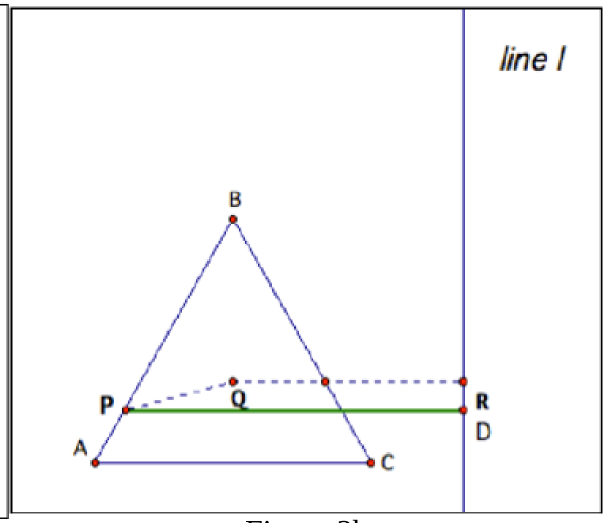

Figure 2b

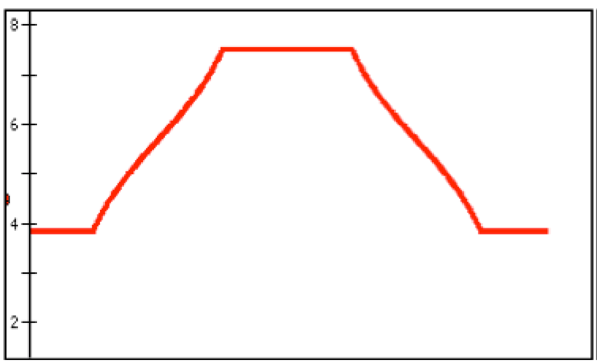

Figure 2c

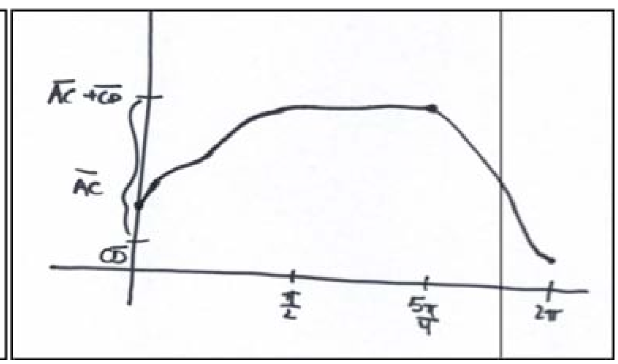

Figure 2d

Figure 2. Given the diagram and a description in Figure 2a the interviewee was asked to generate a function rule for the shortest distance from the point P to the line l as a function of u. A similar question and set of directions accompanied the diagram in Figure 2b. Russ is provided with the computer-generated graph in Figure 2c for the square and he generalizes and generates the graph for the triangle (Figure 2d)

Russ's beliefs about processes. Russ seldom chooses to prove, generalize, or define mathematically in his personal mathematics or in his classroom mathematics. We looked through the data for evidence of how Russ viewed these processes, and found that he referenced them very little unless we directly solicited his views on the processes. When asked, Russ expressed views that seemed to suggest an appreciation for the importance of proof, although he did not make statements recognizing the processes of proving and defining as important to his mathematics or that of his students. Although much of his personal and classroom mathematics seemed to be based on reasoning from examples, he strongly disagreed with the statement that "It is not important to check the validity of the procedure if you have used it before and it has always worked," and he stated that, "I would strongly disagree with that cause um it could be coincidence that it's always worked." Regarding proof, he stated: "I think proofs are always necessary" and "Proofs generally helped me to agree with mathematics … with things that are presented to me." He expressed a view that mathematics was about much more than symbols and calculations: "It's understanding the importance of the symbols and calculations and how, and why, you know, why do I do these calculations."

Russ's core beliefs. Russ's use of processes in his teaching did not reflect the statements he made in interviews related to the use of the processes. We wondered whether, perhaps, Russ's beliefs about teaching could account for these discrepancies. In order to explore this hypothesis, we examined our data to investigate Russ's core beliefs–beliefs that seemed to guide his instruction and that endured throughout our data-gathering.

Russ's most dominant core belief related to his teaching was that his role as a teacher was to support the development of the whole person in his students. This commitment was strong and seemed to regularly underpin his instructional decisions. Russ invariably gave his students positive reinforcement when they responded in class discussions. When those responses were not correct or not on target, Russ would typically respond with something like "I see what you mean." He expressed his reason for one such interchange, saying: "I didn't want to quell their thinking." He elaborated on this perspective:

> *And that's, I guess that's too kind of what I'm trying to do is that there's a baaad, bad connotation when you say mathematics. You either love it or you hate it. And I, I want a lot of loving. I want a lot of loving going on in this math class. So I, I try and push that and maybe sometimes it's by not talking about math. Talking about maybe how math helps me understand the world better. … You know and so it just, it kind of helped them see that um, help them be better decision makers, be better people.*

Russ's commitment to the personal development of his students was also exhibited in his resolve to help his students see that mathematics is useful in their lives. A second core belief was that mathematics should be relevant to his students. He declared: "… what they learn in any class has no meaning unless you can somehow see a connection maybe to life." He carries out this resolution by selecting tasks that are embedded in situations that have real-world features. During our observations of his first year of teaching, every lesson was somehow connected to one such situation: Aunt Edna's Gourmet Cookies set up a business situation in which Russ engaged his students in thinking about x-intercepts and y-intercepts; Turtle Race, to set up talk about slopes of linear functions, drew on the vision of turtles racing each other positioned on their hind legs; and Leaky Faucet involved a data-gathering experiment that provided students with a real-life based function.

A third core belief for Russ was that students should have a "deeper understanding" of the mathematics they encounter. He does not define what he means by "deeper understanding" but offers an example: "But what I'm hoping to do is to get them to understand you know um just, just a little bit deeper you know the three comma four in parenthesis means that on the x-axis it goes over three." This example and our observations of his classes suggested to us that his meaning for "deeper understanding" is different from what we might have expected.

Relationship of Beliefs to Mathematical Processes

We identified what we did as Russ's core beliefs about goals for teaching mathematics both because of what he said and because they seemed to guide what he did in the classroom. We hoped to see a relationship between these beliefs and how Russ used the targeted mathematical processes in his classroom mathematics. His overarching goal of helping his students become better people, his commitment to real-world connections, and his resolve to develop deeper understanding could have led to the need for intense use of mathematical processes. This was not the case, however. Instead, we found that his use of the targeted processes in his classroom conformed closely to the way that he engaged these processes in his personal mathematics. Justifying, generalizing, and defining seems to us to be at the heart of deeper understanding of mathematics, and they were not present as processes in much of Russ's classroom mathematics.

Russ has a relaxed view of mathematics. He does not seem to feel a need to pin down every bit of mathematics; he is content to leave loose ends. This view of mathematics and Russ's way of engaging with the targeted mathematical processes did not conflict with his core beliefs about mathematics teaching. One reason for this lack of conflict is that both his goal of developing the whole person and his goal of having students see that mathematics is useful in their lives are, as interpreted by Russ, not tied specifically to mathematics. For Russ, these goals are focused on student affect rather than on mathematical goals. Russ's view of what it meant to help his students become better people centered on helping them learn to interact appropriately with others–a behavior he expertly modeled. His commitment to real-world connections, which could have engaged students in substantial justifying why a real-world situation could be modeled as he suggested, used real-world settings as incidental backdrops instead of situations that necessitated the mathematics. The lack of conflict for him between his goal of students gaining a "deeper understanding" of mathematics and his ways of engaging in processes becomes understandable only when we consider Russ's limited vision of "deeper understanding." His resolve to develop deeper understanding, which could have required consistent justifying, generalizing, and defining, instead manifested itself in his simply asking the question "Why?" without probing students' responses. Russ's beliefs about teaching did not conflict with the ways in which he used processes. If one's use of mathematical processes does not conflict with his or her beliefs related to teaching mathematics, it is natural that we would see the processes play out in the classroom as they do in one's personal mathematics. This was the case for Russ.

Even though there is no evidence of conflict between the three core beliefs held by Russ and his use of processes in his personal mathematics, as well in his classroom mathematics, we posit that he may be on the verge of discovering a conflict between these two constructs. Even in his first semester of teaching, Russ was already aware of a possible lack of understanding of the mathematical content by his students. He stated that he was not happy with his students' grades and he interpreted this poor performance as evidence of a lack of the type of understanding he hoped to see in his students. He further elaborated, "So I would look at a test and it wasn't that there were a lot of silly mistakes it was a lot of no clue what to do. And so I thought okay maybe I need to start pushing there them to really justify what they tell me in class and not just take their guesses." Russ's solution for this lack of understanding was to ask the question "Why?" to his students in the classroom. However, he did not seem to have a clear plan to deal with the myriad of responses to his queries. He was observed to solicit multiple responses from his students, but then, as a rule, he did not ask his students to clarify their answers. As a result, it is possible that this perceived lack of understanding may persist for his students and may evidence itself in continued poor performance by the students on quizzes and exams. In the future, if Russ continues his pattern of being reflective of the students' performance, then he may notice that the student performance is not improving. This may prompt him to question the sufficiency of his strategy of just asking the question "Why?" Doing so would likely create a conflict between his use of the processes and his belief that students should have deeper understanding of the mathematical content.

Endnotes

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A CASE STUDY OF CHINESE ELEMENTARY TEACHERS' KNOWLEDGE OF CURRICULUM FOR MATHEMATICS TEACHING

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This study investigated Chinese elementary teachers' knowledge of curriculum for mathematics teaching (KCMT). Through qualitative interviews, we identified strands of KCMT: orientations, implementation, connections, and resources. The results show the importance of the sequence of topics in mathematics curriculum for different grades and contents of mathematics curriculum materials at a certain grade and at the previous and following grades. Findings suggest that professional community activities and the alignment of written curriculum with intended curriculum contribute to curriculum implementation. We also formed insights for the connections between KCMT and other components of mathematical knowledge for teaching and resources for obtaining KCMT.

Introduction

Chinese students have repeatedly demonstrated relatively superior performance than their U.S. counterparts on international tests (e.g., Lapointe, Mead, & Askew, 1992). Among numerous explanations, Chinese teachers' profound understanding of subject matter knowledge and markedly different pedagogical content knowledge from the U.S. teachers may account for Chinese students' superior mathematics performance (An, Kulm, & Wu, 2004; Ma, 1999).

Teachers' subject matter knowledge and pedagogical content knowledge as mathematical knowledge required for effective teaching, has been a research focus (e.g., Ball & Bass, 2000; Fennema & Franke, 1992; Hill, Rowan, & Ball, 2005). In studies on mathematical knowledge for teaching, however, knowledge of curriculum for mathematics teaching (KCMT), lacks substantial research.

In this study, we examined Chinese elementary teachers' KCMT, focusing on their perspectives on KCMT, the connections between KCMT and knowledge of content and teaching and between KCMT and knowledge of content and students, and their ways of obtaining KCMT. The following research questions guided this study: 1) What are the perspectives on KCMT from Chinese elementary teachers? 2) How are KCMT and knowledge of content and teaching and knowledge of content and students connected in the cases of Chinese elementary teachers? 3) How can Chinese elementary teachers obtain KCMT?

Theoretical Perspectives

Our theoretical perspectives were derived from frameworks on curricular knowledge (Shulman, 1986) and mathematical knowledge for teaching (Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008).

According to Shulman (1986), curricular knowledge is composed of three components: *alternative* curriculum knowledge, *lateral* curriculum knowledge, and *vertical* curriculum knowledge. Alternative curriculum knowledge means knowledge of *alternative* available curriculum materials for instructing a specific subject or content within a grade; *lateral* curriculum knowledge indicates a teacher's ability to connect the content of a specific subject or content with other subjects or content being studied simultaneously by students; and *vertical*

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curriculum knowledge, which means "familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school, and the materials that embody them" (p. 10). With KCMT, teachers are able to discriminate and give priority to mathematical goals and topics (Kilpatrick, Swafford, & Findell, 2001).

In characterizing categories of teacher knowledge base, Shulman (1987) claimed that "curriculum knowledge, with particular grasp of the materials and programs that serve as 'tool of the trade' for teachers" (p.8). Specifically, curriculum knowledge indicates knowledge about "the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances" (Shulman, 1986, p. 10).

In the most recent efforts to provide a map of mathematical knowledge for teaching, Ball and her colleagues (2005, 2008) developed Shulman's ideas of CK and PCK. In the domain of pedagogical content knowledge, three components were included: knowledge of content and teaching, knowledge of content and students, and knowledge of curriculum (Ball et al. 2005). However, Ball and her colleagues further raised questions bout the nature of KCMT:

> We have provisionally placed Shulman's third category, curricular knowledge, within pedagogical content knowledge …. We are not yet sure whether this may be a part of our category of knowledge of content and teaching or whether it may run across the several categories or be a category in its own right (pp. 402-403).

Methods

Participants

Four Chinese teachers participated in the study. They came from two elementary schools in urban and suburban school districts of East China, teaching children with various socio-economic backgrounds. The four participants were recruited according to the following criteria: first, teaching mathematics at an elementary school; second, having at least ten years of teaching experiences.

Procedures

Prior to the interviews, we adapted a list of open-ended questions (English version) from the interview protocol employed by Kaffuman (2002) and examples of grounded theory interview questions (Charmaz, 2000). During the interviews, the participants were also asked additional relevant questions which aimed to get more details from their answers.

Data Analysis

The interviews were transcribed and translated by the first author into English within one week of the interviews. The transcripts were coded and analyzed using grounded theory (Strauss & Corbin, 1990). From a grounded theory perspective, the transcripts were coded in two steps: first, initial or open coding, which helped us discover views of participants and decide how to analyze the transcripts; second, selective or focused coding, in which the most frequently occurred initial codes were used to sort, synthesize, and conceptualize all the transcripts (Charmaz, 2002).

Through examining the transcripts closely through line-by-line coding, the emergence of themes were pursued and conceptual categories began to be constructed to explain and synthesize the data. After the transcripts of the first two participants were coded and categorized, the categories were compared with each other to synthesize the overall patterns of the interviews. During the coding of the next two participants' transcripts, the categories were refined through comparative processes. When the coding of transcripts and construction of conceptual categories were completed, the categories were integrated into a theoretical framework (Charmaz, 2000; Glaser, 1992; Glaser & Strauss, 1967).

Results

This section describes four central themes emerged from data analysis: orientations, implementation, connections, and resources. These central themes characterize the participating Chinese elementary teachers' KCMT.

Orientations

Remillard and Bryans (2004) found that teachers' orientations toward curriculum materials influenced how they used the curriculum. Orientations toward curriculum refers to "a set of perspectives and dispositions about mathematics, teaching, learning, and curriculum that together influence how a teacher engages and interacts with a particular set of curriculum materials and consequently the curriculum enacted in the classroom and the subsequent opportunities for student and teacher learning" (Remillard & Bryans, p. 364). In this study, "orientations" was used to indicate elementary teachers' stance towards KCMT, composed of their perspectives about mathematics, mathematics teaching and learning, and mathematics curriculum. The results show that participants' orientations centered on three major categories: connotations, implications, and levels.

Connotations. Participants described the connotations of KCMT, composed of principles of mathematics teaching, content areas, methods, techniques, and transition. T1 responded by stating principles of mathematics teaching. T4 also asserted, "We have a national-wide adopted *Compulsory Education: Mathematics Curriculum Standards* (italics added and abbreviated hereafter as the *Standards*)". T1 stated, "I think of content areas in elementary math such as number and algebra, space and graphs … and statistics and probability". T2 mentioned the arrangement of topics in curriculum materials. T1 responded, "Teaching methods and techniques, like, problem posing, practice, communications and discussions … including teacher-student and student-student [discussions]". Participants also made references to transition as an importation connotation of KCMT. For example, T1 reported the transition from elementary mathematics curriculum to middle school mathematics curriculum. Similarly, T3 mentioned the transition from kindergarten mathematics curriculum to elementary mathematics curriculum and the transition from elementary mathematics curriculum to middle school mathematics curriculum.

Implications. Participants shared their orientations in terms of implications of KCMT. Positive implications included a bridge between teaching and learning, a foundation for encouraging students' thinking, and the premise for achieving teaching and learning objectives of the lesson. T1 regarded KCMT as "a medium for the interactions between teacher and students". However, more responses focused on negative implications. For instance, T2 reported an implication of "memorization," and structured ways of teaching with teacher dominance of the class. Similarly, T3 stated an implication of "duck feeding," the outdated teacher-centered style. T4 echoed this point by stating "fixed without flexibility," serving for passing examinations and "not helpful for student development".

Levels. Two levels of KCMT were reported by participants, including KCMT at the macro level and KCMT at the micro level. KCMT at the macro level refers to the sequence of topics in mathematics curriculum. KCMT at the micro level indicates the content of curriculum materials. The most important approaches mentioned by the participants to achieve macro-level KCMT included reading textbooks and teacher manuals for different grades and gaining experience from teaching all grades in an elementary school. Professional community activities in a school or district were also reported helpful.

Major approaches to get micro-level KCMT consisted of familiarity of textbooks at a certain grade and also at the previous and following grades, gaining experience from teaching one grade and participating in teaching and research activities at the grade level. In those activities, as T4 reported, teachers "focus on one copy of the curriculum materials we [teachers] are using and discuss the overall framework and lesson plans for each unit".

Implementation

Implementation of intended curriculum to mathematics classrooms is a key dimension in the participating Chinese elementary teachers' KCMT. Only through implementing the intended curriculum to actual teaching can teachers apply their KCMT.

According to the participants, there is not a direct link between intended curriculum and implemented curriculum. All participants agreed that the intended curriculum or the *Standards* provides a general guidance or blueprint with basic requirements and objectives. The main ideas offered by the intended curriculum serve as orientations for teachers. However, the question is, as T2 said, "how to teach?" T4 also held a similar view: "[The intended curriculum is] not a detailed instruction on how to teach".

In order to transform from intended curriculum to implemented curriculum, participants reported several key factors. Two categories emerged from the interviews: professional community activities and the alignment of written curriculum with intended curriculum.

Professional community activities. Professional community activities include teaching and research activities at the grade, school, or district level. According to T1, professional community activities at the district level are more general than activities at the grade level. T3 also stated that grade-level activities are more practice-oriented: "Teachers in the teaching and research group [at our grade] discuss the content together. For instance, in the treatment of examples, some issues emerged after … [our] discussion, which I ignored or did not realize the importance before".

As stated by T4, the grade-level professional community activities usually last for "half an hour from Monday to Thursday". T1 reported that school-level professional community activities are held "every Tuesday," and the district-level activities occur once "for every semester". It is clear that the most frequent professional community activities are grade-level teaching and research group activities, and the least frequent are district-level activities.

Alignment between written curriculum and intended curriculum. A key emergent issue in curriculum implementation was that teachers can understand intended curriculum by reading textbooks and teacher manuals due to the alignment of written curriculum with intended curriculum. T3 reported, "I do not read the intended curriculum. The textbooks provide objectives for teaching each unit. The objectives of each unit in textbooks represent the objectives of the intended curriculum". T4 also stated, "Aligned with the *Standards*, teacher manuals describe the requirements for each unit. In my teaching, I followed teacher manuals closely to ensure the achievement of the requirements in the *Standards*".

Because of the alignment between written curriculum and intended curriculum, these Chinese elementary teachers were not faced with any challenge in the transformation from intended curriculum to implemented curriculum at the stage of understanding intended curriculum. As reported by the participants, Chinese elementary teachers do not have to read and understand the intended curriculum as much as they read textbooks and teacher manuals. Instead, they read the intended curriculum as a reference book when teachers, as T3 illustrated, “unsure about some points or teachers show disagreements about some issues”.

Connections

Participations described connections between KCTM and knowledge of content and teaching and between KCTM and knowledge of content and students.

Connections between KCMT and knowledge of content and teaching. Two categories emerged from the interviews: knowledge as continuous development, and flexible treatment of curriculum materials. According to the participants, an important influence of their KCMT on knowledge of content and teaching is viewing mathematics knowledge as continuous development in teaching. For example, T3 stated:

> *If you know what needs to be taught in the following classes, you know where you are leading the students …. Let’s say we have beads. If you have knowledge of elementary math curriculum, you will know how to make a string of beads other than separated beads. In that way, you will have a better idea about how to teach in class.*

Furthermore, T1 reported how KCMT is connected with their knowledge of content and teaching: “I learned what prior knowledge is connected to new knowledge. When I teach a new topic, usually I introduce new knowledge by reviewing the prior knowledge”. It can be seen that instead of focusing on isolated bits of mathematics knowledge, these Chinese teachers locate each piece of knowledge in the whole picture of elementary mathematics and view mathematics knowledge as continuous development.

Participants described that their treatment of curriculum materials in teaching mathematics has also been impacted by KCMT. As T3 shared:

> *Since I have knowledge of elementary math curriculum, I will delete some parts and add some other parts as adjustments [in mathematics teaching]. At the sixth grade, we will teach problem solving on fraction and equation solving …. The textbook compilers lowered the learning requirement for computation. Only one-step computation is required in the fifth-grade textbook, such as $2x=10$ or $2+x=10$. Since I am familiar with the math curriculum, I know that problem solving skills and equation solving are important for student learning of algebra such as variables in the future. Therefore I will provide more challenging problems in teaching [at the fifth grade], such as $2x+5=10$.*

Also, T3 “intend[ed] to informally introduce some topics at lower grades”. It is shown that due to the impact of KCMT, participants treat curriculum materials in a more flexible way.

Connections between KCMT and knowledge of content and students. Connections between KCMT and knowledge of content and students were reported in the interviews. According to the participants, understanding problems and difficult points presented in teacher manuals and preparing lesson plans for students help them anticipate common errors and difficulties of students.

T2 shared the experience of preparing lesson plans for students:

> *Now we should be able to, according to the principles of the Standards, prepare lesson plans for students. Lesson plans should consider how students are likely to understand the topic and anticipate their answers to questions. [When preparing lesson plans,] teacher should take into account student thinking, and their possible answers and approaches. Also, teacher may think about the balance between the amount of time for questioning and for students to think about and answer questions. So I think knowing elementary math curriculum helps to understand student learning.*

However, connections between KCMT and knowledge of content and students are not as strong as connections between KCMT and knowledge of content and teaching. Three participants reported that their knowledge of content and students mainly come from their former students at the same grade they taught before. T3 added that asking for suggestions from teachers who already taught the same topic impacts their knowledge of content and students.

Resources

Participants reported resources to obtain KCMT. The most frequently reported resources included textbooks and teacher manuals. All the four participants emphasized the importance of reading textbooks and teacher manuals. T4 stated, "[Elementary teachers should] spend a lot of time reading textbooks and teacher manuals". T4 suggested, "Read textbooks and teacher manuals everyday". Besides time and frequency of reading textbooks and teacher manuals, participants also emphasized the expected goal of reading. As T3 reported, "[Elementary teachers should] read all twelve copies of elementary math textbooks and study the connections between and among mathematical topics, the extent [of teaching] and learning objectives". Furthermore, T2 pointed out the necessity of reading textbooks over and over again:

> *I will read the fifth-grade textbooks again even if I read the fifth grade textbooks before or I already taught the fifth grade. Every time you read the textbooks, you will have different reflections and learn new knowledge. As Confucius said, if you can gain new insights through restudying old material, you are qualified to be a teacher.*

The second most frequently reported resources were videos of mathematics teaching and open classes. Both T3 and T4 said that watching videos of mathematics teaching and attending open classes help teachers gain KCMT, especially new teachers or pre-service teachers. T2 reported reflective journals as another resource, and explained as follows,

> *When I write down my reflections, I will analyze the classes I taught and find theoretical support for the success or deficiency of a class. Every elementary math teacher has to write lesson plans, but it is better to write something more like reflective journals of classes and analyses of open classes. Writing will motivate elementary math teachers to learn and develop knowledge. It is a process of accumulation. Also, teachers may have chances to participate in writing competitions and even publish their reflective journals.*

Other resources mentioned by participants included the *Standards*, teaching and research activities, online resources, books, lectures, colleagues, experienced teachers, courses in teacher preparation programs, other subjects such as reading and science, exemplary lesson plans, and teaching classes while inviting other teachers to attend and provide comments and suggestions.

Discussion

Through qualitative analysis of interviews with four Chinese elementary teachers, we generated a description of KCMT, and identified its strands, including orientations, implementation, connections and resources. As an exploratory study, it serves to broaden our understanding of KCMT.

Conceptually, this case study contributes to deepen our understanding of the role of KCMT in the map of mathematical knowledge for teaching with some evidence that KCMT is connected with other components of mathematical knowledge for teaching. More importantly, it highlights the importance of KCMT and serves as a starting point for studying KCMT.

In addition, the findings from this study may be helpful for professional development of elementary teachers by illustrating important resources for obtaining KCMT. It offers insights for elementary teachers to improve their KCMT by such means as reading textbooks and teacher manuals, watching videos of mathematics teaching, and keeping reflective journals on their teaching.

Limitations were noted in this study. For example, the small number of participants may not represent the population of Chinese elementary teachers. In addition, different from their U.S. peers, Chinese elementary teachers use the national-wide adopted curriculum materials. In fact, some researchers suggested that the lack of a common and coherent curriculum hampers mathematics teaching and learning in the U.S. (e.g., Schmidt, Houang, & Cogan, 2002). Even with the limitations, however, this exploratory study provides an opportunity to get a look into KCMT.

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ASSESSING HOW PRE-SERVICE TEACHERS UNDERSTAND BALANCE THROUGH CLINICAL INTERVIEWS AND A VIRTUAL TOOL

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Our study was enacted in mathematics education classes with pre-service teachers (PSTs). This research focused on videotaped PSTs' interview responses used to assess their understanding of balance when challenged with tasks involving virtual manipulatives. PSTs relied on visual cues to implement procedures that were often inappropriate for the task at hand. When confronted with missing-value balance tasks, 47% of the PSTs attempted an incorrect procedure using direct proportions while others employed an incorrect fractional method. Only one PST systematically solved the tasks, where he invented an inverse proportions model. Most interviewees relied on intuition and qualitative guess and check reasoning.

Introduction

This study investigated pre-service teachers' understanding of the "big idea" (Wiggins & McTighe, 2005) of balance, an idea that transcends the isolated worlds of pure mathematics and science. Our research paper concerns using the context and big idea of balance to assess pre-service teachers' understandings of concepts related to equilibrium. We claim that one cannot understand "balance" without understanding the physical conditions of equilibrium and the mathematical conditions of equality. We argue that studying balance situates mathematics in physical contexts (forces, torque, and center of gravity) and mixes the physics of balancing weights within mathematical realms of systematic reasoning, inverse proportions, vectors, equalities, and product summations.

Literature Review

The *Principles and Standards for School Mathematics* (NCTM, 2000) advocate the need to integrate the mathematics and science disciplines. NCTM argues that students need to experience contextualized mathematics in order to aid understanding of fundamental concepts. Certain topics lend themselves to natural integration such as sound waves and superposition of sinusoidal functions, or motion and rate of change concepts. Some topics are often indistinguishable to classify as "math" or "science" and demand integration. For example, density (mainly taught in the science classroom) could easily be viewed as a mathematics concept with an emphasis on volume and measurement of mass; or velocity and speed concepts taught in the physical sciences are often taught in mathematics as rate problems. For this paper we focus on the topic—balance, which is much more than a topic that is difficult to classify and is, in fact, a big idea. What is a "big idea"? Wiggins and McTighe (2005) defined a big idea as "a concept, theme, or issue that gives meaning and connection to discrete facts and skills" (p. 5). The ideas of balance are found in such things as: mathematical equivalency; general algebraic equations to chemical and physical equations; the center of gravity of an object; and the creation of stability through distribution of weight on each side of a vertical axis.

According to the National Council of Teachers of Mathematics in *Curriculum Focal Points,* students should be able to a) use symbolic algebra to represent situations to solve problems, b) recognize and generate equivalent forms for simple algebraic expressions, and c) model and solve contextualized problems using various representations (NCTM, 2006, p. 36). Problems or tasks involving balance fashion a scenario where learners must: consider the system's center of gravity and conditions for equilibrium; visualize possible rotations; recognize equivalent situations that will create stability; and algebraically represent and model the proposed physical situation. In addition, learners must be able to reason systematically as they problem solve, thus requiring them to "act in the mathematical moment" (Mason & Spence, 1999) as they engage in the big idea of balance.

For this paper, we describe pre-service teachers' ways of thinking and methods of problem solving as they engage in tasks involving balance using a virtual applet. Examples of a balanced beam are illustrated in Figures 1 and 2. Previous research on children's thinking about balance have shown generally four stages of cognitive development regarding rules for distinguishing states of balanced equilibrium (Siegler, 1976; Klahr & Siegler, 1978). The most primitive rule (*Rule 1*) used by children to determine if a beam is balanced involves consideration of the weight on each side of the fulcrum; if the weight on both sides is equal, the beam is balanced and if not, the heavier side dips down. The second rule (*Rule 2*) is slightly more advanced than the first rule and considers the weight's distance from the fulcrum. In general, if the weight is the same on each side, then the weight with the greatest distance from the fulcrum goes down; however, if distances are equal then the beam is balanced. The third rule (*Rule 3*) incorporates both the first and second rules, but also considers situations where the weights on both sides may not be the same and may not be the same distance from the fulcrum. *Rule 3* does not assist with those situations where the greater weight is not on the same side as the greater distance from the fulcrum. When this conflict arises, children usually use the "muddle through" or guess and check process as described by Siegler (1976). Finally, the fourth rule (*Rule 4*), considered by Siegler as the most sophisticated rule, displays the learner systematically realizing the composition rule of summing the products of weight and distance on each side of the fulcrum.

Other methods, but similar to Siegler's *Rule 4*, of solving the problem of balancing a beam or determining if a beam is balanced often includes application of the more traditionally taught summation of torque rule. That is, if the sums of the total torque, $\Sigma_i F_i d_i$, (where F is the perpendicular force on the beam and d is the distance between the applied force and the fulcrum) on each side of the fulcrum are equal and the summation of forces on the beam are equal to zero, then the beam is balanced. Yet another balancing rule, explained by Inhelder and Piaget (1958), involves a double inverse proportion resulting from the amounts of work needed to move two weights, W and W' (on opposite sides of the beam's fulcrum) to heights H and H'.

> "When two unequal weights W and W' are balanced at unequal distances from an axis L and L', the amounts of work WH and W'H' needed to move them to heights H and H' corresponding to these distances are equal. Thus, we have the double (inverse) proportion: W/W' = L'/L = H'/H" (Inhelder & Piaget, p. 164).

Shen (2006), conducted a study on the topic of balance with in-service K-8 science teachers and found that the teachers tended to generate only a limited additive rule (*Rules 1 or 2*), when determining whether a beam was balanced or not, and failed to incorporate the more general multiplicative rule when considering balance (*Rule 4*).

University students often only "experience" mathematics in abstract uncontextualized settings. Because of this, they have difficulties knowing how to solve contextualized mathematical tasks. Mason and Spence (1999) described varying degrees of knowing mathematics: Knowing something is true, knowing how to carry out a procedure, knowing why something is true, and knowing to act in the moment when given a novel problem. Wilhelm, Sherrod, and Walters (2008) examined students' mathematical and scientific thinking as they engaged in project work. Wilhelm et al. found that students had difficulty applying basic mathematics (scaling, ratios, sinusoidal curves) within their project tasks. One piece of the project work necessitated the need for a calculation of an object's velocity as it moved in a circular path. Students seemed to only know that distance and time were involved in this velocity calculation and were unsure whether distance and time should be multiplied or divided. Students had difficulty reasoning systematically about the velocity situation even though all should have been able to relate it to everyday experiences of driving a car and maintaining speed limits of 55 miles/hour or 65 miles/hour. Many also claimed that the distance around a circular path was πr^2 instead of $2\pi r$ and missed an opportunity to reason dimensionally.

Often when novel tasks are posed to students, instead of thinking systematically, they simply rely on visual cues or practiced mathematical procedures that are inappropriate for the task at hand. Davis (1984) as cited in Kirshner and Avery (2004), described how learners used visually triggered sequences to solve problems, "visual cue V_1 which elicits a procedure P_1 whose execution produces a new visual cue V_2 which elicits a procedure P_2,…and so on" (p. 227). Other research has documented that when a task is posed as a missing value word problem, students used a proportional method to problem solve, even when this method did not yield a correct result (Van Dooren, De Bock, Evers, and Verschaffel, 2009). We follow with our research study where participants were given opportunities to apply their background knowledge and experiences to a novel situation using a virtual balance applet.

Participants and Methods

The 17 subjects (13 female, 4 male) in this study were pre-service teachers (PSTs) of junior ranking enrolled in a middle level mathematics methods class taught by the second author in a state university in the southwestern United States. This paper focused on videotaped and transcribed PST responses during a clinical interview assessing PSTs' understanding of balance as they were challenged with tasks involving a virtual manipulative. The first author conducted the interviews with all 17 subjects. The virtual balance applet (Rensselaer Polytechnic Institute, 1998) allowed for the subject to place various weights at different locations on a uniform beam that had a fulcrum located at its center (see Figure 1). No PST had any experience with this balance applet prior to their interview; however, all participants had experiences with physical balances within their math methods class work. In their math methods class, PSTs had built a physical balance and practiced leveling it. Of the 17 interviewed PSTs, 13 had previously taken a physics class (ten college physics, three high school physics). Of these 13, seven had been physics students in Author 1's Introductory Physics class at some point during their middle level certification program but were not her current students during the time of data collection.

The interview protocol consisted of three main tasks that were inspired by Duckworth's (1996) chapter pertaining to "*Figuring out My Own Ideas*" (p. 122-124). The first task had an unbalanced beam with one weight located a set distance to the right of the fulcrum. Each PST was asked to predict, test, and check where a second weight would need to be located in order for the beam to be balanced. The second task had the PST observe the weight arrangement shown in Figure 1, and asked the PST to consider how one would have to move either the red or the yellow object in order to maintain balance if the blue object were moved 30 units to the right.

The third task asked the PST to consider the arrangement shown in Figure 2. The posed question was "If we were to move the blue weight 100 units to the left, how would we need to move one of the other three weights to maintain balance?" The research questions directing this study concern: A) *How do pre-service middle level teachers think about balance and what prior knowledge will they use as they determine ways to achieve or maintain a balanced system?* B) *How will pre-service teachers translate virtual explorations towards a more abstract understanding of equivalence and equilibrium?* Siegler's rules were used in analyzing the PSTs' responses to the balance problems to help determine each PST's level of abstraction achieved regarding balance, equilibrium, or equivalence. Methods of problem solving were coded into categories of proportional, fractional, guess and check, and inverse proportional.

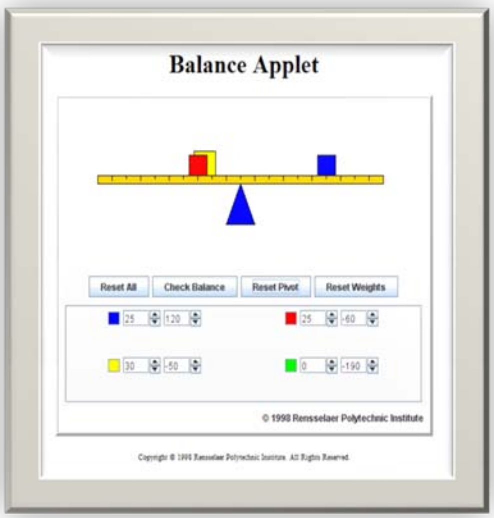

Figure 1. Balanced beam for task 2 of the clinical interview. Rensselaer Polytechnic Institute (1998)

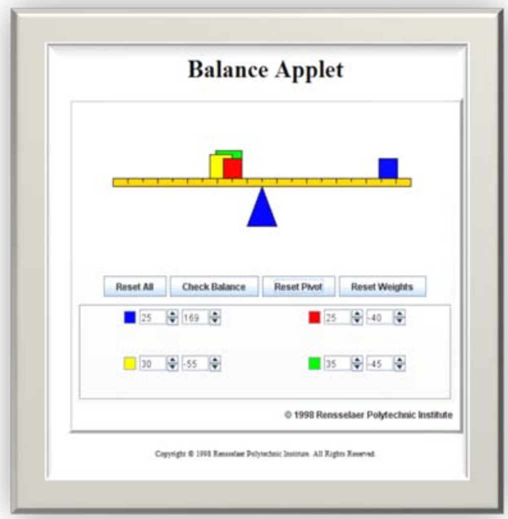

Figure 2. Balanced beam for task 3 of the clinical interview

Data & Analysis

We highlighted representative examples of each of the methods employed by the PSTs to illustrate how they considered their balance tasks, problem solved, and utilized prior knowledge. We also categorized their methods according to Siegler's Rules when appropriate in order to ascertain each PST's level of abstraction as it pertains to balance and equivalence.

Nearly 60% of the participants in the study had previously taken their university physics course, thus giving them prior exposure to balance concepts and conditions for equilibrium. During their interviews, none of the 17 PSTs mentioned the words torque or conditions for equilibrium even though 10 had taken physics within the last two years of their program. Prior to the interviews, all participants had conducted an activity in their mathematics methods class involving the exploration of balance with a physical apparatus (created out of thin cardboard for the balance beam, a pin for the fulcrum, and paper clips for weights).

Each interview began with PSTs' considering an unbalanced beam with one weight (50 units) located a distance of 120 units to the right of the fulcrum. Each PST was asked to predict, test, and check where a second weight would need to be located in order for the beam to be balanced. PSTs chose a weight equal to 50 units placed at a position 120 units left of the fulcrum, which did create a balanced system. This confirmed that all participants had succeeded in correctly utilizing Siegler's most primitive *Rule 1*.

During each of the posed tasks, follow-up questions were asked regarding placement of a second weight that was not equivalent to the weight on the opposite side of the fulcrum. Doing such questioning allowed access into PSTs' level of abstract understanding for those situations where weights and distances were unequal (Siegler's *Rules 2 and 3*). For example with task 1, each PST was asked: *Where would you place a second weight that was not the same weight as the 50 unit weight in order to balance the beam?* Many interviewees chose a lighter second weight to be placed left of the fulcrum. Most (82%) knew that a second weight lighter than the weight on the opposite side would need to be placed farther out from the fulcrum than the heavier weight in order for balance to be achieved. This demonstrated that 15 of the 17 PSTs were within Siegler's *Rule 3* classification level. Many verbally linked their reasoning with prior experiences such as, their math methods balance activity (2 PSTs), a physics course (3 PSTs), seesaw riding (2 PSTs), and daily life (2 PSTs). The following excerpts from the transcribed interviews (names are pseudonyms, R is the researcher) illustrate such previous experiences:

> **Eva:** Maybe if you went a little lighter and put it out a little farther out.
> **R:** Okay, so you think a little farther out if it *(the second weight)* were a little lighter?
> **Eva:** Yeah, we just see-sawed all weekend….for our sorority to raise money….we have like really light people and heavy people.
> **R:** So the heavier person has to go where *(to balance the seesaw)*?
> **Eva:** The heavier person, the farther up *(towards the fulcrum)*, and the lighter person back *(away from the fulcrum)*.

Another PST, Lisa, also correctly stated that a second weight, heavier than the existing 50 unit weight, would have to move farther in, while a lighter weight would have to move farther out. She claimed she knew this would work because of her math methods exploration where she had been "*doing the whole balancing thing. I was exploring with the different paper clips.*"

When not relying on a guess and check method, eight of the 17 (47%) attempted a ratio and proportion method to solve for the location of a second weight. For these PSTs, the visual cue was the missing value, x, for the position of the second weight. For example, one PST's method involved setting up the following proportion: $50/40 = 120/x$ where 50 was the weight located 120 units to the right of the fulcrum and 40 was the added weight left of the fulcrum. This proportion was equivalent to $W/W' = L/L'$, a direct proportion as opposed to the correct inverse proportion. When solving for x, this PST (Amy) arrived at a distance of 96 units; however, she knew instantly that this answer could not be correct since it was not larger than 120 units. She then reasoned that she should add 96 units to 120 units and move the second mass to a position 216 units left of the fulcrum, which also did not balance the beam. The following is an excerpt of Amy's next approach.

> **Amy:** If it is lighter, it needs to be farther away I am going to start at 150 and sees what that does? *(Moving the 40 unit weight to 150 units left of the fulcrum balanced the beam.)* I wonder why that works.
> **R:** Yeah, Why do you think that works?
> **Amy:** I changed the weight by a variable of 10 and then had to move it 30 away. Well, let's see what 1/5 of 120 is Oh! Fractions *(Amy became excited because she thought she had*

figured it out.) So, we take away 1/5 of the distance so 1/5 of the distance from 120 is 30**.** So it is fractional….The mass is proportional, the distance isn't proportional**.** It is a fractional thing…**.**So we have 40/50, 4/5^{ths}**.** Need to find 4/5ths of 120. No, we take away 1/5 from the weight of the block. 1/5 of 120 so you get 5x=120 *(calculating answer).* So it is still not exactly right. In theory it is kind of right, you get closer to the idea.
R: So how do you know it is not exactly right?
Amy: Because you need a change of 30
R: And you got a change of 24.

Amy considered how the 40 unit weight needed to move 30 units farther left of the 120 unit position for the system to balance (to position 150 left of the fulcrum). She claimed that in order to solve the problem, one needed to proportionally think of the masses (W/ W'), but use a "fractional thing" to discover the distance needed to move the 40 unit block. After doing the calculation of 1/5 of 120, she arrived at a "change of 24," which was not the needed change of 30.

Like Amy, others chose to pursue a "fractional method" to discover a missing value position while solving the second balance task (35%). This second task involved the balanced arrangement shown in Figure 1. PSTs were asked if the 25 unit (blue) block on the right were moved 30 units to the right, how would one need to move the yellow (30 unit) block in order to maintain balance. We follow with two representative examples of the fractional method utilized by PSTs.

Jill: I am going to move the yellow over, we will try 15 *(15 units left).*
R: Why are you thinking 15?
Jill: I just move it half. I know that yellow is bigger so I don't think it is going to take as much movement.

Similar to Jill, another PST used fractions beginning with a leftward movement of 1/5 of the 30 unit distance that the blue block had been moved. *"Okay, since trying 1/5 of distance didn't work let's try ½, so 15. So that is -65. Still made blue heavier*". This fractional method mixed with trial and error was continued until the correct 75 unit position (left of the fulcrum) was found for the yellow block. Once this correct position was realized, the PST pondered, "*Now why is that? Trying to think of my physics and that has been so long*".

When another PST, Chris, solved the second task, he came up with an interesting and sophisticated procedure. He made a ratio of the two weights W and W' (25/30) and multiplied that ratio by the 30 unit distance (moved by the blue weight). He arrived at 25 units which he explained to mean was the distance that the yellow weight would have to move to the left for balance to occur. Looking at Chris's method more closely—we find that he was the only PST out of 17 to arrive at Siegler's *Rule 4* level of abstraction. Chris applied the following method: $(W/W')*(\Delta L) = \Delta L'$. He was thinking more in terms of displacement vectors. One could also equate his method to the following, where we begin with the traditional summation of torque rule or product-moment rule. We apply this to the situation in task 2 where we have two objects on the left and one object on the right (subscripts stand for object's color). Summing the torques on the left and right where the beam is initially in balance, we get $W_rL_r + W_yL_y = W_bL_b$. Now subtracting the new summation of torques where only the yellow object is moved to a new position L'_y and the blue object is moved to a new position L'_b, we arrive at the following equation:

$$\begin{gathered} W_rL_r + W_yL_y = W_bL_b \\ -(W_rL_r + W_y\, L'_y = W_b\, L'_b) \\ W_y\,(L_y - L'_y) \;=\; W_b\,(L_b - L'_b) \\ W_y\,\Delta L_y \;=\; W_b\,\Delta L_b, \\ \Delta L_y \;=\; (W_b/\,W_y\,)\Delta L_b \end{gathered}$$

Those that ultimately solved task 2's positioning of the yellow weight arrived at the correct answer through guess and check methods except for Chris who employed a systematic approach utilizing displacement vectors and inverse proportions.

Task 3 used the scenario shown in figure 2, a balanced beam with three weights left of the fulcrum and one weight to the right of the fulcrum. Each PST was asked how one of the weights (located left of fulcrum) would need to be moved if the 25 unit weight (right of fulcrum) were moved 100 units left. This question required displacement of one of the weights across the fulcrum in order to maintain balance. The easiest weight to move would be the red block since it was of equal weight to that of the blue block. However, it still needed to move a total distance of 100 units to the right resulting in a crossover to the other side of the fulcrum, landing it at position 60 units to the right of the fulcrum. Of the 17 PSTs, 7 (41%) correctly positioned the red block at the 60 unit location. Although solving this task in this manner did not require a *Rule 4* abstraction level, it did hone in on ideas of equality as well as displacement vectors. For example, one PST asked permission to cross this imaginary boundary of the fulcrum and was successful at balancing the beam when she placed the red (25 unit) weight 100 units to the right of its original position. She claimed this worked because of "*equal and opposite reactions, just like balancing equations…whatever you do to this side, you have to do to other side…so you wouldn't do something to this side that you wouldn't do to this side because it is not fair*." Where this PST described this balance situation "*like balancing equations*," others made reference to both distance moved and direction, "*So I was correct in wanting to move the red one in the same amount as the blue one; I just need to move it in the opposite direction.*"

While many (41%) were able to solve this task with a simple movement of the red block, albeit across the fulcrum, others (35%) seemed to revert back to a *Rule 1* level of thinking. PST, Samantha, chose to move the red block to position -69 (69 units left of fulcrum) since the blue block was moved to position +69. She seemed to think the system would balance even though there were two additional weights left of the fulcrum besides the red block. Similarly, two other PSTs were unsuccessful as their balance efforts contained a 100 unit leftward movement of the red block instead of the correct vector displacement of 100 units to the right.

As with the other two tasks, a follow-up question required the PST to consider movement of one of the other two weights (yellow or green) that were not of the same weight as the blue 25 unit weight. All PSTs that made this attempt at balance with the other weights made use of the guess and check method except for Chris. He applied the same approach he invented during task 2 for each individual movement. However, when using the technique for the green and yellow weights, he did not arrive at a whole number. His correctly calculated result for the green weight was to displace it 71.4 units to the right. The applet did not allow decimal values, so he first tried a movement of 71 units (which tilted the beam to the left) and then a movement of 72 units (which tilted the beam to the right). He was confident that his technique worked and if the applet permitted decimal answers instead of only whole numbers, the beam would have balanced.

Conclusion and Implications

During the interviews, PSTs vocalized that the “balance” tasks were similar to previous math methods activities, riding a seesaw, physics course lessons, or daily life experiences. Assessing PSTs understanding of concepts related to the big idea of balance showed mainly qualitative levels of sophistication where 82% of the PSTs displayed comprehension within the *Rule 3* stage of Siegler’s analysis. Of these 82% only one PST could quantitatively solve all balance tasks and achieve *Rule 4* abstract mastery.

When posed with three balance tasks during the clinical interviews, most PSTs struggled; none applied the “traditional” rules for establishing equilibrium, but instead relied on intuition and qualitative guess and check reasoning (more than 70%). When other methods were pursued, PSTs relied on visual cues to implement mathematical procedures that were inappropriate for the task (similar to Van Dooren, De Bock, Evers, & Verschaffel, 2009). When confronted with what appeared to be a “missing-value” balance task, 47% of the PSTs attempted an incorrect procedure using direct proportions while 35% made use of an incorrect fractional method. Only one PST systematically solved the tasks, where he invented an innovative model using inverse proportions and displacement vectors.

Our study included 13 PSTs who had taken either a college or a high school physics class, where the presumption was that all had been exposed to concepts concerning conditions for equilibrium ($\Sigma_i F_i d_i=0$, and $\Sigma_i F_i=0$). However, only three PSTs related the balance tasks to their previous physics experiences and only one systematically solved all problems. This suggests that although the teacher education candidates had taken courses in physics as well as other advanced mathematics, they did not fully understand the concept of balance from either the mathematics or physics perspective.

The implications of this study advise that more purposeful experiences should be crafted within teacher education programs that lead to better connections across the disciplines of big ideas such as balance. These experiences should prompt future teachers to act in the mathematical moment and move their mathematical reasoning beyond qualitative connection making towards higher levels of abstraction that can be applied to new and novel situations.

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BEYOND NOTHING: TEACHERS' CONCEPTIONS OF ZERO

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This qualitative study engaged two elementary teachers in a hermeneutic dialogue about how they understood zero and when and why zero became a part of teaching and learning in their classrooms. The data collected revealed two ways in which the teachers understood and worked with the concept of zero: as a set of techniques and procedures and as a philosophical and theoretical concept. The results of this study revealed many potential areas of interest regarding teachers' and students' number sense and understanding of zero.

Introduction

Zero is a foundational, complex, and multi-functional concept in modern mathematics (Seife, 2000; Barrow, 2000; Kaplan, 1999), yet research has shown that pre-service teachers struggle with many of the understandings of zero within elementary mathematics (Wheeler & Feghali, 1983). Ma's (1999) research comparing the mathematical knowledge and understandings of elementary teachers in Singapore and the United States also revealed that teachers from the United States misconceptions related to zero. Furthermore, research involving students demonstrates that they have the same misunderstandings of zero as identified for pre-service and in-service teachers (Piaget, 1964; Pasternack, 2003; Wilkinson Evans, 1983; Baroody, Gannon, Berent, & Ginsburg, 1983; Reys and Grouws, 1975; Kamii, 1981; Allinger, 1980; Whitelaw, 1984; Anthony & Walshaw, 2004). These parallel results suggest that the development of misconceptions about zero may be supported through classroom interactions. Leeb-Lundburg (1977) however, has demonstrated that through engagement in activities that involve exploring quantity and place value, elementary students are able to come to understand many of the roles of zero within mathematics.

Given the importance of zero in mathematics, the lack of understanding by both teachers and students warrants further investigation into the situation. As a starting point for this investigation, this paper details the methodology and results of a new research project that investigated the response of elementary teachers to the questions: "How do you understand zero," and "When and why does zero become a part of teaching and learning in your classroom".

Literature Review

Wheeler & Feghali (1983) investigated many different facets of pre-service teachers' understanding of zero. Paralleling research into students' understanding of zero (Inhelder & Piaget, 1964; Neurwirth Beal, 1983; Pasternack, 2003; Wilkinson Evans, 1983; Baroody et. al., 1983; Wheeler, 1987), 15% of the pre-service teachers Wheeler & Fegahli (1983) interviewed said that zero was not a number, explaining that zero is 'nothing', and that numbers must represent 'something'. More than half of the pre-service teachers habitually said 'oh' instead of 'zero' when reading phone numbers and the digits of license plates out loud which is another common finding in the research related to students' understanding of zero (Baroody et. al., 1983; Allinger, 1980; Whitelaw, 1984). Even in replicating some of the tasks that Inhelder& Piaget (1964) used with children, Wheeler & Feghali (1983) obtained results for the pre-service teachers that were reminiscent of those produced by the children. Wheeler & Feghali's (1983) results demonstrated that the pre-service teachers only considered partitions of numbers involving zero if the problem being solved was set in a real world context, such as fishing. The pre-service teachers also struggled with the idea of using zero, or the null set, as a possible classification within a sorting task.

Ma's (1999) research, involving the comparison of mathematical knowledge of teachers from the United States and Singapore, also gave insights into the teachers' understanding of zero within the contexts of subtraction and multiplication. In the case of both operations, the teachers from the United States demonstrated misunderstandings related to zero in terms of place value and the decomposition of numbers, which echoed the results of Neurwirth Beal's (1983) research involving students. The teachers from Singapore, however, demonstrated greater understanding of zero in both contexts (place value and number decomposition).

In response to this literature, this study embarked on further exploring how teachers understand zero and how those understandings are brought to the mathematics classroom. By choosing to use a qualitative approach, I hoped to uncover what specifically teachers believe and know about zero, and to engage the participants and myself in a process of learning as well.

Methodology

The intent of the research questions required an approach that would engage participants in revealing and constructing personal meaning and understanding of zero, leading to the selection of a hermeneutic approach based on Gadamer's philosophy. Three features of Gadamer's philosophy played an important role in the defining of the methods of data collection and analysis in this research: horizons of understanding, dialogue, and the meaning of words.

Gadamer (1989) defined two inter-related horizons of understanding that each person has for every concept or idea. The historical horizon is defined by the past and traditions that come from the generations before. The present horizon is a composite of all that a person believes, has experienced, and has learned about a concept. The historical horizon is an influence, often unrecognized by the individual, on his/her present horizon. Using dialogue, Gadamer (1989) contends that individuals can explore and come to know both their own present horizons of understanding as well as the present horizons of those that are involved in the dialogue. In a Gadamerian hermeneutic dialogue, the "Dialectic consists not in trying to discover weakness in what is said, but bringing out its real strength" (Gadamer, 1989, p. 367), and thus provides everyone involved in the dialogue with the opportunity to clarify their understandings of others' horizons, while also validating, modifying, and expanding their own. This negotiation of one's horizon is important because of a dichotomy related to words that Gadamer (1989) identifies. Gadamer (1989) presents that a word names an object and everything that object is, however; when a person speaks or hears that particular word he/she is only using that word in the context of a small portion of his/her present horizon. Gadamer (1989) argues that through hermeneutic dialogue both speaker and listeners come to greater understanding as they reveal, explore, validate, and expand the portion of their present horizons being accessed.

Participants

This research involved Nora and Elaine, pseudonyms for the two teachers, who taught in different schools in the same urban school division in central Canada. Nora was teaching a class of 22 students aged seven to ten in a public elementary school in a mostly mid-lower socioeconomic area. Elaine was teaching a class eight to twelve year olds in a mostly lower socioeconomic area with a number of her students requiring additional learning assistance.

Research Methods

Data was collected through three group meetings, an in-class teaching session in each of the participating teachers' classrooms, and an interview with the teacher following one of the in-class teaching sessions. The three group meetings were each three hours in duration and audio recordings of the meetings were transcribed and then verified by the teachers. Through dialogue directed by questioning, the three group meetings focused on the teachers' recollections of learning and teaching about zero, exploring and assessing the teachers' revealed understandings of zero, creating new understandings of zero, and discussing when and how zero enters into, or should enter into, mathematics instruction.

Results and Analysis

The data collected through the group meetings, in-class teaching session, and interview was explored for emerging patterns and themes. These ideas were compared to those found in the literature review, looking for similarities, as well as departures and new perspectives. The insights and ideas of the two teachers were codified according to the broad themes, which form the organizational structure and content of the results.

Starting To Know Zero

At the beginning of our dialogues, Nora and Elaine reflected on their memories of learning and knowing about zero. Elaine commented that she "couldn't come up with a concrete time period when [she] actually [remembered] spending any time talking about [zero]". Nora agreed, but gradually, three themes of knowing zero emerged: zero as a number, zero within procedures, and zero outside the classroom. All three of these themes highlighted how their experiences and knowledge of zero as students were fragmented and dominantly procedural and technical aspects of understanding.

Zero as a number. In elementary school, Elaine learned that zero was the starting point of numbers, but not that it was a number itself. Then in middle school, Elaine was told that zero was the "middle of the integers", like a type of physical divider. With only technical knowledge of zero, these two meanings were irresolvable for Elaine. Without knowing of the social construction and evolution of zero and the integers, as well as not knowing that zero is theoretically a quantity, Elaine's learning about zero, as well as the whole numbers and integers, was relegated to points of trivia to be remembered, but not understood. Elaine's view of zero as not being a number correlates with findings from Wheeler & Feghali's (1983) study of pre-service teachers and with research involving students (Wilkinson Evans, 1983; Anthony & Walshaw, 2004; Wheeler, 1987).

Nora recalled learning that "zero is nothing". Like the pre-service teachers in Wheeler & Feghali's (1983) research, and the students involved in the studies of Baroody et. al. (1983), and Whitelaw (1984), Nora said that defining zero as 'nothing' caused her to believe that zero was not important and that it made sense that it could be ignored. Throughout our discussions, Nora regularly returned to this definition and trying to rectify it was a major motivation for her seeking of a philosophical and theoretical understanding of the concept of zero. Eventually, Nora refined her definition to 'nothing of *some thing*'. This modification parallels the conclusion by the students in Leeb-Luneburg's (1977) research: "Zero is nothing – of something!" (p. 25). By modifying the definition she had been given, Nora was able to create a philosophical understanding of zero that made sense of the technical roles of zero that she had learned.

The students in both teachers' classes also perceived zero as 'nothing' and said it could be ignored. This concerned both Nora and Elaine, and Nora was particularly troubled to hear her students tell the primary researcher that zero was not a number. The students reasoned that if it was a number, then they would have been taught it when they were taught about one to ten and that it would be said in number names (for example, if zero is a number we would say twenty-zero just like we say twenty-one). Although some research had demonstrated students' confusion over the naming of numbers containing zeros (Kamii, 1981; Baroody et. al., 1983), none of the research had provided evidence that students were making assumptions about zero based upon their recognition of what hadn't been said or done by the teacher.

Nora and I were also surprised to find that her students believed that zero, rather than being a number, was in fact a shape. When asked what they had zero of, the students gave replies that pointed out objects in their possession or in the classroom that were circular in shape. Allinger's (1980) research with students, and Wheeler & Feghali's (1983) research with pre-service teachers both noted the equating of the number '0' and the letter 'O', however; none of the research mentioned the equating of zero with a circle.

Zero within procedures. Many of Elaine and Nora's memories of zero related to mathematical procedures they had memorized. Nora recalled learning to "carry the 1" when adding two digit quantities, but noted that it was not until she was an adult that she realized that the '1' actually represented '10'. As a student she had not understood where the '1' came from and that because the zero was not shown, it confirmed her belief that zero could be ignored. Similarly, Elaine spoke of learning to divide questions such as $20\overline{)340}$ and being told to "just knock off the zeros". She did not know why she could do this – just that ignoring the zeros made things easier. These limited and misconstrued understandings are reminiscent of those identified by Ma (1999) with the teachers from the United States' understandings of subtraction and multiplication and of those identified in Neurwirth Beal's (1983), Wilkinson Evans' (1983), Anthony & Walshaw's (2004), and Wheeler's (1987) research involving students.

Both teachers had learned to do computations involving zero as procedures without any basis of understanding. Over time, they had come to theoretically understand some of the computational procedures they had been taught, but many had remained unquestioned until our meetings.

Zero outside the classroom. Elaine and Nora both spoke of how much of their understanding of zero developed outside the classroom. Elaine recalled having an aunt who used the word 'aught' in place of zero, and both teachers spoke of how 'oh' was frequently used when saying the number zero in different contexts such as postal codes which has also been noted in other research (Wheeler & Feghali, 1983; Allinger, 1980; Baroody et. al., 1983; Whitelaw, 1984). This avoidance of the word zero reinforced many of their mathematically incorrect beliefs about zero, including that it can be ignored.

Nora and Elaine also talked about how zero was frequently used in non-numerical contexts. Elaine reflected about zero being associated with people in ways that indicated they were defective or substandard, and Nora provided the example of the novel *Holes* and its central character, Zero, as verification. Both teachers agreed that these experiences gave them the understanding that zero somehow indicated a deficiency or disapproval and that they did not have such a pessimistic (or emotional) attitude towards other individual numbers. Although Allinger (1980) mentions this type of use of the word zero, Nora and Elaine demonstrated that this non-mathematical contextualization of zero had influenced how they view and think about the mathematical concept of zero.

Marginalization And Legitimatization Of Zero

As we moved deeper into the dialogue, both teachers became concerned about what they perceived to be the marginalization of zero and ways to legitimatize it. They identified a number of contexts for the marginalization of zero, some of which are presented below.

Computations. Elaine and Nora both spoke of the need to change the ways of teaching and learning related to computations involving zero. Elaine commented that one problem was that zero was being presented to students in ways that made it seem less significant than other numbers. The result of this was that students either learned to ignore zero, or to memorize procedures to deal with zero when it did occur. Both teachers also spoke of the practice of rote teaching and learning, with memorization and not understanding, being a problem for students' development of computational skill.

Nora and Elaine spoke of the marginalization of students' understanding of zero as being a result of teachers' lack of understanding themselves and of teaching traditions. Both felt that it was crucial that teachers learn to understand zero themselves so that the tradition of teaching for doing and not for understanding could be changed. The processes of teaching and learning about zero were only explored in the research by Leeb-Lundberg (1977), which demonstrated that changes of the type proposed by Elaine and Nora made significant changes in students' understandings of zero.

Nora and Elaine's discussions about teaching and learning demonstrated their ongoing struggle with recognizing and developing theoretical understandings of zero that would either explain or debunk the technical and procedural pieces that made up the majority of their present horizon of understanding.

Language. Elaine and Nora also argued for a change in the way language was being used, both in and out of the classroom. Nora compared society's acceptance of saying 'oh' for 'zero' to how, "when kids are starting to read they get confused with '1' and 'L'… but it's not viewed as acceptable to confuse them". Nora spoke of how zero was not getting the same level of distinction or respect as other numbers do. Both Nora and Elaine felt the practice of replacing zero by 'oh', 'none', or 'nothing' was sending the incorrect message that zero means the same as those three words and is not a quantity. Both teachers called for a stop to the practice of replacing zero with other words. Throughout their discussions, they emphasized specifying zero of 'something', thus strengthening their theoretical perception of zero as a quantity.

Other attempts to legitimatize zero. Nora was concerned that zero was not always included in our world where it could have been. In this regard, she commented: "you have First Avenue, Second Avenue, Third Avenue, but no Zero Avenue". She was also concerned because there was no year 'zero' in our calendar system. Nora was not aware of the use of zero in the Mayan calendar.

Elaine, on the other hand, was more concerned with obtaining a single perspective for zero that made everything work. She found a possibility in her initial knowledge of zero as a starting point. She argued that every context of zero could be related to a starting point. For the integers Elaine explained that zero is actually the starting point of both the positive numbers and negative numbers and that even zero gravity is the starting point of gravity. This then moved the conversation to changes that would need to occur to make zero the starting point, such as making all timers count up from zero, creating a temperature system that starts at zero (she was not aware of Absolute Zero), and the renaming of 'home base' to 'zeroth base'.

These attempts at legitimatizing zero are evidence of both teachers' lack of understanding of the social evolution of zero and of the complexity of its meanings and ongoing development.

Representation Of Zero

Nora and Elaine considered how students might be asked to represent and talk about their understandings of zero. Nora suggested three possible ways in which students could be asked to physically, concretely, or pictorially represent zero: through the absence of objects, as the quantity before one, and through subtraction. In the first case, Nora felt that students could understand zero as the absence of particular objects, such as considering a picture of a farm that showed zero birds. Alternatively, if the picture had one bird in it, the students could be asked how many birds were there before it arrived and then be asked to draw the situation. Finally, Nora addressed students representing and understanding zero through subtraction by having the students remove objects from a given context or picture to lead to the quantity of zero.

Elaine, although appreciative of these ideas, struggled with the notion of students representing zero because of its abstractness. She questioned what would be the concrete aspect (what the students could see or physically touch) of Nora's suggested ways to represent zero. Thus, while Nora became convinced that students are capable of representing and theoretical understanding zero, Elaine was still unsure about whether young students had the ability to develop an abstract and theoretical understanding. Like the two teachers, the research is polarized on this matter. Leeb-Lundburg's (1977) research demonstrated how grade two students were able to construct the types of understandings that Nora proposed, while Piaget & Inhelder 's (1964) research demonstrated that children did not understand zero.

Zero Within Numbers

The final theme that emerged from Nora and Elaine's dialogues focused on the role of zero in place value and number composition. Both teachers argued, based on their experiences as students and as teachers, that students did not understand the full role of zero in our place value system. Elaine spoke of how many of her students 'just lose [the zero]" when they are naming numbers such as in the case where 204 would be read as "twenty-four". This error in understanding was also cited in Baroody et. al. (1983).

Through their dialogues, Elaine and Nora developed an understanding of zero which defined two roles for zero within numbers: as helping to define the magnitude of the numbers (for example two hundred four rather than twenty four), and as defining the quantity of that particular place value position (there are zero tens in 204). Nora and Elaine's concern with what their students understood about zero served as a major impetus for their seeking and proposing theoretical underpinnings for what they technically and procedurally knew with respect to zero.

Discussion

Regardless of the theme, the understandings of zero that Nora and Elaine demonstrated can be classified in two distinct, categories: procedural and technical, and philosophical and theoretical. At the start of our dialogues, there was no interplay between the two categories and most of the understandings that Nora and Elaine revealed and explored were procedural or technical. As Elaine and Nora tried to explain and justify those understandings, they began to recognize their deficiencies related to the philosophical and theoretical aspects of zero and began to construct new understandings to justify or make sense of their procedural and technical knowledge. Through this process, the two teachers explored both determining the axioms and theories that allowed for the mathematics they already knew as well as exploring and trying to guess about the social evolution of zero. Figure 1 below illustrates the relationships that emerged between these two facets of understanding zero.

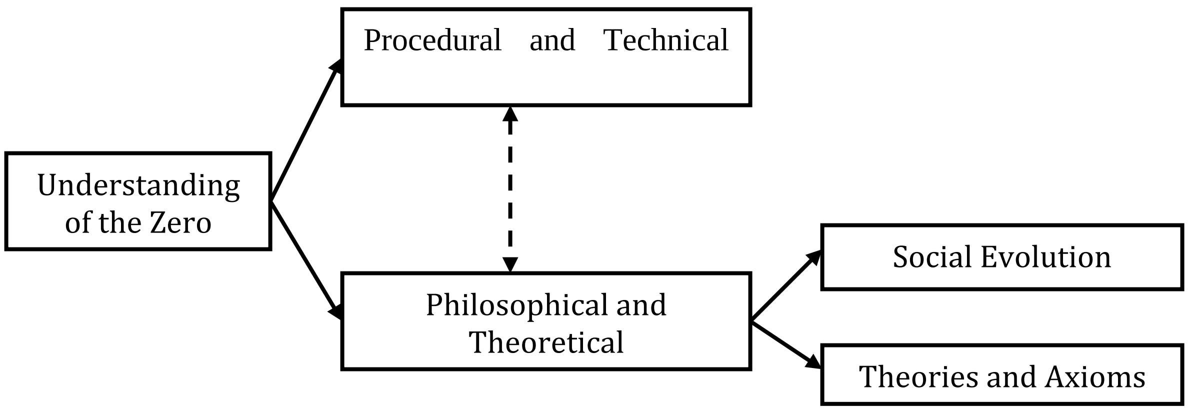

Figure 1. Categorization of Understanding of Zero

In some instances, Nora and Elaine were able to successfully construct philosophical and theoretical understandings that parallel those that developed historically and exist in modern mathematics (Barrows, 2000; Seife, 2000; Kaplan, 1999). There were other cases, however, in which their attempts ended up creating conceptions of zero that do not reflect those of the historical or modern understandings and uses of zero.

Conclusion

This study found that for Nora and Elaine, the answer to the question "How do you understand zero," was a limited set of procedural and technical facts, some right and some wrong, with virtually no philosophical or theoretical understandings to support what those facts. Moreover, as the two teachers attempted to determine and/or define such philosophical and theoretical understandings, they increased their confidence in their knowledge while also creating some theory that "made things work," but was mathematically or socially incorrect. In relation to the second question "When and why does zero become a part of teaching and learning in your classroom," both teachers recognized that zero had never been a consideration in their planning for mathematics instruction. As the study progressed, Elaine and Nora both expressed concern about their prior omissions related to zero, and discussed ways to bring more focus to the roles of zero within the mathematics that they were engaging their students in. As their own personal understandings of zero expanded, so to did their desire to bring zero into their classrooms in meaningful ways.

By using a hermeneutic approach in this study, the primary researcher also noted an expansion of her horizons of understanding zero. She come to realize that the history of zero, especially in terms of socially accepted norms such as not having a year zero in our calendar, is much more than a bit of interesting trivia in one's understanding of zero within modern mathematics. Without knowledge of these socially constructed and arbitrary features of zero, confusion and misunderstandings can abound.

This research demonstrates how the lack of understandings and misunderstandings held by teachers about zero that were reported in the 1980s still exists today. Through carefully planned activities and discussions, however, Nora and Elaine were able to construct richer understandings of zero, just as the students in Leeb-Lundburg's (1977) research did, suggesting that the situation can be changed. It is also notable that my study revealed a misunderstanding of zero held by some students, which has not been noted before, namely the confusion of the number zero with the shape of a circle.

There is still much need for further research into the understandings of zero for both teachers and students, including the procedural and technical understandings and the supporting philosophical and theoretical understandings. Strategies for deliberately and effectively engaging teachers and students in learning about all aspects of zero, including the evolution of zero related to different societies, need to be explored and developed further.

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DEVELOPING AND MASTERING KNOWLEDGE THROUGH TEACHING WITH VARIATIONS: A CASE STUDY OF TEACHING FRACTION DIVISION

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This study examined how Chinese teachers develop students' procedural fluency and conceptual understanding simultaneously when teaching fraction division. Twelve videotaped lessons and corresponding lesson plans from three elementary school teachers in China composed the data of this study. A theoretical framework of teaching with variations was adopted for analyzing the data, which assumes that creating dimensions of variations on critical aspects of learning objects for students to experience is a necessary learning condition. It was found that systematic variations of mathematics problems were assigned to students for generating, clarifying and consolidating their learning of concepts and computational rules. Discussing and solving problems with multiple methods over the consecutive lessons is another essential characteristic in teaching with variations.

Introduction

In school mathematics, it is commonly recognized that students' learning of mathematics knowledge should go beyond procedural acquisition and skill fluency (e.g., National Council of Teachers of Mathematics [NCTM], 2000). The emphasis placed on empowering students mathematically has inevitably led to not only on-going discussions about the relationship between conceptual and procedural knowledge (e.g., Rittle-Johnson & Alibali, 1999; Star, 2005), but also the quest for high-quality instructional practice that can develop students' mathematics knowledge with understanding and fluency. Many investigations of Chinese mathematics classroom instruction suggest that instructional practices in China contain some unique features that may assist our thinking in developing alternative practices (e.g. Huang & Leung, 2004).

Mathematics education practices in China have a long tradition of emphasizing "basic knowledge and skills" (Zhang, Tang, & Li, 2004). It is further believed in China that creativity and higher-order thinking abilities can only be grounded on the "basics" (Wong, 2008), and the key to pursue a high-quality mathematics education is to bridge "basics" and "higher-order thinking abilities" (Zhang et al.). Some researchers argued that a teaching method, called teaching with variations, could possibly lead to meaningful learning in large-size classes (Gu, Huang, & Marton, 2004), and further serve as a bridge from "basics" to "higher-order thinking abilities" (Wong). Huang and Leung (2004) examined how teachers teach the topic of *Pythagoras theorem* from the perspective of teaching with variations. They found that through teaching with variations, students were actively involved in learning mathematics content, solving mathematics problems under teachers' skillful guidance. Recently, based on an experiment about the effect of a mathematics curriculum guided by a spiral variation (*bianshi*) on students' learning, Wong, Lam, Sun, and Chan (2009) revealed that students using spiral teaching materials performed significantly better than their peers using standard textbooks.

Although the previous studies demonstrated a positive potential in improving mathematics teaching by adopting the notion of variations in mathematics teaching and learning, we are not clear about what really happens in classrooms when teachers teach specific content topics over a series of consecutive lessons. In this study, we aimed to examine three Chinese elementary mathematics teachers who taught the same topic of fraction division over a sequence of four lessons. In particular, we focused on these teachers' practices in carrying out their teaching of fraction division with variations over four consecutive lessons.

Theoretical Consideration

To identify and interpret classroom characteristics that are conducive to student learning, we selected a theory of variations espoused by Marton and Booth (1997), which has demonstrated the potential to reveal salient features of classroom instruction (Marton & Tsui, 2004; Watson & Mason, 2006). According to Marton and Tsui, learning is a process in which learners develop a certain capability or way of seeing or experiencing. In order to see something in a certain way, the learner must discern certain features of the object. Experiencing variations is essential for discernment and significant for learning. Marton and Tsui further argued that it is important to attend to what varies and what is invariant to a learning situation. The intended object of learning designed by lesson plans, does not affect the learning of the students in itself, but the way in which it is implemented does. What varies and what is invariant constrain learning and make the learning possible. *Enacted object of learning* is described through necessary conditions for the appropriation of the object of learning (Marton & Pang, 2006). Pedagogically, Gu, Huang and Marton (2004) described the fundamental features of teaching with variations which has been around in China for several decades. This theory of teaching emphasizes building essential connections between new knowledge and previously learned knowledge through guiding students to experience certain variations on critical features of the object of learning. Two types of variations were suggested to help students learn mathematics meaningfully: "conceptual variation" and "procedural variation". Conceptual variation consists of two parts. One is composed of varying embodiments or representations of a concept to form the connotation of the concept. The other means highlighting the substantial features of the concept by providing counterexamples as a contrast. Conceptual variation is meant to provide learners with multiple experiences from different perspectives. Procedural variation refers to the process of forming a concept logically (scaffolding or transformation), arriving at solutions to problems and forming knowledge structure. The function of procedural variation is to help learners acquire knowledge step by step, enrich learners' experience in problem solving progressively, and finally obtain well-structured knowledge.

Recently, Wong et al. (2009) further developed a framework to describe different variations in terms of the functions of a problem. This mode includes four types of *bianshi* (variation): *inductive, broadening, deepening,* and *applying*. By use of '*inductive bianshi*,' rules and concepts are derived through the inspection of a number of everyday situations. These rules are consolidated by a systematic introduction of variations in mathematical tasks. Yet, in the case of "*broadening bianshi*", no new rules and concepts are introduced, and learners broaden their scope through exploring a variety of problems. At a certain point, by further varying the types of mathematical tasks, learners are opened up to more deep mathematics concepts and skills. This is called '*deepening bianshi'*. Mathematics knowledge is then applied to a variety of daily life problems, which can be called '*applying bianshi*".

Essentially, classifications by Gu et al. (2004) and Wong et al. (2009) are closely related. *The inductive* and *broadening bianshi* could be roughly grouped into conceptual variation, while *deepening* and *applying bianshi* basically fit in procedural variation. However, since knowledge development is spiral and connected, all these categories are not static and exclusive, rather they depend on the situation under study. Thus, we regard Wong et al.'s classification as an extension of Gu et al.'s model in this study (hereafter, variation will replace *bianshi*).

In this study, knowledge generation and development will be examined from two levels. At the macro level, we examined the lesson structure over consecutive lessons to see how the intended objects of learning were constructed as a whole. At the micro level, we examined how a particular concept or procedure was developed within or across lessons from the lens of teaching with variations. Taken together, we aimed to answer three research questions:

- What are the intended objects of students' learning over consecutive lessons?
- What are the enacted objects of students' learning through teaching with variations?
- What are similarities and differences in the enacted objects of learning among these teachers?

Methods

Data Resource

The data of this study taken from a larger project, mainly consisted of four consecutive videotaped lessons of each of the three teachers from three different elementary schools in East China, with a total of 12 videotaped lessons. All the teachers adopted the same unified textbook. In addition, lesson plans of the teaching content were used as supplementary and triangulation documents.

Data Analysis

The data analysis is mainly based on lesson plans, videotaped lessons and their transcripts in Chinese. First, we read through the lesson plans which include similar components such as topics, instructional objectives, important and difficult content points, teaching methods and techniques, and post-lesson reflections. It helped us identify the main contents, instructional objectives, and procedures of the lessons. Then, we watched the video-taped lessons carefully, paying particular attention to the content taught, instructional objectives addressed and the procedures of lessons unfolded. Through examining the lesson plans and transcripts, we assured that our understanding was accurate. Thus, we identified the main instructional objectives for Teacher A and Teacher C (see Table 1). After that, we classified all the problems used in the lessons into four categories, with a focus on the ways of solving and discussing these problems (See Table 2 for the case of Teacher A). Finally, we summarized all the dimensions of variations in teaching to identify the necessary learning conditions provided for students' learning. In particular, we focused on the following two aspects:

Intended objects of learning. The lesson plans and lesson structures were examined to describe the intended objects of learning and the pedagogical flow to achieve their instructional objectives. Then, the similarities and differences can be compared with regard to the intended objects of learning.

Enacted objects of learning. At the second level, the analysis focuses on how teachers use different types of problems to introduce, broaden, deepen and apply knowledge progressively. The types of problems were categorized and the ways of implementing these problems were analyzed qualitatively.

Results

The findings are organized into two parts. First, the instructional objectives set in lesson plans (e.g., intended objects of learning) are summarized, and relevant instructional procedures are described. Then, the enacted objects of learning created in lessons through problems solving are illustrated in detail.

Intended Objects of Learning and Lesson Structure

With regard to instructional objectives, Teacher A and Teacher B had essentially the same objectives; yet Teacher C set slightly different aims as showed in Table 1.

Table 1. Instructional Objectives across Lessons

Teacher	Lesson 1	Lesson 2	Lesson 3	Lesson 4
A	Understanding the meaning of fraction division, and the relationship between multiplication and division; Understanding and mastering the computation rules of fraction divided by whole numbers (F/WN)	Understanding and mastering the computation rules of whole number divided by fraction (WN/F)	Understandin g and mastering the computation rules of fraction divided by fraction (F/F)	Mastering the methods of solving word problems by fraction division; Understandin g comparing fractions
B	Understanding the meaning of fraction division, and the relationship between multiplication and division; Understanding the methods to learn a new computation rule; Understanding and mastering the computation rules of fraction divided by whole numbers (F/WN)	Understanding and mastering the computation rules of whole number divided by fraction (WN/F); Understanding the methods to learn (WN/F)	Understandin g and mastering the computation rules of fraction divided by fraction (F/F)	Mastering the methods of solving word problems by fraction division; Understandin g comparing fractions

Thus, basically, all the teachers set the following objects of learning over the consecutive lessons: (1) understanding the meaning of fraction division, and the relationship between multiplication and division; understanding and mastering the computation rules of fraction divided by whole number (F/WN) (in Lesson 1); (2) understanding and mastering the computation rules of whole number divided by fraction (WN/F) (in Lesson 2); (3) understanding and mastering the computation rules of fraction divided by fraction (F/F) (in Lesson 3); and (4) mastering the methods of solving word problems of fraction division; understanding the size of fraction divided by fraction, compared with original fraction (in Lesson 4). In addition, Teacher C emphasized the methods to learn a new computation rule including meaning, rules and sequence in Lesson 1 (See Table 1). As far as the instructional procedure is concerned, all three teachers followed the similar patterns: (1) reviewing the previous lesson content or relevant knowledge before learning the new topic; (2) introducing new topic through solving mathematical problems in everyday life; (3) practicing with a variety of interconnected problems and summarizing relevant key points or contents in the lesson; and (4) assigning homework. There are slight differences across lessons: Lesson 3 and 4 include several times of practice and summary.

Enacted Object of Learning through Solving Problems

Through examining the features of problems used over the consecutive lessons, and ways of implementing these problems, we can map the routes of how each teacher enacted the object of learning over lessons, and the similarities and differences between different teachers. One case of Teacher A was displayed in Table 2.

Table 2. Different Types of Problems Used to Develop Concepts and Computation Rules

Problems	Lesson 1	Lesson 2	Lesson 3	Lesson 4
Inductive variation	*Problem:* If each of five people ate half a cake, how much did they eat in total? *Variation 1:* Can you pose two division problems based on the given information? *Implementation:* Group discussion; multiple representations, and multiple methods.	*Problem:* A pigeon flies 12km in 1/5 hour, how far does it fly per hour? $12 \div \frac{1}{5} = 60$ *Variation 1*: A pigeon flies 2km in 1/5 hour, how far does it fly per hour? Variation 2: $7 \div \frac{1}{4} = 7 \times 4 = 28;$ *Implementation*: Multiple representations and multiple methods.	*Problem:* A butterfly flies 13/14 in 3/10 hours, how far can it fly per hour? *Implementation:* Multiple representations, and multiple methods and justifications.	*Problem:* If 3/8 of a number is 1/4, please find the value of the number. *Implementation:* Multiple solutions, and summary of rules in general.
Broadening variation	*Problem:* If we evenly separate a rope of 4/5 meter into two parts, how long can we get for each part? *Variation 1*: Changing “two parts” into “three parts”. *Implementation:* Group discussion; multiple representations and multiple methods.	*Problem:* A butterfly flies 24 km per 3/4 hour, how far does it fly per hour? $24 \div \frac{3}{4} =$ *Implementation:* Multiple representations, and multiple methods and justifications.		
Applying variation	Variation of problems in terms of formats and functions of the expressions.	Variation of problems from the textbook.	Variation of problems.	Variations of word problems, and comparing values of arithmetic expressions.

Teacher A demonstrated the following features when selecting and implementing problems over lessons: (1) All the concepts or computation rules were introduced through solving and discussing varying word problems; (2) problems were explored through group activities, and multiple solutions were explored and discussed by using multiple representations; and (3) systematic variations of exercise were assigned to students for them to clarify and consolidate the learned concepts and computation rules.

Through exploring these variations of problems, students were expected to develop understanding of the concepts and computations rules (inductive problems and broadening problems), and further consolidating them by applying problems in Lesson 1 and 2. Moreover, in Lesson 3 and 4, after introducing some new concepts through inductive problem solving, the focus was given to the application (applying problems) of all the learned knowledge in the consecutive lessons.

Through these processes, we concluded that Teacher A produced some enacted objects of learning and provided possible learning conditions through creating certain dimensions of variations for students to explore (See Table 3). The dimensions of variations provide possible conditions for students to achieve the enacted objects of learning.

Table 3. Enacted Objects of Learning through Certain Dimensions of Variations

Enacted objects of learning	Dimensions of variations	Examples
Meaning of fraction division	Multiple representations: Word, pictorial and numerical representations.	Word, pictorial and numerical representations of the equations: $5\times\frac{1}{2}=\frac{5}{2}$; $\frac{5}{2}\div 5=\frac{1}{2}$; $\frac{5}{2}\div\frac{1}{2}=5$
Rationale of computation rules of F/WN	Numerical variation, multiple solutions, and multiple representations	Word , pictorial and numerical representations; Numerical variation for word problem: $\frac{4}{5}\div 2\rightarrow\frac{4}{5}\div 3$;Multiple explanations to the rationale of $\frac{4}{5}\div 3=\frac{4}{5}\times\frac{1}{3}$
Rationale of computation rules of WN/F	Numerical variation and multiple solutions, and multiple representations	Numerical variation in the forms of fraction: $\frac{1}{5},\frac{1}{4},\frac{3}{4}$;Multiple explanations to the rationale of $24\div\frac{3}{4}=24\times\frac{4}{3}$
Rationale of computation rules of F/F	Multiple representations; Conjecture and justifications	Multiple explanations to the rationale of $\frac{14}{15}\div\frac{3}{10}=\frac{14}{15}\times\frac{10}{3}$(e.g., conjecture, pictorial explanation, and deductive reasoning)
Application of computation rules of F/F	Different types of problems; Multiple solutions	Computation, judgment, comparison, and word problems;

Let us illustrate how the Teacher A initiated student understanding of the rationale of computation rules of F/WN through constructing several dimensions of variations. First of all, the teacher presented a question: If we evenly separate a rope of 4/5 meter into two parts, how long can we get for each part? By solving this problem, one dimension of variation of *multiple* *solutions* to the problem was presented: pictorial explanations for $\frac{4}{5} \div 2 = \frac{4 \div 2}{5}$; meaning of a number divided by 2; $\frac{4}{5} \div 2 = \frac{4}{5} \times \frac{1}{2}$; and converting fraction into decimals. Then, a second dimension of variation of *changing divisors* (from 2 to 3) was created by presenting a variation of the previous problem: If we evenly separate a rope of 4/5 meter into three parts, how long can we get for each part? By *comparing three methods* in the new situation, the advantages of the computation rule (one of the multiple methods) were highlighted. Thus, students developed understanding and justified the computation rule.

Similarities and Differences in the Enacted Objects of Learning among Teachers

When comparing the enacted objects of learning of the three teachers, although each of them aimed to create a similar enacted object of learning as showed in Table 2, we identified subtle but essential differences. Teacher A created broad dimensions of variations to help students understand why the computation rules work or do not work under different conditions. However, Teacher B always directly focused on some dimensions of variations which help students understand why relevant computation rules work under some particular conditions as stated in the textbook. As a contrast, Teacher C, not only made efforts to create similar enacted objects of learning as Teacher A, but also aimed to create a dimension of variation to help students realize how to learn fraction computation rules by analogizing relevant rules performing on whole numbers. This learning object has a long-term overarching effect on students' learning because it provides a thinking method of extending numerical system and relevant computation rules (i.e., extending operation rules from whole numbers to fractions). Thus, although all three teachers enacted learning objects as displayed in Table 3, they still pursued with some subtle different learning objects that may result in differences of student achievement.

Conclusion and Discussion

The findings of this study suggest that these Chinese teachers had quite similar instructional objectives including developing students' understanding of relevant concepts and computation rules through inductive and broadening problem solving step by step, improving their computation skills through systematic exercises (i.e., applying problems), and summarizing key points over consecutive classes. Classes were organized in a consistent model, including reviewing relevant knowledge, introducing new topics, applying new knowledge and summarizing key points, and assigning homework. Thus, over these consecutive lessons, the teachers may provide students the condition of developing coherent and interconnected knowledge (Leung, 2001). It was argued that through well-designed task sequences, students can learn knowledge, enhance their abilities, and benefit from interactions in the class (Simon et al., 2010). In addition, these three teachers tended to enact the objects of learning to achieve their instructional objectives through creating relevant dimensions of variation focusing on objects of learning. These dimensions of varying problems help students make sense of concepts, understand why computation rules work or do not work under particular conditions, and further deepen and consolidate students' understanding of relevant concepts and computation rules. Based on our framework, developing conceptual understanding and computation skills were enacted simultaneously in these classes.

Theoretically, this study demonstrates that the framework of teaching with variations could be a powerful tool to identify the features of classroom instruction that are conducive to high-quality teaching. It is echoed by the claim that "if the teacher offers data that systematically expose mathematical structure, the empiricism of modeling can give way to the dance of exemplification, generalization, and conceptualization that characterizes formal mathematics" (Watson & Mason, 2006, p.94). This analysis also provides some strategies for enhancing classroom instruction.

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TEACHERS' UNDERSTANDING OF PROPORTIONAL REASONING

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This report provides empirical evidence of pre-service teachers' understanding of proportional reasoning. Eighty elementary pre-service mathematics teachers were given proportional reasoning problems. This paper analyzes their responses to those items. Proficiency in proportional reasoning is considered as one of the critical foundations of algebra. While past research shows the difficulty students have understanding proportional reasoning, results of this study show a concern of a recycling effect where teachers themselves lack a robust, conceptual understanding of proportional reasoning.

Introduction

The purpose of this report is to establish evidence of pre-service teachers' understanding of proportional reasoning, especially reasoning beyond traditional missing-value type problems. In testimony before the Committee on Labor and Education in May 2008, Francis (Skip) Fennell, past president of the National Council of Teachers of Mathematics (NCTM) gave a report of "Foundations for Success", which was work completed in March 2008 by the National Mathematics Advisory Panel. In his testimony, Fennell stated that Algebra is the gateway to higher level mathematics. He also added that research shows that students who complete Algebra II are more than twice as likely to graduate from college as compared to students with less mathematics background.

In the Foundations for Success report, the National Mathematics Advisory Panel noted that a major goal for K – 8 mathematics education should be to develop proficiency in the area of fractions, decimals, percent, ratio and proportions and deemed this area as one of the "Critical Foundations of Algebra" (2008). For this same reason, the NCTM noted in their *Curriculum and Evaluation Standards* (1989), that proportional reasoning "is of such great importance that it merits whatever time and effort must be expended to assure its careful development" (p. 82). In 1988 Lesh, Post, and Behr went as far as to state that proportional reasoning is the capstone of the elementary school curriculum and the cornerstone of high school mathematics and science.

Our youths' understanding of proportional and algebraic reasoning is paramount to our sustenance as a nation to develop the next generation of science, technology, engineering, and mathematical (STEM) careers. In the NCTM's Second Handbook of Research on Mathematics Teaching, Lamon stated "my own estimate is that more than 90% of adults do not reason proportionally – compelling evidence that this reasoning process entails more than developmental processes and that instruction must plan an active role in its emergence" (2007, p. 637). Lamon noted the lack of research within the last 15 years and coupled with the fact that "teachers are not prepared to teach content other than part-whole fractions" goes on to state a critical call for more research in the area of rational numbers and proportional reasoning (2007, p. 632). Therefore, given that many in the mathematics education community note that knowledge of proportional reasoning is essential as a foundation for algebraic thinking, research on understanding of proportional reasoning is essential to understand just where we stand.

Theoretical Perspective

Even though there are relatively few studies aimed specifically at teachers' understanding of proportional reasoning, there have been a substantial number of past research studies on children's understanding of rational numbers, ratio, and proportion as outlined by Lamon (2007). According to Lamon, much of the research from 1970 to 1985 produced an extensive inventory of factors that influence the difficulty children have in understanding proportion problems (2007). Students' difficulty understanding and using ratios, rates, and percents, is largely because they have not acquired a capacity for proportional reasoning. For example, Hart (1981, 1988) and Karplus, Pulos, and Stage (1983) noted children frequently use additive strategies when they should have use multiplicative comparisons.

A research study by Canada, Gilbert, and Adolphson examined pre-service teachers' conceptions of proportional reasoning. Their results revealed pre-service teachers' difficulties with a problem that was posed in three different ways and they concluded that some pre-service teachers had a very limited understanding of proportional reasoning (2008). Another study of pre-service teachers' understanding of fraction problems showed that U.S. pre-service teachers were out-performed by their Taiwanese counterparts and tended to use more intuitive and visual reasoning rather than formal and symbolic reasoning on open-ended problems (Luo, Lo, & Leu, 2009). A study by Lee and Orrill (2009) of twelve middle grades mathematics teachers understanding of fraction division concepts revealed that teachers can reorganize and generalize operations and concepts of fractions.

Since 1988, Susan Lamon has been a leader within the mathematics education community on her work with the analysis of proportional reasoning. She defined proportional reasoning as "the ability to scale up and down in appropriate situations and to supply justifications for assertions made about relationships in situations involving simple direct proportions and inverse proportions" (2005, p. 3). She noted seven core ideas or structures that must be addressed to help children develop a deep understanding of rational numbers and proportional reasoning. Lamon's interrelated concepts, contexts, representations, operations and ways of thinking are shown in the following diagram (2005, p. 9).

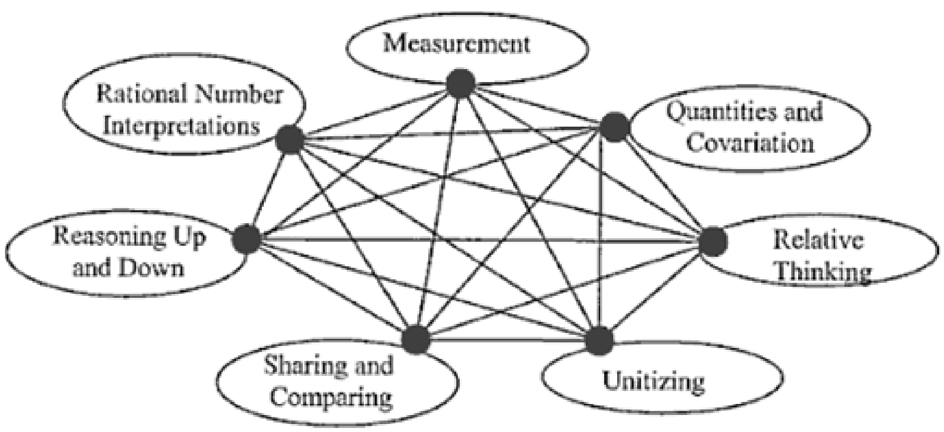

Figure 1. Seven Structures of Rational Numbers and Proportional Reasoning.

This report will briefly touch on a few of Lamon's core ideas. According to Lamon, central to the understanding of the unit in working with fractions is the role of composite units (units of numerosity greater than one) and the idea that ratios and rates can be viewed as complex types of units. Lamon (1993, 1994) suggested that unit building is an important mechanism in sophisticated mathematical ideas. She noted that when this happens, "a part-whole schema comes into play and the individual is able to think about both the aggregate and the individual parts that compose it" (2007, p. 243). Being able to compose and decompose a unit is central to one's reasoning that is needed in the field of rational numbers.

In addition, proportional reasoning is "reasoning up and down" (Lamon, 2007). Classic reasoning up and down problems are those where you are given the situation such as "if 6 individuals can paint a room in 4 days, how long will it take 8 people to paint the room". Students can complete problems like this without using the form $\frac{a}{b} = \frac{c}{d}$.

The mathematical model for proportional relationships is the linear function $y = kx$, where k is called the constant of proportionality. You can think of this as y is a constant multiple of x or that two quantities are proportional when they maintain the constant ratio: $\frac{y}{x} = k$. The constant k plays an essential role in understanding proportionality. According to Lamon (2005), one of the most useful ways of understanding mathematics requires the transformation of quantities or equations in such a way that some underlying structure remains unchanged (invariant).

Method

The participants in this research study were 80 pre-service elementary teachers in their third course of a sequence of three quarter (year-long) courses of mathematics for elementary teachers. All 80 participants were liberal studies majors (elementary education) that had completed the two prerequisite courses. The study was conducted two weeks prior to the end of the quarter. This course represented the last mathematics course that was required for their major.

A questionnaire with 8 open-ended items was used to collect data. The first two items were multiple choice problems and the last six items were open-ended problems that asked participants to complete different proportional reasoning word problems. This report will focus on the results of the six open-ended items in the area of proportional reasoning. Items were adapted from various sources and are noted per item below. The problems were:

#3.) *According to a recipe book, a turkey takes 15 minutes per lb to cook. How long should we cook a 24 lb turkey?* (Fosnot, C.T. & Dolt, M . (2005) cited by Kastberg, S., D'Ambrosio, B., & Lynch-Davis, K., 2010).

#4.) *Glenn mixed 6 ounces of cherry syrup with 53 ounces of water to make a cherry-flavored drink. Ardis mixed 5 ounces of the same cherry syrup with 42 ounces of water. Who made the drink with the stronger cherry flavor?* (Kastberg, S., D'Ambrosio, B., & Lynch-Davis, K., 2010).

#5.) *Brent and Roger decide to start a landscaping business together. Most of the homes in their neighborhood have similarly-sized lawns. Typically, Brent can mow a lawn in 3 hours. Roger usually needs 2 hours to do the same job. They decide to work together on 5 lawns. How long does it take them to finish?* (Sinn, R., Spence, D., & Poitevint, M., 2010).

#6.) *Lorri and Cole are running equally fast around a track. Lorri started first. When she had run 9 laps, Cole had run 3 laps. When Cole completed 15 laps, how many laps had Lorri run?* (Lamon, 1994 cited in Thompson, C.S., & Bush, W.S. 2003).

#7.) *A crew of 8 people can build a concrete wall in 6 days. If four more people join the group from the beginning, how many days will it take to build the same wall?* (Kastberg, S., D'Ambrosio, B., & Lynch-Davis, K., 2010).

#8.) *Hannah and Adele are decorating the gym with helium-filled balloons for the graduation. Hannah can inflate and tie off 7 balloons every 6 minutes. Adele requires 3 minutes to finish 4 balloons. Working together, how long will it take for them to have 25 balloons ready?* (Sinn, R., Spence, D., & Poitevint, M., 2010).

Item #3 can be correctly solved using the cross-multiplication algorithmic approach to set up the proportion and solve for the missing-value. Item #4 was a mixture problem that could be solved by finding the ounces of cherry syrup per ounce of water. This item asks students to compare ratios. Lamon (2005) noted that such items ask students go beyond setting up the proportion and asks students to engage in justification to determine the drink with the more cherry flavor. Item #5, #7, and #8 represent problems in which one quantity is related in an inversely proportional way to another quantity. According to Lamon (2005), items such as this should be understood by students in 7th or 8th grade. To complete this problem, participants needed to understand that as one quantity varies with the other, but in opposite directions. The two scale factors are inverses of each other (Lamon, 2005). Item #6 asked participants to understand the difference between multiplicative and additive thinking where the change in one quantity might be proportional to the change in the other quantity. These items are typical of the upper elementary or middle school curriculum and also are similar to items on the proportional reasoning part of the state's high school exit exam.

Results

The results of the 80 pre-service teachers' responses for the six open-ended items are noted below in table 1 by examining the number of correct responses.

Table 1. Pre-service Elementary Mathematics Teachers Results

Item	Item #3 Turkey Problem	Item #4 Cherry Mixture	Item #5 Lawn Mowing	Item #6 Laps around Track	Item #7 Building a Wall	Item #8 Balloons Problem
Concept	Missing Value	Mixture Problem	Inversely Proportional	Additive Thinking	Inversely Proportional	Inversely Proportional
Correct Response	360 min. or 6 hours	Ardis	6 hours	21 laps	4 days	10 minutes
# Correct Responses (n = 80)	72	57	38	37	31	31
Percentage Correct	90%	71%	48%	46%	39%	39%

In addition to analyzing the results by means of accuracy of the solution, the solution strategy that the participants used to complete the task was noted. For item #3, the problem that asked participants how long to cook a turkey, it is noted that 90% of the pre-service teachers correctly answered this item. This item is a typical missing-value problem that can be solved by setting up the proportion and cross-multiplying. Item #6 that asked participants to solve an open-ended question on additive thinking, only 46% of the pre-service teachers correctly solved this item. In an item analysis of participants' solution strategy used, of the 43 participants that incorrectly solved this item, 38 of the 43 pre-service teachers (48 % of all 80 participants) solved the problem by using a cross-multiplication strategy arriving at an incorrect solution of 45 laps. Below is a one of the responses from a pre-service teacher that was typical of these 38 participants:

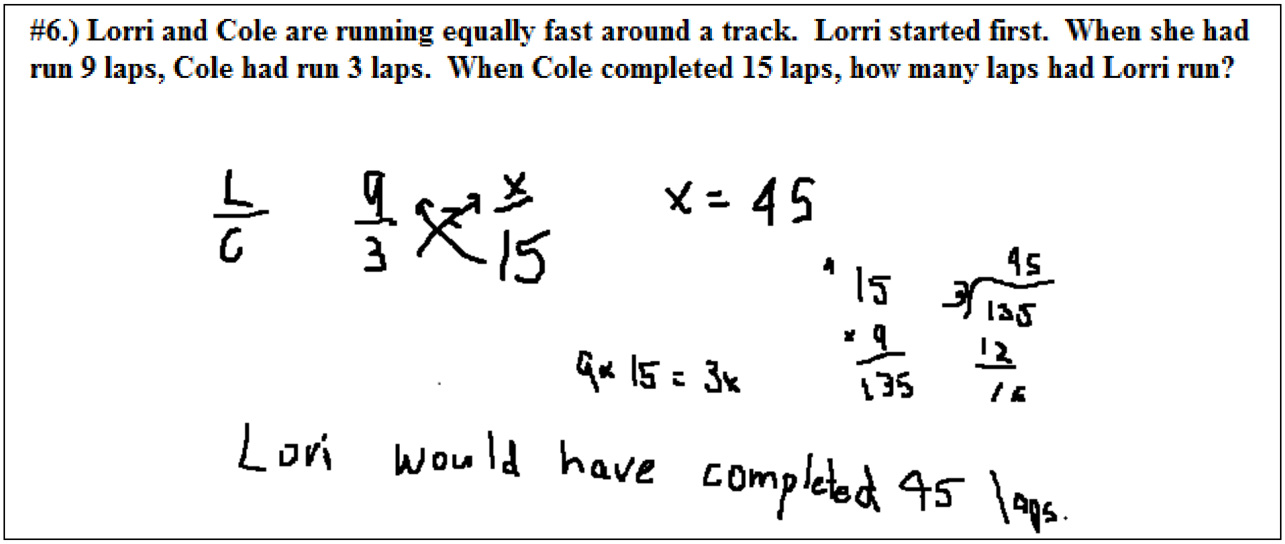

Figure 2. Student C8 Response

For the more difficult items on inverse proportions, some of the pre-service teachers tried solving by the cross product algorithm. In an item analysis of participants' solution strategy used for item #7, of the 49 participants that incorrectly solved this item, 19 of the 49 pre-service teachers (24 % of all 80 participants) solved the problem by using a cross-multiplication strategy arriving at an incorrect solution of 9 days. Below is a one of the responses from a pre-service teacher that was typical of these 19 participants:

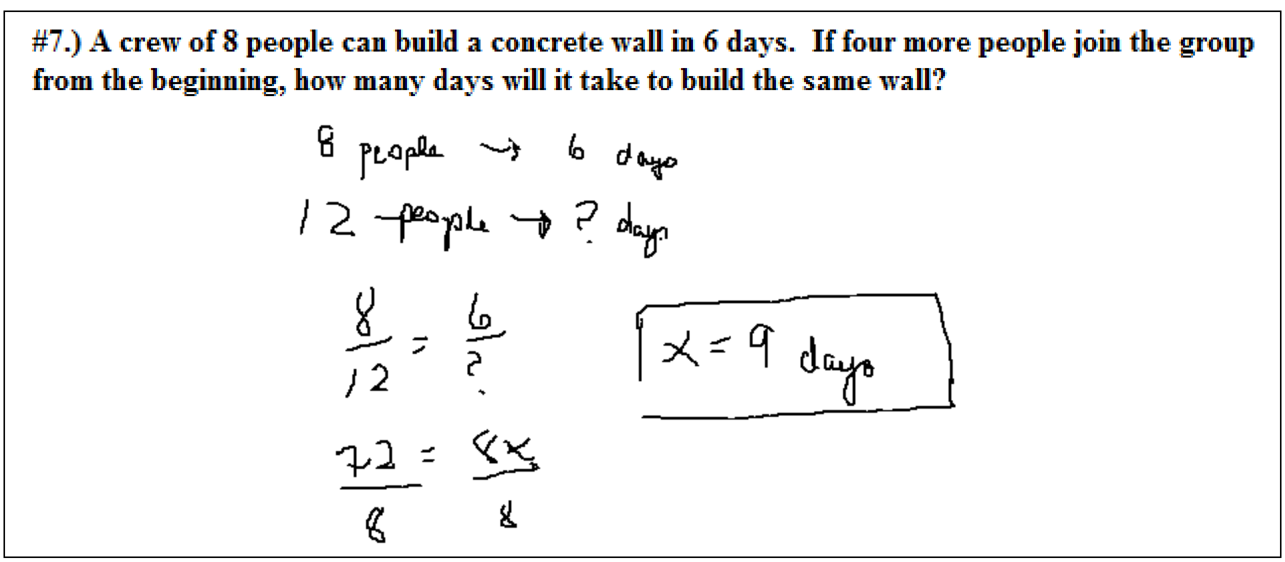

Figure 3. Student C11 Response

Conclusions

An understanding in ratio and proportional reasoning is imperative for our students to develop a solid foundation before they enter Algebra. Given this, in-service and pre-service teachers themselves must have a profound and deep understanding of proportional reasoning that goes beyond applying the cross product algorithm. The main purpose of this report was to provide a snapshot of pre-service teachers' conceptual understanding of proportional reasoning, especially when asked to complete problems beyond the traditional missing-value problems.

The data results from the pre-service teachers' responses is concerning and certainly notes the need for more in depth studies that will inform us of teachers' conceptual understanding of proportional reasoning. Results support that pre-service teachers had difficulty with the more difficult inverse proportional problems. It was surprising to note the number of participants that completed the additive thinking problem (item #5) incorrectly. The item analysis of this problem showed that 48% of the participants had difficulty differentiating between additive and multiplicative thinking. This problem is traditionally part of the upper level elementary curriculum and is a good problem to test whether participants can go beyond thinking of the traditional cross product algorithm. For the inverse proportion problems, some of the students relied on a cross product algorithm to solve. These results support the idea that we need to develop students' conceptual understanding of strategies of proportional reasoning before exposing them to the cross product algorithm.

Results from this study indicate that some of the teachers do not have a full understanding of proportional reasoning. There is concern that pre-service teachers have difficulty discerning when to use or not use cross-multiplication to solve proportional problems. Given that the content of the problems in this study are very similar to the elementary or middle school curriculum that the pre-service teachers will soon be teaching, it is concerning that a recycling effect will happen because teachers themselves lack a robust, conceptual understanding of proportional reasoning. This is in line with Lamon's statement, "teachers are not prepared to teach content other than part-whole fractions" (2007, p. 632).

Certainly more research studies are needed in this area that will inform us of pre-service and in-service teachers' conceptual understanding, especially research that delves into individuals' understanding of proportional reasoning beyond the basic algorithmic approach of cross multiplying. It is noted that an understanding of proportional reasoning is a critical foundation of algebra and the importance of success in algebra is crucial to understand higher level mathematics. Given this, it is imperative that we address the lack of understanding of proportional reasoning with our pre-service and in-service teachers.

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MIDDLE SCHOOL TEACHERS' CONCEPT MAPPING OF THE TERM EQUIVALENT EXPRESSIONS

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Introduction

In order to reach national goals of improved algebraic instruction and enhanced student learning, it is important to continue to examine teachers' knowledge about algebraic instruction regarding equivalence (National Mathematics Advisory Panel, 2008; RAND, 2003). Subject matter knowledge is not sufficient for effective teaching. Prior studies show that teachers' knowledge for teaching mathematics impacts student learning (Hill, Rowan, & Ball, 2005; Ma, 1999). However, there is a dearth of knowledge related to teachers' knowledge about algebraic instruction (Kieran, 2006) and recent studies have identified the need to better understand how such teachers' knowledge develops (Doerr, 2004). Since reformed-based teaching methodology in a mathematics methods class positively influences preservice teachers' knowledge and attitudes (Leonard, Newton, & Evans, 2009), we approached the development of teachers' knowledge of algebraic vocabulary though a meaning making or semiotic perspective. The goal of mathematics instruction using this perspective is "to make and give meaning" (Thompson, Kersaint, Richards, & Hunsader, 2008, p. 4), as signs and representations comprise the essence of mathematics (Hoffmann, 2006).

Given that the goal of mathematics instruction is to construct meaning, middle school mathematics teachers need a robust understanding of key algebraic vocabulary, including the term equivalent expressions. Therefore, we propose that the development of teachers' knowledge of equivalent expressions can be represented through the use of concept mapping, which might afford us the opportunity to see how teachers integrate elements of semiotics and the mathematical processes of communication, connections and representations which are essential to effective instruction (NCTM, 2000). There has been little research on the use of concept mapping as a tool to help promote middle school teachers' knowledge about equivalence. This led us to the following research question: How do middle school teachers develop mathematical, semiotic, and language constructs of the term equivalent expressions through concept mapping?

Theoretical Perspectives

The background in this study draws upon two areas, teachers' knowledge for teaching algebra and concept mapping. Teachers' knowledge, or pedagogical content knowledge, refers to "the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9). Shulman also distinguished between two kinds of understanding, knowing "that" and knowing "why":

> *We expect that the subject-matter content understanding of the teacher be at least equal to that of his or her lay colleague, the mere subject-matter major. The teacher need not only understand that something is so; the teacher must further understand why it is so.* (p. 9)

Teachers who can distinguish the "why" have a robust understanding of pedagogical content knowledge. Insufficient pedagogical content knowledge can impede the teaching and learning processes in algebra. Algebraic instruction is becoming integrated into the elementary grades (Kaput, Carraher, & Blanton, 2007) and is also a central focus of the middle school curriculum (NCTM, 2000). As algebraic thinking is integrated into the grades K-8 curriculum, it becomes increasingly important to understand how to develop teachers' knowledge so that they can help students bridge the gap between arithmetic and algebra. The research base on teachers' knowledge for teaching algebra is limited (Asquith, Stephens, Knuth, & Alibali, 2007; Doerr, 2004). Extant studies report that the transition from arithmetic to algebra is difficult (Kieran, 1992). In studies about lessons on equivalent expressions, novice teachers did not make connections between lessons, nor did teachers use spatial reasoning to show that expressions were equivalent (Even, Tirosh, & Robinson, 1993). Nathan and Koedinger (2000) suggested teachers hold a symbol-precedence view of student mathematical development and concluded that mathematics teachers should become aware of the "range, flexibility, and efficacy of students' alternative mathematical problem-solving strategies and the difficulties students have developing their symbolic-reasoning abilities" (p. 231). Finally, teachers' knowledge about the concepts of equivalence and variable are central to successful algebraic instruction, yet teachers rarely identify student misconceptions as an obstacle to student learning (Asquith et al., 2007).

Concept mapping is commonly used to both illustrate and assess existing knowledge, and can be used to develop deeper understandings about key mathematical ideas (Thompson et al., 2008). Concept maps diagram, classify, and organize relationships between concepts, words, and other elements of knowledge. Concept mapping (also called semantic mapping) is relational in that it integrates prior knowledge, experiences, and beliefs with new information. Research supports the effectiveness of concept maps for reinforcing vocabulary knowledge for a variety of learners (Horton, Lovitt, & Bergerud, 1990), and in many content areas such as mathematics (Monroe & Pendergrass, 1997) and literacy (Levin et al., 1984).

In teaching students new concepts, relating semantic features through the use of mapping approaches has been shown to have positive effects on the comprehension of new material (Baumann, Kame'enui, & Ash, 2003). In studying the vocabulary of junior high students, Margosein, Pascarella, & Pflaum (1982) found significant main effects for using semantic mapping over a more traditional vocabulary instruction approach on both weekly test scores and a standardized vocabulary knowledge measure. Rekrut (1996) described semantic mapping as an effective method for teaching high school students because it makes use of students' prior knowledge and experiences, and it engages them in the learning process. Similarly, an intervention study featuring learning-disabled junior high students performed by Bos & Anders (1990) revealed that those who participated in interactive strategies including semantic mapping outperformed those receiving direct vocabulary instruction on measures of vocabulary and comprehension. They hypothesized that this was because concept mapping strategies emphasize the activation of prior knowledge and also encourage students to share and elaborate on what they already know. Perhaps it is this the fact that the learner often becomes an active participant using prior knowledge to construct meaning that makes concept mapping an effective method for increasing vocabulary knowledge and leading to deeper comprehension of new constructs.

Methods

This qualitative study draws upon a models and modeling perspective of teacher development (Doerr & Lesh, 2003). The development of teachers' knowledge is a complex process where "the essence of the development of teachers' knowledge... is in the creation and continued refinement of sophisticated models or ways of interpreting the situations of teaching, learning and problem solving" (p. 126). Events that foster such model development include a multi-stage process of eliciting new ways of thinking, resolving the mismatches, and finally sharing and sustaining the model. This study is aimed at the initial stages of this model development in an effort to assess whether concept mapping can account for and promote the development of teachers' knowledge.

All participants were enrolled in a six-week summer online graduate course called Literacy Strategies for the Middle Grades (5-9) Mathematics Classroom. The textbook used for the course was written by Thompson et al. (2008) entitled *Mathematical Literacy: Helping Students Make Meaning in the Middle Grades*. Teachers were asked to create a concept map of the term equivalent expressions. They were given two days to post this concept map and asked to create it before beginning the course materials. Midway through the course in Week Four, teachers were directed to solve the Pool Border Problem (Annenberg, 2010), also sometimes called the Tiling Pools Problem (Phillips & Lappan, 1998).

> Explain to the class that they will be working in groups of four to investigate the number of tiles needed for pools of various sizes. (Annenberg, 2010)

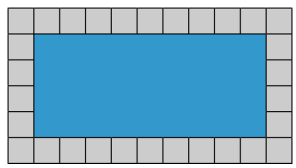

Figure 1. The Pool Border Problem

Week 4 assignments included reading a section on concept mapping and creating a concept map of the term area. During Week 4, the course instructor gave each student feedback consistent with semiotic models of concept maps in the textbook. In the final week, teachers were asked create a final concept map of the term equivalent expressions, and to answer this question: "Compare/contrast your first concept map done in the Icebreaker [first] Module with your new concept map by answering these questions: What changed? Why? How did doing the Pool Border problem influence your new concept map, if at all?"

Table 1. Summary of Teacher Certification

Type of Certification		Total	Gender	
			F	M
Special Education		2	2	
Adolescence Math Grades 7-12 Certification		1	1	
Middle Grades Generalist		1	1	
Math Certification Grades 7-9		2	1	1
Childhood Certification		11	7	4
	Math Concentration	3	3	
	Science Concentration	2	1	1
	Non-Math/Science Concentration	6	3	3

Data analysis was conducted after the course was completed. Each teacher's concept map was examined for correct mathematical content, and then examined for growth and the use of the

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

many suggestions related to semiotics from the textbook including: illustrate relationships between concepts, show special examples, use the term in various examples, give the definition, show examples and non examples, and define relationships (Thompson et al., 2008). Seventeen inservice teachers participated in the course taught by one of the researchers. A summary of the participants' teacher certification appears in Table 1.

Results

Our analysis of the development of the concept maps from the first module to the last showed that the concept maps fell into a continuum between two categories: (a) conceptually well developed, and (b) little to no growth in conceptual understanding. None of the initial concept maps were consistent with what might be considered a rich concept map with multiple nodes. Robust concept maps that demonstrated development included multiple nodes consistent with those modeled in the Thompson et al. (2008) textbook. These maps included nodes for definitions, use of the equals sign, and examples or non-examples. These concept maps integrated examples that referred to the Pool Border problem presented in the middle of the course. These robust concept maps also included nodes related to a semiotic perspective of teaching mathematics. Concept maps with little or no growth and that were in need of further development omitted most of these features and tended to be based in arithmetic.

In Figure 2 and 3, we examine the concept maps from Teacher A, a grades 7-12 mathematics teacher with a B. A. in mathematics education. This example represents the most dynamic growth of any student in the course. This initial concept map does include three different mathematical representations. However, it does not contain a definition, connections to arithmetic, examples of where the term is used, or non-examples. The exit concept map does include most of these elements.

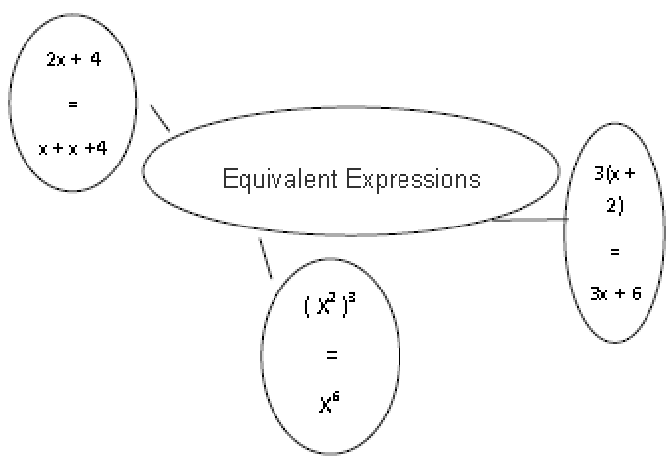

Figure 2. Teacher A: Initial Concept Map

This teacher commented,

> My first concept map only includes examples of equivalent expressions. This does not show that I have an understanding of the term equivalent expressions… My second concept map provides more insight… This change occurred after reading the section in chapter eight about concept maps. I was able to see how concept maps can be more useful than just listing examples. Providing more information such as definitions, properties, and non-examples can

> (1) help students to develop meaning and a better understanding of a concept or term and (2) allow teachers to evaluate where their students' abilities lie with a particular concept. When concept maps are done this way, they can be a very useful tool in the classroom; one that I see myself using to benefit my students' learning.

Teacher A appears to appreciate that the use of concept mapping may result in enhanced student learning and may provide insight into her students' conceptual understanding. It is also interesting to note that her arithmetic example of a false sentence 2 + 5 = 10 appears as a non-example. It is possible that this teacher still needs to think about the transition from arithmetic to algebraic thinking.

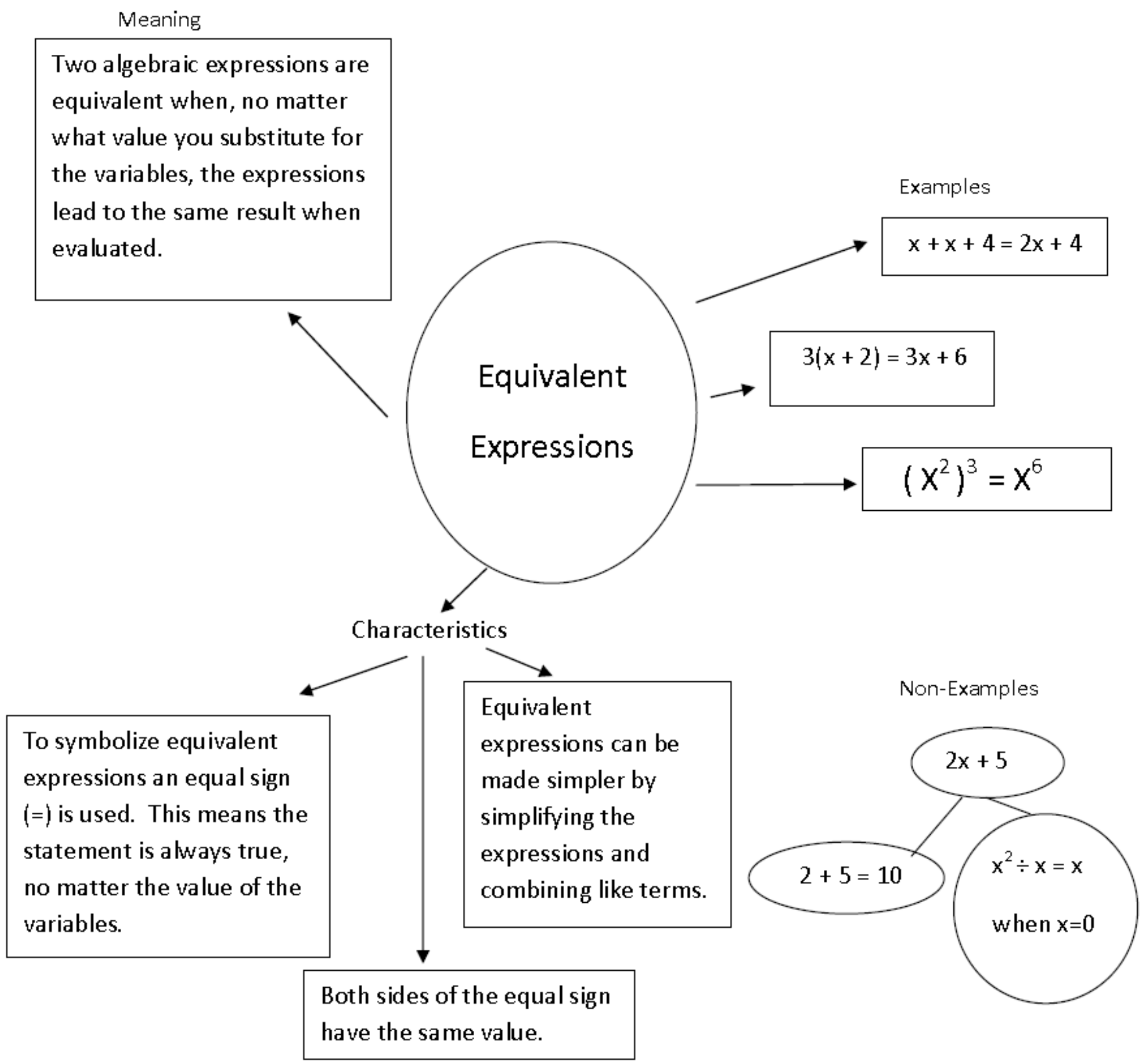

Figure 3. Teacher A: Exit Concept Map

Figure 4 shows a concept map with limited conceptual growth. Teacher B, a childhood science concentrate, began her concept map using examples based in arithmetic and moved to an algebraic interpretation. Her initial definition was based in arithmetic, and she misused the term "Equations" where she should have used "Expressions."

Her exit concept map included a better definition for the term and examples based in algebra. In explaining the difference between her two maps, she stated that "After completing the tiling problem, I realized that equivalent expressions involve the distributive property… [and] Seeing concept maps in the book also helped me." However, many features related to a semiotic perspective are missing including algebraic terminology such as variable, algebraic properties, arithmetic-based examples, and non-examples. Her hierarchical style limited her ability to show connections between nodes. Similar to the concept map of teacher A above, it is possible that Teacher B needs to think more about the transition from arithmetic in the early elementary grades to algebraic thinking in the middle grades.

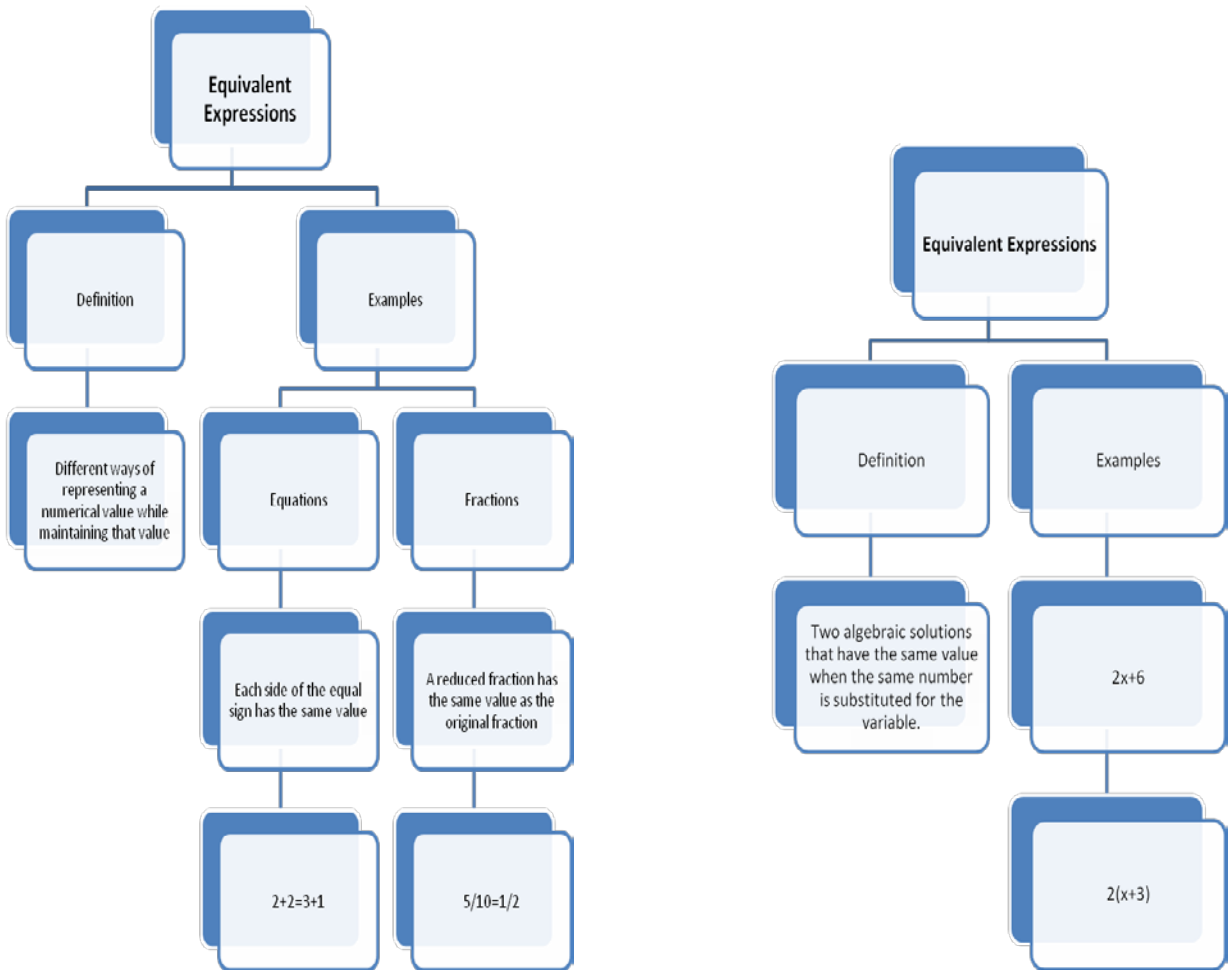

Figure 4. Teacher B: Initial and Exit Concept Maps

In Figure 5, we examine the exit map of Teacher C. This student was a childhood mathematics concentrate with an extension to teach grades 7-9 mathematics. He has worked the last five years in a tutoring center for grades K-9. His concept map omits many of the elements we expected, particularly after completing the Pool Border problem. Concerning his exit concept map, he wrote:

> I took the symbols used in algebraic expressions and linked them to the words that are found in the problems. There is a similarity in the fact that it is like a web where the words in each

bubble are related to each other in one way or another. The tiling pool problem influenced this because the students had to come up with an algebraic expression equivalent to the word problem given to them.

We believe that this student did not develop an appropriately conceptual or semiotic understanding of the term equivalent expressions. We are concerned that the most basic elements covered in the textbook are absent. His map completely missed any mention of the equals sign or equivalence. Student D's comments about his concept map reflect little understanding of the course objectives or the place of equivalent expressions in the middle grades curriculum. In fact, his definitions of key vocabulary are more congruent with instruction in the early elementary grades on the meaning of operations. The Pool Border problem was more about different representations for equivalent expressions than having students find one expression for a given word problem. Though the student referenced the Pool Border problem in his comments, he did not offer any examples related to the problem in his concept map, nor did he allude to the possibility of different interpretations of the given signs.

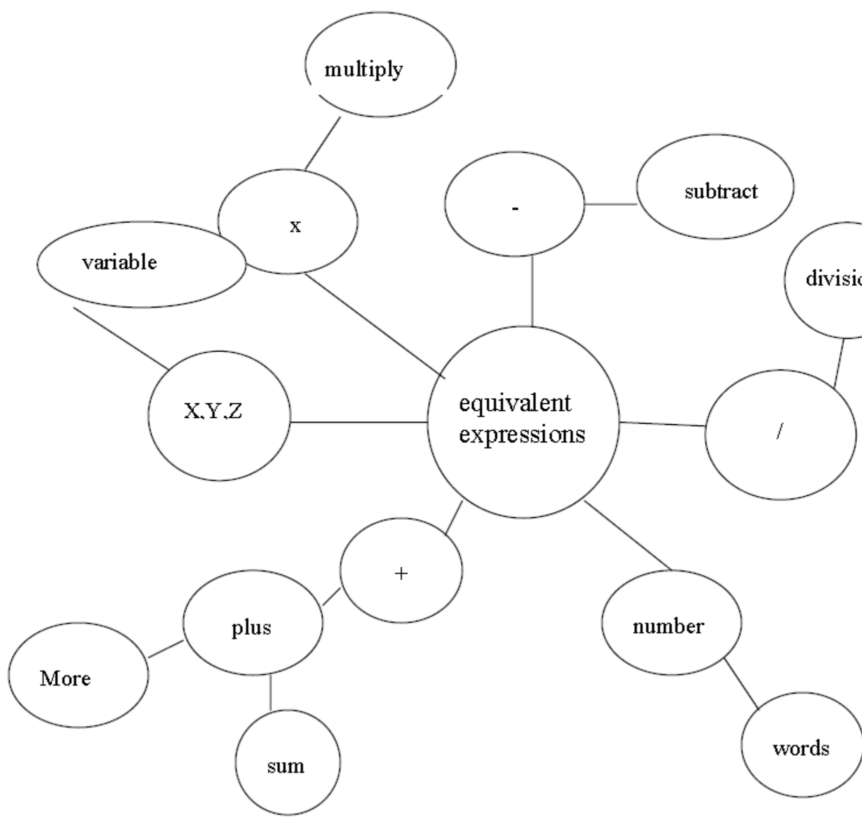

Figure 5. Teacher C: Exit Concept Map

Conclusion

If we want middle school teachers to teach reform-based curriculum where conceptual and procedural knowledge are valued, teachers themselves need a deep understanding of algebraic terminology. We found that iterative cycles of concept maps (Doerr & Lesh, 2003) taught within a reform-based graduate level methods course provided inservice teachers with opportunities to connect mathematical ideas related to equivalence. Consistent with the semiotic perspective advocated by Thompson et al. (2008), concept maps have the potential to pinpoint students' misconceptions. The online course as structured helped some of these inservice teachers develop their model of equivalent expressions. We were able to make very tentative conclusions that inservice teachers with a content concentration in mathematics or science often created multifaceted concept maps. In order to meet national goals of improved algebraic instruction, we believe the use of concept mapping has the potential to immerse all inservice teachers in the language of mathematics. Further research should be conducted in two areas. We believe that these concept maps should be shared by students in the course to further develop their ideas about teaching equivalence. And finally, more attention should be devoted to the development of teachers' understanding about equivalent expressions and the transition from arithmetic to algebraic thinking.

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RECONCILING TWO PERSPECTIVES ON MATHEMATICS KNOWLEDGE FOR TEACHING

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Despite the fact that there is widespread agreement as to the importance of teachers' mathematics knowledge, there is little consensus as to the particular content and structure of that knowledge. In this paper, we expand on our earlier work highlighting the importance of teachers' developing powerful, generative understandings from which an understanding of a body of mathematical ideas and its relation to other bodies can emerge. We will highlight the differences between this perspective and the other primary conceptualization of MKT found in the literature. The paper includes analysis of teachers' work from both perspectives, identifies differences between the two perspectives, and proposes a unified perspective on MKT that capitalizes on the strengths of each.

Introduction

The importance of teachers' mathematical knowledge has been well documented in the literature (Ball, 1993; Ma, 1999; Shulman, 1986). While there is widespread agreement as to the importance of teachers' mathematics knowledge, historically there is little consensus as to the particular content and structure of that knowledge and researchers continue to develop and study more fine-grained conceptualizations of the mathematical knowledge needed for teaching mathematics. In recent years the notion that teaching mathematics for understanding requires special *mathematical knowledge for teaching* (MKT) has gotten a great deal of traction and has become a major research focus in mathematics teacher education.

There are at least two perspectives on MKT present in the mathematics education literature. Ball, Hill and their colleagues have spent almost two decades studying the work that teachers do in teaching mathematics and have identified particularly useful practices, representations and "unpacked" mathematical understandings that allow teachers to interact productively with students in the context of teaching mathematics (Ball, 1993; Ball & Bass, 2003; Ball, Hill, & Bass, 2005). Alternatively, Silverman, Thompson and Clay approach the question of MKT from a different perspective (Silverman & Clay, 2009; Silverman & Thompson, 2008; P.W. Thompson & Thompson, 1994) that places teachers' mathematical understandings –the web of connections developed by individuals that allows them to act and enact within situations he or she encounters – at the center of research on MKT.

In this paper, we will analyze a sample of students' work from both the Ball-Hill and Silverman-Thompson-Clay perspective, identify differences between the two perspectives, and propose a unified perspective on MKT that capitalizes on the strengths of each.

Theoretical Perspective

Ball, Hill and their colleagues have identified practices, particularly useful representations and "unpacked" mathematical understandings that can support teachers as they attempt teaching mathematics to support students learning with understanding. Their pioneering work has succeeded in empirically identifying mathematical reasoning, insight, understanding and skill needed in teaching mathematics for understanding and has developed, tested and validated

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instruments for measuring MKT (Ball, 1993; Ball, et al., 2005; Ball & McDiarmid, 1990). Further, they have succeeded in identifying a positive relationship between mathematical knowledge for teaching and student achievement (H.C. Hill, Rowan, & Ball, 2005; H. C. Hill, Schilling, & Ball, 2004).

As an example of MKT from the Ball-Hill perspective, consider the incorrect application of the algorithm for multiplying two multi-digit numbers shown in Figure 1. In the example, we see a student who forgot to "move the 70 over." Ball, Hill and Bass (2005) argue that while the ability *to do* the multiplication is essential for teachers, teachers need additional knowledge – in this case they would need to recognize that the student was simply multiplying 2 (from 25) times 35 without acknowledging or realizing that the "2" actually represents 2 tens. This MKT provides teachers a starting point for engaging productively with students about mathematical understandings as opposed to a focus on teaching students to implement the rule appropriately.

$$\begin{array}{r} 35 \\ \times 25 \\ \hline 175 \\ 70 \\ \hline 245 \end{array}$$

Figure 1: Incorrect Application of Multiplication Algorithm

In our work, we approach the question of MKT from a different perspective. Rather than focusing on identifying particular mathematical insight, understanding and skill needed in teaching mathematics, we focus on mathematical understandings "that carry through an instructional sequence, that are foundational for learning other ideas, and that play into a network of ideas that does significant work in students' reasoning" (Thompson, 2008). We refer to these understandings as *coherent* understandings: powerful, generative "big ideas" from which an understanding of a body of mathematical ideas and its relation to other bodies can emerge.

As an example of our perspective, consider the case of algebra. Even a cursory glance over a pre-algebra, algebra, or pre-calculus text yields two broad categories of content: (1) those focusing on sets, operations, and their properties (including a set theoretic conception of functions and equation solving) and (2) those focusing on variables as *varying quantities* and relationships between those varying quantities (including rates of change, maxima and minima, and a graphical perspective of functions and equation solving). Further, within each category, students learn a variety of disconnected techniques and procedures[1] for dealing with the variety of problem types that occur. Rarely, if ever, do students have an opportunity to look across problem types, reason with significant mathematical ideas, and ask questions such as "In what ways are $y = x^3$ and $y = \sin x$ the same? How can what we know about $y = x^3$ help us understand $y = \sin x$?"

It has been argued elsewhere that the development of *covariational perspective*, which involves a focus on coordinating changes in one varying quantity with corresponding changes in a second varying quantity (Confrey & Smith, 1994) can support students with engaging productively with a variety of seemingly disconnected functional relationships from category #2 (Silverman & Thompson, 2008; Thompson, 1994; 2008). In this paper, we draw upon the mathematical field of algebra to describe a scheme of understandings within which much of the variety of school algebra content is seen as small variations of the same theme (bringing coherence to the content from category #1). As opposed to learning and remembering when to apply particular techniques for particular problem types, a focus on algebraic structure provides a general framework that allows learners to engage with a variety of mathematical tasks from the 7-16 school curriculum in a consistent and sensible way. Further "new" content is then experienced as an extension of this structure (for example, the inverse of the sine function allowsstudents to answer the question "How do I undo the sine function?"). We argue that this is a coherent understanding of school algebra.

This algebraic perspective places primary emphasis on understanding the algebraic structure of the various sets (integers, real numbers, functions) and operations (addition, multiplication, composition) encountered throughout the school algebra curriculum. Within this context, particular attention is paid to algebraic properties such as the existence of identities and inverses and commutivity. In traditional classes and texts, students are often asked to spend time memorizing each of these properties and matching a particular application of these properties with the name of the property during the first days of each academic year. This surface-level approach diminishes the centrality of algebraic structure within school algebra and mathematics in general. It is these algebraic properties that allow us to perform the actions we want to perform (add), that forces us to narrow the places in which we perform them (one-to-one functions), to create new sets to perform them in (complex numbers), and/or the reason we can not do many the things we would like to do (solve matrix equations with non-invertible coefficients).

Practically every string of symbols in algebra texts (expressions, equations, functions, etc.) is made of quantities from sets and operations acting on those quantities. When viewed this way, these quantities and operations can be seen as emerging organically from students' understanding of arithmetic. For example, consider the following "algebraic" explanations:

Expression	Algebraic explanation
$2x+5$	the quantity x is being multiplied by 2 then 5 is being added to the result
$-\frac{1}{3}x+2$	the quantity x is being multiplied by 1/3, negated, then 2 is being added
$2(x-7)^2-8$	the quantity x is having 7 subtracted from it, being squared, multiplied by 2, then 8 is subtracted

The advantage to the algebraic approach is that once students recognize the algebra in each string, it provides them with a point of entry to topics such as evaluating expressions, solving equations, or finding inverse functions. While evaluating an expression for a particular value of x seems like a trivial step from the algebraic explanation above (perform the operations on a particular value of the quantity, x), solving an equation in which the expression is equal to a particular value or finding an inverse might seem like a stretch for the algebraic perspective. Through our experience working with teachers, we have found that teachers routinely read an equation from left to right and then begin to "work it," saying "add -5 to both sides" or a similar action, giving no mathematical or logical reason for the action. In this way, traditional instruction has a tendency to develop understandings similar to those described by Erlwanger more than three decades ago: (1) students believing there are rules for each type of problem and the challenger of learning mathematics is to figure out which rule to apply to which problem and (2) applications of these rules are not rational and logical and not subject to sense making (Erlwanger, 1973).

In contrast, from the algebraic perspective, solving equations is all about recognizing the quantities, operations, and properties of those operations that can be applied to the quantities, including "undoing" these operations by applying inverse operations and generating identity elements. As an example, the following table demonstrates an algebraic way of thinking about operations being applied to x and applying the corresponding inverse operations to zero to solve the equation $2(x-7)^2-8=0$:

operations on x

Operations	Expression	Inverse operations	Inverse operations
subtract 7	$x - 7$	add seven	$\pm 2 + 7 = 9$ or 5
square	$(x-7)^2$	"un-square"	$\sqrt{4} = \pm 2$
multiply by two	$2(x-7)^2$	multiply by ½	$8 \cdot \frac{1}{2} = 4$
subtract 8	$2(x-7)^2 - 8$	add 8	$(0+8) = 8$

"un-doing" operations on 0

While on the surface this activity may seem virtually identical to the equation solving that goes on in schools, there are two significant affordances to the algebraic perspective. First, it gives students a means of engaging with any algebraic task in the school mathematic curriculum. Rather than seeking (or asking the teacher for) the rule or procedure for each new problem, algebraic situations can be approached by students focusing on and asking the questions, "What are the quantities in this problem?" and "What's happening to these quantities?" Second, this approach allows students to see coherence in the school mathematics curriculum. For example, students can engage in a similar manner as described above with any of the following equations:

- $2(x-7)^3-8=0$
- $2\ln(x-7)-8=0$
- $2\int x-7\,dx-8=0$

As a result, the key challenge for students shifts from learning what to do for each type of problem to engaging sensibly with a given problem and analyzing and utilizing the algebraic structure: "OK, I know how to solve this. First I figure out what the quantities are, what's being applied to them, and what properties can I use to answer the question." When students encounter an operation they are unfamiliar with, several highly mathematical questions arise: "What properties hold? Is there an identity? Does an inverse exist? Can I find it?" It is this algebraic focus in early algebra experiences that can support students as they encounter new and diverse sets like functions, matrices, and differentials and related expressions and equations.

Setting and Participants

In this paper, we will present our analysis of a small slice of a course titled *Algebraic Reasoning* in which we look at solving equations. This online graduate course is a content-based course required for the master's degree program in mathematics education at a university in the Mid-Atlantic region of the United States. The course was designed by the second author and has gone through multiple iterations of testing and revision by all authors. The course is designed to support participants in developing an understanding of algebraic structure and to use this structure to support teachers in developing coherent understandings of algebra. Initial learning activities focused on supporting teachers in unpacking the algebraic structure of the integers under addition and the rational numbers under multiplication, with particular attention being paid to the language of algebraic properties especially the existence of identities and inverses. These activities were then used as a springboard for thinking about solving equations. This transition from operations on numbers to solving equations began with the task of using the language of algebraic structures to read algebraic expressions focusing on an unknown quantity and operations being applied to that quantity, then asked teachers to use the properties they had learned when thinking about solving equations. In the second half of the course participants explore increasingly complex sets including functions, complex numbers, matrices, and finite sets, along with equations with quantities and coefficients from these sets.

Data and Analysis

In the following section, we present two teacher responses to a task that was developed as a starting point for teachers making the transition from studying algebraic properties of various sets and operations to using algebraic properties in a variety of "solving equations" tasks. Prior to working on the task, the teachers viewed a video podcast highlighting the "big ideas" of this transition. Then, the teachers were assigned the task of creating an audio-visual podcast responding to the following task: "Using the big ideas and language you used in our previous work, demonstrate and discuss how you would solve equations. Be sure to demonstrate making zeros and ones, the use of zero-pairs and one-pairs, and to practice using the language of identities and inverses in these problems. Don't forget to read each problem with algebraic meaning." A set of eight linear equations was provided for the students on which to focus their podcasts.

Initial analysis involved reviewing a set of responses to the given task and identifying patterns in the responses. A number of exemplars of each category were transcribed and analyzed in further detail. The responses included below represent two distinct categories of the initial student responses. While throughout the remainder of the course these responses were the focus of small group discussions, individual feedback from the instructor and teaching assistant, and revision, we limit our attention in this paper to the teachers' initial responses.

Category 1: Focus on Coefficients and Canceling with Added Language of Algebraic Properties

As an example of the first category of solutions and explanations, we consider the work of John (all individuals are identified by a pseudonym). In the following excerpt, we see John's discussion of solving the equation $-2x - 1 = -9$:

> *Our first problem is -2x – 1 = -9. Now, we're going to try to solve this equation. What we want to do is get x by itself. ... We have -2x – 1 and we want to get that x to be by itself so we're going to clear out that -2 and that -1 and this is how we're going to do it. We're going to think about "negative one"... we're going to think about "minus one" as being a "negative one" and we're going to think about the additive inverse of "negative one." ... So the positive one that we added cancels with the negative one on the left hand side, because remember positive one and negative one are inverses. They're additive inverses. So when you add them you get the identity. And, of course, the additive identity is zero. ...*

We argue in this example that John is expanding on a "typical" explanation that involves "getting *x* by itself" by examining the coefficients and constants on the left hand side of the equation and canceling each out in an order loosely determined by commonly established order of operations. John does go beyond a typical explanation in that he is incorporating more detailed algebraic language in his explanation. For example, he justifies the step of "adding one to both sides" and provides additional details on "canceling" through clear references to the relevant algebraic properties. While we see these advances as significant both mathematically and pedagogically, we believe they are not evidence of a transition to a more coherent understanding of school algebra. This claim will be highlighted by the contrast between John and the second category of solution and explanation discussed below.

Category 2: Focus on Operations on a Quantity and Properties of the Operations

The second category of solutions and explanations, exemplified by Sarah's below, involves a more substantial shift towards developing an algebraic perspective and using the language of algebra. In the excerpt below, we see Sarah discussing the solution to the equation $3x + 7 = 2$:

> *The first equation I selected was "three x plus seven is equal to two." This is telling us that we have some quantity and then we multiply that by three and add seven and the result is two. ... So we have to find out for what value of x will give us 3x +7 equal to two. There are some operations which are being applied here. The first one we see is that we are multiplying x by three ... The second operation is to add seven, so we first multiply by three and then we add seven. ... In order to solve this equation, we want to find the inverses so that we can use the multiplicative and the additive identity and other properties that we have learned about. Now for three, the multiplicative identity is "one over three." And we know that three multiplied by one over three, its inverse or its reciprocal, is equal to one – the multiplicative identity or the "do nothing" element. Similarly, for our plus seven, its inverse is negative seven and we know that seven plus negative seven is equal to zero, the additive identity. That means it does nothing. With these – seven plus negative seven, that forms a zero pair. And three multiplied by one-third gives me a one-pair. Now let's ... solve this equation. We have 3x +7=2. The first thing I will do is to form a zero pair by adding the additive identity of seven, which we have identified already. And it is an equation so what I'm doing on one side, I have to do on the other. This gives me 3x plus seven plus a negative seven being equal to two minus seven [the right hand side of the equation]. ... From here we have the additive identity zero, so we have 3x plus zero is equal to five and that tells me that 3x is equal to negative five. Now we want to get the single unit of x, so what am I going to do here? I'm going to multiply by its multiplicative inverse, which is one over three. One over three multiplied by 3x is equal to -5 multiplied by one over three. ... What I am doing here on the left hand side – I have one over three times 3x and I'm creating a one-pair and we know that they are multiplicative inverses and they form one. And one multiplied by any number gives me itself, so we have the solution x is equal to five over three or five thirds.*

In this excerpt, we see Sarah using a perspective that we argue is fundamentally different from the typical discussion of solving equations. Rather than focusing on getting *x* by itself by performing actions on the coefficients based on pre-established rules for solving equations, Sarah is reasoning through the algebraic structure by (1) analyzing the operations being applied to the quantity *x*, (2) identifying the inverse operations, and (3) applying those inverse operations in an order determined solely by the operations being applied to *x*. While Sarah's discussion does have room for improvement, most notably, clearly explaining that we're creating zero pairs for a particular purpose and not just "forming them," it does demonstrate a way of thinking that generalizes past the particular task. Whereas John's explanation could be described as telling a reader (or a student) how to solve a particular mathematical task, Sarah's explanation was embedded within a way of thinking that can easily generalize to other problem types. For example, it is easy to imagine Sarah engaging with $\cos x + 7 = 2$ in a similar way, only (possibly) wondering what the inverse of $\cos x$ is.

Discussion

While the distinction between the two categories above is interesting from the perspective of student learning, we wish to close by returning to the previous discussion of the two perspectives on MKT. John and other teachers in this category, who enriched their explanation of their solution method by adding mathematical explanations for the various steps of the solution and connecting it with other mathematical ideas, were drawing on (and possibly developing) MKT. We argue that this "enriched explanation" is analogous to the MKT discussed by Ball and her colleagues: it is mathematical knowledge that can support teachers as they attempt to teach particular mathematical content for understanding. In contrast, consider teachers like Sarah in the second category: they did not simply add mathematical explanations and justifications, but rather capitalized on the algebraic structure to present a solution that is ripe for generalization. While we acknowledge that they are likely not aware of the generalizibility of their explanation, making that evident is a focus for the remainder of the course. Regardless, we believe that without coming to understand particular algebraic scenarios in a way that can be generalized, there is little chance of coming to see the inherent coherence of mathematics or to support it in one's students.

In some ways, this is similar to what Ball and her colleagues refer to as *Horizon Content Knowledge* (HCK), or "knowledge of how mathematical topics are related of the span of mathematics included in the curriculum" (Ball, Thames, & Phelps, 2008, p. 404) or related to mathematics beyond the scope of the K-12 curriculum. While it is similar, we find it necessary to make one clarification: in most cases, teachers are aware of the algebraic properties discussed in this paper and secondary teachers, who have taken abstract algebra are (or were) aware of the connections between the first few sections of their 10th grade algebra text and their university abstract algebra text. What they have not developed is a conceptualization of mathematics that uses the lens of algebraic structure to provide coherence to a variety of related mathematical ideas and content. This, too, is a mathematical understanding, but one of a much bigger grain size than most other perspectives of MKT.

We wish to close this paper by highlighting the synergy between the two conceptualizations of MKT. We see the work of Ball and her colleagues as focusing on language, representations and "unpacked" mathematical understandings of particular mathematical ideas as indispensible. But we also believe that a more global view, one that highlights a teachers' connected web of understandings of which this language and these representations and understandings are a part, is also essential. We believe that a more refined theory of MKT will only emerge if its emphasis is both on the local and global perspectives.

Endnotes

It is important to note that while the mathematics itself is not disconnected, students are offered little opportunity to experience this connectedness. Instead, they experience a disconnected set of facts and procedures.

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TEACHER DIFFERENCES IN MATHEMATICS KNOWLEDGE, ATTITUDES TOWARD MATHEMATICS, AND SELF-EFFICACY AMONG NYC TEACHING FELLOWS

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Supplying urban settings with quality teachers is important for student achievement. We examined the differences in content knowledge, attitudes toward mathematics, and levels of self-efficacy among teachers in the NYC Teaching Fellows program. Findings revealed that high school teachers had significantly higher content knowledge than middle school teachers, and Mathematics Teaching Fellows had significantly higher content knowledge than Mathematics Immersion Teaching Fellows. Lastly, mathematics and science majors had significantly higher content knowledge than other majors.

Introduction

The purpose of this study was to determine differences in content knowledge, attitudes toward mathematics, and concepts of teaching self-efficacy among different categories of alternative certification teachers in New York City. Determining these differences is critical for two reasons. First, it is important for teacher recruitment. If policy makers, administrators, and teacher educators know which teacher characteristics lead to the highest levels of content knowledge, attitudes, and self-efficacy, recruitment can be better focused to yield the best results for students. Second, in teacher preparation, knowing which teachers need the most support, and in what areas, can lead to increased teacher quality through better preparation and focused professional development. This study is a continuation of a previous study that found high positive attitudes and high concepts of teaching self-efficacy among the teachers sampled.

The teacher participants in this study come from two mathematics methods sections of New York City Teaching Fellows (NYCTF) teachers. The NYCTF program was developed in 2000 in conjunction with The New Teacher Project and the New York City Department of Education (NYCTF, 2008; Boyd, Lankford, Loeb, Rockoff, & Wyckoff, 2007). The program goal was to recruit professionals from other fields to supply the large teacher shortages in New York City's public schools with quality teachers. A shortage of 7000 teachers was predicated for 2000, with a possible shortage of 25,000 teachers over the next several years (Stein, 2002). Prior to September 2003, New York State allowed teachers to obtain temporary teaching licenses in order to meet the schools' staffing needs.

Mathematics Teaching Fellows begin graduate coursework at one of several New York universities and student teach in the summer before they start teaching in September. Teaching Fellows receive subsidized tuition and are given a stipend in the summer, and receive full teacher salaries when they begin teaching in September. Over the next several years Teaching Fellows continue taking coursework in education while teaching in their classrooms with a Transitional B license from the New York State Education Department (NYSED) that allows them to teach for three years before earning Initial Certification. Before they begin teaching in September, Teaching Fellows must pass the Liberal Arts and Sciences Test (LAST) and the Content Specialty Test (CST) in mathematics required by the New York State. Those who lack the required 30 mathematics course credits are labeled as "Mathematics Immersion," and those with

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the minimum required 30 credit hours are considered “Mathematics Teaching Fellows.” Mathematics Immersion Teaching Fellows must complete the required credits within three years. Teaching Fellows generally teach in high needs schools throughout the city (Boyd et al., 2006).

Literature Review and Theoretical Framework

Recently there has been an interest in studying the effects of alternative teacher certification programs in U.S. classrooms with a particular interest in teacher quality issues (Darling-Hammond, 1994, 1997; Darling-Hammond et al., 2005; Evans, 2009, in press). Further, there has been specific interest in Teaching Fellows in New York schools (Boyd, Grossman, Lankford, Loeb, & Wyckoff, 2006; Boyd, Lankford, Loeb, Rockoff, & Wyckoff, 2007; Stein, 2002). A majority of these studies investigated teacher retention and student achievement as variables that determine success. These are two of the most important variables, but there is a need to investigate other variables related to success, such as teacher content knowledge, attitudes toward mathematics, and teacher self-efficacy. Suell and Piotrowski (2007) called for a strong academic coursework component for alternative certification teachers. According to them this specialized body of coursework is the key determinant of quality teacher preparation. Previous studies also found that teachers prepared in alternative certification programs, such as the Teaching Fellows program, manage to secure higher scores on tests of content than other teachers (Boyd, Grossman, Lankford, Loeb, & Wyckoff, 2006; Boyd, Lankford, Loeb, Rockoff, & Wyckoff, 2007). However, details about the quality of content knowledge have been sparse. Indeed, there has been a lack of concentrated focus on mathematics teachers specifically.

Aiken (1970) was an early pioneer in researching the relationship between mathematical achievement and attitudes toward mathematics. Similar to Aiken’s work, Ma and Kishor (1997) found a small but positive significant relationship between achievement and attitudes through meta-analysis. This reported relationship between achievement and attitudes, along with Ball, Hill, & Bass’ (2005) emphasis on the importance of content knowledge for teachers, formed the framework of this study. Three research questions guided data collection and analysis as listed below:

1. Are there differences in mathematical content knowledge, attitudes toward mathematics, and concepts of teacher self-efficacy between middle and high school Teaching Fellows?
2. Are there differences in mathematical content knowledge, attitudes toward mathematics, and concepts of teacher self-efficacy between Mathematics and Mathematics Immersion Teaching Fellows?
3. Are there differences in mathematical content knowledge, attitudes toward mathematics, and concepts of teacher self-efficacy between undergraduate college majors among the Teaching Fellows?

Bandura’s (1986) construct of self-efficacy theory framed the study’s focus on self-efficacy of Teacher Fellows. Bandura divided teacher self-efficacy according to a teacher’s belief in his or her ability to teach well, and his or her belief in a student’s capacity to learn well from the teacher. Teachers who feel that they cannot effectively teach mathematics and affect student learning are more likely to avoid teaching from an inquiry and student-centered approach with real understanding (Swars, Daane, & Giesen, 2006).

Methodology

The study utilized quantitative methods. The sample consisted of 42 new teachers enrolled in the Teaching Fellows program. Approximately one third of participants were male. The Teaching Fellows in this study were enrolled in two sections of mathematics methods that involved a combination of both pedagogical and content instruction. The course focused on constructivist methods with an emphasis on problem solving and real-world connections.

Teaching Fellows were given a mathematics content test and two questionnaires at the beginning and at end of the semester. The mathematics content test consisted of 25 free response items ranging from algebra to calculus. The mathematics content test taken at the end of the semester was similar in form and content to the one taken at the beginning. Additionally, mathematics Content Specialty Test (CST) scores for the New York State certification were recorded as another measure of mathematical content knowledge.

The first questionnaire was created by Tapia (1996) and has 40 items that measured attitudes toward mathematics including self-confidence, value, enjoyment, and motivation in mathematics. The instrument uses a 5-point Likert scale ranging from strongly agree, agree, neutral, disagree, to strongly disagree. The second questionnaire was adapted from the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) developed by Enochs, Smith, and Huinker (2000). This survey measures concepts of self-efficacy. The MTEBI is a 21-item 5-point Likert scale instrument with the same choices as the attitudinal questionnaire, and is grounded in the theoretical framework of Bandura's self-efficacy theory (1986). Based on the Science Teaching Efficacy Belief Instrument (STEBI-B) developed by Enochs and Riggs (1990), the MTEBI contains two subscales: Personal Mathematics Teaching Efficacy (PMTE) and Mathematics Teaching Outcome Expectancy (MTOE) with 13 and 8 items, respectively. Possible scores range from 13 to 65 on the PMTE, and 8 to 40 on the MTOE. The PMTE specifically measures a teacher's self-concept of his or her ability to effectively teach mathematics. The MTOE specifically measures a teacher's belief in his or her ability to directly affect student learning outcomes. Enochs et al. (2000) found the PMTE and MTOE had Cronbach alpha coefficients of 0.88 and 0.77, respectively.

Research question one and two were answered using independent samples *t*-tests on data collected from the 25-item mathematics content test, 40-item attitudinal test, and 21-item MTEBI with two subscales using both pre and posttest scores. Research question three was answered using one-way ANOVA on data also collected from the same instruments. For the third research question Teaching Fellows were divided into three categories based upon their undergraduate college majors: liberal arts, business, and mathematics and science majors. Liberal arts majors consisted of majors such as English, history, Italian, philosophy, political science, psychology, sociology, Spanish, and women studies. Business majors consisted of those studying accounting, business administration and management, commerce, economics, and finance. Mathematics and science majors included those studying mathematics, engineering, and the sciences (biology and chemistry). All significance levels in this study were at the 0.05 level.

Results

The first research question was answered using independent samples *t*-tests comparing middle and high school teacher data using the 25-item mathematics content test, 40-item attitudinal test, and 21-item MTEBI with two subscales: PMTE and MTOE. The results of the independent samples *t*-test for the first part of research question one revealed a statistically significant difference between middle school teacher scores (M = 68.42, SD = 15.600, N = 26) and high school teacher scores (*M* = 85.13, *SD* = 16.041, *N* = 16) for the mathematics content pretest with $t(40) = -3.334$, $p = 0.002$, $d = 1.056$. This means high school teachers had higher content test scores than middle school teachers on the pretest. Additionally, there was a large effect size. The results of the independent samples *t*-test for the first part of research question one also revealed a statistically significant difference between middle school teacher scores (*M* = 79.46, *SD* = 15.402, *N* = 26) and high school teacher scores (*M* = 92.63, *SD* = 6.582, *N* = 16) for the mathematics content posttest with $t(40) = -3.230$, $p = 0.002$, $d = 1.112$. This means high school teachers had higher content test scores than middle school teachers on the posttest as well. Additionally, there was a large effect size. For attitudes toward mathematics and concepts of self-efficacy there were no statistically significant differences found between middle and high school teachers on both pre and posttests.

The second research question was answered using independent samples *t*-tests comparing Mathematic Immersion and Mathematics Teaching Fellows data also using the 25-item mathematics content test, 40-item attitudinal test, and 21-item MTEBI with two subscales: PMTE and MTOE. The results of the independent samples *t*-test for the first part of research question two revealed a statistically significant difference between Mathematics Immersion Teaching Fellows' scores (*M* = 68.90, *SD* = 17.008, *N* = 30) and Mathematics Teaching Fellows' scores (*M* = 89.50, *SD* = 7.868, *N* = 12) for the mathematics content pretest with $t(40) = -4.005$, $p = 0.000$, $d = 1.555$. This means Mathematics Teaching Fellows had higher content test scores than Mathematics Immersion Teaching Fellows on the pretest. Additionally, there was a large effect size. The results of the independent samples *t*-test for the first part of research question two also revealed a statistically significant difference between Mathematics Immersion Teaching Fellows' scores (*M* = 80.53, *SD* = 14.460, *N* = 30) and Mathematics Teaching Fellows' scores (*M* = 94.33, *SD* = 7.390, *N* = 12) for the mathematics content posttest with $t(40) = -3.130$, $p = 0.003$, $d = 1.202$. This means Mathematics Teaching Fellows had higher content test scores than Mathematics Immersion Teaching Fellows on the posttest as well. Additionally, there was a large effect size. For attitudes toward mathematics and concepts of self-efficacy there were no statistically significant differences found between Mathematics and Mathematics Immersion Teaching Fellows on both pre and posttests.

The third research question was answered using one-way ANOVA comparing different undergraduate college majors using the 25-item mathematics content test, 40-item attitudinal test, and 21-item MTEBI with two subscales: PMTE and MTOE. Teaching Fellows were grouped according to their undergraduate college major along three categories: liberal arts (*N* = 16), business (*N* = 11), and mathematics and science (*N* = 15) majors. The results of the one-way ANOVA for the first part of research question three revealed a statistically significant difference, $F(2, 39) = 8.582$, $p = 0.001$ (see Table 1) on the pretest. A post hoc test (Tukey HSD) was performed to determine exactly where the means differed. The post hoc test revealed that mathematics and science majors had significantly higher content knowledge on the pretest than business majors, $p = 0.001$ and liberal arts majors, $p = 0.008$. There were no other statistically significant differences. The results of the one-way ANOVA for the first part of research question three also revealed a statistically significant difference, $F(2, 39) = 6.469$, $p = 0.004$ (see Table 1) on the posttest. Again, a post hoc test (Tukey HSD) was performed to determine exactly where the means differed. The post hoc test revealed that mathematics and science majors had significantly higher content knowledge on the posttest than business majors, $p = 0.005$ and liberal arts majors, $p = 0.025$. There were no other statistically significant differences. It was concluded that mathematics and science majors had statistically significant higher content knowledge scores on both pre and posttests than non-mathematics and non-science majors. For attitudes toward mathematics and concepts of self-efficacy there were no statistically significant differences found between the undergraduate college majors on both pre and posttests.

Table 1. Means and Standard Deviations for Content Knowledge

Pre, Post, and CST Tests	Mean	Standard Deviation
Content Knowledge Pre Test		
Liberal Arts (*N* = 16)	70.13	16.382
Business (*N* = 11)	64.45	15.820
Math/Science (*N* = 15)	87.33	12.804
Total (*N* = 42)	74.79	17.605
Content Knowledge Post Test		
Liberal Arts (*N* = 16)	81.19	15.132
Business (*N* = 11)	76.82	14.034
Math/Science (*N* = 15)	93.60	7.679
Total (*N* = 42)	84.48	14.225
CST Content Knowledge		
Liberal Arts (*N* = 16)	255.81	18.784
Business (*N* = 11)	249.64	18.943
Math/Science (*N* = 15)	273.80	15.857
Total (*N* = 42)	260.62	20.184

Since significant differences were only found for content knowledge, as measured by the 25-item mathematics content test, it was decided that a focus on content knowledge differences would be appropriate using another content instrument. The first part of each research question was addressed again by using scores on the CST. It was found using an independent samples *t*-test that high school teachers ($M = 269.25$, $SD = 17.133$, $N = 16$) had statistically significant higher content knowledge than middle school teachers ($M = 255.31$, $SD = 20.372$, $N = 26$) with $t(40) = -2.283$, $p = 0.028$, $d = 0.741$ as measured by CST scores. Additionally, there was a moderate effect size. This means mathematics high school teachers had higher content test scores than middle school teachers on the CST, just as was found using the 25-item mathematics content test.

Further, it was found using an independent samples *t*-test that Mathematics Teaching Fellows ($M = 276.33$, $SD = 16.104$, $N = 12$) had statistically significant higher content knowledge than Mathematics Immersion Teaching Fellows ($M = 254.33$, $SD = 18.291$, $N = 30$) with $t(40) = -3.636$, $p = 0.001$, $d = 1.277$ as measured by CST scores. Additionally, there was a large effect size. This means Mathematics Teaching Fellows had higher content test scores than Mathematics Immersion Teaching Fellows on the CST, just as was found using the 25-item mathematics content test.

Teaching Fellows were again grouped according to their undergraduate college majors: liberal arts ($N = 16$), business ($N = 11$), and mathematics and science ($N = 15$) majors. The results of the one-way ANOVA revealed a statistically significant difference, $F(2, 39) = 6.765$, $p = 0.003$ for the CST scores (see Table 1). A post hoc test (Tukey HSD) was performed to determine exactly where the means differed. The post hoc test revealed that mathematics and science majors had significantly higher content knowledge, as measured by the CST, than business majors, p = 0.004 and liberal arts majors, p = 0.021. Again, it can be concluded that mathematics and science majors had statistically significant higher content knowledge scores than non-mathematics and non-science majors, as measured by the CST. There were no other statistically significant differences.

Discussion

Results indicated that high school teachers had higher mathematics content knowledge than middle school teachers. Further, it was found that Mathematics Teaching Fellows had higher mathematics content knowledge than Mathematics Immersion Teaching Fellows. Finally, it was established that mathematics and science majors had higher mathematics content knowledge than non-mathematics and non-science majors. However, no differences were found between middle and high school teachers, Mathematics and Mathematics Immersion Teaching Fellows, and liberal arts, business, and mathematics and science majors in regards to their attitudes toward mathematics and concepts of teacher self-efficacy.

In the first study it was found that teachers had positive attitudes toward mathematics and high concepts of self-efficacy. Combining the results of the first study with the findings of the present work, an interesting outcome emerged. Teachers had the same high positive attitudes toward mathematics and same high concepts of self-efficacy regardless of content ability. Thus, teachers believed they were just as effective at teaching mathematics, despite not having the high level of content knowledge that some of their colleagues possessed. This is significant since high content knowledge is assumed a necessary condition for quality teaching (Ball et al., 2005). This finding contrasts sharply with reports of research conducted on the relationship between content knowledge and attitudes (Aiken, 1970, 1974, 1976; Ma & Kishor, 1999).

It was not surprising that high school teachers had higher content knowledge than middle school teachers. However, New York State requires a minimum of 30 mathematics credits for both middle and high school teachers. It would be expected that high school teachers may have higher content knowledge due to their experience working with higher level mathematics students. However, this does not explain why high school teachers scored better on the CST and content pretest instruments since this study began at the beginning of their teaching careers, and the teachers did not yet have significant classroom experience. It is speculated that teachers with stronger content knowledge may be drawn more to high school teaching, rather than middle school teaching, and the more rigorous content that comes with teaching high school mathematics.

It is imperative that future research address whether or not there are differences in actual teaching ability among the Mathematics and Mathematics Immersion Teaching Fellows and different college majors held by the teachers. One way to determine this would be to measure student performance in the schools to determine differences in student achievement among students with varying teacher content knowledge, mathematics immersion status, and teacher college majors.

Considering that many alternative certification teachers, such as Teaching Fellows, teach in high needs urban schools in New York City (Boyd, Grossman, Lankford, Loeb, & Wyckoff, 2006), and throughout the United States, it is imperative that policy makers, administrators, and teacher educators continually evaluate the teacher quality in alternative certification programs. NCTM says, “Every student has the right to be taught mathematics by a highly qualified teacher-a teacher who knows mathematics well and who can guide students' understanding and learning” (NCTM, 2005). This study informs teacher education since it was found that high school teachers, Mathematics Teaching Fellows, and those who majored in mathematics and science had higher mathematics content knowledge on several measures. Since New York State holds the same high standards for both high school and middle teachers alike, strategies to better middle school teachers' content knowledge should be investigated and implemented. It is recommended that middle school teachers be given the professional development they need in mathematics content knowledge by both the schools in which they teach and the schools of education in which they are enrolled. Many professional development activities focus on pedagogical knowledge. However, middle school teachers need an emphasis on strengthening content knowledge as well.

Better preparation for Mathematics Immersion Teaching Fellows and non-mathematics and non-science majors is needed. Based upon the results of this study, it is recommended that before teachers from non-mathematics backgrounds are placed in mathematics classrooms, they first be required to earn a minimum of 30 mathematics credits. Further, the 30 mathematics credits should be a prerequisite to enrollment in an alternative teacher preparation program. It is clear from the results of this study that even after a semester in a mathematics methods course teachers from non-mathematics backgrounds are not as knowledgeable as teachers who have stronger mathematics backgrounds. Teacher educators must consider that with today's shortage of highly qualified mathematics teachers and current economic climate, more people are attracted to the profession even if they do not have a strong mathematics or science background. The best ways to prepare career-changers in mathematics content need to be further investigated in future research.

In order to make well informed decisions about teacher recruitment and development, more research is necessary on the growing alternative certification segment of the teaching population. Teaching Fellows currently account for one-fourth of all New York mathematics teachers (NYCTF, 2008). Also, Teaching Fellows grew from around one percent of all newly hired teachers from its beginning in 2000 to 33 percent of all new teachers by 2005 (Boyd, Loeb, Lankford, Rockoff, & Wyckoff, 2007). Unless something is done to better prepare teachers with the rigorous content they need, having teachers who had not majored in mathematics and science related areas teach mathematics could be a disservice to our students. For the sake of urban students who have teachers in alternative certification programs, the certification of high quality teachers must continually be a high priority for policy makers, administrators, and teacher educators. Considering the call for high quality teachers, high stakes examinations, and accountability, now more than ever we need to ensure that the teachers we certify are fully prepared in both content and pedagogical knowledge to best teach our highest needs students.

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TEACHERS' LEVELS OF UNITS AND FRACTION DIVISION

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This study extends research on teachers' difficulties with fraction division by examining underlying difficulties they can have coordinating levels of units. Research on students has demonstrated that coordinating 3 levels of units is central to multiplicative reasoning, including reasoning about fractions, but similar attention to levels of units is rare in research on teachers. The present study uses data from a 40-hour professional development course for in-service middle grades teachers to demonstrate that reasoning with 3 levels of units was a central obstacle for some teachers when using drawings to understand fraction division. The results suggest that the capacity to coordinate two 3 levels structures is necessary for teachers to use word problems and drawings to model fraction division effectively for their students.

Introduction

Research on teacher knowledge has expanded from studies of teachers' subject-matter knowledge of various content areas to the organization of teachers' knowledge for teaching particular content to students (e.g., Ball, Lubienski, & Mewborn, 2001; Ball, Thames, & Phelps, 2008; Shulman, 1986). This expansion reflects increasing awareness that teachers need not only content knowledge that many educated adults have, but also knowledge specialized for teaching particular topics to students. Ma (1999) described several examples of mathematical knowledge specialized for teaching, and elaborating such knowledge remains a central challenge for the field.

The present study uses data from a professional development course to examine knowledge necessary for using drawn models to explain measurement division with fractions—knowledge important for supporting reform-oriented teaching practices. Research reviewed below has demonstrated that teachers often struggle to produce coherent conceptual explanations for the product or quotient of two rational numbers. We extend previously reported results by demonstrating that teachers' capacities to coordinate three levels of units is essential if they are to use drawn models to explain measurement division with fractions. Our results shed new light on what teachers must accomplish if they are to explain fraction division using drawings and suggest important topics on which teacher educators and professional developers should focus.

Introduction

Teachers' Understanding of Multiplication and Division with Rational Numbers

Research on preservice and in-service teachers' understanding of fractions has demonstrated that, beyond knowledge of computation procedures, teachers' understandings of multiplication and division with rational numbers is limited. A main result reported across studies is that teachers can confuse situations that call for dividing by a fraction with ones that call for dividing by a whole number or multiplying by a fraction (Ball, 1990; Borko et al., 1992; Ma, 1999; Sowder, Philipp, Armstrong, & Schappelle, 1998). This confusion is often evident when teachers

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generate word problems to illustrate fraction division (e.g., Ball, 1990; Ma, 1999). Teachers can similarly mistake division for multiplication by a decimal number less than one (e.g., Graeber & Tirosh, 1988; Harel & Behr, 1995).

Only a small number of studies have reported directly on teachers' capacities to use drawn models to multiply or divide fractions or decimals. Borko et al. (1992), Eisenhart et al. (1993), and Sowder et al. (1998) have all reported that teachers' difficulties with basic meanings for multiplication and division can be revealed when they attempt to generate and interpret drawn models. The present study examines an additional underlying difficulty—teachers' capacities to coordinate levels of units.

Unit Structures in Multiplicative Reasoning

Research on multiplicative reasoning has emphasized the central role played by units of various types. For the present study, we focus on two main results. First, conceptual analyses of multiplicative reasoning (e.g., Greer, 1992) have emphasized that when numbers are embedded in multiplication or division situations, each number has a different referent unit. To see this, consider the following two word problems:

1. Carrie has run 1/2 mile. If she walks another 1/3 mile, how far has she traveled?
2. John has run 2/3 of a mile on a track. If one lap is 3/4 mile long, how much of one lap has he run?

In the first problem, 1/2, 1/3, and the answer, 5/6, all refer to the same unit—one mile. In the second problem, 2/3 refers to the distance John has run, 3/4 refers to the miles per lap, and the answer, 8/9, refers to the portion of one lap that John has run. In contrast to Problem 1, each number in Problem 2 refers to a different unit. Some teachers in the present study struggled to identify appropriate referent units in fraction division situations.

Second, further studies of multiplicative reasoning (e.g., Behr, Harel, Post, & Lesh, 1992; Steffe, 1994, 2001, 2003) have highlighted the importance of multi-level unit structures. (We explain multi-level unit structures in the next section.) Nearly all of the empirical research on the role of forming and transforming multi-level unit structures in multiplicative reasoning has concentrated on students. Two studies that have examined teachers are those by Behr, Khoury, Harel, Post, and Lesh (1997) and by Izsák (2008). Both studies have reported significant constraints in preservice and in-service teachers' capacities to form and transform multi-level unit structures when reasoning about situations that call for multiplication of fractions. The present study contributes to this line of research by examining in-service teachers' capacities to form and transform multi-level unit structures in situations that call for division of fractions.

Theoretical Framework

Our theoretical perspective emphasizes cognitive operations for forming and transforming multi-level unit structures. In particular, we make direct use of Steffe's distinction between two levels and three levels of units. Steffe (1994, 2001, 2003) has argued that the capacity to form three levels of units is necessary for students to construct multiplicative schemes and schemes for operating with fractions. We apply a similar perspective to the study of teachers. The discussion below demonstrates that the capacity to form and transform 3-level structures is necessary for using drawn models to make sense of fraction division.

The following analysis of the statement $a/b \div c/d$ focuses on the measurement division question: "How many times (or what part of) c/d is a/b?" We begin with the example $2/3 \div 1/6$. To determine the answer using length quantities, one might start by drawing a unit interval subdivided into three equal parts as shown in Figure 1a. This requires attention to just two levels of units, the whole and the thirds. To determine how many 1/6 are in 2/3, one could further subdivide the thirds into sixths as shown in Figure 1b. Answering the division question requires interpreting the bold line segment simultaneously as 1/6 of the whole and as 1/4 of 2/3. Thus, understanding that the answer is 4 requires attending simultaneously to three levels of units—the whole, the 2/3, and the 1/6.

(a) A 2-level structure for 2/3. (b) A 3-level structure for determining how many 1/6 is 2/3

Figure 1. Determining 2/3 ÷ 1/6

Now consider the more complicated example, 2/3 ÷ 3/4. Here, using length quantities to determine the answer requires coordinating two 3-level structures. One can start again by constructing a 2-level structure for 2/3. The new challenge is that fourths do not subdivide thirds evenly. Thus, one needs to anticipate a finer partition that simultaneously subdivides thirds and fourths. One way to accomplish this is to use knowledge of whole-number factor-product combinations: 3 and 4 are factors of 12. Figure 2a shows 2/3 as 8 one-twelfths, and Figure 2b shows 3/4 as 9 one-twelfths. One now has constructed two 3-level structures that allow multiplicative comparisons between 2/3 and 3/4. Answering the question "How many times (or what part of) 3/4 is 2/3?" requires interpreting the one-twelfth segment shown in Figure 2a and Figure 2b simultaneously as 1/8 of 2/3 and as 1/9 of 3/4. This implies that 8/9 of 3/4 is 2/3.

(a) A 3-level structure for 2/3 (b) A 3-level structure for 3/4

Figure 2. Determining 2/3 ÷ 3/4

Methodology

The Does it Work? Professional Development Course

The present study is part of an ongoing NSF-funded research project called *Does it Work?: Building Methods for Understanding Effects of Professional Development* (DiW). The data came from a 40-hour professional development course on fractions, decimals, and proportions offered in Fall 2008. The participants came from an urban district in the Southeast. Thirteen were in-service mathematics teachers and one came from the district's central office. Of the teachers, one taught in Grade 5, and the others taught in Grade 6 or Grade 7.

We designed the course to prepare teachers for new curriculum standards in their state. Similar to standards developed by the National Council of Teachers of Mathematics (2000), the state standards address content areas, such as number and operations, and processes of mathematical thinking, such as representing and solving problems. The new state standards require teachers not only to compute efficiently and accurately with fractions, decimals, and proportions, but also to reason about fractions, decimals, and proportions embedded in problem situations. We recruited teachers by offering a small stipend and by explaining that the professional development would help prepare them for the new state standards.

A member of the DiW research team facilitated the course, which met for 3 hours one night per week for 14 weeks. Each class focused on a set of tasks that addressed a particular topic. Topics included multiplication and division with fractions, multiplication and division with decimals, and proportional reasoning. Many of the problems created opportunities to reason with

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

multi-level unit structures by requiring teachers to generate and interpret drawn models for multiplication and division of fractions and decimals. Teachers constructed drawn models using either paper and pencil or software called Fraction Bars (Orrill, n.d.) that allowed them to draw and partition rectangles accurately. The instructor moved back and forth between whole-class discussions and group work. The whole-class discussions were lively, and the teachers directed many of their comments and questions toward each other, not the instructor. At the end of most classes, the teachers completed reflections that consisted of short written tasks related to the focal topic for that particular class.

Data and Analysis

We videotaped all class sessions using two cameras. One researcher used the first camera to record the entire class, and a second researcher used the second camera to record written work on the whiteboard during whole-class discussion and on teachers' papers and computer screens during group work. We also administered a multiple-choice pre-test and post-test that we developed (Izsák, Orrill, Cohen, & Brown, 2010) by modifying (with permission) the middle grades Learning Mathematics for Teaching measure of Mathematical Knowledge for Teaching (Hill, 2007). To gather information on how teachers reasoned on the pre-test and post-test, we conducted videotaped interviews with seven. We selected these teachers to represent a range of performance on the pre-test from high to low. (When collecting classroom data, we prioritized these seven teachers.) Finally, we collected all of the teachers' written work.

We began our analysis by examining teachers' answer choices on the pre-test and post-test. We hoped to see clear gains in teachers' performance, but about half the teachers had low scores on the pre-test and post-test, about one third of the teachers had high scores on the pre-test and post-test, and the remaining teachers were in the middle. On closer inspection, we found cases where teachers' answers to particular items moved from incorrect on the pre-test to correct on the post-test, but we also found cases where teachers' answers moved from correct on the pre-test to incorrect on the post-test and cases where teachers' answers moved from one incorrect answer to a different incorrect answer. To understand these results, we turned next to the pre-test and post-test interviews conducted with seven teachers. These interviews revealed aspects of growth and change to which the multiple-choice tests were insensitive—for instance, some teachers appeared more confident in their reasoning on the post-test than on the pre-test—but they also confirmed a high degree of stability from pre-test to post-test for some teachers. This stability was particularly evident on fraction division items. Surprisingly, some of these stable teachers were among the most active participants during the course.

We were puzzled that we did not see more definitive evidence of teacher learning from pre-test to post-test, especially since some teachers gave lucid, correct explanations for measurement division with fractions during whole-class discussions and because fraction division was treated twice, once in the first half and once in the second half of the course. This led us to analyze the video-recordings of the class sessions. Our initial pass through these data, revealed that teachers often struggled with meanings for division, referent units for numbers, and strategies for partitioning lengths and areas in drawn models. We then retraced all of the class sessions a second time to create a more detailed chronological list of problems on which the teachers worked, the strategies they used to solve those problems, and comments and questions they had about their own solutions and solutions generated by other teachers. We looked specifically for data on teachers' meanings for division, referent units for numbers, and strategies for partitioning lengths and areas in drawn models. It was here we noticed that some teachers appeared to reason with just two levels of units, while others appeared to reason with three levels of units. We took one final pass through our data to develop cases for each of the seven interview teachers that focused on their capacities to reason with multi-level unit structures.

Results

Our main result is that some teachers appeared to coordinate only two levels of units, while others coordinated three levels of units, and this distinction played a central role in most teachers' stable performance from pre-test to post-test. In particular, teachers who coordinated only two levels of units had difficulty understanding their colleagues' lucid explanations for measurement division that made use of three levels of units (similar to the explanation summarized in Figure 2). Other teachers could form three levels of units but were still learning to coordinate more than one 3-level structure. Finally, one teacher began the professional development course able to form 3-level structures and to explain measurement division for fractions correctly during the pre-test interview. We illustrate the central role played by multi-level unit structures with cases of two teachers who were among the seven we interviewed and who made regular contributions to whole-class discussions. These teachers were among those for whom we had the largest amount of data. We keep each case brief but emphasize that for both teachers we examined all of our data before making claims about whether they coordinated two levels or three levels of units.

Carrie Reasoned with Two Levels of Units

Initial evidence that Carrie reasoned with two levels of units came from her solution to the Candy Bar problem: *Share two candy bars equally among five people. How much of one candy bar does one person get?* In whole-class discussion, Carrie disagreed with teachers who explained that the answer was 2/5. She used a diagram to argue that getting 1/5 of each bar was the same as getting 2/10 of the two bars; therefore, the answer should be 1/5. Her explanation suggested that she conceived of each bar separately divided into 5 parts (one 2-level structure) and two bars together divided into 10 parts (a second 2-level structure), but not a 3-level structure that coordinated the entire amount of candy (one level), the two separate candy bars inside the total amount (a second level), and the fifths inside each candy bar (a third level).

When the class began to work on fraction division, Carrie reported that she just computed answers to fraction division problems and that she did not have a particular division question that she asked herself. As the class proceeded, however, she did begin to ask the measurement division question. On one reflection that asked teachers to write a story problem for 5 ÷ 1/3, Carrie responded with the following: If I have 5 inches of ribbon, and it takes 1/3 of an inch for a ring, how many rings can I make? Although Carrie was developing meaning for dividing by a fraction, she struggled to coordinate three levels of units one week later. The facilitator had observed Carrie struggle to draw appropriate models for fraction division during the class and so worked with her again at the end of class. They used the problem 2/3 ÷ 1/4. Carrie interpreted the division statement appropriately as asking how many fourths are in 2/3, but she could not use drawings to answer this question correctly. Looking at a diagram that showed a whole divided vertically into thirds and then into 12ths (Figure 3a), Carrie stated that each of the smallest squares were fourths of the whole, when in fact they were fourths of 1/3. She then concluded that there were 8 fourths in 2/3. When the facilitator pointed out that three of the smallest squares made one fourth of the whole and showed Carrie what one fourth in one third looked like (Figure 3b), Carrie interpreted the diagram as showing "three fourths in two thirds." Similar to her interpretation of the Candy Bar problem, Carrie appeared able to think of a whole divided into three parts (one 2-level structure) or a third divided into fourths (a second 2-level structure), but not a 3-level structure that coordinated 12ths, 3rds, and the whole. Thus, articulating the measurement division question is necessary but not sufficient for use drawings.

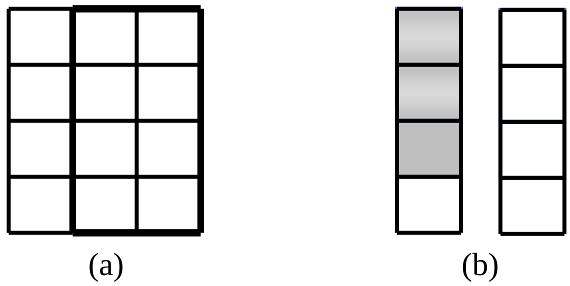

Figure 3. (a) Twelfths, two thirds, and the whole. (b) One fourth in one third
(Reproductions based on screen shots of Carrie's original work)

Donna Reasoned with Three Levels of Units But Still Struggled with Division

Unlike Carrie, Donna began the professional development course with an explicit meaning for division. She stated that division meant repeated subtraction during her pre-test interview, several times during the professional development course, and again in the post-test interview. Donna was able to use her repeated subtraction interpretation to explain problems like 2 ÷ 1/4 for which the answer is a whole number, but she had difficulty explaining problems like 2/3 ÷ 3/4 for which the answer contained a fraction. Donna could construct three levels of units, but she struggled to coordinate two 3-level structures.

The first evidence that Donna could construct three levels of units came on the same Candy Bar problem: *Share two candy bars equally among five people. How much of one candy bar does one person get?* In contrast to Carrie, Donna explained to her fellow teachers that taking 1/5 from each bar was the same as taking 2/5 from one bar. By equating one fifth of total amount of candy with two fifths of one bar, Donna appeared to attend simultaneously to the total amount of candy (one level), the two separate candy bars inside the entire amount (a second level), and the fifths inside each candy bar (a third level).

Further evidence that Donna could construct three levels of units came two weeks after the discussion of the Candy Bar problem. The teachers were working on a task that asked them to examine similarities, differences, patterns in the following set of division problems: 2 ÷ 3, 2 ÷ 1/4, 2 ÷ 3/4, and 2/3 ÷ 3/4. After teachers had worked in pairs, the facilitator convened a whole-class discussion around 2/3 ÷ 3/4. Keith was the first to present, and his solution used a number line similar to Figure 2b. He explained that you cannot "get 3/4 into 2/3" but, if you divide the 3/4 into nine parts, you can get 8/9 of 3/4 into 2/3. Carrie responded, "I am so lost, it is unreal," but Donna seemed to understand Keith when she connected his solution to her thinking "How many times can I subtract 3/4 from 2/3? Not quite one time, 8/9 of one time." Donna then demonstrated to the class how she used two rectangles to approach the problem. She began by drawing two rectangles, dividing the top rectangle into thirds, and dividing the bottom rectangle into fourths. After Donna subdivided the first fourth into three parts, the facilitator interrupted and asked her to explain her thinking. Donna explained, "What I am looking for is a way to name this point right here (see Figure 4). So I realized that right then I had to split these pieces (the fourths) into smaller pieces." Donna's explanation demonstrates the intention to construct three levels of units using the whole rectangle (one level) and the fourths (a second level). As Donna continued, however, she confessed that she did not know how to construct the third level for this problem until she saw another teacher use 12ths.

Figure 4. Donna intends to name 2/3 in terms of fourths

Discussion

The present study contributes to research on teachers' knowledge of fraction division by demonstrating that the ability to form multi-level unit structures can play a central role in their capacity to use drawn representations. The case of Carrie demonstrates that teachers who only construct two levels of units can find it very difficult to understand lucid explanations for fraction division, let alone construct their own solutions. In particular, Carrie could not understand explanations offered by the facilitator and by Keith. This limited opportunities for her performance to change from the pre-test to the post-test. At the same time, the case of Donna demonstrates that the capacity to form 3-level structures is necessary but not sufficient for using drawn representations to explain fraction division. Teachers need to be able to coordinate two 3-level structures, and teachers need to understand division in terms of multiplicative comparisons (see Figure 2). Finally, our results suggest that teacher educators and professional developers should give explicit attention to reasoning with three levels of units so that teachers can be confident using drawn models for fraction division with their students.

Endnotes

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TEACHERS' PRODUCTIONS OF ALGEBRAIC GENERALIZATIONS AND JUSTIFICATION: IMAGES OF SPECIALIZED CONTENT KNOWLEDGE

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This study documents teachers' productions in professional development (PD) of algebraic generalizations and justification that draws on specialized content knowledge (SCK) for teaching. Images of the entailments of teachers' productions that draw on SCK are essential to advance the field's understanding of teacher knowledge. *Twenty teachers' written and verbal productions of two tasks were analyzed. Results show that teachers' productions elaborated on key mathematical work such as correspondences across representations, construction and connection of multiple solutions, and productive interplay of empirical and generic examples that contributed to developing SCK. Implications of this research are ways of working mathematically in PD to advance teachers' SCK.*

Introduction

In order to create classroom experiences where students reason and justify mathematics, teachers need to understand mathematical proof[1] and know how to apply it in practice (Stylianides & Stylianides, 2006). Research on teachers' knowledge uncovers that it is the form of proof, rather then the nature of proof and student learning that teachers notice (Knuth, 2002). Similarly, research on teachers' knowledge of algebraic reasoning shows a number of weaknesses in understanding (Kieran, 2007). Yet, teachers' knowledge of mathematics and their instruction impacts students' learning (Hill, Rowan & Ball, 2005). In order to advance teachers' knowledge of justification and its role in algebraic reasoning teachers need opportunities to construct justifications (Elliott, Lesseig, & Kazemi, 2009; Lo, Grant & Flowers, 2008).

Research on mathematical knowledge for teaching (MKT) suggests that the aim of professional development (PD) programs should advance teachers' specialized content knowledge (SCK) of mathematics that is unique to the work of teaching (Elliott et al., 2009; Suzuka et al., 2009). Although this seems like a logical argument -- teachers should learn the mathematics unique to their work -- researchers have documented competing purposes of mathematics PD where teachers' learning migrates to topics other then mathematics (Wilson, 2003). Confounding the finding that teachers' mathematical learning is often negotiated away, is the fact that there is not an agreed upon knowledge base of the mathematics teachers need to know (Ball, Sleep, Boerst, & Bass, 2009) nor portraits of teachers' deep understanding of algebraic reasoning (Doerr, 2004). Such images could serve professional educators learning how to facilitate PD oriented to SCK and researchers who are trying to build a knowledge base on teacher learning.

This report offers images of teachers' productions of algebraic generalizations and justification as they invoke and develop SCK and suggests ways of working mathematically in PD. In doing so, we address two research questions:

How do teachers' justifications of generalizations orient them to SCK?

What ways can mathematics PD take up teachers' productions with the purpose of supporting SCK?

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This study examines the mathematical productions of teachers who had a long-term commitment to work on mathematics with one another. Analysis of their mathematical work provided insights on critical elements of justification that orients teachers to SCK in PD.

Theoretical Considerations

This study draws from three research areas, (i) justification and proof, (ii) justification for algebraic generalization, and (iii) mathematical knowledge for teaching.

Research on teachers' knowledge of justification and proof has provided insights on their limited understanding of the construct and the need to for enhanced teacher learning opportunities (Knuth, 2002; Martin & Harel, 1989). Teachers' work asks them to: (i) *explain why* something is correct by revealing mathematical concepts underlying a proof, (ii) argue and *communicate* with others within a community (i.e., classroom), (iii) *learn* and support the learning of new mathematical ideas, and (iv) establish truths based on a *deductive system*, their work demands that they understand proof beyond verification (Knuth, 2002; Simon & Blume, 1996).

Although teachers may need this knowledge for their work, further investigation of research on teachers' *use* of justification suggests limited teacher knowledge of the construct in professional education (Lo, et al., 2008; Simon & Blume, 1996). Most research examines preservice and practicing teachers' conceptions of proof or proof schemes given various forms of proofs and non-proofs (Knuth, 2002; Martin & Harel, 1989; Stylianides, Stylianides & Philippou, 2007). There remains a need to understand teachers' productions of justification because these are intellectual resources and practices that teachers draw on when teaching.

Research investigating teachers' use of justification in their work associated within algebraic reasoning is very limited in scope. What research does exist documents a number of deficiencies in teacher understanding of functions, representational use, and the role of generalization (Doerr, 2004; Kieran, 2007). Progress has been made to understand aspects of teacher knowledge in algebraic reasoning, however the field "does not yet have a rich set of portrayals of expertise…" of teachers from a range of settings (Doerr, 2004, p. 285). There remains a need for further teacher development and research on teachers' algebraic reasoning productions.

From a discipline perspective, Carraher and Schlieman's (2007) review of research on algebraic reasoning highlights mathematical representations and the role of generalization. Mathematical representations included pictures, words tables, graphs, and symbols to represent algebraic reasoning. Similarly, Kaput's (2008) research positions generalization as a key strand of algebra. Studies investigating teachers' knowledge in these areas, although mostly focused on teacher beliefs, have found teachers' orientations to algebra inadequate to support the teaching of mathematics called for from research (Kieran, 2007).

Drilling down further into algebraic reasoning, we see a serious lack of research on teachers' understanding of justification for algebraic generalization. Fortunately research on students' understandings in this area provides insights on the import of considering the coordination of justification and generalization and the examination of varying levels of justification (e.g., appealing to authority, empirical arguments, deductive justification) to support generalizations (Ellis, 2007; Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009; Lannin, 2005). In particular, Lannin's work (2005) highlights that justifications provide a view of learners' use and coordination of underlying mathematical relationships, representations, and structure to construct a generalization. By examining the level of justification used to generalize researchers are able to consider learners' reasoning – how the range of evidence available from empirical to deductive is used. It is through the interplay of empirical and deductive justification students gain greater awareness of the limitations of previous levels as new understandings become taken-as-shared, easing the transition to deductive justification (Ellis, 2007; Healy & Hoyles, 2000; Knuth & Elliott, 1998). Research on students' learning has identified the powerful connection between justification and generalization for student learning and the role empirical arguments play in development. We advance justification is an essential element for generalizing especially for teachers who need to understand the underlying mathematical relationships, representations, and structures. Further, they must be able to distinguish empirical from deductive arguments to advance student learning. This study considers how teachers justify their generalizations and the levels of justification utilized in productions constructed during PD.

In previous work researchers have argued that the purpose for doing mathematics in PD is to advance teachers' SCK (Elliott et al., 2009; Kazemi, Elliott, Mumme, Carroll, Lesseig, & Kelley-Petersen, in press; Suzuka et al., 2009). Because SCK is a fairly new idea in the field, researchers and professional educators need images of particular mathematical content oriented to SCK. Ball and colleagues' work on SCK (Ball, Thames, & Phelps, 2008) has helped us theorize what might be entailed in SCK oriented teacher productions for justification of algebraic generalizations. Teachers' work demands that they not only construct justifications to verify solutions, similar to the work of other professionals who use mathematics, call common content knowledge, it requires a repertoire of ways to generalize coordinating ideas across representations, understanding the interplay between empirical and deductive arguments, and knowing what are key connections across generalizations. This is mathematical work that draws on knowledge unique to the work of teaching, SCK. Our current research allows us to examine images of SCK oriented mathematical practice, justification, and specific content, algebraic generalizations.

Project Background

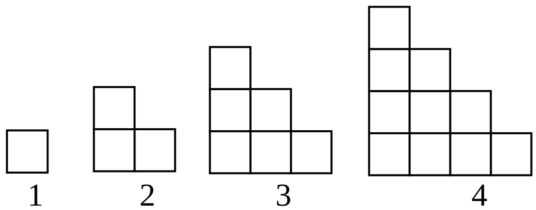
Participants for the study (n = 20) were volunteers participating in *Researching Mathematics Leader Learning (RMLL) seminars* drawn from a pre-existing three-year *Mathematics Initiative* working on mathematics and developing leadership skills. RMLL prepared leaders to facilitate mathematics PD[3] and took place during the second year of the *Initiative*. Teachers in the seminars were becoming, or already were, leaders in their schools, districts, and regions charged with facilitating mathematics PD. All of the participants were current or past full-time classroom K-12 teachers (five high school, seven middle school, eight elementary). Teachers in the study were forthright in suggesting that they had developed particular ways of doing mathematics prior to attending RMLL seminars[4] and saw the mathematical work in RMLL similar to the ways of engaging in the *Initiative*. Teachers brought with them particular conceptual tools – mathematical knowledge and ways of engaging in mathematical work – that were cultivated over the two summers of the *Initiative.*

Data for the study were participants' mathematical productions on two tasks (Figure 1) focused on algebraic reasoning with an emphasis on constructing generalized solutions. During seminars, teachers shared solutions with one another and reasoned to develop collective understandings discussed in whole group. During small and whole group discussions, teachers were asked why a solution worked (justification based on mathematical argument) and how solutions work in all cases (generalization). Teachers commented that they knew they "needed" a justification for their generalization especially when trying to explain solutions to one another (see Elliott et al., 2009b).

The *Staircase* and *Candles* tasks were completed by teachers as homework to prepare for small and whole group discussion in RMLL seminars. Nineteen *Staircase* teacher productions (one teacher did not have a written solution) and 20 *Candles* teacher productions were analyzed.

Tasks One – Staircase

Adapted from Shell Center Materials



1. Build the next two staircases.
2. Describe the 10^{th} staircase.
3. What is the relationship between the staircase number and the number of cubes it takes to build the staircase? See if you can find a rule to represent that relationship.

Task Two -- Candles Task

This task is excerpted from Driscoll (1999).

1. Maria lights two 18 cm long candles at the same time. One candle takes six hours to burn out, and the second takes three. After one hour of burning, what do the candles look like? How much time will pass until one of the candles is exactly twice as long as the other? Explain your answer.
2. Maria lights two 36 cm long candles at the same time. One candle takes three hours to burn out and the other takes six. How much time will pass until the slower-burning candle is exactly twice as long as the faster-burning one? Explain your answer.
3. Maria lights two candles of equal length at the same time. One candle takes six hours to burn out and the other takes nine. How much time will pass until the slower-burning candle is exactly twice as long as the faster-burning one? Explain your answer

Methods

Data used in this study were fieldnotes based on video and audio-taped sessions of small and whole group mathematical discussions including pictures of teachers' work. In addition, teacher journals with recorded mathematical work were analyzed. The first two authors conducted a series of analyses together and separately to come to agreement on the nature of productions. Analysis inventoried teachers' written productions focusing on what representations were used in solutions, brief notes on the ways representations were used, and the number of different models or solution strategies completed. The inventory process captured the broad spectrum of approaches teachers constructed on the two tasks. Fieldnotes and transcripts of video were reviewed, writing memos about teachers' productions shared in small and whole group discussion. Because two cameras were used, two sets of four to five teachers' productions were captured in more detail during small group work. Here analysis focused on the nature of solutions constructed with attention given to: (i) the types of representations used, (ii) the correspondences of ideas across representations, and (iii) the number of solutions constructed. At the same time teachers' journals were examined more closely for productions and to follow the progression of how a production unfolded as teachers' verbalized their reasoning. Finally, we examined the nature of teachers' justification provided in verbal and written form considering the level of justification utilized. Findings are based on the patterns identified in the teachers' written work and discussions of mathematical solutions.

Results & Discussion

Three findings were central in teachers' productions. Because of limited space, data are summarized for our findings. Our presentation will share additional data to support patterns and claims.

Use Multiple Representations To Examine Relationships & Confirm Generalizations Beyond Reproach

One of the most striking themes that emerged across twenty teachers' productions from each task was the use of multiple representations to build justifications for generalizations (recursive and explicit in symbols and words). Most teachers were individually able to construct generalizations, *Staircase* (n = 18), *Candles* (n = 15). Yet, teachers' productions did not conclude as soon as a generalization was found. Teachers used a number of representations, orally and in writing, to either confirm a generalization or help build it. For example, many productions of *Staircase* used a rectangle model, with many constructing more then one visual model, relating it to the expression n(n+1)/2 and other equivalent expressions. Examining correspondences within each task, teachers considered mathematical relationships in the tasks to reason and justify a generalization (either recursive or explicit) as Lannin (2005) suggests is necessary for a deductive argument. Heeding the calls of mathematics reform, teachers showed that they could construct and connect representations (NCTM, 2000). However, teachers went beyond what students might do, teachers' productions, as shown in Table 1, used numerous representations in their individual productions. [The top row shows the number of teachers using the various types of representations.] Although the table does not provide the nature of the representations used and connections, it is notable that a majority of teachers used at least three representations in their solutions and each was means to further their production.

Table 1. Teachers' Representational Use for Staircase & Candles

Staircase	2 T	3 T	5 T	1 T	5 T	2 T	1 T	**Candles**	3 T	1 T	6 T	1 T	2 T	5 T	1 T	1 T
Table	x	x	x			x		Table	x	x	x		x	x	x	
Visual	x	x	x	x	x	x	x	Visual	x	x	x	x	x			x
Expression	x	x	x	x	x			Equation	x		x	x		x		
Words	x			x		x	x	Graph	x	x		x				
Numeric	x	x														

We came to understand teachers' use of multiple representations as a means of exploring mathematical relationships in order to construct justifications of generalizations beyond mathematical reproach in the community. In teachers' verbal productions for *Staircase* and *Candles* there was clear intent to support any symbolic generalization with connections to other representations. Although, teachers knew that for a solution to be a generalization it had to always work, they were not satisfied until they had constructed a justification that made sense to themselves and colleagues examining correspondences across representations. It was through the skilled and strategic use of and correspondences among representations that teachers drew on and develop mathematical knowledge.

Construct Multiple Solutions to Uncover Mathematical Characteristics & Build Justification that Explain Why

Striking in teachers' productions was the volume of solution methods constructed with more then half of the teachers' journals containing two or more unique productions and oral productions sharing numerous methods. Teachers' productions were not "serial sharing" one after another, but allowed teachers to examine relationships among solutions, attend to the nature of change in tasks, and ultimately develop justifications that explained why. Productions of the *Staircase* task illustrated different interpretations of the quadratic pattern manifesting in unique, but equivalent expressions. Discussions centered on how change occurred across cases in the pattern relating to different definitions of variables symbolical represented. Similarly, the *Candles* task illustrated multiple pathways (using tables, pictures, equations) to reason about indirect variation. Here discussions focused on considerations of how solutions were developed, why mathematically they worked, and how solutions provide insights on one another to construct more robust justification.

We have come to understand that the reason for sharing teachers' productions in PD is not *just* to collect unique methods to a task (e.g., to anticipate what students might do with it), but for teachers' productions to uncover mathematically relevant characteristics of tasks by making explicit connections across solutions. Although most teachers were able to construct solutions to the task, and did so quite efficiently, the heart of their mathematical work in PD was to link methods across solutions in ways that opened up understandings and lead to justifications that explained why a generalization was valid. As a result, teachers drew on and developed ways of understanding mathematics that went beyond solving a task, but that positioned them to know and articulate mathematically why ideas were valid.

Elaborate Stumbles With Reasoning to Examine Structures and Validate

In our data there were points in both small and whole group that teachers stumbled in their productions when elaborating challenging generalizations. During these elaborations, teachers returned to empirical cases (a specific *Staircase* or concrete numbers in *Candles*) to reason and elaborate on the nature of the generalization. Examining specific cases allowed teachers to consider underlying key mathematical structures of tasks and ultimately move to deductive arguments that they suggested were essential. For example, it was in searching for the underlying pattern in a unique visual model with a specific case in *Staircase*, and pressing on the relationship of variables and structure of the pattern, that teachers were able to produce a deductive justification using a generic visual model for a generalization. Similarly, in elaborating why a generalization was valid for the indirect relationship in *Candles* teachers worked collaboratively to construct a deductive argument and understand the underlying structure of the task represented in visual models, tables and the more typical system of equations.

We claim that the dance from specific to generic arguments taken up when faced with conceptual stumbles allowed teachers to investigate key mathematical ideas in tasks and verify generalizations. Although researchers suggest that teachers' reliance on empirical justification is evidence of weak understanding (Martin & Harel, 1989; Simon & Bloom, 1996), this dance was productive for teachers to elaborate the mathematical underpinnings of tasks. Furthermore, the less mature (empirical) forms of justification supported the deductive reasoning teachers suggested was needed to validate a generalization. Teachers' work with justification pressed them to distinguish and develop ways of validating justifications dependent on understanding underlying mathematical structures.

Conclusions

Teachers' elaborations of justification, while doing mathematics in PD, have helped us better understand the role that justification and algebraic generalization can play in teacher learning. Because justifications for generalizations needed to convince beyond reproach, explain, and verify, teachers' productions opened up conversations about underlying relationships characteristics, and structures. Teachers developed keen skills of identifying correspondences across multiple representations, exploring connections across solutions, and using and distinguishing levels of justification that serve to explain, communicate, and verify new understanding. Teachers' productions of justifications for generalizations were both useful for the work of teaching and embodied the enhanced notions of justification called for by researchers (Knuth, 2002). This research illustrates how teachers' productions of justification for algebraic generalization support the development of specialized content knowledge (SCK) needed in teaching (Ball et al., 2008). Our work contributes to the limited body of work on specific forms of teachers' SCK. Further research is needed on teachers' justification productions across other groups and tasks to explore the range of potential benefits and drawbacks for developing SCK in PD.

Endnotes

1. The term justification is used to note verbal and written mathematical arguments to verify a solution. When referencing others' work their term is used.
2. Project funded by granting agent. Opinions expressed are the authors and do not necessarily reflect the views of the granting agent.
3. Seminar descriptions are in other publications, see author.
4. Teachers' participation was examined to better understand justification in PD. No causal claims are made about the impact of leader seminars.

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TEACHERS' VISUALIZING, COMMUNICATING AND REASONING ABOUT BASIC THREE-DIMENSIONAL CONCEPTS

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This study used video data to explore how mathematics teachers conceptualize the partitioning of regions by axes in a three-dimensional geometric space. In collaborative settings, teachers approached the problem using both combinatorial and graphical methods, the later being laden with gestures. For some teachers difficulties arose in the transition from two-dimensional to three-dimensional space. For others interpreting three-dimensional models was difficult. While all groups solved the problem posed, many failed to verbalize connections between the multiple representations introduced during their discussions.

Introduction

Teachers need a robust, unified, flexible understanding of fundamental mathematical concepts (Ma, 1999; Ball, Thames, and Phelps 2008) that allows them to "mediate students' ideas, make choices about representations of content, modify curriculum materials, and the like" (Ball and Bass 2000, p. 97). One of the most fundamental mathematical ideas is that of graphing. Graphing usually involves the use of the *xy*-plane and ordered pairs to locate points in two dimensions. The National Council of Teachers of Mathematics (2000, 2006) has advocated the use of graphing starting as early as the upper elementary grades to study patterns and extending through the later high school years to "investigate conjectures and solve problems involving two- and three-dimensional objects represented with Cartesian coordinates" (NCTM, 2000, p. 308).

Despite students' experience with number lines when learning arithmetic, the use of the Cartesian coordinate system is so integrated in the culture of mathematics that it is often introduced as a single, static two-dimensional tool with a particular orientation rather than a blending of two orthogonal one-dimensional spaces. Instead of building on students' foundational knowledge and making transparent the longitudinal coherence of mathematics, teachers place emphasis on conventions, such as the use of Roman numerals and counterclockwise numbering, rather than on actual fundamental concepts. While this is not misleading, it does not convey a deep understanding of the use of two perpendicular number lines intersecting at their origins to form four distinct regions called quadrants.

Purpose of the Study

Teachers routinely share the idea of the four quadrants, but do they understand what these regions, named quadrants, represent or how they are formed? When asked to determine the number of regions formed by the *x*-, *y*- and *z*-axes in three-space, what types of responses and problem solving techniques are elicited from teachers? What does the teachers' reasoning about the regions formed in three-dimensional space reveal about how they conceptualize them? Do teachers realize how the two-dimensional example is related and connected to its three-dimensional counterpart?

The purpose of this work is to take a first look at how teachers conceptualize the partitioning of regions by axes in a geometric space. In particular, in what ways do they respond to the question: *If the axes of the xy-plane divide it into four regions (called quadrants), how many regions do the axes of the xyz-space divide it into?*

Mode of Inquiry

Video and observation data were gathered from groups of preservice and inservice teachers over three years in two different settings. In the first setting, four groups of four to five members worked collaboratively on the question above as part of a mathematics education graduate course at a research university in the northeastern United States. These groups were comprised of a mixture of inservice and preservice teachers. The inservice teachers were all certified, had less than three years teaching experience, and held undergraduate degrees in either Mathematics or Mathematics Education. The preservice teachers all had either a Bachelor's degree in Mathematics or at least 30 credits in undergraduate mathematics courses. In the second setting two college mathematics instructors (each with over three years of experience), seven secondary mathematics teachers (two with 20 years experience and the rest with three years or less), and four preservice teachers worked collaboratively in three groups on the same question. These participants were attendees of a spatial reasoning workshop at an annual state mathematics teacher conference.

In the tradition of multimodal analysis (Jewitt, 2008) and its inclusion of gesturing, tool use, tone of voice, facial expression, and direction of gaze, the research team used the data from the teachers' collaborative problem solving of the question in both settings to capture the essence of the episodes. The video data were analyzed by all members of the research team who discussed their findings until consensus was reached. During the analysis, the data from each group was first observed solving the problem in its entirety, then the problem solving episode was repeatedly watched, often in slow motion or frame-to-frame to determine the exact gestures used by each group member. Patterns began to emerge rather quickly in the teachers' responses and how they engaged with the mathematics problem presented.

A coding schema was employed to unravel the interplay of content and multimodal discourse by classifying gesture and tool use as well as teacher interaction. Coding details are left for future publication. This paper will instead focus on the more general framework that emerged to describe the tendencies of the teachers as they engaged in the mathematical problem that was posed. It will report on how the findings help form a theoretical perspective adopted by the research team related to the domains of mathematical engagement which emerged as they considered teachers' understanding of axes and regions in three dimensions.

Video Episode: Teachers' Collaborative Problem Solving

At some point in the collaborative problem solving activity, all of the groups used gestures and tools to support or supplement the following: their communication, their individual and collaborative reasoning, and their visualization. A more detailed description regarding each group will now be related. Information that was not pertinent to the problem solving episode is omitted.

First Setting: Collaborative Groups in a Mathematics Education Graduate Course

Group 1 consisted of five members. When the question was posed, one of the inservice teachers, Sam, immediately answered the problem. His initial justification was that there had to be eight regions since "two times two times two is eight." Sam received verbal confirmation from only one group member, Barb, who began gesturing as soon as the problem was posed and completed the gesture before responding to Sam. He then immediately launched into a second explanation that relied on counting the ordered triples that were possible. The other three group members showed more response to this explanation; however, confusion arose when one of them discovered she had missed one of the possible permutations. At this point, Sam offered a third explanation that involved extensive gesturing of a three-dimensional box to show how the box's edges could extend to represent three intersecting planes. While Sam was a predominant influence in the discussion, others did contribute and used gestures to consider his ideas. Even though Sam seemed to understand the concept, he failed to verbalize connections between the three different explanations he offered for the problem. While two of his group mates seemed to grasp the visualization approach, two others clung to the ordered triples.

Group 2 also consisted of five members. When the question was posed, the group collaboratively explored a more visual approach. Two inservice teacher members of the group, Matt and Fred, collaborated by using sheets of paper to create a figure that representing the three intersecting planes. Matt understood that there were eight regions and was attempting to explain this to the others. Fred, however, admitted that he was having a difficult time visualizing the situation even with the figure. Two of the other members of the group, inservice teacher Jane and preservice teacher Wendy, studied the figure that Matt and Fred created while joining in the discussion. A fifth member of the group, preservice teacher Ann, also joined in the discussion, but she used two sheets of paper as planes and was focused on her figure rather than Matt and Fred's figure when discussing the problem. While Wendy and Ann seemed comfortable with Matt's eight regions answer, Jane and Fred were still struggling to come to grips with the problem. Even though all the members believed there were eight regions, they were not sure how to write up a justification for this (as was required for the assignment). At that point, Matt then proposed that they consider ordered triples. All group members seemed satisfied with this development although Matt never made an explicit connection between the ordered triples and the more visual exploration of the problem that had been discussed originally. During their discussion the issue was raised, as to whether these regions should be called quadrants or not.

Group 3 consisted of four members. When the question was posed, inservice teacher Lisa, immediately offered that there were eight regions. This received verbal affirmation from one group mate, but the other two offered only confused expressions. Lisa then used a sheet of paper as a plane (holding it vertically) and moved the ruler to represent a line horizontally intersecting the plane. She indicated that four regions would fall on the left hand side of the plane and four regions would fall on the right hand side of the plane, hence there should be eight regions in total. Her group mates agreed with this demonstration. This group spent the least amount of time (just under two minutes) on the problem that was posed.

Group 4 also consisted of four members, three preservice teachers and one inservice teacher, Sonya. Once the problem was posed, Sonya asked Pete to physically represent the situation. He modeled the *xy*-plane with both the *x*- and *y*-axes as well as an intersecting orthogonal *z*-axis. While staring at Pete's figure, Sonya stated there should be four "big" regions. Jen used Pete's model to indicate that there would be eight regions, four where the *z*-coordinate was negative and four where it was positive. Rather than using Pete's figure, Emily used a drawing of the three axes and tried to explain this to Sonya. Sonya, however, referred back to Pete's model later indicating she could not use a drawing to visualize the situation. Sonya continued to discuss the problem referring to Pete's model. Jen spoke up after a while and offered the suggestion that the regions could be thought of in terms of combinations where the *x*, *y*, and *z*-values were either negative or positive. The others agreed and proceeded to write all ordered triples. The group also discussed their unease having eight “quadrants” deciding instead to use the word “regions” even though at one point Jen tentatively formed the word “octants.” Sonya revisited Pete’s model to ensure that all regions were listed, pointing out where each coordinate value was negative or positive. At this point Emily made reference to her drawing again. Emily later mentioned her reluctance to use Pete’s figure stemmed from how the others were labeling the axes contrary to the manner to which she was accustomed. This group spent the longest time on the problem, about 11 minutes.

Second Setting: Collaborative Groups at Mathematics Teachers Conference

Group A consisted of four members. In this group the college instructor, Bob, lead the discussion. Bob immediately solved the problem and then discussed it with the other three members of the group. During this discussion none of the group members made use of gesture. While two of the members indicated that they followed his explanation, the third, preservice teacher Cindy, did not. Bob reworded his explanation using hand gestures to indicate the eight regions that would be formed. Cindy still did not follow and starting using index cards as planes to investigate the situation on her own, eventually asking one of the workshop presenters to confirm her results.

Group B consisted of three inservice teachers and one preservice teacher, Ned. Two of the inservice teachers were sisters, Sally and Sarah. The group did not have a leader but worked collaboratively to solve the problem. Mary (the third inservice teacher) stated there would be six regions; Sally and Sarah agreed. Ned spent this time gesturing to himself and later vocalized that he thought the answer would be eight. The group members individually speculated whether the answer would be six or eight each using different gestures to represent the intersecting planes. They came to the common consensus that the answer was eight only after collaboratively building a model using paper to represent three intersecting planes. At that point Sally pondered, “I wonder why we came up with six?” Mary explained that she was “seeing it flat” using gestures to explain that her mental image was a flat, two-dimensional drawing of the three axes as is typically shown in textbooks (see Figure 1). In their attempt to read the two-dimensional representation in their minds, they confused the properties of this “flat” drawing to the object itself. Sally responded, “Your brain knows its three-space, but your eyes see it in two-space.”

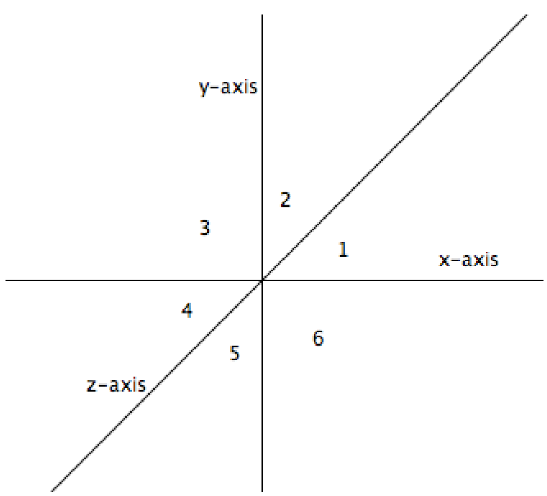

Figure 1. Misconception: Six Regions in Three-Space

Group C consisted of three inservice teachers and one college instructor, Claude. When the problem was initially posed, Claude began using index cards to form two intersecting planes; Theresa, a veteran teacher, did the same using her hands. Nancy first drew the group's attention to Claude's figure then Theresa leaned forward to place a piece of paper on it as the third intersecting plane. Claude proposed that there were eight regions. Henry pointed to the regions formed by the figure and counted them. He and Claude began discussing the problem. Meanwhile, Nancy was still confused so Theresa used her fingers to represent two intersecting axes to explain the situation. When Nancy started to count the regions she hesitated. Theresa then picked up three pens and formed the axes with them. Henry, who still seemed to be mulling the problem over, noticed what Theresa had done and imitated her use of three pens to represent the three axes. Theresa then made connections between her pen figure that represented the intersecting axes and Claude's index card figure that represented the intersecting planes. Claude and Henry also discussed the planes and axes using Henry's figure. Henry struggled to determine the regions since he saw four regions on top, four on the bottom, four on the left, four on the right, four in front, and four in back. The group discusses the problem trying to help Henry see that he was counting the same regions more than once. Theresa started to discuss the sign of the axes leading the group to think about the ordered triples; this convinced Henry that there were only eight regions.

Discussion: Modeling, Terminology, and Connections

Need to Model the Situation

Gestures and tools were often employed by the teachers to help them communicate, visualize, and reason about the number of regions formed in the problem. The teachers most often used their hands or other easy-access materials, like sheets of paper, index cards, and rulers to model intersecting planes. While Groups 3 used a plane and a single axis during their exploration of the problem, Sam in Group 1 hinted at axes in his reference to the edges of a box. Group C represented the situation by modeling the three specific axes alone without any planes. Group 4 represented the situation using all three axes as well as the *xy*-plane. Only Groups 4 and C explicitly labeled the axes in their exploration of the problem.

In Group 4, Sonya had a definite preference for three-dimensional models over two-dimensional figures indicating that she had difficulty visualizing the situation with a drawing. Group B came to the consensus that there were eight regions only after they used a three-dimensional model. Prior to that, misconceptions arose in this group as they attempted to interpret the two-dimensional drawing of a three-dimensional situation. Both cases model what Parzysz (1988) is talking about with regard to the loss of information the more distant the representation is from the actual geometric object. For the three-dimensional situation posed in the problem, a three-dimensional figure was a closer (and hence better) representation than a two-dimensional drawing.

Tie to Terminology: Quadrants

Groups 2 and 4 considered whether it was appropriate to call the regions formed "quadrants" after they determined that there would be eight regions. Sarah and other members of Group B referred to the "quadrants" that would be formed in both three and four dimensions. In general, some of the teachers seemed very tied to this label, even though they recognized that the prefix quad- indicated four. Of all the participants only Jen used the word octants; nobody else made any reference to the prefix oct- when discussing labels for the regions.

Multiple Perspectives, Lack of Connections

As noted earlier, only Groups 3, 4, and C explicitly represented at least one axis when modeling the problem. Sonya in Group 4 and Theresa, along with other members of Group C, were careful to make clear connections between the labeling of the axes, the ordered triples, and the corresponding three-dimensional regions.

Groups 3 and A solved the problem using visual models alone. Other groups also incorporated more combinatorial approaches to solving the problem. While Sam in Group 1, Pete and Jen in Group 4, and Sarah and Ned in Group B mentioned either 2 x 2 x 2 or 2^3 in their combinatorial approaches, none of them verbalized how this idea was connected to the visual models they were also using in their explanations. Similarly, Sam in Group 1 and Matt in Group 2 made no explicit connections between the signs for the coordinates in the ordered triples and the visual models. It is immediately apparent that some of the teachers had multiple perspectives on how to solve the problem, but few communicated to their group mates the connections between these perspectives. These results make one wonder if the teachers really understand and are able to communicate to their students the connection between ordered pairs, axes, and the two dimensional regions called quadrants that form the Cartesian coordinate system.

Lisa in Group 3 demonstrated a fluid understanding of the connections between two- and three-space in her spontaneous use of dynamic gesture to model the outgrowth of eight three-dimensional regions from a two-dimensional plane. Likewise, Theresa in Group C, demonstrated a deep understanding in her flexible modeling, her fluid transition between models, and the explicit connections she made between graphical and combinatorial approaches to solving the problem. In contrast, Nancy in Group C struggled to understand how the two-dimensional plane could be extended into three-space to determine the number of regions there. Emily in Group 4 seemed to envision three-space as a single, static entity with a particular orientation and was resistant to her group's use of a different arrangement of the axes. Nancy's and Emily's responses to the problem seem to indicate that some teachers lack a deep understanding of the Cartesian coordinate system's natural extension into three-space.

Discussion: Mathematical Engagement Domains

During review of the video data, response patterns revealed that engaging with the problem involved three overlapping domains: visualization, reasoning, and communication. These domains are characteristic of mathematical problem solving in general, and geometric problem solving in particular. They underpin the essence of the teachers' engagement with the given problem. Figure 2 models the three-part conceptualization of these domains. A description of each follows.

Visualization independent of reasoning and communication is the act of calling up a mental image without employing it in any way. Visualization alone prohibits the act of communicating ideas to others and does not involve performing any operations on the mental image. Reasoning without communication refers to an individual's internal analyses and interpretations for personal understanding. While these processes may involve symbolic representations (i.e. algebraic manipulations), they preclude the use of mental images. Communication by itself is the act of conveying predetermined or factual information and evoking understanding through the use of linguistic or symbolic means that do not employ the use of mental imagery.

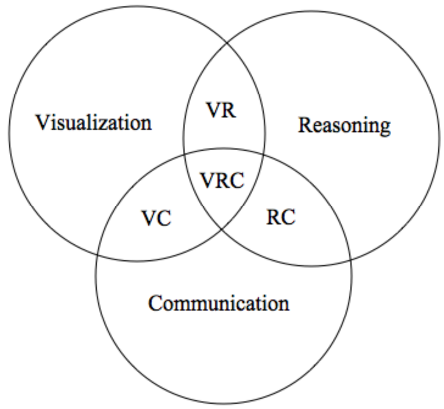

Figure 2. Mathematical Engagement Domains

Intersection of the Visualization and Reasoning Domains

The intersection of the visualization and reasoning domains (VR) occurs when an individual uses imagery while analyzing or interpreting within a given context. Although it is possible to reason about geometric figures without visualization, they are commonly considered in conjunction with mental images (Fischbein, 1993). Teachers in each setting used gestures while making no attempt to communicate with their group members as they reasoned through different aspects of the problem. In Group 1, Barb's immediate response to the problem was to engage in "self talk" in which she was gesturing to herself before she chose to join in the group discussion. Before Nancy directed Group C's attention to Claude's figure, both he and Theresa were engaging in self talk that involved the use of gesturing (implying visualization). In Group A, Cindy disengaged from the conversation that was occurring and initiated self talk using the index cards to help her visualize the problem. In Group B, Ned's self talk as he considered his group mates' assertion that the solution to the problem was six regions, eventually helped him come to the conclusion that the answer was in fact eight regions.

Intersection of the Visualization and Communication Domains

Visualization and communication (VC) intersect when individuals use imagery to share the knowledge (or perceived knowledge) that they possess, in an attempt to elicit understanding from others. Gestures, drawings, and tool usage augment discourse and objectify the internal imagery that the individuals possess. In many instances teachers within a group would attempt to communicate information that they had already reasoned through. The VC overlap manifested as teachers used gestures, tools, and drawings to explain the problem to their group mates. In Group 2, Matt employed Fred's help to create a model of three-space to illustrate his solution to his group mates. At first Lisa and Bob, from Groups 3 and A respectively, gave only verbal explanation for their solutions. When some of their group mates did not seem to follow the verbal explanations, they each spontaneously implemented gestures as a means to better convey their ideas. The inclusion of gestures in the explanation was successful for all of Lisa's group mates, but not for Bob's.

Intersection of the Communication and Reasoning Domains

When participants used discourse, non-iconic gesture (McNeill 1992), written symbols, etc. to share mathematical ideas without employing mental images, they are performing in the intersection of communication and reasoning (CR). While most groups began immediately to gesture or use tools to visualize the problem posed, Sam in Group 1 initially conceived of the solution using a more combinatorial approach. This resulted in members of his group using symbolic notation to solve the problem as they considered the possible permutations of ordered triples. Contrastingly, Group 2 began with a visual approach, but then Matt offered the ordered triple representation of the solution. No gestures, tools, or drawings were used for visualization in either group when they discussed the ordered triples.

Intersection of the Visualization, Reasoning, and Communication Domains

The overlap of visualization, reasoning, and communication (VRC) occurred when participants worked together as they actively explored problems, and communicated with each other via external representations of mental imagery. The VRC overlap was present in all groups. One example occurred in Group 4 when Jen introduced the ordered triples to help Sonya better understand the problem. In an attempt to reason through and connect the visual and symbolic explanations, Sonya reached over to Pete's model using it to indicate the locations of each permutation of the ordered triples. While engaged in the gesturing, Sonya worked to solve the problem, verbalizing her reasoning to Jen. Another example was found with Group B when they worked together to construct a model of the problem as they collaboratively explored and discussed how many regions would be formed.

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THE RELATIONSHIPS AMONG PROFESSIONAL RANKS, TEACHING EXPERIENCE, COURSE TAKING AND TEACHERS' KNOWLEDGE FOR TEACHING ALGEBRA

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This study examined the relationships among teaching experience, professional ranks, number of courses taken, and knowledge for teaching algebra (KTA). Three hundred and thirty-eight in-service secondary mathematics teachers completed a questionnaire of KTA. Multiple comparisons and a structural equation model were used to examine their relationships. It was found that school algebra knowledge (SAK) has a significant and positive effect on teaching algebra knowledge (TAK), but does not have a significant effect on advanced algebra knowledge (AAK). Professional rank and the number of courses taken were found to have a significantly positive effect on SAK while teaching experience was found to have a significantly negative effect on AAK. Teachers' AAK presents negative correlations with their teaching experience and their professional rank. The implications of this study for teacher professional development are discussed.

Introduction

Algebra is an important content topic in school mathematics (e.g., National Council of Teachers of Mathematics [NCTM], 2000; National Mathematics Advisory Panel [NMAP], 2008). Efforts to improve the quality of teaching and learning of school algebra have led to ever-increasing interest in examining and changing educational policy and practice related to school algebra (e.g., RAND Mathematics Study Panel [RAND MSP], 2003). Among many different factors that contribute to the quality of algebra teaching and learning, it is commonly recognized that teachers' knowledge needed for teaching is crucial for developing and carrying out high-quality instruction that can eventually result in student learning outcome (Hill, Rowan, & Ball, 2005; NMAP, 2008). Yet, much remains unclear about the nature of teachers' knowledge needed for teaching mathematics in general, algebra in particular.

What knowledge teachers need to have for teaching algebra is not a question restricted to any single education system. In fact, learning what pre-service and in-service teachers know and are able to do in different education systems has become an important approach to help teacher educators and researchers understand better about the nature of such knowledge that teachers need to have for teaching (e.g., Li, Huang, & Shin, 2008; Ma, 1999; Schmidt et al., 2007). For instance, Ma's study (1999) revealed what elementary mathematics teachers can and should know in mathematics for teaching, shed insight into what we can learn from Chinese practice. By focusing on teachers' knowledge for teaching algebra (KTA), this study is also designed to focus on the case of China where policy and practices in teaching and learning algebra have been promising for students learning (e.g., Li, 2007). In particular, we aimed to examine the

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characteristics of KTA and possible factors contributing to the growth of teachers' KTA in China that are outlined as the following two questions:

1. What are the characteristics of in-service mathematics teachers' KTA?
2. What are the relationships between in-service mathematics teachers' KTA and their highest education degree, professional rank, teaching experience, and number of courses taken?

Theoretical Consideration

Mathematics Knowledge for Teaching Algebra

Drawing on Shulman's (1986) framework of teacher's knowledge, *content knowledge, curriculum knowledge and pedagogical knowledge*, Ball and her colleagues (Ball, Hill, & Bass, 2005) have developed a refined framework defining mathematics knowledge for teaching. This model highlights the kind of mathematical content knowledge that is specific to teachers, and recognizes that knowledge of mathematics for teaching is partially the product of content knowledge interacting with students in their learning processes and with teachers in their teaching practices. Since algebra is the main body of school mathematics and is challenging for students to learn (NCTM, 2000; NMAP, 2008), several researchers have proposed ways of describing teachers' knowledge for teaching algebra (e.g., Even, 1990; Ferrini-Mundy, McCrory, & Senk, 2006). Recently, Floden and McCrory (2007) have developed a three dimensional construct to guide the development of a measure of teachers' knowledge for teaching algebra. According to this model, the knowledge for teaching algebra includes three types of algebra knowledge for teaching: school algebra knowledge, advanced algebra knowledge, and teaching algebra knowledge. *School algebra knowledge* (SAK) refers to the algebra covered in the K-12 curriculum. *Advanced algebra knowledge* (AAK) includes calculus and abstract algebra that are related to the school algebra; and *teaching algebra knowledge* (TAK) refers to typical errors, canonical uses of school mathematics, and topic trajectories in curriculum and so on. The content consists of two major themes in school algebra: (1) expressions, equations/inequalities, and (2) functions and their properties.

Factors Influencing Teachers' Knowledge Growth

Traditionally, education degree earned and years of teaching have been used as main indicators for measuring teachers' knowledge and competence. However, both of these measures are either unrelated or negatively related to improvements in pupil performance (Wilson, Floden, & Ferrini-Mundy, 2002). Some researchers (e.g., Monk, 1994) used the number of courses as one of major indicators for gauging mathematics teachers' knowledge, and found it is positively related to how much mathematics students learn at the secondary school level. Moreover, it was found that teachers' undergraduate mathematics education coursework contributed more to their students gains than did courses in undergraduate mathematics.

In China, there is also a promotion and ranking system through which teachers' professional knowledge has been developed progressively and continuously (Huang, Peng, Wang, & Li, 2010). For example, the professional ranks of secondary teacher include senior (*Gaojie*) teacher, intermediate(*Zhongjie*) teacher, and primary (*Chujie*) teacher. For each level of teacher position, there are specific requirements in political, moral, and academic aspects. For example, the requirements for being a teacher at the senior rank include five or more years' experience as a secondary school teacher at intermediate level or with a Ph.D. degree. Thus, it is reasonable to assume that teacher professional rank reflects the level of teacher knowledge for teaching.

In this study, we considered the background variables including teachers' highest education degree, teaching experience, professional rank, and number of courses taken, as major factors associated with teachers' knowledge for teaching. Thus, we aimed to examine the relationships among these background variables and three components of KTA.

Research Method

Participants

The participants were convenience samples from three different summer teacher education programs. In total, we collected completed questionnaires from 398 in-service mathematics teachers in China. One group of teachers was the participants of a summer course for their master's degree in mathematics education (more than 170 teachers at both middle and high schools) in East China. Another group of teachers was the participants of a training program for using middle school mathematics textbooks (around 100 participants) organized by the textbook publisher in North China. The third group of teachers was the participants of a training program for using a high school mathematics textbook (more than 130 participants) organized by the textbook publisher in South China. Due to blank answers, we discarded 60 incomplete questionnaires. Thus, we have 338 valid completed questionnaires. About 80% of participants held a bachelor degree, 11% of them held a master degree, and the remaining 9% had a secondary school diploma. About half of the participants had an intermediate professional rank (48%), and one third of them held a primary rank (35%), and the rest had a senior rank (15%). 84% of the participants had more than 5 years of teaching experience.

Measure

An instrument was developed based on an existing instrument for measuring mathematics teachers' knowledge for teaching algebra (Floden & McCrory, 2007). The survey consisted of three parts: Background information, multiple-choice items, and open-ended items. Background information includes grades (1=middle, 2= high schools), gender (1=male, 2=female), highest education degree (1=diploma, 2=B.S., 3=M.A.), professional ranks (1=primary, 2= intermediate, 3=senior and exceptional), years of teaching experience (1= less than 4 years, 2=from 5 to 9 years, 3=10 years and above), and number of math and mathematics education courses taken. With respect to the course taking, we provided a list of 22 courses based on literature review (Li et al., 2008), and also asked teachers to add relevant course if necessary (the value of this variable is the total number of the courses taken). The second part included 17 multiple-choice items which were translated from the original instrument of KTA in English. The third part consisted of 5 open-ended questions. Three of them were translated from the original KTA (items 18-20), while the other two questions were developed by the authors based on our literature review and study of Chinese mathematics curriculum. The instrument in English was translated into Chinese by the first author and a Ph. D candidate in mathematics education in the Chinese Mainland independently. A bilingual mathematics educator in the US was then invited to compare the two Chinese versions for deciding a final reversion. The final instrument included seven items (1, 3, 6, 14, 17, 19, & 21) in school algebra knowledge, eight items (4, 8, 9, 12, 13, 16, 20, & 22) in advanced algebra knowledge, and seven items (2, 5, 7, 10, 11, 15, & 18) in teaching algebra knowledge. The following are two sample items:

Multiple choice item (School algebra). Which of the following situations can be modeled using an exponential function?

i. The height *h* of a ball *t* seconds after it is thrown into the air.

ii. The population *P* of a community after *t* years with an increase of *n* people annually.

iii. The value *V* of a car after *t* years if it depreciates *d%* per year.

A. i only; B. ii only; C. iii only; D. i and ii only; E. ii and iii only

Open-ended item (Teaching algebra). On a test a student marked both of the following as non-functions

(i) f: $\boldsymbol{R} \rightarrow \boldsymbol{R}$, $f(x) = 4$, where $\boldsymbol{R}$ is the set of all the real numbers.

(ii) $g(x) = x$ if x is a rational number, and $g(x) = 0$ if x is an irrational number.

(a)For each of (i) and (ii) above, decide whether the relation is a function; (b) If you think the student was wrong to mark (i) or (ii) as a non-function, decide what he or she might have been thinking that could cause the mistake(s).Write your answer in the Answer Booklet

Data Collection

Two university professors who took charge of the teachers' summer programs administered this survey within one lesson duration (45 minutes). Participating teachers were told that this survey is a component of their summer course for researchers to understand teachers' knowledge for teaching algebra. Similarly, two experts from textbook publishing house conducted the summary workshop for training teachers how to use their textbooks. The participants were invited to complete the questionnaire within one section of the workshop (around 45 minutes) as a part of the training program. All the participants returned their completed questionnaires on site. The collected answer booklets were then sent to a coordinator in Shanghai by express mail. The coordinator scanned all the completed questionnaires into PDF files, and sent the files to the researchers.

Data Analysis

Regarding the multiple-choice items, a correct answer was rated as 1 while a wrong answer was rated as 0. With regard to the open-ended items, the rubrics for the first three open-ended items were adapted from the U.S. ones, while the rubrics for the two newly added open-ended items were developed by the researchers. Two coders worked together to code 100 copies from participants of master education program. After that, one coder who was a middle and high school mathematics teacher for many years coded the entire remaining questionnaire (238 copies). Then, the first author cross-checked the coding for 133 participants who attended the workshop of using the textbook at high school level. The agreements for the five open-items are as follows: 89 % for item 18, 98% for item 19, 100% for item 20, 96% for item 21 and 91 % for item 22. The disagreements were resolved through discussion. Then, the coder re-checked all the remaining questionnaires and made necessary corrections.

Results

We used two approaches to analyze the data. First, we examined the means of different components of KTA in terms of background variables by using SPSS16. Second, we estimated a model of teachers' contextual variables and KTA using AMOS16 (Byrne, 2010).

Descriptive Statistics

The reliability (Cronbach's Alpha) of the instrument is .57. The mean of all variables are reported in Table 1 for the 338 participants, as well as by the professional ranks and years of teaching experience.

Table 1. The Means of the Variables by Professional Rank and Years of Teaching Experience

	N=338	Professional rank			Years of experience		
		1	2	3	1	2	3
		N=119	*N=170*	*N=49*	*N=48*	*N=150*	*N=140*
School algebra knowledge	10.69	10.62	10.60	11.18	10.50	10.75	10.70
Advanced algebra knowledge	9.53	9.66	9.50	9.31	9.90	*9.88	9.04
Teaching algebra knowledge	7.54	7.34	7.68	7.55	7.40	7.42	7.71
Knowledge for teaching algebra	27.76	27.61	27.79	28.04	27.79	28.05	27.45

*Note. *p<.05; Asterisks in the column of years of teaching experience represent significant differences between years of teaching experience 2 and 3*

With regard to the relationship between teaching experience and KTA, advanced algebra knowledge (AAK) decreases as teaching experience increases. In particular, the decrease from experience 2 (5-9 years) to experience 3 (10 years and above) (mean difference (MD) =0.84, $p<0.05$) is significant. The SAK and TAK increase as teaching experience increases from experience 1 (i.e., less than 5 years) to experience 2. However, after five years of teaching experience, TAK continues increasing as teaching experience increases while SAK decreases slightly.

Overall, the higher rank a teacher had the higher mean score of KTA the teacher achieved. With regard to AAK, the mean scores decrease continuously with the promotion of professional ranks although there is no statistically significant difference. It is interesting to notice that TAK increases steadily while SAK decreases slightly from primary rank (rank 1) to intermediate rank (rank 2). However, TAK decreases slightly while SAK increases steadily from intermediate to senior rank (rank 3).

Estimating a Model of Background Variables and KTA

The relationships among highest education degree, professional position ranks, years of teaching experience, number of courses taken, school algebra knowledge, advanced algebra knowledge and teaching algebra knowledge were modeled by a series of path models (e.g., Figure 1). We assumed that SAK and AAK interact with each other and both of them impact TAK. Furthermore, KTA (including SAK, AAK and TAK) is hypothesized to be a function of background variables. SAK, AAK and TAK are considered as endogenous variables in the model. The aim of the path analysis is to include the entire set of variables which may contribute to the explanation of the variance in the endogenous variables (components of KTA). As it is impossible to include everything that may impact these variables, an error term is included in the model to be estimated for each endogenous variable (e.g., es1, ea1, and et 1 are the error terms for SAK, AAK, and TAK respectively, see Figure 1). The errors reflect all those unobserved predictors that were not measured in this study nor included in this model.

In addition, these variables of KTA were hypothesized to be influenced by background variables including teachers' highest degree earned, professional rank held, years of teaching experience, and a total number of courses taken. These variables were considered to be exogenous, which are hypothesized to influence KTA.

Initially, all the parameters for the background variables on the endogenous variables were estimated. Subsequently, the path coefficients that were relatively small or not statistically significant were deleted for the model and the model was then re-estimated. The final model is presented in Figure 1.

The Chi-square value for this model was not significant, $\chi^2(3)=1.6$, $p=0.67$. This indicates a good fit of the model. The comparative fit index (CFI)) was equal to 1.00. This index can take value from 0 to 1 with value close to 1 showing a better fit, and value greater than .95 usually indicating a relatively good fit. The root mean square error of approximation (RMSEA) was equal to 0.000. This index takes into account the complexity of the model, that is, the number of parameters being estimated. This index can range from 0 to 1 with a value less than .05 representing good fit (Byrne, 2010). Thus, the indexes suggest that the data fit the model very well.

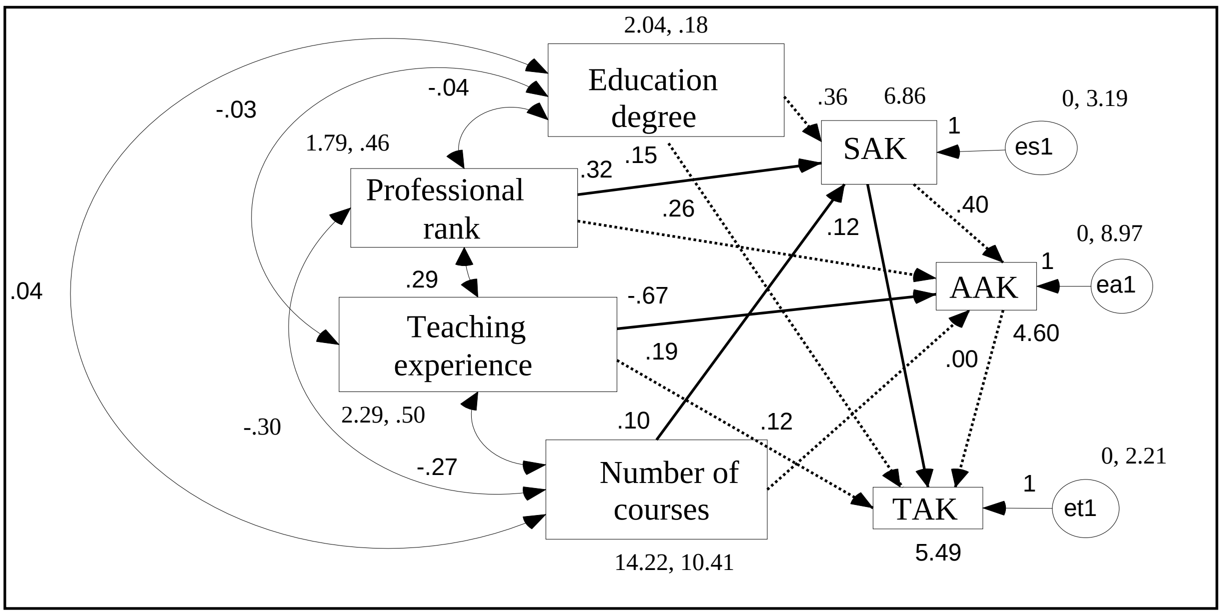

Figure 1. Final model of KTA

(All path coefficients are unstandardized. The dashed lines represent non-significant effects, while the bold lines represent significant effects)

According to the parameter estimates in the final model showed in Figure 1, teachers' years of teaching experience was found to have a negative, direct and significant effect on their advanced algebra knowledge (β=-0.67, p<. 05). It suggests that teachers with more years of experience tend to have less advanced algebra knowledge. Professional rank was found to have a positive, direct and significant effect on school algebra knowledge ($\beta=0.32$, $p<.05$). The number of courses taken was found to have a positive, direct and significant effect on school algebra knowledge (β=0.10, p<.05). Moreover, teachers' school algebra knowledge was found to have a positive and significant effect on their teaching algebra knowledge (β=0.12, p<.05). Teachers' highest education degree was not found to have a significant effect on their KTA.

Conclusion

Teachers with more teaching experience seem to have more *Teaching algebra knowledge* (TAK). However, their *school algebra knowledge* increases with less than 10 years of teaching, then it presents a decreasing trend after 10 years of teaching. Overall, the higher rank a teacher had the higher mean score of knowledge for teaching algebra the teacher achieved. The promotion from primary rank to intermediate rank is mainly associated with the increase of *teaching algebra knowledge* while the promotion from intermediate rank to senior rank is mainly associated to the increase of *school algebra knowledge*. Path analysis also showed that teachers' professional rank has a significant and direct effect on their *school algebra knowledge*.

It was found that advanced algebra knowledge has a negative correlation with the increase of teachers' teaching experience. The path analysis further confirmed that teaching experience has a negative and significant direct effect on advanced algebra knowledge. Moreover, there is a negative correlation between teachers' advanced algebra knowledge and their professional rank. In addition, the number of courses taken was found to have a positive, direct and significant effect on school algebra knowledge, and school algebra knowledge was found to have a positive and significant effect on teaching algebra knowledge. The highest education degree was not found to have any significant effect on KTA.

Discussion

The findings suggest that the number of courses taken (mathematics and mathematics education) has a significant effect on school algebra knowledge; the school algebra knowledge has a significant effect on teaching algebra knowledge. Consequently, the number of courses taken may be related to students' performance (Monk, 1994). This highlights the importance of providing adequate number of mathematics and mathematics education courses in pre-service teacher preparation. However, school mathematics knowledge and advanced mathematics are not closely related to each other, and advanced algebra knowledge does not have a significant effect on teaching algebra knowledge. These call for developing high quality courses for teacher preparation programs so as to enhance the interconnection of different types of knowledge. Because teaching experience has mixed effects on teachers' knowledge for teaching, one should be cautious about how to maximize the positive effects of teaching experience. The advanced algebra knowledge decreases continuously as teaching experience increase, probably due to the amount of time elapsed since taking mathematics courses. Although the teaching algebra knowledge increases continuously with teaching experience, the increase of school algebra knowledge is only associated the cases with lesson than 10 years of teaching experience. Thus, how to maintain or update advanced algebra knowledge, and continuously develop school algebra knowledge is a challenging task for in-service teacher professional development.

The professional rank system is perceived as a powerful mechanism for facilitating in-service teacher professional development in China. It is interesting to notice the pattern that the promotion from primary to intermediate rank is mainly associated with the increase of TAK while the promotion from intermediate rank to senior rank is mainly associated with the increase of SAK. This may reflect the emphases of teacher professional development at different stages in China. At the primary rank stage, the focuses of teacher learning are on studying teaching materials/ textbooks and mastering basic teaching routines through participating in mentoring schema and teaching research activities. These activities may contribute to the development of teaching knowledge (Huang et al., 2010; Ma, 1999). At intermediate rank, teachers further develop their deep understanding of teaching materials and acquire various teaching strategies through participating in various teaching research activities. Since they have already formed certain teaching styles, they may broaden and deepen the understanding of school mathematics. Thus, promotion from intermediate to senior rank is mainly associated with the increase of SAK. However, teachers advanced algebra knowledge weakens although their professional rank advances. This is a serious issue that calls for further research and action. Moreover, the finding that senior teachers do not demonstrate their superiority in teaching algebra knowledge leads to some theoretical considerations. Studies show that expert teachers have a much more interconnected and accessible knowledge structure (Borko & Livingston, 1989) that may not be easy to gauge by a limited number of items.

Although this study produced some interesting findings, the limitation of sampling should be considered. This was a convenience sample, and the cohorts of different groups were not controlled. Thus, any generalization of these findings should be made cautiously.

Endnote

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DETERMINING TEACHER QUALITY IN TEACH FOR AMERICA ALTERNATIVE CETIFICATION

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The purpose of this study was to understand the level of teacher self-efficacy and differences between content knowledge and self-efficacy among teachers of different undergraduate majors in the Teach for America (TFA) program. This present study revisited a cohort of TFA teachers in their second year of teaching and taking graduate education courses at a partnering university in New York. Most studies conducted with TFA teachers have focused primarily on student achievement and teacher retention (Darling-Hammond, 1994, 1997), two of the most important variables. However, examining only these variables is not sufficient if the goal is to increase teacher quality. The following questions guided the current research project:

1. What level of self-efficacy did TFA teachers possess?
2. Was there a difference in mathematical knowledge between undergraduate majors for TFA teachers?
3. Was there a difference in perceptions of self-efficacy between undergraduate majors for TFA teachers?

Undergraduate majors for teachers consisted of liberal arts (N = 8), business (N = 9), and mathematics related majors (N = 5). TFA teachers took the New York State Content Specialty Test (CST) and were given a self-efficacy survey called the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) with two subscales: Personal Mathematics Teaching Efficacy (PMTE) and Mathematics Teaching Outcome Expectancy (MTOE) (Enochs, Smith, & Huinker, 2000). Findings revealed that teachers had high levels of self-efficacy. Mathematics related majors had higher mathematical content knowledge than business majors, but similar self-efficacy levels. Liberal arts majors had similar content knowledge and levels of self-efficacy as mathematics related majors.

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DETERMINING THE LEVEL OF MATHEMATICAL KNOWLEDGE FOR TEACHING OF GEOMETRY OF TWO COHORTS OF SECONDARY MATHEMATICS TEACHERS

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The purpose of our research project was to examine the mathematical knowledge for teaching of geometry at the secondary school level. This project was based on an adaptation of the Knowledge of Algebra for Teaching framework (KAT) developed by Ferrini-Mundy et al. (2008). The participants of this study were preservice teachers enrolled in the one year STEM M.Ed. program 2009-2010 cohort and those enrolled in the 2010-2011 cohort. In particular, we analyzed the preservice teachers' knowledge of content trajectory in geometry when analyzing school children' work. Four questions guided data collection and analysis: (1) what were the levels of the preservice teachers' knowledge of the trajectory of mathematical concepts presumed as central to the study of geometry? (2)What were the levels of the preservice teachers' knowledge of the trajectory of mathematical concepts presumed as central to the study of geometry as it pertains to analyzing and assessing high school students' mathematical work in a geometry context?, (3) How did the specific content knowledge defined in (1) and (2) impact pedagogical knowledge?, and (4) How did the performance of preservice teachers of the 2009-2010 cohort at the end of the program on (1), (2) and (3) differ from that of incoming preservice teachers of the 2010-2011 cohort?

Surveys were administered to the preservice teachers. Based on the responses to those surveys, participants were chosen for an interview. The data from the surveys and the interviews were scored using both qualitative and quantitative methods in order to determine the level of mathematical knowledge for teaching of geometry with respect to the four research questions. Although there has been much research conducted on elementary teachers' mathematical knowledge for teaching of mathematics (Ball et al., 2005), at the high school level much is still unknown about teachers' mathematical knowledge for teaching. This study was a significant attempt toward instrument development for measuring knowledge of teaching of geometry.

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DEVELOP CONTENT KNOWLEDGE FOR TEACHING THROUGH DYNAMIC COURSE CONTENT

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In this paper, we propose that *dynamic course* content has an essential role in developing teacher's content knowledge for teaching mathematics. We are presenting technology - based mathematical conceptual activities designed to improve teachers' *content knowledge for teaching* of abstract algebraic topics including but not limited to the mean value theorem, the Cauchy mean value theorem, and the global minimum of total squared distances among non-intersecting curves (Ball, Thames, & Phelps, 2008). These activities can help to develop teachers' representational fluency of aforementioned topics and deepen their conceptual understanding of these mathematical constructs. Therefore these activities can influence development of teachers' content knowledge for teaching. As mathematics educators in any part of the world we would like to see mathematics teachers who are: a) competent; b) have conceptual understanding of mathematics; c) have strong mathematical content knowledge for teaching; and d) are able and willing to use a curriculum that promotes mathematical reasoning, allows students to develop deep conceptual understanding of mathematics, and gives opportunity to develop students' representational fluency. We believe these goals can be archived through use of dynamic technology, such as GeoGebra, in teachers' training as well as in school mathematics. One of our main goals in this paper is to create technology-based activities (using free open source software Geogebra) as part of the *dynamic course content* that helps to develop teachers' conceptual knowledge. More mathematics concepts, as well as more challenging mathematics can be explored in depth by using technological tools in a mathematics classroom. These *conceptual activities* can also affect teachers' content knowledge for teaching and influence students learning of mathematics (Zbiek, Heid, Blume, & Dick, 2007). The activities presented at this poster are designed to assist teachers to develop representational fluency in order to further improve their content knowledge for teaching of the aforementioned mathematical topics (Becker & Rivera, 2006; Zbiek, Heid, Blume, & Dick, 2007).

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LEADERSHIP CONTENT KNOWLEDGE: A CASE OF PRINCIPAL LEARNING IN ALGEBRA

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Stein and Nelson (2003) argue that to support systemic change, administrators need knowledge of teaching and learning of subject matter *at the classroom level* (about the subject and how students learn it), *at the school level* (how teachers learn), and *at the district level*. This study focuses on the first two categories, examining administrator learning in the context of a content-focused year-long professional development (PD) initiative developed as a joint project between a university and a county with districts spanning from small city to distant rural. Nine secondary administrators from six districts participated in the voluntary PD focused on algebra (conceptualized as the study of patterns and functions). Specifically, we explore the question: what did administrators learn as the result of engagement in the study group about algebra teaching and learning?

The administrators' pre- and post-assessments, transcripts from interviews at the end of the study groups, and content of the local algebra improvement initiatives designed by the principals and teachers for implementation in the following year were analyzed for this report. We examined changes in principal knowledge about algebra (content knowledge at the classroom level) and of algebra teaching (content knowledge at the school level). With respect to content, we focus on defining, representing, and providing examples of function. The results show that principals deepened their knowledge of function - being able to define function, provide examples, and make conceptual connections in a contextual patterning task. For example, on the pre-test, none of the principals were able to construct a definition of function, two principals were able to give an example of function and none were able to provide a non-example. On the post-test, however, five of the eight responses contained a definition that included univalence and did not rule out arbitrary functions, six responses successfully provided an example (and one ambiguous response) and six responses provided a non-example. This change in content knowledge is likely to impact instructional leadership in two ways, giving principals confidence in a core area of the secondary mathematics curriculum and providing a mechanism for principals to continue to enhance their content knowledge through the study of teaching practice. With respect to knowledge about algebra teaching, the analysis suggests that exploring the teaching and learning of algebra supported principals in specifically identifying mathematical practices that supported student-centered learning.

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PRESERVICE TEACHERS' UNDERSTANDING OF MATHEMATICAL CONCEPTIONS VIA VISUAL REPRESENTATION AND REAL-LIFE CONNECTION

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How does environmental context help reveal preservice teachers' thinking about mathematics? To what extent are preservice teachers' beliefs regarding teaching and learning mathematics revealed in visual snapshots of mathematical connections? Do students take pictures related to the mathematics they know or are their insights beyond their formal mathematics? Can a hierarchy be, developed depending on the amount of mathematics a person knows?

This study involved forty-six preservice mathematics teachers and data collection included questionnaires and mathematical snapshots (a two-part Mathematical Pictures activity as part of the coursework in their elementary mathematics methods course). In both the first and second part of Mathematical Pictures, the students were to each independently take ten photographs of mathematics embedded in everyday life and write at least two sentences explaining the mathematical concept in the snapshot. After the preservice teachers submitted the first set of pictures, the researchers introduced the Conceptions Questionnaire, a four point Likert scale questions based on promoting a constructivist approach that focuses on the role of the student, the role of the teacher, and the nature of mathematics. In addition, the preservice teachers encountered creative ways to present and think about elementary mathematics in a methods course before they worked on the second set of pictures.

A rubric was, used to evaluate both set of pictures. Pictures were, rated as "flexible" meaning containing many different categories; "fluent", meaning containing many different ideas within a single category; "elaborative", meaning an in depth view of one category; or "original". The findings from the rubric analysis and questionnaire are, shared in our poster session. The mini study found that the more constructivist the preservice teacher beliefs, the more elaborative and original the mathematical connections made, as seen in their illustrations of the snapshot visuals and captions.

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UNDERSTANDING VARIATION IN MAINTENANCE OF THE COGNITIVE DEMAND OF MATHEMATICAL TASKS ACROSS SCHOOL DISTRICTS

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This poster examines the cognitive demand of written tasks and the extent to which the demand of the task is maintained within a mathematics lesson in middle school classrooms in three, large urban school districts. The nature and cognitive demand of classroom tasks, as well as the implications for student learning have been studied for nearly 30 years. Stein, Grover, and Henningsen (1996) focused specifically on the initial cognitive demand of mathematical tasks and examined how the demand increased or declined across different stages of a math lesson (e.g., the task as set up and the task as implemented). They found that tasks with the *potential* for high levels of cognitive demand often decreased in cognitive demand as they were implemented. I extend this work by exploring teacher characteristics and district factors to explain why the cognitive demand of tasks might initially be low or decline as they are implemented in the classroom.

This analysis uses data collected for a four-year study in which we are collaborating with four, large urban districts that are attempting ambitious, instructional reform in middle-school mathematics. Three of the four districts are using the *Connected Mathematics Project 2* (CMP2) curriculum, and those districts are the foci of this analysis. Each year, we document the classroom practices of approximately 30 teachers in each district by video-recording two successive lessons. These video-recordings are coded using the *Instructional Quality Assessment* (IQA) (Boston & Wolf, 2006). In addition, we conduct semi-structured interviews with teachers that focus on key aspects of the school and district settings in which they work and also assess the teachers' mathematical knowledge for teaching by administering the *Learning Mathematics for Teaching* (LMT) assessment (Hill et al., 2004).

Our analyses showed a significant variation between school districts with regard to the potential of the task as written ($F\ (2, 95) = 5.95$, $p < .01$). We also found that the higher the cognitive demand of the task, the more likely it was to decline over the course of the lesson. Furthermore, there was statistically significant variation in the decline of high-level tasks between the districts ($F\ (2, 95) = 3.24$, $p < .05$).

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"WHAT'S WRONG WITH THIS PICTURE?" COMPLEXITY IN REPRESENTATIONS OF MATH TEACHER KNOWLEDGE

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Research in mathematics teacher knowledge has adopted increasingly sophisticated conceptualizations of it object of study. The research programs espoused by Deborah Ball (e.g. Hill, Ball, & Schilling, 2008) and Brent Davis (e.g. Davis & Simmt, 2006) are used here as characteristic of the current field. Both of these approaches rely on particular modes of representing teacher knowledge, and these modes have their respective and concomitant limitations. In accepting the nature of mathematical understanding for teachers and learners to be a complex phenomenon situated in a dynamic environment this poster uses selected works from a painter and a composer to re-examine some realities and possibilities in addressing complex qualities within representations of math teacher knowledge.

The surrealist painter, René Magritte, produced many paintings explicitly addressing themes of representation. Three of Magritte's earlier works, entitled: 'The Treachery of Images,' 'The Human Condition,' and 'Not to be Reproduced,' each speak to some of the profound challenges and limitations in the human endeavor of representation. These images are used to illustrate some fundamental difficulties that are encountered when attempting to represent teacher knowledge in mathematics.

Composer Igor Stravinsky is perhaps most famous for having written the musical score for a ballet whose 1913 premiere performance instigated a riot among the more typically sedate attendees. The score and music of this same ballet, The Rite of Spring (Le Sacre du Printemps), is used to explore the possibilities of representing complex phenomena within a relatively rich medium of expression (i.e. music). This particular piece of music was written as a description of a human response to the organic process of the spring season, and is a qualified success as a representation of a complex phenomenon. Passages of music excerpted from the full score are selected on the basis of their musical characteristics, which include dissonance, recursion, and parallel asymmetries.

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Chapter 14: Preservice Teacher Preparation (K-8)

Research Reports

Brief Research Reports

Posters

AN EXAMINATION OF THE UNDERSTANDING OF THREE GROUPS OF PRESERVICE TEACHERS ON FRACTION WORDED PROBLEMS

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This study examines the work of preservice teachers (PSTs) on fraction word problems for which drawing a model would be a beneficial strategy. We report on a set of four problems posed to three groups: secondary PSTs in a mathematics methods course; elementary PSTs in a beginning mathematics for teachers course; and elementary PSTs in an upper level mathematics for teachers course. We report the success of those in each group in solving the problems and the degree to which they provided explanations. We also discuss their solution strategies and some of the difficulties PSTs encountered when solving these problems.

Introduction

The importance of fraction concepts and computation in preservice teachers' mathematical understandings across the K-12 curriculum is widely recognized. Although this area of mathematics learning and teaching has been extensively examined, we argue that further articulation regarding the facility of preservice teachers' to navigate fraction word problems is critical in informing preservice elementary, middle, and secondary teacher education programs. Specifically, at all grade levels, K-12, the ability to negotiate fraction word problems by explicitly connecting computational understandings to models and actions elicited by wording in fraction word problems is critical to a teacher's ability to effectively engage students in similar conceptually and computationally engaging fraction problems.

As mathematics teacher educators who work with preservice teachers (PSTs), we view this issue as one of high importance. As such, in our study, we examine the computational strategies used by PSTs in solving worded fraction problems to determine whether they can provide explanations and representations that make sense mathematically, rather than just procedurally. Specifically, in this study we address the following research question: To what extent and mathematical depth do PSTs provide conceptual explanations and representations related to the implied fraction computations in worded fraction worded problems?

Theoretical Perspectives

From Ball (1990) to Newton (2008) there has been an extensive analysis of understanding of fraction concepts by prospective elementary teachers. Understanding of the arithmetic of fractions, especially multiplication and division, is difficult for students. In summarizing past work and setting the focus for future work on rational numbers and proportional reasoning, Lamon (2007) writes:

> *Of all the topics in the school curriculum, fractions, ratios, and proportions arguably hold the distinction of being the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential*

> *to success in higher mathematics and science and one of the most compelling research sites. In the last decade or more, researchers have made little progress in unraveling the complexities of teaching and learning these topics* (p. 629).

In describing the importance of fractions, Wu (2009) states, “Because fractions are students’ first serious excursions into abstraction, understanding fractions is the most critical step in understanding rational numbers and in preparing for algebra” (p. 8). Keiran (2007) describes the difficulty students have connecting word problems and algebraic equations. In prior explorations with middle grades students, one group of researchers (Olson, Zenigami, and Slovin, 2008; Olson, Slovin, & Zengiami, 2009) found that 5th grade students with no formal instruction in multiplication and division of fractions could solve worded problems involving those concepts. In doing so, however, the students made use of the models and actions implied in the wording of the problems. Importantly, these methods for solving the problems were the primary vehicles for computationally and conceptually addressing fractions available at the students’ disposal.

Ma (1999) observed that teachers in the United States tend to be procedurally focused, and that even when they can solve problems they often cannot explain well. For the most part, U. S. teachers in her study felt it was sufficient to justify the steps of the algorithm used, rather than to conceptually examine and explain why the algorithm makes sense mathematically. Green, Piel, and Flowers (2008) observed that reliance on algorithms potentially allows for misconceptions to become more resilient. In fact, it may be that the use of algorithms can act as a shield to avoid further mathematical reasoning and exploration.

Methods

Participants

Participants in the study were all PSTs in licensure programs. Twelve secondary PSTs, seeking licensure for grades 7-12, were enrolled in a secondary mathematics methods course (SMM) designed to develop their pedagogical knowledge and skill for teaching mathematics, and were near the end of their program of study. The elementary PSTs, who sought licensure for grades K-6, were enrolled in Mathematics for Teachers courses and most were in their first two years of study; eleven PSTs were enrolled in the first course (MA1) of the sequence, while sixteen were taking the second course (MA2). Several of the latter students had, during the first course of the sequence, investigated fractions using a variety of models, including Cuisenaire rods. Most of the elementary PSTs had not taken any other mathematics at the college level.

Procedures

PSTs were given about 45 minutes to work on four problems (described below). Due to differences in the nature of the two courses, slightly different directions were given. Directions were given orally as well as in written form. Because the secondary PSTs were enrolled in a secondary methods class they were directed to try to solve the problem “according to how it is written” by using models and sense making rather than direct computations, and to show how to explain to a student who does not understand, or is having difficulty understanding, how to solve it algebraically. The PSTs in the Mathematics for Teachers classes were simply asked to solve the problems and to show their work and explain their thinking.

For each problem the work of the PSTs was examined for correctness of answer using the following coding scheme: 0-no response or simply a restatement of the problem; 1-incorrect solution; 2-correct solution. Explanations for each problem for the elementary PSTs were coded using the following scheme: 0-no explanation; 1-explanation with major flaw; 2-explanation with minor flaw; and 3-correctly justified explanation. Explanations for each problem of the secondary PSTs was coded on a five-point rubric, briefly stated as: 0- no explanation; 1-minimal explanation failing to explain what was done or why it was done; 2-addresses only one of what was done or why it was done; 3-addresses what was done but not fully why; and 4-explained both what was done and why it was done (Olson & Olson, 2010).

Problems Investigated

Four problems were developed to assess the ability of the prospective teachers to solve word problems involving fractions and proportional reasoning. The problems are given below, followed by a discussion of the ideas that guided the development of the problems. “Explain your reasoning and support your answer” was stated at the end of each problem.

1. It takes 3/4 liter of paint to cover 3/5 m^2. How much paint is needed to paint 1 m^2? Explain your reasoning and support your answer.
2. Macy had a distance to swim during practice. When she had gone two-thirds of what she was supposed to swim she had traveled one-half kilometer. What was the total distance Macy was to swim during practice? Explain your reasoning and support your answer.
3. It took Brooke 2/3 of her advertising budget to buy 3/5 of a newspaper column. What part of the advertising budget is needed to buy a whole column? Explain your reasoning and support your answer.
4. Jonnine had a board. She cut and used 2/5 of the board for bracing. She measured the piece used for bracing and found it to be 3/4 foot long. How long was the original board? Explain your reasoning and support your answer.

The following specific ideas guided the creation of the four fraction problems:

1. Problem 1 is referred to as a “common numerator” problem. We maintain that if a student understands the inherent 1:1 correspondence between the numerators, the problem can be reconciled simply by examining the numerators. It is known that common denominators are useful in solving problem but problems where the use of common numerators is helpful to determine a solution are not often investigated. Furthermore, each of the remaining problems could also be solved with a common numerator strategy. For example, in Problem 4 if the fractions 2/5 and 3/4 were replaced by the equivalent fractions 6/15 and 6/8 then Problem 4 has a structure similar to Problem 1.
2. Problems 2, 3, and 4 are similar in structure in two important ways. First, they each can be modeled algebraically as $a \bullet x = b$, where a and b are known values. Second, the first fraction mentioned in each problem has 2 as a numerator. We were deliberate in using fractions with a numerator of 2 because children appear to use models to reach a solution more effectively when division by 2 is involved. We were interested if the prospective teachers would be able to effectively model their solutions and felt these problems might be easiest to model. However, while Problem 3 is similar in structure to Problems 2 and 4, the way the question is asked changes the thinking needed to find a solution.
3. The structure of Problems 2 and 4 suggest that a linear model would match the actions of the words. If a number line is an effective way to understand fractions, then perhaps PSTs would be able to use a linear model to answer these questions.

4. Explanations of models used to solve problems 2, 3, and 4 often use the idea that the first thing to be done is to divide by the numerator of one of the fractions and then multiply by the denominator of the same fraction. Effective modeling of solutions to these problems can help explain why "invert and multiply" makes sense when dividing fractions.

Results

The data collected from the PSTs are summarized in Table 1, organized by group (SMM, MA1, MA2) and by problem. The number of PSTs who scored 0, 1 or 2 for correct solution is reported for each problem. It should be noted that the majority of PSTs in MA1, who had not received instruction in fractions in this course, responded in ways that simply restated the information given, or indicated confusion ("I don't know where to begin"). Several expressed general confusion and misapprehension regarding fraction problems (for example, "I don't understand fractions at all"). PSTs in the other two groups had greater success.

Table 1. Success per problem per group

		SMM	MA1	MA2
Problem	Score	n=12	n=11	n=16
1	0	0	8	0
	1	3	2	5
	2	9	1	11
2	0	0	8	0
	1	0	2	3
	2	12	1	13
3	0	4	8	0
	1	2	3	10
	2	6	0	6
4	0	0	8	0
	1	2	2	7
	2	10	1	9

It was expected that the common numerators in Problem 1 would lead students to use the 1:1 ratio between fourth-liters of paint and fifth-square meters of wall; yet only one of the MA1s and only about 75% of each of the other groups answered it correctly. Though Problems 2 and 4 are similar in structure, only Problem 2 was answered correctly by all SMMs and most MA2s; fewer in both groups answered Problem 4 correctly. Furthermore, while the structure of Problem 3 is similar to Problems 2 and 4 it was answered correctly by half of the SMMs and less of MA2s.

Table 2. Number of students with correct answer and highest level of justification

	SMM	MA1	MA2
Problem	n=12	n=11	n=16
1	4	1	8
2	6	1	10
3	4	0	4
4	7	1	9

Table 2 below presents data showing the number of PSTs in each group who solved the problem correctly and whose explanation was scored at the highest level of justification. While the columns cannot be directly compared because the SMMs were expected to explain with a model and justification that went beyond getting the correct answer, it is of interest that both the MA2s and SMMs solved Problems 2 and 4 with the highest explanation more than Problems 1 and 3. In the discussion we only share student work on Problems 1 and 2 as the work on Problems 3 & 4 is similar to that of Problem 2.

Discussion of Work Samples: Problem 1

In problem 1 the common numerators led some easily to see a 1:1 ratio. One MA2 explained, "Since 3/4 L of paint covers 3/5 sq. m. of wall, 1/4 of a liter of paint covers 1/5 m^2 of wall. One will need 2 more 1/4 L to cover the remaining two 1/5 m^2 of wall that needs paint." A SMM used a diagram, symbolic representations and a written explanation (Figure 1) to describe this use of a 1:1 ratio. Another MA2 used two number lines (Figure 2) very effectively to solve this problem.

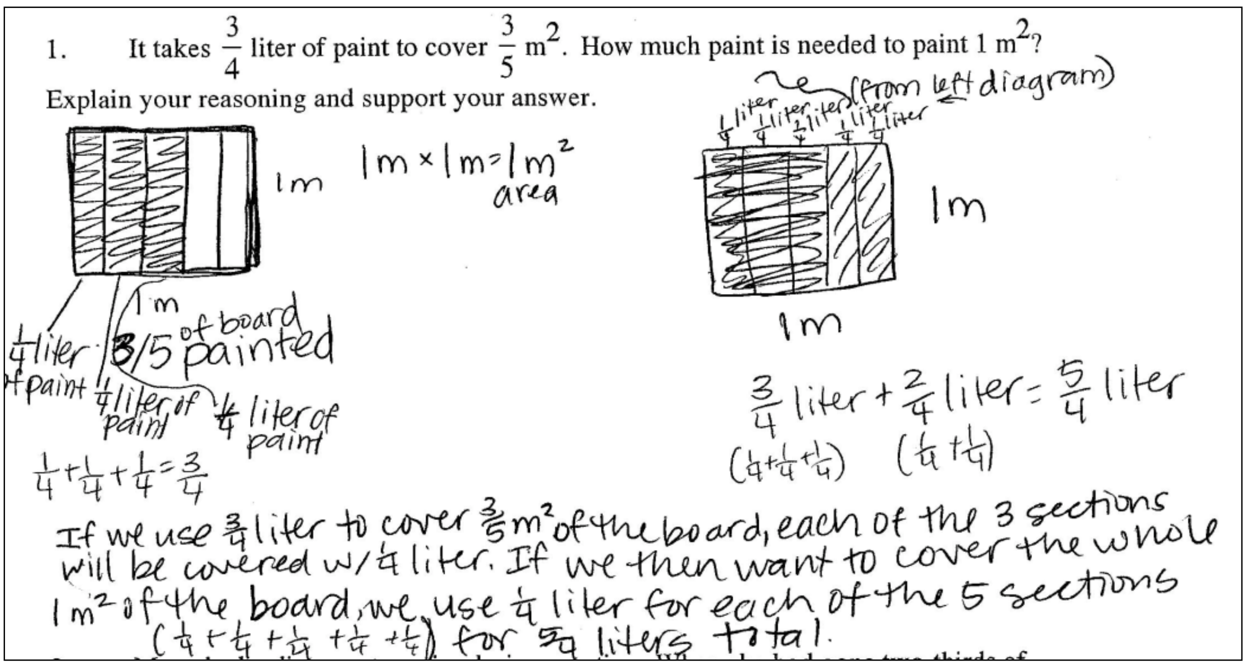

Figure 1. SMM Diagram and explanation

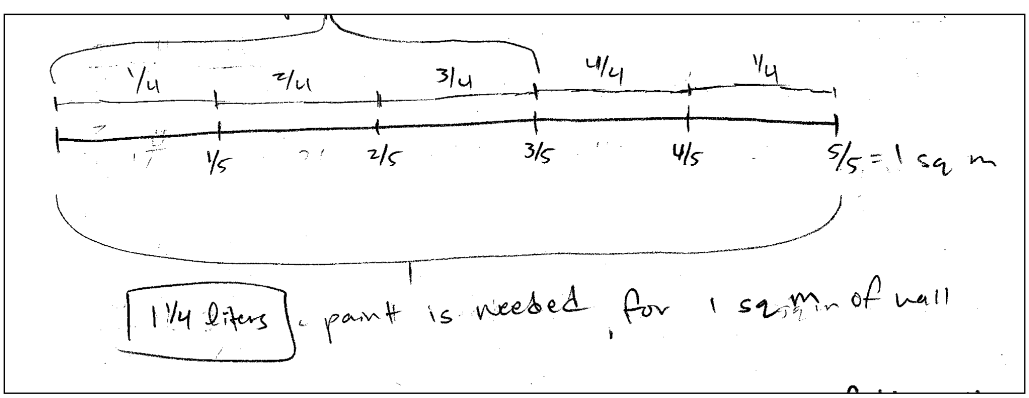

Figure 2. MA2 Simultaneous number lines

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

A variety of algorithms were also used to solve Problem 1, as shown in Figures 3-5. PSTs who solved the problem algorithmically provided no explanation, though some of the SMMs included a diagram. This seems to support the claim by Lesh, Post and Behr (1988) that students use algorithms, or procedural thinking, to avoid reasoning. However, the single MA1 who correctly solved the problem provided a detailed explanation of her work: “To do this problem, I divided 3/4 by 3 because it paints 3/5 of the whole wall. I then realized that 3/4 of a liter paints 1/5 of a wall. This is how I reached 1 1/4 liters to paint a whole wall…”

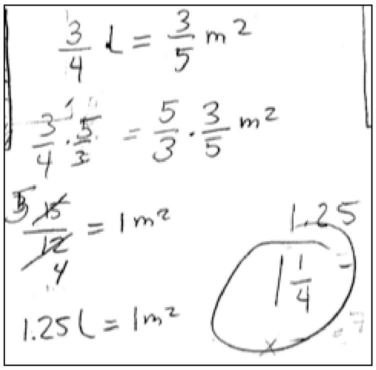

Figure 3. SMM Equation

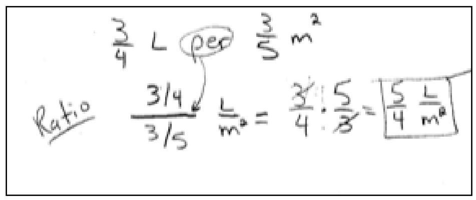

Figure 4. SMM Ratio

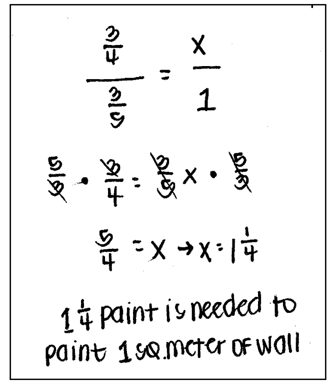

Figure 5. MA2 Proportion

One difficulty with procedural thinking is that if there is not true understanding, algorithms can be applied incorrectly. A few SMMs and several MA2s seemed to choose operations randomly, often beginning with finding common denominators for the two fractions. A MA2 tried several operations (Figure 6), stating that she tried adding the fractions, using “cross multiply” and “straight multiply”. No justification was given for any of these operations, nor did she explain why she chose the answer to the addition problem as the solution to the problem.

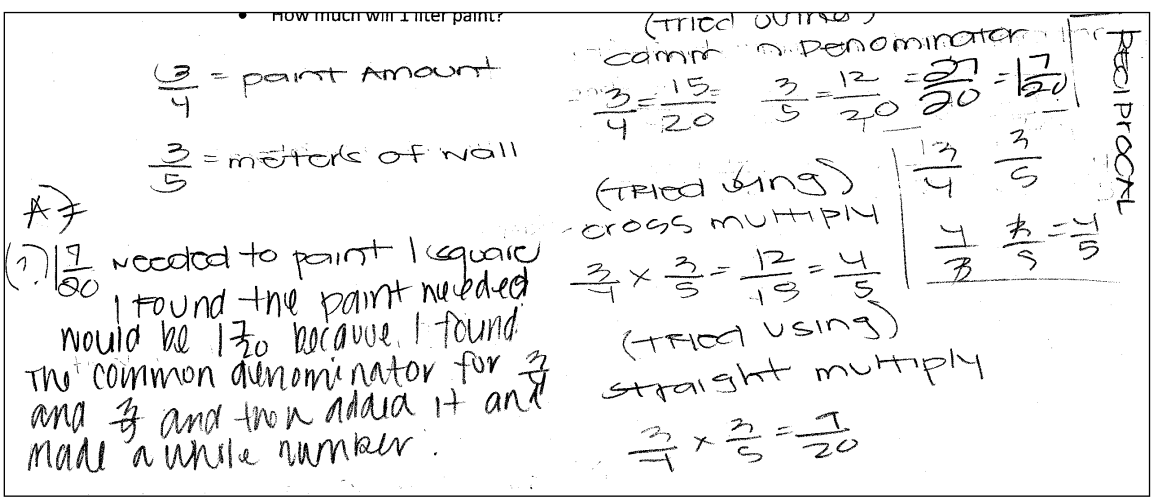

Figure 6. MA2 Three separate operations with no justification

Discussion of Work Samples: Problem 2

The majority of SMMs and MA2s were able to solve Problem 2 and justify their solutions. For the SMMs and MA2s who successfully solved this problem, drawing a diagram as shown in Figure 7 was an effective approach. As they had been directed, SMMs explained the models in more depth; one wrote "If we divide the distance into 3 equal lengths, we can label the total distance of the first two lengths segments as 1/2 km. We split/divide this into the 2 equal segments to be 1/4 km each. Since the last (3rd) segment is the same length, we label it as 1/4 km as well. We add the 3 segments up to equal 3/4 km."

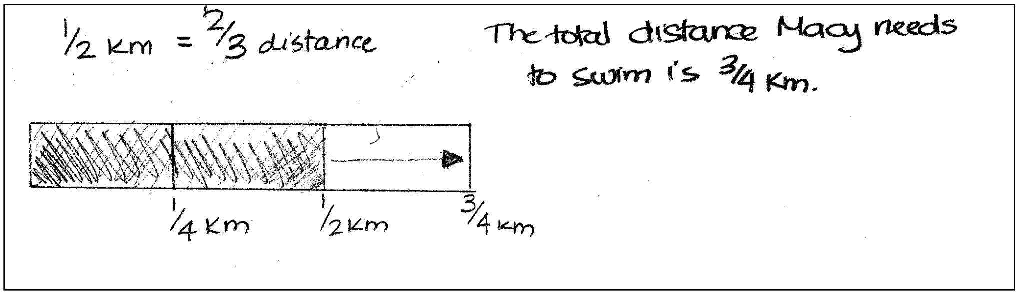

Figure 7. MA2 Diagram and solution

A few relied on an algorithm to solve this problem, and provided no justification (Figure 8). One SMM used a series of operations (Figure 9), similar to the work of the MA2 in Figure 6; there appear to be several "solutions" (1/4, 1/3, and the correct answer, 3/4). Two MA2s also appeared to choose operations at random, for example adding 2/3 and 1/2. This indicates no understanding of the relationship; 2/3 is the portion of the total distance and 1/2 km is the amount of that portion. Like others in each group who solved the problems incorrectly, these two PSTs arrived at an answer that could not possibly be right (1 1/6 km is more than double the portion already swum), but did not question their thinking.

Only one MA1 was able to solve this problem; two others made attempts that might have led to a solution. Eight were unable to even make an attempt, expressing confusion over the use of metric units ("I am having problems with understanding what type of distance she was swimming"), or requesting a formula.

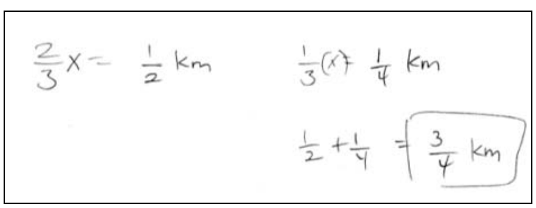

Figure 8. SMM Equation

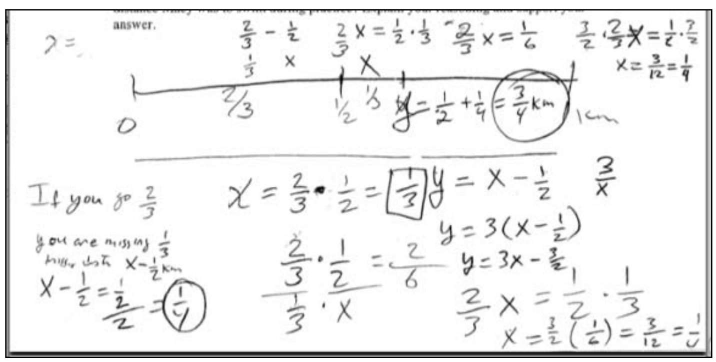

Figure 9. SMM Correct and incorrect "solutions"

Conclusions

It was discouraging to find that a large number of PSTs in this study had difficulty solving these fraction word problems. Most SMMs were successful, but given the fact that all have minors in mathematics, we expected all to be able to correctly solve the problems. About half of the MA2s experienced success, but only one of the MA1s even made an attempt at solving the problems. While it is true that this last group has not had any college mathematics prior to this semester (most are in their first year of college), all successfully completed mathematics up to and including high school algebra. Yet earlier research had found that fifth graders with no formal instruction in fraction arithmetic use the context of the worded problems to find solutions.

Most of the PSTs in this study did not use models to solve the problems. While some drew diagrams or pictures, as directed, many of these simply represent the words of the problem, and are not useful components of the solution. There are exceptions: The simultaneous number lines in Figure 2 represent an excellent model that leads to a solution and demonstrates conceptual understanding of the relationship between the units. However, most chose to solve the problems using algorithms such as equations or cross-multiplying to solve proportions. Some of those who were unsuccessful chose incorrect algorithms, and two (Figures 6 and 9), applied multiple algorithms to obtain multiple solutions; no rationale was offered. Most of the MA1s who did not even attempt to solve the problems stated that they required a formula, or that they just could not solve problems with fractions.

While SMMs were specifically directed to use models and sense making to show how they would help students understand, the explanations related to these problems involving fraction concepts and computations were not at the level we expected of students seeking a mathematics degree with certification to teach secondary mathematics. Many simply wrote and solved an algebraic equation; it is possible that they, like the U. S. teachers in Ma's (1999) study, felt this was sufficient justification.

Several of the MA2s and the single MA1 who successfully solved the first two problems provided models and/or explanations that demonstrated some conceptual understanding, rather than only reliance on algorithms. The second of two mathematics courses emphasizes representations and multiple solutions and the PSTs are required to explain and justify their work. The beneficial nature of that expectation is seen in the differences in the results reported.

Ma's research reveals the lack of what she terms *profound understanding of fundamental mathematics* in U. S. teachers. Teachers need to develop mathematical knowledge that has depth and breadth, to understand the connections among mathematical concepts and procedures, to use multiple approaches in solving problems, and to provide multiple explanations and representations for their students (Ma, 1999). Only teachers who have developed this *profound understanding* will be able to teach for understanding. This preliminary research points to the need for further investigation into PSTs' proportional reasoning, and to strategies that enhance their ability to solve problems and explain their thinking in ways that make sense mathematically and that use models or reasoning beyond algorithms to explain the solution to such problems.

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EVALUATING TAIWANESE PRESERVICE ELEMENTARY TEACHERS' UNDERSTANDING ABOUT THE MEANINGS OF FRACTION DIVISION

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This study investigated 40 Taiwanese preservice elementary teachers' understanding about the meanings of fraction division. This investigation was made through evaluating their performance on representing a symbolic problem of fraction division using words and reasoning how to solve the word problem using a diagram. Results indicated that a significant percentage of the preservice teachers were unable to construct an appropriate word problem for the given symbolic expression. Most word problems constructed by the preservice teachers match with the structure of equal-groups measurement. The most prevalent diagrams illustrated by the preservice teachers can be classified into the area and length models.

Introduction

This paper reported 40 Taiwanese preservice elementary teachers' understanding about the meanings of fraction division. As the National Mathematics Advisory Panel (2008) has identified, "proficiency with fractions" serves as a major goal for K-8 mathematics education because "such proficiency is foundational for algebra and, at the present time, seems to be severely underdeveloped" (p. xvii). Although "proficiency with fractions" is our target, research (Azim, 1995; Ball, 1990a, 1990b; Tirosh, 2000; Simon, 1993) has addressed that the understanding of fractions is a problematic area for elementary teachers especially when involving operations. In fact, while working on the topic of fraction division, preservice elementary teachers often concentrated on only remembering rules and mastering standard procedures rather than demonstrating comprehensive understanding of mathematical procedures (Ball, 1990a). For example, to divide one fraction by another, they only mastered the procedure, "$a/b \div c/d = a/b \times d/c = ad/bc$." Often, they interpreted fraction division as an opposite of fraction multiplication by turning the second factor (divisor) upside down and then multiplying.

To expand our knowledge on this problematic area from a broader perspective, we summarized Taiwanese preservice elementary teachers' understanding about fraction division in this paper. As suggested by mathematics educators (Barlow & Cates, 2007; English, 1998; National Council of Teachers of Mathematics [NCTM], 2000), Taiwanese preservice elementary teachers' understanding level about this challenging topic can be recognized by assessing their performance on word-problem writing and reasoning. The results gained from this assessment can be used to determine what teacher educators still need to know in order to effectively develop and provide better-quality teacher preparation to the preservice teachers.

Theoretical Framework

The theoretical framework for this research is grounded by the discussions of specialized content knowledge by Ball, Thames, and Phelps (2008) and paradigmatic knowledge by Bruner (1985). Specialized content knowledge refers to mathematical knowledge and skills needed uniquely by teachers. Examples of such knowledge include determining the validity and potential of nonstandard solution methods, and knowing the affordances and limitations of different types of diagrams in communicating mathematical ideas. In this study, it refers to

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preservice elementary teachers' knowledge in determining the most valid word problem to represent a symbolic problem and the most potential diagram to reason this word problem's solution. Paradigmatic knowledge focuses on "mathematical models or mathematical structures that are universal and context-free" (Chapman, 2006. p. 216). In relation to word problems, preservice elementary teachers' paradigmatic knowledge can be assessed through analyzing the semantic structure of word problems, the mathematical structures or models that are evoked in a word problem and its solutions.

As suggested by Ma (1999), a deep conceptual understanding of fraction division is built upon a network of prior knowledge. To categorize the developed word problems and their solutions of fraction division into structures and models, we should first conceptualize the meanings of fractions, whole number division, and its inverse operation – multiplication. For the operations of multiplication and division, Greer (1992) has identified four different structures of word problems: (1) equal groups, (2) multiplicative comparisons, (3) area and other product-of-measures, and (4) combinations structures. Of these, the structure of equal groups is the most prevalent one. Multiplication and division problems embedded in the structure of equal groups deal with certain number of groups, all equal size. Multiplicative comparisons problems deal with the situations that one set involves multiple copies of the other. In area and other product-of-measures problems, the multiplication product consists of a two-dimensional unit, such as a two- lengths (length × width) unit for the product of area. The structure of combination problems involves counting the number of possible pairings that can be made between two sets, but rarely involve the need to use the division operation to solve them.

Each of the equal-group and multiplicative comparison structures consist of three models of word problems: multiplication, measurement division, and partition division (Van De Walle, 2007). Of these, the measurement and partition models identified by Fishbein, Deri, Nello, and Marino (1985) must be considered when exploring division situations. For measurement division, one tries to determine how many groups of the intended quantity are contained in the given quantity (object or collection of objects). For partition division, the given quantity (an object or collection of objects) is divided into a given number of equal groups, and the goal is to determine the intended quantity (or size) in each group.

To make sense of fractions, three types of pictorial models are often adopted to represent the concepts of fractions: area or region, length, and set (Reys, Lindquist, Lambdin, & Smith, 2009; Van De Walle, 2007). In an area or region model, the fractions are based on parts of an area or region. Although the versions of area models can be various, Reys and his colleagues think that circular "pie" models are by far the most commonly used area model. It is interesting to know whether this version of area models is more prevalent than others from this study. With length models, lengths are referred instead of areas. Among various versions of length models, the number line has been treated as a more sophisticated length model (Bright, Behr, Post, & Wachsmuch, 1988). In set models, the whole is understood to be a set of objects, and subsets of the whole make up fractional parts.

In this study, through assessing Taiwanese preservice elementary teachers' performance on writing a word problem and reasoning out of a solution to this word problem using a diagram, their specialized content knowledge and paradigmatic knowledge of fraction division were investigated. Their word problems can be categorized but not limited to the following structures: (1) equal-groups multiplication, (2) equal-groups measurement, (3) equal-groups partition, (4) comparison multiplication, (5) comparison measurement, (6) comparison partition, and (7) areas.

Furthermore, their diagrams can be categorized but not limited to the following models: (a) area, (b) length, (c) set.

Methods

The data analyzed in this paper came from a research study on preservice elementary teachers' fraction knowledge. Due to absences, 41 preservice teachers answered the first-stage study and 40 preservice teachers answered the second stage. These participants were from a traditional teacher education university in Taiwan. All preservice elementary teachers in Taiwan have either a content concentration (e.g., mathematics, science, language arts, and social studies) or specialization (e.g., special education, elementary, and counseling education). The participants in this study are all non-mathematics and science concentration majors. They were juniors taking their 2-credit mathematics method course. This is the only required mathematics education related course for non-mathematics and science concentration majors of the teacher education program. The study was conducted at the beginning of this course.

The study in the first stage focuses on preservice elementary teachers' fundamental fraction concepts, and the study in the second stage focuses on their understanding of the meanings of fraction operations. This paper only focused on preservice elementary teachers' understanding about the meanings of fraction division. The data was based on the 40 preservice teachers' performance on a fraction division problem given in the second stage. These preservice teachers were invited to represent a symbolic problem of fraction division using words and reason how to solve the word problem using a diagram as follows:

> Write a word problem which can be solved by using "8 2/3 ÷ 1/4 =?" and model how to solve it by drawings.

The analyses of the preservice teachers' responses were guided by the following two clusters of research questions.

1) How successful were preservice elementary teachers in representing a symbolic problem of fraction division using words? What structures of multiplication and division did preservice elementary teachers use when constructing their representations? How successful were preservice elementary teachers in representing each structure of word problems?
2) How successful were preservice elementary teachers in reasoning how to solve the posed word problems using a diagram? What models of diagrams did preservice elementary teachers use when reasoning how to solve the posed word problems? How successful were preservice elementary teachers in reasoning solutions with respect to each adopted pictorial model?

Results

In this section, we report and discuss two major mathematical processes demonstrated by the Taiwanese prospective elementary teachers. The first process is representing a symbolic problem using words, and the second process is reasoning how to solve a word problem using a diagram.

Representing a Symbolic Problem Using Words

Only 26 of the participating 40 Taiwanese preservice elementary teachers (65.0%) were able to pose a word problem that is meaningful and correct for the given symbolic problem "8 2/3 ÷ 1/4 =?" Taiwanese preservice elementary teachers have been recognized to have excellent abilities to solve fraction division problems (Luo, Lo & Leu, 2008, 2009). For example, all preservice Taiwanese elementary teachers participating in the study of Luo, Lo and Leu (2009) can solve the word problem, “Jim jogged 1 1/2 miles yesterday. This is 3/8 of his weekly goal. How many miles does he plan to run each week? Explain”. This problem can be solved by computing the fraction division “1 1/2 ÷ 3/8.” Thus, in terms of fraction division, Taiwanese preservice elementary teachers’ ability to represent a symbolic problem using words is much lower than their ability to solve a word problem. Still, the Taiwanese preservice elementary teachers in this study have demonstrated a much greater understanding about the meanings of fraction division than their U.S. counterparts in Simon’s (1993) study. Simon found that only 30% of the U.S. preservice elementary teachers were able to create an appropriate word problem for the fraction division of 3/4 divided by 1/4.

As shown in Table 1, although the constructed word problems include various structures of multiplication and division, over a half (57.5%) of the Taiwanese preservice elementary teachers constructed an equal-groups measurement problem to represent the symbolic problem, “8 2/3÷1/4=?” This percentage is much higher than that of the preservice teachers who constructed an equal-groups partition problem (57.5% versus 15.0%). Further, there is a higher accuracy for the word problems embedded in the structure of equal-groups measurement than those embedded in the structure of equal-groups partition (87.0% versus 66.7%). This finding seems contrasting with previous research in fractions and division. The concept of equal-groups partition, which involves the process of forming fair shares, has been treated as the first goal in the development of fraction concepts (Van De Walle, 2007).

Table 1. The structures of word problems constructed by the preservice teachers

Structures	Frequency	Percentage	Correct Frequency	Correct Percentage
Equal-Groups Measurement	23	57.5%	20	87.0%
Equal-Groups Partition	6	15.0%	4	66.7%
Equal-Groups Multiplication	6	15.0%	0	0.0%
Comparison Measurement	1	2.5%	1	100.0%
Comparison Partition	0	0.0%		
Comparison Multiplication	1	2.5%	0	0.0%
Areas or Arrays	1	2.5%	1	100.0%
None (Skip)	2	0.0%		

It is interesting to find that the preservice teachers were more successful in using the structure of equal-groups measurement rather than the equal-groups partition to interpret the meanings of fraction division. Is it possible that their preference in constructing an equal-groups measurement problem rather than an equal-groups partition problem is because the given divisor is not a whole number? When the divisor is a whole number, it is often interpreted as the number of persons in a sharing context. This interpretation has been continually used to introduce fraction concepts in which the whole number is the denominator of a fraction and is often explained as the number of sharing persons. For example, to develop fraction concepts, the very first example provided in the book written by Van De Walle is, “Ten brownies shared with four children.” However, when the divisor is a non-whole number, the interpretation of the divisor as the number of sharing persons does not work in nature since a person is an undividable group. In this study, none of them interpreted the divisor as the number of dividable sharing groups such as “3/8 class” and tried to find the solution for the intended size of quantity for a whole group such as “a whole class.” For those who still successfully adopted the structure of equal-groups partition, rate contexts rather than sharing contexts were applied in their word problems like the sample W.2 shown in Table 2.

Same as the structure of equal-groups partition, 15.0% of the preservice Taiwanese elementary teachers constructed a word problem embedded in the structure of equal-groups multiplication. However, although most of their equal-groups multiplication problems can be led to the intended operation of division and the same final answer like the sample W.3 shown in Table 2, the direct operation embedded in an equal-groups problem is multiplication. Based on "knowledge package for understanding the meaning of division by fractions" proposed by Ma (1999), they only achieved the layer of meaning of multiplication with fractions or meaning of multiplication with whole numbers. They have not yet achieved the understanding of division by fractions. Since the inverse number of the divisor "1/4" given in the symbolic problem is a whole number, it is unknown whether those preservice teachers who adopted this structure tried to avoid the use of a fraction as a divisor while posing a word problem.

Table 2. Exemplary and non-exemplary samples ofword problems constructed by the preservice teachers

Structures	Exemplary (E) or Non-Exemplary (NE) Representation of Word Problems
Equal-Groups Measurement	(E) W.1: A big bag of rice is 8 2/3 kg. How many 1/4 kg bags can be made from this big bag of rice?
Equal-Groups Partition	(E) W.2: The baby brother ran from home to the school in 1/4 minutes. The school which is 8 2/3 meters away from his home. What is the baby brother's rate (how many meters per minute)? *Note.* This word problem is embedded in a rate context, not a sharing context.
Equal-Groups Multiplication	(NE) W.3: Some apples were shared among 4 people. Each person got 8 2/3 bags. How many bags (of apples) are in the beginning? *Note.* This word problem represents the symbolic problem, "8 2/3×4=?"
Comparison Measurement	(E) W.4: The baby brother has 12 snow fakes. His big brother has 8 2/3 times as many snow fakes as him. His big sister has 1/4 times as many snow fakes as him. How many times as many snow flakes does his big brother have as his big sister has?
Comparison Multiplication	(NE) W.5: A fan can run 8 2/3 rounds per hour. If the fan is slowed down into 1/4 of its original speed, how many rounds can be run per hour? *Note.* This word problem represents the computation, "8 2/3×1/4=?")
Areas or Arrays	(E) W.6: The area of a black board is 8 2/3 square meters with 1/4 meters in length. What is the width of this black board?

Reasoning How to Solve a Word Problem Using a Diagram

Overall, 22 of the participating 40 Taiwanese preservice elementary teachers (55.0%) were able to reason how to solve their word problems using pictorial illustration. Since only the data from those 26 preservice teachers who constructed a correct word problem can be considered as reasonable illustrations and solutions, the overall result also means that 55.0% of the Taiwanese preservice teachers could represent a symbolic problem of fraction division using words, and reason how to solve a word problem of fraction division using pictorial models. It is worth to notice that 84.6% (22 out of 26) of the presevice teachers who had successfully represented the problem in words were also able to solve it using pictorial models. Therefore, the presevice teachers who could write a word problem could potentially model the relevant pictorial solution.

Table 3. Pictorial models illustrated by the Taiwanese preservice teachers

Models	Frequency	Percentage	Correct Frequency	Correct Percentage
Area	16	40.0%	11	68.8%
Length	14	35.0%	10	71.4%
Set	2	5.0%	0	0.0%
Real Object	4	10.1%	1	25.0%
None (Skip)	4	10.0%		

Furthermore, as shown in Table 3, the percentage of the Taiwanese preservice teachers who adopted an area model to solve and reason a word problem is approximately the same as that who adopted a length model. It means the Taiwanese preservice teachers' ability or preference in adopting an area model is similar to a length model. This result supports the findings from prior studies by Luo, Lo, and Leu (2008, 2009) that Taiwanese preservice elementary teachers had equal abilities toward both models, which were different from U.S. preservice elementary teachers who were less prepared for the length model. They seem weaker or less preferred in applying a set model than the others.

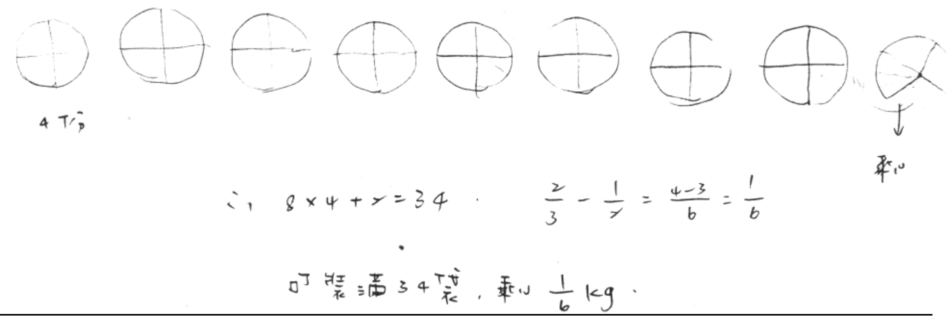
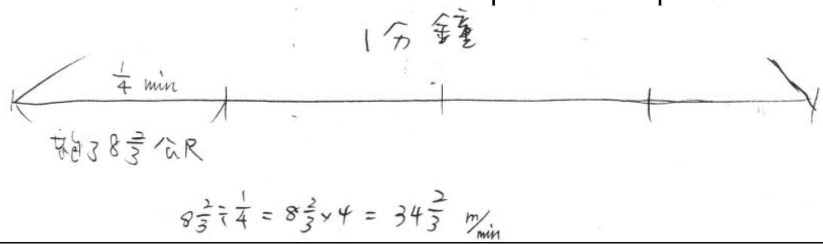
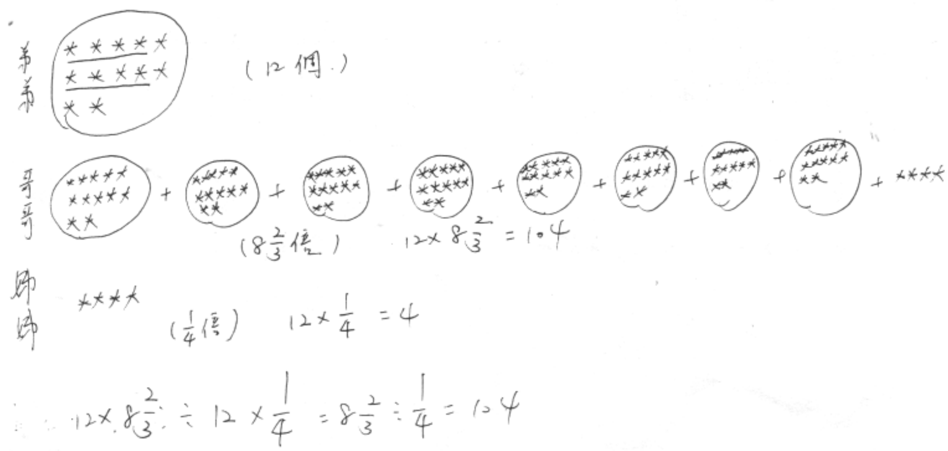
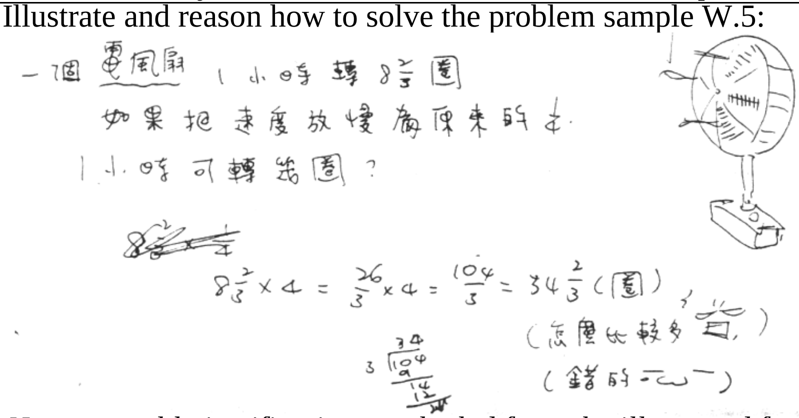
It is interesting to notice that most of diagrams drawn by the Taiwanese preservice teachers did not fully illustrate how to solve a word problem. Those preservice teachers still substantially depended on verbal or symbolic reasoning processes to complete an exemplary solution as samples shown in Table 4. In addition, like the diagram for the problem sample W.1 shown in Table 4, a diagram in a circular "pie" version described by Reys and his colleagues (2009) is still the most commonly used area model. Eight out of the 11 area model-based diagrams (72.7%) are in this version.

Implications

Although prior studies (Luo, Lo, & Leu, 2008, 2009) show that Taiwanese preservice elementary teachers were identified to be well-prepared in their common content knowledge as proficient mathematics problem solvers, it was found that they still need to be better prepared in their specialized content addressed by Ball and her colleagues (2008). Specifically, they have to be more proficient in the following mathematical knowledge and skills needed uniquely by teachers: representing a symbolic problem of fraction division in words, and reasoning how to solve a problem by depending on drawings only. They need to increase their familiarity and flexibility with some less popular structures of fraction division such as the equal-groups partition, comparison measurement, and areas. They need to learn how to reason the solution process using a set model as well as area and length models.

Some findings from this study could serve as starting points for future research. Answers to the following questions, for example, would be helpful in the development of efficient methods for instructing preservice teachers. First, will preservice elementary teachers perform better if they have more experiences in writing and reasoning word problems for fraction division? As previously stated, the Taiwanese preservice teachers attending this study were at the beginning of their mathematics methods course. They might lack experiences in writing word problems or reasoning their solutions using drawings. Thus, the results may more reflect their limitations in writing and reasoning word problems than in understanding fraction division. A follow-up experimental study would clarify whether the performance of writing and reasoning a word problem is determined by experience or by knowledge of fraction division.

Table 4. Exemplary and non-exemplary solution samples with pictorial models reasoned by the preservice teachers

Models	Exemplary (E) or Non-Exemplary (NE) Solutions with Drawings
Area	(E) Illustrate and reason how to solve the problem sample W.1: 
Length	(E) Illustrate and reason how to solve the problem sample W.2: 
Set	(NE) Illustrate and reason how to solve the problem sample W.4:  *Note.* No reasonable justification is led from the illustrated quantities.
Real Object	(NE) Illustrate and reason how to solve the problem sample W.5:  *Note. No reasonable justification can be led from the illustrated fan.*

Second, will preservice elementary teachers choose different structures or models when the numbers given in the symbolic division problem are different? This question is raised from the finding that most word problems developed by the Taiwanese preservice teachers can be classified into the structure of equal-groups measurement. As mentioned earlier, however, in most entry level curricula for division or fractions, this structure is not as popular as the structure of equal-groups partition which can be directly connected to sharing contexts. Further research needs to be undertaken to find what kind of division problems that preservice elementary teachers prefer to construct when the divisor is a whole number. If their problem structures are determined by the characteristics of numbers given in the symbolic division, it means they may have a difficulty to make a coherent transition from whole number division to fraction division. It also means they may need an intervention emphasizing a coherent transition from whole number division to fraction division to develop sufficient abilities to facilitate future students' understanding based on previous knowledge.

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FLEXIBILITY IN PRESERVICE TEACHERS' UNDERSTANDING OF AVERAGE

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A flexible understanding of the notion of measures of central tendency is an important component of educating people in mathematics. Based upon the responses of 27 preservice teachers (PSTs) to an averaging problem, we extended the categorizations of Mokros and Russell (1995) into four different conceptions of average: pairwise balancing, average as total balance point, reasonably close, and algorithmic. Using a U-M-R cycle from the SOLO model, this study analyzes the PSTs' ability to use alternate conceptions of average flexibly. In this study PSTs were presented with student artifacts to facilitate recognition of alternate conceptions and to reflect on which conceptions are most helpful in a particular situation.

Introduction

The notion of average, including the concepts of mean, median or mode, is an important tool in the study of statistics; it provides tools to represent data sets, and it can be especially useful in comparing data sets. A number of researchers (Mokros & Russell, 1995; Konold & Higgins, 2003; Watson & Moritz, 2000) have investigated student representations of average at varying levels of sophistication, however there has been relatively little attention paid to teacher conceptions of average (Shaughnessy, 2007). Effective teaching requires that instructors are sensitive to children's thinking, which includes the flexibility to recognize alternative conceptions of average in others (Thompson & Thompson, 1994). Artifacts of children's thinking can be used to both probe PSTs' currently held conceptions and to introduce alternate conceptions while helping PSTs learn the vital skill of dealing with children's strategies that differ from their own (Philipp, Ambrose, Lamb, Sowder, Schappelle, Sowder, et al., 2007). This study examines preservice teachers' (PSTs) knowledge of different representations of average and their level of flexibility in recognizing and using those representations when examining student work. PSTs' levels of flexibility are categorized using the UMR cycle of the SOLO model (Watson, Collis, Callingham & Moritz, 1995).

Theoretical Background

This study builds upon the work of Leavy and O'Loughlin (2006) that examined 263 undergraduate students enrolled in their first year of an education program in Ireland. They analyzed the PSTs' view of mean in terms of conceptual and procedural understanding through a written think-aloud protocol. Leavy and O'Loughlin (2006) found that the majority of the PSTs had a computational grasp of the concept of mean, as indicated by the relatively high performance on construction activities in which the direct application of the standard mean algorithm could be used. However, the future teachers struggled with problems that involved non-standard applications of the mean, such as creating a data set to fit a mean or working with weighted averages. The researchers concluded that the PSTs' relative lack of flexibility in solving nonstandard problems indicated a deficit in conceptual understanding.

Other authors have proposed multiple characterizations for the conception of average beyond conceptual and procedural. Based on the responses of 26 fourth, sixth and eighth graders, Mokros and Russell (1995) differentiated between five different types of student reasoning about average: mode, procedure, reasonable, midpoint and point of balance. Students were classified as having modal reasoning if they focused only on the most common value when summarizing data. Students had a procedural understanding of average if they relied on the “add and divide” algorithm to find the mean. A student was placed in the reasonable category if they used their own real world experiences to inform their thinking and if they placed the mean toward the middle of the data set. Students with a midpoint conception of average insisted on putting the same number of data points above and below the middle, as would be appropriate for a median. The choice in values was usually, but not necessarily, symmetric. Student who did try to balance higher and lower values around the mean were classified as moving toward a point of balance conception. No students in the Mokros and Russell study successfully completed the point of balance approach.

A common lens in statistical research education through which to categorize understanding is the SOLO model (Shaughnessy, 2007). Originally developed by Biggs and Collins (1982) for analyzing learning in any context, it was adapted to statistics education research by Watson et al. (1995). The SOLO model is a hierarchical structure comprised of five modes of reasoning, and within each of these modes the learner engages in U-M-R (Unistructural-Multistructural-Relational) cycles. At the unistructural level, a learner only gives attention to a single aspect of a task; a multistructural response includes several relevant elements, and at the relational level, the learner recognizes the relationships between relevant components and appreciates the relative importance of each.

Several other authors, including Watson and Moritz (2000) and Groth and Bergner (2006), have studied the concept of average using the UMR framework. In this study, a U-M-R framework was used to examine the PSTs’ flexibility in their conception of mean. Given the difficulty in measuring PSTs’ conceptual/procedural understanding, the U-M-R model provided the researchers better insight into PSTs’ multiple types of conceptions of average as needed to evaluate and guide student thinking. Although each concept of average, such as fair share or pairwise balancing, may have some procedural components, our primary interest was to explore the ways in which PSTs connect the data to a measure of center and how the concepts of average relate to student thinking.

Thus, the primary research question addressed in this paper is if PSTs are able to: (a) use alternate conceptions flexibly depending on the appropriateness of a situation; (b) recognize when another student’s work indicates the presence of a particular conception; and, (c) reflect on which conceptions are most helpful in a particular situation.

Methodology

The participants in our study were 24 PSTs enrolled in the second term of a three-term sequence in elementary mathematics at a large public urban university. Data was gathered through an initial written survey. The surveys were analyzed and then each PST was asked follow-up questions in a 15-20 minute videotaped interview. The interviews were then transcribed and reviewed.

The goal of our research is to understand what statistical conceptions PSTs have before beginning formal instruction, and therefore the surveys and interviews were completed before the PSTs began the probability and statistics portion of their elementary mathematics course. This paper focuses on the Potato Chip Task (first appearing in Mokros & Russell, 1995, p. 23) in Figure 1. Follow-up questions to the Potato Chip Task were designed by the research team, following our analysis of the survey responses. The follow-up questions were intended first to clarify which conceptions of average the PSTs were using when approaching the Chip Task. We chose to use artifacts of student thinking in order to explore PSTs' ability to recognize and work with alternate conceptions of mean. We believe that it is important for PSTs to be able to recognize their students' currently held conceptions of average and to purposefully decide whether to work within their students' current conceptions or to introduce alternate conceptions. The examples of student work provided a tool to assess PSTs' flexibility in dealing with multiple conceptions of average.

Figure 1

Potato Chip Task: Survey and followup questions

3. The average price of a bag of chips is 27 cents. We have seven bags of chips each of which has an empty price tag. Place prices on each of the bags so that the average price is 27 cents.

4. The average price of a bag of chips is 27 cents. We have seven bags of chips each of which has an empty price tag. Place prices on each of the bags so that the average price is 27 cents. **However, none of the bags of chips can cost 27 cents.**

Follow-up Questions

1. How would you help a student who had started number 4 as follows?

25	25	26		28	29	29

2. Below are two student solutions to question 4.

·Which solutions do you agree with and which do you not agree with? Explain your reasoning.
·What solution strategy do you think student (A) used? Student (B)? Explain why you think that.

Student A:

26¢	26¢	26¢	26¢	28¢	28¢	28¢

Student B:

1¢	1¢	1¢	1¢	1¢	84¢	$1.00

The conceptions of average used in this study were observed and categorized by a team of six researchers. Starting with the characterizations of average presented by Mokros and Russell (1995), the transcripts were reviewed independently by the researchers, and then the categorizations were compared and consensus was formed on the primary reasoning models. Several of the characterizations in the literature did not appear in this study; perhaps because of the tasks or perhaps because the subjects were PSTs and not children. For example, the PSTs in this study did not consider average as mode, perhaps because they were already aware of mode as a separate concept. After review, four primary conceptions of average emerged: pairwise balancing, average as total balance point, average as reasonably close, and algorithmic. Pairwise balancing involves picking a pair of numbers equidistant from the average value, whereas total

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balancing takes place over more than two values, such as going 2 below for one value and 1 above for two other values. Although Mokros and Russell used only a single balancing category, we found it useful to distinguish between two types of balancing approaches, as PSTs who demonstrated pairwise balancing did not necessarily have success with total balancing. This was not an issue in the Mokros and Russell study because although there was evidence that a few students were moving toward a balancing strategy, none of the students had fully understood how to complete the balancing to create an appropriate data set for a given mean (1995, p 33).

Our interpretation of average as “reasonably close” relates to Mokros and Russell’s concept of average as “reasonable.” Mokros and Russell classified student solutions as “reasonable” if they were based on the students own experience with real world situations rather than just a number found from a mathematical process. We classify students as having the “reasonably close” conception of average if they are considering the context of the problem and an appropriate level of variability when creating the data set, rather than focusing on a particular mathematical algorithm. For example, students who disapproved of Student B’s solution because the 1 cent prices were unreasonably low were classified as having a “reasonably close” concept of average.

Our final conception of average is algorithmic understanding, which is an extension of the category used by Mokros and Russell (1995). While Mokros and Russell limited algorithmic understanding to a direct application of the “sum and divide” algorithm, we will also include a “working backward” strategy in which the correct total is partitioned into pieces. For example, in this problem the total price for all of the bags would need to be $1.89 for the average to be 27 cents for seven bags. The “working backwards” strategy involves constructing prices that total to $1.89. Mokros and Russell included this strategy as part of “point of balance;” however we do not feel that any balance conception is required to find seven unrelated values that sum to a fixed total. Instead this strategy is a short-cut in the process of the “sum and divide” algorithm, where values can simply be summed.

The flexibility of PST conceptions of mean was analyzed using a UMR framework. At the unistructural level students have a single main conception of mean, such as mean as the outcome of an algorithm. The multistructural level refers to students who have multiple conceptions of mean. At this level, students may consider mean as balancing in one context and mean as fair share or algorithm in another context. However, at the multistructural level students do not articulate an awareness of the relationships between the multiple conceptions, and they may not be able to recognize that they have multiple conceptions. At the relational level students are able to relate multiple conceptions of mean and reflect back on which conception of mean is appropriate for a given situation. We argue that the relational level is especially important for PSTs because it allows them to make purposeful decisions when they choose which conceptions of mean to share with their students. In addition, their ability to relate multiple conceptions may influence how they work with their students’ alternate conceptions of mean.

This framework differs from that used by Watson and Moritz, primarily due to the focus on the flexibility of PST conceptions needed for effective teaching. Whereas Watson and Moritz only allow colloquial usage of average to be classified as unistructural, in the present study any student that indicates only one type of usage of average is classified as unistructural. Watson and Moritz also did not consider "R" as a relational understanding, which we argue is critical for PSTs. Our model also differs from the Groth and Bergner model. While their primary concern was the depth of understanding of the *definitions* of mean, median and mode, we are focused on how the PSTs can respond *flexibly* with different conceptions of mean based on student work.

Of the 24 PSTs who participated in this study, six were chosen as case studies for discussion purposes. Since this research explores the level of flexibility that PSTs have when approaching the concept of mean, case studies were selected to emphasize these different levels. Our analysis focuses on the thinking of Kira and Ursula at the U level, Phyllis at the M level, and Joyce, Greg, and Stacie at the R level. All of the names used in this paper are pseudonyms.

Data and Analysis

Among the 27 PSTs interviewed, the first two authors coded five as unistructural, thirteen as multistructural, and nine as relational. Within each of the levels, the PSTs exhibit a spectrum of abilities associated with each level. The following examples illustrate the representative abilities for each level, as well as some singularities.

Unistructural

Of the five PSTs with unistructural concepts of mean, four relied solely on the mean algorithm and one relied exclusively on the pairwise balancing technique. Our first example, Kira, is representative of the unistructural PSTs that indicated only an algorithmic understanding of mean. To solve the chips task, as she relied on guess and check as her solution method. To check her initial guess, she summed all the numbers and divided by 7; if the resulting average was too high or low, she raised or lowered one of the chip values and tried again. During the follow-up tasks, she realized that the sum of the numbers needed to be 189, without needing to divide by 7 to see the 27 cents.

When Kira was shown the first follow-up task in which a student uses a balancing approach, Kira did not recognize that this is a different tactic than the one she was using. When asked how she could help this student, she replied "just doing a lot of guessing and checking is the way that I would probably help them, but then again I don't know too much about like actually helping a child with probability questions 'cause I haven't done that before."

In response to Student A and Student B, Kira's first reaction was to sum each row of data and divide by seven. This led her to determine that Student A was incorrect and Student B was correct. She did not attend to notions of balance or spread, but relied exclusively on the outcome of the algorithm. When prompted to explain why some students might not like Student B's response, she did note that the data might be unrealistic. This did not seem to contradict her understanding of mean. Kira's conception of mean and interaction with the student responses was typical of the three other unistructural PSTs who relied on the algorithm of mean.

Ursula was unique in that she was the only PST that did not indicate knowledge of the algorithmic method for finding the mean. She stated, "I don't remember if there's some sort of formula to get the average." Instead she relied exclusively on the pairwise balancing technique. Ursula struggled in her approach to question 4, because she saw that her solution did not balance. In her description of the first student's response, her pairwise balancing technique was apparent; "I think it needs to be balanced ... it has to be under 27 and over 27, yeah under and over. I don't know why, it just seems like that would balance it out." When specifically working with the first student's response she said that it balances to 27 "because 25 is two under 27, and 29 is two over. So that's why it would be balanced." Because her concept of average relied on balancing in terms of pairs, she did not have a solution for how to find values for an odd number of items.

Ursula recognized that Student A's response was similar to hers, but also incorrect because it did not balance in a pairwise fashion. She did not think that Student B was correct because it did not balance. She states, “It seems like you could get a right answer with Student B, but not that answer….There’s 1,2,3,4,5 under and two over. But they are so low that it’s like… and these are so high!” Because Ursula only has a pairwise balancing conception of average, she not only fails to identify when students have alternate conceptions of average, but she also incorrectly identifies Student B’s solution as incorrect.

Multistructural

All thirteen of the multistructrual PSTs included the algorithmic concept of mean. In addition to that strategy, ten included the concept of pairwise balancing, one described a general balancing technique, and four indicated the “reasonably close” concept of average.

Phyllis was chosen as an example of a multistructural PST because she indicated algorithmic and “relatively close” understandings of average, although it was unclear that she understood the relationships between them. To solve the problem where none of the bags could equal 27 cents, she used her algorithmic knowledge of average to find the total of 27 times 7, and then she found “different combinations of different amounts” that sum to 189. This might indicate a deficit understanding of average in terms of pairwise balancing, for she was unable to see that 27 was the midpoint of each pair of numbers. In addition, when asked to offer advice to the student with the incomplete pairwise balancing technique, Phyllis could not deviate from her method. She said, “I would try to rework it from the beginning, and try different amounts from 10 cents to 50 cents.” Thus she may have trouble working with a student whose only conception of average relies on pairwise balancing.

In contrast, when evaluating Student A’s and Student B’s responses Phyllis liked Student A’s better because the values was closer to 27. Although Student B gave an exact average, she did not like that the values had a “big jump from 1 cent to 84 cents,” even though her original solution gave values from 3 cents to 79 cents. Thus although Phyllis favored a more algorithmic approach in the first problem, she favored a “reasonably close” view of average for this portion of the problem. She did not articulate a clear reason for her choice. Phyllis was placed into the multistructural category and not the relational category because of her inconsistency in selecting one strategy over another without being able to articulate her rationale for her choice.

Relational

PSTs at the relational level demonstrate multiple conceptions of average along with the ability to relate those conceptions to one another and make purposeful decisions about which conceptions are most appropriate for specific situations. PSTs with a relational understanding of multiple conceptions of average have the flexibility to recognize a student’s currently held conception and to work within that conception if appropriate, or to help the student move beyond that conception if necessary. Nine PSTs were categorized as relational because they offered some indication that one strategy would be more important in a certain situation, however there was a spectrum of abilities in articulating the relationships between the conceptions.

Three of the relational PSTs indicated that the algorithmic conception of average was faster, easier, or more accurate than other approaches. Joyce was an example of a PST that began using a general balancing strategy, but then abandoned it because it was too difficult. She stated, “I did 25, so that’s like -2. 28, and that was plus 3. Because I was going to try and keep track of it. And then I decided that took too long.” Instead she switched to the algorithmic approach of finding values that totaled to 189. These three PSTs were categorized as relational because they specifically indicated understanding of multiple conceptions of average, and they evaluated the merits of those approaches.

Six of the relational PSTs recognized that the pairwise balancing technique indicated in the first sample of student work was inappropriate. Greg is an example of such a PST. He approached the potato chip task using algorithmic understanding; he found seven values that would sum to 189. When asked about the student with the missing solution, Greg instantly recognized that the student was using a pairwise balancing technique, and he evaluated that this type of strategy would not work for an odd number of bags. He states, “What they have done makes sense if you are only using six chip bags, but they need the seventh one, so I think spacing it out just won’t work when it’s an odd number like that.” Although Greg recognizes that this particular technique is inappropriate, he does not believe that the student can modify their strategy into a correct one. He states, “If you want to help them, you’d kind of have to teach them a whole different way.” Thus Greg was able to identify the pair-wise balancing technique, evaluate that it was inappropriate, but he failed to indicated that a more general total-balancing strategy was possible. Instead, he recommended that the student adopt his algorithmic approach to solve the problem.

Stacie is another example of a PST in the relational stage in her concept of average. She initially solved the chip task by finding what the total cost needed to be and worked backwards to find 7 prices that totaled to 189. When given the first student’s response, she recognized that this student was having difficulty specifically because of the pairwise balancing strategy the student had chosen. However, instead of imposing her sum and work backwards strategy as appropriate for the student, she instead recommended modifying the pairwise balancing technique, “I guess telling them that you can add on to any of them.” It is important that unlike Greg, Stacie was not only able to recognize the student’s strategy, but was able to offer advice on how to complete the solution using a strategy similar to the student’s (total balancing), rather than her preferred strategy (algorithmic). Both Stacie and Greg are relational, but because Stacie understands the total-balancing concept of average, she is able to help the student by enhancing the student’s current approach into a more appropriate general balancing strategy.

Six of the relational PSTs stated that the “reasonably close” conception of average was false or unimportant. They maintained that as long as the mean of the values was 27, the values could be anything. This conception is classified as relational because it specifically states that the algorithmic approach to average is more important than “reasonably close.” For example, when asked to give advice to Student A, Stacie offers the following solution; “I guess [I would] explain that it could be more flexible than it having to be exactly off, just like, as long as all of them together add up to 189 the numbers could be anything.” These students recognize that the conception of average as values that are “reasonably close” exists, but prioritize the algorithmic formulation of mean for this context. They distinguish that data values do not necessarily need to be tightly clustered around the mean for the algorithmic mean to hold.

Conclusions

Although having a relational concept of average is helpful when working with students’ currently held conceptions, we have found that not all PSTs operate at that level. PSTs with a unilateral understanding of average may lack the ability to interpret the understanding of others. This may inhibit their ability to communicate with those who conceive of average in alternate ways. PSTs with a multistructural concept of average are much more successful in interpreting student work, and they may also have the stepping-stone skills needed to articulate a full concept of average to students. However they may not have the skills needed to identify when a student is using an inappropriate strategy and be able to guide that student using terminology that still fits within that child's cognitive model. PSTs with a relational conception have the ability to reason using multiple concepts of average, and they may also have the ability to identify and guide students successfully given a particular concept. However having relational understanding of several constructs of average does not imply full understanding of every concept of average.

Our study was limited to PSTs' investigation of a particular task involving the average price of seven bags of potato chips. This research could be extended by examining PSTs' conceptions of average in a variety of contexts. If a situation involves repeatedly measuring a single item, will PSTs use a different conception than for a situation that involves measuring a variety of different items? Do PSTs access different conceptions of average in situations involving sharing and fairness? What types of situations encourage PSTs to attend to variability as well as center? Given different tasks, we anticipate that some PSTs will demonstrate additional conceptions of average beyond the four outlined in our investigation.

One aspect of our research that was especially helpful in probing PSTs' understanding of mean was the use of student responses. These tools allow us a window into whether PSTs recognized alternate conceptions and strategies. They also provided an opportunity for PSTs to demonstrate whether they could use those alternate conceptions to help a student complete the given task, or whether the PSTs would revert to their original conceptions when assisting others. The use of student responses can be very helpful in examining PSTs' conceptions of average and it may also be a valuable tool for teacher educators to use when introducing new strategies to future teachers in their preservice mathematics courses.

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INVESTIGATING PRE-SERVICE TEACHERS' UNDERSTANDING OF AVERAGE IN USING MULTIPLE SOLUTION STRATEGIES

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While research continues to grow towards understanding pre-service teachers' knowledge of statistics, there is still a need to fully define what constitutes strong understanding of averages. This article investigates aspects of statistical content knowledge of 27 pre-service elementary school teachers as they worked on survey and interview tasks that allowed them multiple opportunities to discuss measures of center. Analysis of the data revealed that pre-service teachers could complete tasks requiring the construction of data sets from a given mean by utilizing multiple views of mean simultaneously. A discussion of the advantages and limitations of these solution strategies is presented.

Introduction

In 1989, the National Council of Teachers of Mathematics (NCTM) released the document *Curriculum and Evaluation Standards for School Mathematics*. This document was one of the first to highlight statistics as an area of focus in the K-12 mathematics classroom. As a result, the last two decades have seen an explosion in statistics education research, much of which has concentrated on defining and observing deep understandings of topics in elementary statistics. One such topic is conceptions of average. Current research points to a number of ways in which school children think about averages and suggests further research into how children's conceptions of average develop (Mokros & Russell, 1995; Konold & Higgins, 2003; Watson & Moritz, 2000).

In order to support students' thinking about the representative nature of the idea of average (Mokros & Russell, 1995), it is imperative that their educators possess a rich understanding of data analysis and the roles of central tendency, variability, and distribution in analyzing such data (Shaughnessy, 2007). For this reason, Shaughnessy (2007) has called for further investigations into the statistical knowledge of pre-service teachers (PSTs); a focus of the present study.

Many statistics educators agree that simply knowing the algorithm for and finding the arithmetic mean are not sufficient for developing a rich and flexible notion of average (Watson & Moritz, 2000; Groth & Bergner, 2006). However, varying opinions exist as to what *does* constitute such an understanding. Several authors have taken the stance that a contextual view of measures of central tendency is one key to developing a more complete sense of average. Konold and Higgins (2003) called for young students to build knowledge through experiences with "real data", through which they will draw connections between certain statistical concepts and their applications to context. Watson and Moritz (2000) conducted a longitudinal study with students to discover the nature of their understanding of average, and suggested that emphasizing context, in addition to cultivating an ability to utilize the algorithms for measures of center, could be one critical piece of developing higher levels of understanding. This emphasis should allow students the opportunity to explore connections between a “specific measure of central tendency and possible data sets that could have created it” (p. 48).

Mokros and Russell (1995) presented a categorical framework for discussing students' methods in constructing data sets from an average. They suggest five approaches to solving these problems: average as mode, algorithm, reasonable, midpoint, and point of balance. Average as mode is the inflexible use of the most often occurring value in the data in solving problems related to average. This and average as algorithm (strict adherence to the mean algorithm in solving problems) are not considered to take representativeness of the mean into account, whereas the other three methods are. The midpoint approach involves symmetrically choosing points above and below the mean, often believing that the mean and middle number are synonymous. The balance approach sees the mean as a point of balance, rather than a middle number and takes “into account the values of all the data points” (p. 25). Though the authors explicitly divide the methods by those that see average as representative and those that do not, balance and midpoint are considered the more advanced than average as reasonable. In their study, none of the students were able to construct a data set from a given mean when the mean value could not be used in the data set.

This study aims to look more in depth at the framework presented by Mokros and Russell (1995). Their work focused on young children; the goal of the current study is to determine whether their framework is appropriate for use with PSTs and to modify this framework so as to appropriately characterize PSTs’ thinking about measures of center.

Methods

Participants of this study included 27 PSTs enrolled in a 10-week course at a large urban university in the Pacific Northwest. The course was the second part of a 3-term sequence designed to give prospective teachers the chance to explore different mathematical topics through a hands-on, activity based approach to mathematics. The research team took advantage of an opportunity to investigate PSTs' statistical reasoning prior to statistics instruction in this sequence. Data sources for this project consisted of written surveys and interviews.

A statistics assessment survey was administered to the entire class prior to their statistics unit. Using the framework of Mokros and Russell (1995) as an initial guide to analysis, the current study aimed to gain deeper insight into PSTs understanding of average. The Potato Chip Task shown in Figure 1 was borrowed from their study and forms the basis of the current report. The purpose of using the Potato Chip Task was to see the ways in which PSTs would reason about average and the ways in which they would construct data sets to suit a particular mean.

A) The average price of a bag of chips is 27cents. We have seven bags of chips each of which has an empty price tag. Place prices on each of the bags so that the average price is 27 cents.

B) The average price of a bag of chips is 27cents. We have seven bags of chips each of which has an empty price tag. Place prices on each of the bags so that the average price is 27 cents. **However, none of the bags of chips can cost 27 cents.**

Figure 1. The “Potato Chip Task”

After the surveys were analyzed, PSTs were invited to participate in short (20 minute) clinical interviews. The interviews were designed to further investigate PSTs' thinking about measures of center, and to provide the researchers an opportunity to follow-up on questions that were raised during the analysis of the survey data.

Some of the interview tasks included asking PSTs to further explain their responses to the survey, and some new tasks were created to further explore PST responses to the Potato Chip Task (see Figure 2). The interviews were videotaped and transcribed for further analysis.

How would you help a student who had started number 4 as follows?

25	25	26		28	29	29

Figure 2. Sample Interview Task 1

All 27 interviews were initially analyzed with respect to Mokros' and Russell's (1995) framework, and it was found that some modification to this framework was needed to accommodate the approaches of PSTs. A discussion of these modifications follows. All names used in this report are pseudonyms.

Results

The analysis of PSTs’ work on the Potato Chip Task showed statistical reasoning closely related to the framework of Mokros & Russell (1995). Although their framework was useful in building an initial image of PSTs’ statistical reasoning, many of the PSTs strategies extended beyond the scope of the originally established categories. Though Mokros and Russell (1995) observed that PSTs who viewed average as a midpoint had difficulty transitioning from a given mean to a data set when the mean could not be used, this was not usually the case in the present study. One would expect PSTs, that is, undergraduate college students, to have had experiences working with averages for twelve or more years of their lives. It seems plausible, therefore, that PSTs have multiple approaches to problems involving measures of center and most were able to successfully construct data sets given a specific mean regardless of whether or not the mean could be used as one of the data points. PSTs may, however: (1) be unaware that they hold multiple conceptions of average; (2) experience difficulty applying appropriate conceptions of average in a particular context; and/or, (3) be unable to clearly articulate their different approaches. The authors of this paper argue that PSTs need to be aware of multiple conceptions of average and be able to articulate the differences between solution strategies in order to teach ideas of measures of center coherently.

What follows is a description of the modified framework of Mokros and Russell (1995) for use with PSTs. Described in detail below (Table 1) are the different characterizations of average used by PSTs in this study and how they are similar to and different from those of Mokros and Russell (1995). Excerpts from four interviews are shared in order to clarify categories in the new framework. Modifications are italicized. Average as Backward Algorithm and Average as Cluster Balance are not found in Mokros’ and Russell’s (1995) initial framework. Average as Forward Algorithm, Average as Midpoint, and Average as (Total) Balance were modified from Mokros’ and Russell’s original framework to consider strategies utilized by PSTs. The modified Average as Midpoint addresses a tendency for PSTs to view constructing 27 as the middle number in a pair-wise fashion, often reverting to the algorithm in order to express as much (for example, the average of 26 and 28 is 27). This micro use of the mean algorithm is also seen in Average as (Total) Balance, but the overall solution approach of PSTs differs in that they appear to consider the data as a whole rather than just subsets of it. This transition from subsets to the overall data set is expressed in the modified (Total) Balance category.

Table 1: Modified categories of PSTs' view of averages

Average as Algorithm (as previously defined by Mokros and Russell)

- View finding an average as carrying out the school-learned procedure for finding the arithmetic algorithm.
- Often exhibit a variety of useless and circular strategies that confuse total, average, and data.
- Have limited strategies for determining the reasonableness of their solutions.
- *Use of algorithm in guess and check for choosing data points when creating a distribution.*

Average as Backward Algorithm (developed by research team as a result of analysis)

- *View finding an average as carrying out the school-learned procedure for finding the arithmetic mean.*
- *View finding data sets corresponding to the mean as reversing the school-learned procedure for finding the arithmetic mean. In which case they multiply the given mean by how many data points then choose numbers that sum to this product.*

Average as Midpoint (modified by research team as a result of analysis)

- View average as a tool for making sense of the data;
- Choose an average that is representative of the data, both from a mathematical perspective and from a common-sense perspective.
- Look for a "middle" to represent a set of data; this middle is alternatively defined as the median, the middle of the X axis, or the middle of the range;
- Use symmetry when constructing a data distribution around the average. They show great fluency in constructing a data set where symmetry is allowed but have significant trouble constructing or interpreting nonsymmetrical distributions.
- Use the mean fluently as a way to "check" answers. *Have a tendency to compare pairs of numbers, making sure the mean is the middle of the pair. May* believe the mean and middle are basically equivalent measures.

Average as Cluster Balance (developed by research team as a result of analysis)

- *View the mean as a tool for making sense of the data.*
- *Views the data in the form of subsets/clusters rather than the whole distribution.*
- *Use the mean with a beginning understanding of the quantitative relationships among data, total, and average; they are able to work from a given average to data, from a given average to total, from a given total to data.*
- *Being able to view average in the form of midpoint, but as restricted to subsets.*

Average as *(Total)* Balance (modified by research team as a result of analysis)

- View mean as a tool for making sense of the data.
- Look for a point of balance to represent data.
- Take into account the values of all the data points.
- Use the mean with a beginning understanding of the quantitative relationships among data, total, and average; they are able to work from a given average to data, from a given average to total, from a given total to data.
- *Break problems into smaller parts and find "sub-means" as a way to solve more difficult averaging problem. Besides just comparing subsets of the distribution, the PST is able to generalize from looking at 2 to 3 data to the overall distribution of data points; rather than looking at small subsets of the data, they can articulate how groups of the data relate to the overall distribution.*
- *View balancing through multiple subsets (i.e. 2 or more points) to generate the distribution.*

Average as Backward Algorithm

While many PSTs were able to apply the algorithm of the mean to check their work, it was necessary to consider the knowledge required to reverse the mean algorithm. An individual had to be aware of undoing division in the mean algorithm through a multiplication and then undoing the sum through a method of repeated subtraction; this is highly algebraic and perhaps more readily used by the PSTs given their additional mathematical experiences. The reversing strategy was done by 17 of the PSTs. These took the mean of 27 that was given in the Potato Chip Task and multiplied by 7, giving 189, which they used to redistribute prices to the 7 bags of chips. One example was Lisa.

> Lisa: Uh...I was doing...I know...okay...so I made the 4 bags 1 cent higher and start with that and I got 112. And that was for the four bags. And then...(mumbles)...and then I just did below the 27 cents cause I know I had to get 27 so I get 3 more bags. So I had to find what would equal umm...the 189 from the...and the 112 and 189. And ended up getting 26 and 26. And I decided to figure out um...that was 152 so I subtracted the 189 and 152 from the 189 and that was 25. I think...maybe not...kidding. 164. Yeah 164...25 remain so that was my seventh bag.

As mentioned, Lisa was aware that the total of 189 was needed in order to produce the correct mean. Her method of distributing the numbers to the different bags of chips required a repeated subtraction of numbers from 189 followed by constant checking of her values to make sure the sum remained 189. When distributing the prices of bags, some PSTs considered the idea of “reasonable” values. Lisa appeared to be staying near the value of 27 when subtracting her chosen values from 189. In the context of the Potato Chip Task, some, like Lisa, maintained that it did not make sense to select numbers too far from 27, and none chose negative values or fractions of cents. Therefore, ideas of representativeness and context may play roles in such problems for some PSTs.

Average as Midpoint

The midpoint view of balancing relies on the knowledge that data sets must focus on the mean as a center. Through this view, PSTs tended towards a strategy of looking at pairs of numbers and checking how values related to the mean. 12 PSTs had a tendency to look at how two numbers related to each other (e.g. one number is 1 above the mean and another is 1 below) and in certain cases take the average of those two numbers in order to verify its similarity to the given mean. Gene was an example of one PST who used this strategy. He responded to part B of the survey task as follows: 20, 25, 26, 28, 29, 30, 31. He described his solutions to part A and B of the survey tasks as follows:

> **Gene**: Umm...where 27 was used. And I realized that, you know, by comparing these like 27 in the middle you know you are gonna have 26 and 28 are gonna average to 27. 25 to 29 are gonna average to 27. 30 to 24 are gonna average to 27.[…] except now we couldn't use 27 umm...as one of the price tags. I just started with 28. Umm...and then I just kind of went out from there. So then I, 26 and 29. […] Umm...you know skipping 27. And then 25 and 30. And then umm...31 going up and then, you know, just have figure out the one last number […] Umm...I kinda took a guess at 20 based on umm...you know the differences and

numbers there. And 31 being so much higher. Umm...I sorta took a guess at it. And I figured 20 was probably about right. And so then I plugged in and it worked out.

The strategy Gene used for part B was a midpoint approach, as he explicitly mentioned 27 being the "middle" and then symmetrically distributed his data above and below this value. He moved outwards using pairs that he observed as averaging to 27. The students in Mokros' and Russell's (1995) study using this pairing strategy were unable to complete part B; however, Gene effectively altered his approach by changing his middle number to 28, using two of the same lower values (25, 26) and pushing his upper values up by one (29, 30, 31). Gene's way of assigning the new middle number and finding the remaining value of 20 was done through a "guess and check" utilization of the mean algorithm that he explicitly mentioned. This is consistent with Mokros and Russell's original framework for a Midpoint View where PSTs consistently use the mean fluently to check their answers. It is important to note that his pairing strategy focused solely on looking at two data points at a time. As Gene got closer to finding the final data point, he changed his method to checking through use of the forward mean algorithm rather than simply extending his pairing strategy.

Average as Cluster Balance

While several of the PSTs were able to apply a pairing strategy when finding data points, individuals such as Gene were unable to extend the pairing strategy to the last value. J.Z., on the other hand, extended his pairing strategy differently than Gene. His solution for part B was (28, 28, 26, 26, 29, 26, 26), but he described his process differently.

J.Z.: So, and I had started by going with two bags that had 28 cents and two bags that had 26 cents. And since there is an odd number of bags all together I knew that I was going to have to use something other than 28 and 26 for the last three boxes, or bags. Umm. So I didn't really have a precise way to do that, so I just went with 29 and two 26's. […] 28 and 26, you know, when you add them up together and divide by 2 you get 27, I believe. […] I figured if I was going to use 26, then I would have, if I was going to use 26 twice, then I would have needed something substantially higher than 28 to make it 27.

J.Z. started this problem by seeing a pairing of two values of 26 with two values of 28, each of which was one cent away from the desired mean of 27 cents. J.Z. was averaging each pair using the mean algorithm. This is indicative of the Midpoint View. However, J.Z. considered three values to be his midpoint, viewing the last three boxes as the place where his strategy needed altering in order to solve the problem. J.Z.'s ability to extend beyond looking at pairs of numbers to small subsets that act as a single center indicates a more flexible strategy than the original Midpoint of Mokros and Russell (1995) and a shift towards considering the overall distribution.

Average as Total Balance

The Total Balance view is a modification of Mokros & Russell's (1995) final category. Rather than simply calling it a balance category, we chose total balancing to indicate PSTs who are seeing the overall distribution. While we view Cluster Balance as more sophisticated than Midpoint View, Total Balance indicates an additional level of sophistication from applying a pairing strategy to smaller segments of the data. PSTs who possessed this view of average are capable of extending from individual pairs of data points to balancing subsets of data values to considering the total data set as a balanced entity. One such example is Belle who provided the solution of 26, 26, 29, 26, 28, 26, 28 part B of the Potato Chip Task.

Belle: Umm...I did number 3 the easy way of having if they are all 27 cents then the average is going to be 27 cents. And then for number 4, since none of them can cost 27 cents, I put them on either side. 26 and 28 cents. But because there was an odd number, I needed to balance one more of the 26. So that's why one of them was a 29. So I have 4 26s....so there's a pair of 26 and 28, there's a pair of 26 and 28, and then these three...the 29 balances the 26. The two 26s.

In her description, Belle explicitly introduced the term “balancing” as attributed to the two bags labeled with 26 cents and the single one labeled 29 cents. Belle saw a value farther away from the mean as requiring two values closer to 27 cents in order to balance the data; she had moved beyond her initial pairing method into being able to find the triple. Lastly, she was able to say in her excerpt that the four 26s balance with the three values 28, 28, 29. Her reasoning regarding her final distribution took into consideration how overall groups compared rather than small segments of the distribution.

Table 2 (below) shows the number of PSTs observed to fit into the categories as described in Table 1. Please note that some PSTs strategies fall into more than one category and are tallied as such.

Table 2. Summary of PSTs views of averages pertaining to the “Potato Chip Tasks”

Categories	Number of PSTs
Average as Forward Algorithm	16 (59%)
Average as Backward Algorithm	17 (63%)
Average as Midpoint	12 (44%)
Average as Cluster Balance	6 (22%)
Average as Total Balance	3 (11%)

In summary, many of the PSTs were capable of doing a Backward Algorithm. The Backward Algorithm seemed rather natural for many of the PSTs, perhaps due to more experience working with the forward algorithm or other algebraic manipulations. Twelve of the 27 PSTs were able to apply a Midpoint view of average through the use of a pairing strategy, but returned to using the Forward Algorithm to check their work. Surprisingly only a small percentage of PSTs were able to apply Cluster Balance to complete the construction tasks. Half of the participants who use a Cluster Balance approach were able to explicitly show evidence of Total Balance. The PSTs were rarely seen to relate their pairing or tripling strategies to the distribution as a whole, hence the small representation in the Total Balance category.

Discussion

It should not be surprising that many PSTs were found to have utilized more than one strategy in their work with averages, or that hybrid categories should exist for this population. PSTs have had more experiences in mathematics than grade schoolers, and draw on these experiences in completing tasks like The Potato Chip Task. The primary benefit of this extended experience is that PSTs often find unique ways of creating a solution that is also replicable. Authors such as Watson and Moritz (2000) maintain that this ability to replicate solutions is a necessary skill for a more complete understanding of average. It should be noted, however, that in most instances the PSTs were unable to completely articulate their solution process. This is seen in the above excerpts where all but Belle struggled significantly in coherently describing their methods. This ability to articulate a solution strategy is imperative for PSTs as they plan to become future elementary school teachers, and complete and comprehensible explanations of mathematical methods will be required of them.

This study is but a jumping off point for what we hope will be a basis for further analysis of PSTs' knowledge of measures of center. The preliminary analysis found in this article shows that many PSTs possess the flexibility to utilize different notions of average in conjunction with one another in order to persevere in solving the task. Future studies should investigate how PSTs develop the ability to articulate different strategies for solving mean problems and a recognition of different strategies in examples of student work, as these are important skills for PSTs in order to effectively teach measures of center.

Although this paper is a preliminary study that focuses on PSTs understanding of the mean, it is important to consider how new strategies of thinking of mean may affect how PSTs think of variation. If PSTs are capable of implementing a flexible strategy when solving tasks involving the mean, then this may give educators an opportunity to see how PSTs attend to variation. Therefore, we suggest subsequent studies to investigate PSTs' responses to context-based tasks that highlight the use of these new strategies in conjunction with notions of variability. In order to accomplish this, tasks must be carefully constructed for the PSTs to construct or investigate data around given means with data that is not so artificial. 'Messy'data sets (Mokros & Russell, 1995) might help PSTs consider the conception of the mean as it relates to the variability of real data.

Further investigation must also study the importance of the relationship between context and measures of central tendency. Watson and Moritz (2000) presented a similar problem to the Potato Chip Task when they asked children to develop a data set for the number of children in ten families with an average of 2.3 children. A couple of striking differences are seen, the primary being that 2.3 is not a possible number of children. While the Potato Chip Task asked PSTs to construct data without a value, the value being neglected was reasonable for a bag of chips to be priced, allowing many to initially set this value as the midpoint. In this context, PSTs may find 2.3 a completely unreasonable value to work from. This leads one to question whether the techniques that have been described above are affected by context. It should also be noted that the Watson and Moritz (2000) question had pre-existing values that the students had to take into account (4 children in one family and 1 in another). The question of whether pre-existing data values limit the construction of a data set requires investigation as well.

The future of statistics education depends on discovering what PSTs know, and building on their existing knowledge of central tendency, variability, and distribution in a constructive way that allows them to articulate their ideas thoroughly.

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KNOWLEDGE AND CONFIDENCE OF PROSPECTIVE ELEMENTARY TEACHERS IN CHINA AND SOUTH KOREA

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In this study, we investigated the extent of knowledge in mathematics and pedagogy that 361 prospective elementary school teachers in China and 291 in South Korea have learned and what else they may need to know for developing effective classroom instruction. We focused on both prospective teachers' (PT) confidence about their knowledge preparation and the extent of their knowledge on the topic of fraction division. The results reveal a gap between these PTs' confidence and their knowledge in mathematics and pedagogy for teaching, as an example, fraction division. Moreover, PTs in China and South Korea shared many similarities in their knowledge and confidence but differed in other aspects. The results suggest possible connections with the training programs provided in these two education systems.

Introduction

It is generally recognized that Chinese elementary teachers have a profound understanding of the fundamental mathematics they teach (e.g., Li & Huang, 2008; Ma, 1999). The results have led to further interest in understanding how Chinese elementary teachers are prepared through their teacher education programs and whether different high-achieving education systems in East Asia share similar best practices in teacher preparation (e.g., Li, Ma, & Pang, 2008; Li, Zhao, Huang, & Ma, 2008). While recent studies suggested that high-achieving education systems in East Asia actually have developed and used different policies and practices in preparing mathematics teachers (Leung & Li, 2010; Li, Ma, & Pang, 2008), it is inevitable to question whether prospective teachers are similarly well prepared through different teacher preparation programs practiced in different high-achieving education systems. As a part of a large international research study of prospective mathematics teachers' knowledge development in mathematics and pedagogy, this article focuses on the case of prospective elementary teachers from two high-achieving education systems in East Asia: China and South Korea. Specifically, 361 prospective elementary school teachers in the Chinese Mainland and 291 in South Korea were sampled for a survey to examine their knowledge of mathematics and pedagogy for teaching in general and on the topic of fraction division, in particular.

The selection of the content topic of fraction division is mainly based on two considerations. The first consideration is about the nature of the content topic itself. Mathematically, fraction division can be presented as an algorithmic procedure that can be easily taught and learned as "invert and multiply." At the same time, the topic is conceptually rich and difficult, as its meaning requires explanation through connections with other mathematical knowledge, various representations, and/or real world contexts (e.g., Greer, 1992; Li, 2008). The selection of the topic of fraction division, as a special case, can thus provide a rich context for exploring possible depth and limitations in prospective teachers' mathematics knowledge for teaching (e.g., Ball, 1990; Ma, 1999; Tirosh, 2000). The second consideration relates to some previous studies. For example, several researchers focused on the content topic of fraction division in their investigation of Chinese practicing elementary teachers' knowledge (e.g., Li & Huang, 2008; Ma,

1999) and of U.S. prospective mathematics teachers' knowledge (e.g., Li & Kulm, 2008). Relevant findings can thus be used to help understand possible development in Chinese teachers' knowledge after beginning their teaching career and possible cross-national similarities and differences in prospective teachers' knowledge between the U.S. and the two educational systems in East Asia.

Theoretical Framework

In a recent assessment of prospective teachers' knowledge in mathematics and pedagogy for teaching (Li & Kulm, 2008), a conceptual framework was developed and used to capture both teachers' confidence as well as their mathematics knowledge for teaching (MKT). Different from some other studies that focus on teachers' beliefs and perceptions of mathematics knowledge (e.g., Schmidt et al., 2007), Li and Kulm focused on prospective teachers' confidence of their knowledge preparation and readiness. Results from assessing both U.S. and Chinese prospective teachers' confidence and mathematics knowledge for teaching suggested the importance of such a specification on teachers' confidence (Li & Kulm, 2008; Li, Ma, & Pang, 2008). In particular, these investigations of both U.S. and Chinese prospective teachers' perceptions and knowledge revealed what we did not know before – a gap between what prospective teachers perceived and what they really knew and were able to do. The success in assessing prospective teachers' confidence of their knowledge preparation and readiness prompted us to include a similar component when examining Chinese and Korean prospective elementary teachers' knowledge.

Another component of the framework is teachers' mathematics knowledge for teaching (MKT). Because prospective teachers differ from practicing mathematics teachers in terms of what they can know and learn for teaching, the component is structured in a way that is consistent with others' work (e.g., Hill, Schilling, & Ball, 2004; Shulman, 1986) but different in their specifications (Li & Kulm, 2008). In particular, specific items were selected and adapted to assess prospective teachers' common content knowledge, specialized content knowledge, and pedagogical content knowledge on the topic of fraction division.

In this study, we plan to use the same framework (Li & Kulm, 2008) to examine Chinese and Korean prospective elementary teachers' knowledge preparation. The use of this framework would allow us to examine Chinese and Korean prospective elementary teachers' knowledge in mathematics and pedagogy for teaching in two aspects: confidence about their knowledge preparation, and MKT that is further specified as mathematical knowledge on specific content topics (fraction division) and the knowledge needed when carrying out the task of teaching (teaching fraction division). Moreover, the use of the same framework would also make it possible to compare Chinese practicing elementary teachers' confidence and their mathematics knowledge for teaching (e.g., Li & Huang, 2008) with Chinese prospective teachers' confidence and knowledge reported in this article.

Research Questions

By focusing on the content topic of fraction division, this article is developed to examine Chinese and Korean prospective elementary teachers' confidence and mathematics knowledge for teaching. In particular, we aimed to address the following two research questions:

(1) How do prospective elementary school teachers in China and South Korea perceive their knowledge preparation in curriculum and instruction for their future teaching career?

(2) What similarities and differences can be found between Chinese and Korean prospective elementary school teachers' knowledge in mathematics and pedagogy for teaching fraction division?

Method

Participants and Context of the Study

The participants were prospective elementary school teachers sampled in China and South Korea. In China, the survey was given to both junior and senior prospective elementary teachers in 10 institutions. 361 responses were collected and used for data reporting, with 50.4% of the responses from junior normal colleges (with fewer years of preparations than the 4-year B.A. or B.Sc programs, see Li, Zhao, Huang, & Ma, 2008), and 49.6% from 4-year normal universities/colleges. In South Korea, the survey data was collected from 291 seniors in three universities that all offer 4-year B.A. or B.Sc. preparation programs. These 291 seniors included those with mathematics education as their focus area, and ones with their focus areas in other subjects.

Instruments and Data Collection

Two instruments developed and used in a previous study (Li & Kulm, 2008) were adopted in this study. The first instrument is a survey of prospective teachers' confidence of their knowledge preparation. Many items were adapted from TIMSS 2003 background questionnaires (TIMSS 2003).

The second instrument is a mathematics test that focuses on prospective teachers' content knowledge and pedagogical content knowledge of fraction division. It contains items targeted to prospective teachers' possible difficulties in order to reveal the extent of their knowledge and understanding of fraction division. Some items were adapted from school mathematics textbooks and previous studies (e.g., Hill, Schilling, & Ball, 2004; Tirosh, 2000) and others were developed following the structure of the framework.

Both instruments were translated into Chinese and Korean by persons fluent in both English and the second language. Special attention was also given to specific problem context to ensure their familiarity with Chinese and Korean prospective teachers. The translation was cross-checked by others to ensure the accuracy and feasibility.

All soon-to-graduate prospective teachers enrolled in the selected institutes in China and South Korea were invited to participate in this study. The participants were notified that both the survey and the test were for research purposes only and should be completed anonymously. Mathematics education faculty in different institutes helped administer the survey and the test. Participants were asked to complete the survey first, then the mathematics test.

Data Analysis

All the data for this study were analyzed in the original language of Chinese and Korean. Selected data were translated to English to provide evidence needed in the later sections of this article. To address our research questions, both quantitative and qualitative methods were used in the analysis of the participants' responses. Specifically, responses to the survey questions were directly recorded and summarized to calculate the frequencies and percentages of participants' choices for each category. To analyze participants' solutions to the problems in the mathematics test, specific rubrics were first developed for coding each item, and subsequently, the participants' responses were coded and analyzed to examine their use of specific concepts and/or procedures.

Results and Discussion

In general, the results present a two-sided picture, along the two components specified in the framework that illustrates the importance of examining and understanding prospective teachers' knowledge in mathematics and pedagogy for teaching.

On one side, the results from the survey indicate that (1) many prospective elementary school teachers sampled in China and South Korea were not confident in the preparation they received in mathematics and pedagogy for future teaching careers; and (2) there were more prospective teachers in China than those in South Korea who lack the confidence in their preparation needed for teaching fraction division.

On the other side, however, these sampled prospective teachers' performance on the mathematics test reveals that their mathematics knowledge for teaching fraction division was mathematically sound but less strong in pedagogical content knowledge. The apparent inconsistent pattern in their responses suggests that prospective teachers in China and South Korea did not develop enough confidence with their strong mathematics knowledge only. Nevertheless, the gap is "favorable" as prospective teachers were not satisfied with what they already know in mathematics that is not enough for developing effective teaching. The following sections are organized to present more detailed findings corresponding to the two research questions.

Prospective Elementary Teachers' Confidence of Their Knowledge Preparation in Mathematics and Pedagogy for Teaching

Prospective teachers' responses to the survey show the lack of confidence. The following items are selected from the survey to illustrate prospective teachers' confidence of their knowledge preparation needed for teaching, as related to fraction division.

For item 1: How would you rate yourself in terms of the degree of your understanding of the National Mathematics Syllabus? On a scale of four choices (High, Proficient, Limited, Low), there are only 16% of sampled prospective teachers in China and 20% of those in South Korea self-rated with either "high" or "proficient" understanding of their national mathematics syllabus. Consequently, 84% of prospective teachers sampled in China and 80% in South Korea self-rated with either "limited" or "low" understanding of their national mathematics syllabus.

For item 2-(5): Choose the response that best describes whether primary school students have been taught the topic – *Multiplication and division of fractions.* In a scale of five choices (Mostly taught before grade 5, Mostly taught during grades 5-6, Not yet taught or just introduced during grades 5-6, Not included in the National Mathematics Syllabus, Not sure), 67% of sampled prospective teachers in China and 61% of those in South Korea indicated correctly that the topic is "mostly taught during grades 5-6", while the remaining sampled teachers in both China and South Korea chose other responses. This result, in conjunction with the participants' response to question 1, suggests that the majority of prospective teachers sampled in China and South Korea gave adequate self-assessment of their own knowledge preparation. Some prospective teachers in China and South Korea might even be modest in providing their self-assessment.

For item 3-(4): Considering your training and experience in both mathematics and instruction, how ready do you feel you are to teach the topic of "Number – Representing and explaining computations with fractions using words, numbers, or models?" On a scale of three (Very ready; Ready; Not ready), 45% of sampled prospective teachers in China and 70% of those in South Korea thought they are "ready", while 15% in China and 6% in South Korea chose "very ready" and the remaining 40% in China and 23% in South Korea "not ready." The results indicate that many prospective teachers in China and South Korea were not confident in their preparation for teaching fraction computations, including fraction division. However, there were significantly more prospective teachers in South Korea than those in China who are confident about their preparation for teaching the topic ($\chi^2(2)=14.11$, $p=0.01$). The cross-national difference in prospective teachers' confidence is supported by their performance differences in test items that assess prospective teachers' pedagogical content knowledge needed for teaching, as reported in the following sub-section.

Prospective Elementary School Teachers' Knowledge in Mathematics and Pedagogy for Teaching Fraction Division

Prospective teachers' responses to the mathematics test allowed us to take a closer look at the participants' knowledge in mathematics and pedagogy for teaching, especially on the topic of fraction division. Results indicate that prospective teachers sampled in China and South Korea did very well in computing fraction division. For example, for the problem "find the value of $5\frac{1}{4} \div 3\frac{1}{2} =$ ", 93% of prospective teachers sampled in China and 97% of those in South Korea solved the problem correctly. Even when the problem was changed slightly with a conceptual requirement, their performance is still good. As an example, to find "How much $\frac{1}{2}$'s are in $\frac{1}{3}$?", 88% of those in China and 79% in South Korea solved it correctly. In fact, this problem is a typical problem in mathematics textbooks for elementary school students. The responses of these prospective teachers in the two education systems suggest their adequate preparation in mathematics related to division and their understanding of fraction division.

Moreover, Chinese and Korean prospective teachers' training in mathematics is also reflected in their performance of solving problems that involve fraction division, especially for a multi-step problem. In particular, 75% of sampled prospective teachers in China and 93% of those in South Korea solved the following problem correctly.

> Lee's Pizza Express sells several different flavors of pizza. One day, it sold 24 large-size pepperoni pizzas. The number of large-size plain cheese pizzas sold on that day was $\frac{3}{4}$ of the number of large-size pepperoni pizzas sold, and was $\frac{2}{3}$ of the number of large-size deluxe pizzas sold. How many large-size deluxe pizzas did the pizza express sell on that day?

At the same time, Chinese and Korean prospective teachers were also asked to explain given computations of fraction division. In particular, the problem of "How would you explain to your students why $\frac{2}{3} \div 2 = \frac{1}{3}$?; Why $\frac{2}{3} \div \frac{1}{6} = 4$?" (adapted from Tirosh, 2000) was included in the test. It is found that about 88% of sampled Chinese prospective teachers provided valid explanations for dividing a fraction by a natural number (82% for the second computation). However, the majority (64% for the first computation and 73% for the second) explained with "flip and multiply" that is not recommended for this problem. The remaining valid explanation was dominated by explaining the division procedure (e.g., "dividing a whole into six equal parts, each part is 1/6, four parts should be 4/6. In other words, 4/6 contains four 1/6. Thus, 4/6 ÷ 1/6 = 4.

Because 4/6 equals 2/3, so 2/3 ÷ 1/6 = 4."). Few students tried to draw a picture such as a circle to help with their explanations.

In contrast, more prospective teachers in South Korea were able to provide valid explanations about fraction division (98% for the first computation, and 88% for the second computation). Moreover, they came up with various methods such as drawing (45% for the first computation, 33% for the second computation), using manipulative materials (10%, 5% respectively), using the meaning of division and/or fractions (8%, 4%), finding the common denominator (6%, 23%), number lines (6%, 5%), and using word problems (5%, 1%). In addition, quite many respondents (about 15%) actually provided explanations in two or more ways. Only a limited number of respondents (3% for the first computation, and 10% for the second computation) used the algorithm of "flip and multiply" to explain the computation.

The difference between Korean and Chinese prospective teachers' performance in their pedagogical content knowledge becomes even more dramatic when the fraction division was presented in a way that prevented prospective teachers to use the "flip and multiply" explanation. For example, in solving the following problem:

> During the lesson when you teach the algorithm for "division of fractions" (i.e., $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$), students asked why you change from 'division' to 'multiplication' and flip the second fraction. How would you explain to students?

About 64% of prospective teachers sampled in South Korea and 30% of those in China provided valid explanations. Moreover, among those who provided valid explanations, a higher percentage of Korean prospective teachers (73% of those with a valid explanation in Korea versus 59% of those in China) proved why the algorithm is correct (e.g., $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \div (1 \div \frac{d}{c}) = \frac{a}{b} \times \frac{d}{c}$) while a higher percentage of Chinese prospective teachers (24% of those with a valid explanation in South Korea and 36% of those in China) used specific numbers to verify the correctness of the algorithm. The results suggest that although both Chinese and Korean prospective teachers' pedagogy content knowledge was relatively less strong than their mathematical knowledge on the topic of fraction division, Korean prospective elementary teachers had much better performance in pedagogical content knowledge than their Chinese counterparts.

Chinese and Korean prospective elementary teachers' performances on the mathematics test reveal their strength in mathematical content preparation but relatively less strong in pedagogical content training. This situation is understandable as prospective teachers can be expected to further develop their pedagogical content knowledge once getting into the teaching profession. In fact, existing studies already documented that Chinese in-service elementary teachers have a profound understanding of mathematics they teach, for example on the topic of fraction division (e.g., Li & Huang, 2008; Ma, 1999).

It is important to point out the sampling differences in selecting prospective elementary teachers from China and South Korea in this study. While Korean prospective elementary teachers were all sampled from 4-year B.A. and B.Sc. teacher preparation programs, about 50% of those sampled in China were from junior normal colleges that offered fewer than four years' teacher preparation programs. A recent analysis of Chinese data suggests that these two groups of prospective elementary teachers in China share many similarities in their confidence and performance, with exceptions on a few items that contain complex or abstract mathematics (Zhao, Ma, Li, & Xie, 2010). Thus, further analyses are needed to help us understand better how different types of preparation programs in China may prepare prospective teachers differently in comparison with their counterparts in South Korea.

Conclusion

The findings from this study present two different and seemingly inconsistent sides of prospective teachers' knowledge in mathematics and pedagogy for teaching. The positive side is revealed by the prospective teachers' performance on the mathematics test. Sampled prospective elementary teachers in China and South Korea showed their solid preparation in mathematics content they will teach. At the same time, however, mathematics training that Chinese and Korean prospective teachers received may not be enough to build a strong confidence in their preparation for teaching. While prospective teachers' lack of confidence may partially relate to the influence of Chinese culture in the region (e.g., Bond, 1996; Wong, 2004), the 'gap' itself becomes 'favorable' as prospective teachers were likely cautious for what they may think is enough preparation for teaching. In fact, both Chinese and Korean prospective teachers' performance suggests that their pedagogical content knowledge on fraction division can be improved. As prospective teachers sampled in Korea had higher confidence with better performance on items that assess teachers' pedagogical content knowledge than their Chinese counterparts, it can be assumed that prospective teachers with better preparations in pedagogical content knowledge should become more confident about their readiness for teaching.

Although the discrepancies in Chinese and Korean prospective teachers' performance in pedagogical content knowledge need to be examined further due to sampling differences, the cross-national difference itself suggests possible connections with different training provided to prospective elementary teachers through teacher education programs in these two education systems. Further efforts are needed to investigate what may help Chinese and Korean prospective elementary teachers to develop their pedagogical content knowledge differently through their program studies.

Finally, the results also provide an opposite picture to the case of the U.S. when a group of prospective middle school mathematics teachers were investigated about their confidence and knowledge in mathematics for teaching fraction division (Li & Kulm, 2008). The previous study revealed that sampled U.S. prospective teachers are very confident with their preparation needed for teaching, while their knowledge in mathematics and pedagogy for teaching fraction division is procedurally sound but conceptually weak. The different patterns presented in prospective teachers' confidence and their mathematics performance between the U.S. and the two education systems in East Asia suggest possible influence of the culture in addition to teacher preparation programs and practices. Similar to what we now know about teaching as a cultural activity (Stigler & Hiebert, 1999), caution needs to be taken when learning about teacher preparation practices cross-culturally.

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PRE-SERVICE TEACHERS' LEARNING ABOUT AND FROM STANDARDS-BASED CURRICULUM MATERIALS: THE CASE OF ADDITION STARTER SENTENCES

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Despite the prevalence of mathematics curriculum materials in elementary classrooms, most current mathematics methods courses and texts spend little or no time helping pre-service teachers learn to use curriculum materials. To meet this need, we have designed and studied several activities intended to provide pre-service teachers with opportunities to learn about and from the use of curriculum materials. In this paper, we describe our research related to one of these activities – Addition Starter Sentences. In particular, we focus on the ways in which the activity both supported and constrained pre-service teachers in identifying the goals of the curriculum materials and enacting the materials in ways consistent with those goals. We conclude with implications for mathematics teacher education research and practice.

Introduction

Using curriculum materials is a practice that, for many teachers, is at the core of their classroom mathematics instruction (Remillard, 2005). Nonetheless, despite the prevalence of mathematics curriculum materials in elementary classrooms, most current mathematics methods courses and texts spend little or no time helping pre-service teachers (PSTs) learn to use curriculum materials (Drake, in progress). We believe that a need exists to design research-based activities for the elementary mathematics methods course that allow PSTs to learn about - and from - the use of curriculum materials. We have chosen to focus on the use of *Standards*-based curriculum materials in particular because of 1) their increasing use in districts across the country (Stein & Kim, 2009), 2) their descriptions of mathematics teaching and learning activities that are often quite different from those experienced by PSTs in their own experiences as elementary students, and 3) their explicit attention to opportunities for teacher, as well as student, learning within the curriculum materials (Collopy, 2003; Lloyd, 2009). We have designed these course activities to provide PSTs with opportunities to engage with the educative features (Davis & Krajcik, 2005) of curriculum materials.

In this paper, we present an in-depth look at one of the activities we have designed and our findings related to that activity. We first situate our project in existing literature related to PSTs and curriculum materials, followed by detailed descriptions of the initial design of the activity, findings from the first implementation of the activity, revisions to the activity, and findings from a subsequent implementation. We conclude with implications of our study for mathematics education researchers and teacher educators and a brief description of next steps for this work.

PSTs and Curriculum Materials

We acknowledge that PSTs will not learn to be expert curriculum users in a single-semester methods course. However, it is our goal to position PSTs on a trajectory that will allow them to continue to learn from and about curriculum materials as practicing teachers. In fact, we claim that learning to read and use curriculum materials in particular ways during a methods class can equip PSTs with a generative practice (Franke et al., 2001) that will support them in continuing to learn – about teaching and about children's mathematics – as novice teachers.

Twenty-five years ago, Shulman (1986) suggested the construct of curricular knowledge was an important, but neglected, aspect of teacher education stating that, "If we are regularly remiss in not teaching pedagogical knowledge to our students in teacher education programs, we are even more delinquent with respect to the third category of content knowledge, *curricular knowledge*" (p. 10). Shulman described this curricular knowledge as, in part, about, "the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances" (p. 10). Two years later, Ball & Feiman-Nemser (1988) described two teacher education programs that discouraged the use of textbooks and, as a result, did not prepare pre-service teachers to use textbooks in effective, flexible ways that supported both student and teacher learning. They suggested, "Perhaps beginning teachers can be oriented toward learning *from* teacher's guides and other curriculum materials in a way that allows them to move toward building their own units of study, units that are responsible to subject matter goals and responsive to students" (p. 421).

More recent work related to PSTs and curriculum materials has focused on PSTs' capacities to evaluate, or critique, curriculum materials. Nicol and Crespo (2006) asked PSTs to critique mathematics curriculum materials. The researchers found that PSTs were able to critique materials, but PSTs differed "in terms of: (1) the types of problems selected; (2) how problems were adapted; (3) whether or not problems were developed; and (4) how the participants considered using textbooks in their future teaching" (Nicol & Crespo, 2006, p. 337). Lloyd & Behm (2005) investigated the ways in which PSTs compared and contrasted two textbook lessons (one traditional and one reform-oriented). The researchers found that PSTs looked for aspects of the lessons that were familiar to them. Furthermore, PSTs' fondness for traditional lessons led to misinterpretations of the two lessons:

> *The influence of preservice teachers' familiarity with more traditional instructional material components was, at times, so strong that it led them to inaccurately describe the two sets of instructional materials or to arrive at questionable conclusions about the materials* (Lloyd & Behm, 2005, p. 54).

Methods

In this study, we employed a design research methodology, as described by Cobb et al. (2003). We began with a conjectured trajectory for PST learning from and about curriculum materials and then designed activities to help PSTs advance along that trajectory. Each activity has been implemented multiple times – often by different instructors and at different university sites. In all, six instructors from two universities have participated in implementing one or more of the activities. This variety is a purposeful aspect of our research design as it allows us not only to learn from one another, but also to disentangle the contributions of context, instructor, and activity in advancing PST learning.

Data collected for the study include PSTs' work from the activities and PSTs' reflections on the activities, PSTs' responses to survey items related to their intended use of curriculum materials, instructor reflections on the activities, videotapes of methods classes, and instructor field notes. These data were analyzed through a process of open and emergent coding (Strauss & Corbin, 1998). Using the conjectured learning trajectory and our goals for each activity as an initial framework, data were analyzed independently at the two university sites. As themes emerged and were shared across sites, they contributed not only to the on-going data analysis at each site, but also to subsequent iterations of activity design and data collection. This process is described in greater detail in the remainder of the paper – specifically related to the Addition Starter Sentences (TERC, 2008) activity.

Developing the Starter Sentences Activity

In addition to having a course goal of helping PSTs learn about and from curriculum materials, we have several other course goals (e.g. teaching through problem solving and attention to children's mathematical thinking). Because of limited time in the methods course, we want to accomplish multiple course goals within a single course activity. Therefore, in designing new course activities, we sought out lessons or units from *Standards*-based materials that could address multiple course goals. The Unit 3 Addition Starter Sentences lesson from the 3rd grade *Investigations* (TERC, 2008) series stood out to us because it followed the teaching through problem solving lesson model, provided information about alternate addition strategies, and afforded us a context to discuss children's mathematical thinking.

Addition Starter Sentences (TERC, 2008) begins by asking students to decide which of the following "addition starter problems" is easier for them to solve and why: 100 + 200 = ______, 136 + 200 = ______, 136 + 4 = ______. Next, students are asked to choose a starter sentence to solve 136 + 227 and explain why that would be a good start. After students solve 136 + 227, they share their solution paths. Students then complete a worksheet with five more "sets" of tasks, which consist of three starter problems matched to a final problem.

In the methods course, we enact the lesson as written with PSTs participating as students. We then ask PSTs to read the curriculum materials and discuss the educative features (Davis & Krajcik, 2005). One educative feature we focus on is the information about alternate addition strategies that align with the starter problems: breaking apart by place; adding one number in parts; and changing a number, then adjusting (TERC, 2008). Next, we examine a later lesson in *Investigations* that revisits starter sentences (from Unit 8) focusing on how it builds on the Unit 3 lesson. Finally, we ask PSTs to categorize examples of children's mathematical thinking according to the three strategies and discuss how teachers can support students in using them.

Our hope with the Starter Sentences activity is that PSTs will notice that the curriculum materials provide valuable information about the use of alternate strategies (TERC, 2008). Furthermore, we conjecture that because many teachers view textbooks as authorities (Herbel-Eisenmann, 2009; NCTM, 1991), engaging in a textbook-based lesson about alternate strategies might change how some PSTs value and interpret the role of alternate strategies.

Findings from the First Implementation

Starter Sentences in the Methods Class

Over the course of a semester, Tyminski's methods course met on campus 21 times for 75-minute sessions and nine times at a local elementary school for a field experience. During the field experience, the PSTs had an opportunity to teach three lessons from *Standards*-based curricula. The class consisted of 22 students, all female. Although he regularly included exposure to children's alternative strategies for addition in previous methods courses, this was the first time he did so through the use of curriculum materials. In past iterations of his course, he introduced alternative strategies through the course textbook and its examples of student work (Van de Walle, 2009), and gave PSTs opportunities to solve new tasks using these strategies. Instruction largely focused on the PSTs' abilities to make sense of and use the mathematics of the children's solutions, as well as the types of tasks and teaching approaches useful for generating these strategies. These continued to be his goals for the activity involving the curriculum materials. In addition, he also had goals aligned with the project goals above: for his students to become familiar with curriculum materials designed to support and elicit alternative thinking strategies, and for students to see how ideas are developed in the *Investigations* curriculum. At the time the activity was taught, it was not known (and upon reflection, not even considered) that any of the PSTs would teach this lesson during their field experience.

During the enactment of the Starter Sentences activity in the methods class, it was clear that the PSTs were able to mentally solve the starter sentences and were able to find multiple ways to use the starter sentences to continue solving 136 + 227. When asked which starter problem was easiest to solve, every PST chose 100 + 200. They gave reasons such as "There is nothing on the end" and "You don't have to carry". When asked which starter problem would be a good first step to solving 136 + 227, most PST chose 136 + 200 as the starter problem they would use. One PST reasoned, "Its [136 + 200] just basically the same without the 27". When another PST suggested there were reasons why a child might choose 100 + 200 as a first step, Tyminski asked the class to discus what the child's next steps might be. With this prompt the PST suggested a child might add the 10's together, add the 1's together, and then add all of the partial sums to find the total. Another PST suggested a way she could use 136 + 4 as a first step in a compensation strategy, using 136 + 4 to change 136 + 227 into 140 + 223. The students then selected one of the starter problems and finished solving for the sum. Next, Tyminski introduced the five sets of starter problems for PSTs to solve. In the course of doing so, he was explicit on four different occasions that the PST did not *have* to use the starter problems and it was okay if they did not. All PSTs however, seemed to use the starter problems and were able to use them in a variety of ways and record keep their ideas, which they shared on the board.

The next stage of the lesson involved the PSTs reading the lesson materials from Unit 3 and Unit 8 and discussing what they noticed about the materials and/or their enactment. Tyminski had asked his PSTs to read the materials prior to class, knowing he would not have time to spend reading in class. The students had little to say about the features of the materials or the lesson enactment. The class then discussed and named the three strategies evident in the starter sentences. As the students summarized the Starter Sentences lesson from Unit 8, Tyminski briefly listened to their ideas and summarized what he believed to be the three big additions in Unit 8: asking students to estimate before solving using starter sentences, including problems that would "spill over" into the next place value, and at the end, encouraging students to write their own starter sentences. The rest of the class was spent by the PSTs identifying and making sense of student addition strategies.

As we examined Tyminski's lesson video we realized something important and interesting. Namely, it seemed that his goals and focus in his previous lessons on this topic influenced his enactment of the activity. In spite of his stated goals for his students to become familiar with curriculum materials designed to support and elicit alternative thinking strategies, and for students to see how ideas are developed in the *Investigations* curriculum, he appeared to gloss over the opportunities built into the plan to address these goals. Rather, the majority of the class time was spent either on PSTs making sense of and using the alternative addition strategies presented in the curricular materials or using these experiences to interpret students' work. When his time was limited, he made choices to focus on goals related to the mathematics of children and the mathematics of the PSTs. In and of itself, this did not strike us as problematic. After all, teachers make choices and judgments all the time in the process of teaching. And, the PSTs had been asked to read the materials for two related units and worked through tasks from both. Perhaps, even though explicit attention was not paid to them, the goals of the individual lessons and the big ideas of *Investigations* were as evident to the PSTs as they were to the authors. As we later realized, this was not the case; at least not in the case of Cathy.

Using Starter Sentences in the Field: The Case of Cathy

For the field experience component of the course, Cathy (pseudonym) was placed in a third grade classroom. As a course requirement, Cathy was to negotiate a mathematical topic with her cooperating teacher and teach a lesson on that topic from a *Standards*-based curriculum. Her teacher asked her to teach a lesson on adding three digit numbers whose sum was greater than 1000. Cathy chose to implement the lesson from Unit 8 involving starter sentences.

> *I knew that the students had been learning and practicing adding three digit numbers this semester based on what Mr. "Y" told me. He said that the students were having a lot of trouble with adding four digit numbers and the one-thousandths [sic] place, so I thought that this lesson would be a great way for students to learn how to use simpler ways to add while practicing with the three digit numbers they had been learning to add. My goals were for the students to be able to pick one starter problem and use it to solve the addition problems I wrote on the board and that were in their worksheets that were passed out as an assessment. (Reflection)*

As she enacted the lesson, she began with the lesson's estimation task for 379 + 412, which her students solved by finding exact answers. Then she presented the students with the three starter sentences for this addition problem; 379 + 400 = _____, 300 + 400 = _____, and 379 + 1 =____. The curriculum directed the teacher to remind students they had used starter sentences earlier in the year, and to ask them "Which one of these makes it easiest for you to solve the problem? Choose one of these starter problems, and think about how you would finish the problem. Write your strategy to keep track of it," (TERC, 2008, p. 73). Cathy introduced them by telling her students that the number sentences "were starter problems and they would help them while adding" (Cathy, Reflection). She set the students to work on the task and walked around the room, interacting with the students. Cathy described what she saw in her reflection,

> *While walking around the room to see who [sic] to call on to give me their starter problem, I noticed that all but two students used the standard one number on top of the other way of adding. This is where I felt a bit of panic because students were not doing what I intended or even trying to use the starter problems. I noticed that almost all of the students solved the starter problems and the real question that were on the board instead of picking one starter problem to help them solve the real problem. During this discussion time I called on the students who used starter problems and wrote their methods on the board. (Reflection)*

As she described, the majority of the students used the standard algorithm to solve the problem. One of her students however, used 379 + 400 = ___ to solve the problem by adding the left over 12 to 779. A second student was able to use 379 + 1 = ___ by rewriting 379 + 412 as 380 + 411. Cathy explained these two solution strategies and pointed out how the starter problems helped each student solve the problem. She also had a student explain how they solved the problem using the standard algorithm. "I quickly learned that students who learned how to add the standard way, stuck with that method because it was what they knew and it worked for them" (Cathy, Reflection). Next the students worked on the six sets of starter problems from the Unit 8 lesson. As Cathy walked around to talk with students she noticed that students had solved all of the starter problems as well as the original problem. Only two students, but only one of the same as before, used the starter sentences. In her final discussion, she had the students who used starter sentences share their thinking along with students who used the standard algorithm.

Cathy's goal for the lesson was for students to solve addition problems using starter sentences. However, despite the fact that her students successfully solved the problems posed, she viewed her lesson as a failure because they did not solve them the way she wanted them to.

> *Based on my objectives of each student using a starter problem to solve the real problem, the lesson was a complete failure because that did not happen. I feel negatively about this because I feel that I did not do my job as a teacher since they did not do what I intended. (Reflection)*

Investigations pose tasks for students to explore the process of doing mathematics and supports teachers in interpreting and developing student thinking. Implicit to this notion involves the teacher meeting the student where she is, working from what students are doing, and not "forcing" strategies on students. As evidenced by this excerpt, this idea was not clear to Cathy.

> *I found it very difficult to teach students new concepts especially if what they already do is working for them…Overall I think that my intentions were good, but the execution did not happen. I needed to put more emphasis on why we were using these problems and how they would help students add. (Reflection)*

This is an example of our activity's failure to develop Cathy's curriculum vision (Drake & Sherin, 2009). Cathy clearly focused on implementing the individual lesson's ideas in a literal manner and paid little attention to the bigger ideas and goals of the curriculum. Due to this serendipitous event however, were able to see from her enactment and reflection the message Cathy constructed from the methods class activity. We believe it is reasonable to assume her experiences with the activity in the methods classroom influenced her goals for the lesson and her sense of how the materials should be used. As we realized the message Cathy constructed was not our intended message, we realized we were missing answers to important questions: What message did the other PSTs take away from the methods course activity? What goals would they have had in using this lesson?

Findings from the Next Implementation

Based on the findings described above, as well as our own experiences with implementing the starter sentences activity, Drake and Land made multiple pedagogical changes to the activity in their next enactment. They also made methodological changes in the data collected from PSTs about the activity. They were particularly interested in collecting data related to their PSTs' understandings of the goals of the lesson and the relationship of these goals to children's thinking about and strategies for multi-digit addition.

The major pedagogical change was the addition of a task at the end of the activity. PSTs were asked to construct the "curriculum storyline" by using the curriculum materials to identify the overall goal of the lesson, the key lesson steps (Brown et al., 2009), and the relationship between each of those steps and the lesson goal. The intent of this new activity was to 1) provide greater support for PSTs in identifying and reflecting on the overall goal of the starter sentences lesson, as well as the elements of the lesson and curriculum materials that contributed to this goal and 2) introduce the concept of a curriculum storyline as a strategy, or tool, for PSTs to use in purposefully reading curriculum materials and developing curriculum vision.

Drake and Land also added two reflection questions for PSTs to answer, providing us with insight into PSTs' thinking about their learning about children's strategies for multi-digit addition and the goals of the lesson. Here we share the findings from the question that most directly addressed our findings from the first implementation:

> Depending on your purpose, you might have the following goals for students:
>
> 1. Choose just one strategy and practice it
> 2. Use more than one strategy to solve each problem
> 3. Use more than one strategy across the problem set (i.e., use one strategy for one number combination and a different strategy for a different number combination)
> 4. Learn and master all of the strategies
>
> If you were teaching the Addition Starter Sentences lesson, which of these goals would you have? Why? (You can use evidence from the materials or from your own experiences to support your choice).

Half of the PSTs chose the goal of having children use more than one strategy to solve each problem. The remaining PSTs were evenly split across the other three choices. Most interesting to us, however, was that the PSTs who chose the second, third, or fourth goals above all provided similar justifications for their choices, even though they selected different goals. They all wanted children to develop flexibility in their strategies while still identifying a strategy that worked best for them. Thus, while 15 out of 18 PSTs seemed to hold a consistent understanding of the overall purpose of the lesson – and of the use of alternate strategies more generally – they differed in their interpretations of how they would enact the materials to support this purpose.

Implications and Next Steps

It is our intent to have PSTs learn about and from curriculum materials. Since instructors' future pedagogical choices depend on knowledge of PSTs' current conceptions it is important for instructors to know what ideas PSTs take away from their experiences with curriculum materials. To accomplish this it is crucial to ask them to explicitly reflect on what they have learned from the activities and to see them enact what they have learned in practice. The next steps to this project are reflected in the case presented. We believe we can structure future activities to better examine what PSTs learn from interactions with these types of materials. However, we need opportunities to see PSTs teach with these materials to see the relationship between their intended goals for implementation and their enactment of the lesson.

Endnotes

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PREPARING PRE K-8 TEACHERS TO CONNECT CHILDREN'S MATHEMATICAL THINKING AND COMMUNITY BASED FUNDS OF KNOWLEDGE

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This paper examines the impact of pre-service mathematics methods courses aimed at integrating a focus on children's mathematical thinking and community/cultural funds of knowledge in mathematics instruction. Analysis revealed that many pre-service teachers entered methods courses with the belief that connecting mathematical funds of knowledge was a valued teaching practice. Prospective teachers found their engagement with students' communities to be a positive and meaningful experience that helped them develop mathematics curriculum. However, pre-service teachers also faced challenges, remaining vague about what they might actually do to draw on children's funds of knowledge and at times reifying stereotypes.

Introduction

The field of mathematics education currently lacks a deep understanding of how mathematics instruction might integrate children's mathematical thinking with cultural, linguistic, and community-based knowledge that children bring to classrooms in ways that support learning. On the one hand, research supports the effectiveness of instruction centered on children's mathematical thinking and documents that teachers' knowledge of children's mathematical thinking leads to productive changes in teachers' knowledge and beliefs, classroom practices, and student learning (Carpenter, Franke, Jacobs & Fennema, 1998; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). On the other hand, research documents that historically underrepresented groups benefit from instruction that draws upon their cultural, linguistic, and community-based knowledge (Ladson-Billings, 1994; Turner, Celedón-Pattichis & Marshall, 2008). This research argues that teachers need to understand how students' home and community-based *funds of knowledge* – the knowledge, skills and experiences found in students' homes and communities - can support their mathematical learning (Civil, 2002; Moll, Amanti, Neff & Gonzalez, 1992).

At the same time, as our nation's public schools have become increasingly diverse, the pre-service teacher (PST) population steadily consists of White, middle-class, monolingual females typically underprepared to address such diversity (Hollins & Guzman, 2005). While there is significant research related to preparing teachers to work in diverse classrooms, little of it addresses the specific challenges and resources of learning to teach *mathematics* to diverse learners. We contend that experiences that help PSTs understand the mathematical knowledge and practices of students' communities can enhance their ability to provide effective mathematics instruction for diverse learners.

This research investigated preK-8 pre-service mathematics teachers' work in mathematics methods courses around instructional modules aimed to prepare them to design and implement effective instruction for diverse student populations that builds on and integrates children's multiple knowledge bases. In particular, this study investigated the following research question: What changes do we notice in PSTs' knowledge, beliefs, dispositions, and practices related to integrating children's mathematical thinking and their home and community-based funds of knowledge in mathematics instruction? This paper presents selected results from this research.

Theoretical Perspectives

This work draws upon situated, socio-cultural perspectives of teacher learning (Lave & Wenger, 1991) that frame teacher learning as situated social practice. This study conceptualizes PST learning as a process of identity development (Wenger, 1998); as PSTs engage with the mathematical practices and identities of children and families as children move across contexts and spaces, PSTs move along a trajectory toward becoming mathematics teachers - analyzing, reflecting, and acting on their own practices and identities. In considering potential learning trajectories for PSTs, we draw on Mason's (2008) constructs of *attention*, *awareness*, and *attitude* for examining PSTs' reflections and discussions. In particular, we hypothesize that by participating in modules designed to focus on mathematics and children's multiple knowledge bases, pre-service teachers enter a learning trajectory in which they begin to *attend* to children's mathematical thinking and their cultural, linguistic, and community-based knowledge and experiences. This attention in turn supports PSTs' *awareness* of and attitudes toward students' competencies and knowledge relevant to mathematics. This combination of *attention* and *awareness* prepares PSTs to participate in more complex, field-based tasks in which they work with children in schools and the community. PSTs begin to *make emergent*, and then *meaningful connections* as their understanding of children's multiple mathematical knowledge bases progresses from considering the knowledge bases as separate, to understanding how they might work together to support mathematical learning. As PSTs make meaningful connections, these connections then facilitate *incorporating multiple knowledge bases in instruction*, where PSTs plan problem-solving tasks that incorporate what they know about how children develop understanding of these concepts with what they know about the mathematics used in this community setting.

Methods

Participants

Approximately 200 pre-service elementary and middle school mathematics teachers enrolled in mathematics methods courses at six university sites that represent a diverse range of teaching contexts (i.e., urban, a mixture of urban, suburban, and rural, suburban, and borderlands) participated both in the instructional modules examined in this research and in pre- and post-course surveys.

Survey

Pre- and post-course surveys were administered to participants to gauge PSTs beliefs about the teaching and learning of mathematics and children's cultural, linguistic, and community-based funds of knowledge as resources for teaching. The surveys included 18 Likert-type items, six short answer responses, between two and four instructional scenarios (pre-survey only), and nine questions related to effectiveness of course activities (post-survey only).

Instructional Modules Overview

Instructional modules were created and implemented in elementary and middle school mathematics methods courses with study participants. These modules were informed by research and professional development materials on Cognitively Guided Instruction (Carpenter et al., 1999), community-based funds of knowledge for teaching mathematics (Civil, 2002), and Villegas and Lucas's (2002) six-strand framework for preparing culturally responsive teachers. More specifically, we designed the modules to include opportunities for PSTs to a) identify and build on children's mathematical thinking and funds of knowledge, b) recognize and describe instructional and community practices that support preK-8 students' mathematical learning, and c) adapt curriculum materials and lesson plans to incorporate rich mathematical content, children's mathematical thinking and funds of knowledge.

Focus Module: Community Mathematics Exploration

This paper focuses on one module in particular, *Community Mathematics Exploration*, which required PSTs to engage with the local community (e.g., visit community locations, observe and interact with community members, conduct a community walk) and consider how knowledge gained through that interaction could inform their teaching. The goal was for PSTs to identify mathematical practices and mathematical funds of knowledge in students' communities and build on them in a standards-based mathematics lesson or mathematics problem. In the methods courses, pre-service teachers read articles about what it means to build on children's community funds of knowledge in mathematics classes (e.g., Civil & Kahn, 2001). How PSTs engaged with the community varied somewhat with each site. Regardless, all PSTs engaged with the community in some way, and then designed either a mathematics problem or standards-based mathematics lesson that built on what they learned about children's community and cultural funds of knowledge. In the example discussed in this paper, two PSTs enlisted the expertise of a Latina parent as they engaged in a walking tour of the community. The PSTs then created a third-grade mathematics lesson inspired by what they learned on their community walk. Finally, PSTs reflected on the experience in terms of what they learned about themselves, their students, and their teaching.

Data Analysis

The data analyzed for this paper included artifacts from the methods classes around the module: recordings of class discussions, individual reflective writing, and PSTs' lesson plans or mathematics tasks designed to integrate multiple knowledge bases. Data also included surveys from 194 pre-semester respondents and 130 post-semester respondents.

We examined artifacts from the methods course using analytic induction (Bogdan & Biklin, 1992) with open coding (Corbin & Strauss, 2008). In the analytic process, initial conjectures were made from the existing data record then continually revisited and revised in subsequent analyses. Particular attention was paid to PSTs' developing understandings of the integration of children's mathematical thinking and children's cultural, linguistic, and community-based funds of knowledge. Further, drawing from Mason's (2008) work, we identified patterns of similarity and differences in PSTs' attention, awareness, and attitude with regard to building instruction on children's mathematical thinking and cultural, linguistic, and community-based knowledge.

Survey responses were analyzed both overall and by campus. Particular attention was paid to establishing PSTs' beginning and end-of-course knowledge and beliefs related to connecting children's mathematical funds of knowledge. Analysis of this data was then used to inform the development of initial learning trajectories for PSTs' engagement in this instructional module.

Results

Survey

Across all six campuses, analysis revealed that many pre-service teachers entered methods courses with the belief that connecting mathematical funds of knowledge was a valued teaching practice. More than 90 percent of PSTs agreed or strongly agreed on the pre- and post-surveys with statements such as, "The teacher can better support students' development in math if s/he draws on their home/community resources," and "Home and community activities are good contexts for posing and solving mathematical problems." Despite this agreement, we were able to see movement on these items across the semester, with substantial percentages of PSTs moving from "agree" to "strongly agree."

At the same time, analyses of instructor reflections and PSTs' written work suggest that while PSTs are very supportive of these ideas in the abstract, they tended to be very vague, particularly at the beginning of the courses, about what teachers might actually do to draw on these home and community resources. For example, when asked about the role that children's families and communities play as they are planning mathematics lessons, the majority of PSTs alluded to "making lessons students could relate to" without any specification of how one might learn about students' communities and their out-of-school experiences, or how one might use that knowledge to plan a lesson. Reponses such as "If you're familiar with your students families and communities, you can then incorporate it into your lessons" or "it's important to put math in a context that your students can relate to" were quite common. We conjecture that, through their participation in this and other instructional modules, PSTs acquired specific knowledge about mathematical resources available in children's homes and communities as well as specific ideas for using these resources in instruction. The opportunity to "try out" these ideas may have supported PSTs in beginning to incorporate them into their practices and identities, and in moving along our hypothesized trajectory toward *making emergent connections*.

Community Mathematics Exploration

The following example serves to illustrate how the *Community Mathematics Exploration* module supported some PSTs in identifying mathematical contexts for mathematics lesson development that builds on children's multiple funds of knowledge and in confronting stereotypical assumptions about children. In this example, two PSTs enlisted the expertise of a Latina parent as they engaged in a walking tour of the community. The PSTs then created a third-grade mathematics lesson inspired by what they learned on their community walk and grounded in the mathematical practices and skills used in a familiar "hub" for the local Latino community – a Lavandería/Laundromat. Specifically, the PSTs used information obtained by visiting the Lavandería, such as washer/drier prices, detergent prices, estimation of number of loads, and family budget constraints, to design high cognitive demand mathematical tasks that incorporated problem-solving and computational fluency with whole numbers and fractions.

For example, there were three washer prices: La máquina de una carga / Single load washer: \$1.75/load; Las máquinas de dos cargas / Double load washer: \$3.00/load: Las máquinas de tres cargas / Triple load washer: \$4.00/load. Mathematics tasks were situated in a familial context asking students to help "your mother" determine the cost to wash and dry 10 loads of dirty clothes, including the "maximum and minimum that one might pay." The lesson also included an extension problem incorporating fractions and rates, asking students to use their own family situation to determine the maximum and minimum cost given the parameter that each family member "makes 1 ½ loads of dirty laundry."

While not all PSTs were this successful in developing math lessons that built on children's community funds of knowledge, as will be delineated in the sections below, the example does illustrate the potential of this instructional module to support PSTs' practices related to integrating children's mathematical thinking and their home and community-based funds of knowledge in mathematics instruction. The following paragraphs highlight the learning opportunities and challenges this instructional module holds for developing PSTs' knowledge, beliefs, dispositions, and practices in this way. Specifically, analysis of the artifacts from the *Community Mathematics Exploration* module revealed that PSTs were initially wary and nervous about the activity, but that the module supported them in strengthening relationships with students, feeling more comfortable about engaging with students' communities, identifying mathematical contexts as a resource, and, in some cases, it helped PSTs confront stereotypical assumptions about students.

Learning Opportunities and Challenges for PSTs

Initial skepticism. One challenge for PSTs at the outset of the module was that to engage fully, they first had to acknowledge personal discomfort about the usefulness of such an activity. PSTs expressed initial discomfort, saying things like "Before I engaged in this activity, I was a little nervous. I was concerned that the people I interviewed would be apprehensive and not want to answer my questions fully. I also felt like I would be intruding by asking certain questions…thus, I was unsure of what to expect." Other students expressed their fears about the assignment itself, asking, "Are there specific places we should go?" or "Should what we find be strictly linked only to math?" Still others were explicit about their skepticism, asking, "How is this going to help us teach math?" and reflecting, "Before engaging in this activity, I was not sure if it was going to be relevant or helpful to me."

Strengthening relationships. Once PSTs were able to move beyond their initial discomfort and skepticism, our analysis suggests that the module supported them in a number of ways. First, PSTs saw the activity as a means to strengthen their relationships with their students and the community, reflecting that "The mathematical activity occurring in the community may not intend to shorten the distance between teacher and the students; however, I believe it does." Many PSTs reflected that they appreciated the opportunity to get to know one or more of their students better and argued that this, in and of itself, was a valuable aspect of the experience, noting things like, "Getting to know your students on a personal level is just as important as knowing them on an intellectual level."

PSTs reported seeing mathematics "everywhere" in the community, extending the idea that both mathematical activities and resources extended beyond the classroom walls. PSTs reported being "surprised" at the many ways they could mathematize the students' environments, "We came up with at least twenty promising math explorations that could [get] kids to analyze relationships in mathematical ways." For one PST, a former community activist, this activity allowed her to see her community in multiple "perspectives:" mathematics, kids, and mathematics as a tool for social justice.

> *I actively looked for numbers in places that I normally would not. It was easy to spot numbers in the café or the corner store: prices and money. But in addition, it was fun to*

> *think about the angles of the crooked fencing near the school, or the garbage-to-receptacle ratios in People's Park… In my past, I have spent a lot of time highlighting the gentrification of the [Neighborhood] to City officials, and espousing the positive qualities of the neighborhood and its residents. However, this time, with the "math" and "kid" lenses also in focus, I saw new places for justice. The corner store, with its miles of candy and its spoiled meat from [Big Store Chain], struck me, in particular. It seems to be a place preying upon kids, under the guise of convenience…I'm sure an interesting math lesson exists there.*

In addition, the community mathematics explorations allowed more nuanced views of the communities their schools served. Some PSTs reported previously "avoiding" certain communities because of negative reputation (e.g. violence, pawnshops, few grocery stores and parks). By explicitly investigating these communities, some PSTs found evidence that challenged the one-dimensional negative view of specific communities noting new parks, restaurants, building rehab projects, "ornamental additions" to streets/sidewalks, and the presence of churches (e.g.15 in a 1 mile radius of their school). The *Community Mathematics Exploration* enables PSTs to recognize mathematical practices in the community and facilitate a more nuanced and mathematical view of the community that can benefit mathematical learning and teaching in the classroom.

Comfort. Additionally, the module shows promise in helping PSTs feel more comfortable engaging with their students' communities. By working through their initial fears, PSTs saw both the importance and benefit of connecting with students' families and communities. As one PST said, "This project helped build my confidence as a teacher. It helped me overcome a bit of my shyness and open up to the people that are important to my students. I learned more about my students and gained confidence about speaking to the family members of my students." Another noted her newfound comfort, reflecting, "After completing this informal interview, I feel more comfortable exchanging words between parents of students, especially those parents who have an ESL student in my classroom." Still others found themselves trying to see schooling from the parents' perspective, reflecting, "Before conducting the interview I never realized how difficult it is for the parent of the student. I didn't realize that often the parent is learning and discovering with the child…. Through this experience, I now hope to get parents more involved in my classroom."

Identifying mathematical contexts. The module also supported PSTs in identifying mathematical contexts as a resource for mathematics problem and mathematics lesson development. All of the PSTs that engaged in this activity were able to create a mathematics problem or mathematics lesson that built on what they learned in their *Community Mathematics Exploration*. Many of these problems and lessons required rigorous mathematics and, as such, communicated high expectations for children's learning (e.g., the Lavandería/Laundromat lesson described above).

However, other PSTs struggled with coming up with good, rigorous mathematics tasks that drew on children's multiple funds of knowledge. For example, PSTs noted that many children enjoyed and participated in activities such as shopping and going out to eat and designed mathematics story problems such as, "Johnny was saving money for a toy he really wanted. His mom decided to bring him to the supermarket to empty his piggy bank in the coin machine. He has 75 quarters, 40 dimes, 26 nickels, and 175 pennies. How much money does he have for his toy?" Other PSTs merely described the kinds of activities they imagined doing with students without generating ideas for specific lessons. *Confronting stereotypes*. Finally, in some cases, the instructional module supported PSTs in confronting stereotypical assumptions about children and their communities. For example, one of the PSTs involved in the Lavandería/Laundromat exploration described above noted:

> *When one of the resource room teachers heard about our Community Math lesson project, she exclaimed that it was a great opportunity to "show them how to fix some of their problems. Maybe you can somehow make a lesson that will make parents care about their kids." We felt passionately that this bias against the community was unfair – clearly, parents in the Sunny Hill (pseudonym) community care deeply about their children. As such, we wanted our lesson to be a tiny step in the opposite direction; we wanted our project to recognize (and even celebrate) students' families and values rather than criticize them.*

These PSTs are beginning to develop capacities as culturally responsive mathematics teachers that integrate understanding of mathematics, children's mathematical thinking, and community-based funds of knowledge to promote mathematical advancement for their students.

In other cases, however, this instructional module did not fully support PSTs in moving away from stereotypes about students and their communities. For example, one PST who conducted a community walk in a rural area but did not have an opportunity to speak with any community members reflected, "These students who come from a small community may not have any idea of the problems that are being faced in the real world and teaching them through math could be beneficial. Having students explore ideas of racism, equality, socio-economic status and so forth in math would not only help them learn the relevance of math, but also introduce the students to things they might not even realize are happening."

Discussion

Returning to Mason's (2008) notions of *attention* and *awareness* and our hypothesized learning trajectory for PSTs, our survey results suggest that the majority of PSTs enter their mathematics methods courses pre-disposed to attend to children's family and community-based knowledge and experiences, but lack specific practices for doing so. At the same time, fewer PSTs begin the methods class attending to children's mathematical thinking, but gain both the knowledge and dispositions to attend to and develop awareness of children's mathematical thinking by the end of the course.

This combination of *attention* and *awareness* and attitude toward children's competence prepares PSTs to participate in more complex, field-based tasks in which they work with children in schools and the community. As we see from the results of the module reported above, our PSTs are also beginning to *make emergent connections*. This is evidenced in the fact that all PSTs were able to develop a mathematics problem that builds on children's multiple funds of knowledge, though some struggled with the mathematics involved or with knowing how to connect what they learned about the mathematics activity of the community to instruction.

As PSTs make meaningful connections, these connections then facilitate *incorporating multiple knowledge bases in instruction*. As PSTs *make meaningful connections* they might recognize, as in the example above, that children's family experiences at the Lavandería involve mathematical practices including estimation and reasoning about rates. Next, they *incorporate multiple knowledge bases* by planning problem-solving tasks that incorporate what they know about how children develop understanding of these concepts with what they know about the mathematics used in this community setting.

These results provide an important starting point for understanding how to design instructional activities for elementary methods courses that will support PSTs in learning to connect mathematical funds of knowledge in their own classrooms. Although the *Community Mathematics Exploration* was just one component of the mathematics methods course, it seemed to serve as a catalyst to challenge PSTs' negative views of mathematics and students' communities.

Endnote

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TAIWANESE PROSPECTIVE ELEMENTARY TEACHERS' KNOWLEDGE FOR FRACTION DIVISION

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Prospective elementary teachers must deeply understand fraction division to meaningfully teach this topic to their future students. This paper explores the nature of the subject content knowledge possessed by a group of Taiwanese prospective elementary teachers. The findings provide preliminary evidence that many Taiwanese elementary teachers have developed the knowledge package of fraction division as described by Ma (1989). However, the tasks of representing fraction division either through story problems or pictorial diagrams remain to be challenging. This study contributes to the continued effort to improve prospective elementary teachers' mathematics knowledge needed for teaching.

Introduction

Elementary teachers need a deep understanding of mathematics knowledge in order to optimize their students' opportunity to learn mathematics. One challenging mathematics topic for upper elementary and middle school students to learn is fraction division. Consider the following released item from the 2003 TIMSS as reported in Yoong, Yee, Kaur, Yee, & Fong (2009):

> *A scoop holds 1/5 kg of flour. How many scoops of flour are needed to fill a bag with 6 kg of flour?*

Among all participating 8th graders, only 38% of students answered this question correctly, with the three highest performing participating Asian countries—Singapore, Hong Kong SAR, and Taiwan—ranging from 79% to 75%. Many factors contribute to the low student performance. One factor that has been suggested is the quality of mathematics knowledge held by the teachers. Research studies done in the U.S. and Australia have found that many prospective elementary teachers had difficulty carrying out the fraction division algorithm correctly, and even fewer were able to develop appropriate representations for a given fraction division problem when the divisor is a proper fraction (e.g., Ball, 1990; Li & Kulm, 2008; Ma, 1999; Rizvi & Lawson, 2007; Son & Crespo, 2009). On the other end of continuum, Ma's account of Chinese elementary teachers' knowledge of fraction division has provided a rich description of what deep understanding of fraction division could be expected of practicing elementary teachers. A missing piece of this line of research is what can be expected of prospective elementary teachers. In this paper, we will provide an account of the knowledge of fraction division possessed by prospective teachers from a TIMSS high-achieving country, Taiwan.

Theoretical Framework

Based on interviews with U.S. and Chinese elementary teachers, Ma (1999) proposed a "knowledge package for understanding the meaning of division by fractions" that teachers should have, as illustrated in the Figure 1 below. According to this model, a deep conceptual understanding of fraction division is built upon a network of prior knowledge. At the top layer, it includes concepts of unit, meaning of fraction with multiplication, meaning of division with

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whole numbers, and the conception of inverse operation between multiplication and division. These, in turn, are built upon more fundamental knowledge such as concept of fraction and the meaning of multiplication with whole number, which is built upon the meaning of addition with whole number.

Fishbein, Deri, Nello, and Marino (1985) identified two primitive models for whole number division: a partitive model and a measurement model. For measurement division, one tries to determine how many times a given quantity is contained in another quantity. For partitive division, an object (or collection of objects) is divided into a given number of equal parts (or sub-collections), and the goal is to determine the quantity in (or size of) each part (or sub-collection). Prior studies have identified many challenges students (and prospective teachers alike) face when attempting the above two meanings of division to fraction division, especially when the divisor is a fraction. For example, many find the idea of sharing among “3/4” group to be very unnatural. This observation has led to the suggestion of the introduction of fraction division through measurement context and the emphasis of division as the inverse operation of multiplication. Both were identified as main characteristics of a fraction division unit in Chinese and Japanese mathematics curricula (Li, Chen & An, 2009).

Now, we will use our table to find numerators and denominators that divide equal to 3 (the difference in hours travelled).

Numerator	Denominator	Divide	Answer
27	9	$\frac{27}{9}$	3
15	5	$\frac{15}{5}$	3
9	3	$\frac{9}{3}$	3
6	2	$\frac{6}{2}$	3

So now, using our equations above, we must find numerators that satisfy $\frac{d_1}{t_1} = 3$, $\frac{d_3}{t_3} = 3$,

Figure 1. A knowledge package for understanding the meaning of division by fraction, p. 77 of Ma (1999)

Two unit-related concepts are critical to the development of a deep understanding of fraction multiplication and division. The first one is the idea of unitizing. Steffe (2003) defines unitizing as a mental operation that treats an object or collection of objects as a unit, or a whole. In the measurement division, when considering “How many scoops of 1/5 kg flour are needed to fill out a 6 kg bag?” students need to be able to conceptualize 1/5 kg as a unit, and use it to measure the 6 kg. The second one is the idea of referent unit. Izsák, Orrill, Lobato, Cohen, and Templin (2009) suggest that fraction tasks in the school curriculum can be separated into two categories. One includes those that conserve the referent unit, such as those that explore the relationship between the parts and the whole, compare fractions, and add or subtract fractions. For example, the answers for adding ½ cup of sugar and ¾ cup of sugar will still carry the same referent unit, “cup.” The other includes multiplication and division of fractions, where the referent units change. For example, in the above flour packing problem, the answer 30 will carry the unit "scoops," which is different from the referent unit given in the problem. Teachers need to be aware of these differences and have the ability to deal with referent units appropriately across different tasks.

This study investigates prospective elementary teachers' fraction division concepts with respect to three main areas: meanings of fraction division and multiplication, inverse relationship between multiplication and division, and the two unit-related ideas discussed above. Their solution strategies in three different tasks—number line, story problem, and representations with story problems and pictorial diagrams—are examined. The first two fall under what Ball, Thames, and Phelps (2008) refer to as common content knowledge (CCK). CCK refers to the knowledge needed to solve mathematics problem that is not unique to teaching. This knowledge is distinguished from specialized content knowledge (SCK), which refers to mathematical knowledge and skills uniquely needed by teachers, such as posing story problems and utilizing pictorial diagrams to facilitate student reasoning.

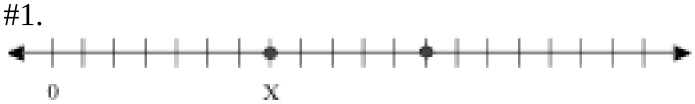
Methodology

The data analyzed in this paper came from a research study on prospective elementary teachers' fraction knowledge. Participants included 45 prospective elementary teachers from a traditional teacher education university in Taiwan. All prospective elementary teachers in Taiwan have either a content concentration (e.g., language arts, math, social studies) or a specialization area (e.g., special education, counseling education). The participants in this study included 28 special education majors, 7 art education majors, and 10 counseling education majors.

A 16-item instrument was administered in two separate sections at the beginning of the mathematics methods course. This methods course is the only required math education-related course for non-math majors in the teacher education program at this university. The first section of the instrument contains 12 items that focus on CCK, including basic fraction concepts (e.g., part-whole, unitizing, ordering, meanings of operations, etc.) and story problems (one-step and multiple steps). The second section contains 4 items that focus on SCK, such as explaining a fraction problem in multiple ways, posing story problems for given computations, and using diagrams to model the solutions of given computations. Because of student absences, 41 students answered the first part and 40 students answered the second part.

The focus of this paper is on prospective elementary teachers' solutions to three fraction division-related items. The first two focus on CCK and the third one on SCK. The three items are listed below.

#1.



What is the value of x?
(a) 7/12 (b) 3/5 (c) 7/10 (d) 7/15 (e) None of the above

#2. Jim jogged 1 ½ miles yesterday. This is 3/8 of his weekly goal. How many miles does he plan to run each week? Explain your solution.

#3. Write a story problem which can be solved by using "8 2/3 ÷ ¼=?" and model how to solve it by drawings.

The analyses of the prospective elementary teachers' responses were guided by the following two main research questions:

3) How solid was Taiwanese prospective elementary teachers' common fraction knowledge? What strategies did they use to solve fraction division word problems? Did their strategy reflect flexibility and connections as depicted by Ma's model?
4) How successful were Taiwanese prospective elementary teachers in their representations of fraction division through story problems and pictorial diagrams? What meanings of division did prospective elementary teachers use when constructing their representations?

Prospective teachers' solutions and explanations for #1 and #2 were coded individually for correctness and type of strategy. Problem #3 was analyzed separately for the story problem they posed and the types of diagrams they used, taking into consideration the correctness and reasonableness of story problems, diagrams, and final answers.

Results

Prospective Elementary Teachers' Solution Strategies for Fraction Division Problems

Similar to the prior studies (Luo, Lo, & Leu, 2008, 2009), prospective elementary teachers in Taiwan are proficient in solving fraction division problems. All but one student answered #1 correctly, and 100% answered #2 correctly in this study. Three main strategies were used to solve #1; one student skipped this problem. These strategies are summarized below.

Strategy 1: (Unit fraction approach: Identifying the quantity associated with the "unit-fraction" first) *6/5 is equivalent to 12 units, so each unit is equivalent to 1/10. Since the unknown quantity is equivalent to 7 spaces, it is equivalent to 7/10.*	*63.4%*
Strategy 2: (Part-whole approach: Conceptualizing the part-whole relationship between the given and unknown quantity) *The unknown quantity is 7/12th of 6/5. Therefore, it is equivalent to 7/12 × 6/5 = 7/10.*	*26.8%*
Strategy 3: (Ratio approach: Setting up a ratio relationship between the given and unknown quantity) *The ratio between the unknown quantity and 6/5 is equivalent to 7: 12. Applying cross multiplication, the unknown quantity is equivalent to 7/10.*	*7.3%*

Four main strategies were used to solve #2. Three of those strategies, the same as those used to solve #1 above, and one additional strategy are listed below.

Strategy 1: (Unit fraction approach: Identifying the quantity associated with the "unit-fraction" first) *1 ½ miles is 3/8 of the total distance, ½ mile is 1/8 of the total distance, so the total distance is ½ × 8 = 4 (miles)*	*14.6%*
Strategy 2: (Part-whole approach: Conceptualizing the part-whole relationship between the given and unknown quantity) *1 ½ miles is 3/8 of the total distance. Let x equal the total distance. Set up the equation 3/8(x) = 1 ½, so x = 1 ½ × 8/3= 4 (miles)*	*68.4%*
Strategy 3: (Ratio approach: Setting up a ratio relationship between the given and unknown quantity) *The ratio between the total distance and 1 ½ miles is equivalent to 1: 3/8. Applying*	*7.3%*

cross multiplication, the unknown quantity is equivalent to 4 (miles).	
Strategy 4: (Partitive division approach: Conceptualizing division as finding the per one quantity) 1 ½ (miles) ÷ 3/8 = 4 (miles).	*9.8%*

Further analysis was conducted to see if prospective elementary teachers used the same or different strategies for solving these two problems. Among 40 who attempted both problems, 60% used two different strategies for the two problems. The following table shows the distribution of the strategies used for these two items.

Strategies for #2

		Part-Whole	Unit Fraction	Ratio	Partitive Division
Strategies for #1	Part-Whole	10	1	0	0
	Unit Fraction	15	5	2	4
	Ratio	2	0	1	0

As the data illustrate all but one prospective elementary teacher who used part-whole approach for the Number-Line problem continued to use the same strategy for the Jugging Problem. But many of those who used unit-fraction approach for the Number-Line problem switched to a different strategy when solving the Jugging problem.

Prospective Elementary Teachers' Representations of Fraction Division

Similar to prior studies, prospective Taiwanese elementary teachers' performance level for representing a fraction division problem was much lower than their ability to solve a fraction division problem. Overall, only 65.0% (n = 40) were able to pose a story problem that was meaningful and correct for the given fraction division computation "8 2/3 ÷ ¼ = ?" Of those teachers posing successful story problems, 21 out of 26, based their problem on the meaning of measurement division, 4 out of 26 based it on partitive division, and the remaining 1 student gave the following area problem: "The area of a blackboard is 8 2/3 square meters. The length of it is ¼ meter, how wide is the board?" Also, similar to the previous findings, the majority of the incorrect story problems were about the computation problems "8 2/3 ÷ 4=?" or "8 2/3 × 4=?" (e.g., Ma, 1989).

Prospective Taiwanese elementary teachers' performance level to represent with a pictorial diagram was even lower. Fifty-five percent were able to draw a pictorial diagram that depicted the underlying meaning of fraction division with various degrees of detail. For example, some students showed only the nature of repeated subtraction for measurement division (Figure 2).

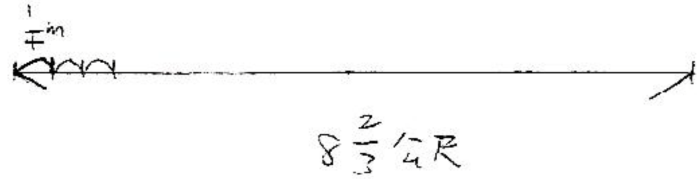


Figure 2. Diagram accompanying the story problem "A long ribbon is 8 2/3 meters. How many pieces of ¼ meter ribbon can be cut from this long ribbon?" by ID#18

In this case, the diagram was used to illustrate the underlying mathematical structure embedded in the problem. This provided a sufficient to justify the division operation later used to compute the answer. However, other pre-service teachers used a different type diagram that showed both the underlying mathematical structure as well as the process of reaching the correct answer. Figure 3 below shows an example of such diagram.

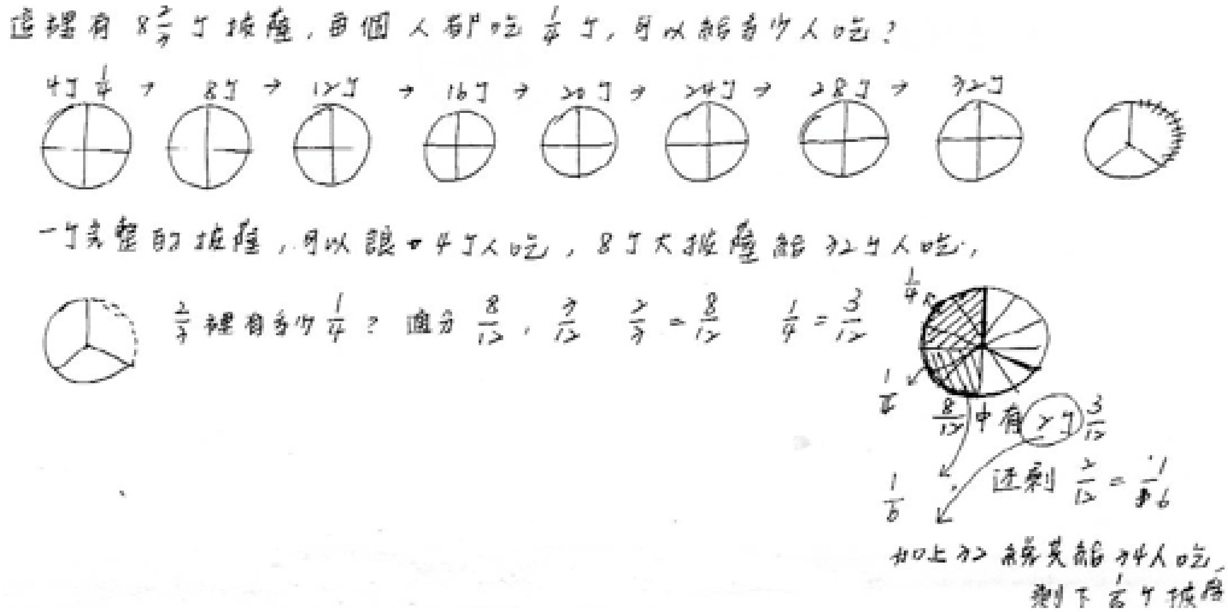


Figure 3. Diagram accompanying the story problem "There are 8 2/3 pizzas. If each person gets ¼ of a pizza, then how many people can get pizza?" by ID#9

The goal of the first type of drawings was to illustrate the underlying meaning of the operation. This is different from the second type of pictorial diagram, where the diagram was used to identify the correct answer in addition to illustrating the underlying mathematical structure of the problem. For example, in the second diagram, one could see the 32 shares in 8 pizzas. In order to figure out how many ¼ pizza are in 2/3 pizza, this prospective elementary teacher changed 2/3 pizza to the equivalent of 8/12 pizzas and noted that there were "two" additional 3/12 pizza (= 1/4 pizza) in 8/12 pizzas, with 2/12 (= 1/6) of a pizza as left over. The majority of the Taiwanese prospective elementary teachers drew the first type of pictorial diagram.

As discussed earlier, one main challenge for solving fraction division problems was to pay attention to the referent units. In the case of measurement division, the dividend, divisor, and remainder will all have the same referent unit, but the quotient will have a different referent unit. In the case of partitive division, the dividend, quotient, and remainder will have the same referent unit, but the divisor will have a different one. Depending on the given story problems, the correct answer could be 34, 35, 34 2/3, or 34 with 1/6 left over, with the appropriate referent unit attached. A close examination of the prospective elementary teachers' solutions indicates that, among those having correct story problems and appropriate pictorial diagrams, all but one had answers reflecting the appropriate use of referent units.

Discussion

The analysis presented above suggests the following characteristics of Taiwanese prospective teachers' knowledge of fraction division. First, they are very proficient in solving fraction division problems commonly seen in upper elementary and middle school levels. Second, there exists preliminary evidence that many of the prospective elementary teachers in this study have developed the knowledge package of fraction division as described by Ma (1999). All three major strategies prospective teachers used to solve the Number Line problem and the Jogging problem —the unit fraction approach, the part-whole approach, and the ratio approach—are built upon multiple pieces of the knowledge package. Being able to use different strategies also demonstrates the flexibility of their reasoning. Furthermore, it is worth noting that, while many prospective teachers solved the Jogging Problem by setting up an algebraic equation, none set up an algebraic equation for solving the Number Line Problem. Rather, they approached the problem through either a unit fraction approach or a part-whole fraction approach. This shows an ability to move between the arithmetic-based reasoning and algebraic-based reasoning, which is a critical goal of mathematics curriculum transition from the elementary to middle school level. Third, similar to the prior studies of prospective elementary teachers in the U.S., the findings of this study also show that Taiwanese prospective elementary teachers were less proficient in posing story problems and using pictorial diagrams appropriately to model a solution strategy than solving fraction division problems directly. Rather than using pictorial diagrams as a tool to "find the answer" of a given story problem, the pictorial diagrams produced by the prospective teachers were used to show the underlying structure of an operation.

Implications

The finings of this study were limited by a small sample of prospective elementary teachers from Taiwan. Nevertheless, the study contributes to the continuing effort to optimize the mathematics learning for all students by focusing on the mathematic knowledge teachers needed to teach effectively. The most recent TIMSS, conducted in 2007, and the Programme for International Student Assessment (PISA), conducted in 2006, indicate that Taiwanese students had a much higher level of mathematics achievement than their American counterparts in a national representative sample of 4th graders, 8th graders, and 15-year-olds (Mullis, Martin, & Foy, 2008; Organization for Economic Co-operation and Development [OECD], 2007). The findings of this study suggest that the roots of such discrepancy in student performance might be in the mathematics knowledge possessed by prospective elementary teachers. The majority of Taiwanese prospective elementary teachers entered their teacher education program with a much deeper understanding of mathematics than their U.S. counterparts. The characteristics of Taiwanese prospective elementary teachers' knowledge of fraction division could provide an expectation benchmark for designing the required mathematics courses for prospective elementary teachers in the U.S.

Even though elementary students in Taiwan are not required to pose story problems or use pictorial diagrams to demonstrate their proficiency with fraction divisions, it is a common practice for elementary school teachers to use both representations when engaging students in reasoning with fraction multiplication and division. This experience alone appeared to be sufficient for many prospective elementary teachers to be able to do such "teacher-tasks" without being taught directly. This seems to suggest that specialized content knowledge could be co-developed with the common content knowledge with a carefully developed mathematics curriculum and an instructional emphasis on reasoning. Nevertheless, a sizeable number of

Taiwanese prospective elementary teachers, despite their strong CCK, were unable to use story problems or pictorial diagrams appropriately. More research studies are needed to identify the conceptual challenges of developing SCK for fraction division and other important mathematics topics.

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WHERE'S THE MATH? EXPLORING RELATIONSHIPS AMONG MATHEMATICS, MATHEMATICAL LITERACY, AND LIETRACY

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This paper considers the heuristic value of using the perspectives "mathematics," "mathematical literacy," and "literacy," individually and together, in discussions of situated geometry-teaching practices with pre-service elementary teachers.

Introduction

Recently, mathematics education researchers Jon R. Star, Sharon Strickland, and Amanda Hawkins (2008) have raised the question "What is mathematical literacy?" Referencing literature in both mathematics education and literacy studies, they have distinguished between two field-specific uses of the term "content-area literacy." According to Star, Strickland, and Hawkins, *literacy* educators tend to read this term as "content-area *literacy*," meaning "reading and writing activities that are embedded within content courses," such as "keeping a mathematics journal, reading newspaper articles related to mathematics, reading the mathematics textbook or other mathematics-related books, writing research papers on topics in mathematics, and writing paragraph explanations to accompany solutions to mathematics problems" (p. 106). As these authors have pointed out, such content-area-literacy tasks may be thematically related to mathematics without requiring conceptual understanding of the discipline (p. 106). In contrast, Star, Strickland, and Hawkins have observed that *mathematics* educators tend to read the term "content-area literacy" as "*content-area* literacy" or as "*mathematical* literacy," which itself has been variously used to mean (1) "mathematical *understanding*, including knowledge of content and the ability to approach mathematical problems (such as those seen in mathematics texts) logically, analytically, and thoughtfully"; (2) "*appreciation* of mathematics, including the ability to recognize when and how mathematics is used in the real world"; (3) "*application* of mathematics to real-world problems, including calculating tips in restaurants, working with budgets, and reading graphs in newspapers"; and (4) "ability to *reason* mathematically, including mathematical communication" (p. 110, emphasis in original).

Speculating on potential consequences of these different uses of the term "content-area literacy," Star, Strickland, and Hawkins have brought into focus challenges that secondary mathematics teachers may face in attempting to integrate content-area *literacy* and *content-area* literacy in their mathematics courses: "…Content-area *literacy* can be faulted for its disconnection with content, while *content-area* (*mathematical*) literacy is perhaps guilty of an insufficient connection to literacy" (p. 111, emphasis in original). According to these authors, current institutional demands on U.S. secondary mathematics teachers to incorporate content-area *literacy* instruction in their courses are especially difficult to meet, given that middle- and high-school teachers have likely had few opportunities to study and practice literacy teaching methods in their teacher-preparation programs (p. 107). But what about U.S. *elementary* teachers whose professional responsibilities include teaching both mathematics and literacy *as content areas*? (U.S. elementary teachers are currently under tremendous pressure to help students in the early grades to develop English language arts skills. Reading abilities, in particular, determine whether or not students are promoted to the next grade.) What conceptual and pedagogical challenges related to content-area literacy might be foregrounded if we consider the particular institutional situation of *elementary* mathematics teachers? How might these insights resonate with the predicament of *secondary* mathematics teachers who teach content-area literacy? Moreover, how might the question of integrating the teaching of content-area *literacy* and *content-area* literacy in mathematics lessons be differently pursued if we think about *pre-service* elementary teachers' designing and delivering mathematics lessons during their teaching internship? (Pre-service teachers in U.S. teacher preparation programs tend to study a great deal of English language arts topics, especially early literacy development.) Are mathematics and literacy always so difficult to teach together? Admiring the theoretical rigor that Star, Strickland, and Hawkins have contributed to conversations on mathematical literacy, we will continue their "explorations" in this paper, as we consider the conceptual and pedagogical challenges of distinguishing among "mathematics," "mathematical literacy," and "literacy" when elementary teacher candidates teach geometry to early elementary students.

Teaching Elementary Geometry

Elementary geometry teaching and learning are understudied by mathematics education researchers, who have favored studies of children's development of arithmetic concepts and skills, a trend reflected in U.S. elementary mathematics curricula and teacher preparation programs, both of which have placed a great deal of emphasis on Number and Operations (Chapin & Johnson, 2006, p. 220; NCTM, 2000). One reason why mathematics education researchers have directed relatively little attention to elementary geometry teaching and learning is that, in contrast to other mathematical topics, U.S. geometry curricula for the elementary grades have historically demanded less mathematical rigor of students. Moreover, these units of study, especially for the early grades, have tended to focus on literacy development, particularly on students' acquisition of geometry *vocabulary*. The National Council of Teachers of Mathematics (NCTM) (2000) "Geometry Standard for Grades Pre-K-2" suggests that early elementary students should learn to "recognize, *name*, build, draw, compare, and sort two- and three-dimensional shapes"; and to "*describe* attributes and parts of two- and three-dimensional shapes" (p. 96, emphasis ours). Similarly, the NCTM "Geometry Standard for Grades 3-5" recommends that upper-elementary students should learn to "identify, compare, and analyze attributes of two- and three-dimensional shapes, and develop *vocabulary* to describe the attributes"; and to "classify two- and three-dimensional shapes according to their properties and *develop definitions* of classes of shapes such as triangles and pyramids" (p. 164, emphasis ours). Put differently, NCTM has proposed that all elementary students should develop lexical and taxonomic fluency related to particular geometry topics, abilities that entail both mathematics and literacy knowledge. Whereas mathematics educators may tend to read NCTM's elementary geometry standards as expectations for *mathematical* performances, these standards can also be read as criteria for evaluating students' displays of *mathematical literacy* or, more broadly, mathematical rhetoric. It is this interdisciplinary potential of U.S. elementary geometry curricula that we will begin to explore in this paper.

Mathematics, Mathematical Literacy, and Literacy

Extending Star, Strickland, and Hawkins's discussion of content-area literacy, we wish to introduce two more terms into the conversation: "Writing Across the Curriculum (WAC)" and "Writing in the Disciplines (WID)." The term "content-area literacy," read as "content-area *literacy*," is a K-12-specific articulation of the tenets of Writing Across the Curriculum, a higher-education literacy movement whose supporters have promoted extensive writing assignments in a variety of university subject-matter courses beyond traditional first-year writing courses (Bazerman et al., 2005, pp. 9, 11, 32-34). Similarly, the notion of "*content-area* literacy" is associated with a higher-education literacy movement, Writing in the Disciplines, whose advocates have sought to investigate writing as it is practiced in specific disciplinary contexts and to use research on disciplinary rhetoric to inform writing instruction (Bazerman et al., 2005, pp. 9-10, 66-104). [WID researchers have tended to share purposes and methods with Writing in the Professions researchers, who have studied writing practices in law, medicine, and engineering, for example (Bazerman et al., 2005, pp. 10-11).] Put differently, both content-area *literacy* and *content-area* literacy are issues of interest to the broad literacy-studies community. Moreover, WAC and WID bear on mathematics education. For example, if mathematics educators are dissatisfied with the contributions of WAC research to the proliferation of content-area *literacy* tasks in mathematics courses, they may wish to draw instead on WID research as they inquire further into *mathematical* literacy. (Issues of disciplinary "reasoning" and "communication" like those raised by Star, Strickland, and Hawkins (p. 110) have been pursued by WID researchers.)

Distinguishing among "mathematics," "mathematical literacy," and "literacy" is itself a rhetorical project shaped, in part, by considerations of audience, genre, and purpose. We do not wish to suggest otherwise. Indeed, in this paper, we contend that evidence of mathematics, mathematical literacy, and literacy may be generated at any moment in a mathematics lesson. For example, imagine a situation in which a pre-service early-elementary teacher introduces the term "square" to students by drawing the four-sided figure on the classroom whiteboard and announcing, "This is a square." From a *mathematics* perspective, we might observe that the teacher is inviting students to engage the initial van Hiele level of geometric reasoning, "visualization," or shape awareness (van Hiele, 1986). If the teacher then draws a second square on the whiteboard, which, relative to the first, has been turned clockwise by ninety degrees, and declares, "This is also a square," we might similarly note that the teacher is encouraging students to work at the next van Hiele level, "analysis," or recognition of shape properties (van Hiele, 1986). From a *literacy* perspective, we might remark that the teacher is showing students that a square is not a letter or a number, that the square figure may be turned without semiotic consequence, which would not be the case for the letters *b*, *q*, *d*, and *p*, or the numbers *6* and *9*. From a *mathematics literacy* perspective, we might think that the teacher is beginning to distinguish for students rhetorical conventions associated with *mathematical* uses of the term "square" from those related to everyday uses of the same term. For example, in general parlance, we might say that a "square" of chocolate retains its "squareness" even when its edges melt and become slightly rounded. In this paper, we will read pedagogical situations through the lenses of "mathematics," "mathematics literacy," and "literacy" not to argue that the boundaries among these terms may be, or should be, determined once and for all but, rather, to claim that these three frames may be usefully mobilized, both individually and in relationship to one another, to open conversations on the purposes, practices, and effects of mathematics teaching.

The Posing, Interpreting, and Responding (PIR) Project

The following scenarios have been informed by our ongoing, National Science Foundation-sponsored study of pre-service and practicing elementary mathematics teachers' posing, interpreting, and responding practices. Our study comprises both cross-sectional and longitudinal study designs. In the cross-sectional study, we have collected pre-service elementary teachers' written imaginings of their pedagogical responses to a variety of teaching scenarios, using survey-style open-response questions, at three different points in their teacher preparation program. Based on these survey responses, we have selected 16 focal participants for two concurrent longitudinal studies. We have followed one set of participants from their first to last year of teacher preparation, and the other set from their last year of teacher preparation to their second year of beginning teaching. In this paper, we focus on two participants in the first longitudinal study, both of whom, at the time of the classroom observations referenced here, were in the early months of their teaching internship in a Midwestern state, whose elementary geometry standards were consistent with those of NCTM. Both of these pre-service teachers had also taken a required course in "geometry for teaching" one-to-two-years prior to the internship.

In the course of the cross-sectional and longitudinal studies, we have gathered evidence of pre-service and practicing elementary teachers' imagined and implemented mathematics teaching practices, a distinction which we have drawn in order to explore relationships among our cross-sectional data sources—various "What would you do?" exercises (imagined teaching)—and our longitudinal data sources—lesson plans and pre- and post-lesson interviews (imagined teaching), and videotaped lessons (implemented teaching). However, we consider our distinction between "implemented" and "imagined" teaching practices to be analytic rather than ontological: Where does text end and reading begin? In keeping with this premise, we present below imagined/observed scenarios of early-elementary geometry teaching, constructed from case studies of two of our 16 focal participants.

Teaching Geometry in Early Elementary Education

Red-tangles, Ovals, and Other (Non-)Shapes

Imagine a kindergarten classroom. Imagine students sitting along the edge of the classroom's oval carpet. Imagine a pre-service elementary teacher, two months into the teaching internship, sitting on the carpet beside the students. Before the teacher are piles of colorful construction-paper cut-outs of geometric shapes. The circles are small and black, about two inches in diameter; the squares are similar in size but red. One kind of rectangle is large and red, a 5x7-inch sheet of construction paper; another is narrow, long, and gray, about six inches in length. Imagine the beginning of a geometry lesson.

Teacher (displaying a large red rectangle for the class): What shape is this?
Students (simultaneously): Red. Rectangle.
Teacher: What shape is this?
Students: Red…tangle.
Teacher: Good. How come it is not a square? Do you remember?
Student 1: A square has four ends, and a rectangle has four ends.
Teacher: They *do* both have four ends. But what is special about the rectangle?
Student 2: There's two big sides and two small sides.
Teacher (displaying a square): Good, and what shape is this, then?
Students: Square.
Teacher: Square. How do we know this is a square?
Student 3: It has four ends. And it is littler.
Teacher: It doesn't matter that it's littler. It could be the same size. Because of why? Student 4?
Student 2 (speaking out of turn): Because they're all the same length.

Student 4: Because they're all the same length
Teacher: Because they're all the same length.

From a mathematics perspective, we might read this scenario as an illustration of a pre-service teacher's engaging students in learning geometry content consistent with the first two van Hiele levels ("visualization" and "analysis") and endorsed by the NCTM standards (p. 96). The teacher invites students to recognize and name two-dimensional geometric shapes and to discuss their properties ("What shape is this?" "How come it is not a square?"). Students generate a sense of "squareness" and "rectangleness" that exceeds a comparison of the particular cut-outs displayed by the teacher ("A square has four ends [corners], and a rectangle has four ends [corners]…[but a rectangle has] two big sides and two small sides"). From a literacy perspective, we might read this episode as an example of a pre-service teacher's involving students in a demonstration of how one might conduct an oral discussion about a topic, a rhetorical practice that students would ultimately stage in their written arguments. We might focus on the situated consequences of speaking out of turn in a discussion. From a mathematical literacy perspective, we might read this exchange as a case of a pre-service teacher's facilitating students' emerging distinction between mathematics-specific and everyday uses of the terms "rectangle" and "square." We might emphasize the quantity-related vocabulary voiced by students ("four," "two," "big," "small," "littler," "length," "same"). However, we might also merge these three frames to consider the question of the "red-tangle." Is "redness" an essential property of a (mathematical) "rectangle," as the metonymy of "red…tangle" in the scenario above would suggest? Is "redness" an essential property of a (mathematical) "square"? If it is possible to confuse a red rectangle and a red square, is it also possible to confuse a gray rectangle and a red square? Is a (mathematical) "square," a (mathematical) "rectangle"? Is "red," a shape? We think that mathematics teacher educators might productively mobilize the perspectives of "mathematics," "mathematics literacy," and "literacy," individually and together, with pre-service elementary and secondary teachers in discussions of the goals and implications of contextualized pedagogical approaches.

Imagine the end of the lesson. Imagine the students sitting again along the edge of the oval carpet, which has both an inner and an outer border, yellow and red. For the past half hour, students have played a game at their work tables, resulting in their each having glued the set of construction-paper cut-outs to a blank sheet of paper to make a picture of a fire truck. Some students have drawn fire fighters and Dalmatian fire dogs in their fire trucks. Imagine the teacher again sitting beside the students on the carpet, a pile of the students' pictures in hand.

Teacher (displaying a student's picture for the class): What shape is this fireman's head? What shape is his head (points to the fire fighter's elliptical face)?
Student 5 (softly): An oval?
Teacher: A circle. His head is a circle.
Student 6: He has hair.
Teacher: And he has hair, doesn't he? What shape is this sun up here (points to the picture)? What shape is this?
Student 7: A circle.
Teacher: A circle. Can you think of any other shapes you see? What shapes do you see in the classroom? What shape is our rug?
Student 8: Circle.
Student 9: Oval.

Teacher: It's an oval. Good.
Student 10: This is a circle right here (points to the carpet's yellow inner oval), and this is a oval (points to the carpet's red outer oval).
Student 11: They're both oval.
Teacher: They're both ovals. You're right. They're both ovals.

This scenario is marked by certainty and confusion ("An oval?" "A circle." "Circle." "It's an oval"). What is a "circle," and what is an "oval"? Do "circle" and "oval" operate in this exchange as geometric concepts, geometric rhetoric, real-world artifacts, and/or everyday words? What might it mean to "understand," "apply," "appreciate," and "reason" (Star, Strickland, & Hawkins, p. 110), both as a teacher and as a student, in this geometry situation? We think that these questions might be usefully raised with pre-service elementary and secondary teachers.

Apples, Squiggles, and Other (Non-)Shapes. Imagine a first-grade classroom. Imagine students sitting at their work tables. Imagine a pre-service elementary teacher, one month into the teaching internship, standing before the students. Imagine the beginning of a geometry lesson.

Teacher: Think of one of the shapes that we talked about this week.
Teacher (after several seconds): Who can think of a shape? Student 1?
Student 1: A Granny Smith.
Teacher: A Granny Smith? Is that a shape?
Students (laughing): No.
Teacher: That's an apple. Raise your hand if you can think of a shape. Student 2?
Student 2: Circle.
Teacher: Circle.

We read this scenario as an illustration of the conceptual and pedagogical challenges of distinguishing among "mathematics," "mathematical literacy," and "literacy," particularly when considering elementary geometry teaching. In this exchange, unlike in the sample content-area literacy activities that Star, Strickland, and Hawkins have presented, mathematics and English language arts are not easily separated and thus difficult to integrate; rather, they are entangled. After all, the questions *What is a shape?* and *How might one teach "shape"?* can be answered from mathematics, mathematical literacy, and literacy perspectives. Moreover, within each perspective, we might plausibly respond to the question *Is a Granny Smith, a shape?* with both *Yes* and *No* answers. In the scenario above, both the pre-service teacher and Student 1 can associate multiple meanings with the word "shape." We can imagine that the teacher, in other rhetorical (even pedagogical) situations, might not treat an apple as never being a shape, or a circle as always being a shape. Similarly, we can surmise that Student 1, at the time of this exchange, was not utterly lost when a classmate proposed that a "circle" is a shape, though Student 1 may have been puzzled as to why both a circle and a Granny Smith must not be included in the same set of shapes. The interfluency of mathematics and English language arts in acts of geometry teaching is an issue that we think mathematics teacher educators might explore with elementary and secondary pre-service teachers. The content-area-literacy challenge is not always one of integrating mathematics and literacy; sometimes it is one of extricating mathematics from literacy, or literacy from mathematics, or mathematical literacy from mathematics and literacy, depending on one's conceptual and pedagogical purposes at the time.

Imagine several minutes later in the geometry lesson. Imagine the students still sitting at their work tables. Imagine a flip chart, supported by an easel, poised before the students. On the flip chart, two long, horizontal, awkwardly undulating lines have been hand-drawn (perhaps with an unsteady hand). Beside these squiggles, the word "parallel" has been written. Imagine the teacher standing beside the flip chart.

> Teacher: Who remembers what two lines are called if they are like this (traces the squiggly lines with two fingers), and they're never gonna touch each other, they're never gonna meet or cross? It starts with a puh, puh, puh (makes the aspirated sound of the letter "p" in English). Student 3, what are they called? Do you remember?
> Student 3: Line.
> Teacher: They're a special kind of line. It starts with a "p." Does anybody remember? Student 4?
> Student 4: Parallel.
> Teacher: Parallel lines. And when we have parallel lines (traces the squiggly lines with two fingers), that means that these lines are gonna keep going, and they're never gonna touch. So are these lines parallel?
> Students (simultaneously): Yes. No.

In this scenario, the pre-service teacher connects for students the oral word "parallel" ("It starts with puh, puh, puh") with the written word "parallel" ("It starts with a 'p'") and a visualization of the notion "parallel" (the drawing of wavy lines). Mathematics educators may recognize the written word "parallel" as a literate performance. But what about the drawing of "parallel lines"? If we regard the hand-drawn parallel lines as phenomenal objects from either a mathematics or a literacy perspective, we might agree with some of the students that these squiggles are not "parallel": They do not meet the requirements of either general or mathematics-specific definitions of the term. Even from a mathematical literacy perspective, we might say that this visualization of "parallel lines" does not adhere to rhetorical conventions of the discipline, as would the symbol "$\parallel$" (vs. "$\perp$"). However, from a *pedagogical* mathematical literacy perspective, we might remember our own experiences with having seen mathematics teachers (perhaps even ourselves) hand-draw parallel lines in a demonstration for students, and we might speculate that the pre-service teacher in the scenario above was reiterating a convention of visual rhetoric for mathematics *teachers*. In other words, we might agree with some of the students that the squiggles *are* "parallel lines." Of course, all of these arguments would be arguments, and we think that staging such a dialogue for pre-service elementary and secondary teachers would be generative, in that it would ask pre-service teachers "Where's the math?" Speaking, writing, and visualizing geometry for students does not necessarily mean that one is teaching mathematics. Nor does it necessarily mean that one is not.

Imagine several minutes later in the geometry lesson. Imagine the students still sitting at their work tables. Imagine that a new page on the flip chart is visible. It is a chart with columns for the names of various geometric shapes; colorful, handmade construction-paper cut-outs of each shape; the number of sides for each shape; the number of corners for each shape; and indications whether all of the sides of each shape are equal in length. Imagine the teacher standing beside the flip chart, capped marker pen in hand.

Teacher: Let's look at our shapes. Let's see if we can find some shapes that have parallel lines. Let's look at our circle. Is our circle gonna have any parallel lines?
Students: No.
Teacher: Why not? Why couldn't our circle have any parallel lines? Student 5?
Student 5: 'Cause it's round.
Teacher: 'Cause it's round. Does it have any sides at all?
Students: No.
Teacher: No. How could it have parallel sides if it doesn't have any sides? What about our triangle? Look at the sides of the triangle (traces three sides of the cut-out triangle with one finger), the straight sides. Do you see any parallel lines? Can our triangle have parallel lines? Raise your hand if you think our triangle could have parallel lines (traces three sides of the cut-out triangle with one finger), two lines that are never gonna touch each other? Do you see any on that triangle?
Students: No.
Teacher: No, because, see, this side and this side (traces two sides of the cut-out triangle with the capped marker pen, extending the line of each side so that they intersect) are gonna touch right here (points to vertex). And then this side is gonna touch with this side (traces two sides of the cut-out triangle with the capped marker pen, extending the line of each side so that they intersect) in the corners. So a triangle doesn't have any parallel lines.

Imagine that the teacher turns over a new page on the flip chart. Imagine that, with the marker pen, now uncapped, the teacher draws a triangle on the new page. Imagine that the teacher then hand-draws lines extending the sides of each triangle, making visible in ink the intersections of these lines. The lines are squiggly, not having been drawn with a straight edge.

Teacher: Can these lines be parallel?
Students: No.
Teacher: No, because they're crossing right here.
Student 6: 'Cause it makes an "x."
Teacher: It does make an "x." That's a good way of thinking about it. If you see two lines and they make an "x," you know that they're not parallel, because they're meeting.

Imagine that the teacher repeats this process for a "square" and a "rectangle," using a ruler to draw the shapes and hand-drawing the two sets of extended parallel lines in different colors of ink. Imagine that the teacher draws a "rhombus," using the straight edge of the ruler, then invites a student to hand-draw the extended lines. Imagine that the teacher repeats this student-assisted process for a "parallelogram."

Teacher: Let's look up at these purple lines. Student 7, do they look like they're gonna touch? What do you think?
Students: No.
Teacher: Raise your hands if you think that these purple lines are parallel.
Students (raise hands)
Teacher: Interesting that this shape is called a parallel-ogram.

In this scenario, the pre-service teacher's oral and visual rhetoric make several conceptual substitutions: Parallel lines are conflated with parallel sides; straight lines with squiggles, intersecting lines/sides with the letter "x," and a parallelogram with a "parallel-ogram." From mathematics, mathematical literacy, and literacy perspectives, we can ask whether such substitutions cross content-area knowledge domains and what effects such crossings might have for particular geometry lessons. In the words of the pre-service teacher in the scenario above, when are mathematics education and literacy education "parallel" projects, and when do they "touch"? When is it conceptually and pedagogically useful to consider general and mathematics-specific rhetorics together and apart? When do literacy terms impede and/or improve mathematics lessons, and vice versa? We think that these questions may be valuably posed to pre-service elementary and secondary teachers.

Where's the Math?

In conversations among mathematics education researchers, the provocative question *Where's the math?* tends to elicit simple answers, like *Nowhere* or *Everywhere*. Nevertheless, we argue that this question can generate complicated responses if pursued from the perspectives of "mathematics," "mathematical literacy," and "literacy," either individually or together. Separate and apart, these three frames can bring into focus the complexity of language-based mathematics instruction, including the perhaps surprising complexity of early elementary geometry teaching.

Endnotes

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CONCEPT-FOCUSED INQUIRY: PROGRESS TOWARDS A VIABLE THEORY FOR MATHEMATICS INSTRUCTION

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Though there are theories used to organize and inform educational research, there exists a gap between theory and classroom practice. In this paper, we propose a general theory of mathematics instruction called Concept-Focused (CFI) which has been specifically designed to inform everyday educational practices in mathematics classrooms. To introduce the theory, the three core principles of CFI are presented. This is followed by an explanation justifying CFI as a viable theory of instruction, as defined by criteria established by Jerome Bruner. The paper ends with a summary of the research framework being used to further develop the theory and a presentation of the pilot year findings showing the impact of using CFI to train secondary preservice teachers.

Introduction

In mathematics education, constructivism or the idea that learning occurs through the construction and integration of new knowledge within a preexisting conceptual framework has been the central learning theory and impetus behind student-centered instruction. The consensus among math educators is that constructivist-based instruction is effective. However, math educators express a common concern that teachers frequently resist the integration of constructivist-based methods in their educational practices. Teachers seem to ignore or have a fragmented perception of the theory, and instead rely too often on their pedagogical knowledge and practical experiences to determine appropriate approaches for classroom instruction.

One approach for bridging the gap between theory and practice is to train teachers to apply an explicit theory of instruction. A theory of instruction "sets forth rules concerning the most effective way of achieving knowledge or skill" (Bruner, 1966, p. 40). This prescriptive function is what differentiates it from learning theories which explain how and why students learn. "A theory of instruction, in short, is concerned with how what one wishes to teach can best be learned, with improving rather than describing learning" (p. 40).

According to Jerome Bruner (1966), a viable theory of instruction (1) identifies the experiences that are compatible with the way students learn, (2) explains the structure of the knowledge within a discipline, (3) identifies the most effective instructional sequences, and (4) addresses appropriate pacing and motivational strategies. Others have described a similar set of principles, each focusing on the application of knowledge and guidance on how to help students learn (Reigeluth, 1999; Merrill, 2002). In summary, "Instructional theories are created as a set of principles and guidelines. They are not rigid sets of rules that must be followed at all cost but are guidelines that help the practioner judge the value of a theory" (Defazio, 2006).

In order to improve teachers' understanding and perceptions of constructivist-based teaching, a theory of instruction designated as Concept-Focused Inquiry (CFI) was developed. Its purpose is to offer a coherent and practical approach to teaching mathematics. In this paper, CFI is introduced by first, describing the core principles that frame the instructional theory. Second, we justify CFI as a viable theory of instruction as defined by the criteria outlined above by Bruner. Finally, we summarize an emerging research project where preliminary findings indicate teachers trained to use CFI specifically improve in their conceptual understanding of mathematics and application of constructivist-based instruction.

Concept-Focused Inquiry

Concept-Focused Inquiry (CFI) is founded on three core principles. The first principle states mathematics is a process of conceptualizing and understanding mathematical objects in our world. This is consistent with the conversation in John Dossey's chapter *The nature of mathematic, its role and its influence* (1992) where he describes Hersh's view that mathematics arises from activity with already existing mathematical objects, and from the needs of science and daily life." (Hersh, 1986 as cited in Dossey, 1992, p. 42). Further, Dossey concludes individuals construct mathematical ideas through experimentation, observation, and experience. The first core principle clarifies the nature of our subject because that has a strong impact on the way in which our subject is approached in the classroom (Cooney, 1985; Thompson, 1992).

The second principle of Concept-Focused Inquiry (CFI) is that in order to teach conceptually, teacher must have a conceptual understanding of the content. Conceptual understanding occurs when an individual can accurately discuss the three attributes of a mathematical concept (macroscopic, modeling and symbolic) and identify the connections between the attributes (Hitt & Townsend, 2007). For example when teachers teach *area* they need to be able to plan lessons that provide students with macroscopic experiences such as a word problem that is applies the notion of area or more elementary an experience where students cover a space to visualize an application of the concept. A macroscopic experience is one that provides a context for the concept and is an occurrence that has the student being able to visualize the concept in an applicable context. Next, students need to draw or explain the situation using their own thoughts and representations. This is where the student creates a record of his/her perception of the concepts. This could be in the form of written text, drawings, or an enactment. It is the students' representation of the concept. Once students have devised a working model of their ideas they are ready to learn the mathematical definitions and formulas, or the more symbolic attributes of area.

Even though the three attributes presented above are described as discrete elements of a concept, the elements are connected. For individuals to engage in a conceptualizing process, they integrate these elements and can explain the relationships between them. Figure 1 depicts this relationship.

The third principle of Concept-Focused Inquiry (CFI) states when individuals explicitly reflect upon the three attributes of mathematical concepts, they achieve conceptual understanding. Commonly teachers and students use terms, definitions, and formulas in classroom conversations but do not truly understand the full meaning. This lack of conceptual understanding may explain why students must often be re-taught fundamental concepts in mathematics. When students do not have opportunities to observe and explain targeted concepts they fail to develop meaningful connections (Bransford & Donovan, 2005). As a result, students resort to the short term solution of memorizing the information for a test. In order to achieve conceptual understanding individuals need to have macroscopic observations, create, critique, and modify their own models of the concept, as well as apply the appropriate symbolic terms and formulas that explain their macroscopic observations and models. In summary, mathematics instruction focused on developing conceptual understanding provides students with opportunities to interact with the macroscopic, model, and symbolic attributes of a concept.

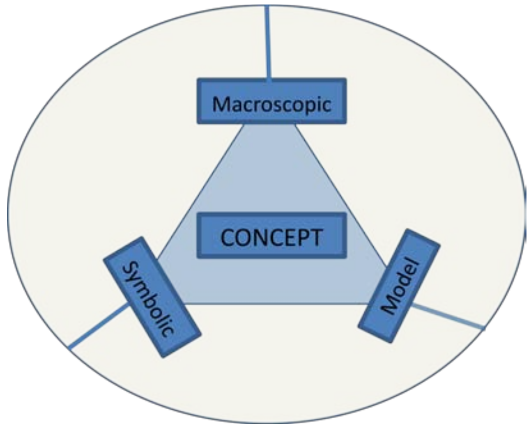

Figure 1. The three explicit and related attributes of concept(s) defined in CFI

The three principles of Concept-Focused Inquiry (CFI) describe the nature of mathematics that is consistent with current views in mathematics education (Dossey, 1992) and make explicit the types of opportunities teachers should make available to students. In general, these ideas are not new. For one, Jerome Bruner emphasized a similar idea in his work on cognitive growth (Clabaugh, 2010). In particular, Bruner states that in order for cognitive growth to occur, students must move through three stages of learning: enactive, iconic, and symbolic.

> *In the enactive state, students begin to develop understanding through active manipulation. Therefore, students at the enactive stage should be given the opportunity to 'play' with the materials in order to fully understand how it works. In the second stage, iconic, students are capable of making mental images of the material and no longer need to manipulate them directly. Here students are able to visualize concrete information. The symbolic is the final stage in which students can use abstract ideas to present the world… Students must go through all of these stages successively in order to connect new ideas and concepts if they are to generate their own understanding* (Clabaugh, 2010, p. 3).

Concept-Focused Inquiry (CFI) is an extension of Bruner's ideas. Bruner explains conceptual learning as a developmental progression where students master one stage at a time. Once a learner fully understands the concept at one stage, they move to the next and the earlier stage is no longer necessary. The three attributes macroscopic, model, and symbolic correlate to the enactive, iconic, and symbolic stages respectively, but the attributes are not viewed as independent of one another or as a stage of development. CFI emphasizes the need for every learner to experience each of the attributes to fully understand a concept. In the next section we make an argument that CFI meets the criteria for a general theory of instruction as defined by Bruner.

Concept-Focused Inquiry, a viable theory of instruction

In *A Theory of Instruction* (1966), Jerome Bruner identifies the criteria of an empirical and instructional theory of education. According to Bruner a viable theory of instruction meets four criteria: identifies the experiences that are compatible with the way students learn, explains the structure of the knowledge within a discipline, identifies the most effective instructional sequences, and addresses appropriate pacing and motivational strategies. In this section, we argue Concept-Focused Inquiry (CFI) is a viable theory of mathematics instruction.

Concept-Focused Inquiry is compatible with the way students learn. According to Bransford & Donovan (2005), in order for learning to occur, students must engage in metacognition, intellectual inquiry, and address prior beliefs and knowledge. These learning processes are embedded in Concept-Focused Inquiry (CFI) because when a teacher provides the opportunity for students to collectively engage with the macroscopic, model, and symbolic attributes of a concept, students share their prior beliefs and knowledge of the concept through their models of the macroscopic experience. When the teacher emphasizes the relationships between among the attributes, it naturally becomes a form of intellectual inquiry. The figure below shows explicitly where the processes are explicitly embedded in CFI.

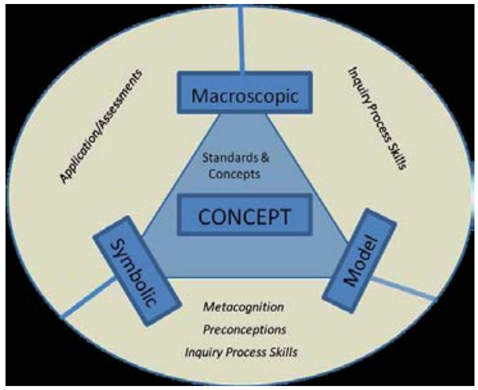

Figure 2. Concept-focused Inquiry and learning processes

Concept-Focused Inquiry explains the structure of knowledge in mathematics. The principles for Concept-Focused Inquiry (CFI) are grounded in the conceptual nature of mathematics. In particular, the first two core principles emphasize the nature and structure of conceptual knowledge. Besides explicitly explaining the structure of knowledge in mathematics, CFI allows educators to operationally define conceptual understanding. That is, conceptual understanding involves the ability to communicate and connect the macroscopic, model, and symbolic attributes of the concept.

Concept-Focused Inquiry identifies the most effective instructional sequences. The third core principle emphasizes an effective instructional sequence. Instruction begins with a macroscopic event. This provides the context for understanding the targeted mathematics concept. It is an event that provides students with a context and mental picture of the concept. Next the students move to the model attribute, where they are asked to explicitly communicate their models for thinking about the macroscopic experience. The teacher probes students to encourage them to make explicit their ideas about the target concept. By emphasizing the students' model, the teacher can identify how the students currently understand the concept, what misconceptions or naïve conceptions are present in the classroom, and what to focus on in order to guide the students' thinking toward a desired mathematical explanation. The teacher uses the students' models to introduce or clarify the abstract, symbolic attributes of the target concept.

Concept-Focused Inquiry addresses appropriate pacing and motivational strategies. Once the learner has been engaged with the macroscopic attribute of a concept, the students create a model to display his/her current understanding of the concept. The accuracy of the models displayed by the students will determine the appropriate pacing of the instruction. That is, when the models are inaccurate or incomplete, the teacher will engage the learners in further inquiry. This is done so students have the opportunity to process the phenomena in more depth and can make adjustments to their models. A teacher using CFI will pace the instruction based on the model attribute. Once the models are developed, the unknown symbolic attributes become part of the instruction. Appropriate pacing in CFI is about not rushing to the symbolic attribute of a concept and instead allowing students to fully experience the opportunity to display a model of their thinking.

According to theories of motivation that are focused on fostering learning, motivation is enhanced when the classrooms practices include providing diverse opportunities to demonstrate mastery; adapting instruction to students' knowledge, understanding, and personal experiences;

providing opportunities for exploration and experimentation; defining success in terms of improvement; emphasizing effort, learning, and working hard rather than performing or getting the right answer; and treating errors and mistakes as a normal part of learning (Stipek, 1996). CFI addresses these suggestions by emphasizing the students' display of the model attribute being the focal point of an instructional sequence. Having the model attribute as the focal point puts the students' knowledge, understanding, and personal experiences at the center of instruction. The emphasis on using metacognition, preconceptions, and inquiry to define the instructional processes further addresses the presence of these classroom practices.

Developing Concept-Focused Inquiry as a theory of instruction

The framework being used to guide the research in order to develop Concept-Focused Inquiry (CFI) is an interpretation of the work outlined by Defazio (2006). In summary, the agenda for developing Concept-Focused Inquiry is organized around a 4-level system. At level 1, creation of the instructional theory, the focus is on clarifying the theory and verifying its substance in existing literature. The work at level 2 is organized around testing the theory's effectiveness. This involves a review of existing literature, particularly those conducted in practice, to verify the theory's appropriateness and effectiveness. At level 3 the theory is tested with quantitative and qualitative data, providing real-world results use of the instructional theory in practical situations. Level 4 is replicating the study on a large scale and/or long term studies to determine further the usefulness and conditions.

At this point in time, the development of Concept-Focused Inquiry (CFI) as a theory of mathematics instruction is in the beginning stages of level 3. In other words, we are concentrating on designing and implementing studies that provide real-world data. We have three projects initiated and actively in progress. Namely, during the Fall of 2009 and again this year, CFI was used as the conceptual framework for a secondary mathematics methods course. The findings from year 1 of this project are elaborated in the next section. The second project initiated Fall 2010 uses CFI as the conceptual framework for a middle level mathematics methods undergraduate course, and finally a third project also initiated in Fall 2010 uses CFI in a year-long professional development program for middle level inservice teachers.

Research using Concept-Focused Inquiry

Subjects/Design. Concept-Focused Inquiry (CFI) has been used as the conceptual framework for a secondary mathematics methods course in a Master of the Arts of Teaching (M.A.T.) program at a mid-sized university during the past two years. Four master-level preservice teachers (1 males and 3 females) were enrolled in the methods course Fall 2009 and student teaching Spring 2010. These preservice teacher completed an undergraduate degree in mathematics, and prior to enrolling in the methods course they completed two graduate level content courses and two education courses, a foundation to education and teaching course and a course on reading in the content area. In addition, two of the four preservice teachers completed courses in educational psychology and diversity prior to enrolling in the methods course. The other two preservice teachers took the educational psychology and diversity course concurrent with the methods course.

The mathematics methods course met once a week over a 16-week fall semester. During class meetings the candidates participated in mathematical activities designed to demonstrate the principles of Concept-Focused Inquiry (CFI) and how to use CFI to design their instruction. Throughout the course the preservice teachers learned how to create lectures, demonstrations, cooperative learning opportunities, etc that sequentially addressed the macroscopic, model, and symbolic attributes of targeted concepts. In addition to coursework, the preservice teachers were immersed in high school mathematics classes for a period of one and two weeks.

In Year 1, the impact of the Concept-Focused Inquiry (CFI) framework on preservice teachers' perceptions of mathematics (Core principle #1) was measured qualitatively using an initial survey and exit interview. Specifically, the preservice teachers were asked to describe in their own words what mathematics means to them, what they believe are the founding principles of mathematics, what they believe students should know about mathematics, and why they believe mathematics may be difficult for some students, and how they might address these difficulties. Initial responses to these prompts were incorporated into classroom conversations, and during the exit interview the participants were further probed. Transcriptions of these selected class periods were included as part of the data set to measure the preservice teachers' perceptions of mathematics.

Table 1. Levels of Inquiry & Learner Activity

Level	Question	Method/Data collection	Interpretation of results	Plausibility of results
0	Provided by teacher	Provided by teacher	Provided by teacher	Provided by teacher
1	Provided by teacher	Provided by teacher	Provided by teacher	Open to learner
2	Provided by teacher	Provided by teacher	Open to learner	Open to learner
3	Provided by teacher	Open to learner	Open to learner	Open to learner
4	Open to learner	Open to learner	Open to learner	Open to learner

Daily lesson and unit plans developed by the preservice teachers were assessed in order to determine their understanding of Concept-Focused Inquiry (CFI) core principles #2 and #3. The preservice teachers completed three daily lesson plans and one 3-4 day unit during the methods course. One of the daily lessons was 'taught' in our class setting to their peers, and at least one of the lessons in the unit was taught in a high school classroom during the second field experience and observed by the university instructor. In both the daily lessons and unit plans, the preservice teachers were instructed to identify specific concepts and explicitly discuss how their lesson addressed the macroscopic, modeling, and symbolic attributes of the target concepts. The plans and instruction were scored using an inquiry rubric developed by Schwab (1962) and modified by Nadelson (2009). In the context of this study the level of inquiry is an indicator of the preservice teachers' ability to apply constructivist-based teaching strategies. The rubric consists of five levels of inquiry-based instructional practices (Table 1). Higher levels of inquiry reflect higher levels of inquiry-based instruction. For example, at the highest level, Level 4, students are required to develop a research question, devise data collection methods, interpret results, and to evaluate the plausibility of the results. In addition the different types and frequency of mathematics processes as defined by the National Council of Teachers of Mathematics (NCTM, 2000) were recorded and analyzed. The processes as defined by the five NCTM Process Standards: Communication, Reasoning & Proof, Connections, Problem Solving, and Multiple Representations.

Findings

Analysis of Concept-Focused Inquiry (CFI) core principle #1 measuring the nature of mathematics indicates the CFI-based instruction changed the preservice teachers' perspectives. On the initial survey, every preservice teacher expressed views indicative of either a Platonist view (2 preservice teachers) where mathematics is a static but unified body of knowledge, a crystalline realm of interconnecting structures and truths, bound together by filaments of logic and meaning" (Ernest, 1988, p. 10) or the instrumentalist view (2 teachers) that mathematics is "like a bag of tools, is made up of an accumulation of facts, rules and skills to be used by the trained artisan skillfully in the pursuance of some external end (Ernest, 1988, p. 10 as cited in Thompson, 1992). During the methods course, these beliefs were challenged by the activity and discussions that took place, and by the exit interview, 3 out of the 4 preservice teachers responded with views indicating mathematics as a dynamic, problem-driven field of activity where people create or developed mathematical thought by experimentation, observation, and experience. These preservice teachers' statements included terms such as concept(s), conceptualization, and conceptual understanding.

Concept-Focused Inquiry (CFI) Core Principles #2 and #3 – Analysis of the three daily lessons prepared early in the semester and the 3 – 4 day unit plan prepared at the end of the course reveal a general increase in the level of inquiry and the number of mathematic process skills used in the plan for instruction. Comparing the average inquiry level of the first sets of lesson designed and the average inquiry level of the lesson prepared as part of the unit showed all four preservice teachers' improving in terms of the level of inquiry, albeit one of them minimally, and only one of the preservice teachers produced lessons using fewer mathematics processes in the unit plan than in earlier lessons.

	Average Inquiry Level			Average Frequency of Process Skills		
Code	Pre	Post	Post-Pre	Pre	Post	Post-Pre
S1	0	1.5	+1.5	1.3	3.3	+2
S2	.6	3	+2.4	3	4.3	+1.3
S3	.3	.8	+.5	2.7	1.3	-1.4
S4	1	2.6	+1.6	2.3	4	+1.7

Two factors must be considered when looking at these values. The preservice teachers selected the topics for the pre lesson plans but the post unit was based on the content they were assigned to teach during the two-week long field experience. It is probable that the cooperating teachers influenced the content of the unit plans. Conversations with the cooperating teachers indicated that they favored more teacher-centered instruction. As a result, the preservice teachers' planning was likely influenced by this view. In light of these factors, it can still be inferred that the preservice teachers developed a relatively strong understanding of constructivist/inquiry instructional practices at the conclusion of the methods course.

After the pilot year, it was determined that Concept-Focused Inquiry (CFI) presents a potentially effective approach for training preservice teachers to plan and implement constructivist-based lessons. We found that by providing preservice mathematics teachers with a theory of instruction, CFI, their instructional decision making for planning and teaching mathematics improved. This is because CFI simplifies the instructional decision-making process. In addition, using CFI in the methods course allowed preservice teachers to "see" connections between different topics as well as instructional approaches.

Concept-Focused Inquiry (CFI) is now being used in two mathematics methods courses--both secondary and middle level-- and one project with middle level practicing teachers. Additional data sources have been added to measure the effects of CFI on the teachers' planning, teaching, and understanding of mathematical concepts. In conclusion, we recognize theory development requires multiple iterations in different contexts. In this paper, we have shared an overview of our work thus far. We have presented the core principles that define Concept-Focused Inquiry (CFI). We made an argument that CFI can be considered a theory of instruction. As the title suggests, CFI is a work in progress, and it is a potential theory of mathematics instruction.

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DEVELOPING A REFERENCE UNITS CONCEPTION: WHAT IT TAKES FOR ONE PRE-SERVICE TEACHER TO GAIN THE KNOWLEDGE NEEDED TO HELP STUDENTS

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Developing a meaningful conception for multi-digit whole numbers is important because it allows PSTs to have the knowledge necessary to help children learn place value in a meaningful way and make sense of multi-digit addition and subtraction. Many PSTs do not have such a conception. Data from a teaching experiment suggests critical events aiding one PST in developing a more meaningful conception of multi-digit whole numbers in the context of addition and subtraction. The critical events involved artifacts of children's mathematical thinking, using manipulatives, and listening to others.

Introduction

Consider the following scenario: A preservice teacher (PST) is asked to comment on the work of a second grader (see Figure 1). The second grader is curious as to why the 1 is put above the 5. Now consider one PST's response to interpreting that regrouped 1:

> this one is part of the number seventeen, and we just carried it over because you can't put the number seventeen here … well, the one is added here because it's part of the seventeen, and, again, you can't write the number seventeen here, so you have to carry it, ahm, over here.

The PST quoted here is not equipped with a conception that would allow her the ability to discuss the regrouping in a meaningful way with the child. The call to teach in a way that allows children to develop a meaningful understanding of the mathematics (including standard algorithms) has become prominent throughout the United States (Kilpatrick, Swafford, & Findell, 2001, NCTM, 2000). The PST above would be unable to help a child understand the standard algorithm with regrouping for addition.

$$\begin{array}{r} 1 \\ 259 \\ +\ \ 38 \\ \hline 297 \end{array}$$

Figure 1. The work of a second grader.
Task modified from (Philipp, Schappelle, Siegfried, Jacobs, & Lamb, 2008, p. 32)

The National Council of Teachers in Mathematics (NCTM) *Principles and Standards* state that meaningful "mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well" (NCTM, 2000, p 11). Creating an environment to promote student learning in a classroom requires a teacher's ability to decompose mathematical content knowledge so they are able to teach in a way that allows the child to develop meaningful understanding of the mathematics (NCTM, 2000; Kilpatrick et al, 2001). In particular, mathematical content knowledge necessary for teaching multi-digit addition and subtraction of whole numbers requires the ability to decompose knowledge of numbers and regrouping. In the example above the 1 should be seen as 10 ones of the 17 ones (a result of adding 8 ones and 9 ones) and as 1 ten (a result of composing 10 ones into 1 ten or decomposing 17 into 1 ten and 7 ones) which is now added to the tens in the tens' column. Thus the 1 in 17 has to be flexibly seen as either 1 ten or 10 ones. This extends further in three-digit numbers where we have to be able to switch flexibly between seeing the 1 in 187, for example, as 1 hundred, 10 tens, and 100 ones.

Upon looking for PSTs' current knowledge of multi-digit addition and subtraction of whole numbers, Thanheiser (2009a) found that 10 of 15 PSTs entering their first mathematics content course for teachers held a conception of multi-digit whole numbers that would be insufficient for a meaningful explanation of regrouping in the context of addition and subtraction. Therefore, mathematics educators need to work with PSTs to help them develop more sophisticated conceptions of multi-digit whole numbers. However, this is not easy (Thanheiser, 2009b).

Theoretical Framework and Background

Although most PSTs are able to compute using algorithms successfully, many are unable to explain the algorithms in a meaningful way (Ball, 1988; Ma, 1999; Thanheiser, 2009a). To help PSTs develop conceptions, which will allow them to work with children in a meaningful, way, it is important to build on PSTs' current conceptions and prior knowledge (Bransford, Brown, & Cocking, 1999). To help PSTs develop conceptions that would allow them to interact in a mathematically meaningful way with a child, we (mathematics educators) first need to identify their conceptions. Then we can create activities to allow them to extend their existing conceptions.

One framework developed to identify PSTs' place value conceptions (Thanheiser, 2009a) includes four categories, two incorrect and two correct.

- Concatenated Digits Only: conceiving of *all* digits in terms of ones (e.g. 389 would be 3 ones, 8 ones, and 9 ones)
- Concatenated Digits Plus: viewing at least one digit in a number incorrectly in terms of ones and not in terms of its value. (e.g. 389 might be 300 ones, 8 ones, and 9 ones).
- Groups of Ones: conceiving of all digits in terms of groups of ones alone (e.g. 389 would be 300 ones, 80 ones, and 9 ones).
- Reference Units: conceiving of the reference units for each digit in the number and relating the reference units to one another (e.g. in 389, the 3 can be seen as 3 hundreds, 30 tens, or 300 ones and the 8 can be seen as 8 tens or 80 ones).

Using this categorization, Thanheiser (2009a) showed that many PSTs enter their first mathematical content course for elementary school teachers with an incorrect conception of place value (either concatenated digits only or concatenated digits plus).

Several different ways to help children develop a meaningful understanding of multidigit whole numbers and the standard algorithms for addition and subtraction successfully have surfaced. One such way is to allow children to develop their own algorithms (Kamii, 1994), another to focus students on listening for differences in explanations (McClain & Cobb, 2001). Both these approaches allowed children to build a meaningful understanding of place value. The later also improved their interaction within the classroom by emphasizing listening for what made methods different. Working with adults is different. They have practiced and used the standard regrouping algorithm for years and we cannot simply expect them to abandon this method. By instead asking them to work with various manipulatives to create ways of adding and subtracting numbers, react to children's mathematical thinking artifacts (see Philipp et al., 2007), incorporate various different strategies and relate those to the standard algorithm, an environment can be created that allows the PSTs to uncover concepts underlying the standard algorithm. In our work, we examine how these various activities affect one PST's conception.

Methods

The data for this study comes from the first three days of a five-day teaching experiment with six PSTs, as well as pre, post, and delayed post interviews of one PST, Isabel. Each day of the teaching experiment consisted of one 150-minute session with six volunteer PSTs from a large, urban, comprehensive state university. Two PSTs had previously taken their education methods course and the remaining four had not. All sessions were videotaped. Field notes were created and used to identify critical events, which were then transcribed. The interviews were all 60 to 90 minutes long and conducted immediately before, immediately after, and approximately eight months after the teaching experiment respectively. All interviews were transcribed. The questions in the interviews were designed to elicit the PSTs' conceptions and coded according to the above framework (Thanheiser 2009a).

The tasks for the teaching experiment were designed based on the pre-interviews and the intent was to address/ build on the PSTs' initial conceptions. The tasks included addition and subtraction of three-digit numbers using various manipulatives, discussion of artifacts of children's mathematical thinking such as video clips of children solving problems, discussion of manipulatives, and an exploration of Mayan numbers.

More specifically, on the first day of the teaching experiment, students used various ways to add 389+475: (a) Montessori digit cards (see Figure 2a); (b) Base ten blocks; and (c) Any other method they could think of (mentally or using paper and pencil). Montessori digit cards provide a way to connect the symbols to their values. Students use the cards to build multidigit numbers. For example, 423 is built by layering the 400, the 20, and the 3 (Figure 2b). Base ten blocks help elicit the correspondence between place values, but as Hiebert et al (1997) point out, this meaning must be constructed and is not inherent in the blocks.

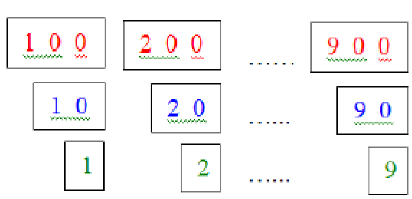

Figure 2a. Montessori digit cards

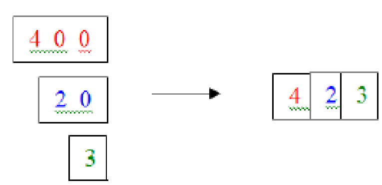

Figure 2b: Representing 423 value with Montessori digit cards

The PSTs also watched one video of a child completing a three-digit addition problem. The video showed a student adding three digit numbers (Figure 3). This child previously used the expanded algorithm correctly with two-digit addition (Figure 4). In the video, the student incorrectly used the expanded addition algorithm while adding three-digit numbers. She added the digits from left to right and lined up the values vertically to get that 638+476=34. The child did six plus four is ten instead of six hundred plus four hundred is one thousand; three plus seven is ten instead of thirty plus seventy is one hundred; and correctly did eight plus six is fourteen (Figure 3).

```
  638
+ 476
 10
 10
 14
 34
```

Figure 3. What a child did to add three digit numbers shown through a video on day 1

```
  38
+ 45
  70
  13
  83
```

Figure 4. To use the expanded algorithm, one looks at each place value and adds it individually. One then adds those values. The child (referenced in figure 3) completed this problem correctly in this manner

The goal of the addition tasks was to help the students develop methods different from the standard algorithm to help explicate and develop their conceptions through explanations of these alternate methods.

In this paper, we analyze one student, Isabel, through the first three days of the teaching experiment using Thanheiser's framework. Isabel's conception was categorized as concatenated digits only on the pre-interview and as reference units on the post interview (Thanheiser, 2009b).We identify the development of her conceptions throughout various tasks and identify critical events that changed her attitude toward learning, demonstrated gaining knowledge, or demonstrated development of a skill (Woods, 1993).

Results and Discussion

Isabel was categorized as holding a concatenated digits only conception in the pre-interview and developed a reference units conception over the course of the teaching experiment. In the delayed post interview, Isabel demonstrated that she still had a reference units conception. A change in conception is not trivial, as mere exposure to correct conceptions was not enough to cause a change (Thanheiser, 2009b). In this section, we will trace Isabel's development over the course of the first three days focusing on three phases. The first phase reemphasizes the findings of Thanheiser (2009b), that exposure alone is not sufficient. The second phase addresses Isabel's changing conception and the third phase is the solidification of her reference-units conception.

Phase 1: Exposure is Not Enough

Isabel saw all digits in terms of ones, rather than in terms of their values at the beginning of the teaching experiment. During the first phase of the teaching experiment, she continued to hold this conception even when exposed to correct conceptions (Thanheiser 2009b). She was aware that her method was different from other students' methods, but did not classify it as being better or worse. "I just think that there is no really right way, like, everyone is thinking about it differently, but it's not like she was wrong or I was really wrong, it's just that everybody thinks about it differently." Although she was hearing the other students, she may not have been paying attention to them or listening for distinct differences to evaluate what these differences might mean (McClain & Cobb, 2001).

Phase 2: A Changing Conception

Part 1: Responding to video. Isabel's first instance of change came in response to the video at the end of day one. In response to the child's solution (see Figure 3) Isabel stated "She didn't even add it properly… she knows it's a ten, but I don't think she is understanding the concept of place values." Upon reflecting about this video, Isabel began to show understanding that her own conception may not be enough (Thanheiser 2009b).

> *She [the child] is just adding them as single digits, like six plus four, and then that's one problem, and then three and two is another problem, and then taking those numbers and making it a third problem or a fourth problem… you need to indent to get the correct place values.*

Through this discussion, Isabel showed, for the first time, an understanding that you need to see each digit of a number as more than a digit. Even though her understanding seems to remain procedurally focused, this was the first instance of her acknowledging that not all strategies have the same validity. This event triggered her change of conception and thus, the researchers classified it as a critical event in the sense that it changed her attitude toward listening to and learning from others.

Beginning with this event Isabel began to pay attention to what others were saying. First, she paid attention to the child's (incorrect) solution. After the video, she began to pay attention to the other students in the teaching experiment as well. One hypothesis as to why she started to listen is that the child held a conception similar to her own but did not arrive at the correct answer. Thus, this may have made Isabel aware of the fact that some additional aspects of understanding are necessary to solve this problem correctly.

Part 2: Attention on others. After responding to the child's incorrect solution (see Figure 3) Isabel began listening to other students in the teaching experiment. The first task following the viewing of the video was to consider teaching multi-digit addition to a class of second graders. The PSTs were asked to react to two scenarios: (a) how they would teach the second graders and (b) how they would teach if they had no materials available. Isabel wrote in response to the first scenario that she would use the blocks because "then they [the children] can understand why we are carrying the ones… because these ones become also a ten … so you put it in the ten's place." This response indicates that she is now aware of the need to understand what the regrouped values mean. In the second scenario, she suggests the use of the standard algorithm because it is "easier to break up into three smaller problems than to see it as one larger problem" (Figure 5a). After some discussion, Jason (another participant in the teaching experiment) presents an expanded algorithm (Figure 5b). Isabel then decides that she would rather use Jason's method to introduce addition because "it is a little more clear for a beginner to see… it's like the actual, you know, that the six represents six hundred." In her response she indicates that she now sees the value of the 6 not merely as 6 (as she did at the beginning of the TE) but as 6 hundred. She is also paying close enough attention to someone else's method to adopt it as what she would rather use and thus the researchers classified this as a critical event in the sense that she demonstrated acquiring knowledge (of Jason's method). Isabel now has an alternate method available to her that she saw as easier for someone who was just learning the ideas for the first time. After the video, Isabel's awareness allowed her to listen carefully to Jason, adopting his method as her own.

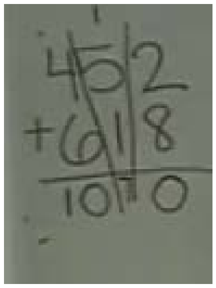

Figure 5a. Isabel's standard algorithm

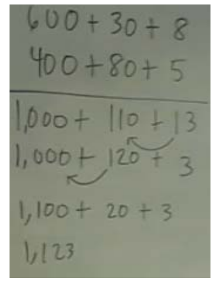

Figure 5b. Jason's expanded algorithm

Part 3: Use of materials. While trying to connect the standard algorithm to Jason's expanded algorithm (Figure 6b), Isabel references base ten blocks.

> Well, eight plus two is ten, and then I guess, like with the blocks it would be ten, it's going to be one ten, so you're going to carry it over to the tens place like fifty plus ten plus you're adding another ten value… and then five plus one plus one will give you seven, so in the problem that's really seventy in that place value there, it's seventy, not seven. And then four plus six is ten, but it's really one thousand seventy.

Throughout the first part, Isabel is comfortable referencing the blocks without having them in front of her and was able to explain the addition by connecting the digits in the ten's place to their value (i.e. 70 for 7 in the ten's place). However, as the explanation carried on, she seemed to lose track of the blocks and fell back to an explanation referring to digits. Even though she attended to placement in the final answer, she did not reference the digits along with their value to support their placement in her final answer. We are able to see that, although she still falls back to a concatenated digits conception, she is beginning to draw on correct conceptions.

Phase 3: Solidifying a Correct Conception

After discussing addition, the PSTs began discussing subtraction. At this point, Isabel moved beyond exclusively relying on the concatenated digits conception. As demonstrated in phase 2, Isabel previously drew on correct conceptions at various points throughout the discussions. In the context of the subtraction problems Isabel was able to draw on these conceptions immediately. For instance, when Isabel was writing an explanation for 527-135 (Figure 6), Isabel used the visual representation of the base ten blocks and stated that you should convert *one hundred* to *ten tens* to aid in the subtraction process (Figure 7). During the discussion, she reemphasized her thoughts: "you can explain if you cannot take away a certain number of tens then you can break one of the hundreds up." By not falling back on her concatenated digits conception, Isabel demonstrated that she had moved beyond the concatenated digits conceptions. She was able to use her developed conception to understand and explain regrouping in the subtraction algorithm, so the researchers classified this as a critical event in the sense that she demonstrated development of her skill set.

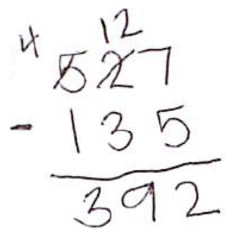

Figure 6. Isabel's standard algorithm for subtraction

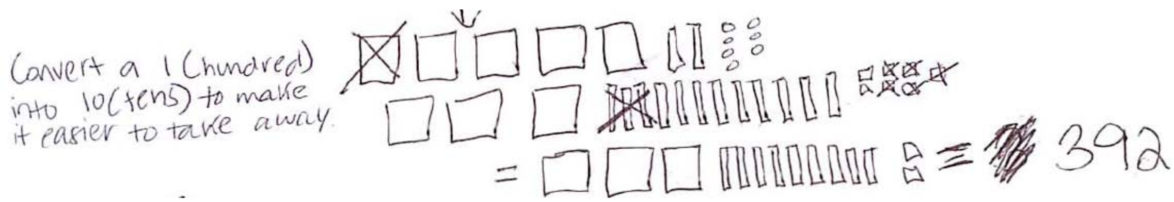

Figure 7. Isabel using the blocks to describe subtraction

Conclusion & Implications

Although helping PSTs develop conceptions that are more sophisticated is hard, there are certain events that seem to aid in developing a more sophisticated conception. Using artifacts of children's mathematical thinking (in this paper, a video) can help a PST to understand that their conception is inadequate. Listening to other students' explanations and solutions provides PSTs one way of developing conceptions that are more sophisticated. Listening is essential for learning mathematics, and this could involve children or fellow students. Either way, a classroom of students will involve many levels of conceptions, so purposeful sequencing of the events is critical to aid in developing these conceptions. Although manipulatives do not hold mathematical meaning on their own they can be used as a tool to communicate mathematical ideas and share mathematical understanding. Essential aspects of all these activities are that (a) they address the PSTs' currently held conceptions (in our case the concatenated digits only conception) and (b) allow the PST to build on those to develop more sophisticated conceptions. For example, the video explicated the in-adequacy of the concatenated digits conception and thus prompted the search for a more meaningful one.

Tracing Isabel's conceptions through the teaching experiment allowed us to see while change of conceptions is possible it may not be fast or easy. We need to allow time and carefully choose activities addressing the PSTs' currently held conceptions to allow them to develop conceptions that are more sophisticated.

Endnotes

1. The analysis of this study was supported in part by a Portland State University Faculty Enhancement Grant.

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EXAMINING PROSPECTIVE ELEMENTARY TEACHERS' CONCEPTUAL UNDERSTANDING OF INTEGERS

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Teachers tend to teach mathematics they are comfortable with, and if their mathematical understanding is limited or based on misconceptions, they are likely to avoid teaching mathematics as much as possible or simply focus their teaching efforts on procedural proficiency. This investigation examined the degree to which prospective elementary teachers had developed a meaningful and conceptual understanding of what integers are and explored their development of models for multiplication with integers that are related to everyday activities. Additionally, this study explored how these understandings informed prospective elementary teachers envisioning of their teaching these concepts to their future students.

Introduction

The success of the current efforts for reform in mathematics education depends largely on teachers' conceptual knowledge and understanding of the mathematics they will teach. By the time prospective teachers begin their education or mathematics methods coursework they have spent years learning mathematics from teachers whose pedagogic practices are predominantly teacher centered and driven by the aim for algorithmic and procedural proficiency. Further, prospective teachers' experiences with mathematics at the university level that tend to focus on college level mathematics do not do much to improve their conceptual understanding of the mathematics they will teach. It is the culmination of these experiences with school and university level mathematics that often develops limited understandings and a sense of low self-confidence with mathematics. Likewise, in prospective teachers, these experiences create difficulties envisioning mathematics teaching for understanding from a perspective that is more aligned with constructivist ideas about learning.

This study aimed to understand the degree to which prospective elementary teachers had developed a meaningful and conceptual understanding of what integers are and to explore their development of models for multiplication with integers that are related to everyday activities. Additionally, this study explored how these understandings informed prospective elementary teachers envisioning of their teaching these concepts to their future students.

Background

Research has indicated that many prospective elementary teachers have developed misunderstandings and/or limited understandings of various basic mathematics concepts (Ball, 1988; Ma, 1999). This limited understanding is often and unfortunately passed on to prospective elementary teachers' future students thus creating a cycle of limited knowledge in mathematics and an inability to critically question the mathematics one is being taught. The National Council of Teachers of Mathematics (NCTM) stated that:

> *Central to the preparation for teaching mathematics is the development of a deep understanding of the mathematics of the school curriculum and how it fits within the discipline of mathematics. Too often, it is taken for granted that teachers' knowledge of the*

> *content of school mathematics is in place by the time they complete their own K-12 learning experiences. Teachers need opportunities to revisit school mathematics topics in ways that will allow them to develop deeper understandings* …. (1991, p. 134).

Despite this admonition from NCTM two decades ago, it remains the case today that most prospective elementary teachers are engaged in mathematics at the university level that includes topics well beyond what they will eventually teach. This common practice may be based on an underlying belief that if prospective teachers are exposed to "higher level" mathematics they will naturally make sense of and deepen their understanding of the mathematics content they will teach to their students.

Conceptual Understanding of Integers. Traditionally, students have been required to memorize rules for operations with integers thus leaving them without the fluency or flexibility to use the mathematics learned in situations different than those in which they first learned them. Students often get confused about which rule to follow and are left to rely on their instincts to solve problems dealing with integers (Bolyard & Moyer-Packenham, 2006; Ferguson, 1993).

When students first encounter negative numbers they are unable to relate them to the models they have previously made sense of with counting numbers because they cannot "see" negative numbers. According to Heibert and Carpenter (1992), the models used to teach counting numbers and fractions should make sense to students so they can remember rules that are generalized for performing the operations. However, working with models that will make sense to students is more challenging with integers because the models can only be used in abstract ways in an effort to relate integers to what students have previously learned. Students struggle with the signs used to indicate positive or negative integers since they are the same signs used for addition and subtraction operations (Stephen, 2009). Models most frequently used to teach integers include the neutralization models and the number line which are often represented by two-color counters or positive and negative charges scenarios (Reeves & Webb, 2004; Nurnberger-Haag, 2007). Further, studies indicate that even when students are introduced to integers in ways that they can relate to, they often do not develop an understanding of the "negativeness" or "positiveness" of the situation in which the integers are embedded (Hackbarth, 2000). This study explores preservice elementary teachers' conceptual understandings of integers and multiplying them when they gain experiences with both abstract models and "real world" scenarios that depict multiplication of positive and negative whole numbers.

Methodology

Data Collection

Journal entries, participant short-answer responses, and video-taped classroom discussions constitute the data sources for this study. Three different groups of prospective elementary teachers comprised the participants for the study and data were collected over two semesters in two sections of a senior level mathematics course in the fall (N = 32) and one section of the same course (N= 29) during the spring. In both the fall and the spring, the instructor for the course collected the data as an ongoing and normal part of the coursework and assignments.

Data were collected after the prospective teachers had participated in a variety of activities focusing on what integers are and experiences aimed at developing their understanding of operations with integers in everyday contexts. Participants were asked to respond to the following questions and/or prompts:

- What are integers?
- Where do we encounter them or in what ways are integers used in the “real world”?
- How would you teach each of the following scenarios with integers?

    (+)x(+), b. (+)x(-), c. (-)x(+), and d. (-)x(-)
- Write a story or scenario for each of a through d above.
- Reflecting on your own learning experiences answer the following:
    - Were you taught to multiply integers by using the “rules”?
    - Did you question these rules when you were taught them? Why or why not?
    - How did this classroom experience make you feel?
    - Do you feel like you can now teach these concepts better through an inquiry approach?

Setting

The participants involved in this study were prospective elementary teachers at a university in the Southwestern region of the United States in a teacher preparation program wherein perspective teachers take four mathematics content courses and three additional courses focused on elementary mathematics methods. A belief in the tenets of social constructivism was the underlying philosophy and theory that drove the decision to engage students in activities and experiences that supported problem centered learning approaches to teaching and learning mathematics (Wheatley, 1991; Wheatley & Abshire, 2002; Van de Wall, 2004).

The course in which this study was conducted served as the last mathematics methods course students took in their final semester prior to their student teaching internship. While the course focused on teaching mathematics in grades four through eight, an overarching goal of the course was to further develop the *mathematical power* (NCTM, 1989) of the prospective teachers through engaging them in continual problem solving, critical questioning of the ways in which they had been taught mathematics, and integration of social justice arts. The activities and experience designed for this course were purposively selected in order to improve the mathematics content understanding of prospective teachers, to help them make connections in and among mathematics and other content areas, to engage them in mathematics in every day contexts (specifically issues of social justice) (Gutstein & Peterson, 2006), and to provide opportunities for them to develop their ability to solve non-routine problems, with the goal of improving prospective teachers overall *mathematical power* (NCTM, 1989).

Data Analysis

Data were analyzed using an interpretive framework. The initial phase of data analysis was an exploratory examination of student journals and video recordings. The researchers hoped to develop an overall idea of what student perceptions of integers were. This initial exploration was followed by several subsequent readings and viewings, that were analyzed and began to form common trends and categories. After several cycles of analysis through the data and the formation of themes, the researchers began to identify answers to the following questions. (1) How did these preservice teachers describe integers? (2) What methods did they develop for teaching multiplication of integers? (3) How did their understandings inform their ideas about teaching these concepts to their future students? Data were then summarized and one final read through was conducted to verify that the questions of the study had been answered. During the final analysis cycle, examples were extracted for use in illustrating the results.

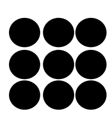
Research Results

The results of this study indicate that after having had multiple experiences with integers through discussion about where integers are encountered in the everyday activities of life, solving non-routine problems with integers, working with a variety of models typically used to teach integers and operations with integers (two-color counters, positive and negative charges, and number line models) and creating problems (stories and scenarios) with integers for each operation, the prospective teachers in this study could discuss what integers are and where they are found in the “real world.” They seemed comfortable with multiplication of two positive integers and multiplication of one positive integer with one negative integer; however, they remained hesitant in their abilities to demonstrate an understanding of multiplication of two negative integers (particularly in writing a “real world” story problem). A small minority of students seemed to be cautious about using approaches and models aimed at conceptual understanding of operations with integers and instead believed that the use of “gimmicks” or “sayings” will help their future students memorize the “rules” for multiplication of integers.

Analysis of the data collected revealed that all participants could accurately discuss what integers are and where they are likely encountered in everyday activities. Overwhelmingly participants indicated that integers can be encountered in the game of golf (above and below par), the game of football (yards lost or gained), borrowing money and/or debt with credit cards, temperature, above and below sea level, and a variety of board games. All seemed to be ways that made sense to the prospective teachers and contexts they could relate to.

The second focus in data collection asked the participants to discuss specifically how they would teach multiplication of integers to their future students. They were asked to address all possible combinations of positive and negative integers in multiplication (i.e., (+)x(+), (+)x(-), (-)x(+), and (-)x(-)). Most responses to this question included participants citing one or more of the models that had been presented in class (typically two-color counters or a line number model). Analysis of this data revealed that in all cases wherein a participant indicated a model for teaching multiplication of integers, she/he did so correctly. Additionally, many participants also indicated that they would use repeated addition or multiplication by grouping relying on the context of everyday activities with integers. However, approximately half of the participants who cited everyday activities with integers and combined it with an array, two-color counters, or number line model did so correctly across all four combinations of multiplication with positive and negative numbers. The following is an example of one participant’s response that included both models and everyday contexts. Maggie wrote:

> a.) + x +
> Arrays, two-color counters, lattice multiplication, groups of items, drawings.
> Sally had 3 baskets with 5 apples in each basket. How many apples does she have in total?
> b.) + x –
> Story problems. Teach – x + first and then commutative property.
> 4 friends have $-20 in each of their bank accounts. What is the total of their bank accounts put together?
> c.) – x +
> Two-color counters, pictures.
> -3 x 3



> (Maggie drew open circles with a capital letter "R" in each one to indicate red for negative)
> You scored -1 (1 birdie below par) on every hole on an 18-hole golf course. How far below par are you?
> d.) – x –
> Every time I go to the casino I lose $25. I don't go for 3 days. How much $ did I save? (Student journal)

Although many students such as Maggie were able to create story problems for multiplication of integers (i.e., the casino problem above), during discussions that proceeded writing these scenarios, many of them expressed concern about the mathematical validity of the (-)x(-) word problems. Debates ensued about whether or not saving or maintaining money is the same as gaining money. The most common concern was some form of the idea that if you begin with $0 and don't go to the casino for three days, you are still at $0, not at $25. Is a problem such as Maggie's accurately portraying a negative integer times a negative integer? Many students emerged from the class believing that it does; however, for some, concerns pushed them to ponder the matter further. Although this issue seemed to cause significant dissonance for most students in class, after discussing the matter for a few minutes, all but one class seemed temporarily satisfied with the idea of "saving money" as a positive integer answer and so discussion of the matter was terminated. However, in one class, the following example was proposed:

> Your friend paid for your lunch three times this week. Lunch costs $5 each time. In other words, at the end of the week, when you pay your friend back your bank account will decrease by $15, or 3 x -5 = -15. Your friend decides to take away two of those lunch charges, so she relieves you of two $5 charges, or -2 x -5 = 10. Your account will decrease by $5 instead of $15, so you have gained $10.

Participants in this particular class then began asking whether it was necessary to find a "real world" example of a negative integer times a negative integer, because it seemed to be the case that in the "real world" it would make more sense to simply use the terminology of a positive integer times a positive integer. Ultimately, one student proposed that although it may not be common to phrase "real world" multiplication problems in the way this example illustrates, the point of the example would be to portray the logic behind a negative times a negative in the "real world" (i.e., there is a "real world" scenario for negative integer multiplication). Although only one class seemed to debate the matter for a longer period of time than the others, students from all the classes continued to be uncomfortable with the idea of finding a contextual example for (-)x(-), as many of them revisited and questioned this concept periodically throughout the rest of the semester.

Unfortunately, about one-fifth of the participants indicated they would rely solely on methods of memorization to teach multiplication with integers. Shelly discusses the fact that she believes that teaching "rules" is still a necessary part of what she will need to do as an effective mathematics teacher

> I know that we should be allowing the students to explore math and come up with their own rules and definitions without actually teaching them the rules; however, sometimes

you have to teach the rules and then create opportunities to prove they are correct. I found this poem online while trying to research why a – x – = +. I think this is a great tool to use when teaching negative integers.

When good things happen to bad people, that's bad. (+ x – = –)
When bad things happen to bad people, that's good. (– x – = +)
When bad things happen to good people, that's bad. (– x + = –)
When good things happen to good people, that's good. (+ x + = +) (Student journal)

Jennifer likewise indicates that she would not necessarily teach the "rules" but instead found a way she thinks is better for teaching multiplication with integers. She believes she will use the "Dating Game."

The Dating Game!

a.) I like you, you like me – it's positive.
b.) I like you, you don't like me – it's negative.
c.) I don't like you, you like me – it's negative.
d.) *I don't like you, you don't like me – it's positive.* (Student journal)

The final focus of this research study was to examine the perceived impact of experiences in a methods course aimed at improving prospective teachers overall mathematical power on their vision of how they would teach mathematics as it pertains to multiplication of integers. Asking the prospective teachers to write about whether they feel like they can teach integers and multiplication of integers through inquiry approaches revealed in some cases the resilience of their misunderstandings and challenges with mathematics. For example, Julie discusses her own learning of the "rules" for the multiplication of integers:

As I mentioned in class, I don't really remember being taught those rules. The only thing that really sticks out of my head is when I was taught that you take the bigger number's sign when multiplying. I remember having a hard time learning these rules. I couldn't understand why you would need to know this equation in real life. I believe that I was my biggest downfall in mathematics, not being able to relate math to real life experiences. (Student journal)

Brittany echoed common sentiments about frustrations with learning mathematics in her journal:

Thinking more critically about how to teach (-)x(+) and (-)x(-) and story problems that have students doing this mathematics was at first mindboggling to me. For so long I just knew the rule and it was hard for me to think about why I thought this. Doing the (+)x(+), (-)x(+), and (+)x(-) were much easier than (-)x(-). When it came to creating a problem, for (-)x(-) I was frustrated because every "everyday" situation I thought of wouldn't work for this problem. (Student journal)

The experience of learning with the goal of conceptual understanding in mind, and being asked to reflect on their own mathematics learning, seemed to have a positive impact on most of the participants. As they envisioned how they would eventually teach their future students, most students seemed to believe a "deeper understanding" should be a necessary component of their future classrooms. Allison wrote:

> *Growing up, I was always very frustrated by math and got through it purely by memorization and repetition. However, even though I was constantly confused by the many varying rules, I never once questioned them or thought to ask why they were applicable. I think this is mainly because my math was taught to me in fairly traditional manner with little social justice or authentic applications or connections looking back. I wish that I had learned differently because I feel that it would help me teach, but I'm also thankful that I leaned the way in which I did because it will help me understand the many different perspectives of my students. I do think it is better to teach integers through an inquiry approach because it allows the students to develop their own individual understanding and methods prior to being subjected to the "right" way*. (Student journal)

Anne continued with this sentiment by stating;

> *I feel like I can teach these concepts better now that I have analyzed the meaning behind them. Having this deeper understanding will allow me to ask probing questions that will cause the students to analyze and consider these concepts. Using story problems will help the students put these concepts into a context that makes sense to them.* (Student journal)

Although some participants indicated that they were unsure whether they could teach in ways that would foster more conceptual understanding due to their own frustrations with mathematics, all students revealed a belief that mathematics is better understood, better taught, in the context of everyday scenarios. Kim states "relating math to real life scenarios help everyone understand it." The participants expressed overwhelmingly that with integers in particular, the use of everyday contexts helped them examine their own misconceptions and limited understandings of multiplication with integers.

Discussion

Teachers tend to teach mathematics in a way that is comfortable to them (Sowder, Phillip, Armstrong, & Schappelle; 1998). If they are only exposed to algorithmic and procedural methods and do not develop a solid understanding of mathematics, they may be left with very few options but to teach in traditional ways. When the idea of developing the *mathematical power* of prospective teachers is infused in a mathematics methods course it may be possible to not only improve the content knowledge of prospective elementary teachers but also provide an opportunity for them to envision teaching mathematics for conceptual understanding. The results of this study indicated that providing prospective elementary teachers the opportunity to examine their own mathematics understanding through challenging them regularly to make sense of mathematics in everyday contexts can be powerful. While some of the prospective elementary teachers who participated in this study still indicated that they would rely on memorization based methods for teaching their own students multiplication with integers, the overwhelming majority indicated the power of using everyday contexts or scenarios for teaching mathematics. They also indicated that through the use of everyday contexts and a variety of models for understanding integers, they feel more empowered not only in their own mathematics knowledge but also in the ways they now envision teaching their future students.

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THE INFLUENCE OF A REFORM-BASED MATHEMATICS METHODS COURSE ON PRESERVICE TEACHERS' BELIEFS AND ATTITUDES

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We report the results of two studies aimed at examining two elementary preservice teachers' beliefs and attitudes before and after taking reform-based methods courses at an urban university in the northeastern U.S. It was learned that practices such as those modeled in reform-based mathematics methods courses lead to higher gain scores in the attitudinal survey given to one cohort of preservice teachers.

Introduction

The preparation of mathematics teachers has become an increasingly important area of interest to mathematics educators and policymakers in recent years. In the United States the National Science Foundation (2005) and the National Research Council (2000) agree that reforming teacher preparation in postsecondary institutions is a central factor in enhancing the capacity to improve mathematics education for all students. At the same time, the quality of K-16 mathematics education is a critical component to broadening the diversity of students in the undergraduate STEM pipeline. Quality mathematics education is especially needed in urban schools where there are high concentrations of minority students and underachievement is exacerbated by diminished access to highly qualified teachers of mathematics1 as compared to non-urban and low-concentration-minority schools (Ingersoll, 2002). The conversation about highly qualified teachers is generally focused on knowledge of content (Ball, Hill, & Bass, 2005). However, two important factors that are often overlooked in the discourse on highly qualified teachers are beliefs and attitudes about teaching mathematics and students' ability to learn mathematics. In our work with preservice teachers who are more likely to enter the teaching profession in urban settings, these beliefs and attitudes are seminal to student success.

Purpose

Based on data collected on elementary preservice teachers' mathematics backgrounds and attitudes toward mathematics, the first author and teacher-researcher in this study believed her primary role was to influence preservice teachers' beliefs about teaching and learning mathematics. In 2004, she conducted a qualitative study (Study 1) to examine elementary preservice teachers' beliefs before and after taking the reform-based methods course at an urban university in the northeastern U.S. Then in 2005, a second instructor (and co-author of this paper) conducted a mixed-methods study (Study 2) to examine teacher beliefs and attitudes before and after taking a similar reform-based methods course at the same university. Thus, the purpose of these studies is to report on changes in preservice teachers' beliefs (Study1 & Study 2) and attitudes (Study 2) before and after taking reform-based mathematics methods courses. The main research question that guided this study was: How do preservice teachers' attitudes toward mathematics (ATM) compare before and after taking a reform-based mathematics course? We also examined beliefs, which were part of a larger study (Newton, Leonard, Evans, & Eastburn, under review).

Theoretical Framework

This research study is framed by sociocultural theory (Vygotsky, 1987) and self-efficacy theory (Bandura, 1997). The sociocultural perspective purports that collective and individual processes are directly related (Cobb & Yackel, 1996) In this context, an instructor guides learning and enculturation as students are taught the skills and concepts needed to function and become productive citizens in society (Vygotsky, 1987)..In this perspective, learning is understood as a process of "enculturation into a community of practice" (Cobb, 1994, p. 13). As a community of practice, we developed a set of ideas to strengthen preservice teachers' mathematics identity. Mathematics identity is one's belief about his/her "(a) ability to do mathematics, (b) the significance of mathematical knowledge, (c) the opportunities and barriers to enter mathematics fields, and (d) the motivation and persistence needed to obtain mathematics knowledge" (Martin, 2000, p. 19). Many of the preservice teachers we taught had negative experiences with mathematics teachers and/or the content. Our premise was most of these preservice teachers did not fully understand the mathematics and were fearful of teaching it. Teacher "identity and awareness both mediate action and pedagogy" (Gonzalez, 2009, p. 23). By attending to identity, we focus on preservice teachers' beliefs and attitudes about teaching and learning mathematics and their roles as teachers of mathematics (Gonzalez, 2009).

Beliefs, attitudes and values combine to form a person's belief system, which influences teacher behaviors and decision-making in the classroom (Parajes, 1992). These belief systems, which we call educational beliefs, encompass beliefs about confidence to perform a specific task (self-efficacy) and confidence to impact student performance (teacher efficacy) (Parajes, 1992). Thus, self-efficacy is also an important construct in the examination of teacher beliefs. Bandura (1997) developed what is now commonly known as efficacy theory.

According to Bandura (1986), teacher efficacy beliefs consist of two substructures: personal self-efficacy and outcome expectancy. Personal self-efficacy is defined as the perceived judgment that an individual has about his or her capacity to teach, whereas outcome expectancy reveals the teacher's perception of the students' ability to learn from his/her teaching and is often referred to as teacher efficacy in the literature (Parajes, 1992). Bandura's (1997) theory suggests efficacy is malleable, and he described four factors that contribute to its development: mastery experiences, vicarious experiences, verbal persuasion, and physiological and affective states. Mastery experiences are those obtained from actual practice. How one feels about a particular subject influences how one teaches it. Teachers who express poor attitudes or disaffection with mathematics are more likely to avoid planning or teaching these subjects (Trice & Ogden, 1986).

Negative emotions may contribute to low efficacy, triggering avoidance behaviors and inhibiting teaching performance (Jesky-Smith, 2002). On the other hand, teachers with high efficacy are more likely to engage students in inquiry and student-centered teaching, which have been linked to higher student achievement (Swars, 2005). Teachers need opportunities to engage in this kind of inquiry throughout the preservice stage, taking advantage of a growing set of practice-based materials (e.g., video of classroom instruction, case studies of student thinking, and student work samples) to engage in thinking about not only what they need to know, but also "an appreciation of what it takes to use it wisely in context" (Lampert & Ball, 1998, p. 36).

Methodology

In order to build a learning community, the first assignment was for preservice teachers to write about their history and past experiences with mathematics. Then she engaged preservice teachers' in a reform-based mathematics methods course, which she believed would positively impact preservice teachers' beliefs about teaching and learning mathematics. Teacher educators, who are reflective of their own practice, often draw upon ethnography as the primary means of data collection (Cochran-Smith & Lytle, 1990). Ethnographies take on many forms including qualitative research, case study research, field research, or anthropological research (Erickson, 1985). Our concept of the ethnographic method is also aligned with Goodall's (2000) notion of the *new ethnography*, which deals more directly with the interpersonal aspect:

> *writing based on interpersonal relationships gains authenticity from the quality of the personal experiences, the richness and depth of individual voices, and a balance between engagements with others and self-reflexive considerations of those engagements* (p. 14).

In light of this work, we used *autoethnography* to engage in reflexivity and interpret our respective roles as instructors in our methods courses (Goodall, 2000).

Setting and Participants

The College of Education where the study took place is situated in an urban city in the northeastern United States. The number of students enrolled in the College is approximately 2,100 per year. In their junior year, prospective teachers who are admitted to the early childhood/elementary certification program are enrolled in practicum and methods courses. Each practicum is taken in conjunction with specific methods courses in the core curriculum (i.e. reading/language arts and social studies methods and practicum; mathematics and science methods and practicum). Practicum students are assigned to urban K-8 classrooms where they team-teach in groups of four two days a week.

Results

Due to the limitations of this paper, we are only reporting changes in preservice teachers' attitude scores. The qualitative analysis on teacher beliefs is currently under review and was presented at a previous conference (Newton, Leonard, Evans, & Eastburn, under review).

Comparison of Pre-Post Attitude toward Mathematics Scores

Preservice teachers' scores on the ATMI were compared (Tapia & Marsh, 1996) before and after taking the methods course. As shown in Table 1, the results of a paired samples *t*-test (two-tailed) reveal a statistically significant difference in attitude pretest and posttest scores for preservice teachers in Carol's methods course.

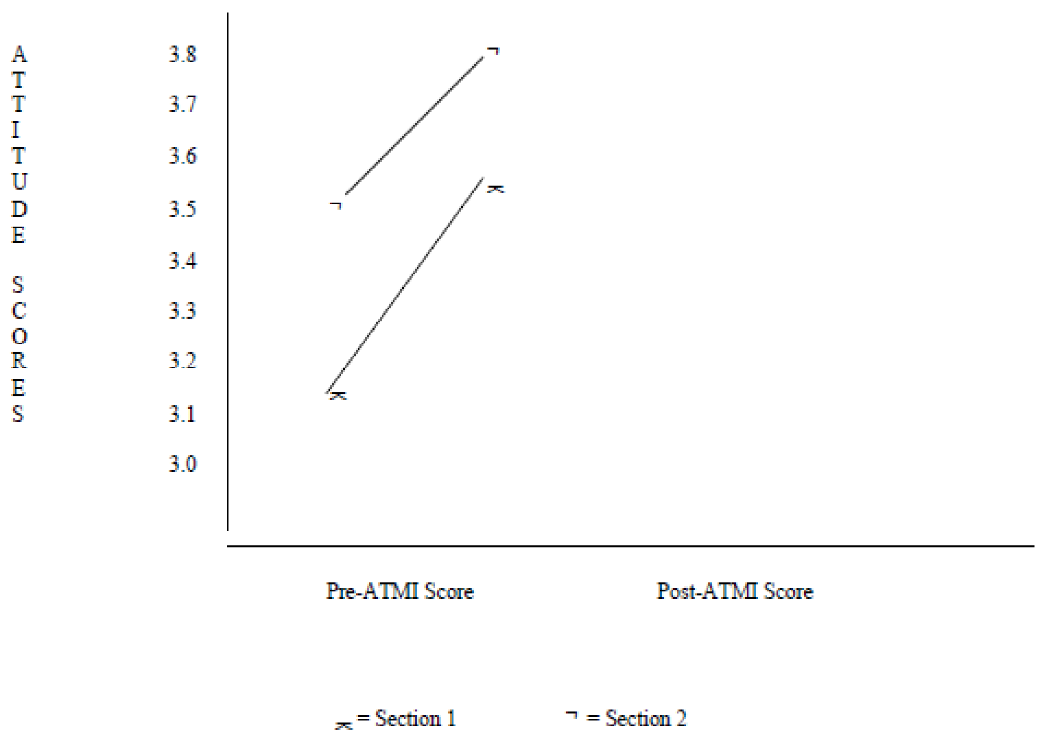
Table 1: Results of Preservice Teachers' Pre-Post Attitude Scores by Section (Carol)

Attitude	Mean	Standard Deviation
Section 1 ($N = 28$)		
Pretest	3.16	0.663
Posttest	3.54*	0.632
Section 2 ($N = 24$)		
Pretest	3.56	0.607
Posttest	3.77*	0.702

* $p < 0.05$ (two-tailed)

For Carol's first section, there was a statistically significant difference between pretest scores (M = 3.16, SD = 0.663) and posttest scores (M = 3.54, SD = 0.632) for the attitude test, $t(27) = -5.108$, $p = 0.000$, $d = 0.59$. Further, for Carol's second section, there was also a statistically significant difference between pretest scores (M = 3.56, SD = 0.670) and posttest scores (M = 3.77, SD = 0.702) for the attitude test, $t(23) = -2.491$, $p = 0.020$, $d = 0.29$. Thus, in both cases, there was an increase in attitudes over the course of the semester. Effect sizes were small to medium.

Figure 1: Results of Attitude toward Mathematics Inventory Scores by Section (Carol)



Discussion

The results of this study of two teacher educators' practices and the development of preservice teachers' attitudes toward mathematics support several findings. Both sections of Carol's classes experienced significant increases in positive attitude toward mathematics. Analyses of qualitative data support Skemp's (1976) claim that attitudes in mathematics are reflective of the teaching-learning environment. It is clear from the cross sample of journal reflections that the reform-based mathematics methods course allowed preservice teachers in this study to make connections between research and practice and methods and practicum. Additional studies that examine development of preservice teachers' attitudes in different types of learning environments are needed to increase understanding about the relationship between the cognitive and affective domains. Results from this study suggest that educational beliefs can be positively enhanced during a methods course.

Experiences reported in Bridget's class illustrate the next finding: the microteaching of peers had a profound influence on preservice teachers' beliefs. Microteaching was a salient part of the reform-based course. This finding is unique since it is absent from the literature on preservice teachers' beliefs. Another element of the course that influenced preservice teachers' efficacy was the Kay Toliver videos. These preservice teachers' belief statements confirm that videos can provide preservice teachers with the vicarious experiences needed to change their beliefs. While the literature suggests that *mastery experiences* are also important they were only mentioned by one student at the beginning and end of the course. This may have more to do with the writing prompts. This finding suggests future studies should ask preservice teachers to reflect on practicum more specifically in connection to the methods course.

In summary, it was learned that practices such as those modeled in reform-based mathematics methods courses lead to higher gain scores in the attitudinal survey given to one cohort of preservice teachers. Hill, Rowan, and Ball's (2005) claim that reform-based mathematics instruction produces high quality early childhood/elementary teachers of urban students. Thus, pedagogy in the classes in this study was changed to engage preservice teachers in inquiry-based activities in order to enhance their conceptual knowledge and understanding. Future studies should examine the relationship between increased content knowledge, attitudes toward mathematics, and concepts of teacher efficacy. Improving both the mathematics content knowledge and attitudes of beginning teachers who choose to work with diverse populations should translate into improved mathematics learning and achievement among poor and urban students if teachers are also attentive to these students' learning styles and culture (Leonard, 2008; Martin, 2007). The results of this study are promising. However, studies are acknowledged that span practices learned in teacher education programs to practices actually performed during induction that are needed to determine whether attitudes toward mathematics learned in reformed-based mathematics methods courses are sustained across time.

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DEVELOPING PRE-SERVICE ELEMENTARY TEACHERS' UNDERSTANDING OF COMPARING FRACTIONS USING A BENCHMARK

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With the *Curriculum Focal Points* (NCTM, 2006) and other standards advocating for students to compare fractions using reasoning, it is increasingly important for teachers to understand this type of reasoning as well. This is not only for teachers to foster students' development of understanding fractions as quantities, but also "to distinguish appropriate student strategies from those based on faulty reasoning" (Lamon, 2005, p.113). Though studies have documented the importance of using reasoning strategies when comparing fractions as opposed to using procedures (Behr et. al., 1984; Lamon, 2005), research has lacked in documenting the ways in which teachers develop this knowledge for themselves.

To determine the ways in which teachers develop this knowledge a semester-long classroom teaching experiment was conducted with a class of thirty-three pre-service elementary teachers. The study was conducted in a content course focusing on mathematics for teaching elementary school. Specifically, the research question was:

In what ways do classroom mathematical practices develop related to rational number?

This poster will highlight the results from this study regarding pre-service teachers' development of comparing fractions using a benchmark. The results indicated that there were three different solution strategies that were presented by the pre-service teachers. These included comparing fractions to 1/2, comparing with an additive within relationship, and comparing using a whole number dominance strategy by only comparing numerators. After two days of instruction, the class determined that the only valid strategy of these was the strategy of comparing the fractions to 1/2.

By exploring misconceptions in conjunction with correct strategies the class was able to develop an understanding of why the benchmark strategy was acceptable, understood when the benchmark strategy would fail, and was able to expand this to include that any fraction can be used as a benchmark. Thus, allowing pre-service teachers to explore problem situations in which their reasoning can come to the forefront of classroom conversations supported their understanding in determining which reasoning strategies were mathematically correct or incorrect.

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EXPLORING MIDDLE SCHOOL PRESERVICE TEACHERS' MATHEMATICAL PROBLEM POSING

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Problem solving plays an important role in the teaching and learning of mathematics. The National Council of Teachers of Mathematics [NCTM] *Principles and Standards for School Mathematics* suggested that teachers integrate problem solving into the context of mathematical situations by choosing specific problems, because they are likely to prompt particular strategies and allow for the development of certain mathematical ideas. However, the activity of problem generation is typically a more challenging and creative task than obtaining the solution itself (Kilpatrick, 1987). Problem posing allows students to generate or to reformulate new problems based on a given situation or problem (Gonzales, 1994). In addition, NCTM encourages teachers to incorporate problem posing as a means of instruction to engage students with problems involving higher-order mathematical thinking. Therefore, this study analyzes the effect of "What-If-Not" (WIN) strategy (Brown & Walter, 2005) on the types of problems pre-service teachers (PTs) generate based on a give problem. In addition, using a reflective journal, this study investigates PT's perceptions on their problem reformulation process.

An open-ended mathematical problem was presented to 39 middle school PTs at a southwestern public university. Based on the given problem, PTs posed a new problem with the same or higher level of difficulty. Later, after three sessions of problem posing interventions, PTs were asked to reformulate the original problem using the WIN approach. A classification scheme adapted from Lavy and Bershadsky (2003) was used to categorize the problem posing statements in terms of problem types. Finally, content analysis was conducted on journal entries to investigate PT's perceptions of their problem reformulation process. Results revealed that even when a wide range of types of problems were posed, the changes made were not significant. Major emphasis was given to numerical changes. However, the overall PT's problem posing skills improved. Also, the findings from their reflective journals enabled the researchers to study the differences in PT's perceptions in creating their own problems. Implications for mathematics teacher education are described.

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FOSTERING PROBABILISTIC INTUITION THROUGH MULTICULTURAL GAMES OF CHANCE

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The *Principles and Standards for School Mathematics* (2000) strongly emphasizes the importance of teaching probability for students at all levels. Traditional instruction is often insufficient, and even has negative effects on students' understanding of probability (Shaughnessy, 1992). Reform-based teaching methods call for inclusion of diverse games of chance to promote an individual's understanding of probability.

For teachers to develop a strong, coherent and intuitive background for probabilistic reasoning among their students they need to possess strong content knowledge and pedagogical knowledge in this domain; moreover they should be aware of and be able to understand, identify, and overcome their own misconceptions related to probability (Fischbein & Schnarch, 1997). However, there is a dearth in literature that concerns teachers' understanding of probability that can inform preservice and inservice teacher education programs. We devised a research study that attempts to fill in part of this gap by investigating preservice teachers' (PSTs) understanding of probability as they explored multicultural games of chance (e.g. Lu-Lu, Ampe, Songish).

We will refer to the probabilistic thinking framework developed by Polaki and colleagues (2000) to analyze preservice teachers' understanding of probability and their responses to the tasks based on the games. We engaged PSTs in a 4-week probability module that exposed them to multicultural games of chance. PSTs explored these games, collected, recorded and analyzed experimental data. Data included written student work, audio / video recordings of student engaging in simulations of the games, and pre test and post test responses. PSTs completed simulations of games, collected and analyzed experimental data, and engaged in meaningful conversations related to sample space, experimental and theoretical probability, and in certain instances, the *law of large numbers*. Preliminary data analysis highlighted PSTs' difficulties associated with connecting and extending their prior knowledge of probability to new real-life contexts. In our poster, we will feature descriptions of games, probability tasks, and findings related to two of these games – Lu-Lu and Ampe.

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INFLUENCES OF MATH CONTENT COURSES ON ELEMENTARY PRESERVICE TEACHERS

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There is a general consensus that mathematical content knowledge (M-CK) is crucial for enabling elementary school teachers to effectively teach mathematics (Ball, 2003). However, it has been suggested that M-CK is not sufficient for elementary school teachers – it must be accompanied by mathematics pedagogical content knowledge (M-PCK) (Ball, 2003). In order to identify coursework that may promote M-CK and M-PCK, this study investigates confidence of M-CK and M-PCK of elementary preservice teachers (PSTs) who have completed mathematics content coursework designed for elementary teachers. Confidence with respect to M-CK and M-PCK was explored by comparing survey responses of PSTs who completed the identified math content courses (content, C-group) with those who did not complete them (non-content, NC-group). The survey included Likert items adapted from the *Fennema-Sherman Mathematics Attitude Scale* (Mulhern & Rae, 1998) and open-ended content problems designed to uncover both M-CK and M-PCK. This poster focuses on 19 PSTs (9 C, 10 NC) who completed both pre- and post-surveys prior to and after completion of a required math methods course. Likert items were analyzed using standard statistical techniques. Analysis of open-ended problems and additional interview data is in progress. Analysis of the Likert items indicated that the C-group reported greater M-CK confidence both pre- and post- methods course than the NC-group, but that the NC-group showed greater *increases* in M-CK confidence. Further, the results suggest greater M-PCK confidence by the C-group than the NC-group both pre- and post-methods course *and* that the C-group showed greater *increases* in M-PCK confidence than the NC-group. A conjecture is that PSTs going into a math methods course with greater M-CK (C-group) may increase their M-PCK confidence, and that this increase could be greater than those going in with less M-CK (NC-group). The NC-group's gains in M-CK confidence and relative lack of gain in M-PCK confidence from pre- to post-methods course may suggest that their attention was focused on mathematics content over teaching methods or student learning. A possible implication is that math content courses designed for elementary PSTs may enhance learning outcomes of math methods courses by providing sufficient M-CK to allow the PSTs to focus their attention, during the methods course, on the teaching methods and the student learning related to the mathematics. Without these math content courses, the PSTs' attention toward student learning and mathematical teaching may be diluted.

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INTERACTIONS BETWEEN A PRE-SERVICE ELEMENTARY SCHOOL TEACHER'S CONCEPTIONS OF DECIMAL VALUES

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Research has shown that pre-service teachers struggle with the concepts of decimal numeration and decimal place values (e.g., Stacey et al., 2001). In order to help pre-service teachers develop a stronger conceptual understanding of decimal numbers in teacher preparation courses, we must continue to try to understand their current conceptions about decimals and how those conceptions play out in various contexts (Bransford, Brown, & Cocking, 1999).

Two incorrect conceptions have been identified in the literature: 1) associating decimal numbers with negative numbers (Stacey et al., 2001) and, 2) increasing powers of ten for each consecutive place value to the right of the decimal point (Zazkis & Khoury, 1993). Our research considered the interactions of these two conceptions.

The data analyzed for this report are drawn from a 60-minute one-on-one, task-based interview with one pre-service teacher, Sarah. The analysis focused on characterizing Sarah's conceptual understanding. Results of this analysis suggested that Sarah holds the two incorrect conceptions described above and that these misconceptions can interact to produce correct answers on decimal value comparison tasks. For example, in her interview, she correctly indicated that .01 is less than .9. However, her reasoning was as follows: 1) both numbers are less than zero (associating them with negative numbers) and 2) that .01 is "one more" than .9 (like 10 is one more than 9 in base ten). Combining these two conceptions, she argued that .01 is less than .9 because the value comparison is reversed. She used similar reasoning to justify that .606 is less than .66.

This case study is reminiscent of previous research results that have established the idea that correct answers are not necessarily an indicator of correct conceptual understanding (e.g. Erlwanger, 1973). This case adds to the research literature because this particular interaction of conceptions is previously unreported. This result can be used to inform future research and improvements to teacher education on decimal numbers.

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PRE-SERVICE TEACHERS' DETACHED VIEWS OF MATHEMATICS

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In his review of research on mathematics teachers' beliefs and affect, Philipp (2007) emphasizes the importance of teacher affect in mathematics teaching and learning. But there is a lack of research on teachers' affect. While there are many dimensions of affect, I focus on the relation of subjectivity between mathematics and teachers as learners of mathematics. I ask the following questions: What do pre-service teachers (PSTs) perceive as their relationship with mathematics? How do they perceive themselves as learners of mathematics? To what degree do they find personal meaning and purpose in learning mathematics?

These questions arise in the literature on teacher affect and critique of current education policy and discourse in many Western countries (e.g., Greene, 1988). In the field of mathematics education, there is a loss of professional-ethical autonomy, and teachers are increasingly represented as objects rather than as subjects in policy discourse (Neyland, 2004; Walshaw, 2004). Therefore, it is important to understand how PSTs construct their subjectivity in relation to learning and teaching mathematics.

This is an on-going investigation into pre-service elementary school teachers' development of ethical selves as learners and teachers of mathematics. The first phase of data were collected from 22 PSTs in a mathematics methods course, including drawings of what mathematics was to them, definitions of mathematicians, and reflection papers on their schooling experience with mathematics.

The most salient theme was that a significant majority of these PSTs alienated themselves from a meaningful and purposeful process of learning mathematics. They regarded mathematics as an external instrument for agendas set by others. They judged their learning and achievement in mathematics by recourse to external evidence such as grades and comparisons with peers. At stake was their power to pronounce personal meanings and purposes. I will share selected teachers' drawings of mathematics and reflections on their learning experiences. I will also discuss the implications of the findings in relation to prevalent mathematics education discourses, in which the Platonic view of mathematics is the norm.

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PRE-SERVICE TEACHERS' KNOWLEDGE OF FRACTION OPERATIONS: A COMPARATIVE STUDY OF THE U.S. AND TAIWAN

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A recent international survey of student achievement, Trends in International Mathematics and Science Study (TIMSS), shows that that several Asian countries continued to greatly outperform the United States in math. Fourth-grade students in Hong Kong and eighth-grade students in Taiwan are the world's top scorers in math, according to the survey (TIMSS, 2007). However, studies of international comparison among pre-service teachers in math achievement are limited.

The present study focused on the comparison of Taiwanese and US pre-service teachers in procedural and conceptual knowledge (Hiebert & Lefevre, 1986). We were seeking to understand the strength and weakness of pre-service teachers' knowledge of fraction operations in both countries.

The participants in the U.S. sample were enrolled in a teacher education program at a mid-western university. Fifty pre-service teachers, who were preparing to teach children in grades K-8, participated in this study. They were in their third year of study. The Taiwanese sample was similarly selected. The Taiwanese sample consisted of 47 pre-service teachers enrolled in a teacher education program at a university in central Taiwan.

Results indicated that Taiwanese pre-service teachers performed better in procedural knowledge on fraction operations than American pre-service teachers. No significant differences were found for conceptual knowledge on fraction division. Further, the correlation in this study showed that both Taiwanese and American pre-service teachers' conceptual and procedural knowledge of fraction operations were weak.

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REAL-LIFE CONNECTIONS: PROSPECTIVE ELEMENTARY TEACHERS' PERCEPTIONS REFLECTED IN POSING AND EVALUATING STORY PROBLEMS

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The study explored perspectives of a group of prospective elementary teachers' on real-life connections through posing story problems and evaluating the quality of sample story problems. The goal was to identify prospective teachers' perceptions about teaching and learning mathematics in relation to real-life connections in an explicit way and to gain deeper insights into their current perceptions.

The analysis of data from 71 prospective elementary teachers, while they engaged in the posing and evaluating story problems, revealed the following results:

- *Positive beliefs with insufficient specifics*. Participants demonstrated predominantly positive beliefs about the purpose and effective use of real-life connections. However, the criteria that the participants proposed included very few specifics about their understanding of what it might take for teachers and students to incorporate real-life connections into their teaching and learning processes.
- *Themes on the paradigmatic-narrative spectrum.* In categorizing the themes identified by participants as "criteria for real-life connected story problems" I relied on the paradigmatic-narrative framework Chapman (2006) and Depaepe et al. (2009). Analysis indicated that the narrative-oriented responses were more dominant than the paradigmatic-oriented responses among the participants of this study.
- *Reality: How real is real?* Although, participants possessed strong beliefs about importance of reality in story problems. However, their story problems and their evaluation of peers' problems revealed a vast discrepancy between how reality was defined and accepted by the them.
- *Utilitarian view.* The majority of participants perceive that real-life connection has a value because of its usefulness. Considering the role of story problems as the interplay between mathematics and reality, the participants' perspectives were extremely unbalanced.
- *Noted gaps.* The story problems participants posed did not reflect their perceived beliefs on real-life connections. This suggests that prospective teachers were aware of the importance of utilizing real-life connections in their teacher education programs at the general level, yet they were not fully exposed to the teachers' thinking process for the implementation of real-life context in mathematics education.

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RETHINKING "UNLEARNING" TO TEACH MATHEMATICS: QUESTIONS FOR MATHEMATICS TEACHER EDUCATORS

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In Deborah Ball's 1998 piece "Unlearning to Teach Mathematics", she presents the case that preservice teachers do not enter their programs as empty sponges that will absorb and expel only information that is presented to them in teacher preparation. Instead they are a group of diverse learners who, before ever stepping into the classroom as teachers, have previous knowledge and have had prior experiences that can influence their views on teaching (Ball, 1988). However, at several points in this article, the previous experience is presented as obstacles for the preservice teacher classroom. This poster hypothesizes what a teacher education classroom might be like if, instead of obstacles, these previous experiences are treated as opportunities for learning about teaching mathematics.

This poster begins by briefly looking at what current research tells us about teacher preparation and teacher learning, based on Sowder's chapter in the second handbook of research on mathematics teaching and learning (Sowder, 2007). Next, we present a sample of preservice teacher responses to mathematics teaching scenario prompts (Brakoniecki, 2009). This poster asks teacher educators to imagine these preservice teachers are in their mathematics methods courses and how they could incorporate their beliefs and knowledge about teaching mathematics into their methods courses. Lastly this poster presents different ways of responding to the preservice teachers' responses to the prompts: one set if the classroom treats these experiences as obstacles, and one set if the classroom treats these experiences as opportunities.

Endnotes

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Chapter 15: Preservice Teacher Preparation (Secondary)

Research Reports

Brief Research Reports

Posters

PRESERVICE TEACHERS' INITIAL EXPERIENCES IN SHIFTING FROM "LEARNERS/DOERS OF MATHEMATICS" TO "TEACHERS OF MATHEMATICS"

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Successful teacher preparation programs provide learning experiences that help candidates make the shift from "student" to "teacher." In this paper we explore a process for providing candidates such experiences. Utilizing the MATH process, prospective high school mathematics teachers explore rich tasks by solving the task, analyzing sample student work, designing a solution key and modifying the task. We use their engagement in these explorations and reflections on the process to analyze the development of candidates' Content Knowledge and Pedagogical Content Knowledge.

Introduction

[I] solved the problem algebraically and hadn't thought of another way doing so
[A Teacher Candidate, EDT 430]

Successful teacher preparation programs provide learning experiences that encourage candidates make the shift from "student" to "teacher." This is not an easy transition, particularly for future school mathematics teachers. This transformation is often complicated by the fact that candidates' interest in mathematics and self-efficacy regarding mathematics arises from their personal success in solving complex problems. Over the past several years, we have developed a five-step approach for examining rich content tasks with preservice teachers. The approach, which we refer to as the Mathematics as Teacher Heuristic (MATH), is designed to encourage candidates to shift their mathematical view from a student orientation to one embracing teacher-oriented perspectives. The MATH process requires candidates to revisit rich mathematics tasks from points of view rarely considered in content-oriented coursework. Candidates complete a five-step process that includes (a) solving a rich task as a student; (2) assessing student work samples associated with the same task; (3) constructing "classroom ready" solution keys for students; (4) developing scaffolded instructional materials addressing student misconceptions (gleaned from earlier analyses); and (5) reflecting on the process. Each step of the MATH process encourages candidates to reconsider rich tasks from increasingly teacher-centric points of view. Assessing student work and constructing solution keys are activities that encourage candidates to consider rich tasks from the perspective of younger learners.

Ultimately, the instructional materials that candidates create as part of the MATH process are shared with practicing teachers and their students. By creating work for audiences beyond the university classroom, candidates find the work more meaningful. As they construct solutions and materials for classroom teachers and their students, candidates recognize that "finding an answer" is no longer the central purpose of their mathematical work. Rather, problem solving is situated in the context of helping students learn mathematics (a teacher-oriented behavior rather than a student-oriented one). As candidates construct learning materials, they discuss mathematics content in the methods classroom from a learner's viewpoint. The approach provides a useful basis to analyze the development of candidates' Pedagogical Content

Knowledge (PCK) (Shulman, 1986), combination of Content Knowledge (CK) and Pedagogical Knowledge (PK).

In this paper, we analyze the work of one group of pre-service teachers as they explored a rich mathematics task using the MATH process. In particular, we examine candidates' analyses of student work, their solution keys, their modifications of the task and their reflections and insights on the MATH process as a whole.

Related Literature

Models for understanding the transition from "student" to "teacher" focus on dissonance and motivation in school settings, factors that occur regularly for in-service teachers in their day-to-day practice (Clarke & Hollingsworth, 2002; Edwards, 1994). Clarke and Hollingsworth (2002) and Loughran (2002) stress the importance of self-reflection in the development and evolution of teacher knowledge, beliefs, and attitudes. We believe that pre-service teachers need to experience similar dissonances in university coursework order to accept the need for change and reflect on their change process. Providing such opportunities for change and reflection is, perhaps, more difficult in the case of pre-service teachers at the very beginning of their course of study since they have fewer authentic teaching experiences to draw upon. Studies show that two main sources for dissonance in initiating candidate growth are in methods courses and student teaching.

Brown and Borko (1992) argue that there are three important issues in the process of “learning to teach;” namely, (1) the influence of the content knowledge, (2) novice’s learning pedagogical content knowledge, and (3) difficulties in acquiring pedagogical reasoning skills. Furthermore, they assert that “one of the most difficult aspects of learning to teach is making the transition from a personal orientation to a discipline to thinking about how to organize and represent the content of that discipline to facilitate student understanding” (p. 221). In a case study of one teacher’s successful transition from student to teacher, Velez-Rendon (2006) identified a complex interplay between many factors during this transition including “the learning background the participants brought with her added to her knowledge of the subject matter” (p. 320). Similar to Brown and Borko’s (1992) results, Velez-Rendon concluded that the preparation of the subject matter for instruction was challenging for the student teacher.

Another related issue is the “tension between participants’ views of themselves as adult learners of mathematics and their practice with young children” (Brown and Borko, 1992, p. 215). Ball (1989) studied the role of a methods course for elementary mathematics teachers in helping candidates to learn to teach. Ball highlights the importance of content knowledge and the experiences of pre-service teachers as learners of mathematics. “Unless mathematics teacher educators are satisfied with what prospective teachers have learned from their experiences as students in math[ematics] classrooms (and most are not), this highlights a need to interrupt, to break in, what is otherwise a smooth continuity from students to teacher“ (p. 4). In this sense, we see Ball’s suggestions to break with experience as an opportunity to create a dissonance.

Loughran (2002) explored the development of knowledge through effective reflective experience from student teacher to experienced teacher, comparing candidates’ views of teaching and learning with those of practitioners. Candidates' views regarding teaching and learning typically equated learning “with gaining right answers" (p. 41).This naive view of teaching contrasts markedly with teacher comments that emphasize the importance of“student . . . opportunities to be active and think about their learning experiences” (p. 41). Loughran's study illustrates the importance of giving prospective teachers opportunities to face their views, reflect and reconsider them. Swafford, Jones, Thornton, Stump, and Miller (1999) echo these findings for inservice teachers. Swaford et al. recommends the creation of environments for teachers that improve their content and pedagogical knowledge through reflection and collaboration.

Moving from the back of the table to the front of the blackboard is not a smooth or simple journey. Pre-service teachers carry the baggage of their previous experiences obtained as students. Those experiences affect not only their journey and their content knowledge but also their beliefs, views, attitudes, and values. A candidates' affective domain influences how they utilize their own knowledge (content knowledge (CK), pedagogical knowledge (PK) and pedagogical content knowledge (PCK)) in classrooms and in their journey. How, then, can we can help our pre-service teachers to distance themselves their student identity during their methods courses? We believe that a partial answer exists in reflective problem solving activities embedded within the MATH process.

Methodology

Participants

Participants in this study were candidates enrolled in EDT 430, the second course in a year-long methods sequence designed for prospective secondary mathematics teachers. Twenty-three of the 29 participants were undergraduates; six were graduate students who had returned to school seeking teacher licensure in secondary mathematics education. In the previous methods course (EDT 429), candidates spent significant time solving rich mathematics tasks as students. The intention of problem solving experiences in the first semester was two-fold, namely (1) to reintroduce the content of secondary school mathematics to candidates and (2) to communicate to candidates the notion of "rich" mathematical tasks.

In the follow-up course (EDT 430), we revisit a subset of these rich tasks with teacher candidates using the five-step MATH process. In the paragraphs that follow, we explore data generated through the exploration of one such task, namely the Bridgewater Problem. We analyze data generated from the task using a blend of quantitative and qualitative methods.

Candidate Tasks

Candidates solved the problem presented below in the first semester methods course.

> **The Bridgewater Problem.** At 1:00 p.m., two hikers began walking, the first from Amityville to Bridgewater, the second from Bridgewater to Amityville along the same path. Each walked at a constant speed. They met at 4:00 p.m. The first hiker arrived at Bridgewater 2.5 hours before the second hiker arrived at Amityville. When did the second hiker get to Amityville?

Using the MATH process, we asked the candidates to complete the following tasks related to the Bridgewater Problem either in pairs or individually.

Task 1: Begin by solving the Bridgewater problem with a partner, without referring to any outside materials. Show all work on separate paper. You are welcome to write work by hand.

Task 2: Examine student solutions. Identify student solutions that you would consider as (a) primarily numerical; (b) primarily geometric; (c) primarily algebraic without overt use of functions; and (d) primarily functional (overtly uses functions in solution).

Task 3: Using one of the student samples as a starting point, construct a formal, answer-key quality solution to the Bridgewater Task that relies primarily on one of the following approaches: (a) numerical; (b) geometric; (c) algebraic (no functions used); or (d) functional approach (includes calculus-based approaches).

Task 4: Construct a modified version of the Bridgewater Task worksheet that fosters student exploration using either a numerical or geometric approach (no functions or significant equation solving required). Your worksheet should be no longer than 2 single-sided pages.

Task 5: Reflect on what you've learned while completing this activity. Identify one "big idea" you can take away from Mathematics Project 1 that will help inform your future teaching of mathematics at the high school level.

Figure 1. Tasks

Results

In our analysis, we focus on Tasks 2 – 5 of the MATH process assignment. As previously mentioned, each step of the process encourages candidates to reconsider rich tasks from increasingly teacher-centric points of view. In Task 2, candidates begin to think about the problem from another person's perspective. Task 3 requires candidates to internalize the idea of other perspectives and incorporate other viewpoints into their own work. Task 4 provides candidates with an opportunity to reconfigure the problem to guide/scaffold exploration by the student.

Analyzing Student Work

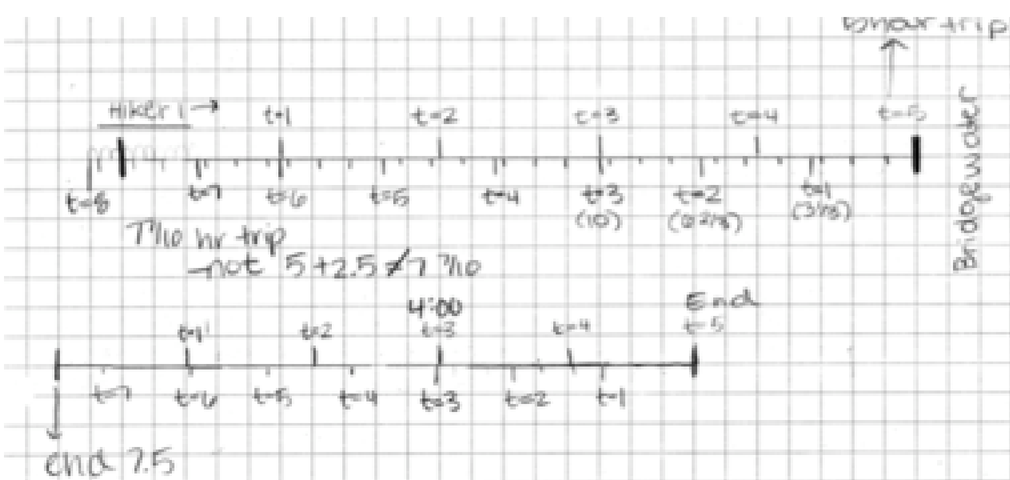


Figure 2. Student solution to Bridgewater task highlighting a geometric number line approach

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

Whereas teacher candidates tend to solve the Bridgewater task in fairly similar ways - typically using rates and variables to construct functions and equations that may be solved to determine an answer - high school students use a wide array of approaches. Candidates' extensive experience with symbolic forms and function in their recent studies of calculus make it difficult for them to "see" numerical or geometric approaches such as those depicted in Figure 2.

Reflection on the Analysis of Student Work

> *We had never had an experience like this where we were critically thinking about students' solutions, and this skill can be very helpful as we become teachers in the future.* [Teacher Candidate, EDT 430]

Although many candidates were familiar with the notion that students solve problems in different ways, nevertheless many expressed surprised upon seeing the variety of solutions that students created for the Bridgewater problem. Nineteen of 20 reflections included comments expressing the sentiment that "after examining the student work it occurred to me that not only are there multiple ways to solve the problem, but there are different ways within the same basic approach."

As candidates take initial steps towards becoming a teacher, they begin to realize that nothing is clear-cut and that there are no "cookie cutters." Eight of 20 groups discussed difficulties they experienced during the interpretations of student work. Until this point candidates had only considered their own work and thought processes. The transition to teacher isn't easy. As one candidate noted, "Many of the solutions were hard to follow at first because of their seeming lack of logical procession...we did not expect to see the vast array of methods that the students used, none of which approached the problems in the same way that we did." It was apparent that students' reasoning was different from that of the prospective teachers. Eleven groups mentioned a similar issue. Some groups discussed their way of thinking or their solution and compared this to the students' ways of thinking or solutions. Other candidates mentioned the need to be more aware of different solutions not just think about their own.

"We solved the problem algebraically and hadn't thought of another way doing so. The project showed me that as future teacher I need be aware of any and all possible solutions for a problem, because I'm bound to have at least one student who takes that approach". [Teacher Candidate, EDT 430]

Candidates also discussed more practical issues regarding the analysis of students' work. Five groups were surprised to see how much time it took to grade students' assignments and the need for rubrics: "In a sense the task introduced us to the reality of being a teacher, such that we are going to have to face the fact that the grading and the understanding of each individual student's work will take a significant of time."

Creating an Answer Key

As prospective teachers, the participants rely on their previous experiences as learners when developing solution keys for students to follow. Although candidates were told that "classroom ready" keys would be shared with students and their teacher, the candidates' work resembled exam or graded homework solutions more than materials produced for younger learners. Several common weaknesses common in candidate solution keys including the lack of student guidance and scaffolding, complicated or incomplete use of mathematical notations, and ambiguous text accompanying mathematical notation. Several examples are provided in Figures 3-5.

Lack of Guidance/Scaffolding. Prospective teachers did not provide detailed explanations of how they built the mathematical model to solve the problem. Participants also failed to provide a detailed explanation regarding what the mathematical formulas or variables stand for in their models.

Incomplete Mathematical Notation. In the solution key in Figure 3, the candidate explains variable t, however, the reader is left to guess what m, n and D represent.

Call the time that it takes George to get from Amityville to Bridgewater t. Therefore, Peter's time will be t +2.5. By our distance formula, we know the following:

D=tm

D=(t+2.5)n

Now solve these two equations for m and n:

$$m = \frac{D}{t}$$

$$n = \frac{D}{t + 2.5}$$

Figure 3. Solution Key with incomplete mathematical notation

Problems with Mathematical Notations. Many prospective teachers failed to pay adequately close attention to labeling the units. They also chose to use complicated mathematical terms instead of contextual names. Figure 4 shows the work of a candidate who chose to label the columns of their table with vague descriptors.

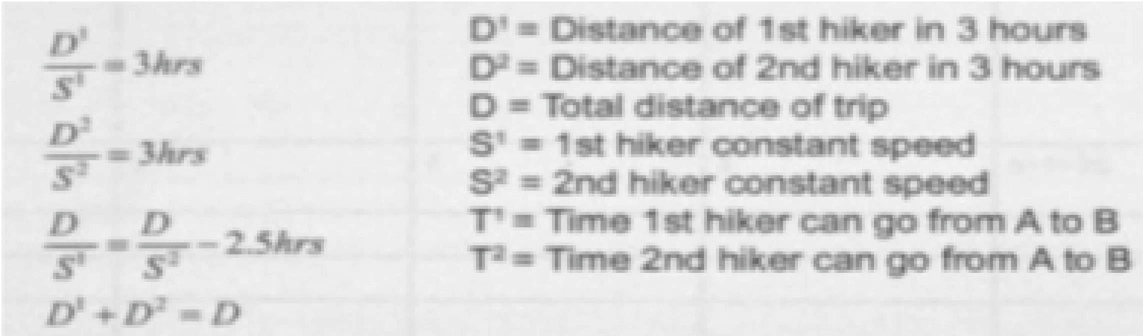
Now, we will use our table to find numerators and denominators that divide equal to 3 (the difference in hours travelled).

Numerator	Denominator	Divide	Answer
27	9	$\frac{27}{9}$	3
15	5	$\frac{15}{5}$	3
9	3	$\frac{9}{3}$	3
6	2	$\frac{6}{2}$	3

So now, using our equations above, we must find numerators that satisfy $\frac{d_1}{n_1} = 3$, $\frac{d_2}{n_2} = 3$,

Figure 4. Solution Key with Vague Mathematical Labels

The use of non-standard variables and inconsistent use of variables were the other issues that might complicate the answer keys that prospective teachers provided for their future students. Figure 5 shows an answer key with variables defined using superscripts. Here the use of superscripts is ill-advised since they look like exponents, which will likely confuse students.



$$\frac{D^1}{S^1} = 3hrs$$

$$\frac{D^2}{S^2} = 3hrs$$

$$\frac{D}{S^1} = \frac{D}{S^2} - 2.5hrs$$

$$D^1 + D^2 = D$$

D^1 = Distance of 1st hiker in 3 hours
D^2 = Distance of 2nd hiker in 3 hours
D = Total distance of trip
S^1 = 1st hiker constant speed
S^2 = 2nd hiker constant speed
T^1 = Time 1st hiker can go from A to B
T^2 = Time 2nd hiker can go from A to B

Figure 5. Solution Key with Complicated Mathematical Notations

Reflection on the Creating a Solution Key

In discussing their experiences in the development of the solution key, candidates identified the main struggle as deciding how to cater to the diverse body of students without providing multiple keys. As one candidate noted, "we also learned that it would be difficult to create an answer key based off of one method. It is impossible to use all the methods to make an answer key, but by choosing only one method it feels as though we are leaving out so much great work that the students could learn from.' 'Clearly, the creation of a solution key provided another experience for the candidates in thinking not just about how they would do a problem but how others would do the problem and how they would evaluate and assess that work.

Modification of the Worksheet

We believe that when preparing a worksheet for students, candidates were more pedagogically oriented than was the case with their solution key preparation. For instance, candidates asked students to explain their answers in their revised materials. One group asked students to use technology to solve the problem. Another group included a rubric for their students to show how different parts of the problem were going to be evaluated differently. Two groups focused on helping students to extend the solution for different rates. Nevertheless, there were still some problems with some of the worksheets that prospective teachers created. For instance, many worksheets included vague questions, variables, or tables. A particular instance was a worksheet asking students to create respective formulas for the hikers but was not clear which distance was to be used: “Now create two formulas using distance and time to find George and Peter’s rates.” Not using units, the choice of seemingly arbitrary distances, and the use of non-standard variables were recurrent problems.

Candidates’ Reflection on Modifying the Task

In reflecting on their worksheet design some candidates discussed their struggle to balance the information provided in the worksheet to guide students while not revealing the answer but also serving the students with diverse learning styles and at different achievement levels with clarity.

> *Revising the worksheet was a challenge for me…I tried very hard to make sure that students are given hints, examples, and even answers if they look hard enough.* [Teacher Candidate, EDT 430]

> *The creation of the new hand out we made forced us to consider the student perspective. This allowed us to read what we created as a student might, which allowed us to troubleshoot our worksheet by making it more clear what we wanted student to do through the use of unambiguous language and very clear directions.* [Teacher Candidate, EDT 430]

The last quote above provides evidence of the change that some pre-service teachers underwent through the creation of a worksheet with students firmly in mind.

Discussion

Recall that Brown and Borko's (1992) identified three important issues associated with the process of "learning to teach" namely, the influence of the content knowledge, novice's learning pedagogical content knowledge, and difficulties in acquiring pedagogical reasoning skills. The task in which the participants in this study were engaged promoted development in each of these areas. The candidates Content Knowledge (CK) is engaged not only in solving the problem themselves but through the analysis of numerical, geometric and functional approaches to solving the problem in addition to the algebraic approach on which most of them relied.

Their Pedagogical Content Knowledge (PCK) is engaged both in their attempts to create a solution key and their reconfiguring of the problem as an exploration task. For both of these tasks the candidates had to engage with the idea of how someone else is thinking about the problem, how to anticipate other peoples responses and how to design a task which can account for many approaches not just their preferred one. "Not only can students learn from teachers, but we both learned a lot from these students."

Finally, Pedagogical Reasoning Skills are engaged throughout, particularly in Task 2, the analysis of student work, where the candidates have to reason through and understand a students' thinking and think about how they could discuss the solution with the student. "Even though we are becoming teachers all of our classes so far … have never let us experience the role as a teacher … This was the first real chance we had to look at student work." These pre-service teachers were beginning to be able to see themselves getting up from the chairs and going to the in front of the classroom.

We believe that all five tasks in this project taken together provide a powerful mechanism for providing a candidates with a dissonant experience which forces them to reflect on their own understanding of mathematics but, more importantly, provides a framework to move prospective teachers from doers or learners of mathematics to being teachers of mathematics.

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PROSPECTIVE HIGH SCHOOL TEACHERS' PROBLEM SOLVING APPROACHES THAT FOSTER THE USE OF TECHNOLOGY

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The aim of this research report is to analyze and document problem-solving approaches that prospective high-school teachers exhibited while using a dynamic software (Cabri-Geometry) to represent and explore textbook problems. Research questions that helped us structure the development of the study include: What type of questions do teachers pose and pursue during their interaction with the problems? To what extent their initial problem solving strategies are enhanced through the use of the tools? To what extent does the use of the tool help them identify and examine possible instructional routes to guide the development of their lessons? Results indicate that the use of the tool helped them visualize, explore, and present a set of new mathematical relations that open up new avenues to frame learning activities for students to construct mathematical knowledge.

Introduction

The National Research Council (2001) suggests that mathematical tasks are important for teachers and students to develop mathematical proficiency. Here, learner's mathematical proficiency is conceived of as the learner's development of a network of five interwoven and interdependent strands: (a) conceptual understanding that involves the learners' construction of a web connections and meanings of concepts and ideas that allow them to develop a coherent an integrated mathematical knowledge; (b) procedural fluency that refers to the efficient access and use of procedures, rules and skills in problem solving activities; (c) strategic competence that involves the learners' tendency to formulate questions, conjectures, and new problems that need to be represented and explored in different ways and by using diverse media including computational tools; (d) adaptive reasoning refers to the learners' tendency to look for different arguments, including formal ones, to support and communicate mathematical relations or results; and (e) productive disposition that refers to the learners tendency to conceptualize mathematics as worthwhile discipline and useful and helpful to comprehend social and disciplinary phenomena. In this context, an important goal for teachers, while working on tasks, is to discuss that the extent to which the process involved during their interaction with the problems help them develop and reflect on mathematical activities associated with the proficiency' strands.

In this context, the significant development and availability of computational tools such as dynamic software and hand calculators open up several ways to promote problem-posing activities in mathematical classrooms (Santos-Trigo, 2007).

Regardless of the particular tools that are used, they are likely to shape the way we think. Mathematical activity requires the use of tools, and the tools we use influence the way we think about the activity…[Understanding] is made up of many connections or relationships. Some tools help students make certain connections; other tools encourage different connections (Hiebert, et al, 1997, p.10).

In this study, we document and analyse the extent to which the use of a dynamic software helps teachers represent and explore textbooks problems dynamically. In this context, there is interest to discuss ways in which teachers identify or construct a set of mathematical relations as a result of exploring the task dynamically. In particular, we document the extent to which the use of the tool helps teachers identify and explore potential instructional routes to structure the development of their lessons.

Conceptual Foundations

The conceptual support of a problem-solving approach relies on recognizing that teachers' development of mathematical teaching knowledge could be promoted within an intellectual community that fosters an inquiring or inquisitive approach to mathematical tasks or activities. Tasks are considered a key component in promoting students' construction of conceptual webs or networks, and developing a set of flexible strategies to be engaged in problem-solving behaviors. (Gueudet 2007) stated that: "Grounding a teaching design [focusing on knowledge organization and the development of different forms of problem solving strategies] should comprise a variety of tasks, allowing development of different solutions, in order to foster a form of students' mathematical autonomy … Working with technological devices such as dynamic geometry environments, but also computer algebra systems … can be helpful in fostering flexibility" (p. 242). Learning mathematics and developing mathematical knowledge are processes that can be framed in terms of a set of dilemmas or problems that need to be represented, explored, and solved through the use of mathematical content or resources. A mode of inquiry involves challenging the *status quo,* and it demands a continuous reconceptualization of what is learned and how knowledge is constructed. Jaworski (2006) stated that "[in a community of inquiry] participants grow into and contribute to continual reconstitution of the community through critical reflection; inquiry is developed as one of the forms of practice within the community and individual identity develops through reflective inquiry" (p. 202). This process also can be used as a vehicle to construct new relations or to identify new routes to reach or reconstruct classic mathematics results (Santos-Trigo, 2008). This type of teachers' interaction becomes relevant to foster and develop a problem-solving approach that involves:

> *Seeing the mathematical content in mathematically unsophisticated questions, seeing underlying similarity of structure in apparently different problems, facility in drawing on different mathematical representations of a problem, communicating mathematics meaningfully to diverse audiences, facility in selecting and using appropriate modes of analysis ("mental", paper and pencil, or technological), and willingness to keep learning new material and techniques* (Cohen, 2001, p. 986).

The use of computational technologies offers teachers and students the opportunity to participate in activities for:

> *(a) gaining insight and intuition, (b) discovering new patterns and relationships, (c) graphing to expose mathematical principles, (d) testing and especially falsifying conjectures, (e) exploring a possible result to see whether it merits formal proof, (f) suggesting approaches for formal proof, (g) replacing lengthy hand derivations with tool computations, and (h) confirming analytically derived results* (Borwein & Bailey, 2003, cited in Zbiek, Heid, & Blume, 2007, p. 1170).

In this context, the use of computational tools becomes important for the community, not only to better identify and explore mathematical results, but also to discuss pedagogical paths associated with the potential instructional routes that can be useful for teachers to guide or orient their instructional practices.

Methodology

Six high school high school teachers participated in 3 hours weekly problem-solving sessions during one semester. The aim of the sessions was to select, design and work on a set of problems that could eventually be used in actual instruction. In this process, all the participants had opportunities to reveal and discuss their mathematical ideas openly and use both dynamic software and hand-held calculators to solve the problems. The idea was that the participants became familiar with the use of the software by representing themselves directly the mathematical objects dynamically. In addition, they also explored possible instructional routes to frame their lessons. While solving and discussing all the problems, themes related to curriculum, students' learning, and the evaluation of students' mathematical competences were also addressed.

During the development of the sessions, the participants worked on the problems individually and in pairs. And later, they presented their work to the whole group. In general, the instructional activities were organized around a particular pedagogic approach in which the participants were encouraged to use an inquisitive approach to deal with the problems. As Jaworski (2006) indicates:

> *It [the instructional approach] is a social process in the sense that a participant is a member of a community (e.g., of teachers, or of students learning mathematics) with its own practices and dynamics of practice which go through social metamorphoses as inquiry takes place. It is an individual process in that individuals are encouraged to look critically at their own practices and to modify these through their own learning-in practice* (p. 2002).

Data used to analyze the work shown by the participants come from electronic files that they handed in at the end of the sessions and videotapes of the pair work and plenary sessions. In addition, each participant, at the end of the semester, presented the collection of problems and results that they had worked throughout the sessions. Here, it is important to mention that, in this study, we are interested in identifying and discussing problem solving approaches that emerge while the high school teachers use computational artifacts rather than analyzing in detail individual or small groups performances.

Results

We focus on analyzing what the participants showed while working on a task that is representative of the types of problems discussed during the sessions.

The task: Given two intersecting lines in the plane, find the geometric locus of all points whose sum of the distance from each point, on that locus, to the two lines is equal to a given length.

Comprehension of the task

All the participants read the statement and began to identify relevant objects to make sense of the problem. The questions that they posed and explored in this process involved: What does it mean that the sum of the distances from a point on the locus to the lines is constant? How can we calculate the distance from a point to a line? Can the given length be any given number? What happens when the two intersecting lines are perpendiculars? The discussion of these types of questions led the participants to represent and visualize the problem in different ways. Indeed, those who used the software to represent the problem differed in ways to introduce the given length. Two participants drew initially a segment to represent the given lengths while two defined the constant from calculating the sum of the distances between a point (assuming that it was situated on the locus) and the two given lines (Figure 1 & 2). Two participants followed an algebraic approach in which it was not necessary to identify the constant length explicitly.

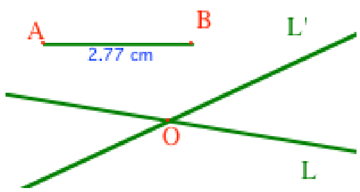

Figure 1. Defining the given length as segment d(A, B)

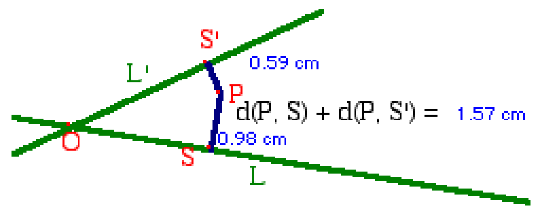

Figure 2. Defining the constant length as the sum of the distances from P to both lines

Formally, the two ways to identify or introduce the constant represent different initial conditions of the statement. Figure 2 relies on considering point P on the locus and from that point to calculate the constant, while in figure 1, the constant is given by the length of the segment and the problem is reduced to find the locus related to that length.

Initial approaches and particular cases

The participants who used the dynamic software to represent the objects embedded in the problem, showed diverse paths to approach empirically the problem. For example, Hugo constructed Figures 3 and 4 to identify empirically the solution. In Figure 3, L and L' are the given lines and point O their intersection. The length of segment AB is the given constant. On segment AB, he situated point Q such that d(AQ) + d(QB) = d(AB). From point P, he drew a perpendicular line to L' and a circle with center P and radius AQ. This circle intersects the perpendicular at point C. He also drew a circle with center C and radius QB and a perpendicular to L from point C. The circle and the perpendicular intersect at point D. The loci of points C and D when point P is moved along line L' are parallel lines (*m* and *m*') to L' and line *m*' intersects line L at point D_2. From D_2, he drew a perpendicular line to L that cuts line *m* at point C_2. From C_2, he drew a perpendicular line to L' that cuts line L' at P_1. From the construction $d(C_2, D_2) + d(C_2, P_1) = d(Q, B) + d(Q, A) = d(A, B)$. That is, point C_2 is on the locus. What is the locus of point C_2 when point Q is moved along segment AB? This question led Hugo to identify that segment EE_1 is a partial solution to the problem (Figure 3). Later, by using arguments of symmetry, Hugo identified the rectangle EE_1FG as the complete solution to the problem (Figure 4).

Victor used the dynamic software to draw lines L and L' and a point Q in angle LOL'. He drew the perpendicular segments from point Q to both lines and found that the sum of the lengths of the segments was 4.04 cm. Hence, he used the number 4.04 as the constant (given length) and identified point R (by moving it) such that d(R, B) + d(R, B') = 4.04 with segments RB and RB perpendicular to L' and L respectively. Later, he drew a line QR which intersects line L and L' at point S' and S, and point P on that line, and he observed that when point P is moved along segment SS', it holds the condition of the locus. That is, segment SS' is the partial solution to the problem.

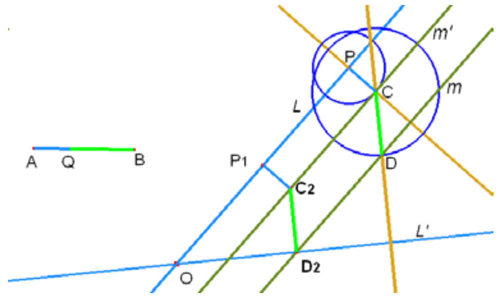

Figure 3. Partial solution to the problem.

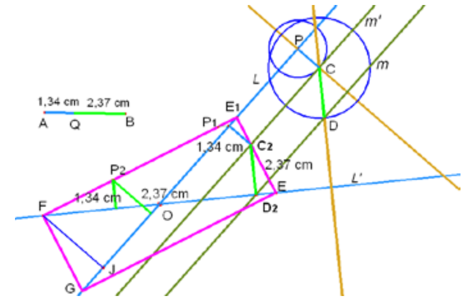

Figure 4. The locus is a rectangle.

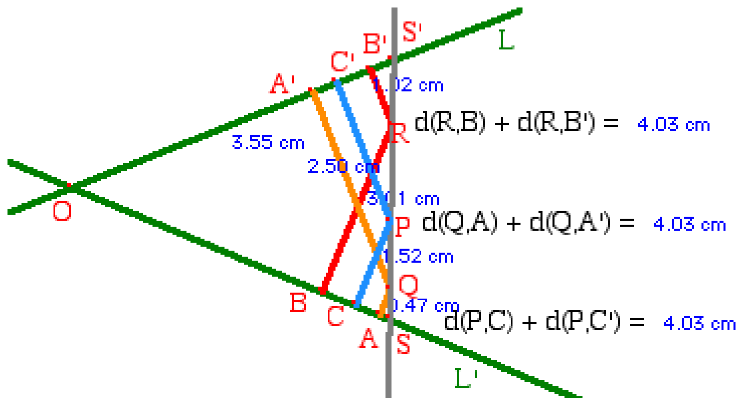

Figure 5. Empirical approach to the problem

Initial representations and approaches to the task

Two participants who relied on an algebraic approach to the problem began by assuming that lines L and L' coincide with axis X and Y of a Cartesian system and that the condition of the problem could be expressed as $|x|+|y|=k$. They graphed this expression and concluded that the solution was a square. Similarly, James explored the case in which both lines are perpendiculars and their intersection is the origin O. Here, he assumed that the constant was 4 units and located two points B(4, 0) and C(0,4) on the locus. He noticed that all points on the hypotenuse of right triangle PBC also were part of the locus and concluded that the solution to the problem was the square BCJH (Figure 6).

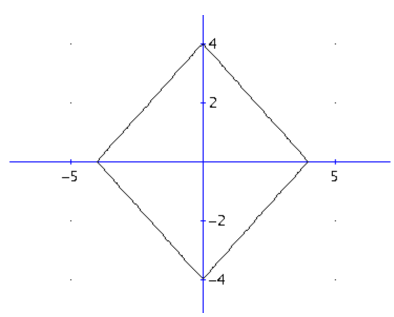

Figure 6.Graph of $|x|+|y|=4$.

The use of particular cases helped Cesar conjecture that the locus was a rectangle (Figure 7). To validate his conjecture, he drew a circle with centre O (intersection of the lines L, L’) and radius OD (D a point on L). The intersection points of the circle with the lines L and L’ are the vertices of rectangle ABCD whose diagonals are BD and AC. Based on this construction he found that for any position of point P on the perimeter of rectangle ABCD, then d(P, G) + d (P, H) is a constant, in this case 5.09 units (segments PG and PH are perpendiculars to L and L’ respectively.

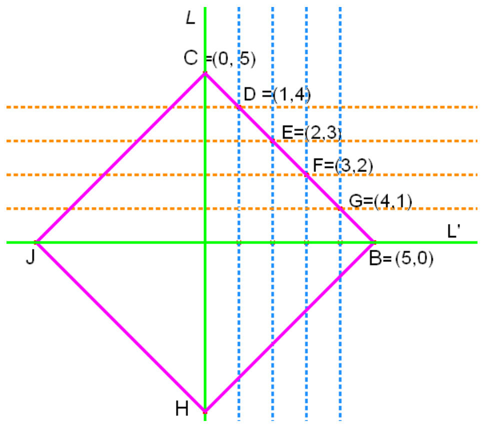

Figure 6. Using particular cases

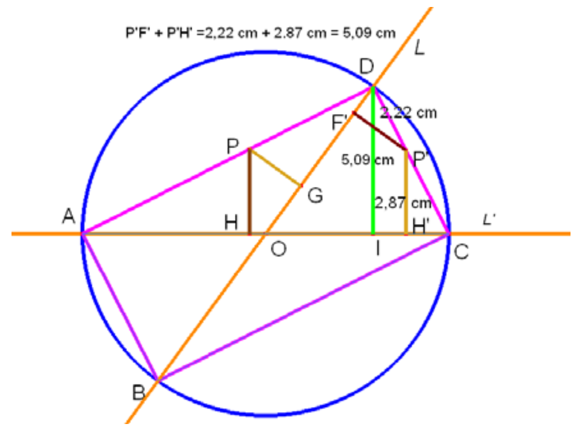

Figure 7. The solution is a rectangle

The identification and exploration of mathematical relations

At this stage, all the participants recognized that the locus was a rectangle; however, they were aware that it was important to provide an argument to support their conclusion. To this end, Hugo and James found that the length of the given constant was the length FI that corresponds to a height of triangle DOG. Based on this observation, Hugo mentioned that to prove the conjecture it was sufficient to show that “for any isosceles triangle ABC with AB$\neq$ AB = BC, it is held that for any point on segment AB, then the sum of d(P, E) + d(P, D) is constant”. To show this, Hugo relied on using that triangles ABC and PFP_1 are similar and that triangles PBF_1 and PEP_1 are congruent and also that BF is the height of triangle ABC (Figure 10).

The participants also provided an algebraic proof by calculating the sum of the distances from the point to both lines and making the sum equal to four. In this case L is axis X y L’ is the line y = x (Figure 11).

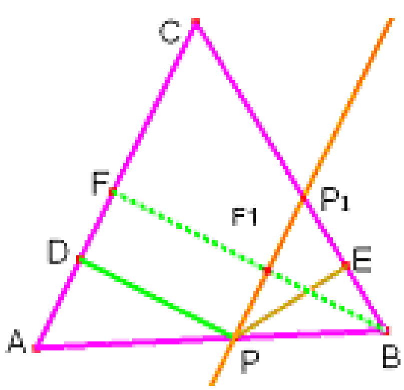

Figure 10. d(PD)+d(PE)=BF

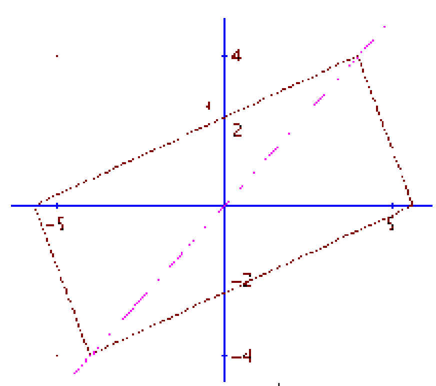

Figure 11. Graphic of: $\frac{|x-y|}{\sqrt{2}}+|y|=4$

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

Connections and generalizations

Dring the plenary discussion, the participants identified and explored three related problems: (i) Given two lines (L, L') that get intersected at point O, find the locus of points P whose difference of distances to the lines is a constant k. (ii) Given two lines that get intersected at point O, determine the locus of the points whose product of the distances from the point to both lines is a constant *k*. See figure 12. (iii) Given two lines L and L' that get intersected at point O, find the locus of the points P whose quotient of the distances from that locus to both lines is a constant *k*. The participants explored the three cases and identified interesting mathematical relations. For instance, Figure 12 shows that the locus in case (ii) is a hyperbola.

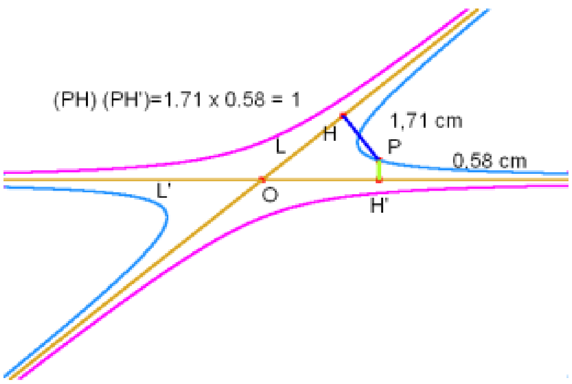

Figure 12. Finding the locus when product of distances to the lines is constant.

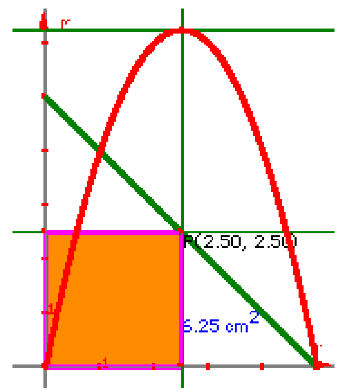

Figure 13. the square is maximum rectangle inscribed

Another connection emerged when James presented to the group the case where the locus was identified as the hypotenuse of a right triangle. Here, the lines were perpendiculars and their intersection point O was the origin of the coordinate system (Figure 6). Hence, the participants noticed that the constant (5 units) could be associated with the perimeter of a family of rectangles inscribed in that triangle. That is, from each point on the hypotenuse it could be drawn a rectangle and asked from that family of rectangles (having the same perimeter of 5 units) where to locate that which has the maximum area. The response was when the coordinates of the point on the hypotenuse were (2.5, 2.5).

Discussion

The participants recognized that the initial comprehension of the task was crucial to represent the problem in terms of mathematical objects. Two empirical approaches emerged during the process of making sense of the problem statement: One in which the participants assumed one point on the locus and from that point the constant length was determined as the sum of the lengths of the perpendicular segments to each line, and the other in which the participants relied on the given length to identify a point on the locus. The latter approach became important to construct a dynamic representation of the problem in which concepts of perpendicularity, parallelism, and loci appeared relevant to find the solution. At this stage, the participants pointed out that it is essential for teachers to spend significant time discussing with their students not only the ways in which they make sense of the problem; but also the strengths and limitations of emerging approaches. For example, when the participants presented the algebraic approach, they focused on discussing where to locate the Cartesian system, so that the algebraic operations involved could be simplified. They recognized that the dynamic representation of the problem offered them an opportunity to identify relations that were not present in the algebraic approach. For example, they noticed that the given length (constant) represented the height of an isosceles triangle that was formed with the diagonals and one side of the rectangle (the locus). In addition, when they moved the given lines to form a right angle, they noticed that the locus was a square. The participants recognized that the presentation and discussion of the individual approaches to the problem within the group led them to conceptualize the problem as an opportunity for them to visualize and explore diverse routes to solve the problem. For example, the initial empirical representation was important to visualize the locus as a segment, the dynamic approach became relevant to identify the rectangle as the locus and later the algebraic approach led them to examine the relation between the dynamic and the algebraic approaches. In general, all the participants agreed that working on the problem and discussing the emerging approaches was an interesting activity that could help them to identify crucial decisions or problem solving stages that they need to take into account during the organization of their instructional activities. In particular, they mentioned that the use of dynamic software offered them the possibility of identifying a particular mathematical relation (invariance or locus) that later could be explored numerically. Here, the dynamic approach unveiled interesting relations that were not visible in the algebraic approach to the problem. As a consequence, the participants agreed that their students should have opportunities to use computational tools to represent and examine relations from diverse perspectives. In general, several problem-solving strategies (dynamic models, use of particular cases, searching for loci, etc) seemed to be enhanced with the use of those tools. In addition, the use of the dynamic software helped them construct an argument to support conjectures or relations that emerged during the empirical approach to the problem. Finally, we argue that high school teachers need to participate directly in problem solving activities where they have the opportunity of revising and extending their mathematical knowledge and also discuss the advantages of looking for different ways to approach mathematical tasks. In particular they need to identify the potential in using various computational tools during the diverse problem solving phases that include representing, exploring and supporting mathematical relations.

Endnotes

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COMPARING PRESERVICE TEACHERS' PEDAGOGICAL IDEAS WITH THEIR PRACTICES AND THEIR STUDENTS' BELIEFS

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In this article we describe the results of a research that had the purpose of finding out the beliefs of preservice teachers and how these relate to their own practice and to their students' beliefs. This study was part of a special trial course in a teachers' college which had the aim of improving the content and pedagogical knowledge of a group of students about to become teachers. Here we will focus on the pedagogical aspects. The classes of this course were analyzed, several questionnaires were applied and some classrooms presentations of this group of student teachers were observed. We also applied a questionnaire to their students receiving the lessons, about their beliefs. The study revealed that although the student teachers had a good pedagogical knowledge, their practices were mostly incompatible with it until they were given the opportunity to reflect on this knowledge and their own beliefs, and employ them in their practice.

Introduction

Professional development is a key aspect to improve education. Many studies have shown a strong correlation between the teachers' knowledge and conceptions, and students' achievements. With new principles, standards and approaches brought everyday into education, the preparation of teachers becomes even more crucial.

One way to improve education is to provide preservice teachers with the necessary elements for their future practice. Preservice teacher's math education is complex since it involves not only different kinds of knowledge, but also beliefs, competencies and attitudes. Recognizing this, we initiated an educational project in a teachers college in Mexico which consisted of transforming one of the last courses of its program of studies. The student teachers in their last year of formal training already have a knowledge of mathematics and mathematics teaching, but the former is limited and based on procedural understanding, and the latter is composed of theoretical knowledge from previous years of instruction and conceptions and beliefs formed throughout their own experience. So it was felt necessary to introduce a special course that would give an opportunity to build on their previous knowledge and to restructure their conceptions and beliefs through group discussions, reflection and to put them into practice. The research project attached to it had the purpose of finding out the general knowledge and beliefs of these last year student teachers, how these influence their practice and the benefits of this didactical intervention.

The general term Pedagogical Content Knowledge (Shulman, 1986) refers to a complex mixture of knowledge related to many components like content, pedagogy, organization of topics and problems, student conceptions, models, representations, activities, curriculum, etc. Some facets of this teachers' knowledge are more closely related to the mathematical content, like understanding and extending students methods of solution, deconstructing one's own knowledge into its elemental parts to make it more evident or knowing the structure and connections of

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mathematical concepts and procedures. Ball and Bass (2000) associated this special knowledge with the term: Mathematical Knowledge for Teaching.

Based on different frameworks and methods of inquiry, there have been a number of research studies connected to teachers' professional development projects in different countries. Amato (2006), within a mathematics teaching course for student teachers, conducted a study to improve their relational understanding of fractions, by playing games. In a study investigating the Pedagogical Content Knowledge (PCK) of elementary school teachers in the topic of decimals, Chick, et al. (2006) proposed a framework with three categories: 1. *Clearly PCK*; 2. *Content Knowledge in a Pedagogical Context* and 3. *Pedagogical Knowledge in a Content Context*. Like some other authors, Seago and Goldsmith (2006) studied the possibility of using classroom artifacts like students' work and classroom videos to assess and promote MKT. In a collaborative action research, Cooper, Baturo and Grant (2006) uncovered some characteristics of instructional interactions that lead to positive results in student learning.

Teachers' beliefs, on the other hand, direct their planning, content presented, decisions and evaluation related to their classroom. A very strong relationship has been observed between teachers' practices and their beliefs and conceptions about the subject matter and the learning-teaching process. Thompson (1984) showed a correlation between the teachers' ways of instruction and their beliefs about math. He also reported consistencies and inconsistencies with respect to how teachers believe math should be taught and their own practice, which shows that this research field is inherently complex. Some of these discrepancies can be accounted for with other factors like the social context in which the teaching is being done.

To give an overview, we cite some of the beliefs about math and its teaching that appear in the extensive literature, many of them repeatedly in several articles, in different forms.

Teachers' beliefs: ❖ Math is a hard subject. ❖ Math doesn´t change. ❖ Math consists of calculations and procedures. ❖ Students should memorize the rules. ❖ There is only one correct method and answer. ❖ The objective is to obtain the right answer...

Students' beliefs: ❖ Math does not relate to the real world. ❖ There is only one way to solve problems. ❖ Math is just a set of rules to follow. ❖ Exercises can and should be solved by the techniques given in the book or by the teacher. ❖ Problems can always be solved applying one or several operations...

Ernest (1989) suggested that beliefs are the main regulators of the behavior of the teacher in the classroom. He proposes three key elements that have an influence on the teaching of math: i) the teacher's mental contents, specially his beliefs system; ii) the social context of the pedagogical situation, in particular, the limitations and opportunities provided and iii) the level of the teacher's thinking processes and reflection.

In an article about cognitively guided instruction, Carpenter et al (2000) stressed the importance of the teacher's knowledge about the mathematical thinking of children. The authors identified four levels of teachers' beliefs that correlate with their mode of instruction. A brief explanation of these is: I. They believe that math has to be taught explicitly and therefore they show procedures and ask the students to practice them. II. They start to question this explicit mode and therefore they give to the students some opportunity to solve problems by themselves. III. They believe that students can have their own strategies so they provide problems and the students report their solutions. IV. The previous mode of working becomes more flexible, in which the teacher learns from his students' productions and adapts his instruction to this knowledge. For a review of the literature on teachers' and students' beliefs, conceptions and practice we can refer to Leder, et al. (2002) or Ponte and Chapman (2006).

In addition to the quality of the mathematics content and pedagogy knowledge held by teachers, effective learning requires students to construct their own knowledge through exploration and interaction. According to Askew, Brown, Denvir, & Rhodes (2000), this process is productive if the next four components meet some necessary requirements: (a) *Tasks* are challenging, meaningful and interesting; (b) *Talk* facilitates learning and includes all sorts of teacher and students interactions; (c) *Tools* cover a range of modes and types of models; and (d) *Relationships and norms* help towards a social construction of knowledge. We used this framework to analyze the classes we observed of teachers during this study.

Methodology

A special pilot course was designed and given during the last semester of the formal courses to a group of 21 students in a teacher's college in Mexico City (ENSM – Escuela Normal Superior de Mexico), preparing for teaching at the secondary school level. Each six hour class of this course was divided into two thirds of selected contents (fractions and decimals; mental calculation and estimation; algebraic thinking and functions; and probability and statistics) and one third of pedagogical elements (an approach that combines content and pedagogy would have been preferred but we decided to "separate" these due to lack of time for implementation). In this article we will describe only the pedagogical section of the course.

The content of the nine pedagogical sessions given was: 1) Discussion and probing further the responses given by the student teachers in the initial questionnaire. 2 and 3) Discussion of pre-read pedagogical articles and the ideas involved (for example: Cooper, et al. 2006; McDonough and Clarke, 2002; Schwartz, et al. 2006). 4 and 5) Worksheet design and discussion about its elements. 6, 7 and 8) Demonstrative classes given by three of the student teachers to their peers, and their analysis. 9) Discussion and probing further the responses given by the student teachers in the final questionnaire. In the remaining six weeks during the semester, the student teachers went to secondary schools to give practice classes.

To collect data, we applied an initial and a final questionnaire to the student teachers about some pedagogical elements and beliefs. The initial questionnaire consisted of 22 questions about characteristics of good teachers; about mathematics; and about elements related to the four components of Askew, et al. described before. The final questionnaire contained only three questions asking their perspective about: i) What they learned from the course; ii) How their beliefs and attitudes changed; and iii) The changes they observed in their own practice and what else they would like to achieve. In addition, we observed each of the three practice classes given to secondary students by eight (chosen at random) student teachers; one at the beginning (before the course started), one at the middle and one at the end (after the course finished). We also applied at the beginning, a questionnaire about their beliefs to one of the groups of students that received these practice classes, similar to the initial one given to the student teachers.

Results

Teachers' responses to the initial questionnaire

We will describe here only the responses to some of the items of the initial questionnaire (some other answers will be given in other subsections below). Most of them coincide that students should participate actively, discovering concepts and be the ones talking and giving explanations, and that teachers should only direct, organize and, find out about the existing knowledge and uncertainties of the students. Some of the bad practices they mentioned are: to be very directive and not very flexible, to ignore students' doubts and to give poor explanations.

When they were asked what they dislike about math, many responses were along the lines that some of the topics have to be taught in a traditional operative way so they could be understood, like geometric formulas or number algorithms, and that they have to be practiced. Almost all stated that mathematics is very important for every day life and that to construct and comprehend mathematics, real life examples and manipulating concrete materials were necessary. With respect to class organization, they prefer equally different forms, giving advantages and disadvantages (some replicated next): *Team work* ("not all of them work"; "hard to finish activities"); *Whole class* (many stated "disorder" and "hard to control"; "some don't work"; "making fun of the others' answers"); *Individual work* ("don't understand"; "don't ask questions"; "the advance is uneven"). Thus, overall the student teachers demonstrated in writing a reasonable blend of pedagogical knowledge, and a very constructive set of beliefs about the roles of the teacher and the students. Some weaknesses they mentioned are: to be able to capture the attention and interest of the students, and that their evaluation is centered mostly on exams.

Students' responses to their questionnaire

As before, we will describe here only the responses to some of the items of the questionnaire (instead of giving proportions, we will give representatives of the most common responses). The students see a good teacher as someone who "explains in detail" and "asks about doubts". Some of the things a teacher does wrong are: "only explains in one way"; "only explains the subjects once"; "doesn't explain well so that we can understand"; "doesn't pay attention to the students"; "make the problems bigger with detours". About half claim they like math because "makes me think"; "it is useful", but others don't because "the operations are complicated"; "I don't understand them"; "they are difficult"; "teach us things that we are not going to use in our lives"; "they are really boring", although the great majority accepts that math is useful for everyday life. They would like activities "creative and enjoyable to stimulate students"; "entertaining to draw attention"; "interesting so I don't get bored and fall sleep"; "with lots of examples"; "easy to understand". With respect to class organization, they prefer team work, giving advantages and disadvantages (some replicated next): *Team work* ("only some work"; "some only play"; "there is not agreement"); *Whole class* (as the teachers, many stated "disorder"; "only the ones in front participate"; "not everybody understands"); *Individual work* ("you don't understand the work"; "you don't have support"; "we have many mistakes"). These are indications that teachers do not manage correctly these forms of organization.

These students' conceptions and beliefs (the teacher has to explain; math is difficult and boring; would like interesting and enjoyable activities) contribute also to shape teachers' practice since they are the expected behavior of teachers by their students.

Discussion about the elements of worksheets design

The student teachers designed some worksheets and then analyzed them in class. Among the aspects they saw as inadequate and listed "for improvement" were:

▪Objectives and instructions are complex, directed more to other teachers than to the students.
▪They are only repetitive exercises that emphasize memorization.
▪They contain a lot of text.
▪Centered on mechanical aspects.
▪Figures are used for aesthetic reasons and not as helpful illustrations. ▪ Correction by the teacher is based only on results.
▪They are very long.

With these in mind, they discuss the characteristics of a good worksheet and how it should be presented in the classroom (they were instructed to think further about the level the worksheets should have on issues like orientation, feedback, difficulty, reasoning, etc.)

The student teachers recognized that they had very little knowledge about worksheet design, giving previously to the students only “lists of exercises or problems” and that this activity gave them elements that have to be considered to really support the learning and teaching process. In the demonstrative classes given by three of the student teachers and in their last practice classes to secondary students, we found they designed very much improved worksheets. The main new feature was that they weren’t repetitive exercises anymore but instead, they were a sequel of a few problematic situations to analyze.

Analysis of demonstrative classes

In the three sessions programmed for this, a student teacher presented a demonstrative class and the others gave their comments organizing them into *strengths* and *weaknesses*.

Teacher P gave a class on fractions.

Some of the strengths mentioned were: ▪Shows confidence. ▪Gives the purposes of the class. ▪Utilizes several representations for the fractions. ▪Uses both individual and team work. ▪Interacts with the teams, helping those who request it.

Some of the weaknesses mentioned were: ▪Tried to cover too many contents (proper and improper fractions, their localization on the number line, equivalence of fractions and decimal representation). ▪Didn´t use concrete materials. ▪The activities weren’t challenging. ▪Introduces the topics with a lecture and passing the students to the blackboard. ▪Didn´t take advantage of the comments given during the lecture to open a discussion. ▪During the interaction with the teams, gives only his conception or point of view. This teacher, although demonstrated some effort, showed reluctance to change his practice and to apply new approaches, due to his beliefs. Teacher L gave also a class on fractions. He had all the positive comments that were listed above for teacher P, and in addition had: ▪Did use games and concrete materials. Also, he had very few comments to improve like: ▪The instructions to the games should be more clear so all the students understand them.

(Teacher F class is commented briefly afterward.)

This activity of demonstrative classes was very helpful since the student teachers saw a class presented and then heard from others, good and bad qualities of teaching. This make them reflect on their own practice, which is a requisite for change. It also shows that these students about to become teachers, have the pedagogical knowledge needed but the real problematic issue is to be able to implement it.

Comparison of initial pedagogical beliefs expressed and actual practices

1. Compatible.

Defines it as a good practice and follows it.

- Most of the teachers believe that classroom or homework activities shouldn´t be challenging and follow this. Some of the reasons they give are: “The students don’t have enough knowledge.” “We have to give it slowly so they can understand.” “Don’t make complicated exercises because we can discourage them.”
- Teacher J. P. expressed that he works in class in a traditional way and prefers to give the whole explanation himself and then give some practice exercises to the students. This is exactly what he does in class.

- Teacher L expressed that it is important to use materials and consistently did this, using for example ribbons or other concrete material and visual computer software.

2. Incompatible.

2a. Defines it as a bad practice but follows it anyway.

- Teacher A wrote as a bad practice: "teaching the class to the blackboard and forgetting the group". However, most of the time his class followed this technique with very little interaction with the students.
- Teacher L wrote as a bad practice: "give exercises based on memorization". However, in his first observed class, he gave as homework to solve 18 similar equations of the form ax+b=c.
- Teacher O wrote as a bad practice: "to tell the students all the procedure so they just perform the operations". However, in his class he posed a problem to the students but almost immediately he took over, giving the procedure that had to be followed.

2b. Defines it as a good practice but doesn´t follow it or does the opposite.

- Most teachers expressed that it is important the use of materials "to help students construct their knowledge" and "to motivate them and center their attention". However, most of them had a very limited use of materials.

2c. Other incompatibilities observed.

- When asked if they prefer to explain first to the students, or let them find out by themselves, most agreed in choosing the second form ("the students should be the ones to investigate and explain"). However, most of them, were very directive teachers in their first practice classes, giving very little opportunities for the students to ask questions or to explain.
- Most student teachers answered in the questionnaire that it is important to learn math because they are useful in everyday life and they stated explicitly that they should show the connections with real life. However, in their initial practice class, they very seldom use examples based on real situations.
- Also the majority stated in the initial questionnaire that activities have to be "well structured", "gradual", "dynamic", "with a purpose", "allowing different strategies of solution", "gradual in complexity", "understandable", "motivating thinking", "generate conflict", "attractive", "based on problem solving"... However, these were not characteristics seen in their practice classes.
- Teacher F, in his demonstrative class of areas in geometry (given to his fellow classmates) employed a worksheet with nice drawings of composed figures, handed out paper figures and other materials and formed teams to work. His peers gave, in general, very positive comments on this class. However in his later presentation on the same subject to actual students, he didn't employ the materials and instead, used the blackboard extensively and moved quickly into formulas. So he is not congruent with what he shows he believes and his actual practice.

Classroom observations: changes observed

We give only two representative examples, but in general the teachers improved in all four aspects: tasks, talk, tools and norms.

- The initial class of teacher A could be summarized simply as talking and writing on the blackboard, with the students listening and copying. He was rated either unsatisfactory or poor in the aspects of teaching considered (tasks – C, talk – C, tools – C and norms – D, on a scale from A to D). In his second class after the course given, he designed worksheets, organized his students to work in groups of three and went to each group to discuss ideas. Due to the interest of the students in the activity, the discipline was much better.

- In the initial observation teacher L was rated B (good) in tasks and tools (with the use of many materials), but only C (unsatisfactory) on conversation (students asking questions only about what they didn't understand from the materials) and C on norms. He was a directive teacher but at times, tried to explore his students' ideas. After the course he used materials and software to motivate discovery by the students and his talk with them improved noticeably, being much more responsive. The norms also improved considerably, due to the same techniques used.

Teachers' responses to the final questionnaire.

In the final questionnaire, teachers expressed what they thought they obtained from the course. They stressed the pedagogical aspect as being very important, while before, they didn´t think much of it. Some of their comments were: "It allowed me to see the general value of knowing the pedagogical side since it is a facilitator for us in class." "I finally realized all the aspects a teacher should cover in his classroom... it is not only to present the subjects to the students... it is the planning, the looking for activities..." "Before, I placed my emphasis on the content. Now I can tell that the pedagogy gives you tools to be better and more effective." "I learned the meaning of pedagogy... it is a series of activities that have to be planned with regard to the attitude and capacity of each student, it is searching the way for the student to acquire his knowledge." "We have to let the students express their own ideas..." "We have to motivate students to write and validate their own methods and to propose activities that require thinking and not mechanization..."

About how they felt their beliefs were changed they wrote: "Yes, they changed. Much of this I already knew but didn't put it into practice because my teachers didn't do it either..." "Before I had a vague idea of the role of the teacher... Now putting into practice what I learned, I realized that we can achieve a more solid knowledge if we try to have the students analyze what they learn." "Some how they changed because I had the idea that the teacher is the one that is right, that his way of teaching doesn't change and that the students have to adapt. However we saw that the practice of teachers must be very varied." "It changed because my worksheets have different characteristics now." "A lot. I believed that math was unique, not related to real situations. Now I support myself with... and based on life situations." It is interesting that they measure their beliefs according to their own practice.

Finally about the changes they saw in their practice they mention: "Before I didn't think about the needs of the students, I would see them as a group that I had to face..." "... I changed some aspects, I restrain myself telling everything to the students, try to make more questions, especially to those that don't participate so they realize their mistakes and can correct them." "... give them more freedom to solve problems." A big concern however was: "To find activities for the students to understand better, since it is very difficult to do so." and also: "To know how to put real math challenges at their level to guide their learning."

Some comments however showed some of the reality of Mexico: "Having a group of 40 to 50 students doesn't allow you to put all this into practice and I haven't seen a teacher that does it..." "The readings are studies in other countries with very different conditions from Mexico" Obviously, a very strong influence is the practice of others teachers they have observed.

Some student teachers had difficulties in adapting to these "new" ideas (somewhat conflicting from what they observe in normal classes) especially with respect to interacting with students and discerning their thinking, even that there is sometimes a good disposition for it. One possible factor is their limited general knowledge, but another obstacle could be the social context they live in, inside and outside the classroom. As Chapman (2002) expresses: "… belief structure can be complex and simply exposing mathematics teachers to alternative beliefs or contexts may not be sufficient to alter their teaching accordingly." And looking more broadly at the social context she adds "Thus while a teacher may be willing to change her/his teaching, she/he may not be willing, consciously or subconsciously, to change other aspects of her/his identity."

Conclusions

We found in general a good pedagogical knowledge among the student teachers and their expressed beliefs are in accordance with this knowledge. However, their practices were in general very traditionally oriented. It seems like their knowledge and beliefs articulated do not influence their actual planning, presentation and evaluation of their classes.

The normal instruction they received is in general very theoretical and their practice is in front of other teachers of the "old generation", so they tend to imitate them even though their knowledge is contradictory to these practices. So it is not enough to let them know the new approaches in teaching. What we did in this special course is open spaces for them to discuss, analyze and design activities and ask them to bring this into their practice.

In a normal set up, the student teachers go and observe inservice teachers in their classrooms and then under their supervision they prepare and give some practice classes. This mode to proceed however, perpetuates the same kind of behavior in instruction, which in the great majority of cases is of directive type, the teacher in a dominating role, giving mostly procedures that have to be memorized and then followed in the practice exercises.

We observed that their practice could change if they are given the opportunity to examine their beliefs, change them accordingly and apply the new techniques taught in their courses.

This suggests that a new approach in training students to become teachers should be undertaken in Mexico, trying not to let the old practices influence their more current knowledge and beliefs. It is not easy to improve the practice of teachers, but it is clear that, to do so, theoretical aspects have to be combined with demonstrative and practical sessions in which they hear comments and make them reflect about their own practices and the obstacles they have.

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"WHEN WILL I LEARN TO BE A MATHEMATICS TEACHER?": ALTERNATIVELY CERTIFIED TEACHERS, A CASE STUDY

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The New York City Teaching Fellows (NYCTF) was started in 2000, in part to address "the most severe teacher shortage in New York's public school system in decades" (NYCTF, 2008, p. 1). Two-thirds of new mathematics teachers in the New York City Public School System are TFs. In this paper we present a case study of a first-year Mathematics TF. Through this case study we examine and interrogate the relationship between the theory, rhetoric and promise of Alternative Certification (AC) programs and the reality of the course an AC candidate navigates through her first year of the training program and of teaching.

Introduction

The New York City Teaching Fellows (NYCTF) was started in 2000, in part to address "the most severe teacher shortage in New York's public school system in decades" (NYCTF, 2008, p. 1). NYCTF is the largest program in the U.S. that provides an alternative route to teaching certification. Two-thirds of new middle and high school mathematics teachers in the New York City Public School System are "Teaching Fellows." The NYCTF program is one of many Alternative Certification for Teaching (ACT) programs which seek to address the dearth of teacher candidates in traditional programs. We present a case study of the first-year of teaching of mathematics Teaching Fellow (MTF). Our aim in presenting this case study is to examine and interrogate the relationship between the theory, rhetoric and promise of ACT programs and the, sometimes consonant, sometimes not, reality of the course an ACT candidate navigates through her first year of the training program which coincides with her first year of teaching. We focus on her family and educational background, her beliefs as a novice teacher, preparation to teach mathematics, and first year experience teaching middle school mathematics in a high needs school in New York City (NYC) and place all of these factors in relation to claims made about the type and quality of candidate ACT programs claim to attract. We situate the case study using results from a larger observational and survey study of novice MTFs in order to substantiate our claim that our case study candidate is reasonably typical. Our analysis, and situating, of the case study is from a perspective which supports reform-oriented approaches to teaching of mathematics. Our data collection and analysis was guided by two questions:

1. How does an individual navigate through their first year of teaching in a large AC program?
2. What is the relationship between the framing of the AC program and the realities on the ground?

Theoretical Framework

There is a growing body of literature on ACT programs and teachers (Boyd, Grossman, Lankford, Loeb, & Wyckoff, 2006; Feistritzer, 2004; Grossman & Loeb, 2008) that reflects the growth of ACT programs in the U. S. (Feistritzer, 2004). Based on a summary of peer reviewed research in this area, Wilson, Floden, & Ferrini-Mundy (2002), conclude that high-quality ACT programs claim to include "high entrance standards; extensive mentoring and supervision;

extensive pedagogical training in instruction, management, curriculum, and working with diverse students; frequent and substantial evaluation; practice in lesson planning and teaching prior to taking on full responsibility as a teacher; and high exit standards" (p. 201).

Indeed, some researchers (e.g., Peterson & Nadler, 2009) and early advocates of ACT policy, find that ACT programs too often resemble the traditional, university-based route they were intended to replace. They argue that to be truly alternative, ACT programs should be "streamlined" - allowing candidates "early entry" into the classroom after taking the bare minimum of practical education courses (i.e., 12 to 18 credit hours). Advocates of streamlined ACT policy make several interrelated assumptions, namely, that: (1) the reduction of entry requirements, attracts academically talented candidates who otherwise would not consider teaching, (2) these candidates possess strong-subject matter backgrounds, relevant professional experience, or both, and (3) because of their strong backgrounds, ACT candidates require much less preparation than do candidates in traditional education programs (see also, Raymond, Fletcher, & Luque, 2001).

ACT programs generally promise intensive induction and mentoring to ACTs as they begin teaching. This induction is designed to help ACTs "learn to teach on the job", to meet their individual needs, and to help with program retention (Kardos, 2004). However, research indicates that there is a gap between espoused and enacted induction support for ACTs, with many early-entry programs failing to fully provide promised support in the field (Humphrey et al., 2008; Johnson & Birkeland, 2008). Humphrey et al. (2008) note that, "mentoring, like other training components, varies considerably across and within [ACT] programs, not just in terms of the amount of mentoring provided, but in its quality as well" (p. 518).

Further complicating matters, many ACTs begin teaching in difficult teaching contexts - in high needs schools, with children that other teachers avoid (Humphrey et al., 2008; Johnson & Birkeland, 2008). In such contexts, new ACTs, and other novices for that matter, struggle with issues of instruction and classroom management. They often feel isolated and unsupported by administrators in these contexts (Goodnough, 2004; Humphrey & Wechsler, 2007; Kardos & Johnson, 2007).

In many district and school contexts, new teachers also are being asked to implement the same student-centered pedagogies (e.g., classrooms where students play an active role in the development, discussion and evaluation of ideas) that experienced teachers are expected to implement. Indeed, research indicates that as pre-service students, both traditional and ACT candidates articulate student-centered ideals about teaching mathematics (Flores, 2006; Noyes, 2007). However, many are unable to realize this vision and adopt traditional teaching methods (e.g., lecturing, emphasizing student memorization) once they become practicing teachers.

Additionally, in contrast to claims of advocates, there is mixed evidence that ACT policy enables early-entry programs to attract the "best and the brightest" to teaching (Shen, 1997; Wilson, Floden, Ferrini-Mundy, 2002). Based on an analysis of nationally representative survey data, Shen (1997) finds that, if anything, ACTs are slightly less well qualified than traditional candidates on average. In contrast to rhetoric by advocates which posits that ACT programs attract professional and experienced career changers, large numbers of ACTs are recent college graduates (Shen, 1997). Such ACTs tend to be exploring teaching as a possible career or to view teaching as a temporary (Shen, 1997).

Finally, while there is some evidence that ACT policy has helped diversify the teacher population, both the general population and that of mathematics teachers, the differences between the ACT and traditional teacher population are generally small (Shen, 1997, 1999). Like the general teacher population, ACTs are predominantly white and female. At the same time, a large proportion of ACTs begin teaching in high needs schools where they are outsiders to the communities in which they teach. Martin (2007) argues that well-known ACT programs instill a missionary zealousness amongst many such ACTs who are positioned as “saviors” of students of color. He claims that ACTs in these programs are taught to focus so narrowly on raising student academic achievement that they are blinded to social-historical forces (e.g., racism) and a consideration of student experiences and desires which are at play in their classrooms.

Methodology

This study was one of eight case studies of MTFs completed over a one-year period. Kelly's first year experiences and preparation in the NYCTF program was fairly representative of many MTFs, recent college graduates in particular.

Study Context

Over the course of the study, Kelly completed state-required master's coursework at Borough University (BU), one of four partnering universities that provide pre-employment training and Master's coursework in education for MTFs. In her first year, Kelly taught at a large middle school in a NYC neighborhood predominantly populated by African-American, Caribbean, and Hispanic families. NYC DoE data indicates that approximately three-fourths of student families receive public assistance at this school [http://schools.nyc.gov/SchoolPortals]. The school was in "restructuring year 2" which means that it had been failing for two years when Kelly started working there. The class that was the focal point of our study was, within the context of the school, a high track class. Kelly also taught two lower track courses in her first year. The high track class consisted of 17 Caribbean and African American girls, 4 Caribbean and African American boys, 3 Latinas, 4 Latinos, 1 Asian girl, and 1 Asian boy.

Data Sources

Working in teams of two, we collected observational data in our eight case study classrooms on an average of twice a month. In a typical observation, one of us took jottings while the other videotaped instruction. We also placed digital audio recorders on the teacher or in front near them. The jottings, audio and video data were used to create in-depth fieldnotes of each observation. Some of our case study teachers taught 90-minute block schedule periods, while others taught 45-minute periods. After each observation, we asked our case study teachers to reflect on the lesson that we had just observed.

We conducted in-depth interviews with the eight case study participants at the beginning and end of the 2006-2007 schoolyear. In the summer of 2007, we collected in-depth survey data from more than 85% of pre-service and inservice MTFs (Kelly became a second-year Fellow) at BU. Our survey study collected data from MTFs at the four NYCTF programs for mathematics. Interview and survey questions concerned the Fellows' educational background and preparation, ongoing Master's coursework, induction support, and teaching mathematics in NYC public schools. Having completed her first-year of teaching, Kelly filled out an inservice survey.

Data Analysis

Fieldnote Analysis

The coding scheme that we used to analyze our fieldnotes was produced in collaboration with a research team of approximately a dozen mathematics educators, most of whom were collecting data on the seven other case study MTFs. Our schema included such codes as: classroom management, teacher math questions, and opportunity for meaning making. Specific codes allowed us to address our research questions by providing us with a detailed “thick” description of week-by-week progress of Kelly through her first year. The coding of the lessons and the reflections allowed us to examine and characterise important dimensions of her progress such as preparedness, influence of college classes, and scale and effect of mentoring from various sources. It also gave us an informed position to compare the reality of one candidate's experience of an AC program (situated in a larger group to control for “typicalness”) with the claims and promise of AC programs.

Interview Analysis

For the purposes of this paper, we focused on parts of the interviews and reflections that dealt with: (1) Kelly's background and preparation to teach in the NYCTF program, (2) experiences working in her school, particularly her induction experiences, and (3) her beliefs about the nature of school mathematics and effective mathematics teaching.

Survey Analysis

For the purposes of this paper, we focus on findings from survey data from 55 preservice and 42 first-year MTF respondents at BU - these respondents comprised some 95% and 80% of these two cohorts respectively. Although the surveys contained a range of issues, we examined only the parts of the survey that addressed the three categories outlined above. The design of the surveys, informed in part by interviews and the initial observational component of our larger study, allowed us to examine the representativeness of our 8 case studies to the MTFs who completed the surveys (e.g., their use of required textbooks, beliefs about students) and to compare these key participants' ideas to the aggregate data of the entire cohort.

Results

Kelly's Background and Preparation to Teach

Kelly, our case study subject, came to her position as an urban middle school teacher after having been away from exposure to any mathematics instruction for the six years she was in college and graduate school. While she reported receiving the highest possible score on the AP Calculus exam in high school, she also admitted that she was not interested in studying mathematics in college. Statistics 1 was the only mathematics course she completed in college. Our survey data indicates that Kelly was not atypical in this regard. While BU fellows completed 1.66 basic math courses (college algebra, basic statistics) on average, a large number did not take calculus or any advanced courses that typically follow calculus courses after college. Kelly graduated from a large Eastern State University with a double major in International Relations and Religious Studies. She was interested in pursuing a career with the intelligence services but postponed pursuing this option in order to complete a Masters degree in Islamic Studies.

Kelly's Preservice Views of Urban Mathematics Teaching

Before teaching Kelly was enrolled in a six-week Summer Course at Borough University (BU) in which she took courses in lesson planning and classroom management. In an interview conducted a week before she began teaching Kelly described her readiness and gratitude: “I think that [BU] is amazing at getting us ready for that first day of school.” Kelly did not have much educational or practical mathematical background to draw on as she entered her first year. While

Kelly appreciated the BU preservice preparation, she expressed concern about her lack of exposure to mathematics content or mathematics specific teaching methods prior to becoming a teacher of record. She declared that her primary goal was to gain control of her classroom. Her secondary stated goals were to help students overcome mathematics anxiety and help them in relating mathematics to their real lives. This was the most common response for first year MTFs, though those at BU felt somewhat better prepared than their colleagues at other university partners in terms of classroom management. On the first year survey, 25.5% of BU (preservice) respondents cited management as one of the “most effective aspects” of their instruction, compared to 13.8% of MTFs at other university partners.

Early in the school year

Observations of Kelly’s teaching early in the school year highlighted a number of significant issues which continued with Kelly through the year. First, despite the reform-oriented attitudes evinced in her pre-service interview, her teaching was teacher-centered and she controlled mathematical explanations. Second, she privileged students' correct answers leaving them unchallenged but regularly and immediately challenged incorrect answers with comments such as “interesting,” or, “Are you sure?” It can certainly be argued that these are common issues of concern for any novice teacher (Moyer & Milewicz, 2002) but these issues are often addressed early in student teaching leading to the implication for the NYCTF program that the summer school teaching may be inadequate as a student teaching experience for the Fellows. On the other hand, Kelly had not taken mathematics since high school nor had she received much training in mathematics pedagogy in the NYCTF program. As discussed above, she was typical of other preservice MTF's in this regard.

At this early stage Kelly was struggling (at varying levels) with both content and content specific pedagogy including: listening to students, questions she asked students, tasks she designed and used in class. Her lesson plans also stuck closely to the required textbook. These, of course, may be viewed as common struggles among novice teachers. On the other hand, there were already very few management issues in Kelly’s class as she had developed a positive relationship with her students. In a post-lesson reflection Kelly stated: “I am feeling much more confident as the weeks go by [and I am] much more relaxed with my position as the teacher, and giving the students more responsibility in their own learning environment. She further observed, in the same interview, “I still want to work on the students being more reliant on each other and less on me for answers [and] I am still struggling with making the material interesting.” She felt that “there are only a few management issues such as getting out of their seats.”

At that stage Kelly might be considered to be in a reasonable position to develop as a teacher. Management was not really an issue in this class (although it still was in her other classes) and she had every reason to look to her university classes, her school's mathematics AP, her school's mathematics coach, and her university supervisor for help in improving basic aspects of her teaching. However, in her reflection on this lesson she wrote, “no one has helped me with much of anything in terms of math content.”

The Middle Part of the Year

Kelly's journey through the middle part of her first year was marked by a recession in instructional support she received from the school administration and the university representative with whom she worked.

Support

One of the central stories of Kelly's journey through her first year, a central story of MTF's more generally, is a consideration of the support she received. As a Teaching Fellow Kelly can expect to receive support in developing her teaching from at least four distinct sources

(i) *University classes.* Kelly was critical of ongoing Master's coursework she was taking during the year at BU. She was looking to these classes to help her develop not as a teacher in general but specifically as a mathematics teacher but ended up frustrated: "I'm not taking any classes on how to teach math. So, how am I supposed to be a good math teacher? They purposefully talk about other subjects because we're like, well you guys know about mathematics so let's talk about English, let's talk about science. Let's do a debate on technology and science classroom, or special education in the classroom … you want us to be well rounded, we're not even well rounded in our subject area yet." This represented a shift; as mentioned previously, Kelly had been extremely positive about her summer school experience at BU and had felt well-prepared.

Following the Summer program Kelly had taken one class in Special Education, one class in Literacy in the classroom, and one in Web Design. She stated, however, that she felt these course did not help her learn how to teach mathematics.

(ii) *University supervisor.* Initially Kelly had a positive relationship with her University supervisor. He had taught in Kelly's Summer Program at BU. On the survey, Kelly wrote: "he came in and helped with my classroom setup. Also suggestions on how to deal with bad behavior." However, the relationship lapsed as the school year progressed with Kelly commenting: "I'm not falling on my face, so he's got other things to do. Because he spends a lot more time with Ms. Brown than he ever did with me. He was here a lot in September. I really haven't seen him since. He stops in to say hello, but he even, he's supposed to see me once a month for Borough University but the last three times he's just stopped in for lesson plans so he can write it out because he trusts me."

This story in its essence is reported throughout Kelly's support network. Because her classroom management skills were good and she had a positive relationship with the students she had gained the reputation of being a successful teacher. Kelly's practice however, remained unexamined. She lacked the support she needed to improve in areas she felt less secure.

(iii) *APs in the school.* There were two Assistant Principals (APs) in the school who worked with Kelly to some extent. Her somewhat cynical summary of the extent to which they helped with her teaching is "They saw me in September or October. They realized I was competent … and then they've never helped me at all." In fact the relationship improved and in the second half of the school year the AP for mathematics was coming into her class occasionally: "I asked [him] can you stop in and see how things are going because, so he's stopped in a few times and gave me some pointers." He was giving her specific advice about the mathematics instruction that she considered to be useful. "He was telling me about how to develop from one idea to the next, connect, because I wasn't connecting … I was reviewing graphing and he said this is a good opportunity to do positive and negative numbers. He said that they're not good at it, so if you are creating tables [of values] you should also try, find time to stick in a little revision on adding and subtracting integers. I was like, oh I never even thought about doing that." The APs thought highly of Kelly considering her highly motivated, very effective, and very dedicated. Indeed, she was characterized as an "excellent" teacher.

Kelly had indeed possessed some critical skills as a teacher. She was dedicated, organized, and successful in establishing good interpersonal relationships with the students. Somewhat more surprising was the APs comments about Kelly's content knowledge and her content knowledge for teaching. He said: “What has impressed me is her knowledge. We have her teaching one of the more advanced eighth grade classes and teaching them high school. [I am impressed with] her knowledge in conveying the lesson, getting her lesson across to the kids.” We have observed in a number of Kelly's lessons that this was an area of concern for both Kelly and us.

(iv) *DoE mentors:* By law, the district is required to provide the fellows with mentor teachers. Kelly had a DoE mentor who visited her weekly. Most of these observations were short, this mentor only observed Kelly teach one full lesson per month. On the second-year survey Kelly wrote, “My mentor was great at helping me with personnel things--days off, death in the family, can I change schools, etc…” Kelly also felt that the content of courses she had taken was “useless” to her and believed she would have been better served by engaging in discussions that concerned her classroom needs.

At the mid-section of Kelly's first year of teaching we observed a highly committed teacher who showed good presence in the classroom and willing to learn. It was also clear early in the year that Kelly was experiencing typical difficulties that many novice teachers endure regarding command of content knowledge for teaching, pedagogical skills such as questioning, eliciting and responding to students’ explanations. Unfortunately, what we see develop through the year was that inadequate or insufficiently applied support system which did not allow Kelly to further develop as a teacher. The New York City Teaching Fellows enter the classroom with minimal training but under the assumption that there is a battery of support systems in place which are supposed to ensure that the kind of teacher development that would take place in the mentoring teachers entering through a traditional route are also in place for these alternatively certified teachers. Among the triumvirate of basic pedagogies: norms, tasks, and discourse (Henningsen & Stein, 1997), Kelly's training (together with her background and personality) was sufficient to help her achieve reasonable success with the norms. However, she continued to struggle with tasks and discourse components.

Observations of Kelly’s teaching in the latter part of the school year showed that she successfully established norms concerning small group work which allowed her to engage in extensive discussions with individual groups with only minimal disruption from outside. The tasks she used in her class were, however, highly procedural. Furthermore, her modes of discourse and question techniques tended to “funnel” (Wood, 1998) students toward a correct answer.

Discussion

In this paper we described the experiences of a new Teaching Fellow in a New York City public school. The story we tell is one of a teacher who we believe had the potential to be an effective Middle School teacher but was failed by the support systems expected to be in place to help her reach that potential. We highlighted that despite having taken a full set of university courses, and presence of at least three individuals in support/mentorship roles, instances of Kelly being guided to carefully examine her practice by a mentor were rare. Striking about Kelly’s case is that she herself was fully conscious of the failure of the system, as she asked: “When am I going to learn to be a mathematics teacher?”

Currently, a large number of New York City public schools are faced with the challenge of staffing positions and retaining their existing staff. In order to successfully address this challenge it is important that all aspects of training and support for incoming teachers are in place so to allow for effective experiences for both teachers and students are developed. Our findings suggest that such a system is not currently in place for some new Teaching Fellows in the New York City public school system.

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CHARACTERIZING MIDDLE AND SECONDAY PRESERVICE TEACHERS' CHANGE IN INFERENTIAL REASONING

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In recent years, national standards have given greater emphasis to K-12 statistics. Unfortunately, many teachers have not had an opportunity to learn statistical content during their college coursework. Therefore, teachers are least prepared to teach statistics and probability according to the College Board of Mathematics and Statistics (2001). As a result, teachers have difficulties in both understanding and teaching core ideas of statistics. Moreover, there has been limited research about preparing teachers to teach statistics.

The following study provides a needed baseline of how well prepared preservice teachers are to teach inferential statistics from a reasoning perspective. A cohort (n=33) of preservice middle and secondary teachers enrolled in a statistics course designed specifically for preservice teachers participated in the study. Three parallel assessments were administered to the cohort to gauge their inferential reasoning progress throughout a semester (e.g. *pre-assessment*, *midcourse clinical interview,* and *post-assessment*). The assessments consisted of tasks that could be solved both formally and informally. The change in the preservice teachers' statistical reasoning was characterized from cognitive perspective using a two-stage Structure of Observed Learning Outcomes (SOLO) taxonomy (Mooney, 2002; Biggs & Collis, 1982). General agreement exists that students' learning and statistical reasoning progresses through a number of hierarchical levels and cycles. The two-stage taxonomy follows the work of Watson, Collis, Callingham and Moritz (1995) who claim that the first stage is primarily focused on developing conceptions (informal approaches), and the second stage consolidates and applies the statistical concepts (formal methods). The participants' opportunity to learn was analyzed in terms of the statistical content covered in the course and the emphasis placed on reasoning. While students tended to incorporate more statistical concepts as they progressed through the semester, coordination of statistical concepts in their argumentation was lacking. Plausible explanation for these results will be discussed.

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DIFFERENTIATION MODEL FOR MATHEMATICS PRESERVICE TEACHERS TO ENGAGE ALL STUDENTS IN THE LEARNING OF MATHEMATICS

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This case study details the creation of a differentiation model of instruction that helps mathematics preservice teachers write lesson plans that address the needs of all classroom students. The study follows the model as it was developed over three years. Observations of highly qualified classroom teachers while they taught mathematics in a manner that was able to reach all students in a classroom were the start of the study. Professional development sessions that focused on readings about differentiation and discussions as to how practicing teachers differentiate in their classrooms revealed an automatic practice that sprang from teacher observations of students' behavior causing the teacher to change an element in teaching or the requirements for a student. How to translate that automatic practice into a mode for student teachers was a perplexing problem.

The framework used in the study was informed by Response to Intervention (RtI) program, with some adjustments. In our model, student teachers have a three level framework in which to plan classroom engagement with mathematical content: students at grade level; students who needed a small amount of intervention by the teacher; those students who have an individual education plan and stipulated needs the teacher was required to address.

The new method was reviewed and used by in-service teachers. They used the format for their classes and identified a key missing element – a level of planning for the accelerated students. Thus, a four level method to differentiate emerged that student teachers would use to plan their lessons. Student teachers would be required to address and prepare for all the needs of students prior to teaching a lesson. The new method was tested by a student teacher who gathered data by comparing post testing on two units of instruction in which two of the laws of sines were the content. The first testing was preceded by a lesson taught in the fashion of the cooperating teacher. The second unit was taught using a review session that addressed the four levels method of differentiation. The student teacher identified only three levels of students in the class, levels zero, one, and two since this was a pre-calculus class there were no level three students in the class. When conducting a summation review of the scores between the two tests, there were solid point gains from the first to the second test. There were outlier differences that needed further data exploration to support the differences between the two tests. The t-test results noted that for the whole class of 22 students there was a significant difference between the two unit tests scores in favor of the differentiated teaching. When broken down into the three levels and t-tests conducted on each group the results varied. For level zero with six students, there was not a significant difference between the two tests. For level one with eight students, there was a significant difference between the two tests. For level two with eight students, the difference between the two tests was just slightly significant with a *p value* of .056.

Chapter 16: Teacher Professional Development (K-8)

Research Reports

Brief Research Reports

Posters

IN-SERVICE MIDDLE GRADES TEACHERS' USE OF DOUBLE NUMBER LINES TO MODEL WORD PROBLEMS

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This study investigates in-service middle grades teachers' approach to word problems that can be modeled with double number lines, which serve to represent quantities and their linear relationships arising from word problems. Our data consist of videotaped PD seminars with the participating teachers. We generated a thematic analysis supported by a retrospective analysis and constant comparison methodology. Our main result is that the teachers' approach to the double number lines method evolved naturally, without overt intervention by the PD facilitator who taught from a constructivist viewpoint. Participants who attended to the measurement units inherent in the quantities and relied on iteration strategies had a more sophisticated interpretation of double number lines than the others.

Introduction

This study is part of the Does it Work? Building Methods for Understanding Effects of Professional Development (DiW)[1] project that has as its main purpose the exploration of the connections between teacher professional development (PD) in mathematics, classroom practices, and student achievement. PD focused on middle grades teachers and sought to support their development of reasoning about rational number using drawn representations. This particular study is informed by recent research on teacher knowledge frameworks (Hill, Rowan, & Ball, 2005; Ma, 1999), which outline the elements that PD needs in order to help teachers learn more about the content they teach and how their students learn.

Our theoretical framework is based on Lamon's study of ratio and proportion (1995), and Lobato's and Siebert's (2002) work on composed units of differences, which we regard as the basis for the double number lines representation. According to Lamon, "a ratio is a comparative index that conveys the notion of relative magnitude" (p. 171). Proportion, on the other hand, is about an equivalence of two ratios. "A student is reasoning proportionally when that student presents valid reasons in support of claims made about the structural relationships that exist when two ratios are equivalent" (pp. 172-173). The processes of unitizing and forming composite units are crucial components in proportional reasoning. Lobato and Siebert (2002) introduced the terminology "composed unit of differences," which refers to a phrasal quantity such as "50 miles in 3 hours", "12 degrees every 5 minutes", or "3 cans for every 7 batches". Such phrasal quantities, serving to demonstrate an existing relationship between pairs of quantities, may be represented using double number lines, where, the first number line is to be partitioned into units of the 1st quantity, and the 2nd number line is to be partitioned into units of the second quantity under consideration. The conceptualization and sense-making process may induce a reference to units-coordination (Olive, 1999; Steffe, 2002) and quantitative reasoning.

In this paper we investigate in-service middle grades teachers' approaches to mathematical tasks that can be modeled using double number lines. This representation models pairs of quantities and their linear relationships arising from word problems. In particular, we looked at the processes our research participants went through, and the mathematical ideas to which they referred in interpreting double number lines. To be more specific, our research question was "How did the in-service middle grades teachers participating in the professional seminars come to understand double number lines for representing pairs of quantities?"

Context and Methodology

Little is known about how the facilitator, teachers, and program elements that comprise a PD system (Borko, 2004) interact with classroom instruction that is shaped by further interactions among teachers, students, and content as embedded in instructional materials. To study this interaction, DiW offered a PD course to middle grades teachers in two large, urban school districts. The context of our study was one InterMath (http://intermath.coe.uga.edu) PD course on Rational Numbers, a 40-hour PD course aimed at developing teacher content knowledge in Rational Number concepts (fractions, decimals, and proportions) appropriate to the middle grades by engaging teachers in open-ended problems that are explored with technologies such as spreadsheets and Fraction Bars software.

This course included 15 teachers who met weekly for 14 weeks in three-hour PD seminars led by an experienced middle school teacher who had previously been a statewide trainer for the InterMath project. All course meetings were videotaped using two cameras, one focused on the seminar instructor and the other on the teacher participants. The videotapes were combined using picture-in-picture technology to create a restored view (Hall, 2000) of the learning environment. The corpus of PD video data were reviewed, along with their daily lesson summaries to generate possible themes for a more detailed analysis. These PD videos were revisited many times in order to generate a thematic analysis (with retrospective analysis using constant comparison) from which the following results emerged.

Results

The first problem to be modeled using the Double Number Lines (DNL) model was introduced in the fifth class meeting 7:08:39 PM (Figure 1).

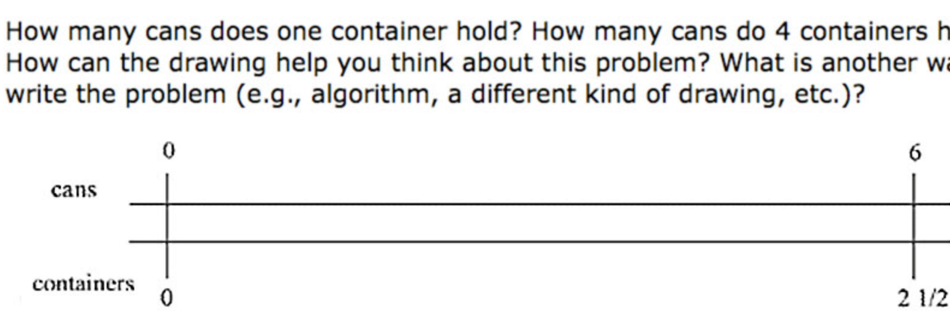

Figure 1. Cans and Containers Problem

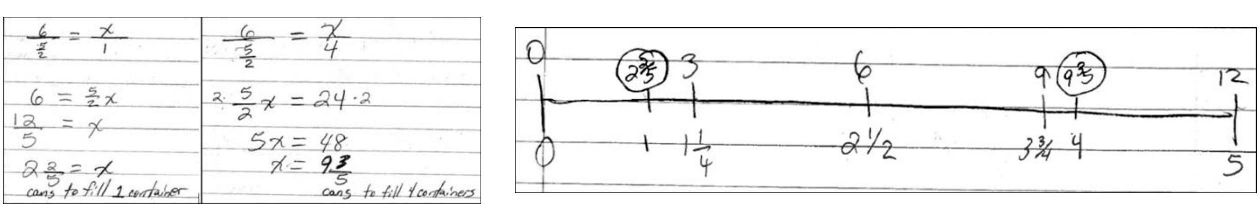

Figure 2. Corey's Algebraic (left) and DNL-Based (right)- Solutions to the Cans and Containers Problem

Although he did not explicitly use the phrase DNL, Corey suggested the use of DNL and presented a solution based on a DNL. Corey first worked the problem with the traditional cross-multiplication algorithm and then developed a DNL-based representation (Fig.2).

Corey explained that he represented both quantities on a single number line with left endpoints 0 and 0; and right endpoints 6 and 2 ½. He marked the midpoints as 3 and 1 ¼. He then extended the number line and marked 9 and 3 ¾ as the new right endpoints. Jacinda, another participant, emphasized the necessity to extend the number line once more and mark 12 and 5 as the new right endpoints. The facilitator reproduced Corey's solution on the board using a single number line, similar to the computer-based Figure 3 below.

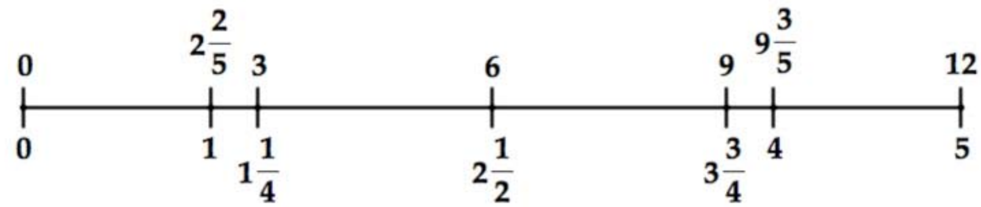

Figure 3. Computer-based DNL for the Cans and Containers Problem

Before this conversation started, Jacinda said that she had solved the problem using "proportions," meaning cross multiplication. As she gave directions to the facilitator to guide the drawing of Corey's DNL, she used the phrase "extend the line" many times. We could deduce that Jacinda's proportional reasoning was based on developing a set of coordinated pairs of units by iterating both units. We name this type of reasoning *DNL in mind*. In the next class meeting (class 6), the facilitator introduced the problem "*7 containers hold 4 gallons of juice. How many gallons does one container hold?*", which triggered Jacinda's willingness to set the problem as a proportion – which backs up our hypothesis on how Jacinda relates such pairs of quantities (with different measurement units and numerical values and for which a DNL model could apply) via proportional reasoning (PR). Jacinda's interpretation of PR lies in her iteration strategy. The simultaneity refers to the fact that the pair of units that are coordinated via iteration meet at the same point on the DNL at each iteration. In week 8, the facilitator introduced another problem for which the DNL could be used: "*1/4 serving is 2/3 of a candy bar. How much is one serving?*" Jacinda stated "I saw this as a proportion "1/4 over 2/3 equals 1 over x". She then solved the problem using the same proportional reasoning she had used in previous class meetings: She iterated the ¼ four times and obtained 1; she also iterated 2/3 four times and obtained 8/3. She explained, "Each ¼ represents 2/3, so substitution yields 2/3 plus 2/3 plus 2/3 plus 2/3" (Fig.4). Jacinda *substituted* a *unit* for another *unit* with which to be coordinated. In a sense, Jacinda treated the first quantity (1/4 serving) as a placeholder for the second quantity (2/3 candy bar). In this case, substitution stood for the fact that the quantities – that could be iterated at the same time – should meet at the same point on Jacinda's *DNL in mind* (Fig.4). Also, Jacinda's description, "Each ¼ *represents* 2/3" is similar to some other teachers' willingness to connect the pair of quantities under consideration with a *verb*. A reference to a verb, such as *represents* in Jacinda's statement can be thought as crucial in the sense-making process of solving a problem with a DNL.

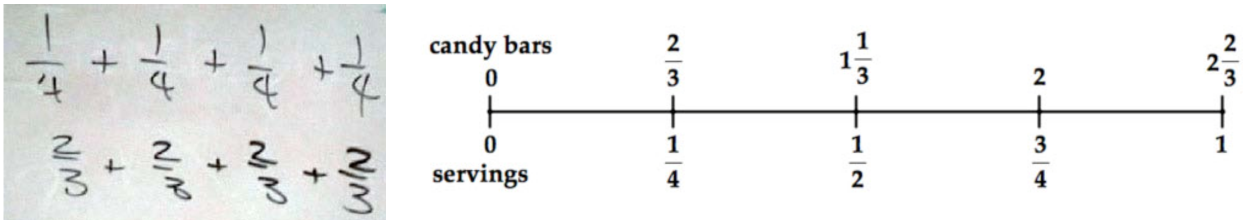

Figure 4. Jacinda's PR Based on Substitution (right) and DNL in Mind (right)

In the week 9 meeting, the facilitator asked: *Think about a "how much in one group?" story problem for 0.75 (dollars) $\div$ 1.25 (liters) and show your process in following DNL model."* Woodrow was bothered by the fact that both number lines were the same length. He raised this issue saying, "they put both of the values [.75 and 1.25] right at the same distance from 0. That's kinda weird." This statement indicated that Woodrow had not attended to the measurement units attached to the quantities, leaving him unable to reconcile the "equality" of .75 and 1.25. The notion of measurement units had, in fact, not been discussed by the participants or the facilitator in the InterMath sessions. In this problem, the verb *costs* connected the quantities 1.25 liters and 0.75 dollars. A reference to an equivalence and a verb is necessary using DNL to reason about two quantities meaningfully. The following protocol, which is an abridged transcript, illustrates the discussion on reconciling the quantities 1.25 liters and 75 cents.

Protocol I: Discussion between Woodrow and Jamal (from PD seminar on 11/10/08)

1. Jamal: I was thinking about it another way too. I was thinking like, if I were looking at it in terms of groups. If I could take the 1.25liter (line) and break it into a certain number of groups and take the 75 (cent line) and break it up into a certain number of groups to where it would balance out equally. What I'm trying to say is, cause I noticed when you asked earlier, you said, this line represented 1.25 liters and this line represented 75 cents. They look equally. It's really not – I don't think it's really saying they are *equal*. It's saying that the 1.25 is broken up into a certain amount that it equals out to 75 cents. Do you know what I'm trying to say?
2. Facilitator: Um hmm.
3. Jamal: So, it's not that they're equal, it's just that you get 1.25 liters for that amount. But, to say how much 1 liter would cost, you've got to find out what it is -- how it's broken up.
4. Facilitator: Okay, that's a good start, so how is it broken up?
5. Jamal: I mean, I guess you would have to kind of investigate and play around with it.

The fact that Jamal was working to understand the relationship between the explicitly stated pair (.25 liters and .15 dollars) may indicate that PR-based interpretation of DNL had not been achieved by Jamal at this stage. In fact, it appears that Jamal was not reasoning with composed units and was not coordinating the two sets of values in a meaningful way. When the facilitator asked why they should partition the first number line into quarters, Jamal replied, "Just to make it a nice round number because it stops at one and a quarter." This may have indicated that Jamal wanted to partition *only* the first number line, reinforcing the notion that he was not reasoning with coordinated units. Jacinda provided a word problem that would model the given DNL: "If

1.25 liters cost 75 cents, how much does 1 liter cost?”, which is then discussed by the participants.

As the discussion continued, Bree suggested using common multiples to determine partitions for the lines and determined the multiple could be twenty-fifths. Bree went on to explain that both lines could be divided into 25 parts, then one could determine how many twenty-fifths fit into 0.75. However, she was unable to create this representation when prompted. Bree did, however, seem to make sense of what the facilitator drew. We argue that the facilitator’s *equipartitioning of both number lines* strategy played a role in Bree’s sense-making process (Fig.5).

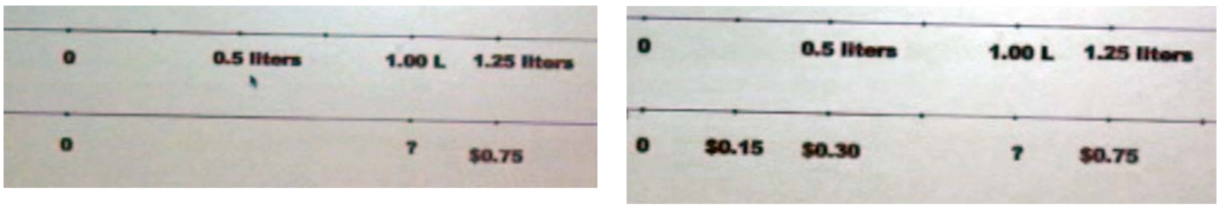

Figure 5. Facilitator’s Computer-based DNLs

Building from the computer-based DNL, the participants could see how the top number line could be broken into quarter liters. The next decision to be made was how to partition the bottom number line. Jacinda, who had previously reasoned by adding on the DNL, supported the idea of subtraction as illustrated in the following protocol (abridged transcript):

Protocol II: How to partition the second number line (from week 9 session)

Jacinda: Well mathematically we know that that space would have to be point fifteen. I know that the difference between [point 6] and [point 75] is zero point fifteen. So we find that by subtracting [inaudible] on the number line there.

Facilitator: What happens if we did not know?

Corey: Ok it’s going be over point five but less than point seven [This triggered the facilitator to add more points to the second number line as shown in Figure 5]

Jacinda: So if you were trying to explain this to your students you first draw a double number line up there and then you divide your first number line into five equal parts.

Facilitator: Why five equal parts?

Jacinda: Actually from zero to one is four spaces. So the bottom number line’s spaces have to match up. So you ask them now how are you gonna divide zero point seventy-five into five equal parts so that it matches up like that. So just divide it into five equal parts.

Facilitator: What do you guys think? How do I know it is fifteen cents?

Jacinda: You would have to kind of mathematically solve that somehow. So you would just say divide this [75 cents] by five.

Based on this protocol, we can hypothesize that Jacinda wanted to use the same number of partitions for both number lines so that the partitions matched up, indicating that she was relating the values of the top and bottom number lines. However, like Bree, Jacinda did not emphasize the pair of units – .25 liters and .15 dollars – had to be iterated at the same time. Further, it was not clear that the DNL was being used to represent quantities rather than to position two related numerals. Jacinda later commented that the DNL model required first finding the answer to the problem then verifying whether the answer was correct using the number lines. This indicated that the model was failing to support proportional reasoning for Jacinda. Instead, it was providing a drawing of the precalculated answer.

In class 11, the facilitator posed a new investigation "One batch of a certain shade of purple paint is made by mixing 3 pails of blue paint with 2 pails of red paint." Teachers were asked to represent the problem using a table and a DNL. Once the teachers had worked on the investigation, the facilitator drew two number lines on the board: the top line in blue and the bottom line in red. Jamal opened the conversation saying, "Show one of them increasing by 3 and one increasing by 2. The first one would be counting by multiples of three and the second one you can count by multiples of two."

In his first attempt to create a DNL model for this situation, shown in Figure 7a, Jamal placed the numbers on the number lines in an unevenly spaced manner. Figure-7a does not represent a DNL because the same multiples of two and three are not vertically aligned (e.g., 24 and 16 are 8 multiplied by 3 and 2, respectively, however, they are not vertically aligned). This indicated that Jamal, like Jacinda, was struggling to coordinate his knowledge of proportions, his approach to solving this problem, and the use of the DNL as a model for this situation. As Jamal drew his model on the board, Corey shared his work with Jacinda and the facilitator at his desk, as illustrated in the following protocol.

Protocol III: Corey explains his DNL (from week 11)

Facilitator: So what's this here [pointing to the vertical line segments passing by both number lines].

Corey: This is one batch, two batches, three batches, four batches [pointing to these vertical line segments respectively]. Number of batches. This is the blue pail [pointing to the first number line] this is the red pail [pointing to the second number line]. The total will be five [pointing to the first parts of the number lines] because you gotta have five cans to make one batch.

Facilitator: So what do these one, two, three, four represent? [pointing to the vertical line segments passing by both number lines]

Corey: Number of batches (Fig.6). For instance, for one batch, you need one two three [counting the spaces with his finger] pails of blue and one two [counting the spaces with his finger] pails of red. And then for two batches you need one two three four five six pails of blue [counting the spaces with his finger] and then one two three four pails of red [counting the spaces with his finger]. In this you can see the increase, the increments [pointing to both number lines].

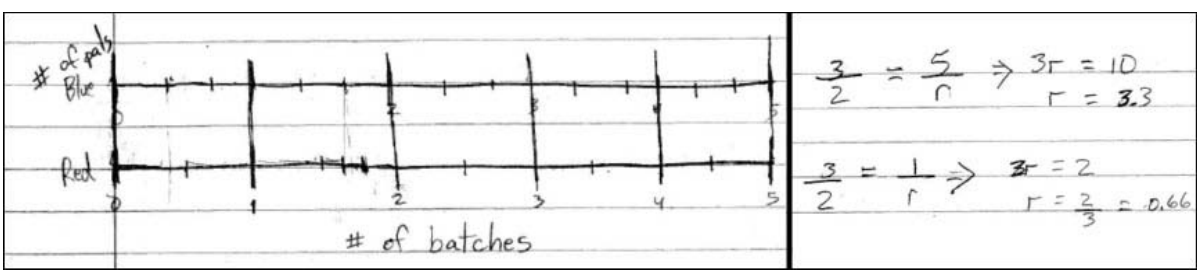

Figure 6. Corey's DNL (left) and Algebraic (right) Solution

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

Corey's diagram was the first evidence we observed of a teacher in this course iterating pairs of units coordinated by number of batches (the vertical line segments in his drawing). Corey not only coordinated the pair of iterated units with each other, but he coordinated this pair with another unit: number of batches at each iteration. Corey's coordination scheme can be diagramed as follows:

- First Iteration: 1 unit of 3 units of blue pail AND 1 unit of 2 units of red pail → 1 unit of 5 COMBINED units
- Second Iteration: 2 units of 3 units of blue pail AND 2 units of 2 units of red pail → 2 units of 5 COMBINED units
- Third Iteration: 3 units of 3 units of blue pail AND 3 units of 2 units of red pail → 3 units of 5 COMBINED units, and so on.

While this conversation unfolded, Jamal got help from another participant as he tried to draw his DNL model on the board. In his second figure, Jamal relied on a *same-multiples-of* coordination strategy in placing the red and blue numbers on the corresponding number lines.

Protocol IV: Jamal's DNL conception evolved (from week 11)

Jamal: What I did on the first number line… I did… multiples of three. Second number line I did it by multiples of two. I put 2 right under 3 because 3 pails of blue and 2 pails of red create one batch. So each one of those will be a batch. [hand gestures indicating vertical lines] We know that it takes 3 pails of blue and 2 pails of red to make 1 batch (Figure 7b).

Facilitator: How can we use these number lines to answer some of the questions we had? For example, how many pails of red paint will you need if you have 24 pails of blue paint?

Jacinda: Just read the 24 and read the corresponding number below [upon this, the facilitator draws a dashed vertical line segment passing through 24 and 16 – Figure 7b].

Facilitator: Does this double number line make sense?

Corey: I am being comfortable with it. Again I'd use the batches. It's actually kind of similar to what [Jamal] did. But I actually showed the splits between each one. I, for instance, where he has the 3 and the 2, I'd put a line straight down and say that's one batch. And I'd divide the blue up to three parts up to the one batch; and the red into two parts. It's basically the same thing.

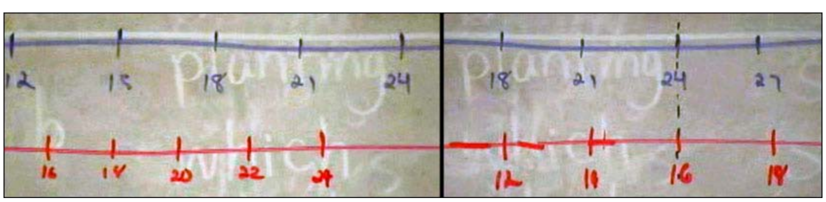

Figure 7. Jamal's First (left) and Second (right) DNL Attempts

The facilitator then asked the whole group what they think about double number lines. In contrast to earlier conversations about this, the participants were now beginning to verbalize an understanding of the important aspects of the model for reasoning about proportions, though it was clear that the teachers were not seeing all of the mathematics of PR in the DNL. For example, teachers were still struggling to coordinate units in their reasoning with them.

In week 13, the facilitator introduced two more problems (Cookies Problem and Reservoir Problem) for the teachers to investigate with the DNL model (Figure 8).

1) Andrew has a recipe for cookies that calls for 3 cups of brown sugar to make 7 batches.

> a) How many batches can Andrew make with one cup of sugar?
> b) How many cups of brown sugar does Andrew need for one batch?

2) Erik is putting oil in his scooter. When he had 3/4 of a liter in the reservoir, he realized that the tank was 1/2 full. How many liters will the fully filled reservoir hold?

Figure 8. Cookies Problem and Reservoir Problem

With the Purple Paint Problem in week 11, Corey first used the DNL method, then provided the algebraic solution. For the Cookies Problem and the Scooter Problem, on the other hand, both Jacinda and Corey first solved the problem algebraically, then illustrated their solutions using a DNL. This was consistent with our earlier observations of Jacinda who, despite understanding composed units, was unable to determine values with the DNL. We noted that Corey, who also calculated first, continued to discuss DNLs as tools for estimating relationships. Interestingly, with the Cookie task, he moved away from the coordinated drawing of the previous week to rely on an estimation-like diagram (Figure 9). As he created his DNL for the Cookie task he drew a connecting line at 1 on the top line explaining, “Here, where we said there was one batch, you can draw a line to about 2 1/3.” Then, he drew a box around 1 and 2 1/3. Then he repeated this process for all of the relevant iterates. Unfortunately, his move away from the coordinated DNL was not discussed.

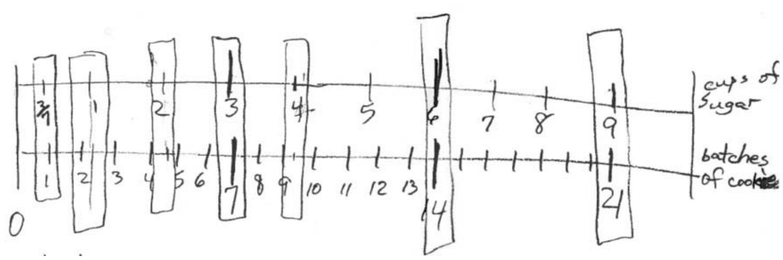

Figure 9. Corey’s DNL for the Cookies Problem

When the facilitator reconvened the whole group, Jamal volunteered to share his solution for the Cookies Problem. His use of the DNL model showed greater sophistication compared to his earlier efforts. He coordinated his 1st number line in increments of sevenths, and his 2nd number line with increments of thirds, in such a way that the 7 and the 3 were aligned (Figure 10a). He explained that he broke the first line into thirds, and the second line into sevenths. He emphasized that he first solved the problem algebraically using a proportion, then he drew his DNL. He continued, “7 batches would equal up to 3 cups of sugar.” His last statement is in agreement with our previous theories concerning Jamal, in that he feels the need to “connect” the two quantities via a verb, which was essential in fully making sense of DNL.
Jamal relied on a unit-coordination strategy in his reconciliation of the pair of quantities “7 batches and 3 cups;” he was able to iterate the pair of units *7 and 3* that were connected via the verb *equal*. Moreover, Jamal was aware of the unit coordination at the second level, which was

the iteration of the pair of subunits: 1/3 and 1/7. This unit coordination strategy can be derived from Jamal's statements as illustrated in the following protocol:

Protocol V: Jamal's Unit-Coordination based on subunits (from week 13)

Jamal: If we do one-seventh, two-sevenths, three-sevenths, four-sevenths, five-sevenths, sixth-sevenths, and this is one whole [marks the numbers 1/7, 2/7, 3/7, 4/7, 5/7, 6/7, 1 on the "sugar" number line]. So for one batch you need three-sevenths cup of sugar [connects the 1 from the top number line with the 3/7 from the bottom number line]

Facilitator: So for one cup of sugar how many batches will you need?

Jamal: For one cup of sugar it is gonna take two and a third [connects the 1 from the bottom number line with the 2 1/3 from the top number line – Figure 10b].

Facilitator: So you use the same number line to answer those questions.

Jamal: Yes you use the same number line to answer all the questions. Because when you had it all split up you can just kinda eyeball in it and that should give you the answer.

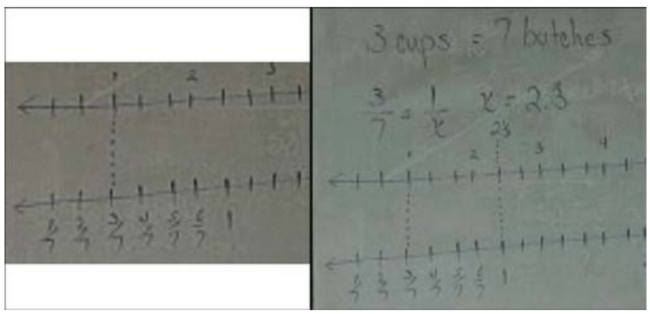

Figure 10. Jamal's Unit Coordination Based on Subunits

Jamal's unit coordination (UC) scheme can be diagramed as follows:

- First Level UC: 7 units of batches AND 3 units of cups → "7 batches would equal up to 3 cups of sugar" → coordination of the pair of units "7 and 3"
- Second Level UC: 1/3 unit of batch AND 1/7 units of cups → coordination of the pair of subunits "1/3 and 1/7"
- Iteration of the Subunits: Three 1/3 unit of batch AND Three 1/7 unit of cup → "for 1 batch you need 3/7 cup of sugar" → coordination of the three-times-iterated pair of subunits "1/3 and 1/7"

Conclusions and Discussion

In this study, we observed DNLs being used in a way that was algorithmic. Often, teachers first used cross-multiplication to find an answer to a task that could be modeled with a DNL, which Lamon suggests, "does nothing to promote proportional reasoning" (1995, p. 167). Then, they used the DNL to draw their algebraic solution. In only one of the six tasks the group worked on together was a solution generated with DNL. And then, only one of the 15 teachers used the DNL in this way. We interpret this lack of using the representation to support problem solving as indicating that teachers were not comfortable in reasoning about proportional relationships and/or they had not developed an understanding of representations as tools for generating solutions and not just for illustrating them.

Despite issues with reasoning with representations and making mathematical connections, we did note that teachers were demonstrating multiplicative reasoning in the ways they conceptualized the relationships of variables in the problems. Several teachers consistently referred to phrasal quantities, which we interpreted as verbalized versions of what Lobato and Siebert (2002) call “composed unit,” which is critical for making sense of slope. In Lobato and Siebert’s research study, Brad appeared to construct ratio by forming the composed unit “10 cm in 4 seconds”, which, later was iterated by Terry to show that “walking 30 cm in 12 seconds is the same speed as walking 10 cm in 4 seconds” (p. 107). Our findings agree with Lobato and Siebert’s in the sense that our teachers felt the need to connect quantities of different units using a verb, which can be thought of as setting the stage for proportional reasoning, just as Terry does with reference to the *be* verb in the quote above.

In representing the situation with double number lines, an awareness of the equivalence of such phrasal quantities corresponds to the iteration of the pairs of quantities on the double number lines simultaneously. The process of looking at simultaneous iteration of the units and making use of phrasal quantities and connective verbs corresponds to teachers’ construction of ratios, in our research. For instance, in the Purple Paint Problem, Corey’s DNL provided the first evidence in this class of a teacher iterating pairs of units coordinated by numbers of batches, as shown in the vertical line segments in his drawing. He was not only able to coordinate the pair of iterated units with each other, but he was able to coordinate this pair with another unit, *batches*. A reference to a verb connecting the pairs of quantities was also established.

Endnotes

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LISTENING TO LEARN: FOSTERING PRACTITIONER PEDAGOGICAL CONTENT KNOWLEDGE WITH THINKER-DOER TASKS

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Problem solving forms the core of mathematical endeavor (English, Lesh & Fennewald, 2008). From 2006-2010, a team of mathematics educators from X University designed and implemented a professional development experience for mathematics teachers in four school districts called X University's Partnership for the Enhancement of Teaching Mathematics (XUPET-Math). This paper highlights some findings in using Thinker-Doer tasks (Hart, L., Najee-ullah, D. & Schultz, K., 2004) in professional development settings and how teachers grew in their subject matter knowledge and pedagogical content knowledge using this professional development tool. We discuss teacher growth along a telling-listening continuum, and an individual-collaboration continuum.

Introduction

Problem solving forms the core of mathematical endeavor (English, Lesh & Fennewald, 2008), and a great deal of research has focused on the ways in which students develop as problem solvers (Schoenfeld, 1992; Hiebert, J., Carpenter, T., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A. & Wearne, D., 1996). However, our experience with practicing teachers has indicated the need for helping classroom teachers develop as problem solvers themselves so that they can understand and model the practices they teach to their students.

From 2006-2010, a team of mathematics educators from X University designed and implemented a professional development experience for mathematics teachers in four local school districts called *X University's Partnership for the Enhancement of Teaching Mathematics* (XUPET-Math). The XUPET-Math experience comprised a series of coordinated experiences involving lesson study, interviewing students and questioning techniques. One activity in particular—Thinker-Doer tasks (Hart, L., Najee-ullah, D. & Schultz, K., 2004)—focused on the development of problem-solving strategies and problem-posing through cooperation, communication, and content knowledge development. This paper highlights some of our findings in using Thinker-Doer tasks in professional development settings and how teachers grew in their subject matter knowledge and pedagogical content knowledge using this professional development tool.

Theoretical Framework

To provide a structure for our work with teachers and to provide a framework for understanding our progress in fostering the professional development of teachers as problem solvers and problem posers, we offer the model depicted in Figure 1. As Figure 1 suggests, we considered our work with teachers along two basic dimensions: (1) a telling-listening continuum, and (2) an individual-collaboration continuum. We decided to focus on these two dimensions based on an initial needs-assessment conducted with district curriculum supervisors and teacher leaders in the partnering districts. We explore each dimension of the model in more detail in the following paragraphs.

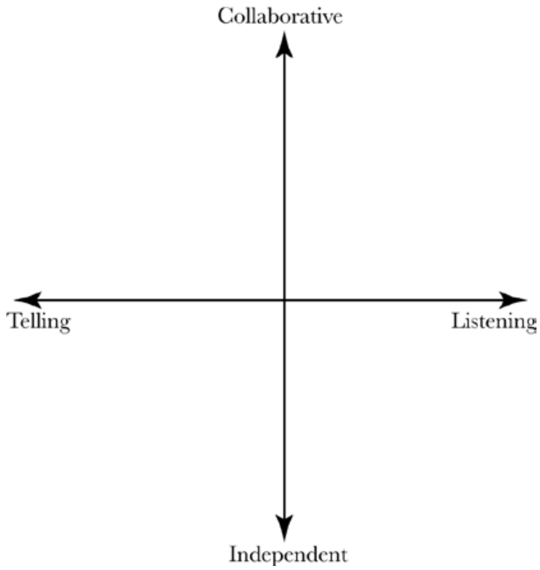

Figure 1. The Telling-Listening / Individual-Collaboration Plane

Telling Teachers and Leading Questions

In three of the four partnering districts, classroom teachers were actively transitioning from traditional textbook series to reform-oriented teaching materials (e.g., *Mathematics in Context* [*MiC*], *Connected Mathematics* [*CM*]). Teachers in the fourth district were already using reform curricula - with elementary teachers using *Investigations in Number, Data, and Space* and middle school teachers using *CM*. Research indicates that the social constructivist orientation of reform materials poses significant challenges for teachers - particularly for those with extensive experience using traditional, teacher-centered materials (Smith, 1996). Because reform materials emphasize student knowledge construction and student-initiated problem solving strategies, successful implementation of the materials requires teachers to ask questions that help students develop their *own* hypotheses, methods, and generalizations. When such questioning is unfamiliar, teachers may inadvertently undermine student problem-solving development by posing questions that lead students to approaches favored by the classroom teacher.

As Wilson and Goldenberg (1998) note, many teachers feel uncomfortable when students struggle—although arguably it is in this struggle where learning occurs. Indeed, in their quest to "help" students "learn" mathematics, classroom teachers inadvertently convince many that "there is always a rule to follow in mathematics" (Dossey, Mullins, Lindquist & Chambers, 1988, p. 102) and mathematics problems must be solvable in five minutes or less (Schoenfeld, 1988). Indeed, such beliefs and practices remain pervasive among students and teachers in school classrooms today. With these obstacles in mind, the XUPET-Math instructional team decided to focus a considerable portion of XUPET-Math sessions on the development of teacher questioning strategies. In terms of the Telling-Listening / Individual-Collaboration Plane depicted in Figure 1, we aimed to encourage teachers to "slow down" and listen to others' mathematical thinking. As Pirie (1996) notes, "mathematical communication can take place effectively only if all participants are prepared to adopt both roles, to listen actively as well as to talk" (p. 105).

Conversations, Collaboration, and Problem Solving

Given the social constructivist orientation of reform teaching materials, a second aim of the professional development was to strengthen teachers' collaboration and communication skills—both central components of mathematical problem solving and that are embedded in the curriculum reform materials that they use with their students. For instance, students using the *CM* materials "present and discuss their solutions as well as the strategies they used to approach the problem, organize the data, and find the solution" (CM, 2010). Furthermore, they "pose conjectures, question each other, offer alternatives, provide reasons, refine their strategies and conjectures, and make connections" (CM, 2010).

Features of Thinker-Doer Tasks

The Thinker-Doer problems were used as a tool to help teachers focus on listening to students as they talked about their initial thoughts and plans as well as their strategies for solving rich tasks—all as a vehicle for informing teachers' instructional planning. The initial implementation of Thinker-Doer tasks followed the procedures outlined by Hart, Schultz and Najee-ullah (2004) (as described below), and later evolved to foster the collaborative aims of XUPET-Math.

Thinker-Doer Pairs

The process of completing a set of Thinker-Doer problems begins with the selection of two challenging rich mathematics tasks (of equal difficulty, if possible). The instructor divides teachers into two groups and gives distinct but related tasks to each group. Teachers retreat to different rooms to solve their problem collaboratively. After the groups solve the task, they discuss problem-solving strategies, alternative methods and the mathematics involved in their solutions, becoming "experts" of their own problem. Each person in the two large groups is now a "Thinker" for their particular problem.

When both groups finish solving their problems, they reconvene and partner with one of the teachers from the other group. The pair of teachers takes turns solving each other’s problem. The "Thinker" encourages his or her partner (the "Doer") to read the unfamiliar problem aloud and asks if the partner understands the question or has any ideas about how to start the problem. The Doer's job is to try to solve the problem by thinking out loud. The Thinker does not interfere with the Doer's thought process. Instead, the "experts" are to listen carefully to their partner's problem solving strategies and mathematical explanations—only asking for clarification for their own understanding. When finished, the two partners change roles and tackle the other problem.

We found that listening rather than telling was a huge obstacle for our teachers; and it wasn't until the second or third set of Thinker-Doer problems that the teachers really started to listen to their partner's explanations. This multi-part process (large group initial solving and solving the two tasks in pairs) took a significant amount of time, but we felt that it was vital in helping our teachers learn more about what it means to really listen to their students even if the teachers believe that they know a better approach for solving a problem.

Thinker-Doer Triads

As our teachers became more familiar with the Thinker-Doer process, we reflected further on the model of the Thinker-Doer pairs. We found that some teachers were intimidated by solving challenging problems aloud as their partner recorded notes of their struggle. These teachers were often reluctant to share their mathematical strategies or explanations which in turn defeated one of the primary purposes of the Thinker-Doer problems. Since we wanted to model problem solving as we would want to see it in the teachers' classrooms, we changed from Thinker-Doer pairs to Thinker-Doer triads. This process starts in a similar manner as before with three groups of teachers solving three different but related problems. Unlike in the original model, teachers are typically given time to think about and solve the problem individually before convening with the larger group. We found that individual thinking time was important to get each person engaged and committed to a particular strategy. Once the groups become "experts" of their own problems, we form triads of teachers selecting one from each group to work together. One expert poses his or her problem to the other two teachers. These two teachers work together to solve the problem, and are encouraged to think aloud communicating their own problem solving strategies. The teachers then rotate through all three problems, taking turns as the expert for their problem and the Doer of the two additional tasks. We found that this revised configuration further encouraged mathematical communication and helped to relieve teachers' anxiety of not knowing how to solve the problem.

Methodology

As part of the XUPET-Math professional development, participating in-service teachers completed three Thinker-Doer activities. The teachers completed a numerical self-assessment associated with each Thinker-Doer task, along with a written reflection. The reflection encouraged teachers to think about the mathematics content, student thinking, and ways to differentiate instruction to meet all students' needs. The self-assessment consisted of nine statements, each rated by the participant using a Likert rating scale from 1 (little or no knowledge/agreement) to 5 (complete agreement/competence). The following is a listing of the nine statements:

1. I personally believe or feel comfortable with the level of mathematics in the Thinker-Doer problems for the grade level(s) I teach.
2. I personally believe or feel that my students will be able to solve these problems rather comfortably.
3. I personally believe or feel that I could predict what my students might be thinking when they are solving these problems.
4. After sharing with my Thinker-Doer partner(s), I understood the mathematics content more clearly than before.
5. After sharing with my Thinker-Doer partner(s), I understood what my students might think about the problem more clearly than before.
6. After this Thinker-Doer session was completed, I learned something new or came to a better understanding about mathematics content.
7. After this Thinker-Doer session was completed, I learned something new or came to a better understanding about ways to explore mathematics content.
8. After this Thinker-Doer session was completed, I learned something new or came to a better understanding about ways to predict how students or others may think about solving mathematics problems.
9. After this Thinker-Doer session was completed, I learned something new or came to a better understanding about ways to explain/facilitate mathematics learning.

In addition, teachers responded to the following writing prompts for each Thinker-Doer task.

Prompt 1: When I solved my problem as a Thinker I thought of the following things (i.e., mathematically with regards to math content and processes as I attempt to recount what I did while solving the problem).

Prompt 2: When I observed my Thinker-Doer partner(s) solve my original problem I thought of the following things about the way s/he worked through the problem as compared to my approach.

Prompt 3: If I had to prepare a problem similar to this in my grade level range (e.g., 3-6), my students' content knowledge and mathematical understanding could vary, I might expect the following differences in content knowledge or mathematical understanding, and would adapt or differentiate instruction to meet my students' needs in the following ways.

Results

Likert-Style Self-Assessment Data

The following table summarizes the mean scores for each of the following questions for Summer 2008 participants in one of the four partnering school districts.

Table 1. Mean Self-Assessment Scores for Thinker-Doer Activities

		Mean Score, Questions 1 through 9 from Self Assessment								
Grade Level	**No. of Assessments**	Q 1	Q 2	Q 3	Q 4	Q 5	Q 6	Q 7	Q 8	Q 9
3-6	56	2.7	2.1	3.1	4.2	3.6	4.3	4.2	3.9	4.0

As evidenced by the responses for Questions 1 and 2, most teachers were slightly uncomfortable with their own knowledge level of the mathematics content of the Thinker-Doers, and they did not feel that their students would be able to solve the problems comfortably. However, after completing the Thinker-Doers, these same teachers felt that they understood the content more clearly than before, they learned new ways to explore mathematics content, and they better understood ways to explain mathematics, as evidenced by the mean scores of 4 or higher on questions 4, 6, 7, and 9.

Question 3 asks whether the teachers felt that they could predict what their students might be thinking if they were to attempt to solve the problem. The mean score for this question was 3.1, and the scores that teachers reported ranged from 1.0 to 5.0.This indicates that some of the teachers felt very confident in their ability to predict what their students might be thinking while solving such problems, while others felt that they have little knowledge about what their students might be thinking. Question 5 then asked if after sharing information with their Thinker-Doer partner, they felt that they could more clearly understand what their own students might think about when solving the problem. The mean score for this question was 3.6, which was an increase over the mean of 3.1 on question 3. This could indicate that the discussions taking place between partners and in groups helped the teachers to understand various ways that different people think about solving problems. These insights were also evidenced in the written reflections, which will be discussed in the next section.

Analysis of Written Reflections

For the Summer 2008 workshop, teachers wrote a total of 53 reflection papers. The written reflections from the Thinker-Doers were examined against the objectives that were established for the XUPET-Math project and the Telling-Listening / Individual-Collaboration Plane.

One of the main goals of XUPET-Math was to document how teachers grew in their subject matter knowledge and pedagogical content knowledge using the Thinker-Doer professional development tool. Many of the teachers reflected on new understandings that they gained from acting as the "Thinker" and the "Doer." Many of the teachers mentioned how this activity helped them see the importance of peer collaboration. The following excepts identify teacher growth along the individual-collaboration continuum.

"I believe this would be a terrific activity to use during Intervention, when I am working with a small group of students. By hearing the students discuss and 'think out loud,' I feel the teacher could gain so much insight into why/where a student is having difficulty with a problem."

"I like the idea of partners working together and sharing their thoughts aloud. We do not do enough of this; we need to help every student have a voice in their learning. Being the teacher, I give too much aid to my students and this lesson showed me the power in being the listener and encourager over being the instructor."

Other teachers discussed pedagogical strategies in their written reflections. The following excepts identify teacher growth along the telling-listening continuum.

"This activity is a wonderful tool to help teachers understand the students' thought process! I will definitely try this kind of activity to better assess the students' needs by listening!"
"I had asked my partners repeatedly, 'What does that number mean, what is it representing?" That is a technique that requires the student to slow down and focus on what is being asked rather than finding a numerical solution."

"It was fun to see that just questioning the students seemed to be the only thing that I needed to do to assist them. It is an ah-ha moment for me, that I don't have to give hints over and over to assist."

Discussion

Trends in the Likert-style and qualitative data, suggest teacher development along the two dimensions highlighted in the Telling-Listening / Individual-Collaboration Model. Figure 2 suggests this development visually. Most teachers—before they began working with Thinker-Doer tasks—identified with those in TD Set 1. These teachers strongly believed in the importance of telling learners (like their Thinker-Doer partners) when they were making mistakes and how to correct them. Teachers who completed Thinker-Doer tasks in their first iteration (where they were grouped in pairs and had to work independently while solving the new problems) identified with those in TD Set 2. At this stage, teachers practiced better listening skills (although they still relied somewhat on telling) but they still struggled with working collaboratively when solving problems. The Thinker-Doer pairings did little to help teachers progress along the Individual-Collaborative continuum. Teachers who regularly practiced Thinker-Doer tasks in their final iteration (where they were grouped in triads and were able to work with one partner when solving the problems) identified with those in TD Set 3. Here, they became much better at listening and providing scaffolding rather than telling while also appreciating the importance of discussing ideas with a partner.

Preliminary findings from open-ended responses suggest that the Thinker-Doer tasks (especially in the final iteration) fostered teachers' development as listeners. Teachers indicated that listening (rather than telling) was initially quite difficult but that the process became easier with practice. This is suggested by the left-to-right ordering of teacher placement in Figure 2. Likewise, collaboration was fostered through the implementation of the Thinker-Doer protocol. The revised Thinker-Doer triads made this collaboration more effective (and less daunting) for teachers, as suggested by the upward trend from Set 1 and 2 to Set 3.

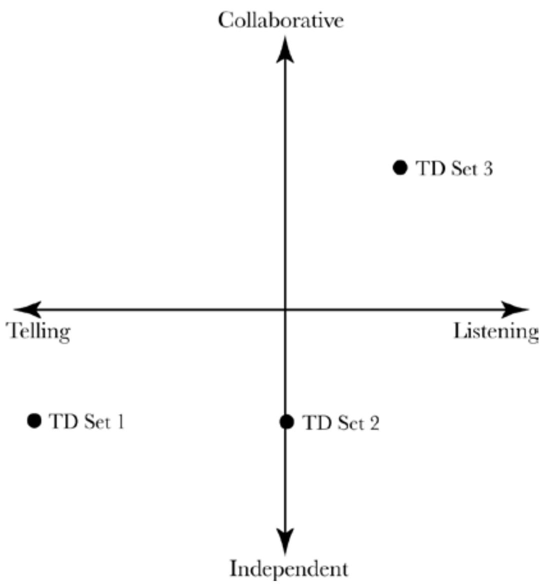

Figure 2. Development of teachers along the Telling-Listening / Individual-Collaboration Continua

Conclusion

Thinker-Doer tasks served multiple purposes—they provided a context for both the development of teachers' subject matter knowledge as well as their pedagogical content knowledge. The tasks were used as launching points for subsequent discussions of specific content that teachers needed to know as well as ways in which the content could be delivered in their classrooms. In addition, the Thinker-Doer tasks gave teachers experience developing communication skills that they themselves would use when teaching students in an inquiry-based classroom. Teachers learned to listen as others solved problems and how to provide useful feedback that didn't detract from the solvers' experiences. This was not easy for all teachers to do and they expressed in their reflections the difficulty of using these experiences to make changes in their own practices.

Teachers also came to value the importance of working collaboratively when solving tasks. They began the project with the view that learning mathematics is an solitary process—that each person learns at his or her own rate and building on his or her experiences. While these claims are true, they ignore the importance of social constructivism and the knowledge that can be attained when working with someone else. Teachers realized the value of collaboration first-hand in moving from the Thinker-Doer pairs to the Thinker-Doer triads, and these experiences helped to establish a norm for group problem solving in their own classrooms.

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TASK PROBLEMATIZATION: THE CASE OF THE FILLING TASK

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This paper focuses on the development and problematization of a task designed to foster spatial visual sense in prospective and practicing elementary and middle school teachers. We describe and analyse the cyclical stages of developing, testing and modifying several 'task drafts' related to ideas around dilation and proportion. The task in its present form incorporates numerous considerations, including choices around materials, wording of questions and prompts, pedagogical structure, and sequencing of experiences.

Introduction

Research shows that increased attention to spatial, visual and kinesthetic approaches to learning mathematics helps students make connections between various representations of the underlying concepts (e.g. Goldin, 1998). Battista, Grayson, and Talsma (1982) found that pre-service elementary teachers often have poorly developed spatial visual reasoning skills for geometry, and have substantial anxiety about working with these approaches. Whiteley (2004) claims that visualization and mental imagery is central to mathematical reasoning. As such, we recognize a need for teachers to accrue experiences with tasks and activities that make visible the underlying mathematical ideas and the connections between and among them. Both the NCTM and the CBMS stress the importance of spatial visual reasoning. In particular, the CBMS (2001) recommends teachers develop competence in visualizing, including becoming familiar with projections, cross-sections, and common two- and three-dimensional shapes. They also identify the importance of spatial visual reasoning in problem solving with 2- and 3-D objects.

In regard to these competencies we found that examples in the literature were often limited in scope (e.g., they focused specifically on mental rotation, or on shifting between 2-D and 3-D interpretations of a drawing) or they were developed for young students. For instance, Del Grande (1990) categorized skills that are key to children's development of 2-D and 3-D geometry understanding: 1) spatial perception, 2) visual discrimination, and 3) visual memory. Clements (1999) suggests that children need opportunities to develop their visual and spatial senses, for example, by rotating objects in their minds. In considering older students, Eisenberg and Dreyfus (1991) reported that visual reasoning was often avoided (even by university mathematics students) in preference to algebraic reasoning. They theorized that a significant reason for this is that visual thinking requires more cognitive effort.

Despite the important contributions of these researchers, we did not find a structure of experiences that we believed would help teachers develop the spatial visual reasoning necessary to support students in spatial visual tasks and problem solving. In response, we set out to design a task that would foster such experiences. In line with the expressed content-based learning goals of the NCTM and the CBMS, our task addresses concepts in dilation and proportional reasoning.

Framework for Task Design

Sierpinska (2004) identifies task problematization (debating variations of a task and discussing the effects of such variation on the learning or research results) as central to studies in mathematics education. She proposes that design research is needed to clarify what is essential or arbitrary about tasks, as such knowledge would allow replication of task-based research results, and would inform the design of task sequences. An example of task design research is provided by Gadanidis, Sedig, and Liang (2004), who brought Human Computer Interaction (HCI) and mathematics education expertise together to examine an online mathematical investigation from two design perspectives - pedagogical and interface. We relate the cyclical process of design, develop, test, analyse, (re)design, (re)develop, (re)test, described by Gadanidis et al. (2004) to our design of visual spatial instructional tools. We introduce the idea of "task drafts", which are discussed and analysed at each stage of the design cycle, and as such provide a mechanism within which task problematization can occur.

The design cycle begins with task designers' consideration of broad principles, such as the importance of multiple representations and active participation, and computer interface design principles related to visual encoding and organization (Gadanidis et al., 2004). Such principles inform decisions about the inclusion of particular questions, prompts, materials, etc., and about how to orchestrate the task. At subsequent stages of development, the focus of the designer shifts to the responses of participant testers. Responses may reveal conceptual difficulties with the mathematical content, or they may bring to light underlying problems of the design. In the testing stage, participant actions and responses – particularly their non-actions and non-responses - enrich the task designers' understanding of task elements and how they may interact to further the intended learning goal. In regard to the interactions of task elements our research draws on Sinclair's (2003) research on the design of technological learning tasks, which revealed the importance of carefully structuring the links between prompts/questions and pre-constructed dynamic geometry sketches to support learning. That is, in the technological setting the task designer creates opportunities for students to investigate by including appropriate affordances – perhaps an onscreen button to add a line or measure an angle, or capabilities for revealing information in a novel way. The designer must also set prompts and questions to help learners using those affordances to notice, interpret, and extend their thinking. Relating these ideas to our study, we extend these perspectives to the design of 3-D hands-on investigations. For example, the materials we selected (hollow polyhedral shapes) needed to allow the learner to take appropriate action (e.g., holding, turning, moving, filling) in response to a prompt or question, which in turn needed to guide the learner as they acted. Throughout the design process we focused on developing links between prompts and models to support growth in spatial visual reasoning. This paper details three testing stages and task drafts in our on-going cycle of task problematization.

Methodology

The Task – Learning Goals

The motivation behind our task drafts draws on three themes: 1) 'big ideas' – the spatial visual concepts and experiences foundational for teaching elementary school mathematics; 2) making connections – experiences to foster and extend teachers' thinking by linking ideas; and 3) strategies – enabling deep thinking about spatial visual concepts to recognize and address misconceptions. Our task drafts aim to (re-)develop spatial and visual skills, as well as develop relationships and connections among ideas related to dilation, ratio and proportion – 'big ideas' in school curricula (NCTM, 2000). We sought to foster connection-making among multiple representations and also between dimensions by including a kinesthetic component to the task, which incorporated manipulating 3-D models, and 2-D technological objects. The technology component was designed to provide opportunities to see dynamic change and develop imagery for such change. Through this component we aim to foster spatial visual strategies that allow learners to ‘see’ diagrams as encoding both 2- and 3-D information, and to explore both 2-D diagrams and 3-D objects.

The Task – Scope and Content

Specifics of *The Filling Task* are detailed below; however we briefly describe the scope and content here. The task has two distinct components – a GSP examination of dilation, and a 3-D exploration of cross-sections of polyhedral. Each of these subtasks was tested at three stages in the research process. The GSP task on dilation was intended to draw attention to the idea of center, to broaden the usual curricular treatment of scale diagrams, and to allow the user to investigate changes in length and area dynamically. The 3-D model task evolved out of a discussion around imaging cut surfaces (cross-sections), which we anticipated would be a new and interesting channel through which spatial visual skills could develop. Included in the appendix are the written prompts that accompanied the 3-D activity, as well as snap shots of a selection of the GSP screens, each from the third task draft.

The Participant Testers

Our task design occurred over three testing stages with pre- and in-service elementary and middle school teachers. The stages each lasted approximately two hours during which participants worked in small groups of two to four, and field notes were collected. Written responses to prompts and questioning were also collected and used to inform our analysis and task refinement. In the first phase of testing, 18 pre-service elementary teachers engaged with *The Filling Task*. The second testing phase occurred with 24 practicing elementary and middle school teachers. In the third session, our refined task draft was tested with another group of 18 pre-service elementary teachers. Participants in each of the testing phases were novices with respect to GSP, and our task design reflected the need to familiarize participants with basics of the software. The majority of participants at each stage were anxious about their understanding of mathematics in general, having not studied mathematics beyond the high school level.

Discussion: The Filling Task Testing Stages

Stage 1 Testing – The Initial Task Draft

Our initial task draft of the 3-D model included two sets of materials from which to produce cross-sections: objects to *slice* and objects to *fill*. The slicing material included plasticene shapes (e.g. pyramids, prisms), that could be cut parallel to the base to produce similar shapes. The filling materials included clear plastic shapes and water, which can show cross-sections and truncated 3-D shapes from multiple points of view. For the slicing approach, we observed that participants found it hard to create 3-D shapes with plasticene, and while the softer play dough could be shaped using plastic moulds, students had difficulty getting a clean cross-section cut (i.e. one that clearly showed the shape). The filling approach also posed some difficulties. Participants had difficulty identifying surface shapes because of the effects of surface tension, and some of the provided shapes were simply too small for making observations.

The GSP task included blank sheets on which participants were encouraged to construct and dilate shapes. Although instructions were provided, we noted that most participants needed more scaffolding and were often distracted from issues of dilation by the challenges of constructing shapes with the software. Despite the challenges, the GSP task did lead to a discussion of the idea that the slices through the tetrahedron models could be thought of as dilations. This was an important connection for participants, and it motivated subsequent changes to our task design.

Stage 1 Testing – Task Revisions

As a result of the first testing session, we made a number of modifications to the GSP task. In resonance with Sinclair's (2003) observations regarding directing student attention, and because of participants' struggles with constructing polygons, we decided to shift to the use of pre-constructed sketches. The intent was to lessen the demand on participants and to help maintain their focus on dilation. The revised screens were designed to lead the learner through sequenced interactive investigations of ratios in connection with dilation.

Although slicing had engaged the students, the difficulty of getting a clean slice in the 3-D model task led us to focus on filling (despite our concerns about surface tension) as a more appropriate activity. Based on this testing session we also made decisions about the timing and focus of the activity. We determined that 1) we should work with just a few shapes: triangular and square-based pyramids; 2) to develop the idea of proportional reasoning, we should have the participants pour water gradually into a (stabilized) 3-D shape and examine the resulting shapes; and 3) we should make connections to and between a scale factor's effect on side length and area, and the effect of tilting a pyramid on volume and surface area.

Thus, we developed the 3-D model task to focus on filling large, transparent plastic polyhedra with water; we emphasized tilting the objects; we focused questions on proportional reasoning; and in light of the interest participants showed in making connections between the filling activity and the GSP demo we deliberately linked the two tasks.

Stage 2 Testing – A Revised Task Draft

The revised GSP task included eight pre-constructed GSP sketches that addressed linear and area relationships of scaled figures through dilation. Instructions and prompts were included onscreen to focus attention on the sketches. Sketches were designed to be done in sequence, to ensure that participants learned GSP procedures at a modest pace, and to build mathematical ideas. Printouts of the screens were provided for recording purposes.

The GSP activity focused on the dilation of a triangle – it is easily manipulated by dragging and is a natural 2-D analogue of a pyramid. All screens were designed in light of the idea that visual cues such as colour, orientation, location on the page, etc. play an important role in learners noticing relevant mathematical ideas (Gadanidis, et al., 2004). Architectural measurements of segments and calculated ratios were included, as was a slider that participants could use to control the scale factor; these were intended to help participants connect a visual sense of dilation with the results they would get from applying a formula. We included "hide/show" buttons that toggled between the strictly visual representations and the numerical relationships. Text instructions and prompts were also provided on-screen. In resonance with Sinclair (2003), we acknowledged that text prompts are key to supporting mathematical understanding during technological investigations, as is directed dragging in the specific case of dynamic geometry software activities (Arzarello, et al., 1998). Such directed dragging was encouraged on several screens of the GSP activity. Appendix A presents the first screen of the activity, and exemplifies the design considerations mentioned. Despite the encouragement, we found participants were hesitant to explore shapes on their own. As before, participants had little experience with GSP. Our revisions to the prior task draft incorporated interface and pedagogical considerations (Gadanidis, et al., 2004), and motivated the inclusion of pre-constructed sketches, so there was little need for participants to know GSP. Nevertheless, participants' use of dragging appeared restricted and hesitant, rather than directed and purposeful. We also noted that participants were moving step by step through the screens but were not making connections between them; they seemed to lack an overall picture of the activity.

The second task draft of the 3-D model involved filling square- and triangular-based pyramids. The focus was on observing and comparing side lengths and areas of the surface shapes while the pyramid was in the upright position and then in a tilted position. Participants were encouraged to find how many different types of shapes they could create by filling and tilting the pyramid, and what their findings implied for cross-sections of other 3-D shapes. In this stage of testing, the 3-D model task succeeded the GSP activity. We noted that participants had difficulty making connections around scaling, similarity and proportional reasoning. They became distracted with pushing calculations and seeking formulae, rather than trying to use the model to see the possible connections and overall proportional changes. In the case of the triangular based pyramid, the filling activity linked closely to the work set up for the GSP task, providing a concrete model of dilation, with successive water surfaces acting as the "dilating triangles" and the bottom vertex as the center. While we observed that some participants could see the GSP sketch in two and (simulated) three dimensions immediately; for others, the activity of filling the pyramids helped make the connection. Further connections to proportional reasoning were supported by access to measurements provided in the GSP sketches.

Stage 2 Testing – Task Revisions

Despite the fact that a small number of participants recognized the link between the two subtasks, we decided in our revisions to reverse the order and place the 3-D model task first and the GSP dilation task second. While our intention is that subsequently learners will work back and forth between the two contexts, we contend that the hands-on investigation is better preparation for the GSP task, than the GSP task is for working with the 3-D models. Specifically, the change was motivated by the views that 1) children initially learn mathematical ideas from their 3-D experience of the world and only later are schooled into thinking in 2-D, and 2) elementary teachers often lack mathematical background and investigative skills, and we found that these deficiencies were more easily addressed during the hands-on investigations than when participants were working with the software. These considerations are analogous to interface considerations described by Gadanidis, et al. (2004) who contend that the arrangement of ideas intended to be communicated "mediates student engagement" (p.275).

Another change was to introduce a "guide" to the GSP dilation activity. Each screen was named, e.g., Explore Transform, Scale Factors, and these titles, with a short description of that section of the activity and several prompts were provided as a one page addendum to the activity. Though instructions were still embedded in the sketches, we developed the guide to provide a "big picture" and reveal the structure of the task. Instructions for the GSP task underwent considerable revising and we hoped that future task drafts would be more successful at directing participants towards the intended learning goals. Though not strictly part of the task draft, the revision included development of an introductory GSP session, which included opportunities to use and explore items in the construct, transform, and measure menus.

Stage 3 Testing – A Revised Task Draft

As indicated, the 3-D model task preceded the GSP task in this stage. A selection of written prompts for the 3-D model portion of this task draft are included in Appendix B, while snap shots of some of the GSP screens appear in Appendix C. Starting with an exploration of the 3-D model encouraged participants to take a 'playful' approach to the task – they explored creatively and even initiated looking at extreme cases (e.g., by using very large angles for tilting). Although use of a model can reaffirm the belief that mathematics should make sense, physical constraints can be problematic when one is aiming to generalize. Our response at first was to improve our materials, but then during subsequent task drafts we reasoned: Noticing mathematically significant details requires ignoring imperfections - thinking about the perfect tetrahedron, or the imagined line of symmetry of an isosceles triangle in the water surface. A key aspect of visual interpretation is that we ignore almost all features of our current environment, and focus on the few features that we anticipate 'matter'. This implies that an important strategy for working towards generalizing is acknowledging imperfections in physical objects and explicitly identifying the properties that one wants to attend to across cases, and ignoring most of the 'mathematically irrelevant' bits. In this testing stage, our participants learned what features were distracting and which were relevant to their reasoning.

Participants in this stage were also more confident using GSP than in the previous testing stages, which we attribute both to the introductory GSP session and to the atmosphere of exploration stimulated by the 3-D investigation. As in earlier sessions, the link between the GSP and the 3-D model tasks was noticed by participants. One example was quite striking - a participant looked at the very first sketch – two dilated triangles and a point labelled "center of dilation" – and said to her partner, "Do you not see lines here connecting these here and here?" pointing to the center point and corresponding vertices in triangles and actually connecting them with her ruler held in front of the screen. She was referring to lines which were not visible on the screen, yet she was "110% sure it's a pyramid." Overall, we found that participants made connections between the two sub-tasks more explicitly; we attribute this to the revisions, which provided clearer opportunities to notice the relationships. However we continue to see significant difficulty with the 'scaling of area' and are developing modifications to build up these links.

Through this lengthy (and still unfinished) process we have problematized a spatial visual task designed to help teachers broaden their understanding of dilation, ratio and proportion. Variations of the task have been tested and discussed with respect to their effects on learning. The task in its present form represents numerous decisions around materials, wording of questions and prompts, pedagogical structure, and sequencing of experiences.

Concluding Remarks

We designed *The Filling Task* to foster connections between 2-D and 3-D representations, to support proportional reasoning in geometric contexts with spatial visual skills, to give context and experience with cross-sections of objects, to support development of abilities in spatial investigations, and to provide a rich network of connections and representations that could act as a launching pad for extended explorations. The analysis of ratio and area concepts and measurements, and the idea of dilation as a transformation with a center in the GSP task can be seen as foundational for a rich exploration of the filling problem. Alternatively, the filling experience in the 3-D model task can be seen as a preparatory, kinesthetic experience to the more visual, symbolic, and numerical work in the GSP task.

Through our process of testing and refining task drafts, our research sheds light on the combined use of models and technology to foster connections and mathematical understanding. Due to space limitations, this paper offers a mere glimpse at the cyclical process of task design, but it highlights what we consider are some important pedagogical considerations. In particular, our task illustrates how engagement with dynamic software can complement a 3-D exploration, for instance, by drawing explicit visual-numerical connections and by offering tools that allow the learner to 'sit back and watch' a 2-D realisation of their 3-D activity.

An important goal of ours was to design a task through which participants may develop a flexible imagery, in which changes can be explored mentally, and other variations not seen externally can be imagined. We continue to modify the task towards this end. We hope that our analysis of multiple 'task drafts' will inform the process of task problematization in mathematics education research and teaching.

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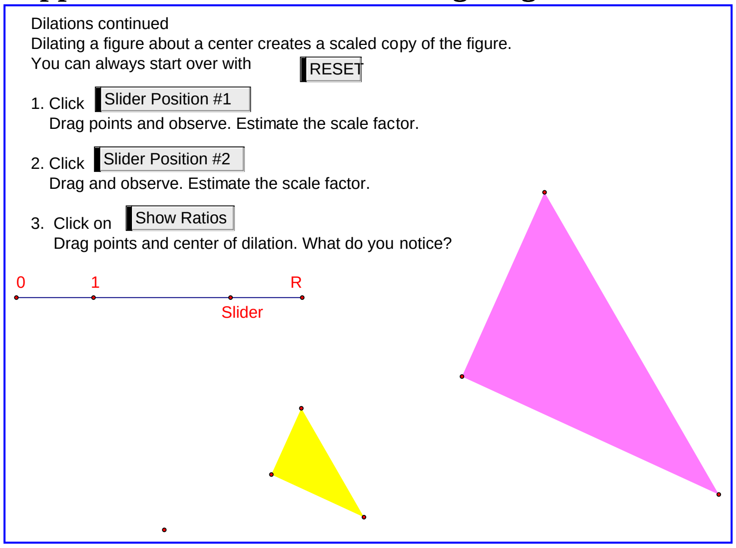
Appendix A – GSP Task, Testing Stage 2 - Screen 1



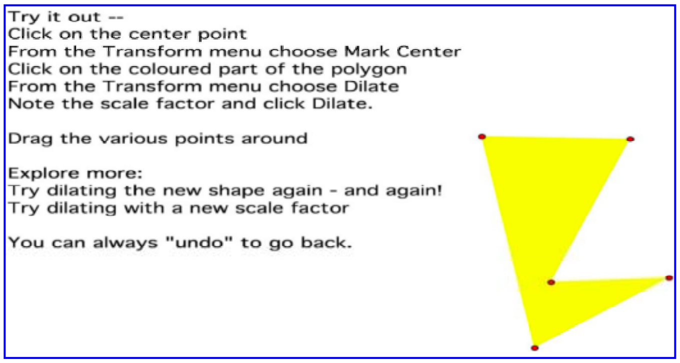
Appendix B – The Filling Task 3-D Model, Testing Stage 3

Selection of written prompts provided as a hand-out.

1. Using the materials provided, embed the bottom vertex of a pyramid in play dough so that the top is parallel to the table.
    a. Pour in a small amount of water – record as best you can the shape of the surface of the water **from the top** (e.g. with a picture, verbal description…).
    b. Pour in an additional amount – record the shape.
    c. Pour again – record.
    d. Compare the shapes you have drawn as to their **lengths of sides, surface area,** and **volume.**
2. Repeat #1 for the same pyramid tilted at different angles.
3. Describe in your own words what was happening to each of the dimensions of the water's shape as (i) more water was added, and (ii) some water was poured out.
4. Reflect on steps 1 and 2 above.
    a. In what ways, if any, did the shape of the surface area change when you tilted the pyramid?
    b. In what ways did the shapes' measurements change?
5. Do the observations you made hold true for new shapes? Why or why not?

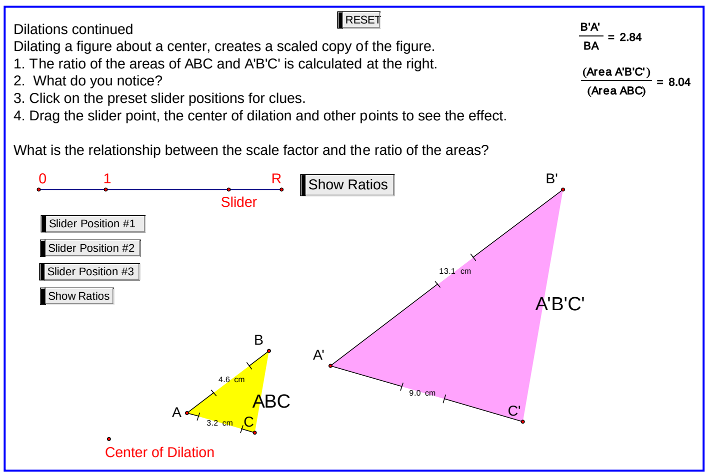
Appendix C – The Filling Task GSP, Testing Stage 3

Screen 2



Prompt: This sketch was made with the dilation tool of the transform menu. See how it works on a general shape. Could this be done with any shape? How do you know?

Screen 6



Prompt: We have looked at how various lengths scale. How does the area scale? Please explain. Can you predict a formula for how the area scales when the sides scale factor of the sides is r? Please explain the reasoning behind your prediction.

A MIXED METHODS INVESTIGATION IDENTIFYING CHARACTERISTICS OF INFLUENTIAL FACILITATORS OF ELEMENTARY MATHEMATICS PROFESSIONAL DEVELOPMENT

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This paper provides results from a research study that adds to the literature on professional development in elementary mathematics by focusing on the role of the facilitator in influencing teachers to be engaged during professional development experiences. An exploratory sequential mixed methods design was utilized to answer the central research question: How do United States elementary school teachers perceive an influential facilitator of elementary mathematics professional development (EMPD)? Phase one of this study explored teacher perceptions through a dual-phenomenological design, which informed the second phase of the study, the implementation of a survey to elementary school mathematics teachers on a larger scale.

Introduction

When examining the research related to mathematics professional development, there is a lack of focus in the literature on the facilitator's role during professional development experiences. A facilitator of professional development represents a movement away from the traditional, or additive, approach to professional development where trainers transmit or tell information and teachers are passive recipients. A facilitator of professional development acts as a guide for teachers as they develop new knowledge through a variety of experiences. This paper describes results from a study analyzing teacher perceptions of influential facilitators of elementary mathematics professional development (EMPD). For the purposes of this study, professional development was defined as any teacher learning, district mandated or not, that is dedicated to improving teacher quality. Because this study focused on elementary mathematics teachers, Simon's (2000) call for a change in knowledge and beliefs to be aligned with reform efforts was used as criteria for determining improvement in teacher quality. This definition of professional development excluded experiences that are additive in nature (Smith, 2001). When determining whether a facilitator is influential, in essence, we are examining the characteristics of the facilitator that motivate teachers and enable them to learn from the professional development experience. If teachers are unmotivated during professional development, they are unlikely to gain anything from the experience. An exploratory sequential mixed methods design was utilized to answer the central research question: How do United States elementary school teachers perceive an influential facilitator of EMPD?

Literature Review

Visions and goals for mathematics instruction call for a shift in beliefs about the nature of teaching and learning mathematics (Wilkins, 2008; Cady, Meier, & Lubinski, 2006; NCTM, 2000) along with a need to increase teachers' level of pedagogical content knowledge in mathematics (Hill, Ball, & Schilling, 2008). This transformation in mathematics instruction can be described as a movement away from instructional practices focusing on the transmittal of rules and procedures and towards instructional practices that allow students to construct meaning and understanding about mathematics as a dynamic system of concepts (Romberg, Carpenter, &

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Dremock, 2005). For these changes to occur, teachers require extensive professional development experiences designed to critically examine current instructional practices and provide them with the tools to implement reform practices in mathematics instruction (Loucks-Horsely, Hewson, Love, Mundry, & Stiles, 2003).

Current literature in mathematics education identifies characteristics that are necessary for high-quality mathematics professional development to occur. An effective professional development model is ongoing and situated in practice, focused on mathematical content, has student learning as the ultimate goal, and leads toward the development of a community of learners (Desimone, Smith, & Ueno, 2006; Weiss & Pasley, 2006; Guskey, 2003; Smith, 2001; NRC, 2001). Professional development should be transformative in nature, meaning that it is designed to alter or improve existing instructional practices. This type of professional development in mathematics differs from additive methods where teachers are given activities or tools to add to their existing practice (Smith, 2001). While the literature relating to effective practices in mathematics professional development is extensive, there is little focus on the role of the facilitator during these experiences. Just as a well designed mathematics curriculum can be negatively affected by poor instruction (Ball & Cohen, 1996), a well designed professional development model can be negatively affected by poor facilitation (Weiss & Pasley, 2006; Gibson, 2005). In a review of 133 peer-reviewed sources related to professional development, only 4% of the sources focused specifically on the facilitator’s role in mathematics professional development. Of this 4%, very few articles were identified as empirical research examining the role of the facilitator in elementary mathematics professional development (Sztajn, Hackenberg, White, & Alexsaht-Snider, 2007; Weiss & Pasley, 2006).

The theoretical framework guiding this study was grounded in motivation theory and adult learning theory. It is imperative that teachers be motivated during professional development in order to learn from the experience. The facilitator can play a key role in developing this motivation. This study primarily centers on Bandura’s social cognitive theory (1986). In this theory, Bandura identified three constructs: (1) environment, (2) self, and (3) behaviors; which act in a symbiotic manner to influence motivation. Embedded in this theory is the notion of self-regulation, or peoples’ ability to control learning through the combination of academic learning skills and self-control. Self-regulation focuses on the need for a person to recognize their own ability to learn, however, this recognition is dependent upon perceptions of the world around them. The rationale for examining teacher perceptions in this study is two-fold. First is the underlying principle set forth by Husserl and then refined into method by Moustakas (1994) that truths can be derived from the examination of perceived truths to identify common themes. Second, according to Bandura’s social cognitive theory (1986), perceptions play a key role in influencing motivation. A number of studies have shown how perceptions influence motivation (Watt & Richardson, 2007; Groth & Bergner, 2007).

Pellicer and Anderson (1995) identify the need for facilitators of professional development to build experiences around the tenets of andragogy as a means of motivating teachers to be engaged. Malcolm Knowles first introduced the theory of andragogy, or the study of how adults learn, as compared to pedagogy, the study of how children learn (Knowles, Holton, & Swanson, 1998; Knowles, 1994). In this theory, Knowles stipulates that certain conditions are necessary for adults to thrive in a learning environment. These conditions include the need for adults to be actively involved in learning tasks; the need for learning to be relevant to past experiences; and the need for self-direction (Brown, 2006; Merriam, 2001). According to this theory, adults prefer professional development experiences where they are able to give and receive feedback and can actively test ideas rather than passively receive information. Adults are also self-directive in that they are present, rather than future orientated. This point is crucial for professional development experiences because teachers have to believe that what they are learning is useful to their current practices (Merriam, 2001). Theoretically, facilitators who utilize motivation theory and andragogy when implementing professional development experiences are more likely to have teachers learn from the experience (Pellicer & Anderson, 1995). However, a lack of empirical evidence exists determining if these principles alone will ensure teacher motivation in elementary mathematics professional development (EMPD).

Methodology

The central research question guiding the focus of this study was: How do United States elementary school teachers perceive an influential facilitator of elementary mathematics professional development (EMPD)? This question was examined using a mixed method methodology with an exploratory sequential design (Creswell & Plano Clark, 2007). In this study, an exploratory design was necessary due to a lack of empirical work identifying qualities of influential facilitators of professional development in elementary mathematics. Phase one of this design examined how South Carolina teachers describe influential facilitators of professional development. This question was examined qualitatively through a phenomenological design (Moustakas, 1994) where data was gathered through semi-structured interviews. In this first phase, two subgroups of teachers were interviewed to determine their perceptions of influential facilitators of professional development. While the results of phase one provided extensive data on how teachers in certain areas of South Carolina perceive influential facilitators of EMPD, it was unclear if these results would generalize to a larger sample of teachers in South Carolina or across the United States. It was necessary to use the results from the first phase of this study to inform the second phase of this study, which examined teacher perceptions of influential facilitators of EMPD across the United States. In this second phase, the results from phase one were mixed to develop a survey instrument that was administered to a larger sample of teachers to determine if their perceptions of influential facilitators were similar to those identified in phase one.

Sampling Techniques

The participants in this study were elementary school teachers who had experience with transformative mathematics professional development. These participants included kindergarten through fifth grade teachers who teach all content areas, those who only teach specialized areas such as math and science, and special education teachers who work in either inclusive or self-contained settings. Phase one contained two subgroups of teachers, one group who had experienced transformative professional development that is ongoing (labeled Group I) and the other group who had experienced transformative professional development that is isolated into a one or two day session (labeled Group 2). Each subgroup was comprised of 10 participants. While more teachers have experienced isolated professional development that is transformative in nature, very few have experienced a model that is ongoing (NCES, 2006). Ball (2002) recognized the need to examine isolated professional development experiences that are devoted to improving teacher quality in addition to the ongoing model recommended by researchers in professional development. For the purposes of this study, it was necessary to complete a dual phenomenology examining teacher experiences of professional development models that were ongoing in addition to those that were isolated to determine what teachers perceive as common characteristics of influential facilitators.

Phase two participants were identified through convenience and snowball sampling techniques. These potential participants were contacted through gatekeepers from a variety of areas in education, including state agencies, district personnel, curriculum publishing companies, university professors, and state and national educational organizations. A total of 49 gatekeepers participated in this phase of the study. The researcher provided each gatekeeper with access to the survey instrument, who then distributed it to participants. A total of 652 participants responded to the electronic survey. However, this sample was narrowed down to 565 participants after a cursory examination of the responses. Due to the nature of electronic surveys, many potential participants began the survey and did not complete it for various reasons, including not meeting the criteria necessary to be a participant or clicking on the electronic link without intending to complete the survey. These potential participants were omitted from the sample used during data analysis. Examples of sampling issues relating to electronic surveys are not uncommon and have been documented in the literature relating to survey research (Franklin, 2008; Davidson, 2008; Dillman, Smyth, & Christian, 2008).

Data Collection and Analysis

The researcher conducted 20 semi structured interviews with the participants identified for phase one. Each interview was approximately 45 minutes to an hour in length. The data from each subgroup in phase one was examined separately in a primary analysis by isolating and extracting significant statements from interview transcripts. These statements were then used to create meaning units, which were clustered into common themes (Moustakas, 1994). Following this analysis, a secondary review identified common themes between groups.

Mixing occurred between phases to develop the survey instrument used in phase two. Qualitative data that had been organized by theme prior to data reduction were quantitized to determine how often each idea occurred. A total of 116 items were developed based on the themes that emerged in the qualitative phase of the study. These items were reviewed and reduced by two experts in the field of instrument development. Based on their recommendations, the items were reduced to a total of 65. The resulting survey was piloted to 35 elementary school teachers in South Carolina prior to the phase two implementation.

The final survey was distributed electronically to 49 gatekeepers in areas across the United States who, in turn, distributed it to elementary school teachers. The reliability coefficient was computed to establish the internal consistency of the survey instrument. Following this examination, construct validity and the dimensionality of items were analyzed using principle components factor analysis (PCA) (Thorndike, 2005; Ott & Longnecker, 2001). Once factors were identified and labeled, the researcher conducted a series of *t* tests to determine if participants in phase two agreed with the findings from phase one.

Results

Five themes emerged in phase one that outlined the essence of how teachers perceive an influential facilitator of EMPD: (1) Credibility, or qualities which allow participants to feel confident that the facilitator is qualified and capable to conduct professional development, (2) Support, or assisting and reacting to participants in a way that develops trust, (3) Motivation, or the facilitator's rationale for conducting professional development, (4) Management, or the way the facilitator presents activities or tasks, and (5) Personality, or the demeanor of the facilitator.

Participants across samples identified a barrier or wall that exists between the facilitator and participants, mostly due to the nature of professional development as being something that teachers are required to attend. If a facilitator wants to influence participants to alter instructional practices, this barrier must first be addressed. The five themes emerging from the data provide a framework for facilitators to understand why this barrier exists. In order for the barrier between teachers and facilitators to deteriorate, it is necessary for the facilitator to possess the qualities inherent in all of these themes.

The reliability of the scale in phase two was .941. Principle components factor analysis (PCA) was conducted for data reduction and to examine the dimensionality of the items. A factor extraction yielded six factors with eigenvalues greater than one. The Kaiser-Meyer-Oiken measure of sampling adequacy yielded a high level of compactness (.945), further, Bartlett's test yielded significant results ($p < .001$) therefore, factor analysis was appropriate for these data. Factor rotation was determined through a comparison of the scree plot and the amount of variance explained by each factor. Based on these results, five factors were rotated using a Varimax rotation procedure. Each of the five factors yielded an interpretable factor solution. Factor 1, labeled Support, accounted for 36.42% of item variance and was defined by 16 of the scale items. Factor 2, labeled Personality, accounted for 6.89% of item variance and was defined by six of the scale items. Factor 3, labeled Management, was defined by seven of the scale items and accounted for 5.15% of item variance. Factor 4, labeled Knowledge, accounted for 4.05% of the variance and was defined by six items. Factor 5, labeled Connections, was defined by five items and accounted for 3.6% of the variance. Factors 4 and 5 were comprised of items that related to the theoretical construct of Credibility. The five rotated factors accounted for a total of 56.10% of the variance. These results indicate a relatively high level of reliability and validity in the survey instrument.

Five one-sample t tests were conducted on the survey responses to determine the extent to which participants agreed or disagreed that the items were characteristics of influential facilitators. The five factors identified through PCA were used in this analysis. Each had a scale of 1 (Completely Agree) to 8 (Completely Disagree). The test value was the midpoint of a summative score calculated for the items clustered under each factor. The alpha was set at .05. All of the t tests yielded significant results ($p<.001$) with large effect sizes. Based on these results, it is evident that participants in phase two agree with the findings in phase one to a significant extent. Therefore, the findings from phase one were generalizable to a larger sample of teachers in phase two of the study.

Conclusions

Although extensive research has been conducted on professional development in mathematics, a review of the literature revealed a lack of research related to the role of the facilitator in EMPD. The current study was conducted in response to this gap in the research literature. Five themes emerged from the analysis of the phase one sample: (1) Credibility, (2) Support, (3) Motivation, (4) Management, and (5) Personality. The characteristics clustered under each of these themes were identified as influential by teachers across all samples. The characteristics related to Credibility included the need for the facilitator to have both classroom experience at the same grade level as participants and experience with the topic they are discussing, they must have knowledge of both content and pedagogy, they must provide specific data supported the practices they are promoting, and they must act in a professional manner during sessions. The characteristics clustered under Support included the need for a facilitator to build trust by providing extra help before, during, and following professional development experiences. The facilitator must also react to participants in a positive way during professional development by answering questions and treating them as equals. Characteristics related to Motivation pertained to the facilitator's rationale for conducting professional development. Facilitators should want to do their job so they can help teachers and students. They should believe in the methods they are promoting. Management characteristics included the way a facilitator moves around the room and engages with participants. Facilitators should hold teachers accountable during professional development to demonstrate that everyone can provide relevant information. The facilitator should also be organized in their management of materials and their methods of disseminating information. Finally, characteristics relating to Personality included the need for facilitators to connect with participants by having an easygoing demeanor and a sense of humor. They should be approachable and display a sense of confidence without seeming arrogant.

This need to possess characteristics across all themes in order to be influential holds implications for research relating to motivation theory. Research has been conducted on the roles people play in influencing motivation (Vogt, Hoecevar, & Hagedoren, 2007; Watt & Richardson, 2007; Groth & Bergner, 2007). Role models have been shown to increase motivation through such qualities as perceived competence, perceived prestige, or perceived connections (Bandura, 1986). Sztajn and colleagues (2007) identified the importance of perceived trust between teachers and facilitators of ongoing professional development. In the current study, a multiplicative effect that can be termed "perceived influence" was found to potentially increase motivation. It is important to note that each of the themes described in these findings must be present for a facilitator of EMPD to be considered influential by participants. If a facilitator has perceived credibility but does not exude perceived support, then that facilitator will not be influential for these samples of K-5 teachers. Future investigations of this multiplicative effect are necessary to determine if facilitators who are perceived as influential increase teacher motivation during professional development and to determine the implications that increased motivation holds for instructional practice.

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DEVELOPING LOCAL THEORIES OF INSTRUCTION AND THE IMPACT ON TEACHING CHALLENGING CURRICULA

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This study focuses on a form of teacher's pedagogical content knowledge instrumental to enacting challenging tasks in ways that elicit and refine students' mathematical reasoning. The teacher, chosen for how she consistently engaged student reasoning while enacting challenging tasks, showed evidence of possessing a local theory in that she articulated the ways student thinking developed over time, the processes by which that thinking developed, and the resources that facilitated the development of student thinking. Her knowledge informed how she revised and enacted challenging tasks in ways that elicited and progressively refined and formalized student thinking around integer addition and subtraction.

Introduction

In the U.S., a number of curricula have been developed that depart from conventional notions of curricula and that build from decades of research in the cognitive sciences of how children learn mathematics (Schoenfeld, 2006). Although these curricula were intended to support teachers' efforts to engage students with challenging tasks, there has been little evidence that simply introducing these curricula changes teachers' practices (cf. Collopy, 2003; Remillard & Bryans, 2004). Consequently, it is likely that teachers need to develop new forms of knowledge in order to enact the tasks in the materials in ways that elicit and build from students' mathematical reasoning.

There has been much discussion about the kinds of knowledge that impacts teachers' ability to engage children in ways that help them learn with understanding, with a particular focus on the role of *pedagogical content knowledge* (PCK) (e.g. Grossman, 1990; Shulman, 1986), which Gess-Newsome (1999) describes as "the transformation of subject matter, pedagogical, and contextual knowledge into a unique form – the *only* form of knowledge that impacts teaching practice" (p. 10, author's italics). More recently, Hill, Ball, and Schilling (2008) have attempted to refine notions of PCK into various components, including situating PCK in particular mathematical domains. Thus, the most relevant knowledge is conceived as *transformed* while teaching and has some explanatory power with respect to how students learn in particular domains. The research on teacher knowledge, however, has primarily shown an association between forms of PCK and student learning, with a need to further elaborate how PCK functions to help teachers organize instruction to elicit and refine student reasoning through the use of challenging tasks.

I explore teacher knowledge in terms of a *local theory of instruction* as a form of PCK situated within a particular instructional sequence. A local theory of instruction is an empirically-tested set of conjectures about how students learn in a specific mathematical domain, or what diSessa and Cobb (2004) call *domain specific learning processes*, with respect to a particular instructional sequence. The testable conjectures include both the processes by which students learn in a particular mathematical domain *and* the resources that influence or support those processes.

This study focuses on one teacher who consistently elicited and organized instruction around students' reasoning. The goal of the study was to explore the nature of the teacher's knowledge and how it helped her to enact tasks from challenging curriculum materials. The research questions that guided this study were:

1. What knowledge was evident in the teacher's practices and reflection on those practices, particularly with respect to her understanding of how student thinking developed across an instructional sequence related to integer addition and subtraction?
2. How did the teacher's knowledge help her to use challenging tasks to elicit and build from students' mathematical reasoning?

Methods

Video data were collected and analyzed and used to stimulate the teacher's reflection on her enactments of an instructional sequence based on integer addition and subtraction. The teacher, Ms. Graham, was selected because she had the highest rating of the ten teachers in a broader study in an analysis of classroom discourse in terms of the extent to which she engaged students' in conceptual discourse around challenging tasks. Ms. Graham was using materials from the Connected Mathematics Project (CMP) curriculum (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006), which was designed as a problem-based curriculum using funds from the U.S. National Science Foundation. The CMP materials differ from more conventional texts in that they are comprised largely of tasks that are designed to be implemented at a high level of cognitive demand.

Description of Phases in the Instructional Sequence

The analysis of the sequence focused on eight tasks divided into three phases that roughly constitute a progression from informal to informal and that focused on the early development of resources that formed the basis of subsequent activity. In the first part of the sequence, the tasks constituted a phase in which students' reasoning based on prior experiences with integers was elicited. In the next part of the sequence, there was a focus on developing representations and models that facilitated the development of student strategies in subsequent tasks. In the third phase, students focused on patterns in the operations they had carried out earlier in the sequence to develop conjectures and ultimately strategies for integer addition and subtraction.

Results

Eliciting Students' Informal Reasoning

Ms. Graham used the tasks early in the sequence to elicit students' reasoning around integers, in part to begin the process of collectively discussing ideas and to build from students' prior experiences with integers. She stated:

> *These opening discussions are to help students tap into what they already know… to help them draw upon that as a basis for a foundation… what I want them to do in those beginning discussions is make connections … to either problems that they've done before or contexts* (Video-Stimulated Interview [VSI], August 28, 2009).

In several classroom episodes from the first tasks in the sequence, Ms. Graham engaged students in several free-ranging discussions, in which she repeatedly asked students to express their reasoning and to comment on other students' reasoning. These discussions were intended to be informal and to provide opportunities for students to make connections to their prior knowledge and to begin more formal consideration of the properties of integers. She did not attempt to formalize the ideas raised by the students, and instead emphasized the sense-making aspects of the early tasks, allowing students to state claims without directly evaluating them.

Developing Models and Representations

Ms. Graham described the role of models and representations in developing student reasoning beyond the informal phase. She described how these models and representations served as resources in terms of coordinating the discussion between number sentences and operations and in terms of establishing the basis of claims made about integer addition and subtraction. For example, early in the sequence Ms. Graham focused on developing number sentences that could be subsequently used to make conjectures about the relationship between integer addition and subtraction. The goal, as Ms. Graham stated, was to have students “write multiple number sentences that represent the same situation so that we can draw on those examples later when they’re trying to make their generalizations about how the operations work” (VSI, August 28, 2009). She also discussed the role of the number line and chip board models in allowing students to develop and justify conjectures, and the need for students to have informal experiences with the models. She stated for example that it was important to get students “to talk about the number line and make sense of it and mess around with it, (e.g.) where is big or where is small or what’s happening over here with these negative numbers” and emphasized the number line early because “I knew the number line was going to be used as a model later in terms of adding or subtracting” (VSI, June 4, 2008). Several episodes from the fourth (of seven) lessons showed how Ms. Graham focused students on the actions represented by number sentences and the connections between the number sentences, the number line, and the operations used to arrive at an answer.

Generating and Refining Conjectures

Ms. Graham used tasks in the latter part of the sequence to help students reflect on and refine the conjectures they had stated throughout the sequence. For example, she had the students record their conjectures, from which she selected a sample to collectively pose to the students to discuss their accuracy. In her class, the students spent 60 minutes (February 4, 2009) reflecting on the conjectures around subtraction. She stated that she emphasized students’ conjectures because the early part of the sequence did “not formally or explicitly draw kids’ attention to what they were actually doing and help them to draw generalizations” (Joint Interview [JI], March 25, 2010). The goal of the emphasis on student conjectures was to “support [the students] towards writing …about what they’re noticing” (JI, March 25, 2010), as a precursor to developing efficient yet comprehensive strategies. In the latter part of the sequence, Ms. Graham continued to focus on helping students generate efficient yet sensible strategies for adding and subtracting integers. Several episodes in the latter part of the instructional sequence showed how Ms. Graham explicitly engaged students in the processes of refining their thinking and language with respect to integer operations and how she continually relied on the resources developed throughout the unit, such as the number line and chip board models.

Discussion

Ms. Graham showed evidence of having a local theory of instruction and that, furthermore, this local theory helped her to orchestrate student engagement from challenging tasks in ways that elicited and refined their reasoning around integer addition and subtraction. First, she described a broad learning trajectory in terms of the sequence of tasks and how the sequence provided students opportunities to connect to their prior experiences, to develop the tools and experiences to subsequently make sense of integer addition and subtraction, to generate and refine conjectures, and ultimately to develop sensible and efficient strategies. Second, Ms. Graham noted a variety of resources that were influential in helping students to make sense of integer addition and subtraction. Notably, she emphasized the use of the chip board and number line models throughout the sequence, initially to help students to develop an understanding of integer operations and then to use as tools to describe and justify their emerging conjectures. Third, Ms. Graham described the processes by which students developed an understanding of integer addition and subtraction. She described the need to 'mess around' with models and with integer operations, alluding to the necessity of engaging in pre-formal experiences that would serve as a 'foundation' for subsequent work in the sequence.

Ms. Graham's knowledge of the instructional sequence informed how she enacted the sequence. She adapted tasks over the years to better establish and utilize the resources in the instructional sequence and to provide greater opportunities for students to build from their own reasoning. For example, she adapted a task in which students were to notice and develop conjectures around patterns of grouped addition or subtraction problems. The written task separated the problems by whether the operands had like or unlike signs. She revised the task by ungrouping the problems in order to focus students on the operations used to solve the number sentences rather than on the more superficial characteristic of whether the operands had different signs. She felt that this was a more productive focus in light of the development of algorithms that followed and in light of the variety of conjectures the students had developed previously. This adaptation helped students to observe a variety of possible ways to group the problems, each way related to some characteristic of integer operations, including but not limited to differences in signs. Furthermore, this adaptation showed evidence of how Ms. Graham revised the instructional sequence based on her understanding of how tasks influenced the development of student thinking across the sequence.

Implications

The development of Ms. Graham's local theory co-emerged with her close observations of student engagement with challenging tasks and her revisions of those tasks that helped her to elicit and refine students' mathematical reasoning. The local theory, as it developed, guided her discussions of when to formalize students' ideas and when to let them 'mess around' with contexts and models. Her theory helped her to recognize key ideas in students' contributions and to focus discussion on the ideas, while the theory's focus on progressive formalization allowed her to hold off establishing formal terminology and procedures until the end of the sequence. These practices related to her theory provided students' opportunity to engage with challenging tasks in ways that explicitly elicited and developed their mathematical reasoning. Finally, she developed her local theory by observing how students engaged with tasks from the materials over multiple enactments, suggesting that such knowledge takes years to develop and requires close attention to student thinking.

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DISCUSSION OF LEARNING GOALS AND STUDENT DEVELOPENT DURING A COLLECTIVELY PLANNED DIVISION LESSON

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This study examines how teachers talk and think about student development and how lesson goals are determined during a joint planning session. Lesson study is utilized in this research to highlight the considerations teachers make regarding cognitive, language, and social development of students. Likewise, how teachers collectively plan a lesson to advance student development is analyzed in relationship to learning goals for the lesson.

Introduction

Lesson planning is often an individualized activity (Bage, Grosvenor, & Williams, 1999); however, lesson study takes this individualized activity to a group (Fernandez, 2002) where minds collectively make an agenda for action (Lampert, 2001). Lesson study is a joint decision making process utilized for professional development, with the purpose of improving the practice of teaching. The process is used in mathematics education to improve teaching through goal setting and collective decision making. In the process of lesson study, one lesson is planned and analyzed over an extended period of time, typically spanning three to four weeks. During this process the group collectively plans one lesson and one teacher in the group teaches the lesson, while the others observe and take field notes (Fernandez, 2002). Finally, the group meets back together to discuss the observation, reflect on the lesson, and make changes if necessary.

Theoretical Perspectives

The lesson planning process is an important phase in the teaching cycle, during which learning goals should be made explicit (Hiebert & Grouws, 2007). When teachers are explicit and design goals to promote the understanding of concepts, students are able to benefit by developing mathematical skills. These learning goals should be determined based on students' needs. Therefore, in the lesson study process, teachers should set mathematical goals for students and be explicit about their desired learning outcomes for students.

When determining learning goals, the developmental levels of the students should be at the forefront of teacher thinking. Developmentalaf levels refer to a multidimensional pathway of interactions involving "physical, social, emotional, cognitive, and linguistic" elements in which students progress toward goals (Horowitz, et al., 2005, p. 94). These levels lead the teacher to make a hypothetical learning trajectory for students, which allows students to progress developmentally along the trajectory (Simon, 1995).

One form of development teachers should consider when planning lessons is cognitive development. Cognition is defined as the knowledge a child possess at a given time (Horowitz et al., 2005), meaning cognitive development is the advancement of knowledge to progress toward goals and the obtainment of knowledge. To support cognitive development, teachers should know what individual students know and do not know in a particular content area. After consideration of the cognitive abilities of all students, teachers should select the appropriate materials, design the lesson format, and create a learning environment that promotes meaningful understanding and cognitive advancement.

In addition to considering the cognitive development of students, teachers should also consider the language development of their students (Horowitz et al., 2005). Content is taught through language, so students should be able to make connections in their learning between the language being use and the content (Bransford, Brown, & Cocking, 1999). When connections in language use are made, students are able to develop by advancing their abilities with language.

Finally, teachers should consider the social development of their students when planning mathematics lessons (Horowitz et al., 2005). The relationship students have with their peers directly impacts their learning, so fostering learning environments where students come together to work collaboratively promotes development. Rogoff (1990) explains that "children are already engaged in a social activity when they actively observe and participate with others" (p. 16). As a result, when children come together in a learning environment, the opportunity for social development exists; teachers should be cognizant of development when planning lessons.

This study examines how teachers consider these developmental aspects of learning, specifically when collaboratively planning through the process of lesson study, with the aim of setting achievable learning goals for students. The lesson study process allows for analysis of teacher thinking on an in-depth level. Therefore, this process highlights teacher considerations and makes teacher thinking explicit. Knowing what teachers consider allows for the determination of areas where teachers would beneficial from additional professional development support to improve teaching. As a result, this study will answer the following question:

> How do teachers jointly discuss learning goals along with cognitive, language, and social development when collectively planning an elementary division lesson?

Methodology

Data for the study comes from a larger three-year Math Science Partnership Project involving five school districts and twenty-two schools in a western state. The project was a collaboration among the state department of education, the school districts in five counties, a university, and a regional professional development program. The project engaged practicing elementary teachers in three years of professional development, focused on math and science content and pedagogy.

In the three years of the project, teachers involved were held accountable for maintaining an understanding of current research literature in the area of mathematics education. These teachers were responsible for reading current research articles and discussing best practices collaboratively. As a result, many of the teachers were focused on improving their instructional practices based on their new knowledge from the professional development sessions, readings, and activities.

Teachers were immersed in algebraic thinking and spent time solving their own mathematical problems. This practice allowed the teachers to gain an understanding of the problem-solving process from the students' perspectives. Pedagogically, teachers were given opportunities to collaborate and work on improving their teaching practices through multiple methods. Lesson study is one format that was used for professional development for the teachers.

When the data for this study was collected, teachers had completed two-and-one-half of the three years of professional development of the program. The teachers in the study varied in experience, age, and background. The participants consisted of one female first-grade teacher, one female second-grade teacher, one female third-grade teacher, one female fourth-grade teacher and one male sixth-grade teacher. Teaching experience ranged from less than three years to more than fifteen years, and all teachers worked at the same rural elementary school.

Data included a video of the lesson study session along with a video of the planned lesson being taught. A debriefing session after the lesson was also recorded on video. Data was analyzed using a grounded theory approach with the purpose of describing the thinking and considerations of teachers and adding to the theory on learning goals and student development (Corbin & Strauss, 2008). The purpose of utilizing grounded theory was to lead to a theoretical explanation about the thinking that occurred throughout the lesson study process.

Results

Results indicate that teachers considered cognitive, language, and social development distinctly when collectively planning, but also simultaneously considered language and cognitive development, and language and social development; this was all done while thinking about the lesson goals and the purpose of the lesson, along with lesson activities that would help students achieve these goals.

From the three developmental areas analyzed, cognitive development was discussed most frequently and in the greatest detail by the teachers in the group. The teachers discussed cognition and learning goals simultaneously, as they relate to student knowledge. The first one-and-a-half hours of the lesson study planning session were devoted to determining an appropriate student learning goal based on areas of mathematical difficulty. The group determined the content area of measurement was a continual struggle for student across grade levels; however, the group ultimately determined that students often struggle with problem solving when learning division. As a result, they decided to teach a third grade problem posing lesson, allowing students to create and solve division word problems.

After determining the lesson content, the teachers considered advancing student knowledge to help students develop conceptual understanding of division. To plan the specific lesson, the teachers relied on their knowledge from previous experiences and thought about student cognition by considering how students learn and the necessary prerequisite lessons and knowledge that would be needed. The teachers discussed the progression of taking the students from concrete to abstract, based on learning goals and desired outcomes. To do this, the teachers discussed activities that would promote development, based on the understanding of the students:

> Maria: You know that sub sandwich problem they had last year? Do you think third graders could do it? I bet they could, especially if they are doing all of these other activities. It is something like all of these kids go on a field trip and there are different versions and each version can feed a different number of students, so each group like gets this number of sandwiches. It is a set amount for each of the three groups. One group is five, one is seven, one is three, or something. What is the fraction that each one gets? That is not using whole numbers, but you can maybe have a problem that is similar using whole numbers. You know? And then asking if each student should get….which group would you rather be in? Did any of the groups have any remaining Skittles or something? You know, whatever your problem is and then they would have to work the problem to solve it. And they can actually pass things down or something. Would that work for them?
> Laura: (Nods head).
> Maria: And then they would have to add as maybe a group. I don't know.
> Jonie: So, are you having? I guess I misunderstood because I was visualizing having them pose a problem and create a story problem, but you are more ….
> Laura: That is the ultimate goal.

As the teachers planned the problem posing activity, they considered their previous experience with activities related to the concept of division and discussed transitioning students from reading problems to writing and solving problems.

The teachers went beyond writing out a set of activities for the lesson; they considered cognitive development by thinking about how the students would approach lesson tasks. The teachers worked together and utilized their past experience and knowledge of the students to determine areas where common misconceptions would arise as students conceptualized division.

> Greg: They try a lot and they get it mixed up with multiplication, so if you could somehow show one that is multiplication, what is the difference between. They would be looking at like a Venn diagram or something.
> Tracy: What would be another thing that kids might do when they would do a division problem?
> Laura: They could start with what would be a factor instead of a product.
> Tracy: So that might be another misconception.
> Fatima: What is that?
> Laura: They might start with the factor instead of the product.
> Laura: I am thinking of the unknowns.
> Fatima: They might not know which numbers to put in which place in the problem.
> Laura: Exactly.

This discussion demonstrates the ability of the teachers to think about learning from the child's perspective. The group considered cognition in greater detail by considering how they would address misconceptions during the lesson.

> Tracy: Do they understand what a division situation is? Do they understand? Do you want to use a story context?
> Laura: But, I am still going to have misconceptions.
> Tracy: So the question is instead of having to teach every little group, how do you get them to a whole class discussion? Make the misconception explicit and then see if there is a conceptual change through discussion. Do you see what I am talking about?
> Maria: I know exactly what you are saying.
> Tracy: How do you?
> Maria: How do you stop the whole class when they are on their own reasoning process in their own small groups?
> Maria: Do you then at that time say, "Everyone, some of our friends are having, you know whatever, we want to give them some help here," and all of a sudden they're thinking about their own little problems and then do you stop the whole class at that point in the middle? Because then, you are like mid thought for a lot of kids when you are stopping to help someone else. You might see another group that is being very successful and you might be able to take that group and say you know I seem to think that you really understand how to write them well, maybe you can come over and we can work with this other group.

To address the misconceptions, the teachers thought about specific misconceptions students commonly have with division and made a plan to address the misconceptions and advance the understanding of students. This process highlighted teacher thinking about learning goals for students in relation to anticipated areas where students would have trouble meeting the objectives. Together, the teachers in the lesson study group jointly thought through the entire process of the lesson and made specific plans to ensure that students would understand the lesson content.

During the process of planning the mathematics lessons, the teachers repeatedly discussed language development as it relates to learning mathematics. They discussed student background knowledge with language and considered how to provide language support to ensure students would understand the language used when teaching division with problem posing.

The process of lesson study allowed the teachers to come together to discuss the background knowledge of students and ways they would support student learning of unfamiliar terms. Teachers set specific learning goals related to language for the students by jointly determining that students should verbally use mathematical terms in the lesson.

> Tonia: Yeah, that is going to be something you could introduce in a previous lesson? I would think, because that is a standard.
> Fatima: They need to know the vocabulary.
> Tonia: They need to know the vocabulary.
> Fatima: But if it has already been introduced is that when you explain your problem? I expect to hear you using those terms.

The teachers scaffolded the vocabulary in the lesson by deciding to incorporate specific terms into previous lessons and learning. They determined that if students were already familiar with specific terms they would expect students to use those terms in the lesson. In this way, the teachers thought about a logical progression of language development and incorporated support for students to allow them to develop until mastery, which would be demonstrated through the verbal use of newly learned terms in the lesson.

Additionally, the teachers discussed specific terms students would need to understand when learning division. Specifically, they discussed the terms divisor and dividend and made a plan to support language development for the students.

> Tonia: We do talk about it a lot, divisor and dividend.
> Tracy: I think if they get the idea, what is the meaning of divisor and dividend, they can describe, then maybe they can introduce the vocabulary.
> Greg: Up here we already said….that is the thing we brought up is that students confused it.
> Maria: Well, you need to have an interactive writing lesson prior to that where you have all labeled, like done a diagram of a division problem, you know the three ways of writing a division problem or something.
> Fatima: But I would say, when you write your problem, be sure you identify your divisor and dividend.

After determining that students would need support for the language in the lesson, the teachers further discussed student language development by considering ways they could support the development of terms for the students. The teachers decided they would utilize an interactive writing lesson utilizing the terms divisor and dividend in the week prior to the lesson, to scaffold the learning and support students in their language development.

The teachers went beyond determining how to support the acquisition of new language and discussed student understanding of word meaning in relation to conceptualization of the lesson goals. They thought about multiple meaning words students use, but may confuse.

> Maria: No, but I have had kids who have had, have had, there have been multiplication problems that have said *altogether*, how many….
> Tonia: Well, multiplication is addition.
> Maria: Right, well that is my, that is when you bring up, you say, “Well, that word is used for both of these problems because aren’t we repeatedly adding when we multiply?” And then, they are like, “Oh yeah, you know, okay,” so it works for both addition and subtraction.

In this situation, the teachers discussed the term *altogether* and came to a joint decision that students often neglect to fully consider the context and meaning of the word. Instead, students associate the term *altogether* with addition and do not think of multiplication as a process of repeated addition. By making this point explicit in their discussion, the group demonstrated their cognizance in considering students’ language use.

The teachers considered language by thinking about how to support development for students learning new terms and how to clarify meanings of words with multiple definitions. They jointly planned to address the language needs of the students by supporting students through scaffolding based on lessons that would be taught prior to the problem posing lesson.

Social development was also discussed and considered by the teachers in the lesson study process. The teachers thought about including activities that would promote social learning through the use of student groups and joint decision making. Additionally teachers planned to foster an environment where students would be engaged by working with other students.

The teachers considered the development of students on a social level by planning lesson elements that would require students to justify their thinking and reasoning with their peers. This would hold students accountable to their group members.

> Laura: My thought about this is that if they are given a bag and they have written the problem on big pieces of chart paper, then we come back whole class. They haven't solved it and nobody has solved the problem and everyone is analyzing the problem and the way it is written. So then, first of all, the group decides…is it a division problem?
> Tracy: (Nods) Mmm, hmm.
> Laura: And the group decides….is it clearly written? And the group follows a rubric possibly to evaluate this problem. Nobody has solved it. Not even the group that did it yet. Or maybe they have. I don’t know, but they are not showing. They are not solving the problem. They are modeling the problem.

The teachers planned to advance the social development of students by having them jointly make decisions about their mathematical thinking. Through this process, students would work with each other to evaluate a problem, given a rubric. This situation would hold all students accountable for learning and would foster a social context where all students would be involved through discussion.

In the lesson planning process, teachers determined student decision making as a group would be coupled with solving mathematical problems, which resulted in a simultaneous consideration of cognitive and social development. The teachers planned to create a context for social development, and within this context they planned to enhance cognitive development through the use of problem solving as a social activity.

> Maria: Because our last math program did that a lot. They would say, “Have students write a multiplication problem,” or, “Have students write division problems,” and then they would just write out a problem, but they didn’t even know if it was solvable. They don’t even know if maybe they worded it in a way that makes no sense, so I just thought that step with them as a group, having to solve their own problem prior to having to give it to another group of students and then to explain their thinking, (Tracy nods) their own reasoning with the problem solving process to you first. It is like I was saying.
> Tracy: So, you want them to explain it to you?
> Maria: They would, well, my thing is if they, they explain it to the whole class everyone is going to know how to solve the whole problem and it takes the fun out of being able to test their friends.

In this instance, the teachers considered the social interaction of students and planned for learning as a direct result of peer interaction. The planned process combined the consideration of social development and cognitive development because students would be interacting with their peers while discussing mathematical concepts and problems.

Additionally, the teachers combined the consideration of cognitive development and language development. This occurred as the group considered the relationship between specific words the students needed to know and the concepts and meanings associated with those words.

> Tracy: So, is this a partitive question or a quotitive? They don’t need to know the words, but I don’t know if that distinction is important for third grade, but you can base if off to look at dividing.
> Maria: What are you doing with the number?
> Tracy: One is putting it into groups and the other is seeing how many groups go together.
> Laura: Exactly, and that is what I did with that information. Do I now how many groups or do I know how big my groups are? Or, whatever.

The teachers discussed the terms partitive and quotitive and determined that third graders do not need to know the specific words, but students should have an understanding of the meanings behind the terms. By considering the relationship between the terms and the meanings, the teachers jointly considered language development and cognitive development simultaneously, as the terms relate to the problem posing lesson.

Discussion

Throughout the lesson study process, the teachers adhered to the recommendation by Horowitz et al. (2005) and routinely considered the cognitive, language, and social development of the learners. At times, the teachers considered components to development independently and at other times development in one area was considered in conjunction with elements of development in another area. Throughout the discussion, the consideration for the development of the students along with the learning goal was apparent.

When considering cognitive development, the teachers thought about ways to support students by providing scaffolding to advance cognition. They considered the background knowledge of students and planned lesson activities that would allow for attainment of the learning goal. Additionally, the teachers thought about mathematics content areas where students would need more or less support to understand the concepts being taught.

When discussing the language development of students, the teachers understood that many elements of a lesson are taught through language (Bransford, Darling-Hammond, & LePage, 2005), so they discussed how they would support the development of students in this area. The teachers jointly thought about key words, vocabulary, and ways to provide a context in which learning could take place through language. Once they had determined what the students would need to know, they thought about how they would scaffold the learning, so that students would understand the language at the time of the lesson.

Likewise, the teachers thought about ways to provide opportunities for students to work together and interact through discourse (Cobb, Boufi, McClain, & Whitenack, 1997). The teachers determined that small groups and whole-class discussions would take place, so that students could learn from working with each other. This situation would then foster a cooperative learning environment (Rogoff, 1990) where students could utilize the thinking and language of their peers to advance their social development.

This process of lesson study as a professional development tool demonstrates that teachers are making considerations about the development of their students as they jointly plan lessons. The group set goals, collectively planned the lesson, and worked with each other to refine the lesson in the planning process (Fernandez, 2002). In this way, the teachers took a commonly individualized activity of lesson planning (Bage, Grosvenor, & Williams, 1999) and extended it to a joint planning session that allowed for collaboration and decision making with the purpose of increasing student learning and development.

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ENHANCING LANGUAGE, ENHANCING LEARNING: AUGMENTING MATHEMATICS TEACHERS' CAPACITY IN THEIR LINGUISTICALLY DIVERSE CLASSROOMS

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Language is essential to mathematics learning and demonstrating understanding. This study reports on a year-long professional development program designed to enhance urban mathematics teachers' ability to address issues related to language and mathematics in their classrooms. Twenty-three teachers from one urban district participated in the program. We report on the math-related language strategies they developed as well as issues that arose as they made deliberate efforts to develop their pedagogy in this regard.

Introduction

Language is recognized as a vital mediator of learning (Vygotsky, 2002; Wertsch, 1991). Recommendations related to mathematics instruction for all students, including the growing population of English language learners (ELLs) (NCELA, 2006), prompt explicit attention to academic language, rich mathematical discourse, and justification, a language-intensive mathematics practice (Brenner, 1998; Moschkovich, 2002, 2007). To accomplish this goal, teachers must find ways to build off students' everyday language practices and scaffold the development of their students' mathematics register.

In his seminal work on mathematical language, Pimm (1987) notes the importance of increasing student awareness of different registers and the recognition that "the *grammar*, the *meanings*, and the *uses* of the same terms and expressions may vary within them and across them" (p. 109). Pimm follows this recommendation with the statement, "In my opinion, a major question yet to be explored by the mathematics education community is *how* the awareness may be fostered and cultivated" (p. 109). Aligned with Pimm's ideas, we contend that an important first step is to increase *teacher* awareness of language issues inherent in mathematics education. This awareness, and concomitant set of teaching skills, is critically important in schools with linguistically diverse student populations. Without this, teachers may inadvertently deny students' access to participation in the lesson (including meaning making of concepts) (Brenner, 1998) and teachers may not attend to language-related learning goals (Lucas, Villegas & Freedson-Gonzalez, 2008), which impact students' long-term success with mathematics (Echeverria, Vogt & Short, 2010) and short-term success on State tests.

Data on teachers and teaching suggests that such comprehensive skill sets are not widespread among teachers (NCES, 1999), and perhaps particularly mathematics teachers. Preservice teacher programs rarely address such issues of language in-depth (Valdés, Bunch, Snow, Lee & Matos, 2005), and when they do, it is not often with attention to the subject area. A similar situation may be present in inservice training as well. Indeed, the teachers in this study reported minimal to no prior experience with language-focused professional development linked to the teaching and learning of mathematics. Of 23 teachers, 9 reported having some PD related to academic language or ELLs, 9 reported having some PD related to the academic content of the institute (algebraic or proportional reasoning), and only 3 of 23 teachers reported having both.

This research documents the design and results of the ACCESS Project, a one-year professional development program for teachers of mathematics in one urban district. ACCESS stands for Academic Content and Communication Equals Student Success. ACCESS aimed to increase teachers' awareness of mathematical language and pedagogical repertoire for bridging everyday and mathematical language and to foster the development of students' academic language (here, the mathematics register) and promote their mathematical proficiency. In addition, we sought to develop in collaboration with teachers, a broader repertoire for working on these issues with students in mathematics classrooms.

In this paper, we document language-related mathematics instructional practices that the group of ACCESS teachers developed to foster their students' increased awareness of the mathematics register/academic language and support improved mathematical performance. We also document some aspects of the process and questions that arose for them as they made deliberate efforts to implement ideas from the one-week professional development program in their classrooms. This is the challenging work of teacher learning – the shift of "having knowledge" to "knowing in practice" (Kazemi & Hubbard, 2008).

Theoretical and Conceptual Frameworks

This research draws on diverse perspectives. Sociocultural theory (Vygotsky, 2002; Wertsch, 1991), with its contention that learning is mediated by language, along with theories of language (e.g., linguistics, sociolinguistics, and functional linguistics) (Halliday, 1987; Pimm, 1987; Schleppegrell, 2007) provide frameworks for focusing on classroom discourse. In particular, we focus on the mathematics register. Halliday (1978) described the mathematics register as "meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself)" (p. 195). He further states, "It is the meanings, including the styles of meaning and modes of argument, that constitute a register, rather than the words and structures as such" (p. 195).

To guide our work designing and implementing the ACCESS Program, we developed a conceptual framework comprising three principles or "pillars" that research has found to be critical for the development of students' mathematical proficiency. These pillars are: appropriate and effective development of students' academic language; student engagement in mathematical practices of justification and collective argumentation; and access by all students to rigorous mathematics (Truxaw, Staples & Ewart, 2009). Although only the second one focuses explicitly on language, the first and third pillars have an important language component as well.

Math Discourse Pillars

Students' academic language. As noted above, a key aspect of developing students' academic language is to create a bridge from everyday, informal language toward academic language and use of the mathematics register (Halliday, 1978; Pimm, 1987). This goal is pertinent for all students, but particularly so for students whose first language is not English (Cummins, 2000; Schleppegrell, 2007). Too often, attention to language in mathematics classrooms focuses merely on vocabulary. Rather, it should include attention to how language is used to express mathematical ideas (functional linguistics) and the development of the mathematics register (Pimm, 1987; Schleppegrell, 2007).

Justification and collective argumentation. Student participation in justification, meaning making, and argumentation have been implicated as critical components for supporting students in learning mathematics (e.g., Brenner, 1998; Hiebert et al., 1997; NRC, 2001; Wood, Williams

& McNeal, 2006). There is evidence that participation is particularly effective in supporting the learning of lower attaining students or ELLs (Boaler, & Staples, 2008; Moschkovich, 2002).

Access by all students to rigorous mathematics. Teachers need to ensure that all their students, who vary in their prior mathematical background, language ability and other characteristics, have access to participating in cognitively demanding mathematical activities. Inequitable access leads to inequitable learning opportunities and learning gains (Cohen & Lotan, 1997; Gee, 2003).

An earlier paper reported on the ACCESS Project teachers' growth in professional *knowledge* related to the weeklong summer PD (Truxaw, et al., 2009). This research builds on those results, following the teachers as they worked to move beyond *knowledge* to *knowing in teaching practice* (Kazemi & Hubbard, 2008). We address two research questions: 1) what language-related instructional practices develop among the group of teachers as they move from their *knowledge of* language to *knowing in practice?* 2) What issues – pedagogical or otherwise – did the teachers find themselves facing as they engaged this work?

Data Sources and Modes of Inquiry

Context

The research took place over one year in an urban school district in the northeastern United States where 45% of the students spoke a non-English home language, over 95% of students qualify for free/reduced priced meals, and 96% of the students are categorized as "minority students" (State Strategic School Profiles, 2008). The project was supported by a Teacher Quality Partnership Grant that allowed for 45-hours of summer PD for 23 teachers (grades 4 -10) from four schools in the district. Nineteen of these teachers participated in the academic year component as well that comprised a modified form of lesson study where teachers (3 hrs/month), organized in grade-band teams, collaborated to develop, implement, and debrief higher order thinking (HOT) lessons that used pedagogical strategies related to each of the three pillars. Surveys conducted at the end of the summer PD indicated that teachers felt they had learned useful information with respect to language and mathematics (knowledge of) and were confident that they could promote the development of students' academic language (Truxaw, et al., 2009).

Although our focus is teachers' practices, student mathematical performance was monitored throughout the project. Open-ended, single-item pre-post tests (released items from State assessments) were administered to students. These prompts required higher order thinking and justification. Prompts were double-scored according to State criteria; discrepancies were resolved by a third scorer. Across all grade-levels, mastery increased and scores of 0 decreased. To gauge whether growth in the project classes was beyond what could be expected in a typical year, we administered pre-post tests in non-project classes. Overall, project classes in the same grade levels performed better than non-project classes. For example, grade 5 class scores of 0 decreased markedly (74% in the pre-test as compared to 10% in the post-test), as compared with non-project classes (83% to 62%) and mastery scores increases (0% to 24%), as compared with non-project classes (3% to 15%). These results suggest that the instructional practices used by the project teachers had a positive impact on student mathematical performance.

Data Collection and Analysis

To document language-related instructional practices, we examined 25 archived HOT lesson plans developed by teams of teachers (see http://www.crme.uconn.edu/lessons/). We reviewed the objectives of the lesson to determine if language objectives were included, and if so, the type of language objective. We used categories established in the literature related to supporting ELLs, including: key vocabulary, language functions, language skills, and grammar or language structures (Echevarría, Vogt, & Short, 2007). An example of a language function objective is "Students will be able to express the likelihood of various events using everyday language." An example of language skills is "Students will be able to restate the problem in their own words."

To further document language-related instructional practices, we reviewed selected field notes, written documentation of meetings, audio recordings, and transcripts from teacher collaborative sessions, focus groups, and teacher interviews. Initial themes had been developed in an ongoing manner as the project and data collection were in process. To refine these themes, standard qualitative techniques (Creswell, 1998; Strauss & Corbin, 1990) were used. Two researchers reviewed data to finalize themes.

Results and Discussion

Analysis of the data revealed that the teachers developed and used an expanded repertoire of language-related practices that were associated with students' improved mathematical performance and enhanced verbal and written justification. We report our findings in terms of the work teachers did in planning instruction and implementing instruction. The first category focuses on language objectives and task design. The second category focuses on pedagogical strategies used as the lesson unfolds. We then turn our attention to some of the questions and issues teachers faced as they advanced their practice.

Planning Instruction

Language objectives. The HOT lesson plans teachers developed collaboratively included both content and language objectives. An analysis of the language objectives demonstrated all lessons had appropriate language objectives that addressed a range of language-related learning goals. Not surprisingly, the category of language objective that occurred most frequently among the 25 HOT lesson plans was "key vocabulary" – 84% of the lesson plans included one or more language objectives related to this theme. Although the teachers reported that developing language objectives outside the "key vocabulary" category was challenging (focus groups), 92% of the HOT lesson plans included one or more language objective outside that category as well. These language objectives included focus on everyday language necessary to explain contexts of problems and mathematical language necessary to express and justify mathematical ideas.

Overwhelmingly, the teachers noted that the explicit inclusion of language objectives, including sharing them with the students, helped to enhance their teaching and student learning. For example, Mr. Borreca, a fourth-grade teacher said, "I like having the objectives written out. I never thought I would like that, to be completely honest…writing out the content and the language, but I do it … if for anything else, it focuses me" (mid-year focus group).

In meetings, developing language objectives was not automatic. Generally, it was difficult to identify these objectives at the beginning of planning (in contrast to the content objectives). It often took some time working on the planning before it became clear what the linguistic challenges might be, and which ones would be appropriate to address during the lesson.

Deliberate task design. As the teachers worked in their collaborative teams, they deliberately designed certain features into tasks to promote the development of students' language. A specific strategy, evidenced on many of the HOT lesson student handouts, was the use of "check points" – a strategy that had been introduced during the summer and which the teachers found very useful for promoting language. Check points were built into tasks. When students hit a checkpoint, they called the teacher over. All members of the group needed to be prepared to explain the work and respond to questions, which encouraged group conversation and explanations. A more individualized strategy was the inclusion of “hint” and “think” cards in lesson materials. *Hint cards* supported language by asking questions like, “What does the word *pair* mean?” or, “When you wear a pair of shoes, do you wear one shoe or two shoes?” *Think cards* offered challenges like asking students to explain alternative strategies and solutions.

In addition, full components of tasks – including class warm-ups – were designed with particular language goals in mind. Some of these focused on *building context,* an important strategy identified in the literature (Echevarria et al., 2007), which was often motivated by the teachers’ pursuit of the *access for all* pillar. As the teachers started to think about building context, this often led them to think about the bridge between the language students would readily bring to the context and what needed to be developed to be successful with the lesson.

For example, as the teachers started a unit on probability, they wanted to find ways to help students make sense of notions such as “a probability of .8”. They noted that students find little meaning in this phrase, but do have a sense of “pretty likely.” They hoped to connect students’ already extant notions of probability (embedded in their daily lives and everyday language) and extend that to the more precise, mathematical quantification of likelihood. They designed an introductory activity with a “probability continuum” from 0 to 1. They used white boards or large papers and had students generate as many words as they could that represented ideas related to likelihood (or probability). This component of the lesson focused students on everyday words that had to do with likeliness. The teachers felt that it “helped them [the students] understand better what probability was all about” (Ms. Vargas, end-of-year written reflection).

Implementing Instruction

A main focus during the implementation of instruction was getting students to produce language to express mathematical ideas. Teachers encouraged the use of language frames, a technique to scaffold verbal and written responses. For example, in a lesson using pattern blocks to represent fractions, the language frame, “I think ____(shape) is ___(fraction) of the whole because _____,” supported students’ ability to explain and justify their responses while still requiring that they work through the mathematics. Particularly the grades 4-6 teachers began to recognize that literacy strategies, already familiar to them as they were in self-contained classrooms, could be useful in math instruction. For example, the reinforcement of time/sequence words (e.g., first, then, next, finally) was a literacy strategy used to help students organize their ideas, moving toward academic language in their written explanations.

In addition to literacy strategies, teachers actively sought to promote student discourse. Attending to student explanations both in whole class discussions and in small groups supported the development of language, mathematical understanding and justification. To support whole class discussion, teachers practiced “talk moves” (Chapin, O'Connor, & Anderson, 2003) and asked open-ended, “HOT questions” (described by teachers as *questions that go beyond skills and procedures to deeper understanding of concepts*, mid-year focus group), for example, “Why?” “How do __ and __ compare?” “Do you agree/disagree … and why?”

All of the strategies reported here are present in the literature. What is important to note is that these are the strategies that teachers gravitated towards in their first year of deliberate work towards developing students’ language. They found these the most accessible and useful, and were able to integrate these on a fairly regular basis. It is also interesting to consider the resources they drew upon to enhance their pedagogy. The teachers reported using materials from the summer institute (e.g., books, strategies, graphic organizers, web links, etc.) when they were preparing HOT lessons, and they highly valued the collaboration of the yearlong activities, using one another as resources. In addition, we found that the elementary grades teachers began to see connections between the work in mathematics class and the other content subjects. For example, they drew on literacy strategies such as sentence frames, KWL charts, and word walls.

Issues Arising

We now turn our attention to some of the issues that came up for teachers as they worked to advance their practice with respect to language. Teachers acknowledged sometimes becoming overwhelmed while trying to unpack language aspects of a mathematics lesson. They grappled with how and when to introduce vocabulary. They dealt with how to provide easy access to language (e.g., through highly engaging contexts such as playing card games) and then, how to move on *from* the initial context toward the use of the language in mathematical context. There were times when they felt that they had spent too much time on language aspects of the lesson, neglecting mathematical content. Other times, they recognized that *not* attending to language sufficiently may have impeded the understanding of mathematical content.

Collateral teacher content learning. An unexpected outcome of the teachers' focus on language was a deepening of their own thinking about the mathematical *concepts* at hand, and not just the language, and what it would take for their students to fully understand the ideas. As the teachers planned, their mathematical understanding – or lack of understanding – posed a productive challenge for them to work through in order to successfully design and implement the HOT, language-rich lesson. For example, in the probability lesson described above, teachers explored for themselves the multiple ways that we represent likelihood. As they discussed the language and developed the continuum activity, they began to realize how the different representations of likelihood, which included verbal representation, decimal representations, percent representations, and fractional representations, were indeed distinct ways of knowing and understanding probability. Whereas they were fairly fluid moving among these, their students were not, and they needed to account for this in their teaching.

As a second example, a group of grade 9 teachers worked on a World Wealth and Population lesson (adapted from Gutstein, 2006). They started with a mathematical goal. They wanted students to reason about which region of the world was the wealthiest by using a data table that included the population and gross domestic product (GDP). They decided it would be productive to have students do this informally first (before computing any percentages or ratios). They pursued this because it promoted sense making (2nd pillar) and they expected it to promote language use (1st pillar). The teachers then looked at the language demands of doing this kind of work, which required the teachers to think about what a "good" response would be. They realized that students had to coordinate two quantities (population and wealth) and be able to talk about the *relative* relationship between these and another population-wealth pair. Cognitively, this was certainly beyond the demands of comparing the unit rate or the percentages which only required comparing two quantities which, although proportions, were compared absolutely. In examining these language demands, the teachers unpacked the mathematics in the lesson in a way that they may not have if they had remained focused on the content.

Another example of teachers deepening their own thinking about mathematical concepts through a focus on language came from a grade 4-5 lesson study team. The teachers developed a HOT lesson related to identifying patterns in the classic handshake problem (i.e., if x number of people shake hands with each other, how many handshakes take place?). A language objective was "students will be able to *explain* the way the pattern worked." In planning the lesson, the teachers decided to use yarn strung between the students as they modeled the handshakes – one string of yarn for each handshake. This created a visual and tactile model of the connections and helped to illustrate the progression of the pattern. Students were able to translate the yarn representations to diagrams and tables – all that assisted them as they worked to uncover and *explain* the strategies they used and the patterns they noticed. Mrs. Shaw reflected that in order to come up with representations that supported explanation, she had needed to analyze the mathematical concepts inherent in the task. "I have to develop from my weaknesses because I know the kids will have some of those same weaknesses … I have to work through it to make sense of it" (mid-year focus group). Although Mrs. Shaw did not see herself as naturally mathematically strong, the focus on language provoked her to understand the mathematics more deeply so that she could ask her students HOT questions. "When the next person comes in, *why* is that number [of additional handshakes] increasing by 1 each time? They have to *look* at what is causing this increase, rather than, 'Oh, I know what comes next.' Well, *why* is that? *Why* is it working that way?" (Mid-year focus group).

For these teams of teachers, focusing on language enhanced not only the language, but also the mathematics – for both students and teachers. Mr. Barreca, for example, said, "I feel I have improved my basic content knowledge of math" (end-of-year written reflection), and, "I'm able to look at math more conceptually…more holistic, understandable" (end-of-year focus group).

Conclusions

This investigation documented language-related strategies developed by one group of urban mathematics teachers as they moved from their learning *about* issues related to language and mathematics to knowing-in-practice. The study identifies promising instructional practices, particularly those that might be most accessible to mathematics teachers as they begin this kind of work. The study also suggests some of the anticipated issues that might arise. Though not surprising in retrospect, we did not anticipate that a focus on language would lead to a deeper understanding of the content for the teachers. This reinforces the importance of having professional development focused on language issues *in mathematics,* as the details of a general strategy must be worked out within a particular mathematical context and set of understandings.

Endnotes

All teachers' names used are pseudonyms.

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HOW-I-TEACH-MATH STORIES INTEGRATED/EXPERT IDENTITIES

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This paper describes and examines a type of stories of mathematics teaching practice that elementary teachers began to tell as they continued to implement new mathematics pedagogies after learning about them from a professional development course. These stories, how-I-teach-math stories, *emerged as teachers began to view themselves as the kinds of teachers who routinely taught using the new pedagogies. The significance of how-I-teach-math stories and implications for teacher professional development are discussed.*

Introduction

Professional developers working with experienced teachers have yet to attend to something that preservice teacher educators know: the development of identity is important to becoming a teacher (Britzman, 2003). However, how experienced teachers' develop new identities as a result of mathematics Professional Development (PD) is relatively unstudied. In this study teachers told stories about teaching mathematics using new practices they had learned in a PD course. As they implemented the new mathematics teaching practices over time, some teachers began to tell a kind of story I call *how-I-teach-math* stories—general narrations of how they generally teach mathematics and students tend to respond (as opposed to narrations of specific events). The teachers began to tell these stories about their uses of the new mathematics teaching practices at various times and in various ways. These narratives functioned to claim some new mathematics teaching practices as the teachers' own—to identify them as teachers who use particular practices. I examined teachers' *how-I-teach-math* stories as narrative performances of teachers' emerging integrated/expert identities (that is, the ways they identify that integrate their new ways of being mathematics teachers and their identities as experts). I seek to uncover ways that these narratives, though characterized by certainty, can be resources for mathematics teacher learning.

Theoretical Framework

Mathematics reform documents (e.g. NCTM, 2000) advocate methods that are likely different from those teachers have experienced. One result has been a proliferation of PD workshops, the effects of which are unknown (Wilson & Berne, 1999). The work of teaching impacts teachers (Waller, 1932), and teacher preparation impacts the identities of preservice teachers (Britzman, 2003). However, the effects of the implementation of new mathematics pedagogies on the identities of experienced teachers are unknown. How are teachers' identities different after they have integrated new ways of teaching mathematics (and new ways of identifying themselves as mathematics teachers) into their practices?

Identity has been operationalized as narratives one tells in order to be perceived as a certain kind of person (Holland *et al*., 1998; Sfard & Prusak, 2005; Sfard, 2006, 2007). Through narratives (and indeed all discourse, but narratives are a particularly rich site for identity construction) people perform stories of their experiences and invite their audiences to respond in certain ways (Juzwik, 2004). Especially powerful are *direct identifying statements*, which are statements about oneself that are reified (that is, treated as objects that are true regardless of time), endorsed (meaning that they are believed to be true) and significant to the storyteller's self-understandings (Sfard & Prusak, 2005). Because "*how* something is said is a part of *what* is said" (Hymes, 1974), I examined the rhetorical and theatrical tools teachers use to tell these stories. Crespo & Juzwik (2006) studied the performative aspects of teachers' *expository* stories about mathematics teaching. Expository stories are final draft stories told without invitation for dialogue. *How-I-teach-math* stories are a subset of expository narratives.

Oslund (2009) has described ways in which integrating new mathematics teaching practices into their repertoires required teachers to integrate previous identities with newer ones. For example, a teacher who identified as one who gave a lot of assistance to students desired to become a teacher who stood back and watched students struggle before intervening. In order to do so, she needed to integrate her identity as a teacher who *gives* with an identity as a teacher who *steps back*. In this paper, I report on how teachers began to identify as experts after they had spent various amounts of time working toward these integrated identities.

Specifically I ask, "How do the performances of *how-I-teach-math* stories told by elementary teachers help us to better understand the identity development of experienced teachers learning to implement new mathematics pedagogies?"

Methods

Participants and Context

Participants are four elementary teachers who participated in a workshop on using Complex Instruction (CI) (Cohen, 1994; Cohen & Lotan, 1997) to promote rigor and equity in mathematics and had reported using CI as a result. The weeklong PD workshop, *Designing Groupwork in Mathematics,* was facilitated by Dr. Lisa Jilk of Teachers Development Group and was offered during two different summers at a university. The workshop was about using CI, which is a particular form of cooperative groupwork, to teach mathematics. What makes CI different from other programs of cooperative groupwork is its attention to *status* in small groups. Teachers learn to identify status issues that interfere with learning in groups and are taught several interventions for disrupting status so that all students can engage in mathematical tasks (for example, using tasks that require multiple abilities, setting norms and roles for group interactions, and publicly assigning competence to students with low status).

Teachers attended the workshop voluntarily. The participants for this study were volunteers from among both cohorts of the workshop who reported having implemented CI to teach math after the workshop and desired to continue doing so. They were teachers of grades one, three, four, and five in three different school districts (one urban, one suburban, and one rural). This study spanned eight months during which the teachers were teaching mathematics using CI.

Data generation and analysis

Data was generated in multiple contexts in order to attend to gain a comprehensive understanding of teachers' identities. Teachers told stories about teaching specific math lessons with CI in individual interviews. They participated in focus group discussions where teachers told stories about teaching math with CI to each other, raised problems of practice, and shared ideas for teaching. They also presented CI teaching stories to a group of teacher educators who had not participated in the PD. These teacher educators had heard about CI and were interested in learning more.

Sessions were audio-recorded and transcribed. Narrative portions were arranged into idea units, lines and stanzas (Gee, 1991), highlighting their poetic aspects. Transcripts were analyzed for parallelism, alliteration, shifts in point-of-view, the rhythm of the syllables, subjects and objects of idea units, and other poetic aspects of language. I also studied the syntax and grammar of teachers' idea units. In this paper I report only on teachers' use of verb tenses to reify their stories and identify as expert CI/mathematics teachers.

In the following section I present one *how-I-teach-math* story about teaching mathematics with CI from each teacher and report on the timeline for which this type of story emerged in the teachers' storytelling. Three of the teachers told how-I-teach-math stories about how they move through whole CI lessons or parts of CI lessons. However, in the case of Glynnis, she only told a how-I-teach-math story about one small part of one CI lesson. In the discussion section I theorize these stories as narrative performances of the teachers' emerging CI/mathematics teacher identities—that is to say that the teachers began to identify as "teachers who use CI to teach mathematics" through the reification of experience into integrated/expert identities.

Generalized Narratives

Patricia

Patricia had been using CI for one school year when the study began. When I walked into her classroom for her first individual interview she began to talk about CI before I had an opportunity to ask a question. I interrupted her and asked if I could turn on the audio recorder. When the recorder was on, I said, "So tell me what you were going to tell me." She began:

> *I use the Group Solutions book (Goodman, 2000). I walk through it from the very beginning. Actually, I don't walk through it from the very beginning. I start with jobs. I describe the jobs, and we model the four different jobs. The jobs are not listed as they are in the [the CI course materials]…the information on the back tells them what to do, and the model conversation that they can have is still the same on the back. But on the front I just gave the little picture, because first graders know that, so that everybody's involved and understands. The jobs rotate. The groups actually rotate too. About every six weeks I change desks, so they may end up with different people. Some kids say, "Well, I was material manager two times in a row," but it all works out in the end…we review [the roles] almost every week. And there'll be an emphasis one too. Like, I'll say, "The questioner is the person who begins, and then we go clockwise." That way, we talk more about the questioner and we ask the other people what the questioner is doing so that they're more aware of, a little bit of observation, with it…At the end, of the activity we all get together and we talk about how we solved things*

In using the word "jobs", Patricia is talking about what was called *roles* in the CI course. Roles are an important instructional strategy in CI. Students are given roles that are specifically designed to engage them in the mathematics at stake in a *groupworthy task*.

This is a *how-I-teach* story about using CI to teach mathematics. At times, Patricia tells narratives using present tense terms (I walk… I describe…) indicating how she always or usually is when doing CI lessons. Remember that direct identifying statements are *reified*; that is, that they are spoken of as detemporalized objects (Sfard & Prusak, 2005). In this story, Patricia has reified verbs by using present tense, thus identifying herself as the kind of teacher who routinely carries out these practices as a part of teaching. She identified as a CI teacher.

Jonathan

Jonathan had also been teaching mathematics using CI for a year when this study began. Jonathan told the story that follows during the first focus group meeting. (This meeting happened after the first set of individual interviews, during which Jonathan did not tell a how-I-teach-math story about CI). Teachers were asked to come prepared to share stories of using CI to teach mathematics. They took turns sharing the stories they prepared. Jonathan said:

> *I might give the group, inside a laminated or clear sheet, this page with the other one to the back, so they can see them both. They're on a clear sheet so they don't really write on them. One of the things I've been working with is this idea of putting the math skills that that they're going to be working on the page, and then the groupwork skills that they're going to be working on below that, and then some of the materials they might need, and then I put the directions there…To back up and tell you how I do this. I don't do a task every week. I don't always do a task every month. I tend to do a task toward the end of a unit, because for me it's a really nice way to bring together a lot of new skills they've learned and share them out. Sort of showcase. Look at this, we've done, we can do this now. We can do this. We can do this. And then other people will go, yes. Yes. No. Yes. You know? And so I find it very powerful to do it like that…they're group one. I don't like names. They're strictly groups one to seven. Everybody writes their name, whatever seat they're on. And as they finish the task they are required to call me over. They don't know who in the group's going to have to explain it. And so I'm, you know…I have a little stamp, and I'll stamp [their task card], and then they can go on to the next task.*

Notice that Jonathan moved back and forth between generalized and specific talk about his teaching in his first how-I-teach story about using CI to teach mathematics. Like Patricia, much of Jonathan's language was in the present tense when he refers both to how he teaches mathematics with CI (I don't, I tend to, I find it) and what his students do (They don't know, everybody writes). By using present tense Jonathan identified as a teacher who does CI and does it in a certain way.

It is interesting that Jonathan did not tell a *how-I-teach* story about CI in our first individual interview but did tell this type of story in the first teacher group. Within the teacher group there were teachers who had only been using CI for a couple of months and teachers who had been using it for over a year (including Jonathan). One way that Jonathan and Patricia identified themselves as the *old-timers* to CI (Sfard & Prusak, 2005) was to tell stories that identified themselves as people who use CI and who use it in an individualized way.

Joanna

Joanna was one of the teachers who had been teaching mathematics using CI for only a couple of months when the study began. In her first individual interview, she said:

> *[The students] don't want to follow the rules like, make sure you've talked this over with everyone in your group before you ask the teacher for help. There's always, three, somebody from three different tables jumping up that, you know, that impulsiveness they want to ask me it.*

Joanna had used CI to teach mathematics a couple of times and told this story that included the type of generalized language that is a marker for a *how-I-teach* story. However, it is not really a how-I-teach story (which are about how teachers teach math using CI) but a story about how students respond to her attempts at teaching mathematics using CI. She only spoke in this generalized manner about her students. She did not talk about herself in a generalized way, or identify herself as a certain kind of CI teacher. She continued the story differently, however:

> *We have talked about this since the beginning of the school year. We've repeated it since the beginning of the school year.*

When she continued the story to include herself in the action, she switched back to speaking about the past. "We have talked" sets her story in a specific time. Her use of past participles indicates it has happened more than once. However, she is not detemporalizing her behavior, so this is not a how-I-teach story.

However, when presenting her work to teacher educators who had not been to the PD (the last data generation session of the study), Joanna told a *how-I-teach* story about using CI to teach math. She said:

> *I start out modeling with tasks from Group Solutions Too (Goodman & Kopp, 2007), the other book. It gets my kids accustomed to the norms and roles by using those tasks…but I try to use other tasks too. I try to use tasks from Balanced Assessment (Dale Seymour, 1999). Something in the classroom that I'm curious about are the norms, what we just call group rules that I have in the classroom…those are some of those things are things that we instill throughout the school day in everything that we do. They're very familiar with them but I keep them posted when we do any kind of a [CI] task...The kids, in the sharing process, they'll raise a hand and they'll ask a team, "What do you mean by that. I really don't get it. Can you explain that to me?" There's just a lot of dialoguing that's going on...I rotate [the groups] in my grade book. I go by whatever their seating arrangement is. I just pass the cards out to each group and I just put a sticky note on and I just say, "This time you're the organizer. This time you're the captain. This time you're the recorder/reporter." I just pass them out. They just hold on to them and it helps give little reminders about what their responsibility is within their group.*

As both Patricia and Jonathan did in their how-I-teach stories, Joanna spoke of herself primarily in the present tense, indicating that these are things she continues to do. While she had spoken this way about her students at the beginning of the data-generation time period, she had not spoken this way about herself until this point in time (the end of the study).

The audience to which Joanna told this story was different than the audience of the individual interviews (where I was the audience) or in the CI/math teacher group meetings (in which other teachers who had been using CI to teach math for various time periods were the audience). In this context, with instructors who had not taken the CI course, Joanna identified as a CI teacher who generally used CI in a certain way. She identified as a CI expert in relation to her audience for these stories. In that context she had more experience with CI than her audience did, which was not true for her in the other contexts.

Glynnis

In her presentation to teacher educators who had not participated in the CI PD, while Joanna was sharing materials she used to teach math with CI, she mentioned that she created different names for some of the roles than teachers had seen in the CI course. Glynnis interrupted and suggested that the names for the roles in the CI course were problematic. She said:

> *I try to tell the kids that everybody has a role and none of them are more important than the other. That they each are all very important roles. Because otherwise they're like, "Oh, she wants me to be the captain," and the status is already there. Because the 4th grade, especially, they've had years and years of it.*

Glynnis only told this one short segment of a story of teaching mathematics with CI in which she reified her actions using present tense verbs. In this story, Glynnis spoke in generalized terms about how she dealt with one possible problem with the roles. Glynnis did not tell *how-I-teach* stories about how she did any other part of the process, including preparing tasks, organizing her classroom for instruction, or implementing instructional strategies. Again, it is notable that she only told this type of story about CI teaching in a context where she was someone who had used CI and there were people in the audience who had not. In that context, in this small part of a story, she identified herself as a person who typically dealt with this issue in this way. In doing so she identified herself as an expert in dealing with possible conflicts that might come about because of the role cards.

Discussion

I have shown a particular type of narrative that teachers told about their use of CI in mathematics—the how-I-teach-math stories. These narratives function to claim some new ways of teaching as the teachers' own and reify the teachers' experiences into integrated/expert identities. In the discussion that follows, I posit how-I-teach stories to be markers of teachers who identify as "CI teachers" with integrated/expert identities.

Integrated/Expert Identities:What it Means to Identify as a CI Teacher?

As stated in the theoretical framework, direct identifying statements are statements about a person that are reified, endorsed, and significant. A statement is said to be reified if talk about actions or processes have been replaced with talk about detemporalized objects. For example, "I gave the students a problem solving task" is not reified because it talks about an action that a teacher has taken, but "I am a problem-based teacher" is a reified statement because the teacher is then describing a characteristic of herself as true regardless of time (Sfard & Prusak, 2005).

The stories in this chapter are not direct identifying statements because in these stories the teachers have not made succinct, reified statements. Reification is defined as turning actions or processes into objects (Sfard, 2008). In these stories the teachers are not talking about objects; they are still talking about actions. However, they are not talking about specific actions. They are describing general ways they act regardless of time. For example, instead of saying, "We modeled the first four jobs," Patricia stated, "We model the first four jobs." She identified as the kind of teacher who models the roles with her students on a regular basis as opposed to having done that at one specific point in time. She could have identified even more strongly had she said, "I am a modeler."

If we conceptualize narratives on a continuum (as opposed to categorizing narratives that are reified and those that are not) it is logical that a teacher's narratives would move along that continuum in the process of her identity work. For example, a teacher might first say about a CI math lesson, "I modeled." She may later speak of her CI math teaching practice saying, "I model," and finally say directly, "I am a modeler."

Integrated identities. Persisting in the implementation of CI required the teachers to integrate new ways of identifying as teachers with their prior ways of identifying. How-I-teach stories about CI in mathematics are one indicator that some teachers were able to accomplish this integration of identities.

For example, in the theoretical framework I described how Joanna had struggled to integrate her identity as a teacher who gives with an identity as a teacher who steps back to allow children to struggle through a math problem. Eventually she discovered that, as a teacher who stepped back, she was also being a teacher who gave a new resource to students—the resource of each other's competencies. At the same time, Joanna continued to modify her use of CI in order to align it with her ways of identifying, including using tasks from a book received during the PD, Group Solutions Too (Goodman & Kopp, 2007), because she felt that the tasks were short enough that she did not need to step in during groupwork to intervene. By the end of the study she told how-I-teach stories about CI in math class. In one story, she stated, "I start out modeling with tasks from Group Solutions Too (Goodman & Kopp, 2007)". In doing so, she was identified as a teacher who does this. She was also performing her newly integrated identity: the teacher-who-gives and the teacher-who-steps-back. Because these types of narratives are indicative of integration of two identities, I call them integrated identities.

Expert Identities. In telling how-I-teach stories about their use of CI to teach mathematics the teachers were identifying as having expertise in one or many areas of teaching with CI. This is not to say that an outside observer would necessarily label each teacher as an expert CI teacher. I came to see these particular types of expository narratives as expert identities because of the contexts in which the stories were told. Patricia and Jonathan, the two teachers in the group with the most experience using CI to teach math, told how-I-teach stories at the beginning of the study. They used these stories to identify as the teachers in the group with expertise to share. Joanna and Glynnis told how-I-teach stories about using CI in math when the audience was a group of instructors who had not yet taken the CI math course. Therefore, in that context, Joanna (and to a lesser extent, Glynnis) identified as having some expertise to share with novices.

Implications

How-I-teach stories seem to be characterized by certainty and a sense of stability. On the one hand, they represent a transition from one mathematics teaching identity to another--part of the goal of PD events that seek to change practice. On the other hand, when teachers identify with such certainty teacher educators may become concerned that their certainty will impede future learning and reflection. I posit that how-I-teach mathematics stories could be a fruitful area for generative discussions about practice and student learning. For example, how-I-teach-math stories could be leveraged as potential questions for teacher inquiry. Perhaps in a next phase of the PD teachers could inquire into their own practices by collecting data on student learning to study the effects of the behaviors identified in their how-I-teach stories.

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INCORPORATING A MEASURMENT LEARNING TRAJECTORY INTO A TEACHER'S TOOLBOX FOR FACILITATING STUDENT UNDERSTANDING OF MEASUREMENT

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A learning trajectory for measurement provides a detailed explanation of the ways in which children think about measurement and prescribes tasks that will motivate students to think in more sophisticated ways. This paper reports on the potential use of a Measurement Learning Trajectory (MLT) as a means of supporting a teacher's understanding of her students' conceptual knowledge of measurement concepts. We found this tool useful in supporting the teacher's growing knowledge of her students' ways of thinking and her ability to evaluate and develop tasks to assess student understanding.

Introduction

A proficiency in measurement is essential to students' understanding of the mathematics they will encounter in their future lives. The NCTM (2006) *Focal Points* document identifies measurement as one of the content areas that form the foundation for future mathematics learning. Lehrer, Jenkins, and Osana (1998) suggest that length and area measure "constitute the building blocks for developing a mathematics of measure" (p. 152). Although a conceptual understanding of measurement concepts is essential to students' future mathematical understanding, Clements and Battista (1992) found that many children use measurement tools in a rote fashion and fail to understand the concepts behind the tools. Similarly, Lehrer (2003) concluded that the conceptual aspects of measurement are often left "unpacked" resulting in a procedural view of measurement (p. 188).

This paper describes how one teacher, with the support of a researcher used a Measurement Learning Trajectory (MLT) for length and area (Clements & Sarama, 2004; Barrett & Clements, 2003) to uncover her own students' conceptual understanding of measurement ideas. Table 1 and Table 2 show the portions of the length and area trajectories that were used in this study.

Theoretical Framework

As students develop and learn mathematics, their thought processes become more sophisticated. These more sophisticated levels of thinking are central to Simon's (1995) notion of hypothetical learning trajectories (HLTs). Clements and Sarama (2004, 2009) view the HLTs as, "descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking" (p. 83). The use of research-based accounts of students' more sophisticated levels of thinking allows teachers to use an HLT to assist in assessment and instructional decisions.

Once a teacher establishes the level at which a student is operating, an informed decision can be made as to which tasks may be helpful in motivating the student to move towards developing more sophisticated levels of thinking. A cycle of task implementation, analysis of student thinking, and development of tasks to address the student's misconceptions is critical to the

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development of instructional interventions meant to motivate growth from one level into the next level. In the hands of a teacher, initially supported by a researcher, an HLT can be used as a tool to engage a teacher in the process of what Steffe (1991) found to be essential to the understanding of students thinking processes. This process of "hypothesizing what the learner might learn and finding ways of fostering this learning" will assist the teacher in her developing understanding of the strategies students use (Steffe, 1991, p. 83).

Table 1. A Hypothetical Learning Trajectory for Length Knowledge (Adapted from Sarama & Clements, 2009)

Level	Observable Operations	Mental Actions on Objects
End-to-End Measurer (EE)	Compares a train of short objects to an object. May use incongruent objects.	Expects that length is composition of shorter lengths.
Unit Relater and Repeater (URR)	Composes length by combining parts. Attends to unit size. Composing, often not reversibly.	Iterating a mental unit along a perceptually available object. Allows for *counting-all* addition schemes.
Length Measurer (LM)	Keeps identical units. Attends to zero position. May partition units. Can integrate partitioning actions and grouping actions. Employs broken ruler.	Well-developed scheme for linear composition and partitioning from units to *composite units* and also *units of units* of length. Units sub-divisible.
Conceptual Measurer	Can accurately estimate length, including a system of subunits. Operates with flexible arithmetic on collections of objects. Can coordinate operations among figures.	Has an interiorized length scheme that enables the child to mentally partition into a given number of parts, or project iterations onto objects. Enacts a multiplicative scheme to operate on units of units of units.

Table 2. A Hypothetical Learning Trajectory for Area Knowledge (Adapted from Sarama & Clements, 2009)

Level	Observable Operations	Mental Actions on Objects
Area/Spatial Structuring: Primitive Coverer and Counter (PCC)	Draws as above. Also, counts corrects aided by counting one row at a time and, often, by perceptual labeling.	Counts all objects once and only once and use of rows as an intuitive structure or explicit application of labeling as marker, allows child to keep track.
Area/Spatial Structuring: Partial Row Structurer (PRS)	Draws and counts some, but not all, rows as rows. May make several rows and then revert to making individual squares, but aligns them in columns.	Applies a row as a composite unit repeatedly, but not necessarily exhaustively, as its application remains guided by intuition.
Area/Spatial Structuring: Row and Column Structurer (RCS)	Draws and counts rows as rows, drawing with parallel lines. Counts the number of squares by iterating the number in each row, either using physical objects or an estimate for the number of times to iterate.	Applies a row as a composite unit repeatedly and exhaustively to fill the array.

Methodology

A veteran fifth-grade teacher (Kathy) in a rural school in the Midwestern United States was encouraged to assume the lead role in the cyclical process of hypothesis, testing, and reflection pertaining to two of her students (Eliza and Isaac). In order for the teacher and the researcher to better understand the students' knowledge and misconceptions concerning measure, they used Steffe's (1988) notion of teaching experiments in which a researcher generates and tests hypotheses about a student's thinking strategies and then tests those hypotheses in individual interviews (Steffe & Thompson, 2000) to investigate students' understanding. With the support of the researcher, the teacher used the measurement learning trajectories (MLTs) for length and area as a research-based framework to guide her reflections about her students' thinking processes and to develop tasks to use in predict/check cycles with two of her students.

To gain a clear picture of the students' understanding and the teacher's use of the MLT, the researcher analyzed video recordings of the teacher and researcher working with two focus students in sessions that were referred to as "teaching episodes." On an alternating basis, the two focus students each participated in six 20-minute teaching episodes for a total of 12 weeks. Also analyzed were the teacher's and the researcher's ongoing and retrospective analysis of the students' thinking strategies in the form of written reflections and audio recordings of the collaborative meetings between the teacher and researcher. These collaborative meetings included analysis of the videos of the teaching episodes, as well as the discussion of the predict/check cycles of the teaching episodes. The 12 weeks of video and audio recordings, as well as the written reflections over this time period comprise the data sources for this paper.

Results

We first describe a brief sequence of teaching episodes that focused on Eliza's understanding of length as a compilation of equal sized units and her misconceptions concerning the ruler as a tool for measurement, which resulted in her inability to understand the correspondence between tick-marks and units. Next, we describe a sequence of teaching episodes that focused on Isaac's lack of understanding of the array structure of area. We use both Kathy's and my own accounts of the student's thinking and the narrative of our thinking about the student's progress or struggles as we conducted the sequence of sessions to describe how the students were thinking about the particular task. By describing the thinking of myself as researcher, of Kathy as both teacher and professional development learner, and of the students directly, we portray the situation that is fundamental to this research on the use of a MLT as a tool for professional development.

Eliza

On the initial assessment (January 13 and 14, 2009) Eliza successfully completed End-to-End (EE) tasks as well as the Unit Relater Repeater (URR) tasks. She was unable to attain the correct answer to perimeter task, which would indicate thinking at the Length Measurer level (LM). She did complete a blank ruler by paying close attention to equal spacing of units and subunits. Kathy thought Eliza's attention to spacing and her accurate estimate of an inch was evidence that she was operating at the URR level.

Tasks were designed to check that Eliza was indeed thinking at the URR level for most tasks by presenting tasks that required the use of thinking at the prior level (EE). Eliza was able to consistently complete tasks at the EE and URR levels, however, we had yet to offer Eliza sufficient opportunities to complete tasks that required thinking at the LM level, thus we presented a task that required the use of a broken ruler. Kathy predicted that Eliza would be able to use the broken ruler, numbered from 7 to 11, broken on both ends, to determine the length of a 4-inch strip. However, Eliza explained her answer of “five inches” by counting tick marks from 7 to 11.

When Eliza was then given an unnumbered ruler to use to measure a 5-inch strip she called the five-inch strip “six” by counting ticks. Kathy then asked her to use a traditional ruler to measure the 5-inch strip in order to motivate Eliza to confront her misunderstanding. Eliza aligned the strip with the beginning of the ruler and read off the number that aligned with the end of the strip, which was five. Kathy asked, “What do you think of that? Do you think the paper ruler is not correct?” Eliza aligned the unnumbered paper ruler with the end of the traditional ruler to see that they were the same. Kathy asked Eliza to show us how she got five using the traditional ruler and Eliza explained by counting the tick marks, however, this time she did not count the 0^{th} tick mark. Kathy then asked her to use the traditional ruler to help her mark off the five inches that she had measured. Eliza placed ticks along the segment, including one at the beginning of the strip. When she was done, she said, “But that’s six!” Kathy asked why that was six and Eliza responded, “Because of the zero” and pointed to the beginning of the ruler.

After Eliza’s consistent use of the tick counting strategy while using a variety of unnumbered measuring tools and her confusion with the role the zero plays on a ruler Kathy decided to try a task that the MLT described as useful in motivating students to confront their strategy of tick counting. Eliza was asked to use an unnumbered ruler to draw line segments of decreasing length from four inches to one inch. Due to Eliza’s reliance on tick counting, Kathy predicted that Eliza would use the ticks to help her draw segments, but that Eliza would not count the 0^{th} tick and succeed due to the previous statements Eliza had made that indicated she was aware of the 0^{th} tick causing her measurements to be off by one. However, Eliza counted ticks, including the 0^{th} tick to draw her segments, thus she made segments that were always one inch less than they should have been. When she had to draw the one-inch segment, she said, “I can’t.” This prompted Kathy to ask why and when Eliza did not respond, Kathy asked, “Did you run out of lines?” Eliza smiled and said yes. Kathy and the first author (JM) had planned on focusing Eliza’s attention on the fact that a ruler is made of inches placed end-to-end, therefore, we asked Eliza to use a special ruler (one-inch strips placed end-to-end and taped together) that Kathy suggested Eliza make prior to this episode to measure her line segments. Using this tool to measure, Eliza surprised us by correctly counting ticks on this ruler instead of the spaces. When we asked how she got “four” for the 3-inch strip the first time, she explained that she had counted the 0^{th} tick by accident. We were not satisfied with this strategy, as we wanted to motivate her to see and count the individual inches, which would have been evidence of the LM level of thinking.

Isaac

After analysis of the initial assessment, we had placed Isaac at a Partial Row Structurer (PRS) level. In most instances when asked to find area, he either wrote in a number with no explanation or drew in individual tiles that were not uniform in shape and size, but seemed to follow some row or column structure in various portions of the figure. Kathy predicted that Isaac would quickly move into the Row and Column Structurer (RCS) level.

The first task we describe asked Isaac to look at part of a five by six grid with the bottom right portion erased from view. Because one complete row and one complete column were in view, we hoped this would motivate Isaac see the row and column structure of the grid. Kathy predicted that Isaac would get frustrated when she did not give him a pencil to complete this task so she gave him a pencil at the beginning of the task. Even with the pencil, Isaac touched each square and imagined square with the eraser and counted along each row and got 28 tiles. Since there were 30 tiles in the original drawing of the 5 x 6 array, he was now short by 2 tiles. We took this as evidence of thinking at the level of (PRS).

Next, Kathy asked him to try to fill in the missing tiles. He drew individual squares and was less accurate in the open spaces and ended up drawing 33 tiles (See Figure 1). To motivate the need for uniform tiles, Kathy gave him a square tile to use to iterate along the rows. Kathy asked if the tile was the same size as the squares on the grid and he said that it was not the same as all of them (because his squares were often too small or odd shaped). Kathy asked, "Does it make a difference that some of the squares you drew in were not the same size?" He said, "Kinda." Kathy asked, "Would it be helpful to make them all the same size?" He said "yeah" and she asked him to explain why it would be helpful. Isaac replied, "So you could get the exact measurement." Kathy asked him if there would be another way to partition the space instead of drawing individual squares and he could not think of any way. He used the tile and iterated it along each row and got 30. Isaac's drawing and iterating along the rows of the rectangle confirmed our claim that he was indeed operating at the PRS level.

Kathy became convinced that Issac's inability to represent uniform area units was a result of his need for a ruler to draw in the tiles. She predicted that a task that used rulers in some way would help Isaac "see" the uniformity of the tiles and thus, begin to think of area as rows or columns of area tiles. JM showed Kathy a task that had two sets of two unnumbered rulers placed perpendicular and there was a three by eight rectangle on one set of rulers and a six by four rectangle on the other set of rulers (See Figure 2). Isaac was asked to compare the areas of the two rectangles. We decided that if he arrived at the area by using the formula, Kathy would ask him to show her the 24 units. Kathy predicted that he would again draw in individual tiles, but do a better job of making them uniform because the rulers were there for a guide. She thought that Isaac viewed a ruler as a tool for precision and that would motivate him to be careful with his drawing of the area units.

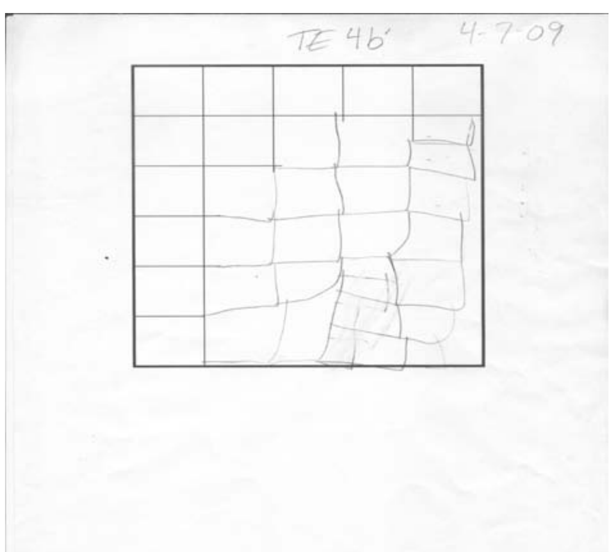

Figure 1. Isaac's Record of the Ink Blot Task

Figure 2. Comparing the Area of Two Rectangles Task

Kathy asked Isaac to compare the areas. He said, “Well this one has three here (pointing to the side with three units) and eight here (pointing to the side with eight units).” He thought the four by six rectangle was bigger. “The big one is 24 and this one (3 by 8) is ….24.” Kathy asked if that surprised him? He said no. We asked him to show us the 24 square inches by drawing them in. He did not understand what we meant by this so Kathy went back to a prior task that involved a completed 3 by 4 grid to show him what she meant.

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He then drew tiles in one at a time and left gaps and did not attend to the size of each tile. He ended up with 27 squares. Again, Kathy thought he would do better if we emphasized the ruler so she gave him a regular ruler to see if he could use that to help him. He did not think the ruler could help and said that although he drew in 27 tiles, he knew that the area was really 24. His drawing led us to believe that he was still operating at the PRS level because he continued to draw non-uniform individual square tiles.

To motivate Isaac to see rows as rows and iterate rows we designed a task that presented Isaac with a pile of square tiles. Some of them were taped together forming rows of six tiles and others were loose. Kathy asked him to use the tiles to make a rectangle with a length of six and a width of seven. Kathy predicted Isaac would start out using the individual squares and would get three rows into the construction before he realized he could use the strips. If this did not happen Kathy planned to point out the strips of six and ask him if he thought he could use those to help him.

He began by placing a set of six squares on the left and right side of an open space. He then placed seven individual tiles along the bottom in between the left and right sets of six tiles. Along the top he placed a set of six tiles and one loose tile. Kathy gave him a ruler and had him check how long the top was. He found that the sides were six and the top and bottom were nine. He took the top strip of six away and placed seven individual tiles in between the two columns of six and re measured. To remind him of his goal Kathy said, “Ok, I want a rectangle that’s six by seven.” Isaac removed two loose tiles, one from the top row and one from the bottom row, so that he had seven across the top and bottom. He used the ruler and said, “That’s seven inches long.” Kathy asked, “So what would the area of that be?” He said “42 squares.”

Kathy then asked him what the perimeter was. He individually counted around and touched the table beside each tile as he counted correctly. He did not use the algorithm, which is how he apparently got the 42 squares that he found for area because there were no tiles in the interior of his rectangle. This told Kathy that Isaac did not seem to understand how the figure and the area formula were connected. Because he struggled to make the rectangle and did not indicate that he understood the row and column structure of area, we thought he was still operating at the PRS level. Isaac apparently was struggling to see the area in terms of rows and columns. However, if we were to do this task again, Kathy would ask him to make a complete rectangle from the start. We did not anticipate that he would see the square tiles and think in terms of perimeter.

Conclusions

Eliza's confusion concerning the role of the zero on the ruler and her unwavering reliance on counting tick marks to determine length when using a nonconventional ruler surprised Kathy. Isaac's inability to see the row and column structure of area units, frustrated Kathy, as she believed we could design tasks that would motivate Isaac to confront the misconception and quickly move into the next level. Although I was there to help analyze these episodes with Kathy and discuss her predictions of Eliza's and Isaac's responses to the tasks, Kathy began to make her own decisions about what tasks to use next and even created some tasks based on her own professional knowledge of teaching measurement. The brief descriptions above show instances where Kathy took the lead role in choosing and developing tasks, as well as asking questions that probed students' understanding. For Kathy, this was an opportunity to investigate ways to prompt her students to confront their misconceptions and attempt to increase their level of thinking without teaching students in a "show and tell" manner. Although there were times when Kathy struggled with her desire to "just tell them how to do it", she continued to work with these students using a more critical lens for viewing her role as a facilitator of knowledge and creator of appropriate and useful assessment tasks.

Discussion

This brief example of Kathy's ability to use the MLT to guide task creation to assess her students' understanding of length and area offers insight into the practical use of this research-based framework. Although some teachers might be able to incorporate the ideas of the MLT quicker or in more meaningful ways than Kathy did and others might not have the same level of success Kathy found with the MLT, this example offers a glimpse of the potential impact the MLT could have on a teacher and the way she views the quality and the role of assessment tasks in her desire to increase her students' competency in the area of measurement. If further research finds that Kathy's growth and development is typical, then we know what to expect of other teachers in a similar context. However, if it turns out she is unusual in her ability to grow, then we will have to make changes to our methods for introducing and encouraging teachers to use the MLT in practice.

Endnotes

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MATHEMATICS COACHING AND ITS IMPACT ON STUDENT ACHIEVEMENT

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This investigation is to determine whether a classroom-based mathematics coaching professional development model has a positive impact on student mathematics achievement in low performing elementary and middle schools. Relevant literature is practically void of research connecting a professional development program, particularly on coaching, to improved student mathematics achievement. Findings suggest that this mathematics-coaching model has made large and significant impacts at all grade levels and each level effect sizes range from 0.5 to nearly 0.9. These results may provide a framework for discussion of why so many 'proven-to-work' research-based instructional strategies fail to yield desired results in student mathematics achievement.

Introduction

In 1983, *A Nation at Risk* was published describing the perils in education that must be addressed for school systems to succeed. The report clearly represented the condition of the United States educational system as dismal. A Nation at Risk calls for refining mathematics curriculum and instruction for which it has been calling now for over a century. Federal funding sponsors such as the National Science Foundation and the United States Department of Education have spent hundreds of millions of dollars over the last 30 years producing only small gains in mathematics achievement scores (NAEP, 2008; Hiebert et al., 2003).

Within a parallel timeframe, learning theories were further refined to better suit the teaching and learning of mathematics. The genre of research focused on cognition and instruction has provided insight into the teaching and learning of mathematics (Cobb, 2001) suggesting ways in which student thinking may be enhanced. Progress was made in not only understanding how children learn but also in how to effectively teach to promote understanding (Franke, Kazemi, & Battey, 2007) motivated by the desire to break away from the traditional textbook.

With findings and funding in place, professional development became key to changing practice. Experts in professional development have studied and designed quality experiences for teachers with promising results (Darling Hammond & McLaughlin, 1995; Shifter & Fosnot, 1993; Sparks & Loucks-Horsley, 1990). With all of these trends and forces converging, it was hoped that all mathematics classrooms would soon transform into research-based, theory-driven spaces where teachers and children would engage in mathematics using reform-based best practices. However, there still remains little gain in mathematics achievement.

Mathematics instructional coaching is fairly new as a model of school-based professional development. However, there are instances of coaching in the literature even though it is not referred to as such (e.g., Borko, Davinroy, Bliem, & Cumbo, 2000; Breyfogle, 2005; Franke, Carpenter, Fennema, Ansell, & Behrend, 1998). Coaching emerges in a variety of structural approaches and has not been studied extensively. Coaching is grounded in the notion of capacity building for change and having highly trained practitioners *working with their colleagues* to support learning. As a professional development model, the method capitalizes on collegial interactions to improve practice (Becker, 2001). The use of such a model in the mathematics education community as a means to enhance teaching and learning has gained momentum (Davenport, 2008).

The Mathematics Coaching Program—Our Model

The design and structure for our mathematics-coaching model are original while the content of the professional development is based on our conceptual framework that is well grounded in theory and research. The model has four sections: 1) MCP Conceptual Framework, 2) Coaches' Professional Development, 3) Design and Structure for Implementation, and 4) Accountability and Quality Control. The MCP conceptual framework is representative of what is valued in the MCP, how we organize those values, and how they together combine to make a whole. The fundamental goal of the MCP is to help students become active and informed agents of their own mathematical learning, making learner-responsive mathematics education the core of our work and the framework.

The foci for all MCP professional development sessions include: a) deepening mathematics content knowledge; b) deepening pedagogical knowledge and sharing research-based best practices for mathematics learning and teaching; c) attending to socio-cultural elements of mathematics teaching and learning; d) focusing in on student thinking and learning how students can learn mathematics; and e) learning to transfer knowledge to classroom teachers through coaching and teaming; f) using the NCTM Process Standards in all facets of teaching and learning mathematics; g) defining coaching as teaming with teachers in classrooms 4 days per week; and h) working with data to enhance program elements and progress.

Pedagogy in the MCP is student-centered, student-responsive, problem-based, and provides many opportunities for students to use all the NCTM Process Standards. In our work over the last 20 years we have become strongly grounded in several specific learning theories and research-based instructional strategies. Theoretically, we use a social cognitive, social constructivist, and social justice perspective in the teaching and learning of mathematics. Specifically, our work is grounded in Cognitively Guided Instruction (CGI) (Carpenter, Fennema, Franke, Levi, Empson, 2000; Carpenter, Fennema, Peterson, Chiang & Loef, 1989); using teaching experiments (Cobb, 2000) and rethinking mathematics teaching and learning to close achievement gaps using a social justice perspective (Gutstein & Peterson, 2005; Frankenstein, 1987).

The MCP instructional approach is drawn from scholars who focus on the critical features of instruction (Hiebert, J., Carpenter, T., Fennema, E., Fuson, K., Wearne, D., Murray, H., Oliver, A., & Human, P., 1997), teacher content knowledge and content knowledge needed for teaching (Ball, Hill & Bass, 2005; Leinhardt & Smith, 1985; Lampert, 1990; Ma, 1999; Hill, Shilling & Ball, 2004), and teacher pedagogical content knowledge (Leinhardt & Smith, 1985; Ball, 2000; Pinar, Reynolds, Slattery & Tubman, 1995; Shulman, 1986, 1987; Wilson, Shulman & Richert, 1987). These scholars recognized the complexity of teacher knowledge and the role of both content knowledge and pedagogical content knowledge (PCK) in effective instruction. Furthermore, PCK is about how to teach the content and in that how to understand students' thinking, cultural identities and how they impact learning, and learning styles and preferences.

The MCP instructional approach also incorporates Lipping Ma's (1999) notion of addressing teachers' content simultaneously with student learning. Ball & Cohen (1999) also acknowledged professional development should be situated in the context of practice. In addition to these recommendations, research on professional development in mathematics education and in education in general provides insights as to the qualities of effective professional development

(Darling Hammond & McLaughlin, 1995; Shifter & Fosnot, 1993; Sparks & Loucks-Horsley, 1990). All of this is in alignment with recommendations from the National Council of Teachers of Mathematics (NCTM).

The Design and Structure of the Coaching Model

The MCP requires mathematics coaches to team in classrooms with a small group of four teachers everyday for six weeks, and all coaches must participate in professional development four-days per month. At the end of a coaching cycle, the coach begins working with a new group of four teachers while maintaining lessened contact with previously coached teachers. The MCP provides support to coaches for approximately 300 hours per year for up to three years.

The Study

Our overarching goal is to investigate how professional development must be conducted so that teachers learn, internalize, implement, and perpetuate successful teaching and learning of mathematics leading to improved and sustained student achievement. In this paper we intend to determine whether mathematics coaching using the model described herein has a positive impact on student mathematics achievement and if so, is the impact significant?

About the Participants

The Mathematics Coaching Program's effort is situated within some of the lowest-performing urban and rural schools from across the state. Statistically these schools are rife with disproportionate numbers of students in poverty, minority subgroups, disabilities, and/or students with limited English proficiency reside. The State Department of Education administrators privileged the mathematically lowest-performing schools in the state to be first in line to receive the professional development they sponsored. To determine which schools would be on the most-eligible list, schools had to be in School Improvement Status and Title I served. Additional mostly undisclosed state criteria were used to narrow the pool, one of which was size of district.

Since the state department was funding schools across the state to participate in either a mathematics-coaching program or a literacy-coaching program, the list of eligible schools would, by Principal's choice, result in schools having no coach, a literacy coach, or a mathematics coach. School principals hired their own coach from within their building or otherwise.

MCP teachers. Teachers in participating schools self-select whether he/she is willing to work with a mathematics coach. All teachers who teach mathematics from grades K through 8 are invited to participate. Once willing teachers are identified, the mathematics coach takes time to prepare for the first six-week cycle. Teachers are required to supply the coach with copies of curricular materials, textbooks, pacing guides, and a list of students. There are only two requirements that teachers must abide by while working with the coach. First, teachers are required to team teach with the coach at all times and second, teachers are required to meet with the coach at least one time per week for planning and debriefing. Enforcing these two requirements allows the program to have a chance at building capacity across the staff and subsequently having principals seeing the benefits and sustaining the program after the project is over.

Facilitators. Facilitators are professional educators with a mathematics background who the MCP would be able to hire for four days per month. People who are able to make such a commitment include but are not limited to college/university professors, professional development specialists, educational service center professionals, district curriculum supervisors, and retired mathematics teachers or MCP graduates. Two days per month, the facilitators join their assigned coaches at the MCP Professional Development sessions so that they learn exactly what the coaches are learning and expected to do. The other two days per month, the facilitators call small group meetings in their local area to provide a layer of support that has proven to be priceless to the success of this program.

Students tested. The state Achievement Test (AT) is given to all public school students in this state and all students in community schools (those public schools that are run by private entities). The AT is optional for all private schools – including parochial schools. However, most all of these schools choose to take these exams as well. There are no exemptions from the test. Students with IEP's have testing accommodations, or, in severe disability cases, alternative assessments are given.

Methods

Data were collected from the state testing website from school report cards. The data collected were from all schools on the state's eligibility list, of non-coached, literacy coached, and MCP coached schools. Specifically, the data used for this study included grade level percent passing rates at or above proficient levels. To determine whether the MCP had an impact on student mathematics achievement a structural equation model was used to measure effect size. All student mathematics achievement test data for all schools at all grade levels tested in grades 3-8 in the first four years of the program were used. Thus, there were hundreds of schools at each grade level providing the numbers needed for credible results. To test for significance, two-tailed multi-year T-tests were conducted. The results of these measures are displayed in Table 1.

Results

Effect Sizes and Significance Levels.

In Table 1 there are many data points to analyze. First, the data are reported in grades 3 through 8. For each grade level there are data reported from two groups, the schools that participated in the Mathematics Coaching Program and those that did not. In each grouping the various numbers of schools, mean scores of mathematics achievement at or above proficient, and standard deviations are presented. The final two rows show P-Values, and Effect Sizes for each grade level.

Table 1. Mathematics Coaching Program Data Effect Size and Significance

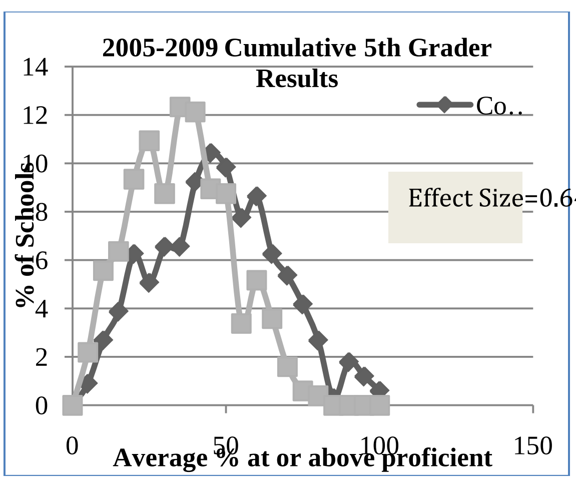
	Grade	3	4	5	6	7	8
	N_schools	367	388	336	230	142	137
MCP Coached	Mean	67.81226	62.89021	44.82768	59.13304	50.39577	52.72774
	SD	17.32337	18.45494	20.23823	21.68486	19.81219	19.93282
	N_schools	545	541	503	358	274	275
Non-Coached	Mean	58.7167	52.68096	33.33817	42.15028	36.83358	38.42836
	SD	17.32246	17.31087	16.009	17.88441	15.65568	15.86556
	T-Dist	7.775715	8.532308	8.739036	9.908033	7.090264	7.32055
	DF	785.2781	800.9471	603.2424	421.2725	234.3475	224.3051
	P-Value-2	**2.36E-14**	**7.14E-17**	**2.32E-17**	**6.11E-21**	**1.57E-11**	**4.39E-12**
	Eff_Size	**0.52463**	**0.573172**	**0.644083**	**0.871668**	**0.787777**	**0.824046**

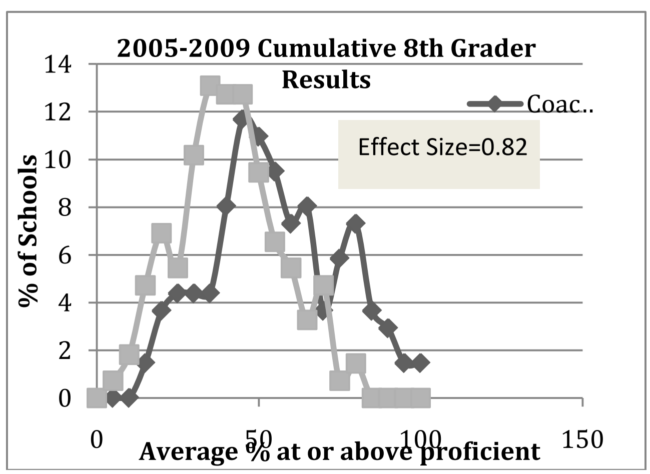
The data in Table 1 were from a large number of schools accumulated over four years. This large number reduces the concern of non-randomization of school choices and the control of student population and teacher variations. If there are two schools that have major contextual or constitutional differences, the outlier effect can be "averaged out" with a large number of schools having diverse contexts and backgrounds. A second observation shows that the mean values are greater in the coached group at every grade level. Studying the means further shows that grade 5 has the lowest mean value in each group. This is true in all 5th grades statewide. One of MCP's foci for Year Five is to examine Grade 5 more closely so that we may better understand this phenomenon.

Effect size has been widely used to evaluate the impact of an education treatment. Generally the evaluation of impact in terms of effect sizes ranges from small (<0.3), medium (<=0.5), large (>0.5), and very large (>0.8). In Table 1, the effect sizes for the MCP data are all large. This finding alone leads the researchers to conclude that the Mathematics Coaching Program as a whole does have a positive impact on student mathematics achievement.

With the effect sizes all indicating a large positive impact on student mathematics achievement, it is also important to note that the respective p-values were all at significant levels, which is the natural outcome of large size samples. To view the impacts pictorially, a sample of two grade level charts is presented in Figures 1 and 2.

In Figure 1, the data represent hundreds of schools having 5th grade students as well as whether schools have a mathematics coach or not. Each school has an average percentage of 5th graders who scored at or above proficient on the mathematics achievement test. This figure shows two line graphs, one representing the non-coached schools and the other representing the MCP schools. Looking at the first graph, the line shows a large percentage of the non-coached schools aligning with the lower percentage of students at or above proficient. This non-coached line also indicates that there were 0% of schools scoring more than 85% of its students at or above proficient. The coached data show that more schools scored higher percentages of students at or above proficient. There is an observable spike on the high end of the achievement scale. This spike shows a sizable rise in the percentage of schools having an 85% to 100% of students scoring at or above proficient.





Figures 1 & 2. Percentage of schools compared to percentage of 5th and 8th grade students in those schools averaging at or above proficient levels in mathematics

Data in Figure 2 are gathered and organized the same way as in Figure 1. The data represent hundreds of schools having 8th graders scoring at various averages of at or above proficient on the mathematics achievement test. In this case, the results are similar in that greater percentages of schools in the non-coached schools have larger percentages of 8th graders scoring at or above proficient at the low end of the scale. The MCP coached schools at this grade level are showing greater percentages of schools having even greater numbers of 8th graders scoring at or above proficient. This greater percentage shows a much greater effect size. The results of both graphs also show that the changes of student performances happen rather uniformly at low through high performance schools. This suggests that the coaching model benefits schools at all performance levels.

Discussion

Many professional development providers maintain long held traditions including presenting too many ideas to an auditorium full of classroom teachers all day everyday for one or two weeks during the summer with the expectation that the teachers learned what was intended, interpreted learnings as intended, implemented ideas as intended, and that all children will not only learn the mathematics but will achieve highly on high-stakes testing. The effectiveness of this approach has long been challenged. After 30 years of trial and error, American students' mathematics achievement in national and international theatres remains the same—high achievers remain high, low achievers remain low (NAEP, 2008; Heibert et al, 2003).

The move towards designing job-embedded, sustained experiences for teachers in order to advance teacher learning in ways that alter classroom practice has been well grounded in the community's best thinking on transformational learning about teaching (Ball, 2000). However, relationships between such models and student achievement, in large scale, have not been widely explored. In this paper we presented empirical data on the positive impact of Mathematics Coaching Program and student achievement in settings where traditional professional development plans, nationally, had not been successful. These results fully position us to challenge traditional views on teacher development approaches and argue that providing teachers with information is not sufficient to improve practice. Teachers need to be provided with a classroom-based professional development system that is ongoing, consistent, and long term.

Indeed, the MCP Model for Professional Development provides ample evidence that student achievement in low performing schools can improve at significant levels. Therefore, we argue that our approach merits attention and further study. Our goal in this paper was to establish whether the Mathematics Coaching Program, as a whole, had a positive impact on student mathematics achievement, and if so, was the impact significant. The findings show that the MCP has had a large positive impact on student mathematics achievement at all grade levels 3 through 8 and that the impact was significant at all grade levels. Considering that our work is conducted in some of the lowest performing schools in the state gels some hope for other schools that are otherwise not expected to improve.

Having established a positive relationship between our PD model and student achievement our research team is now engaged in a critical study of several critical issues associated with explaining these results, namely: a) What do mathematics teachers and coaches need to know and be able to do to help struggling learners become successful? b) What MCP elements are most critical to the learning process? c) How are the underrepresented subgroups of students performing? d) What other statistical models may be used to dig deeper into the MCP database to produce additional credible findings in a non-randomized classroom-based program? e) Does this program increase mathematics and pedagogical knowledge among the participating teachers and coaches, and if so, in what ways does this happen? f) What is happening in MCP classrooms and how do we know if the mathematics coaches are following the program as intended?

These findings also give way to variations of professional development approaches that can make a difference. Further, it may be the case that when examining elements of the Mathematics Coaching Program researchers from across the nation may create hybrid models of this program by making context-specific adaptations for others challenged by school mathematics.

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MATHEMATICS COACHING: IMPACT ON STUDENT PROFICIENCY LEVELS AFTER ONE YEAR OF PARTICIPATION

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This paper reports research results on the impact of a mathematics coaching Professional Development Program on student proficiency levels. The program prepares classroom teachers to serve as mathematics coaches for their respective schools. This study examined state achievement test results for schools with first-year coaches in their buildings. Data were analyzed for grades 3-8 and compared for schools participating in the mathematics coaching program with similar schools not involved in the program. The number of students at each proficiency level – limited, basic, proficient, accelerated, and advanced – was investigated and compared using data from the year prior to participation in the program with data from the end of the first year of participation. Results show significant gains.

Introduction

Mathematics education in the United States has been unfavorably compared to that of other countries for decades (Miller, Sen, & Malley, 2007; Nelson, 2003). There is evidence that a majority of the students in U.S. schools lack both an understanding of and an appreciation for mathematical literacy and by the time many of them reach high school, mathematics is one of their least favorite subjects and they fail to see its relevance to any aspect of their lives. Calls for the need to change this pattern have been voiced by scholars and professional organizations. Some evidence indicates that the quality of teacher knowledge may have an impact on students' performance.

Elementary mathematics education sets the foundation for each student's future success. However, most students at this level are taught by teachers with limited mathematical preparation as few were probably required to take more than two or three mathematics content courses and maybe one mathematics content pedagogy course (Fennell, 2007). Most elementary school teachers, through no fault of their own, lack the necessary mathematical content knowledge or pedagogical knowledge to effectively instruct students so that they have a thorough understanding of mathematical concepts. The Virginia Math and Science Coalition Task Force Report (2005) confirmed this for teachers in Virginia in a study designed to present a case for using mathematics specialists in their elementary schools. In a separate study, it was found that the mathematical content knowledge of teachers significantly affected student gains at the first and third grade levels regardless of years of teaching experience (Hill, Rowan, & Ball, 2005). Literature confirms that teacher preparation programs as well as professional development need to emphasize both mathematical content knowledge and mathematical content pedagogical knowledge (Cavanagh, 2008; Hill et al., 2005; Li, 2008; Reys & Fennell, 2003). In light of these suggestions, calls for designing professional development programs that target teachers' subject matter understanding while increasing their sensitivity to children's thinking have been wide spread (Becker & Pence, 2003; Bruce & Ross, 2008; Cavanagh, 2008; Hill et al., 2005; Li, 2008; Murray, Ma, & Mazur, 2009; Reys & Fennell, 2003). Despite this, the relationship between coaching professional development programs and student achievement remains mostly unexplored (Murray et al., 2009). The purpose of the research reported here was to address this gap.

This report is part of a larger longitudinal research project that closely studies the link between a Mathematics Coaching Professional Development model on teacher and student learning. While there are several studies examining how mathematics specialists or coaches impact teachers (Becker, 2001; Bruce & Ross, 2008; Gerretson, Bosnick, & Schofield, 2008; Lord, Cress, & B. Miller, 2008; McGatha, 2008; Murray et al., 2009; Olson & Barrett, 2004), a concern remains regarding the impact of coaches on student achievement (Murray et al., 2009). In general, researchers assume a positive impact but literature provides little in the way of empirical evidence to support this assumption.

Theoretical Framework

Job-embedded professional development programs for coaches are grounded in a constructivist orientation (Becker & Pence, 2003). The focus of existing programs is on development of mathematics coaches who work collaboratively with teachers, administrators, parents and community with the overall goal of improving mathematics achievement for all students in their school. Collaborative classroom coaching has shown to help teachers rely more on their own knowledge and judgment than that of textbooks and encourages them to be more receptive to trying new strategies. Such a pathway facilitates transitioning from traditional to more reform-based methods of instruction (Becker & Pence, 2003; Lord et al., 2008).

While there are few empirical studies with respect to coaching professional development and its impact on student achievement, what little research exists points at some common perceived benefits of these programs. Collaboration is the most often mentioned strength of using a mathematics coach. Most teachers appreciate the opportunity to share ideas, techniques and strategies and as a result are willing to move their instruction towards more standards-based methods (Bruce & Ross, 2008; Gerretson et al., 2008; Murray et al., 2009). Movement in this direction had a positive effect on student achievement (Gerretson et al., 2008). In their discussion, Bruce and Ross (2008) state, “When a teacher receives positive and constructive feedback from a respected peer, there is even greater potential for enhanced goal setting, motivation to take risks, and implementation of challenging teaching strategies.” (p. 348)

Literature divides mathematics specialists along three conceptual models. The first type is the Lead Teacher Model where the classroom teacher has extra duties as a mathematics resource person. The second type is the Specialized Teaching Assignment Model where a classroom teacher’s only responsibility is to teach mathematics. The third type of model is the Mathematics Coach Model where the specialist works collaboratively with other teachers but is not responsible for his or her own class (Gerretson et al., 2008; Pitt, 2005). The Coach Model represents the type of mathematics specialist used throughout the entirety of this research project.

Roles and Responsibilities of the Mathematics Coach

In general, a coach works with administrators, teachers, parents, and the community providing leadership and direction for the mathematics program of their school. Literature (Fennell, 2007; McGatha, 2008; Reddell, 2004; Virginia Mathematics and Science Coalition Task Force, 2005) summarizes roles and responsibilities of mathematics specialists into the following categories: (1) Implementing standards and research; (2) Planning and facilitating professional development; (3) Working collaboratively with classroom teachers; (4) Analyzing and interpreting data; (5) Organizing and coordinating mathematics resources; (6) Communicating with parents and community. This paper focuses on the collaborative working relationship between the mathematics coach and the classroom teacher. Of the six categories of responsibilities for mathematics specialists, collaboration with classroom teachers would seem to necessitate the greatest attention. Collaboration includes such activities as co-planning, co-teaching, modeling a lesson, and coaching. The coach and classroom teacher can work together to plan a lesson discussing instructional methods, questioning techniques, and possible areas of concern for student learning. Reflecting on a lesson prior to its implementation can help teachers anticipate difficulties that may occur. They can also work together to teach the lesson and upon completion of a lesson, the teacher and coach can then reflect on its overall effectiveness and, for future classes, adjust it as necessary. The strength of an effective mathematics coach is with their expertise in mathematical content knowledge and mathematical content pedagogy (Feger, Woleck, & Hickman, 2004; Powell, 2009). By working collaboratively with classroom teachers, the mathematics coach can pass on their knowledge with the goal of improving instructional practices and ultimately, student learning. However, these assumptions, while intuitively sensible, remain theoretical due to lack of empirical data.

Background

In this work we consider "Mathematics Coaching Program" (MCP), which has been in schools throughout Ohio in partnership with The Ohio State University. MCP is in its fifth year and has grown from a K-6 program to a K-8 program. Initial and current focus of MCP is to work with teachers from the lowest performing school districts, as determined by Ohio Achievement Test (OAT) results, with the goal of improving their mathematical content knowledge and pedagogy. To achieve these goals, MCP recruits teachers from participating school districts and, through ongoing professional development, prepares them to serve as mathematics coaches in their respective buildings. Professional development consists of two full-day monthly sessions for eight months and then five full-day sessions in May. The focus of these sessions is to provide research-based activities to engage coaches in mathematical and pedagogical exploration. Coaches also have two additional full-day sessions each month to meet with their facilitator groups to discuss experiences as well as problems and concerns. Facilitator groups include coaches from different school districts and the group facilitator. To progress through a complete cycle of training, coaches remain with MCP for three years at which time the opportunity usually exists to become a facilitator.

Coaches who participate in MCP have no classroom responsibility of their own. They cannot be assigned more extra duties than other classroom teachers. Their primary focus is to work collaboratively with teachers in their building to help them gain the knowledge, confidence, and necessary instructional practices needed to effectively teach mathematics. Coaches work with a maximum of four teachers each time for a period of six weeks spending part of each day in the classroom of each teacher with whom they work. At the end of the six-week period, coaches begin working with a new set of four teachers. At the time of this study, MCP trained first, second, and third year coaches from a variety of low performing school districts. Ongoing research investigates the impact of the program on teacher and coach knowledge development, change in instructional practices, and student achievement. The study reported here aimed to address the relationship between MCP and student achievement.

Methodology

This study analyzed student achievement data from eighteen schools with twenty-one first-year MCP coaches. Fourteen schools had one coach, two of the K-8 schools had three coaches each, and one coach served at two schools. Results of the MCP-coached schools were compared with eighteen non-MCP coached schools similar in grade span, enrollment and typology. Data were organized according to grade level, including students from grades 3-8, with the number of subjects at each grade level indicated in Table 1. It should be noted that only six of the eighteen MCP-coached schools and eight of the eighteen non-coached schools contain subjects from all grade levels.

Table 1. Grade-Level Number of Students: MCP Coached vs. Non-Coached

	2007-2008		2008-2009	
Grade	MCP	Non-MCP	MCP	Non-MCP
3	1,113	872	1,116	1,002
4	1,207	927	1,073	911
5	736	927	764	948
6	542	936	538	1,015
7	1,674	1,291	1,567	1,255
8	1,719	1,325	1,691	1,300
Total	6,991	6,278	6,749	6,431

Table 2. School Typology: MCP Coached and Non-Coached

Typology Description of Schools	#MCP	#Non-MCP
Urban/Suburban, very high median income, very low poverty	3	0
Urban/Suburban, high median income, low to above avg. poverty	4	1
Major Urban, very high poverty	8	15
Urban, low median income, high poverty	1	1
Rural/Small Town, moderate to high median income, below avg. poverty	2	0
Rural/Agricultural, high poverty, low median income	0	1

Table 2 displays the school typology as designated for their district by ODE.

Student achievement data were obtained from Ohio Achievement Tests from the 2007-2008 and 2008-2009 academic years. A group of first-year coaches from the above-mentioned schools joined MCP in the fall of 2008. The 2007-2008 data served as baseline data for subjects from participating schools prior to MCP and the 2008-2009 data were for subjects from the same schools at the end of the their first year with MCP. The 2007-2008 and 2008-2009 OAT results of subjects in similar schools were used for comparison with non-MCP schools. Specifically, OAT results for both MCP and non-MCP schools were analyzed according to grade level and proficiency levels as defined by the Ohio Department of Education (ODE). The five proficiency levels, in order from lowest to highest achieving, include limited, basic, proficient, accelerated, and advanced. This study focused on the number of students at each level of proficiency in the year prior to MCP and at the end of one year of participation in the program. For comparison, the number of students at each level of proficiency was also analyzed for similar non-MCP schools. For each school and grade level, state report cards provided a percentage of students at each proficiency level. The number of students at each proficiency level and grade level was determined by multiplying the percentage given on the report card by the grade-level student enrollment as provided by ODE. These values were then totaled for each grade level and proficiency level combination and divided by the total number of students for the respective grade level to obtain a weighted average percentage of students at each of the five proficiency levels. These calculations were performed for the baseline data and the post-MCP data for both coached and non-coached groups. By subtracting the baseline data percentages from the post-MCP data percentages, the relative change was determined for each of the grade and proficiency level combinations for both coached and non-coached groups. Finally, using these values, the non-coached relative changes were subtracted from the coached relative changes to determine an overall relative change for each grade and proficiency level combination. A positive result indicates that the relative change for coached data was greater than that of non-coached data and the reverse is true for a negative result. Using these overall values, the following three data summaries were determined:

1. Relative change on "Proficient and above" by grade level.
2. Average of relative change for each of the five proficiency levels for all grade levels combined.
3. Average of relative change on "Proficient and above" for all grade levels combined.

It is noted that "Proficient and above" refers to the proficient, accelerated, and advanced levels combined. With regard to the OAT, student achievement at any of these three levels of proficiency satisfies necessary state requirements.

Results

Figure 1 summarizes the results of this study. For all grade levels combined, the average relative change for students achieving the proficient level or higher was 4.65% greater for MCP-coached schools than non-coached schools. At the opposite end of the spectrum, the average relative change for students achieving only the basic or limited level was 4.46% greater for the non-coached schools. Combined, this represents more than a 9.00% difference between MCP coached schools and non-coached schools.

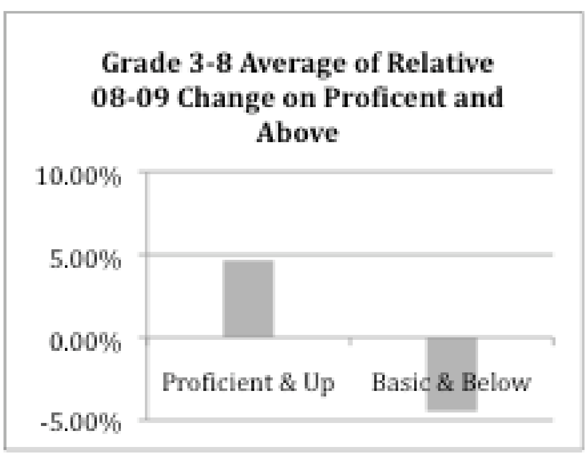

Figure 1

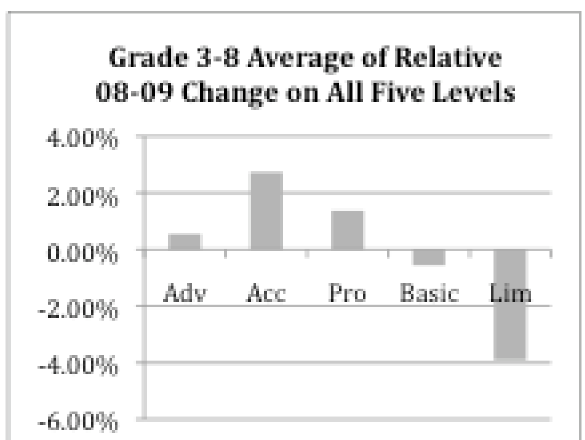

Figure 2

Figure 2 further breaks down the results summary into each of the five proficiency levels. For all grade levels combined, proficient (1.36%), accelerated (2.73%), and advanced (0.55%) levels' average relative change was greater for students from MCP-coached schools than non-coached schools. The average relative change for basic (-0.56%) and limited (-3.91%) was greater for non-coached schools.

Figure 3 provides a summary by individual grade level of the relative change of students achieving proficient, accelerated, and advanced levels of proficiency. Relative changes were greater for students in MCP-coached schools than non-coached schools for grade 3 (0.55%), grade 4 (6.75%), grade 5 (10.07%), grade 6 (7.63%), and grade 8 (4.87%). However, the relative change for grade 7 (-2.00%) was greater for non-coached schools. Again, the average relative change across all grade levels (4.65%) was greater for students from MCP-coached schools than non-coached schools.

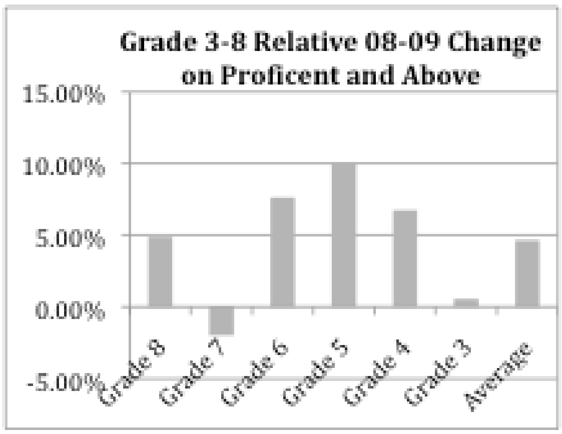

Figure 3

Discussion

In this study, we examined whether a mathematics coaching program (MCP), focused on improvement of teacher knowledge and instructional practices, would have a positive impact on student proficiency levels. Results provide encouraging evidence that even after only one year with the program, students at MCP-coached schools performed better on statewide achievement tests than students at non-coached schools. The statistical significance of the results is reported in Table 3.

Table 3. Statistical Significance and Effect Size of Relative Change

Grade	Relative Change	P-value (2-tailed, $\alpha = 0.05$)	Effect Size
3	0.6%	0.435	0.03
4	6.8%	0.000	0.42
5	10.1%	0.000	0.65
6	7.6%	0.000	0.43
7	-2.0%	0.001	0.12
8	5.4%	0.000	0.30
All Grades	4.5%	0.000	0.27

Data analyzed represented a weighted average percentage of students at or above the proficient level for grades 3-8. Since data for this study did not include individual student data, two sample t-tests were used for each grade level. The p-value gives the statistical significance of the relative change at each grade level indicating that they were all statistically significant except for grade 3. Further, it can be stated that MCP made positive impacts on student achievement for grades 4, 5, 6, and 8 with moderate to large effect sizes. It is noted that grade 7 results did not indicate a positive impact on student achievement. At present there is no obvious reason as to why grade 7 deviated from the general trend, but this certainly could be a focus area for future studies. Results at the grade 5 level offer an interesting perspective as well. On the 2008-2009 OAT, fifth grade students had the lowest passage rate of any grade from 3-8 yet the results of this study showed that their relative change was statistically significant with the largest effect size of all the grades. One reason for this result could be that at the time of this study and relative to the other grade levels, they had the most room for improvement. The average relative change of 4.5% was also statistically significant with a moderate effect size of 0.27. These results provide evidence of a positive impact of MCP on student achievement.

Conclusion

Lack of empirical evidence with regard to professional development programs for coaching and their impact on student achievement served as a guiding force for this study. Indications are that coaching programs utilizing classroom-embedded professional development focused on standards-based research had a positive impact on student achievement. While this report is only a small piece of a very large research project, its results are certainly encouraging. However, much more research and empirical evidence in this area is necessary before any definite conclusions are possible.

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MATHEMATICS COACHING AND ITS IMPACT ON URBAN FOURTH GRADE STUDENTS' MATHEMATICS PROFICIENCY ON HIGH STAKES TESTING

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This study examines mathematics coaching, as defined by the Ohio Mathematics Coaching Program (MCP), a classroom-embedded professional development model that focuses on student thinking in low-performing schools. Fourth grade students at eleven MCP-coached schools all in their second year in the program, in a major urban school district in the Midwest participated in pre-and post testing. The pretest consisted of items released from the prior year's state achievement test and the posttest was the official state achievement test. Results indicated significant gains in all content strands.

Introduction

Students from low-performing schools generally struggle learning mathematics. It has been a national trend to use coaching as a form of professional development to assist teachers in enacting research-based instructional strategies shown to work with low-achieving students. The Mathematics Coaching Program (MCP) was designed for such a purpose. The idea was to place a mathematics coach into four teachers' classrooms each day for six weeks to find out how students could learn, as they were not learning the way they were being taught. Since the Ohio Department of Education (ODE) organizes mathematics into five content standards, this provided a challenge to any professional development program trying to positively influence mathematics achievement in low-performing schools.

For the past two years, I have worked with MCP at The Ohio State University examining proficiency data for our coached students as a collective. I wondered whether or not MCP was having an impact at the student level in each of the five mathematics content standards: Measurement; Number, Number Sense, and Operations (hereinafter Number); Patterns, Functions and Algebra (hereinafter Algebra); Data Analysis and Probability (hereinafter Data); Geometry and Spatial Sense (hereinafter Geometry).

Literature Review

Mathematics coaching is a fairly recent arrival to the realm of professional development, one whose available evidence is promising for classroom implementation (McGatha, 2009). Rather than having an outsider come into a school once or twice a semester to impart new content-knowledge or 'best practices' upon a group of teachers, the rationale behind coaching is to have someone in the building on a regular basis, helping shape teaching strategies and acting as a support person. Many coaching models do not pull students out of the classroom, but rather insert another qualified teacher into the classroom to help guide the teacher in new pedagogical methodologies. This person also can act as a level of support for content knowledge as well.

Staub, West, and Bickel (2003) address the issue of content-focused coaching in the first chapter of Staub and West's book *Content-focused coaching: Transforming mathematics lessons*. This text, along with works by other researchers (Neufeld & Roper, 2003), suggest an organizational structure for developing a content-focused mathematics coaching program. They note that coaching should provide structure for ongoing professional development aimed at:

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

helping teachers design content-specific lessons; examing methodology in terms of student thinking; is based on a core philosophy; enriches pedogogical content knowledge; and encourages commincation skills. Coaching models should help to empower the teacher to use current (reform) practices in the classroom, to change their instructional style without fear of criticism, and to embrace collaborative efforts on teacher-teacher and teacher-coach and teacher-administrator levels (Neufeld & Roper, 2003). Each of these levels of communication is paramount in developing a successful coaching program, and Neufeld and Roper (2003) provide instruction to not only teachers and coaches, but administrators and district personnel as well.

In developing a mathematics coaching program, all personnel involved need to collaboratively address several key elements: the content to be learned, how and why that content should be taught, and why the content should be taught in a specific manner. (Staub, West, & Bickel, 2003). In addition, the teacher and coach should both reflect on what they know about the given content and how that content relates to other mathematics content. Possessing limited content knowledge can significantly impact the effectiveness of curricular reform efforts (Olson & Barrett, 2004). The result of this reflection and collaboration should help determine a shared vision of how the mathematics classroom should look (Becker, 2001).

Perspective

Unlike some other programs, MCP does not focus on solely mathematical content knowledge or procedural knowledge, rather they seek to promote rich connections between both types of knowledge (Baroody, Feil, & Johnson, 2007) while assisting teachers better their own pedagogical content knowledge. As such, MCP does not focus on any one specific mathematical content standard set forth by ODE. They feel that addressing all mathematical content, and the inherent mathematical processes involved, are the only way to build true mathematical understanding. Using Baroody, Feil, and Johnson's (2007) reconceptualization of mathematical content and procedural knowledge as my basis, I sought evidence of improvement in mathematical understanding from students in MCP coached classrooms. Assuming that MCP's methods work, achievement scores in each of the mathematical content standards should show improvement, rather than in one or two.

Methodology

Participants

This project is a study of 110 fourth grade students chosen from eleven schools that participated in the Mathematics Coaching Program in the 2006-2007 Academic Year. Ten students were selected at random from each school. These schools are all from a major urban school district in the Midwest and were in their second year of participation in the program. The students are of different racial backgrounds and different socioeconomic standing. Of these 110 students, 13 were excluded from future analysis due to lack of reported pre-test data. One school lacked pre-test data, and, thus, had its ten students excluded. Either the test was not administered to these students and/or their scores were not reported. After exclusions, I had a sample of 97 fourth grade students from 10 schools: 47 male, 50 female; 31 Caucasian, 66 minorities; 81 receiving free- or reduced-lunches, 16 had no reduction in cost. Given the small number of students with no reduction in lunch cost, that factor was not used in analysis.

Data Sources

The student data included a pretest and a posttest. The pretest, administered to the students in Fall 2006, and is comprised of released items from the 2005 Fourth Grade Ohio Achievement

Test (OAT). Each pretest item was graded collaboratively by the mathematics coach and the students' teacher. The coach then entered the data into a spreadsheet that was submitted to MCP. The posttest is the official OAT administered in spring 2007, as graded by ODE. The students' scores are provided to MCP in fulfillment of program requirements.

Each score on both the pretest and posttest is broken down by mathematical content standard. Students received a raw score for each of these standards. ODE used these scores below, at, or above standard. Using ODE guidelines, I did the same with the pretest results.

Instruments

All items from both the pre- and post-tests are official items from the OAT. Table 1 presents the results of the reliability analysis of the raw scores from both pre-test data and post-test data, as broken down by standard, when run through SPSS:

Table 1. Reliability Statistics for Pre- and Post-tests (SPSS)

Test	Cronbach's Alpha	Cronbach's Alpha Based on Standardized Items	N of Items
Pre-test	.81	.82	5
Post-test	.85	.87	5

Data Analysis

Cronbach's Alpha was used to determine the reliability of the instruments. Significance for trial refers to differences between the pre-test and post-test forms of the instruments. The repeated measures MANOVA and repeated measures Univariate Analysis of Variance were used for each dependant variable. ANOVA was used to determine whether there were any significant main or interaction effects for trial, gender, ethnicity, or school as they relate to mathematics content standards. The Pearson product moment correlation was used to determine the correlation between gender, ethnicity, school, pre-test and post-test raw scores, pre-test and post-test standard indicators. Only the 97 students who had both pre- and post-test data were used in the data analysis.

Results

The Multivariate Analysis revealed two significant 2-way interaction effects: within subjects by trial by school ($F_{(45, 271.5)} = 1.45$, $p = 0.039$), and within subjects by trial by ethnicity ($F_{(5, 60)} = 2.36$, $p = 0.051$). The Multivariate Analysis within subjects did not reveal any 3-way interaction effects or any 4-way interaction effects.

The ANOVA for standard raw score by trial revealed significant interaction effects for Number, Algebra, Data, and Geometry. The ANOVA for standard raw score by trial by school revealed a significant interaction effect for Data ($F_{(45, 271.5)} = 2.96$, $p = 0.005$). Additionally, the ANOVA for standard raw score by trial by ethnicity revealed a significant interaction effect for Number ($F_{(5, 60)} = 7.52$, $p = 0.008$). Only significant interactions were reported in the table.

Table 4 reports the means and standard errors for the Data Standard by trial by school. In the Data Standard, most schools showed modest gains. School 2 showed a slight drop from (M = 4.63, SE = .80) to (M = 4.60, SE = .75). Two schools however showed noticeable gains: School 3 rose from (M = 3.85, SE = .89) to (M = 7.78, SE = .83) and School 5 rose from (M = 1.87, SE = .89) to (M = 5.55, SE = .83).

Table 2. Multivariate Analysis Within Subject Effects

Within Subjects Effect	Value	F	Hypothesis df	Error df	Sig.
Trial	.20	48.45	5.00	60.00	.000***
Trial x School	.38	1.45	45.00	271.50	.039*
Trial x Ethnicity	.84	2.36	5.00	60.00	.051*

Note. * $p < .05$. ** $p < .01$. *** $p < .001$

Table 3. Analysis of Variance for Trial Raw Scores by Gender, by School, and by Ethnicity

Source	Standard	SS	df	MS	F	p
Trial	Number	876.00	1	876.00	234.72	.000***
	Algebra	56.91	1	56.91	17.57	.000***
	Data	55.09	1	55.09	27.19	.000***
	Geometry	26.59	1	26.59	16.09	.000***
Trial x School	Data	54.00	9	6.00	2.96	.005***
Trial x Ethnicity	Number	28.07	1	28.07	7.52	.008***

Note. * $p < .05$. ** $p < .01$. *** $p < .001$

Table 4. Mean (M) and Standard Error (SE)
for Data Analysis and Probability Standard by Trial and by Schools

Standard	School ID	Trial	*M*	*SE*
Data	2	1	4.625	.804
		2	4.604	.750
	3	1	3.850	.887
		2	7.783	.827
	5	1	1.867	.887
		2	5.550	.827

Table 5 reports the means and standard errors for Number by trial by ethnicity. For Number both ethnic groups showed noticeable gains. Caucasian students rose from ($M = 3.55$, $SE = .36$) to ($M = 10.37$, $SE = .64$) and minority students rose from ($M = 3.18$, $SE = .25$) to ($M = 7.77$, $SE = .45$).

Table 5. Mean (M) and Standard Error (SE) for Number by Trial and by Ethnicity

Standard	Ethnicity	Trial	*M*	*SE*
Number	Caucasian	1	3.55	.36
		2	10.37	.64
	Minority	1	3.18	.25
		2	7.77	.45

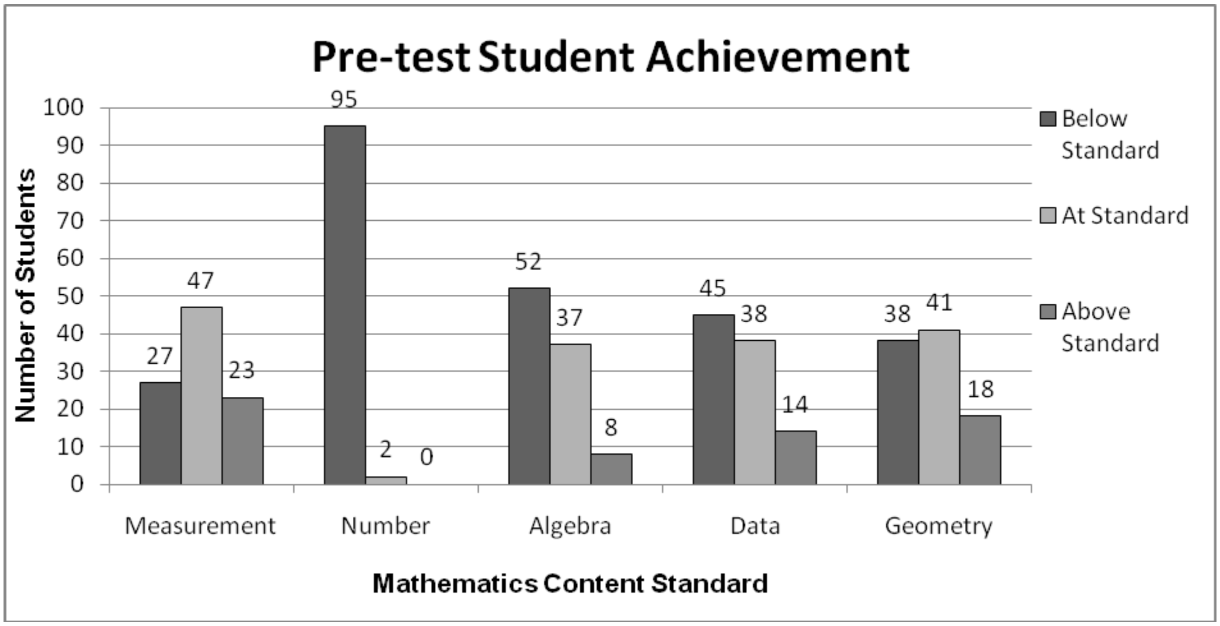

Figure 1. Number of students below, at, or above standard by content standard (pretest)

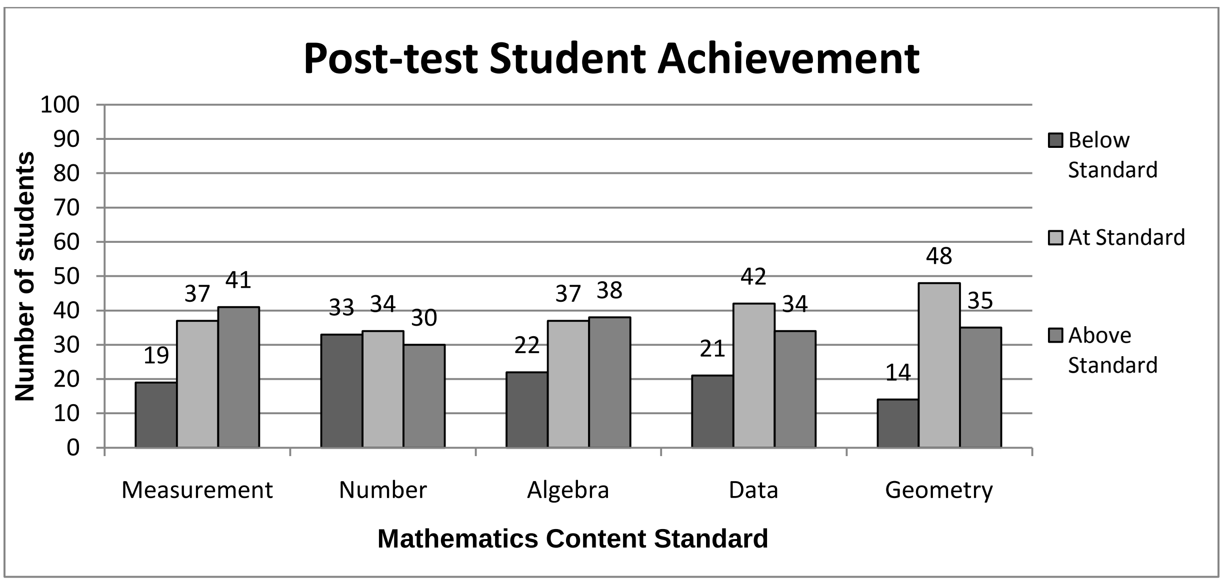

Figure 2. Number of students below, at, or above standard by content standard (post-test)

Figure 1 illustrates the number of "below", "at", and "above" standard students by mathematics content standard on the pre-test. As seen in Figure 1, the number of students deemed "below standard" on Number is disturbingly large. Figure 2 illustrates the number of "below", "at", and "above" standard students by mathematics content standard on the post-test.

The numbers of "below standard" students dropped in each of the five content standard areas. The number of "above standard" students also increased in each of the five content standard areas. The only content standard area in which there were more "below standard" students than "above standard" students was in Number.

Table 6 shows the Pearson Product Moment Correlations. There were significant positive correlations at the .001 and .01 levels between the pre-test raw scores from Measurement, Number, Data, and Geometry and the post-test raw scores from all five content area standards. The pre-test raw scores from Algebra had fewer significant correlations to post-test raw scores. These scores were only significant at the 0.05 level with the post-test raw scores for Number (*r* = .23, *p* = .023) and with post-test raw scores for Data (*r* = .22, *p* = .035). Neither school nor gender showed any significant correlations to any other variables, so they were left off of the table.

Table 6. Pearson Product Moment Correlations for Pre- and Post-Scores for Measurement Number, Algebra, Data, and Geometry Standards, and Ethnicity

Variable		1	2	3	4	5	6	7	8	9	10
1. Pretest Measurement	*r*										
	p										
2. Pretest Number	*r*	.58***									
	p	.000									
3. Pretest Algebra	*r*	.44***	.35***								
	p	.000	.000								
4. Pretest Data	*r*	.63***	.55***	.39***							
	p	.000	.000	.000							
5. Pretest Geometry	*r*	.49***	.37***	.41***	.47***						
	p	.000	.000	.000	.000						
6. Posttest Measurement	*r*	.42***	.38***	.19	.47***	.34**					
	p	.000	.000	.063	.000	.001					
7. Posttest Number	*r*	.53***	.35***	.23*	.56***	.38***	.53***				
	p	.000	.000	.023	.000	.000	.000				
8. Posttest Algebra	*r*	.52***	.38***	.11	.42***	.22*	.49***	.58***			
	p	.000	.000	.280	.000	.034	.000	.000			
9. Posttest Data	*r*	.52***	.43***	.22*	.52***	.45***	.63***	.60***	.62***		
	p	.000	.000	.035	.000	.000	.000	.000	.000		
10. Posttest Geometry	*r*	.45***	.32**	.15	.34**	.33**	.54***	.56***	.51***	.52***	
	p	.000	.001	.150	.001	.001	.000	.000	.000	.000	
11. Ethnicity	*r*	-.13	-.09	-.03	-.26*	-.17	-.27**	-.37***	-.11	-.23*	-.31**
	p	.197	.402	.804	.010	.090	.007	.000	.278	.022	.002

*Note. * p < .05. ** p < .01. *** p < .001*

Ethnicity showed significant negative correlations to four other variables. There was a significant negative correlation (r = -.27, p = .007) between ethnicity and Measurement, (r = -.37, p = .000) between ethnicity and Number, (r = -.23, p = .022) between ethnicity and Data, and (r = -.31, p = .002) between ethnicity and the Geometry. Caucasian students achieved higher in each of these four content area standards than their minority counterparts. There was no significant correlation between ethnicity and Algebra.

Conclusions

My goal for this project was to determine MCP's impact on each of the five mathematical content areas. We appear to have made great strides from pre-test to post-test in all areas with students in low-performing urban schools as evidenced by the dramatic decrease in the number of 'below standard' students and corresponding increase in the number of 'above standard' students in each standard. These gains are a testament to the commitment of our staff and coaches, as well as the teachers and administrators who participate in the program. Without support and resources from both MCP and each institution, we would not have made such gains.

Every program must recognize its weaknesses if it wants to continue to improve and grow. With MCP, it seems that our weakest content area is Number. Admittedly, this is probably the most challenging standard to address directly, but it is also probably the most important standard to address. Given the interconnectedness of procedural and conceptual knowledge, it is of paramount importance to instill a better sense of number in our students in order to give them the opportunity to excel in all areas of mathematics. Without a fundamental understanding of number and operations, their success in other areas will be limited by this factor. Additionally, there was a negative correlation between ethnicity and several of the content areas. While that indicates in this instance that Caucasian students did better in these areas than their minority counterparts, this does not necessarily translate to the general population. In the random sample, minority students outnumbered their Caucasian counterparts more than two-to-one.

In future examinations, alternate methods of sampling will be considered, either based on population or trying to balance the groups in equal numbers. Certainly, there were a number of limitations involved in this analysis, several of which have been corrected for future use. The first limitation was with the data collection tool itself, and that has been rectified. Secondly, inter-grader reliability may have been an issue for the pre-test scores. MCP uses the pre-test grading predominantly as an exercise for the teacher and coach to discuss student thinking. Additional limitations arose from my method of sampling students. The random number generator used to select the samples did so without regard to the completeness of their data, balance of ethnic or gender groups, and balance of their prior achievement. Also, this data presents only one year's worth of data, while each school can participate in MCP for up to three years.

There is also a question regarding the ability to generalize the findings to other settings. In future studies, with better controls, I think this problem could be alleviated. As generalization to other settings was not my initial aim, I didn't include these factors in my design.

Implications

The results of research bodes well for MCP and its associated classrooms. We now know that the student-level data that we collect can be analyzed to our benefit and the future benefit of our coaches, teachers, and students. Findings suggest fruitful venues for future research, including:

- Analysis structured to balance groups (gender, ethnicity, achievement) according to population breakdown to look for specific correlations.
- Individual items analyzed on the pre-test to determine specific content areas of concern.
- A longitudinal study of student growth throughout a school's involvement in the program to examine number sense and abstraction results.
- An examination of whether or not MCP's instructional philosophy has an impact in other academic content areas.
- A comparative study of MCP and other similar mathematics projects.

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TEACHER NETWORKS AND THE ROLE OF MATHEMATICS COACH: HOW INSTITUTIONAL FACTORS INFLUENCE COACH CENTRALITY

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Currently, the innovative practice of content-focused coaching in mathematics is being implemented by districts with very little understanding of its effectiveness. Theoretically it seems plausible that allowing teachers to have access to expertise in the form of mathematics coaching can assist teachers in developing ambitious mathematics instructional practices. This study aims to analyze mathematics coaching through the use of social network analysis in order both to uncover important aspects of the institutional setting and to clarify how those aspects relate to teachers' utilization of coaching.

Introduction

Over the past two decades, ambitious mathematics reform goals and standards for student learning, grounded in research on student learning in specific mathematical domains, have been proposed (e.g., see National Council of Teachers of Mathematics 1989, 1991, 1995, 2000). Meeting these goals and standards requires considerable learning on the part of practicing teachers, many of whom learned to teach under a different paradigm (Stein, Smith, & Silver, 1999). The necessary learning is transformative, requiring fundamental changes in deeply held beliefs, knowledge, and routines of practice (Thompson & Zeuli, 1999).

While it is challenging to support a small group of teachers in improving their classroom practice, it is even more demanding to provide learning opportunities for large numbers of teachers (Coburn, 2003; Gamoran, 2003; Elmore, 2004). One proposal for facilitating teacher learning at scale is to provide them with access to a more knowledgeable other in the form of a mathematics coach (Penuel, Riel, Krause & Frank, 2009). Many districts are implementing coaching designs as part of their instructional improvement efforts. In fact, a recent report on teacher development in the United States indicates that the adoption of school-based coaching programs is increasing rapidly, with about 46% of the teachers polled reporting that they participated in some form coaching at their school (Darling-Hammond, Wei, Andree, Richardson & Orphanos, 2009). Coaching as an instructional improvement strategy is currently being used by districts with very little understanding of its effectiveness.

One way to analyze the efficacy of mathematics coaching is through the use of social network analysis. Penuel and colleagues (2009) argue that social network analysis of teachers in a school offers a method for understanding how teachers' interactions relate to instructional change. In particular, investigating the extent to which the coach is central within networks is a first step toward uncovering important institutional structures that support teachers' access to instructional expertise. A number of studies document that teachers' instructional practices are shaped to a considerable degree by the materials and resources that they use in their classroom practice, the instructional constraints that they attempt to satisfy, and the formal and informal sources of assistance on which they draw (Cobb, McClain, Lamberg & Dean, 2003; Coburn, 2005; Stein & Spillane, 2005). Cobb and Smith (2008) indicate that the findings of these studies call into question an implicit assumption that underpins many reform efforts, “that teachers are autonomous agents in their classrooms who are unaffected by what takes place outside the classroom door” (p. 4). If empirical work has shown that teachers’ practices are shaped by the settings in which they work (Cobb & Smith, 2008), then it is reasonable to assume that coaches’ practices, too, are shaped by the settings in which they work. By combining information about the coaches’ positions within teachers’ social networks with interview data, we can uncover important institutional structures and resources that influence the extent to which teachers’ have access to expertise.

Methodology

This study is part of a larger, four-year longitudinal study in which we are collaborating with four, large urban districts that are attempting to support all middle-school mathematics teachers’ development of ambitious instructional practices. One of these districts (District B), on which we focus in this study, has implemented mathematics coaching as a key strategy to support middle-school mathematics teachers’ learning. District B designed its mathematics coaching program to support the implementation of an inquiry-oriented, NSF-funded, middle-school mathematics curriculum. As part of the design, principals in all middle schools in the district selected a mathematics teacher leader who would teach mathematics for half of the day and spend the other part of the school day working to provide learning opportunities for teachers. The selected mathematics coaches then received extensive professional development on the curriculum and how to work with teachers in their schools. District leaders intended that coaches would: 1) provide one-on-one instructional support to their colleagues in the classroom, 2) act as a resource for the principal on matters of the mathematics content, and 3) deliver professional development to middle-school mathematics teachers on district-wide PD days. The district intended the coaches to be a central mechanism in assisting teachers when they had questions about teaching mathematics.

This study draws upon online survey and interview data collected during the second year of implementing the coaching program. Our purpose in analyzing these data was to explain variation in how the coaching design was enacted in the seven schools participating in the study. The schools were selected to represent the range of all middle-schools schools in District B in terms of their capacities for instructional improvements. In October, we interviewed 15 district leaders to determine the current district design for mathematics coaching. In particular, we asked about the hiring of coaches, the envisioned role of coaches, the types of activities in which coach were expected to engage with teachers and principals, the professional development plan for coaches, as well as other supports provided for coaches. Between January and March, we collected data on how the design was being realized in the seven middle schools. In each school, we interviewed and surveyed the principal, the mathematics coach, and 3 to 5 randomly chosen mathematics teachers for total of 32 mathematics teachers in the district. In addition, all the participating teachers completed an online network survey. In interviews and in the surveys, we asked about participants’ perceptions of the role of the coach and about their interactions with the coach. Similarly, we asked the coaches about their interactions with teachers and principals, as well as how their work is supported.

The online network survey asked teachers to whom they have turned for advice or information about teaching mathematics. In addition, we asked about the frequency, influence, and the types of advice or information they sought from the people they identified. We then examined the teachers' social networks in an attempt to characterize the centrality of the coach within each of the seven schools. We used UCINET social analysis software (Borgatti, Everett, & Freeman, 2002) to create sociograms in order to understand the extent the teachers turned to the coach for advice in each of the seven schools. Because our focus is on the influence of individuals within the teacher networks, we defined centrality of an individual as the in-degree centrality, or the ratio of the number of ties directed toward that individual to the total ties that could be directed toward that individual (Hanneman & Riddle, 2005).

Using in-degree centrality, we calculated coach centrality in each of the seven schools. Table 1 contains information about the numbers of teachers within the schools, the number of survey respondents, and the calculated measure of coach centrality. We used this information to assign schools into three categories of coach centrality: central, somewhat central, and not central.

Table 1. School Teacher Counts and Measures of Centrality

School	# Math Teachers	# Teacher Responses	In-degree centrality	Coach Centrality
1	12	4	1	Central
2	9	6	0.333	Not
3	20	14	0.286	Not
4	12	8	0.25	Not
5	14	9	0.667	Central
6	6	6	0.833	Central
7	7	7	0.571	Somewhat

We coded all of the interview transcripts for the 32 teachers and their coaches and instructional leaders across the 7 schools. We created a coding scheme with categories that were developed from a review of the literature on coaching and supporting teacher learning, and from an initial reading of the interview transcripts (e.g., based on the questions that were asked and the participants' responses). The coding scheme was designed to characterize the following aspects of coaches' work in the schools: 1) activities in which the coach engages with teachers individually and in groups either in or outside the classroom, 2) additional responsibilities assigned to the coach beyond assisting teachers, 3) activities in which the coach engages with the principal, 4) evidence that teachers view the coach as a legitimate source of instructional expertise, 5) consistency of communication related to mathematics instruction or instructional improvement, and 6) the support the coach receives from the district and within the school to improve his or her practices. The first author coded teachers, coaches, and other instructional leaders (e.g., principals, assistant principals) interviews in order to take into consideration different people's accounts of the role of the coach. After coding all of the participants' interviews and considering coach centrality within teacher networks, we looked for patterns in the variation across the seven schools.

Case Selection

We conducted a cross-case comparative analysis because we wanted to uncover contextual conditions of the school (Yin, 2003) that support or impede a coach's efforts to support teachers' development of increasingly effective instructional practices. By comparing teachers' networks and aspects of the school setting, we uncovered some of the factors that appear to influence the extent to which mathematics teachers turn to the coach for advice or information about mathematics, and the extent to which the coach is able to support those teachers. The coach was unable to do the coaching work in two schools (one had health problems and the second was assigned to teach full time) and we did not have sufficient data from a third school (less than 50% of the mathematics teachers in the school competed to the online network survey). We therefore focused on four schools in order to examine centrality of the coach within teachers' social networks (2 central: School 5, School 6; 1 somewhat central: School 7; 1 not central: School 2).

Results

Despite the variation in coach centrality, we found similarities across the four focal schools: most of the teachers indicated that the coach had visited their classrooms and that they met weekly as a math department with the coach in attendance. Differences included: the reasons why coaches visited classrooms, whether or not the coach and teacher engaged in a follow-up conversation after the classroom visit, how often the mathematics department met, the purpose of the meetings, and whether the coach led the meetings. In addition, we found differences in structural aspects of the school (e.g., the size of the school) and the principals' views of their role in supporting coaches and teachers.

First, we report the structural aspects of the school that seem to support a coach in being central in the teacher network at that school. By structural, we mean aspects that are related to organizational characteristics of and organizational arrangements in the school (e.g., the size of the school, the physical proximity of mathematics teachers' classrooms, whether the department head is also the coach), and to characteristics of school personnel (e.g., the years of experience of the teachers, the grade level the coach teaches). We found that it is easier for the coach to be central in smaller schools primarily due to the fact that there are fewer teachers who might need their support. Although the mathematics teachers in all four schools met weekly as a department, the focus of the meetings and who led them differed. In the schools in which the coach was more central, the coaches led the meetings and were therefore able to set the agenda. In these same schools, the meetings frequently focused on issues of instruction (e.g., lesson planning, looking at student work) rather than on administrative issues (e.g., planning for Saturday school). We also found that the grade level the coach taught seemed to have implications for which teachers turned to her for advice. Recall that the coaches in District B also taught students for half of the school day. In all four schools, same grade-level colleagues identified the coach as someone they go to for advice. A significant proportion of teachers who taught at other grade levels said that they go to another same grade-level colleague before turning to the coach. Finally, physical proximity was also a factor – some teachers indicated they went to the coach because her classroom was nearby. These different structural aspects of the school influenced how frequently teachers interacted with the coach and thus the extent to which the coach was central within the teacher networks.

The centrality of the coach was also affected by the relationship between the coach and principal, how the principal envisioned the coach's role, and how the principal supported the coach's work. The principals have been given responsibility for hiring the coach, and all four principals indicated confidence in their selection, citing the coach's capabilities as a mathematics teacher and, in some cases, also describing their abilities as a coach (e.g., the coach is a good listener). However, the principal's vision for the coach's role and the ways in which they interacted with the coach differed. The principals all describe the role of the coach as that of supporting teachers; however, the principals in the two schools where the coach is more central were able to articulate how the coach was expected to assist teachers in more detail. In addition, principals in schools in which the coach was more central tended to assist their coach by: 1) procuring resources that the coach needed to support the teachers, 2) assisting the coach in identifying teachers to work with, 3) meeting with coaches on a regular basis, 4) assisting the coach with issues that arise as a result of being a teacher leader, 5) negotiating a vision of how instructional improvement in mathematics will occur in their school, and 6) sharing responsibility for enacting that vision. Furthermore, in the schools with more central coaches, the principals attended the mathematics department meetings regularly, pressed for an instructional focus in these meetings, and spent time in teachers' classrooms observing instruction and providing feedback. These principals were therefore in a better position to have conversations with their coach about teachers' instructional practices and could assist the coach in making decisions about teachers on which to focus and in what ways.

Teacher social networks are emergent phenomena that are continually regenerated in the course of ongoing interactions (Penuel et al., 2009). In other words, the emergence of teacher networks is influenced but not determined by organizational structure and social relationships, such as that between the coach and principal. We acknowledged the role of (institutionally situated) personal agency in our analysis by also examining the reasons why teachers might choose to turn to the coach. These reasons include the teacher's perceptions of the coach as a mathematics teacher, the teacher's perceptions of the coach as someone who can assist them in improving their instruction, and the prior history and level of trust between the coach and teacher. Teachers were more likely to interact with the coach when they felt the coach had a certain amount of expertise to offer the teacher, when they felt comfortable talking to her or identified her as someone who was a good listener, and when they had a history of working with the coach.

Discussion

The reasons why the teachers in our study did or did not seek advice from the coach have implications for some of the criteria that school or district leaders might use when selecting or hiring coaches. We found that certain structural aspects of the school, such as the size of school and who led mathematics department meetings, influenced coach centrality. In the schools in which coaches were central, the principal that structured time for coaches to meet with teachers and pressed them to work in meaningful ways on instructional issues. When teachers worked with the coach on instructional issues in department meetings, they were more likely to go to the coach outside the meetings, presumably because they had a history of doing that type of work together. These findings have implications for school leadership. By arranging the school day to allow for teachers to meet with one another to work on issues of instruction and encouraging the coach to set the agenda on key instructional issues and lead the meetings, school leaders can better support the role of the coach. This finding also has implications for how the district might support the coach. The district needs to not only continue to provide professional development specific to mathematics for coaches, but should also support them with in learning how to facilitate teacher meetings, and in coming to understand what types of activities might best support teachers' development of ambitious instructional practices.

Our findings have implications for districts when they create and implement designs for coaching. It is important for district leaders to negotiate the intended role of the coach with principals. Principals and district leaders need to discuss the coach's primary responsibilities and whether or not any secondary responsibilities might be added. We found that in those schools in which coaches were central, the principals were able to describe the coach's role and how they should work with teachers in some detail. These principals saw the role of the coach as assisting *all* teachers in improving their instructional practices over time whereas principals in the schools in which the coach was not as central saw the role of the coach as working with weak teachers. In all of the schools in our study, the coaches might have been more central in teacher networks if the principals had not assigned additional responsibilities that took the coach away from working directly with teachers. In each of the schools, coaches described extra duties (e.g., serving as department head or coordinating tutoring programs) that they carried out during the time that they were supposed to be working directly with teachers in their classrooms. District leaders can hold principals accountable for ensuring that the coaches' responsibilities do not grow beyond those intended by the design.

Our findings indicate that principals described their role in supporting coaches in different ways. In the schools with central coaches, the principal assisted the coach and collaborated with her to support instructional improvement. The principals and coaches in these schools shared responsibility for supporting teachers in implementing the adopted curriculum effectively. The principals in these schools attended mathematics department meetings and could monitor whether the agenda focused on instructional issues. In addition, these principals spent a considerable amount of time observing classroom instruction and could therefore discuss with the coach the types of assistance they believed teachers needed. In contrast, in schools where the coach was not central, discussions between the principals and coach were more likely to be about compliance issues (e.g., whether teachers were turning in lesson plans each week). Again, principals can be supported by being held accountable by district leaders for assisting coaches in assisting teachers to improve their instruction.

Another way that principals can support coaches is to assist them with achieving legitimacy in their school. Our findings indicate that teachers' decisions to seek advice from the coach are influenced by the extent to which they perceive the coach to be a legitimate pedagogical authority. We found that trust is an important factor related to gaining legitimacy. If the principal expects the coach to take on an evaluative role, or uses information from the coach in an evaluative manner, then this undermines the teachers' trust in the coach. Principals impact coach legitimacy significantly, and district leaders need to support principals in creating the appropriate conditions for the development of trust between teachers and the coach in their schools.

All of the schools in our study were, to some extent, under pressure to raise student achievement scores. To prevent this demand from influencing the implementation of the coaching design adversely, it becomes important for district leaders to assist principals in appreciating that the work of the coach is integral to the goal of improving the quality of instruction and thus student achievement. In the absence of this support, principals' responses to accountability pressures might result in differences in the responsibilities assigned to the coach, what the teachers perceive themselves to be responsible for, and the issues that are the focus of mathematics department meetings. It is therefore important that district leaders develop relationships of trust with school leaders by holding them responsible primarily for improving the quality of instruction.

In the study reported in this paper, we investigated the implementation of a district's design for coaching by examining how aspects of the institutional setting influenced the extent to which the coach was central in teacher networks. We identified several factors that district leaders should consider when designing and implementing coaching programs. With regard to future research, it will be important to investigate whether interactions between teachers and the coach give rise to opportunities for teachers to improve their instructional practices. Future research should also seek to identify the specific types of activities (e.g., modeling, co-teaching) in which coaches might engage with teachers that involve significant learning opportunities for teachers. Finally, it will be important to investigate what the coach needs to know and be able to do in order to provide high quality support for teachers. Understanding what high quality coaching looks like will allow districts to be able to hire better candidates from the start and know what types of professional development and other support those coaches will need.

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A CHILD'S VIEW: WHEN DOES HE/SHE START LINKING THE WORLD OF MATHEMAICS?

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The study consisted of individual interviews with approximately 100 children ranging from PK (4 years old) through 6th grade (12 years old). Each child was presented with two slide presentations. The eleven photos on each presentation were the same and were of various snapshots that most children would see in their everyday life i.e. buildings, railroad tracks, object made with Lego, etc. The first presentation asked the child to just describe what they saw in the photo such as a traffic sign, trees, blue sky, etc. The second presentation asked questions such as "Do you see any math in the picture" or "Do you see any math in the carriage? This particular slide was a horse-drawn carriage which had a big hotel in the background.

The concept of patterns is introduced to children in PK, yet when presented a picture of a Lego object, only a sixth grader reported the photo illustrated patterns. When asked if there was any math used to create the object in the photo, one 3rd grade student answered, "No, you just use your imagination and have fun." Most of the other students reported similar answers.

The focus of the current study was to try to determine if some of the math concepts are actually being taught at times which the child is not developmentally ready. With the study being limited to approximately 100 students, it is not possible to make generalizations to all children but it can be the vehicle to develop some further studies.

These children did not attend the same schools so it was very interesting to review the results of a number of the children. It became very apparent that they most likely had been studying a concept in their math class. Several thought there was measurement in every situation. Several others thought there was geometry in every photo. The slides will be shared in the presentation as well as the results of the interviews. Title: A Child's View: When does he/she starting linking the world to math?

In the first show the child was asked to just explain what they saw with no mention of math. After the first slide was completed, the child was asked to view the slide show again. This time he/she was asked to explain it they saw any math in the photo? For example PK children are making patterns yet, in the pilot test only a 6th grade child made the connection in the photo. Although the study involved a small sample (approx 100) which cannot allow generalizing, information can create a curiosity which may lead to larger studies.

EXPLORING CHILDREN'S MATHEMATICAL VOICES AS INPUT FOR IMPROVING THE TEACHING OF MATHEMATIC S

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In this poster presentation we will report on the findings resulting from the analysis of teachers' interviews with children during a professional development summer workshop. As part of the two-week summer workshop, teacher participants (grades K-9) interviewed children (identified as struggling students) who would be coming into their classrooms in the following school year. This activity required teachers to develop interview tasks for their student, interview the student, transcribe and analyze the interview, and plan for future instruction. The research questions investigated include: (a) To what depth do teachers analyze the mathematics of children when asked to find strengths in children's mathematical understanding? (b) To what extent do teachers use the findings from their interviews to plan meaningful experiences for the child?

The theoretical framework used for planning this experience is based on the works of Schifter (2001) and Little (2004) in which teachers' analysis of student work grounds teacher learning in professional development experiences, and Simon (1995) in which he emphasizes the need to understand students' thinking in crafting their hypothetical learning trajectories.

It is clear from our preliminary analysis of the written reflections that it is difficult for teachers to engage in non-evaluative listening and that, upon reflection, teachers became aware of the challenges. One particular challenge was overcoming the urge to provide corrective instruction during the interview. As one teacher wrote, "*Throughout this interview it was evident to me that the decisions I made as an interviewer and observer were critical to the information that I gained…It is pertinent that I think more critically about the unplanned questions I asked during the interview.*" Additional findings will be featured on the poster.

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LEADERS' SENSE MAKING OF FRAMEWORKS FOR FACILITATING MATHEMATICAL WORK IN PROFESSIONAL DEVELOPMENT

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Researching Mathematics Leader Learning (RMLL) is a five-year project investigating mathematics PD leaders' understandings and practices associated with fostering mathematically rich learning environments for teachers in PD. RMLL researchers' current focus is on connecting practices for orchestrating discussion (adapted from Stein et al, 2008) and fostering sociomathematical norms (adapted from Yackel & Cobb, 1996) to the development of teachers' specialized mathematical knowledge for teaching (Ball et al., 2008) through skilled and strategic use of mathematics tasks in PD.

This poster provides data from research on leaders' understandings of project frameworks as inventoried with a series of instruments. Profiles of leaders were constructed based on analyses of data from a cohort of leaders (n = 35) using responses to: (1) a Learning Mathematics for Teaching & Leading Survey (LMT), comprised of items selected by RMLL staff, (2) a pre and post seminar (quantitative and qualitative) questionnaire, and (3) a pre and post seminar scenario elaborating the project's frameworks in the context of mathematics PD. The coordination of these three instruments provides a unique means to purposefully organize data and identify patterns in leader responses with an associated LMT score.

As a result of generating these profiles, researchers were able to better identify the impact of leaders' sense making of these frameworks and how they drew on these resources in their work on mathematics tasks in leader seminars. Contrasts within leader profiles brought to light additional factors at play in leaders' work as participants in seminars and in their experiences facilitating PD. For example, further examination of PD participation and facilitation was required for leaders who took hold of important aspects of PD, such as attention to sociomathematical norms and developing teachers' MKT, yet had relatively low LMT scores. Given how little is known about the learning and work of mathematics PD leaders, these findings help further identify the resources drawn upon in the work of facilitation and how these ideas are taken up in leader development seminars designed to promote practices associated with productively engaging teachers in mathematical work.

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THE "FARMER DAN" PROJECT: IMPLEMENTATION OF JAPANESE LESSON STUDY IN RURAL MATHEMATICS CLASSROOMS

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In 2006, a team of middle school mathematics teachers received a grant to implement Japanese Lesson Study within their classrooms. The middle school had recently increased the length of 7th and 8th grade mathematics classes from 44 minutes to 88 minutes each day in order to improve mathematics achievement test scores. To effectively utilize the additional class time, teachers were interested in developing lessons that addressed problem solving and meta-cognition, as well as increasing relevance to students. In rural areas, science, technology, engineering, and mathematics (STEM) jobs are not as prevalent as they may be in urban areas, making it challenging for mathematics teachers to connect the content to students' lives. The team identified a topic for the lesson, area and volume of geometric solids, that spanned grade levels and was an area of poor student performance. The teachers then developed a lesson that incorporated word problems, locating resources, and writing about mathematics, as well as a connection to local farming and hay production. Further, the lesson reversed this school's traditional lesson cycle of lecture, practice, and re-teach; instead presenting a situation and investigation to encourage the students to search for the needed information as they progressed.

Following a modified version of Japanese Lesson Study to develop and improve the lesson, the teachers followed a cycle of teaching, modification of the lesson, and reteaching. While one teacher presented the lesson to the class, the other team members acted as non-participant observers. After the lesson, the team met to discuss lesson modifications based on observations and student input through consensograms and plus/delta charts. This cycle was followed through three implementations with different students. As a result of this project, teachers improved collaboration through professional development, and students benefited from different pedagogical techniques and assessments.

WHEN PROFESSIONAL DEVELOPMENT PRODUCES TEACHER CHANGE: A CASE STUDY OF MRS. G

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Professional development is often credited with promoting teacher change but evidence of success is sketchy at best. Classrooms exist, however, where goals for teacher change are realized and active student learning is the focus. Studying these environs may highlight strategies that could be replicated in teacher preparation and professional development. While collecting data for a connected classroom project, such a case came to light. The larger project sought to use Texas Instruments Navigator™, a wireless system, to open communication between students and teacher in mathematics classrooms, specifically Algebra I.

The four-year IES funded project aimed to promote student learning by enhancing classroom practices such as enriched classroom discourse, quality and levels of questioning, self-regulated learning and utilization of formative assessment. Recognizing that simply the presence of new software does not ensure teacher change, the project designers implemented many avenues for reflection and professional development. The professional development opportunities of the larger project include: making ideas relevant to participants, allowing time to change, offering ongoing support and encouraging participants to reflect on their learning. Formal professional development, led by practicing high school teachers who were also Teachers Teaching with Technology (T^3) instructors, was offered at summer institutes. Professional development sessions were also provided at T^3 international conferences. Additional support included a listserv, technical support and telephone interviews.

"Human motivation is a complex phenomenon, so it follows that mastery orientation is dependent on many…factors, not necessarily explainable by a single theory," (Owens et al., 2005). The conclusions of this study are that if teachers are going to engage in life-long, they must be treated as professionals and given a voice in their own educational pursuits. In the words of Mrs. G, "Learning sets people on fire…[but necessarily] the professional development must be relevant."

Endnotes

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VIDEO CLUB AS A TOOL TO INCREASE TEACHERS' PROFESSIONAL VISION OF STATUS ISSUES

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This poster describes an effort to use a video club to foster teachers' consciousness of status issues within interactions in secondary mathematics classrooms. A video club is a gathering of professionals for the purposes of viewing, analyzing, and discussing video of practice.

This poster follows closely the theoretical framework put forth by Sherin and van Es (2009) whose study focused on the impact of video club on teachers' *professional vision*, a concept introduced by Goodwin (1994). Sherin and van Es instituted video clubs that aimed to foster teachers' professional vision of interactions in which student thinking was visible.

In contrast, the video club of this project endeavored to use professional vision as a lens through which to address issues of equity within classroom interactions. In particular, we concerned ourselves with teachers' consciousness of *status*, which is described by Cohen and Lotan (1997) as "an agreed-upon social ranking where people believe that it is better to be in the high than in the low state" (p. 64). Status mediates participation by creating a classroom environment where participation rights may be distributed inequitably among students. According to Cohen and Lotan (1997), unequal participation is of particular importance, because learning correlates with participation; that is, the more a student participates, the more he or she will learn.

The data for this poster was gathered from a video club that included 14 secondary mathematics teachers from three urban high schools in a large school district in the Pacific Northwest. Three data sources were analyzed: notes from video club conversations, teacher interviews conducted at the start and end of the school year, and video of teachers' lessons at the start and end of the year.

The results of our analysis indicate that teachers exhibited an increasing consciousness of status issues in their video club contributions as the year progressed. When viewing a clip, teachers were more likely to comment on unequal participation patterns among students. Similarly, they were more likely to hypothesize about actions that a teacher might take in order to ameliorate such patterns.

The analysis points to several important findings. First, video clubs are a useful tool for improving teachers' professional vision of status issues. Second, the study establishes the viability of several video club design elements including a large participant roster and a multiplicity of participating schools.

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Chapter 17: Teacher Professional Development (Secondary)

Research Reports

Brief Research Reports

Posters

CHARACTERIZING PIVOTAL TEACHING MOMENTS IN BEGINNING MATHEMATICS TEACHERS' PRACTICE

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Although skilled mathematics educators often "know" when interruptions in the flow of the lesson provide an opportunity to modify instruction to improve students' mathematical understanding, others often fail to recognize or act on such moments. These moments, however, are key to instruction that builds on student thinking about mathematics. Video of beginning secondary mathematics teachers' instruction was analyzed to identify and characterize "pivotal teaching moments" in mathematics lessons and to examine how such moments play out during the lessons. Recognizing pivotal teaching moments and effective responses to them is an important first step for learning to capitalize on such moments.

Introduction

Accumulating evidence suggests that it is possible to identify instructional practices that lead to student achievement gains in mathematics. The list of such practices includes engaging students in using and discussing rich tasks, building on students' current thinking, and helping students connect mathematical ideas (e.g., Fennema et al., 1996; Stein, Smith, Henningsen, & Silver, 2000). This identification has prompted the study of how to best support teachers in implementing these ideas in their classroom.

One idea that has emerged from this work is the consideration of how teachers can be supported in learning to focus their attention on the features of classroom practice that are most important to student learning—student ideas, evidence of student understanding, and the relation between student thinking and pedagogical decisions (e.g., Santagata, Zannoni & Stigler, 2007; van Es & Sherin, 2008). Ball and Cohen (1999), for example, discuss the notion of teachers *learning in and from practice* that involves teachers "siz[ing] up a situation from moment to moment" (p. 11) and using what they learn to improve their practice. Van Es and Sherin (2008) use the term *noticing* to describe teachers' attending to important elements of instruction that support student learning and then reasoning about them in order to make instructional decisions. Although the specific language researchers use to describe the practice of keying in on important moments during instruction varies, the idea is the same—teachers need to pay attention to students' ideas and consider how to use these ideas to advance student learning.

Although skilled teachers and teacher educators often intuitively "know" when important mathematical moments occur during a lesson and can readily produce ideas about how to capitalize on such moments, these moments frequently either go unnoticed or are not acted upon by others in the profession, particularly novices (Peterson & Leatham, 2010). This raises the question of how teacher educators can help novice teachers recognize important moments during their instruction and use them to support student learning. Characterizing the circumstances likely to lead to such moments is an important first step in making them visible.

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

In this study, we define a *pivotal teaching moment* (PTM) to be an instance in a classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend, or change the nature of, students' mathematical understanding. This is related to Remillard and Geist's (2002) idea of *openings in the curriculum* in the context of teacher professional development, which they characterize as moments in which teachers' questions, observations, or challenges require the facilitator to make a decision about how to incorporate into the discussion the mathematical or pedagogical issues that are raised. The nature of the facilitator's decision determines the extent to which the teachers' ideas advance the learning of the group. Similarly, when a PTM occurs in a classroom lesson, a teacher first needs to recognize it as such, and then make a decision about how to handle the interruption. Depending on the action taken by the teacher, the PTM may or may not support the development of students' mathematical understanding. Because teachers may or may not act on a PTM, determining whether an event is such a moment is independent of the teacher's actual decision in response to the interruption. Consequently, an event can be characterized as a PTM even if the teacher does not notice it. The distinction here is that we are not trying to capture whether the teacher *did* modify instruction to support student understanding, but, instead, whether the events provided an opportunity for the teacher to do so.

To better understand the nature of these PTMs and how teachers might act on them in ways that support student learning, this study uses videos of beginning teachers' classrooms to investigate the following questions:

1. What are characteristics of pivotal teaching moments faced by beginning secondary school mathematics teachers during classroom instruction?
2. What types of decisions do beginning mathematics teachers make when a pivotal teaching moment occurs during their instruction, and what is the relationship between these decisions and student learning outcomes?

Perspectives

The idea that there are important moments or events within a mathematics lesson that a teacher needs to notice and act upon is grounded in a particular vision of teaching. Consistent with reform recommendations for mathematics education (e.g., National Council of Teachers of Mathematics, 2000), this vision is one in which teachers build on student thinking during instruction; this is accomplished through the teacher continually reflecting on the mathematical ideas that underlie students' comments or solutions and then using these comments or solutions in ways that help the class as a whole construct meaningful understandings of the mathematical ideas. Inherent in this vision of teaching is a need for teachers to reflect-in-action (Schön, 1983) in order to adapt instruction as it unfolds in response to students' current understandings (Ball & Cohen, 1999). In order to do so, teachers need to both *notice* important mathematical moments when they arise and have the *mathematical knowledge* and *dispositions* necessary to act on them in ways that support student learning.

In their Learning to Notice Framework, van Es & Sherin (2002) highlight three main components of teacher noticing in the context of analyzing artifacts of classroom practice: (a) identifying important aspects of the situation, (b) reasoning about these aspects, and (c) connecting what is observed to more general ideas about teaching. Although related, we view the type of noticing that teachers engage in post-instruction as quite different than that in which they must engage during instruction. During instruction, teachers have the benefit of neither time to systematically analyze student thinking and consider alternate interpretations of it, nor colleagues to help draw attention to important moments in instruction. Instead, teachers need to quickly draw on their own mathematical knowledge to recognize moments that may be mathematically important and then make decisions about how these moments could be capitalized on either to help address the goals of the lesson or to make connections to larger ideas in mathematics. Given the cognitive demand of this work and novice teachers' need to simultaneously pay attention to developing other more routine aspects of their practice (Berliner, 2001), it is not surprising that they are often unable to recognize these important mathematical moments and, if recognized, have difficulty making pedagogically sound decisions about them.

Although capitalizing on PTMs may be difficult for a novice teacher to accomplish, an important first step is recognizing that such moments exist. Without this awareness, teachers may experience *inattentional blindness* (Simons, 2000)—a phenomenon described in the psychology literature as a failure to focus attention on unexpected events. In the context of teaching, a teacher's failure to recognize that a student's ill-formed idea may be mathematically significant may be a result of a failure to recognize that this could be the case. This is also related to the idea of *framing* (Levin, Hammer & Coffey, 2009)—the way in which a teacher makes sense of a classroom situation. From this perspective, whether teachers notice the value in student thinking depends on how they frame what is taking place during instruction. If, for example, a teacher views a student error as something that needs to be corrected, he or she is unlikely to consider the mathematical thinking behind the error or whether the error could be used to highlight a specific mathematical idea. On the other hand, a teacher who views an error as a site for learning is more likely to consider both the mathematics underlying the error and how it could be used to develop mathematical understanding.

Modes of Inquiry

As part of a larger research project examining teacher learning using practice-based teacher education materials, over 45 hours of video of classroom instruction were collected from six beginning teachers. All of the teachers were graduates of an NCTM (2000) *Standards*-based secondary mathematics teacher education program that focused on teaching mathematics for student understanding. At the time of the data collection, they were teaching in grades 8-12 in a variety of school settings. The data included recordings of two classes per day for three consecutive days of instruction. We focused on beginning teachers because we hypothesized that the decision-making process involved with PTMs may be more obvious with this group than with skilled teachers. That is, skilled teachers may recognize a PTM and make the decision to act so quickly and smoothly that the interruption wouldn't be easily observable by someone who wasn't intricately familiar with the teacher's plan for the lesson. Furthermore, it is rare that a written lesson plan would have enough detail to note any but the most obvious changes in direction. Beginning teachers, on the other hand, were predicted to have a slower response time—slow enough that, although the thinking process itself wouldn't be visible, the fact that they were making a decision would be. This hypothesis was supported during our classroom observations. The analysis of the data reported here involved three phases: (a) reducing the video data to potential sites for PTMs, (b) identifying PTMs, and (c) characterizing PTMs.

Two graduate students who were experienced secondary school mathematics teachers completed the initial reduction of the video database. On their first pass through the video, they eliminated from the database any non-instructional activities, such as listening to school-wide announcements and completing individual assessments. On the second pass, they narrowed the video of instructional time down to episodes that had the potential to include PTMs. They individually identified any episodes that they felt had potential and then conferred to verify that the episodes warranted further exploration by the full research team. At the end of the reduction process, 65 episodes totaling 3 hours and 57 minutes remained in the database. The process of identifying PTMs involved a research team of the authors and, during much of the process, a fourth researcher. All were experienced observers of mathematics teaching. The episodes were first watched by the team to determine whether they agreed that a PTM had occurred during the episode. If it was agreed that one had occurred, the episode was transcribed for further analysis. At the end of this stage of the process, 28 episodes totaling 2 hours and 38 minutes remained in the database. Next, the four researchers individually watched these episodes and marked on the transcript the locations of the PTM, the teacher decision made in response to the PTM, and evidence of the outcome. At weekly research meetings, these individual notations were compared and discussed until agreement was reached.

As a result of this work, the research team was able to identify a preliminary coding scheme that was used to characterize the PTMs, the potential they had for advancing students' mathematical understanding, the teacher decision, the way in which the decision was implemented, and the likely impact on student learning. The researchers first coded the videos individually using the video analysis program Studiocode (SportsTec, 1997-2010) and then merged their video coding timelines for comparison. Again, differences were discussed until agreement was reached and the coding definitions were modified to reflect the refined use of the codes. Figure 1 gives the final labels used in the coding.

<table>
<tr><th colspan="2">Pivotal Teaching Moment</th><th colspan="2">Teacher Decision</th><th rowspan="2">Likely Impact on Student Learning</th></tr>
<tr><th>Type</th><th>Potential</th><th>Action</th><th>Implementation</th></tr>
<tr><td>Extending</td><td rowspan="5">High
Medium</td><td>Ignores or dismisses</td><td rowspan="2"></td><td>High positive</td></tr>
<tr><td>Sense-making</td><td>Incorporates into plan</td><td>Medium positive</td></tr>
<tr><td>Incorrect math</td><td>Pursues student thinking</td><td rowspan="3">Effectively
Moderately
Poorly</td><td>Low positive</td></tr>
<tr><td>Contradiction</td><td>Emphasizes meaning of the mathematics</td><td>Neutral</td></tr>
<tr><td>Confusion</td><td>Extends math or makes connections to other topics</td><td>Negative</td></tr>
</table>

Figure 1. Coding labels used to categorize pivotal teaching moments and the corresponding teacher decisions and likely impact on student learning

Examples of Episodes Containing Pivotal Moments

Episode 1

A class of Algebra 2 students have worked in small groups on a task that describes how a soccer ball is kicked and asks them to "sketch the path of the soccer ball and to find an equation for the parabola that models it." Verbally, the teacher had prompted them to first construct a graph, including axes, and then use the graph to find the equation. The teacher asks a student to draw her graph on the board. Although hesitant because she's not sure her graph is correct, the student draws her graph, which does correctly model the situation (although the teacher does not say this). The teacher then asks if anyone drew a different graph. A second student says that she did, and is invited to draw it on the board. This student draws a picture of the situation, complete with a soccer player and a ball flying through the air, but without axes or coordinates. A third student questions the graph, and subsequently imposes axes on the second student's "graph." The second student objects, claiming that all the extra lines make it confusing. After one more (correct but different) graph is drawn on the board, the second student again expresses confusion about the situation because there are three different depictions displayed on the board. The teacher responds by saying, "Then [student name], you don't have to do it this way. You can do it that way," implying that all three representations are equally valid. Another student says, "So you pretty much are saying we could do it whatever way in the world," to which the teacher responds, "You can do it a ton of different ways and still get the right answer."

Episode 2

As an Algebra 1 teacher is at the board discussing how to write an equation in the form $y = mx + b$ for the graph of a linear function, a student asks how one would know what number to add or subtract, that is, how to find the "b" in $y = mx + b$. The teacher explains that the number that is added or subtracted is always the value at which the graph crosses the y-axis. He uses a graph with a y-intercept at (0, 1) as an example, saying, "See, this is a positive one [points to the graph], so this is a positive one [points to the equation]." He then refers to another graph that was previously drawn on the board to show a second example with a y-intercept of negative one. At this point, a student asks if there can be "two dots"—referring to two different y-intercepts. The teacher responds, "There won't be two dots on the y-axis. That's a good question, though, you're thinking bigger," and then continues with his prepared explanation.

Episode 3

In their Algebra 2 class, students have worked on simplifying the expression $(16x^2y^2)^{-1}(xy^2)^3$ as part of their homework. Following a brief small-group discussion time, a student is selected to present her solution to the class. After she writes her solution on the board, another student expresses confusion. In the ensuing discussion, the teacher asks, "Did you just start a different way?" to which the student responds, "Yes. Her second step was, like confusing, but that's cool." The teacher asks whether she ended up with the same answer; the student confirms that she did. The teacher then invites her to explain what she did differently than the first student. After other students push her to do so, she writes her solution on the board and points out what she did differently. The teacher highlights the differences in the two solutions and confirms that the two solutions are both "fine."

Preliminary Framework for Looking at Pivotal Moments

When Pivotal Moments Occur

Pivotal teaching moments seem most likely to occur when students are actively engaged in the mathematics lesson—regardless of the nature of the classroom instruction or the curriculum used. That is, they can occur in classrooms where students are doing mathematics themselves and sharing their thinking with their classmates (as in Episode 1), as well as in classrooms where students are listening to the teacher present information and asking questions about what they are hearing (as in Episode 2). Five different circumstances leading to PTMs that we noted in our data are elaborated on in the following paragraphs.

One circumstance that seems to guarantee a PTM is when students make a comment or question that goes beyond the mathematics that the teacher had planned to discuss. For example, in Episode 2, the teacher was focused specifically on explaining how the “m” and “b” in the equation $y = \mathrm{m}x + \mathrm{b}$ can be found from the graph of a linear function. By wondering if it were possible to have more than one *y*-intercept, the student *opened up the possibility of extending the lesson* to make a connection with the definition of a function.

Other PTMs occur when students are *trying to make sense of the mathematics* in the lesson. Verbalization of this sense-making often provides opportunities to clarify or highlight critical mathematical components of the lesson. For example, a student who is trying to conceptually understand what is being presented as a purely procedural explanation may raise a question about why a procedure works. The student’s push for meaning is an indicator to the teacher that the lesson isn’t making important connections and is a prompt to begin to do so. A variation of this also occurs in lessons in which sense-making is the focus. In the process of trying to make sense of the mathematics, students may over-generalize or misconstrue aspects of the mathematics that, when surfaced, can provide learning opportunities for the entire class.

PTMs often occur when *incorrect mathematical thinking or an incorrect solution* is made public. In some cases an error does not qualify as a PTM—as when it is based on an incorrect calculation or something else that isn’t likely to interfere with or provide opportunities to improve students’ mathematical understanding. In other instances, however, the error is likely to affect what students take away from the lesson. We see this in Episode 1 where the student’s incorrect representation of the situation provided an opportunity to clarify the difference between a mathematical representation of a situation and a picture of the situation—a common idea with which algebra students struggle (Wagner & Parker, 1993). Not addressing the incorrect representation created the very real possibility of students leaving the lesson thinking that a picture is a valid mathematical representation, thus reinforcing this common misconception.

The occurrence of *a mathematical contradiction* also seems to generate a PTM. This can be as straightforward as two different answers to a problem that clearly should have only one answer, or as complex as two competing interpretations of a mathematical situation. Regardless, the contradiction creates an opportunity for the teacher to bring to the students’ attention the nature of mathematics that makes such contradictions unacceptable, as well as critical aspects of the mathematics at hand that can help them determine which of the options holds up under scrutiny. The very process of this scrutiny—the justification needed to support the different options and the making of a decision—can create a powerful learning opportunity.

Finally, a student’s *expression of mathematical confusion* can also lead to a PTM. It is important to distinguish general confusion—when a student expresses that they don’t know what is going on or they cannot follow what someone has just explained—from mathematical confusion. Confusion seems to lead to a PTM when a student can articulate mathematically what they are confused about. In Episode 3, for example, the second student was able to point to the second step of the first student’s solution as being her point of confusion. This gave a mathematical focus to the confusion—the properties of exponents that were used in that step—and created an opportunity for the teacher to refocus the students on the meaning behind the procedures they were applying to the problem.

Potential of Pivotal Moments

We drew on aspects of Stein and colleagues’ (2000) work on cognitive demand of tasks to think about the potential of PTMs to have a positive impact on students’ understanding of mathematics. Some PTMs, such as the one in Episode 1, seem to have an inherent high level of cognitive demand attached to them. Others, such as that in Episode 3, seem to involve a lower level of cognitive demand. Initially, we hypothesized three levels of potential for PTMs—high, medium, and low—but as we analyzed the data further, the episodes that led us to generate the low category no longer met our evolving definition of a PTM. That is, if the potential for learning was low, it was not a PTM. High potential PTMs involved rich mathematics and, frequently, connections among mathematical ideas. Often they provided a gateway to discussing important mathematical ideas that were not part of the planned lesson, but were an important part of the mathematical terrain the students were traversing. Medium potential PTMs often related to attributes of mathematics, such as its usefulness or coherency, and provided the opportunity to better understand procedures, definitions, or concepts.

The decision a teacher makes about how to respond to a PTM—ranging from to ignore or dismiss it, to using it as an opportunity to extend or enhance the planned lesson—shapes whether the potential of the PTM is actualized. The resulting outcome of the PTMs ranges from significantly increasing student learning opportunities, to having a negative impact, as seen in Episode 1. In that episode, the teacher's decision to pursue and support a student's thinking came at the expense of the mathematics in the lesson. As a result, rather than achieving the high potential of the PTM, the teacher's decision led to the PTM having a negative impact on student understanding. One of the things we are interested in exploring is whether certain PTMs—such as those arising from incorrect mathematics—are more likely to have a negative effect on student understanding and, thus, must be dealt with more proactively by a teacher.

As can be seen in Figure 2, we are currently conceptualizing these events in the classroom as triples that include the pivotal moment, the teacher decision, and the likely impact on student learning. We are particularly interested in using our data on beginning teachers to determine whether there seem to be types of PTMs that are more likely to lead to improved student learning opportunities. It may also be that there are types of PTMs that are not likely to do so; if this is the case, knowing so would allow teachers to more accurately and efficiently make decisions that would minimize unproductive use of class time.

Figure 2. Triple of a pivotal teaching moment, the corresponding teacher decision, and the opportunities they provide for student learning

Conclusion

The intent of this work is to provide a means to capitalize on pivotal teaching moments that occur during classroom instruction. Teachers are faced with a myriad of decisions on a moment-by-moment basis in their classrooms. We argue that PTMs, by their very nature, prompt teacher decisions that can have a high impact on the learning that goes on in a classroom. As such, they are worthy of analysis so that high-leverage PTMs and effective responses to them can be identified and shared with teachers. This is particularly useful in the case of novice teachers who have less experience and knowledge to draw on as they make their decisions. Being able to recognize a PTM and effective responses could improve beginning teachers' ability to act in a way that would increase their students' mathematical understanding.

While this work is an important first step, we recognize that strong mathematical knowledge for teaching is key to responding to PTMs in ways that enhance students' mathematical understanding. For example, to productively respond to a PTM, a teacher needs to decipher the mathematics underlying a student's response and consider how it fits with the goals of the lesson or broader goals of the course. Future work around conceptualizing the *mathematical knowledge for teaching* (Ball, Thames, & Phelps, 2008) required to act on these moments in a way that supports student learning will provide important insights into the process of becoming the kind of mathematics teacher advocated by the NCTM (2000).

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QUANT PROGRAM EVALUATION AND REVISION BASED ON AN ANALYSIS OF TPACK GROWTH AMONG THE PARTICIPANTS

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Quantifying Uncertainty and Analyzing Numerical Trends (QUANT) is a yearlong professional development program for high school mathematics teachers focused on statistics, probability, and data analysis. QUANT is designed to develop teachers' statistical proficiency for teaching using technological pedagogical content knowledge *(TPACK) as a framework. This paper describes the program, its goals, and results from an exploratory pre-post investigation and a participant survey, both designed to evaluate the program's effectiveness and to revise the program itself. Results show that program participants gained content knowledge, pedagogical content knowledge, and increased their confidence and desire to incorporate technology into their instruction.*

Introduction

As many U.S. states are moving toward requiring four years of mathematics for *all* high school students, in accordance with the National Teachers of Mathematics (NCTM) *Math Takes Time* position statement (2006), it is vital for teachers to be prepared to teach subjects that go beyond the concepts taught in algebra and geometry classes while engaging students in tasks that promote reasoning, communication, and connecting mathematics to the world outside the classroom. Thoughtful implementation of data collection, data analysis, and statistical activities is one of the best ways of providing students with such experiences. Teaching mathematics in a way that assists high school students to develop *statistical reasoning*, however, presents challenges distinct from teaching other forms of mathematical reasoning (Ben-Zvi & Garfield, 2004; Cobb & Moore, 1997; Groth, 2007). One of these challenges involves a subtle yet profound shift in focus with regard to *context*. Cobb and Moore note that mathematics often purposely ignores context to focus on the underlying structure of a problem situation. In statistics, however, context is what provides much of the meaning for a given problem. Despite this, Ben-Zvi and Garfield (2004) assert that students can struggle with the context involved in a statistical task. This is often due to the amount of complex mathematics involved as well as students' focusing too much on the numbers and calculations without truly engaging with the task. In addition, the concept of *variability* plays a critical role in statistics, as does the ability to *formulate a question* in a way that attends to the context and anticipates variability (Franklin et al., 2007). A final challenge is that many practicing high school mathematics teachers have had little formal education in data analysis, probability, and statistics.

In recent years, leading professional organizations and national initiatives have begun to recognize the need for statistics education in Grades K–12 (American Diploma Project, 2004; Common Core State Standards Initiative, 2009; Franklin et al., 2007; NCTM, 2009b). Concerning the need for learning statistics and probability, NCTM (2009a) states,

> In our increasingly data-intensive world, statistics is one of the most important areas of the mathematical sciences for helping students make sense of the information all around them, as well as for preparing them for further study in a variety of disciplines (e.g., the health

sciences, the social sciences, and environmental science) for which statistics is a fundamental tool for advancing knowledge. (p. 73)

The construct of *quantitative literacy,* or *numeracy,* varies from author to author. For Paulos (1988/1990), numeracy was the ability "to deal comfortably with the fundamental notions of number and chance" (p. 3), coupled with logical reasoning. Steen (1990) took a broader view of numeracy as grounded in pattern and including dimension, quantity, uncertainty, shape, and change. For us, *quantitative literacy* is a form of general literacy that includes: (a) numerical reasoning, (b) a working knowledge of measurements and indices, (c) statistical reasoning (a la Franklin et al., 2007), (d) all of Steen's modeling components, (e) the ability to link contexts with appropriate mathematics (modeling), and (f) facility in the *mathematics register*, academic mathematical speaking and writing (Schleppegrell, 2007).

The needs of teachers in the areas of data analysis, probability, statistics, and quantitative literacy are a serious concern for those who care about mathematics education in the United States. These needs are precisely what are driving the ongoing research and development of the *Quantifying Uncertainty and Analyzing Numerical Trends* (QUANT) project.

The QUANT Professional Development Program

QUANT is designed to develop *statistical proficiency for teaching*. The RAND Mathematics Study Panel (2003) identified "developing teachers' mathematical knowledge in ways directly useful for teaching" (p. 7) as a priority for their research and development program targeted at enhancing the mathematical proficiency of all students. Although the concepts, strategies, and reasoning processes differ between mathematics and statistics, a key premise of our theory of action is that the National Research Council's (2001) components of mathematical proficiency—*conceptual understanding*, *procedural fluency*, *strategic competence*, *adaptive reasoning*, and *productive disposition*—are critical dimensions of statistical proficiency as well. Following Foley, Strayer, and Regan (2010), we define statistical proficiency for teaching as teachers' knowledge and skills that are useful in promoting the statistical proficiency of their students.

The mechanism that QUANT uses to develop this statistical proficiency for teaching is technological pedagogical content knowledge (TPACK) (Neiss et al., 2008). QUANT addresses TPACK in the areas of (a) measurement and data collection, (b) data analysis and descriptive statistics, (c) combinatorics and probability, and (d) statistical inference. These content foci are combined with a pedagogical and classroom implementation focus on selecting, setting up, and enacting high cognitive level tasks (Boston & Smith, 2009; Stein, Smith, Henningsen, & Silver, 2009). The technology addressed and used in QUANT includes data collection, data transfer, memory management, lists, spreadsheets, and interactive statistical software.

QUANT is a yearlong professional development (PD) program for high school mathematics teachers. The goals of this PD include the following:

- To engage teachers in investigating cognitively demanding mathematical and statistical tasks as a way to learn mathematics and statistics.
- To help teachers learn how to set up and maintain cognitively demanding mathematical and statistical tasks in their classroom instruction.
- To develop teachers' capacity in the areas of data collection, data analysis, probability, and statistics.

- To develop teachers' comfort level and facility in using handheld data collection and data analysis technology, and connectivity and statistical software, as tools for instruction.

The QUANT program begins with an intensive 2-week summer institute that explores data collection, data analysis, probability, and statistics concepts and problems using technology. The program includes online support and daylong follow-up workshops during the ensuing school year. During the 2008–2009 implementation phase of QUANT, the *mathematical tasks framework* as a tool for teaching and learning mathematics was introduced to the QUANT participants during the third such follow-up workshop (Boston & Smith, 2009; Stein et al., 2009). Because Boston and Smith demonstrated that using the mathematical tasks framework significantly altered mathematics teachers' decisions to select and implement high-level tasks, the QUANT team decided to incorporate the framework's use in the program.

The activities incorporated in the PD stress the four components of statistical problem solving articulated in the *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report* framework (Franklin et al., 2007):

1. Formulate questions.
2. Collect data.
3. Analyze data.
4. Interpret results.

With these components in mind, the activities are focused on hands-on, inquiry-based, technology-enhanced, content-rich tasks in data analysis, probability, and statistics. The instructional sessions model a learning environment that uses technology to "support teachers' efforts in fostering students' understandings and intuitions by engaging them in conceptual conversations" (Knuth & Hartmann, 2005, p.162). These conceptual conversations are aimed at creating an environment that encourages reasoning and sense making in the mathematics classroom.

QUANT institute activities are aligned with NCTM's (2009a) assertion that focusing on reasoning and sense making does not need to be an extra burden for teachers. In fact, a focus on reasoning and sense making provides context for students who struggle because they find mathematics unconnected to their life outside the classroom. The QUANT institute models NCTM's belief that "with purposeful attention and planning, teachers can hold all students in every high school mathematics classroom accountable for personally engaging in reasoning and sense making" (p. 6).

Context of the Research

The QUANT project employs an iterative research and development design to address the research question: *How can professional development improve the statistical proficiency for teaching of high school mathematic teachers?* The PD materials and resources and their effectiveness are objects of research, as are teacher variables. The research reported herein is part of this overall research agenda. As with other exploratory evaluations of QUANT, the purpose of the studies reported here is to provide feedback to refine the materials and instructional methods. The QUANT team has hypothesized that improving teachers' use of high cognitive level tasks in data analysis, probability, and statistics will have a positive impact on the statistical proficiency and quantitative literacy of their students. There are many steps from the proposed intervention to the anticipated student outcomes. In keeping with Foley, et al. (2010), Figure 1 provides a framework for our theory of action.

Three exploratory QUANT programs have been offered to date. Each involved a summer institute:

- QUANT 1 at Ohio University in Athens in June 2008 involved 5 practicing teachers, 9 teacher candidates, and 4 teacher educators.
- QUANT 2 at the Ohio Resource Center in Columbus in August 2008 involved 10 practicing teachers and 5 teacher educators.
- QUANT 3 at the Ohio Resource Center in Columbus in June 2009 involved 8 practicing teachers.

Each of these program included implementation and follow-up during the ensuing academic year. Additional QUANT programs are planned for Summer 2010 through Spring 2011. This paper focuses on an analysis of pretest and posttest scores and an exploratory survey to evaluate the QUANT 1 and QUANT 2 programs, both of which were conducted from Summer 2008 through Spring 2009.

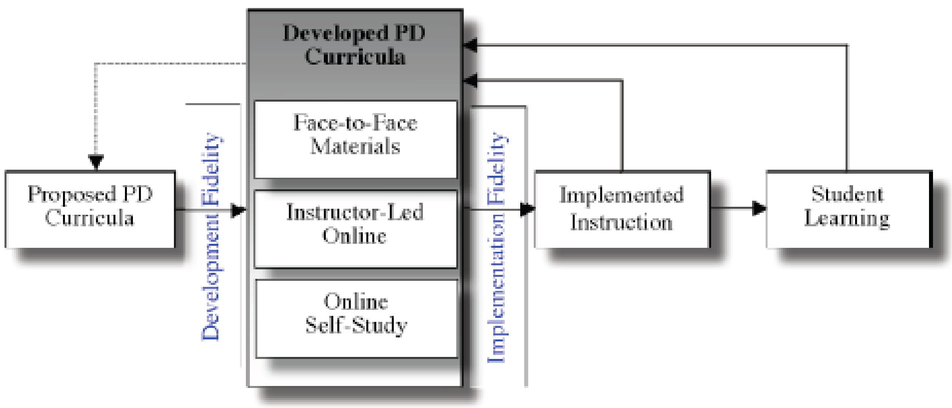

Figure 1. Model illustrating the interaction among the current and anticipated objects of research in the QUANT project

In Firgure 1, the left-to-right arrows show a chain of events from developer planning to student outcomes. The arrows looping back to the left represent the ongoing development that is based on feedback from teacher practice and student attainment. This feedback is being used to refine the professional development materials, and even the premises of the design itself, as needed to achieve the desired outcomes. (Figure and caption from Foley et al., 2010, p. 3)

The Studies

This paper reports on two separate research studies. One of the studies consisted of a quantitative analysis of pretest and posttest scores from QUANT 1 and QUANT 2 participants to evaluate change in content knowledge and pedagogical content knowledge. The other study was a fellowship research project based on a survey addressing what QUANT participants acquired from the PD program, together with an analysis of the responses (Regan, 2009). The survey collected both quantitative and qualitative data that were use to identify (a) QUANT participants' confidence in teaching statistical and probabilistic concepts; (b) confidence in using technology in the classroom; and (c) factors that influenced confidence and implementation of QUANT materials.

Methodology

The sample for the pretest and posttest analysis included nearly all of the QUANT 1 and QUANT 2 participants. It should be noted that one of the pretests and one of the posttests for QUANT 1 were not included in this analysis as they were from participants that attended only one week of the institute. The pretests were administered in the afternoon on the first day of the summer institutes, after a few brief introductory activities. Questions included single-answer multiple-choice items, multiple-answer multiple-choice items, open-response numerical items, open-ended essay items, and graphical interpretation items. The posttest was identical to the pretest and administered on the final day of the summer institute. The data were analyzed using a *dependent t-test* as well as a *sign test of the median*. Only QUANT 2 participants' change in pedagogical content knowledge was analyzed because the pretest and posttest of QUANT 1 contained no items assessing pedagogical content knowledge. The QUANT 2 items addressing pedagogical content knowledge were scored using a continuous scale from –1 to 1; a score of –1 meant a correct answer on the pretest and an incorrect answer on the posttest, a score of 0 meant no change, and a score of 1 meant an incorrect answer on the pretest and a correct answer on the posttest. External evaluator Reed and a graduate research associate Regan assigned these scores, given them a somewhat subjective quality.

The survey was emailed to participants a week before the final follow-up meeting and participants had the option of returning by email or bringing a hard copy to the follow-up meeting. Of the 24 participants who attend QUANT 1 and QUANT 2, 19 surveys were returned. The qualitative data were analyzed to find the demographics of QUANT participants as well as their confidence in teaching statistical and probabilistic concepts, their confidence in using technology, and factors that influenced their implementation of QUANT materials. Descriptive statistics of the variables corresponding to the survey were calculated, and the data were used to determine whether there was a significant growth in confidence as a result of the institute. The qualitative data were analyzed for themes about how QUANT participants had used the materials they received at the summer institute as well as why some participants had not implemented some materials. For the quantitative analysis, *Wilcoxon Sign Rank tests* as well as the *Kruskal-Wallis tests* were used to determine significant differences in attitudes of the ordinal data.

Results

Results from the analysis of pretest and posttest scores showed significant gains in content knowledge for participants of QUANT 1 and 2 and significant gains in pedagogical content knowledge for participants of QUANT 2. The sign test of median showed a .0017 probability of results occurring without the institutes. In addition, the 8-item subscale assessing pedagogical content knowledge showed growth at the .05 level of significance. This subscale included items to assess the participants' knowledge of the mathematical tasks framework, criteria for high-level tasks, the four steps for statistical problem solving, and the importance of context in statistical problem solving.

The results from the quantitative analysis of the survey data showed significant gains in confidence when teaching statistical related concepts and incorporating technology in one's classroom. The data also showed that participants were generally confident when implementing the materials they had obtained during the QUANT institute. Various tests revealed that confidence levels did not significantly depend on gender, years of teaching, type of school (public or private), location of school (rural, suburban, or urban), or type of teacher (pre-service, in-service, or college faculty). While analyzing the qualitative data, two main concepts emerged: technology and learning community. Many participants made comments about how the QUANT institute helped them feel comfortable using technology and incorporating it into their classroom.

> *I am really glad I had the opportunity to work so much with the TI-nspire calculators. Since I felt that using technology was a weakness for me, I really feel prepared to use the calculator for activities with my students, especially for data representation and such* (Answer to survey question reported in Regan, 2009).

Yet, it should be noted that many participants were concerned with the availability of that technology outside of the institutes. As for the learning community aspect, many participants mentioned how they enjoyed the sense of community involved with the institute. Many participants shared thoughts about how they enjoyed collaborating with other educators in similar fields and positions as well as with the QUANT team. These results align with those reported in by Reed (2009). Some of the materials that teachers found helpful and useful were the technology and compatible software, the textbooks and reference materials, and the QUANT activity worksheets. In response to a question about what has hindered the implementation of materials, participant answers included student knowledge base, statistics did not fit into their course's objectives, lack of technology, and lack of time to incorporate QUANT materials and activities.

Implications for the Revision of QUANT

The research reported in this paper focused primarily last two goals of the QUANT program:

- To develop teacher capacity in the areas of data collection, data analysis, probability, and statistics.
- To develop teachers' comfort level and facility in using handheld data collection and data analysis technology, and connectivity and statistical software, as tools for instruction.

In regard to developing teacher capacity in the areas of data collection, data analysis, probability, and statistics, the analysis of pretest and posttest scores shows that the QUANT program can help educators significantly increase their statistical content knowledge and pedagogical content knowledge. The survey collected data that showed evidence of participants increased comfort level and capability of incorporating handheld data collection and data analysis technology.

Some recommendations for future QUANT institutes to make it more effective include:

- To include discussions about how activities can be implemented into non-statistical classrooms.
- To include discussions on how to adapt activities to meet the needs and knowledge base of one's students.
- To explicitly discuss how to maintain the cognitive level of tasks.
- To discuss possible funding options for technology and other materials.
- To add tutorials on how to write grant proposals.

Discussion

The QUANT team used the results from these studies to revise the program for 2009–2010: the QUANT materials developed specifically for the program, the resources given to the participants, and the methods used to measure teacher change as a result of their participation in the program. Some of the changes that were made included increased time to discuss implementation plans and concerns, an online resource site for all QUANT participants, as well as giving all participants a copy of *Implementing Standards-Based Mathematics Instruction* (Stein et al., 2009). "The 2009 QUANT institute was not just a replication of the 2008 pilot institutes; it was an improvement" (Wagner, 2009, p. 10). The QUANT team is now the process of revising the QUANT program for 2010–2011. One of the challenges for the QUANT teams is adapting the *Survey of Preservice Teachers' Knowledge of Teaching and Technology* (Schmidt et al., 2009) to practicing secondary school teachers and to the TPACK content associated with the QUANT program. Specifically, the team intends to measure TPACK in the areas of

1. Formulating questions and designing statistical studies,
2. Measurement and data collection,
3. Data analysis and descriptive statistics,
4. Combinatorics and probability,
5. Interpreting results and drawing statistical inferences.

The QUANT program for 2010–2011 will include two formats. One will be the original format: 2 weeks of face-to-face instruction along with follow-up workshops and online support. The other format will replace 3 days of the summer institute with six 3-hr instructor-led online modules. Ultimately, the QUANT program will be offered in three delivery formats: (a) face-to-face, (b) instructor-led online, and (c) online self-study. The future challenges for the QUANT team will be to refine the materials to fit these multiple formats, while keeping in mind the recommendations mentioned above as well as to develop student assessments to evaluate whether and to what extent the QUANT program has a positive effect on student learning. Through ongoing formative research the QUANT program will continue to improve its ability to assist teachers in their development of statistical proficiency for teaching and hence help students in their development of statistical proficiency.

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SUPPORTING STUDENTS BY SUPPORTING TEACHERS: COACHING MOVES THAT IMPACT LEARNING

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Mathematics coaches are being hired in an effort to improve students' mathematics learning by supporting teacher development. However, little is known about the activities of coaching and how those activities translate into teacher learning and better instruction This paper unpacks the coaching moves of one high school mathematics coach as she facilitates conversations with a collaborative team of teachers. Our analysis points to particular coaching moves that engage teachers in a process of guided participation. These moves include maintaining focus, assigning competence, and generalizing teachers' stories. Initial findings indicate that this work has a positive effect on teachers and their students. This research has direct implications for those interested in shaping the work of coaching in secondary mathematics.

Introduction

The National Mathematics Advisory Panel (2008) reported that many schools across the country are calling for math specialists, including mathematics coaches, to increase students' mathematical understanding and academic achievement, despite a lack of clarity in both research and practice about how coaches effectively support teachers to improve instruction. As part of a larger research study and professional development project aimed at reforming high school mathematics instruction, researchers implemented a coaching model for algebra and geometry teachers in two urban high schools in the Pacific Northwest. This study looks in-depth at the work of an instructional coach in an effort to name specific coaching moves that impact both teacher and student learning. We use a socio-cultural learning framework to frame the analysis.

This paper first provides a brief overview of the coaching literature followed by a description of the larger research project in which this study is situated and the theoretical framework used in this work. Results are discussed along with implications for coaching in general, and mathematics coaching in particular. This work has the potential to aide those seeking to understand this relatively new and increasingly popular educational role.

Literature Review

The concept of coaching stems from the professional development literature that indicates the most effective way to support teacher change is through personalized and differentiated development. For example, in a study using teachers' self-reported changes, Garet and colleagues (2001) claimed that "professional development focused on academic subject matter (content), gives teachers opportunities for 'hands-on' work (active learning), and is integrated into the daily life of the school (coherence), is more likely to produce enhanced knowledge and skills" (p. 935). In addition, the authors reported that professional development is most effective when it spans time (see also Hawley & Valli, 1999). Such results have influenced K-12 leaders to implement instructional coaching as a means for improving teaching and learning.

The literature on coaching, however, is thin. Much of what has been published is either anecdotal or conceptual. Few empirical studies exist that describe rigorous methodology. Terms are often left undefined and ambiguous. Additionally, little effort has been made to explicitly connect coaching to theoretical frameworks of learning. Research on mathematics coaching, particularly in secondary mathematics classrooms, is even more scarce. The National Council of Teachers of Mathematics reported only seven studies in the past 20 years focused on mathematics coaching, with no common claims across the studies (NCTM, 2009).

The existing literature provides some insight into the sorts of activities expected of coaches. Typically, the main job of an instructional coach is to meet with teachers, although the content and form of these meetings is often not well-defined. The assumption is that when coaches and teachers meet, teachers are subsequently better able to support student learning. For example, Knight (2006) worked with many coaches and claimed that effective coaches “spend the bulk of their time working with teachers on instruction” rather than completing clerical or non-instructional tasks (p. 37). Gibson (2006) demonstrated how individual meetings between a coach and a teacher after observations enabled the teacher to improve reading instruction. In another school, coaches met daily with grade-level teams or individual teachers (Dempsey, 2007). While no distinction is made between working with individuals or groups, meetings with teachers through observations and follow up debriefing is most often the crux of the research literature about coaching.

The meetings between coaches and teachers often take the form of observations and feedback cycle. Dole’s (2004) found that the coach feedback from to be critical for improving teachers’ practices. Shanklin (2007) also recognized the key role of observation and feedback in coaching for the improvement of practice. Observations and feedback sessions appear in the literature as the crux of the work of coaches. However, the content of both observation and feedback is vague and rarely discussed.

Research concerning instructional coaching in mathematics suggests that coaches should meet with teachers, observe classrooms, and offer feedback as they help teachers improve instruction. However, the content of these meetings and the focus of the observations have not been thoroughly explored. Coaching has not been comprehensively studied or explicitly connected to theories of teacher learning. The research presented in this study attempted to bridge this gap by investigating specific coaching moves of a successful coach working with a secondary mathematics teaching team.

Purpose of Study

The work discussed in this paper is part of a larger research study and professional development project aimed at high school mathematics reform. The overarching goals of project are to: 1) train teachers to use powerful pedagogies aimed at disrupting typical hierarchies of status that often negatively impact students’ participation and access to rigorous tasks with a broad range of students in urban schools, 2) build and sustain site-based teacher learning communities that take up problems of practice and, 3) restructure what it means to be a mathematics teacher and learner in secondary math classrooms for the purpose of increasing students’ engagement and learning.

In this larger project, teachers engage in several professional development opportunities that focus on developing understanding of students’ mathematical thinking and learning. Teachers attend a monthly Video Club (Sherin & Han, 2004) focused on teacher moves that support equitable student participation and mathematical understanding. The math teachers also work collaboratively within their departments. They are provided a daily common planning time to plan instruction and reflect on their practice. The expectation is that the teachers will develop a professional learning community as a result of their participation. Such communities have been found to positively impact student learning (McLaughlin & Talbert, 2001; Vescio, Ross, & Adams, 2008). Finally, an instructional coach works with the teachers involved in this project. She spends one day a week in each school facilitating common planning time and engaging in one-on-one coaching with teachers. Through these different learning opportunities, it is expected that teachers will be better equipped to implement ambitious pedagogies designed to specifically address issues of academic and social status that often effect students' decisions for when and how to engage in the learning opportunities provided. These practices are grounded in the belief that all students are intellectual resources and capable of academic success (Jilk, 2007). In this study we analyzed specific coaching moves made by the instructional coach, Jill, while she worked with mathematics teachers during their daily common planning time.

Participants

Fourteen secondary mathematics teachers from three public high schools participated in the larger research project in which this study is located. For the purposes of this paper, we focused on one particular school, Lotus High for several reasons. First, the teachers at Lotus made significant changes in their professional interactions during the first year of this project. They increased the amount of time they collaboratively discussed mathematics, their students, and how to teach mathematics. Second, the entire team of teachers continued their participation in the second year of the project, and another department member chose to join the team. Finally, preliminary analysis of student achievement data at Lotus indicates that 70% of students passed algebra and geometry compared with only 40% when this project first began. Additionally, student enrollment increased in upper-level mathematics courses.

Lotus High is a traditional comprehensive urban high school with the following student demographics: 55% Black, 25% Asian, 10% Hispanic, 6% White and 64% free and reduced lunch (NCES, 2010). Lotus's proximity to a major research university with a focus on urban education makes it a prime location for intervention-focused research. The school has experienced many reforms over the years. The entire school staff worked with a motivational framework as well as a variety of professional development foci from its district in 2008-2009. Additionally, a new principal, known for successfully improving schools, was placed at Lotus with the goal of improving standardized test scores.

The math department at Lotus High School consisted of seven teachers, although only four were involved in the work discussed in this paper. These four teachers, Jennifer, Natalie, Carly, and Akira were all relatively new, each having less than ten years of teaching experience. Both Jennifer and Carly were new teachers to Lotus High, and Akira joined the department during the second half of the previous year as a long term substitute. Jennifer and Carly are both White women. Natalie and Akira are both of Asian-American descent. Natalie is female and Akira is male. Jennifer and Natalie taught algebra, and Carly and Akira taught geometry.

The principal investigator of the study served as the mathematics coach and was employed by a nearby university. Prior to joining this research project, this coach served as a consultant and now works as both participant researcher and coach. She has significant expertise in mathematics education and equity-focused instruction but was never formally trained as a coach.

Data Collection

Several data collection strategies were used in this study. Relevant to the findings discussed in this paper are two main sources of data: videotapes of common planning time and teacher interview transcripts. Common planning time, facilitated by the coach, was videotaped three times during the 2008-2009 school year. Each video tape was approximately one hour long. In addition, teachers were interviewed twice during the school year using a semi-structured interview protocol. These interviews focused on how teachers understood the reform work in which they were involved and how their ideas about teaching and learning mathematics changed over time.

Theoretical Framework

The literature suggests that coaching is a mechanism to help teachers improve their instructional practices, but theories of learning do not explicitly undergird this research. In this paper, the concept of guided participation, originally developed by Rogoff (1990) is used to understand the work of coaching in this context.

Guided participation regards learning as a process that requires both guidance from a companion and participation in culturally meaningful activities. Contextual factors are always considered. Rogoff (1990) asserted that “development is assumed to proceed throughout the life span” (p. 11). She defined development as increasing ones’ ability to effectively manage the problems of everyday life and contribute to community. This could also serve as a definition of learning. Therefore, although guided participation was initially used to make sense of student development when guided by teachers, it is apt to apply it to this context to understand teacher learning alongside the guidance of an instructional coach.

Guided participation suggests that learning involves two processes. First, “building bridges from present understanding and skill to reach new understanding and skills” (Rogoff, 1990, p. 8). This is similar to Vygotsky’s (1978) concept of the Zone of Proximal Development. Learning requires guidance, support, and challenge from a companion (or companions) to help bridge a person’s current knowledge with new knowledge and understandings.

The second process involved in learning is participation in a culturally valued activity. Here, participation refers to active involvement in building shared understandings among participants. Rogoff (1990) explained, “from guided participation involving shared understanding and problem solving, children appropriate an increasingly advanced understanding of and skill in managing the intellectual problems of their community” (p. 8). The same could be said of teachers involved in guided participation - by developing shared understandings and solving problems with others, teachers appropriate more advanced skills to manage problems of practice.

Thus, learning, defined as increasing ones’ ability to manage everyday problems, occurs when one participates in building shared understandings of culturally meaningful activity with the guidance of someone who can help build bridges from current understandings to new understandings. Additionally, Rogoff (1990) cautions that learning takes place in a larger context which cannot be overlooked. This context involves individual effort, social interactions, and societal influences. She explains,

> *My stance is that the individual’s efforts and sociocultural arrangements and involvement are inseparable, mutually embedded focuses of interest. Rather than examining context as an influence on human behavior, I regard context as inseparable from human actions in cognitive events or activities. I regard all human activity as embedded in context; there are neither context-free situations nor de-contextualized skills* (p. 27).

This framework for learning, involving guided participation within a context of individual, social, and societal influences can be applied to teacher learning. Under such a framework, a coach guides the learning of teachers and provides activities in which teachers participate. This study investigates how an instructional coach engaged in such work.

Findings

> ***Interviewer*****:** What do you see Jill's (the coach) role having been in the common planning time?
> ***Natalie*****:** I think she was our guiding light. 'Cuz she really was! Like, "Okay. We want our students to be 'here'. So what do we need to do to get them 'here'? Here are some of the things that I'm thinking is good. I don't know. What do you guys think? Oh, okay, let's try it. Oh, it didn't work. Okay. So what else?" And she…it was like we know "This is our goal", and so it was nice to have a goal. Not "nice"—it was *helpful* to have a goal that we need to get to. She…I don't know! She just kind of strips everything from it and just says, "Okay. What do we want the students to know? What do they need to know? What will they need to do to show us that they know this, and what can we do to back that up, or what do we do to make sure that they do that?"

In many ways this exchange exemplifies the findings of this study. The teachers reported that Jill, their instructional coach, successfully engaged them in guided participation, and her presence and involvement positively affected their work. Jill built bridges through discussions with teachers in which she asked questions and stated observations. She also developed shared understandings with teachers, particularly around the implementation of the particular pedagogical approach they were using to mitigate issues of status in their classrooms (Cohen, 1994). According to the teacher interview data, all members of this team came to value their work with Jill as a critical tool for improving their practice and continually expressed a desire to learn more.

It must be noted that in addition to the lack of turn over within this team, the context of Lotus High may have contributed to the beliefs these teachers had about their success with students and their work as a collaborative team. The new administrators at this school worked with the entire staff to implement changes aligned with the goals of complex instruction, and both the principal and assistant principal showed significant support for this professional development project.

As Rogoff (1990) suggests, it appeared that effective coaching that built bridges and developed shared understandings within a context that was conducive for positively impacting teacher learning. However, this does not explain how such work took place. A closer analysis of the data in this study offers some possible explanations. Coding the data for the types of coaching moves made throughout all video tapes, three themes emerged: the coach (1) maintained focus, (2) assigned competence and (3) generalized teachers' stories into larger principles of teaching and learning.

Maintaining Focus

During the first year of this project, the math team at Lotus made great strides in their learning. They met daily to discuss planning and instruction, adopted many of the practices of complex instruction, and exerted great efforts to change their teaching practices. The team expressed a sense of accomplishment both in individual interviews and during common planning times. As a result, conversations often took on a celebratory feel, where teachers talked about using their new learning to change classroom practices during the next school year. Admittedly, community celebrations are critically important for developing coherence in any group and motivating continued engagement. However, it is also common for such conversations to deter teachers from the work at hand and quickly turn into sharing and storytelling rather than critical reflection and thoughtful planning (Little, 1990). Therefore, rather than simply chatting about their day, Jill provided teachers with both direction and support to move away from sharing and storytelling towards meaningful discussions about students, mathematics, and instruction.

To illustrate what is meant by maintaining focus, we offer the following two examples. During a common planning meeting at the end of the first semester, the mathematics teachers strayed from a conversation about the end-of-course exam content and began imagining future classroom practices based on current successes. Although exciting, this conversation drifted from the important issue at hand. Jill first allowed the excitement and celebration to continue for a few minutes before bringing the teachers back to thinking about student learning at the end of the term. This coaching move kept the teachers' conversation on topic rather than spiraling into an imagined world without crushing the enthusiasm being expressed.

Another focusing move that Jill frequently made again related to the ways in which the teachers attributed their problems and frustrations to larger systemic structures. Although issues such as the school's attendance policy, rules for placement in honors courses, or administrative discipline procedures, often affect what happens in teachers' classrooms, Jill would not let the team use them as excuses for lack of student achievement or to derail discussions about changing instruction. Instead, Jill acknowledged teachers' concerns and shifted the conversation back to mathematics classrooms where teachers had more agency over the changes they were seeking.

This finding is significant, because it suggests that structures for teachers' collaboration may not be enough to create an environment where efficient teacher learning happens. Jill's focusing moves allowed this group of enthusiastic and sometimes frustrated teachers to engage in reflective dialogue focused on student understanding and learning rather than getting distracted by their own celebrations, stories or frustrations.

Assigning Competence

To assign competence means to tell someone what they have done well as it is directly related to the activity at hand. A statement of assigning competence is specific and public (Cohen, 1994). Jill frequently assigned competence during team meetings. She named particular instructional moves teachers made as evidenced from her classroom observations. This coaching strategy did two things for teachers' learning. First, it highlighted specific teaching moves that Jill valued and wanted teachers to take up as part of their practice. Second, it distributed expertise across the team such that teachers recognized and valued each other's intellectual contributions. It is assumed that when people's contributions are valued, they are more likely to participate and continue to engage in their learning communities (Cohen, 1994). Below is an example of how Jill assigned competence to Natalie during a common planning time meeting.

> *You are pushing more for clarification and they (the students) are not pushing back as much. So the more you do it, the more they get used to it. You are also holding them more accountable for your follow up. Like when you said, 'I want you to do this,' and then you would come back and check in with them about what you told them you wanted them to do. You are getting more [participation]. There's been a shift in your personality at the front of the room. Like, 'I expect this,' and 'I really expect it,' not just, 'I'm saying it because I have nothing else to say.'*

In this statement, Jill assigned competence to Natalie by referencing a specific instructional move she used in her classroom, that of holding students accountable for engaging with a task and explaining mathematical ideas. Jill described the teaching move and how it affected students, and she discussed the experience in the public presence of the entire math team. One might easily imagine this conversation taking place between Jill and Natalie as part of their observation debrief in the privacy of a one-on-one meeting. However, when Jill assigned competence to Natalie, she made this artifact of Natalie's practice available to the entire team. Holding students accountable in such as way now belongs to everyone and can potentially be used as part of the teams' repertoire of practices. Simultaneously, Natalie is more likely to contribute during team meetings, because her ideas and practices were valued and validated by Jill. Natalie likely feels that she has something important to offer her colleagues as they work collaboratively to improve their teaching.

Generalizing Stories

The third coaching move Jill consistently used was one of generalizing teachers' stories into larger principles of teaching and learning. "Generalizing, or making claims that extend beyond particular situations, is a central mathematical practice" (Jurow, 2004, p. 279). In this situation, the act of generalizing helped the team make connections between the particulars of their own classroom contexts and the instructional strategies they were working to implement.

During one of the collaborative team meetings, the math teachers discussed conversations they had with students about their expectations for behavior and participation. After some of the teachers shared particulars from these events, Jill connected this practice of whole-class discussion to creating classroom communities focused on equitable participation and assumptions for competence. At another point in the same conversation, teachers talked about specific students who questioned the reason for homework. Jill again pointed out how this concern was related to the norms for learning that the teacher team was creating across all math classes. This move reminded teachers that although homework was a particular issue they needed to address, it was couched within a larger framework with particular goals for student participation and learning. Teachers' stories then shifted from isolated events to connected activities focused on the goal of creating equitable and engaging classrooms, a conversation which the entire team could then take up and address, regardless of specific occurrences within their own classrooms.

Conclusion

Jill, the coach in this study, engaged teachers in guided participation. She learned about their individual teaching practices and built bridges between those and a larger set of pedagogical practices known to effectively engage students. Jill actively created a community of teacher learners who developed shared understandings around a set of ambitious pedagogies. She did this in many ways, but most prevalently through her facilitation of common planning time which included maintaining focus, assigning competence to teachers' ideas and instructional moves, and generalizing specific stories of classroom events. Perhaps these moves contributed to comments such as the one made by Natalie who said, "Every Tuesday when she (Jill) steps into the building there is this magic that happens in the classroom."

Despite its popularity in practice as a means for improving students' mathematical achievement, instructional coaching is an underexplored area in research. This study looked in-depth at the work of one coach who was perceived as successful by the mathematics teachers with whom she worked in a high school where students' mathematical achievement was on the rise. This research provides a learning framework and three specific coaching moves that have great potential to positively impact both teachers and students' learning.

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PROFESSIONAL DEVELOPMENT: A CASE STUDY OF MRS. G

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This report is a case study of a teacher who participated in a project to investigate the effect of a wireless communication system on student algebra achievement. The longitudinal study aligns her instruction with the pedagogy promoted in the professional development sessions provided by the project. Classroom observations and teacher interviews were key data for analysis of classroom discourse, levels of questioning and formative assessment. The results of this study indicate Mrs. G's practice was consistent with the project PD.

Introduction

One of the significant concerns regarding education reform today is that the American education system is "always reforming but not always improving," and the most alarming aspect is we have "no mechanism for getting better" (Stigler & Heibert, 1999, p. ix). Professional development (PD) is often credited with promoting teacher change but evidence of success is sketchy at best. Studying environments where goals for teacher change are realized and active student learning is the focus may highlight strategies that could be replicated in teacher preparation and PD. While collecting data for Classroom Connectivity in Mathematics and Science (CCMS), such a situation came to light.

The larger project sought to use Texas Instruments TI-Navigator™, a wireless system, to open communication between students and teacher in mathematics classrooms, specifically Algebra I. The author was assigned to collect data from project participants based on geographical convenience. One such participant was perceived to be excited about the project and open to the new technology. Students demonstrated an enthusiasm for participation and learning. The seasoned teacher demonstrated an authentic passion for student understanding. The author was motivated to make a rich description of this classroom scene.

The four-year IES-funded project[1] aimed to promote student learning by enhancing classroom practices such as enriched classroom discourse, quality and levels of questioning, and utilization of formative assessment (CCMS, 2005). Recognizing that simply the presence of new technology does not ensure teacher change, the project PIs implemented many avenues for reflection and PD. Clarke (1994) gives a framework for effective professional development. The professional development opportunities of the CCMS project were aligned with this framework and include: making ideas relevant to participants, allowing time to change, offering ongoing support, and encouraging participants to reflect on their learning. Formal professional development, led by practicing high school teachers who were also Teachers Teaching with Technology (T^3) instructors, was offered at summer institutes. PD sessions were also provided at T^3 international conferences. The PIs of the project provided direct instruction focusing on the theoretical and pedagogical issues addressed by appropriate implementation of the new technology. The gap between theory and practice was more effectively bridged by T^3 instructors who provided real-life examples of classroom activities. Additional support included a listserv, technical support and ongoing telephone interviews.

Theoretical Framework

"Human motivation is a complex phenomenon, so it follows that mastery orientation is dependent on many…factors, not necessarily explainable by a single theory," (Owens et al., 2005, p.10). However, this paper is framed on social constructivist theory, whereby individuals build knowledge based on prior experiences, and that knowledge is constructed in a social environment. Much emphasis has been placed on students' learning in exploratory environments. Unfortunately, the same theory has often been neglected in teacher learning (Clement & Vandenberghe, 2000; Sowder, 2007). Construction of knowledge is greatly hindered, in an environment where one person is the holder of knowledge and continually dominates the conversation, and the other person simply follows along. Critical thinking occurs when both parties are held responsible for asserting and justifying new ideas and opinions. "For learning to be mutually beneficial, *especially among adults*, all parties must engage in critical thinking" (Nyikos & Hashimoto, 1997, p. 508, emphasis added).

Modes of Inquiry and Data Sources

Transcripts and videotapes of Mrs. G's classroom observations, post observation interviews and telephone interviews serve as data. The researcher looked for indicators from Mrs. G's interviews of her perception of implementing the pedagogy supported by the project PD in her classroom practices by employing formative assessments and initializing classroom discourse by using questioning. Evidence of these constructs in her classroom transcripts were compared with the transcripts of the project PD sessions. Although it is impossible to isolate which elements of the PD are credited with changes in Mrs. G's classroom practices, an alignment of the PD sessions, Mrs. G's comments about her implementation of the technology and the pedagogy supported by the PD sessions, with evidences from her classroom observations are indicators of success of the PD. The author conducted a classroom observation after Mrs. G left the project to ascertain whether she continued to employ the aforementioned constructs.

The research question for this paper is: How do Mrs. G's classroom practices align with the PD provided by CCMS project? Studying a seasoned teacher who continues to learn about desirable constructs in her classroom can add to the knowledge base of successful PD. It is worth examining PD that has been implemented so that the education community can glean from it something that works and attempt to replicate it in subsequent PD opportunities.

Data Analysis

Classroom Discourse

In the PD sessions, specific references were made to utilizing the technology to instigate classroom discourse. An example of this is comments made by Stephen Pape, co-principal investigator of the larger project at the opening day of the summer institute, "And the immediate feedback, and the discussions that you can have after it, are what got me into this grant. I am so excited about that data that you can get…Now this is some math problem, but you have these data now. What kind of conversations would you have about that?"

In keeping with a social constructivist paradigm supported by the PD sessions, participants were encouraged to participate in small and large group discussions. Participants brainstormed ideas projecting how they might use the TI-Navigator™ to foster discussion. Among several comments was Mrs. G, "What I find a lot is there's only one right answer in math. And when they see this, they're going to know that there's more than one right way to present your answer."

Comments about classroom discourse made by Mrs. G during her tenure in the project include this at the conclusion of her first year in the project, "It has assisted and validated the way I have been teaching. I feel it's a battle to get them to think on their own, and Navigator has fostered more discussion from quieter kids that normally you wouldn't get." In the middle of her second year in the project, she made this observation about classroom discourse, "I probably would only interact otherwise with those who are more vocal. It fosters more discussion about the 'hows' of doing a certain problem. It allows you to interact with the quiet ones as well."

Levels of Questioning

Embedded in the conversations supporting classroom discourse was the notion that the teacher fosters a desirable environment with the type of questions she asks. Levels of questioning, while embedded within discourse, were treated as an additional construct. It was suggested that the questioning techniques employed by the teacher inform the classroom climate for discourse. A T^3 instructor addressed the issue of changing her questioning as a result of having the TI-Navigator in her classroom, "With Quick Poll and later with Class Analysis, it changed my way of questioning. Instead of saying, 'Well, how did you get that?' That's too accusational. Even though I wasn't meaning that. I now change it to say, 'How would someone get that?' It alleviates that feeling of, 'I'm the only one that was wrong, and how am I supposed to explain it? I have no idea.' But it also gives the students who frequently or most often get things right, it gives them an opportunity to look at perhaps someone who has not done it the same way they did, and they can help talk through what would be the possibility of getting that."

In a post-project interview, Mrs. G offered the following comments regarding questioning in her classroom, "I usually question to assess what they already know on a topic before we start a new topic or to connect something we have learned to something we are going to be learning. I question for understanding. I question for keeping students on task… and to basically assess whether or not they are understanding…it directs what I'm going to do next. Whether I keep going, or back up to re-explain or back way up…re-teach something they need before we go on."

This research employed a deductive analysis of the data, using a coding system developed in an emergent and initial analysis of the data (Pape, 2009; Pape et al., 2008). NVivo software was used in a line-by-line analysis of the data so that every utterance from the transcripts was counted. The constructs in Table 1 are a sub-group of the constructs from the CCMS project (Pape, 2009). Table 1 reports counts for multi-directional discourse and levels of questioning. In defining each utterance, the author of this report used the codebook for which she was co-author. The abbreviated codebook definitions are listed below. The recorded counts in Table 1 have been standardized to a 60-minute lesson. The final column in Table 1 represents a post-project visit. This was a two-day visit, but only the first day was transcribed. The second day of the observation, students were playing a review game that required them to work in teams. The teams selected a question that was appropriately challenging for themselves, and they worked on it together. Although the entire lesson was an excellent example of classroom discourse, levels of questioning, and formative assessment, it was not transcribable due to the nature of game.

Definitions for Table 1

Initiation, response, evaluation [IRE]. Typically, a 3-conversational turn sequence during which (a) the teacher initiates by asking a question, (b) a student responds, and (c) the teacher evaluates the student's response.

Uptake. Uptake of correct and incorrect responses or student comment or question refers ways in which the teacher "takes up" (i.e., explores, engages with, discusses, critiques, reasons about, provides rationale to support) responses and comments as objects of classroom discourse.

High order cognitive load question. Elicit responses that may involve manipulation of information and ideas in ways that transform their meaning and implications—combining facts and ideas to synthesize, generalize, explain, hypothesize, or arrive at some conclusion or interpretation. (The code "st" refers to students.)

Low order cognitive load question. Elicit recalling and stating information or known facts; carrying out a simple algorithm, math procedure, or problem-solving steps to complete a task.

Teacher press for elaboration, explanations, and justifications. Teacher presses students to elaborate their ideas or to make their reasoning explicit—a request for deeper thinking.

Teacher press for involvement. Teacher strategies for increasing involvement for all students.

Student-to-teacher mathematics comment. All mathematics statements from a student to the teacher including direct response to a teacher question.

Student-to-teacher mathematics question. The direction of the mathematics question is from a student to the teacher. (The code "wpc" refers to words per comment.)

Student-to-student mathematics comment. All mathematics statements between students.

Student-to-student mathematics question. The direction of the mathematics question is from a student to another student.

Teacher-to-student mathematics comment. All mathematics statements from the teacher to a student or students including lecture about mathematics content.

Teacher-to-student mathematics question. The direction of the mathematics question is from the teacher to a student.

Authentic question. Open-ended, no specified answer by question source.

Recitation question. Pre-scripted answers are known by question source.

Discussion of Table 1

Initiation, response, evaluate (IRE). Typically, IRE is an undesirable pattern in a classroom because it usually consists of low cognitive load questions and may be used as more of a lecture masked as student involvement. The evaluation component deems the teacher as the mathematical authority. So, an increase in IRE would be a disappointing result. The NVivo results show that Mrs. G increases from 14.4 episodes of IRE in year one, winter of 2006, and 4.67 episodes in year one, spring of 2006, to as many as 29 episodes of IRE in year three, winter 2007. However, Mrs. G employed nontraditional uses in her IRE pattern. An example of transcripts that were coded as IRE, yet were used as a discourse generator is:

Mrs. G: Oh, okay. Let me get so I can see all your numbers. Okay, all right, so talk to me about this one compared to the other two. Say something about that line. Carla, just say something. There's no right or wrong, I'm just asking you to make an observation.
S: It's a straight line.
S: It's a positive slope.
Mrs. G: It's another positive slope. Good.
S: It's a straight line.
Mrs. G: Yeah, it's a straight line.
S: Instead of like one more down, it goes like up.
Mrs. G: Oh, give it another name; more up.

Table 1. Quantitative Summary of NVivo Coding Results

Code	Year 1 Winter 2006 2 days	Year 1 Spring 2006 5 days	Year 2 Spring 2007 4 days	Year 3 Winter 2007 3 days	Post project 1 day
IRE	14.4	4.67	21	29	18.76
Uptake	3 23%	1.88 14%	8.75 27.34%	7.67 16.24%	1.3 11.24%
High order	3 0.67 st	0.8 0.268 st	5 0.25 st	8.3 0.67 st	0.67 0
Low order	49.58 1 st	27.1 4.15 st	87.28 6.75 st	105.3 8.3 st	81.74 21 st
Press elab	4.7	2	5	7	0
Press involve	10.72	2	4.5	10.67	0
S-T MC	72 5.77 wpc	40.5 7 wpc	116.25 5.5 wpc	160.3 4.8 wpc	142 4.8 wpc
S-T MQ	3.35	4.7	7.25	10	14.7
S-S MC	2.345 4.86 wpc	2.28 10.76 wpc	3 4.92 wpc	2.3 2.43 wpc	1 1 wpc
S-S MQ	0.34	0.27	0.5	0.67	0
T-S MC	29.14 23.6 wpc	26.4 23 wpc	31.83 18.4 wpc	40.42 17.17 wpc	73.7 22.7 wpc
T-S Q	50.59	20.52	82.75	102.3	67.67
Authentic question	22 2.1 st	13.94 4.56 st	34.75 7.25 st	43.67 8.67 st	32.16 13.4 st
Recitation	42.21	13.69	56.5	71.3	50.92

Uptake. Mrs. G spent a large percentage of time taking up student comments and questions as objects of classroom discourse. This exercise validated student contributions and therefore encouraged student input. The first and third observations show nearly one fourth or more of the class time is spent on comments and questions instigated by students, thereby validating student contributions and allowing them to be holders of their own knowledge.

Types of questions. The recitation questions were all posed by the teacher, but the authentic, higher cognitive load and lower cognitive load questions were occasionally posed by a student. That count is represented in the table with an “st.” The highest percentage of student questions is found in the authentic code. This is reasonable since authentic questions are ones in which the answer is not known. It is interesting to note that the teacher frequently relinquished her authority by asking authentic questions, placing herself as a learner along with her students.

Teacher press for involvement and teacher press for elaboration, explanations, and justifications. Mrs. G pressed her students for explanations and justifications of their comments, more so in the first and last observations. She also used many verbal cues to keep her students involved in the lessons. Some prompts that Mrs. G used to press her students to be involved were: I want to ask Monica because she’s falling asleep, and I want to wake her up; Somebody who disagreed, can you tell me what you got for an answer? Dave, since you brought it up; 14-15 people. Are the rest coming? Where’s everybody else? Okay, talk to me about what the word maximum means. Jonathan; I’m asking you anyway, especially because you didn’t do it.

Comment and question directionality. This is a combination of the question and comment constructs from Table 1 addressing all instances in which the teacher makes a comment or question to a student (T-S MC, T-S MQ) a student makes a comment or question to the teacher, or another student.(S-T MC, S-T MQ, S-S MC, S-S MQ). An interesting observation is that Mrs. G's students made many times the comments that she made; however the words per comment (wpc) were fewer for the students, indicating that their comments were probably not as sophisticated as those of the teacher. In year one, winter of 2006, the students made 72 mathematics comments to the teacher, and the teacher makes 29.14 mathematics comments to the students. In this case, the students made 2.47 times the number of teacher comments. In year one, spring of 2006, the students made 1.53 times the number of teacher comments. In year two, spring of 2007, this number jumped to 3.07 times the number of teacher comments; and in year three, winter of 2007, nearly four times the number of teacher comments. In year three, winter of 2007, the number is reduced to just over twice the number of teacher comments. Note that in IRE, the response is always coded as a student-to-teacher mathematics comment; however, her low levels of IRE would not account for a great percentage of the student-to-teacher comments, ranging from 11.5% in year one, spring, to 20% in year one, winter 2006 . These percentages were calculated by dividing the number of IREs by the number of student comments. Likewise, the teacher-to-student questions are somewhat accounted for in IRE, but only ranging from 22.8% in year one, spring 2006, to 28.5% in year one, winter 2006. These percentages were calculated by dividing the number of IREs by the number of teacher-to-student questions. The student-to-student comments and questions may not be adequately represented by the count in Table 1 because the students had time every day that they worked at their desks on a warm up, reviewing homework or exploring a new idea, and took advantage of time to discuss something quietly with a neighboring student. These instances were viewed on the videotapes, but were not audible, so did not appear on the transcripts; therefore, these comments were not counted.

Formative assessment. After the principal investigators and T[3] instructors spent some time in the PD sessions establishing the importance of discourse, the next big idea was, what does the teacher do with the knowledge gained from listening to the students? Assessment is a key reason for providing an environment where students can speak freely about their mathematics. As students expose their ideas, they also expose their misconceptions. Once the misconceptions are brought to light, they can be rectified. Stephen Pape made these comments at the summer, 2005 institute regarding formative assessment:

> *You have this information, and you have to think about which road to take. One of the questions we have in the research is about the critical junctures in a lesson when you use the Navigator. When will you make that decision in a lesson about when to use… Quick Poll? Nothing preplanned; but you're noticing something, and you say, 'Oh, let me get this data.'…I think that's going to be interesting, and perhaps it makes your job somewhat more complex…We always ask questions, but we're getting that information back* (S. Pape, 2005).

Mrs. G made many references to using formative assessment to evaluate student understanding. She claimed that formative assessment has traditionally been her practice; however, the TI Navigator made the students' needs more visible. In every telephone interview and post-observation interview, Mrs. G remarked that she did change the course of the day's lesson to accommodate student learning. The following comments came from her post-observation interviews: After her first semester in the project:

> *I taught the material yesterday. They weren't as comfortable as I wanted them to be. I was shocked that there was still so much work to do because my plan was to go on to some word problems. . . . I can't write a plan in advance. I mean, it changes* (Mrs. G, 2005).

After her first year in the project:

> *We backed up from what we originally planned because after taking a quiz, I realized the kids were not ready to write equations of lines from scatter plots. Well, some of them, those five who got it, they're solving system of equations right now* (Mrs. G, 2006).

After two years in the project: "I try to get as much student involvement and feedback as I could. I could get a better picture of how they were doing." After three years in the project: "I decided to whip out that TI-interactive calculator [i.e., TI- Navigator™]. I hadn't planned that. I could see they were struggling."

Conclusions

The results of the qualitative analysis and Mrs. G's comments indicated that the PD offered by the larger project was implemented in her classroom practices. Participation in the CCMS project was not Mrs. G's first exposure to the constructs of classroom discourse, levels of questioning and formative assessment, she indicated that the PD sessions of the summer institute and yearly conferences aligned themselves with her belief system. When asked to compare the PD of her own choosing such as CCMS, with the professional offerings mandated by her administration, Mrs. G stated:

> *PD needs to be relevant to your profession. [As a mathematics teacher] I am looking for PD that is relevant to me. How can I use this in the classroom? And how is this going to improve student learning and understanding? Because those are my goals. My goal as a teacher is…to have students understand what I teach them. The generic PD that gets handed down through the county...is just a process that I didn't see the relevance for. It was time consuming, and I did not see the relevance for it in my classroom. Whereas, I sought out the Navigator project because I saw Navigator as something that would help me understand student learning and improve student learning. It was going to be a motivator for students. It was going to allow students to talk more about math, so that PD was meeting my goals and my requirements for teaching mathematics. The PD sessions we had at Ohio State and Denver and other places were all research based. Nobody was saying, "You have to do it this way." The way it was presented allowed you as a professional to choose whether or not it was [going to be useful in your classroom]. There have been other PDs that I have chosen to do that I felt were beneficial…They are probably geared more toward my style of teaching, and that is why I gravitate towards them: investigative, discovery, hands-on approaches to learning math* (Mrs. G, 2009).

Recommendations for Practice

This case study can inform pre-service and in-service instructors to build programs that nurture a desire for teacher learning. Current trends that are an "emphasis on correcting deficits rather than encouraging professional growth" are listed as an impediment to staff development sessions (Clarke, 1994, p. 41). The result is a most frustrating situation where teachers are not treated as professionals (Romberg, 1988), but rather bludgeoned with speakers, workshops, consultants, new curriculum, new materials, new math, new activities and an overwhelming offering of other potentially useful, but thoughtlessly mandated, programs. "The most common form of staff development…in the United States… continues to be the one-shot in-service seminar in which an external expert makes a presentation, with little active involvement and no follow-up" (Clarke, 1994, p. 42). Furthermore, "two thirds of U.S. teachers state that they have no say in what or how they learn in the PD opportunities provided to them in schools" (Bransford, Brown, & Cocking, 2000 p. 193)

Teachers must be allowed to have a choice of which PD opportunities they will undertake. However, there may be many in this profession who have chosen it for their passion for teaching and not a passion for learning. The latter must be cultivated in pre-service teachers. It is possible that many teachers do not know how to choose appropriate PD for themselves because the expectation is that they will be told what to do. In order for in-service programs to be effective, pre-service programs must incorporate the expectation that its graduates will continue to seek appropriate learning opportunities for themselves. Pre-service teachers must be educated in the opportunities available to in-service teachers and must leave their programs armed with a plan to continue their education in an area that is relevant to them. Perhaps assertive teachers with a plan will be less likely to fall prey to generic sweep of stagnating workshops.

Endnotes

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LEARNING BY TEACHING: USING A MODEL TEACHING ACTIVITY TO HELP TEACHERS LEARN TO USE COMPARISON IN ALGEBRA

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With current reform efforts and demands on teachers, quality professional development is more important than ever. There is a growing interest in practice-based professional development, an approach that aims to provide opportunities for teachers to learn mathematics and to make connections between the mathematical ideas, students' ways of thinking about those ideas, and related pedagogy. In this paper, we explore one type of practice-based professional development task that has not received great attention in the literature - where teachers write a lesson plan, teach the lesson to a "class" of fellow teachers, and discuss the demonstration lesson as a group.

Introduction

With current reform efforts and demands on teachers, quality professional development is more important than ever (Borko, 2004). Research suggests that professional development experiences should be grounded in particular content and in the practice of teaching (Borko & Putnam, 1996; Smith, 2001). Yet, much is still needed to understand and implement effective strategies for educating teachers (Borko, 2004).

Large-scale studies of professional development in mathematics and science suggest some general guidelines for creating quality learning experiences for teachers. Specifically, Garet and colleagues found that teachers' knowledge and skills are positively impacted by professional development that is focused on content, includes active learning opportunities, and is aligned with other school goals and activities (Garet, Porter, Desimone, Birman, & Yoon, 2001). In addition, Penuel and colleagues found that when activities were focused on specific content and were perceived to be consistent with other goals, teachers' felt more prepared to help students engage in inquiry. Time for planning was also related to implementation (Penuel, Fishman, Yamaguchi, & Gallagher, 2007).

Recently there has been a growing interest in *practice-based professional development*, an approach that aims to situate teacher learning within the profession of teaching (Silver, 2009; Smith, 2001). This approach utilizes artifacts of teaching, such as curriculum materials, lesson plans, and student work, to create Professional Learning Tasks (PLTs). These tasks accomplish particular learning goals by engaging teachers in aspects of their work and drawing on their experiences as teachers (Smith, 2001). However, Silver (2009) suggests that empirical evidence is lacking for theoretical claims about PLTs. In particular, there is a need to better understand whether and what teachers are learning from the tasks, what features of the task facilitate this learning, and how this learning might transfer to the classroom.

Model Teaching

The focus of this paper is on a professional development activity that we refer to as *model teaching*. Model teaching occurs when participants collaboratively plan a lesson, teach the lesson to a "class" of fellow teachers, and discuss the demonstration lesson as a group. Although model teaching has not been focused on in the literature, it is closely related to another, more extensive literature - *microteaching*. Microteaching, where preservice teachers teach lessons to their peers

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as a part of methods courses, has a long history in teacher education with roots in the early 1960s (Kasten, 2008). In its earliest form, the lessons were taught to small groups of actual students, but peers were later used in order to make the practice logistically feasible for regular use (Davis & Gregory, 1970). Microteaching provides an opportunity for preservice teachers to practice teaching skills in a simplified environment; hence, the class size is small, and the lessons are short and often focused on a particular pedagogical skill (e.g., student involvement). Typically, the lessons are videotaped for subsequent viewing and analysis by the presenter and a supervisor, but a number of variations to the practice have emerged over the years, including variations on the mode, style, and source of feedback. Research on microteaching suggests that participants enjoy the activity and feel it has value. The feedback after teaching is especially important, both by the observers and in the form of self-analysis (Kasten, 2008; MacLeod, 1987). However, the research has not been entirely conclusive (MacLeod, 1987). Some have argued that these environments may have been oversimplified, that "what was stripped away may have been the very aspects of teaching that make it difficult" (Grossman & McDonald, 2008, p. 190-191).

Several differences exist between microteaching and model teaching. First, microteaching has most often been used with novice teachers (MacLead, 1987). As such, microteaching is often a way for novice teachers to work on 'gross' teaching skills such as clear directions, presentation style, and student involvement (Kasten, 2008), whereas model teaching is intended to help experienced teachers explore more nuanced aspects of teaching. Related, the microteaching environment is meant to reduce the complexities of teaching so that the preservice teacher can practice particular skills and receive feedback on them (MacLeod, 1987). On the other hand, model teaching is expressly meant to help teachers consider and practice a new (and more advanced) skill, while simultaneously considering the complexities of teaching. Teachers are encouraged to embed the skill in a complete lesson in order to consider how it might be integrated into daily practice. Also, the debriefing period following the lesson is not aimed so much at the particular teacher and his or her execution, as much as it is a collaborative reflection about the lesson's ability to effectively use the new skill to help students learn.

Current Study

As suggested by Garet et al. (2001), the professional development activities in the current study were grounded in particular content (i.e., the teaching of algebra) and included multiple active learning opportunities, all of which were designed to help teachers integrate the new materials into their current practices and to merge them with regular classroom materials (e.g., texts). In particular, the goals of the professional development were to: 1) introduce teachers to comparison as a tool for fostering flexibility in algebra; 2) train teachers to use materials created for this purpose; and 3) assist teachers in finding ways that these materials could be easily implemented in their own classrooms.

Star and colleagues have identified comparison as a particularly effective means for promoting the development of students’ flexibility in mathematics (Rittle-Johnson & Star, 2007; Star & Rittle-Johnson, 2009). Within their work, several practices for the effective use of comparison in mathematics instruction were found to be important. These instructional practices include using side-by-side presentation of problems and solution methods; engaging students in subsequent discussion of these multiple solution methods to highlight the similarities and differences among problem solving techniques; helping students evaluate and compare the accuracy and efficiency of different solution methods; and making connections to the underlying mathematical principles. The current study investigated how teachers learned to implement these practices within a practice-based professional development institute.

The study is guided by two research questions. How do teachers implement the model teaching activity? In what ways do teachers' implementation of the model teaching activity foster the integration of mathematics content, student thinking, and pedagogy?

Method

Participants

In July of 2009, a one-week professional development institute was held for a group of middle and high school algebra teachers local to a large city in the northeastern part of the United States. The teachers were participants in a pilot year of a multi-year research project on using comparison to teach Algebra I. Of the 13 teachers, 8 taught in high schools and 5 in middle schools. All of the teachers had taught either algebra or pre-algebra courses in their schools. The range of schools varied from public urban and suburban schools to private suburban. The number of years teaching ranged from 2 to 25, with an average of almost 10 years.

Professional Development Structure

Daily overarching questions were used to frame the discussions and organize the activities during the week of professional development. In particular, on Monday the overarching questions were: What are some benefits of promoting multiple strategies for solving problems? What are some benefits of using comparison to promote the use of multiple strategies? On Tuesday the questions were: What sorts of comparisons are useful for learning algebra? What might these comparisons look like within the algebra curriculum? Which comparison problems can be easily woven into your instruction? On Wednesday: What kinds of questions/activities will facilitate the use of these comparison problems? How might the materials work together to foster flexibility with algebra? On Thursday: How might comparison might be implemented within your own classroom? On Friday, the guiding questions were: What difficulties do you anticipate/concerns do you have about the use of comparison? How will the study be conducted? Typically, each day included an activity designed to address the guiding questions and promote active learning opportunities. The activities ranged from solving mathematics problems to watching videos to planning and enacting lessons.

The Model Teaching Activity

During the professional development week, several activities were included to introduce teachers to the new teaching methodology, underlying educational research, and comparison curriculum materials. The activities were meant to provide teachers the opportunity to explore new ideas and make connections to their own teaching practices. Each activity was followed by a group discussion. The focus of the current paper is the activity that occurred on Thursday – the model teaching activity. For this activity, teachers chose side-by-side comparison problems from the curriculum, referred to in the curriculum as Worked Example Pairs (WEPs), and worked in small groups to design a lesson plan. (The 13 teachers divided into five small groups for the purposes of this activity.) Each group then co-taught the lesson to the other participating teachers, who played the role of students. The presenting teachers selected their own topics from those that were part of the curriculum, which included finding the slope of a line, solving a system of linear equations, factoring, and solving quadratic equations. All five of the teacher presentations were videotaped, and observational notes were taken at the time of presentations.

Prior to designing their own lessons, the teachers participated in a demonstration lesson taught by one of the professional development leaders. Within this lesson, characteristics of effective implementation were modeled. In particular, desired implementation of the WEP included three phases: *understand* (e.g., what are the two methods?), *compare* (e.g., what are the similarities and differences between the methods?), and *make connections* (e.g., why is one method more efficient?). The presenter did not narrate the lesson; instead, he enacted the lesson as if he was the teacher and the participants were his students. Before he began, he emphasized that his lesson was just one way to implement the materials.

Analysis

To begin the analysis process, two of the authors independently observed all five demonstration lesson videos. Subsequent analysis of the videos proceeded in three phases: writing detailed descriptions of the lessons; identifying features of the implementation; and coding the lessons for mathematics content, student thinking, and pedagogy. After an initial phase of independent analysis followed by discussion (iteratively for the first two videos), subsequent analysis were completed by one coder and checked by the other.

In particular, the detailed descriptions were used to characterize different phases of the lesson, use of WEPs, and aspects related to enactment of the lesson (e.g., did the teachers narrate or play the role of a teacher?) These characterizations were then used to identify trends in the way that mathematics, student thinking, and pedagogy were addressed within the model teaching activity. Video were reviewed as often as necessary to understand and confirm the trends.

Results

The results for this study are organized according to the research questions. Before describing these results, however, it seems important to note that teachers' reactions to the activity were quite positive. Semi-structured interviews conducted with each participant before and after the professional development week revealed that the majority of them viewed the model teaching activity to be the most meaningful part of the week. Several teachers noted that this activity helped them to think about the implementation process and how to “blend the [materials] into the lesson.” One of the teachers noted, “The part I enjoyed the most was watching other teachers implement the [materials]....anything from how they wrote it on the board to the questions they just asked the class in general.” Another teacher commented that the teacher presentations "were incredibly helpful, and it was the springboard for a lot of ideas that are sort of percolating right now."

Implementation of the Model Teaching Activity

Our first research question focuses on teachers' implementation of the model teaching activity. Three types of implementation features were examined in this study: lesson sequence, use of WEPs, and enactment of the lesson.

Lesson sequence. With regard to lesson sequence, a general trend was found that included: warm-up problems to review what students already knew about a mathematical topic; introduction and discussion of the WEPs; opportunities to apply the methods presented in the WEPs and draw conclusions about them; and an “exit ticket” problem to assess understanding of the methods and general conclusions. Not surprisingly, this sequence was quite similar to the one demonstrated by the professional development leader.

Slight variations in lesson sequence among the five presentations were primarily due to personal teaching styles and also classroom environment where the participating teachers would be using the curriculum materials. For example, one of the teachers had students of different levels of achievement in her class and she had to be conscious in how she would adjust the comparison materials and pedagogy to fit the needs of her class. Another teacher was working with non-English native speakers and for her, the major challenge was to think about ways to integrate this new approach and be able to support meaningful discussions without adding an additional load on students' language comprehension. Another variation involved the use of a second, teacher-made example to extend students' thinking about the particular methods being examined in the WEP.

Use of WEPs. During the use of the WEPs, all of the groups included the *understand*, *compare*, and *make connections* phases. These first two phases were quite similar for all groups. In the *understand* phase, all of the groups asked students to read through the example and understand how each method worked. A small difference was that some of the groups asked the students to read the example and discuss the solution methods with their partner before beginning the whole class discussion, while other presenters began a whole class discussion right after their students read through the problem. In the *comparison* phase, all five groups asked their students about the similarities and differences between the solution methods.

The most variability in the lesson sequence occurred during the *make connections* phase. For example, three of the groups were focused on helping students understand why and under what circumstances a particular method might be preferable. For example, after discussion about which method was more efficient and strategic for a problem that involved factoring with a leading coefficients greater than 1 (e.g., $2x^2 + 14x + 24$), a presenting teacher in one group said, "I want to ask a follow-up question. How does a factoring of GCF here affect the length of the solution?" In these three groups, these teachers wanted students to pay attention to the affordances of certain methods, given particular problem characteristics. Again, classroom environment seemed to also play a role for this questioning phase. For example, one teacher explained that his target class was quite advanced and consequently, he spent most of his time in the *make connections* phase. In two other groups' implementation of the *make connections* phase, the teachers were concerned with helping their students generalize mathematical ideas related to the WEPs. For example, one group asked the students to state general rules about the slopes of horizontal and vertical lines, based on the discussions following the use of the WEPs.

All five groups included an "exit ticket" problem at the end of their lesson. Perhaps due to the time limitations for each of the presentations, the teachers generally did not discuss these problems in detail.

Enactment of the lesson. The last aspect of implementation examined in this study was the enactment of the lesson. The most notable finding here is that none of the five groups stayed "in character" throughout the lesson, although the groups differed in the extent that they stopped to make comments. The comments generally fell into one of six categories; most typically, teachers stepped "out of character" to explain their approach or view about something in the lesson. For example, one teacher stopped to explain that she tells her students the slope of a vertical line is undefined rather than telling them it has no slope. In a few cases, teachers stepped "out of character" to explain what students in their classes typically think or do. As one teacher noted, "You also have to remember to remind the kids to bring the GCF down, because they do pull it out, work on the expression, and then forget it. That is a common thing." Teachers also stepped "out of character" to explain their rationale for decisions within the lesson, to describe what might happen in part of the lesson they did not enact, to explain the context of the lesson, or to comment on logistical issues such as time or materials.

The other notable finding about the enactment is that the lessons tended to become shorter with each group. The first lesson was the longest (49 minutes), during which the teachers enacted most details of the lesson. The last one was the shortest (18 minutes), during which the teacher focused primarily on the implementation of the WEP. It was unclear whether this change was a result of teachers becoming tired near the end of the day, the teachers gaining confidence in how comparison might be implemented and therefore not finding it necessary to enact all components of the lesson, or both.

The Integration of Mathematics, Student Thinking, and Pedagogy

Our second research questions concerns the ways that teachers' implementation of the model teaching activity fostered the integration of mathematics content, student thinking, and pedagogy. On the whole, it appeared that the model teaching activity did foster the desired integration. Our analysis of the five model lessons indicated that content, pedagogy, and student thinking were regularly addressed by teachers and were often intertwined.

Signs of this integration were seen in teachers' enactment of the lessons and seemed to be especially influenced by the fact that all teachers served both as presenters and (in others' presentations) as "students." In other words, when in the role of "students," teachers did act like students -- not by modeling student misbehavior but rather by illustrating ways that students might think about or approach the given problems. Although the "students" often gave correct answers, they also asked questions that revealed student misconceptions and difficulties. Initially (as the first group taught its lesson) there was some discomfort among presenters when “students” asked questions that revealed points of confusion, but teachers seemed to grow comfortable with it as the activity continued. The teachers seemed to want to know how their peers would handle common points of confusion, such as dividing by zero and understanding the difference between no slope and a slope of zero. To illustrate, a "student" asked how one can know the graph of a particular equation is going to be a vertical line. The presenting teacher responded by suggesting a table of x and y values where x was always 3 could be useful for showing that "every single time x is going to be 3, it makes no difference what y is." She then asked the "student" to give her "some points for y." Instead, the "student" demonstrated confusion by saying, "But there is no y." After pausing, the presenting teacher again encouraged the "student" to provide some values for y so that she could show "what is going to happen." Once the table of values was created, the presenting teacher plotted the points on a graph to demonstrate that the points formed a vertical line.

In addition to asking questions, participants playing the role of the student also made comments that revealed what students struggled with, such as being able to manipulate multiple negative numbers in the same problem. The presence of "students" appeared to play a particularly critical role in connecting pedagogy, content, and student thinking. Note that teachers were not given explicit instructions on how they should behave when in the role of "students"; rather, the group seemed to spontaneously recognize the value (to teachers' learning) of having "students" model the thinking and misconceptions of actual students.

Another second way that implementation of the model teaching activity showed integration of content, pedagogy, and student thinking was in teachers' spontaneous parenthetical comments while teaching. Recall that the model teaching activity began with the presentation of a demonstration lesson by one of the leaders of the professional development; in this presentation, the leader did not step out of character or offer any parenthetical comments. Yet presenting teachers frequently offered parenthetical asides to their peers, as a way to explain teachers' thinking and underlying rationale for pedagogical decisions. For example, after introducing the elimination method for solving a system of equations, one presenter asked the "students" to describe which of two equations had been modified and how, but she did not have them actually solve the system. Before moving to the next problem she commented on this decision and how she would use this approach to help focus her own students on the process instead of the answers. She said, "And I would actually, in the lesson - sorry, this is just an aside - at this point, I wouldn't finish that one. Because my whole point, with the next couple of these, is just getting the kids to think about what am I changing and why am I changing it."

Discussion

Our interest in the present study was in exploring one type of practice-based professional development task - where teachers write a lesson plan, teach the lesson to a "class" of fellow teachers, and discuss the demonstration lesson as a group. Despite the prevalence of practice-based professional development tasks in mathematics teacher professional development, empirical evidence is largely lacking about the effectiveness of these tasks (Silver, 2009). The model-teaching task is particularly under-explored, despite the apparent potential of this task to help teachers make connections between mathematical ideas, students' ways of thinking about those ideas, and related pedagogy.

Our results indicated that the model teaching task was powerful and useful in the context of our week-long professional development on comparison. First, this task provided us with a useful means of assessing teachers' learning from the professional development. Recall that the task was used at the end of the professional development, as a culminating activity. Our analysis of the model teaching lessons showed that teachers appeared to understanding the importance of comparison as well as how to implement our comparison-based lessons using our three-phase discussion model. Additionally, their lesson sequences indicated that they were able to integrate the WEPs into a typical classroom routine, making it more likely they will do so within their own classrooms.

Second, the model teaching task showcased the variety of ways that teachers could implement our materials. This activity gave teachers an experience for thinking about what examples would fit their lesson goals and then planning step-by-step the lesson sequence followed by the practice presentation. In addition to providing teachers with a comprehensive lesson planning and implementation stages, the teaching activity offered a unique opportunity to learn from each other’s implementation approaches. As one participant noted, "It was good for me to get up and do it myself, and it was beneficial to me to see how other people do it. It was also beneficial to work in the team that I did, not simply to do it by myself." In sum, participants were able to demonstrate and share their own customization of our curriculum, which also provided evidence that personalization of the curriculum was not necessarily at odds with implementing our approach with fidelity.

Finally, the model teaching activity seemed to foster the integration of several aspects of teacher knowledge. In particular, having teachers act as "students" served as a powerful way to highlight issues of student thinking, to identify student misconceptions, and to consider pedagogical strategies for addressing particular student difficulties. In fact, teachers seemed to spontaneously focus on what students might be thinking about the mathematics, as evidenced by their questions and comments. Prior research suggests that getting teachers to focus on student thinking can be challenging (Perry & Lewis, 2009). Research by Lin (2002) and van Es and Sherin (2010) indicates that videos and narrative cases can be effective in helping to shift teachers' focus from teacher behaviors to student thinking, but it generally takes time for this shift to occur. It may be that, by observing the lesson as a "student," a greater and more immediate focus is placed on student thinking. As one teacher noted, "I think it is good that we are doing this because it is really making me think more about what it looks like to the kids."

Prior research on practice-based professional development has emphasized the importance of using artifacts from teachers' *own* classrooms in order to help teachers reflect on practice and learn new pedagogies or mathematics (Lin, 2002; van Es & Sherin, 2010). However, such artifacts (e.g., videos) may not be available in the initial stages of a professional development program. The current study describes a professional development activity that allows teachers to simulate what they do on a daily basis, in order to help them learn to incorporate new ideas into their practice. The unique features of this task seem to forefront student thinking, an important aspect of teacher knowledge, as teachers learn the new ideas. Such an activity holds promise for helping newly acquired knowledge be accessible and usable in the classroom.

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MATHEMATICS TEACHER RETENTION: FOUR TEACHERS' VIEWPOINTS ON STRESS IN THE PROFESSION

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Up to half of teachers leave the profession by the end of their fifth year and teachers of mathematics are found to have one of the highest departure rates when compared to teachers in other disciplines. This paper studies four mathematics teachers from diverse backgrounds and teaching situations and reports their attitudes and beliefs on teacher stress, mathematics teacher retention, and their feelings about the needs of mathematics teachers, as well as other information crucial to resolving the mathematics teacher retention problem in the United States.

Introduction

It has been reported that the retention rate of teachers is a continual problem. This is more prevalent in the areas of mathematics and science and in urban settings (Ferrini-Mundy & Floden, 2007; Rotherham & Mead, 2003). The most recent Teacher Follow-up Survey indicates that Special Education teachers are the most likely to leave the profession with secondary level teachers of core subjects such as mathematics, English, and social sciences as the next highest number of departures (Cox, et al., 2007). Ingersoll (2003) reported on the prior version of the Teacher Follow-up Survey in which secondary mathematics teachers held the highest percentage of departures.

Studies have been conducted that attempt to find the reasons that teachers are leaving the profession. The results of these studies are mixed and, at times, contradictory. For example, Ferrini-Mundy and Floden (2007) and Cwikla (2004) suggest there is not enough preparation in mathematics content for college students who want to teach mathematics after graduation. However, Paul (2005) reports that the current undergraduate mathematics courses required for a degree in mathematics can be a "filter" that eliminates potential mathematics teachers. This is a commonly discussed issue in teacher preparation, and it is also noted that currently a third of students in grades 7-12 do not have a teacher with a major or minor in mathematics (Reys & Reys, 2004).

Stressful situations occur in all areas of teaching. The Teacher Follow-up Survey revealed 32% of teachers who changed schools stated "poor working conditions" as a reason for their move. Additionally, over 37% of teachers who left the profession stated they were going to "pursue a job outside of teaching" (Cox, et al., 2007). A teacher's stress can be elevated by poor student behavior (Geving, 2007), lack of administrative support (Lambert, O'Donnell, Kusherman, & McCarthy, 2006; Blase, Blase, & Du, 2008), and the plethora of additional tasks required of teachers such as extra duties like hall monitoring, bus duty, and bathroom patrols (Brown, 2005).

Studies conducted on teacher stress and retention rarely focus on one specific content area. Stress is generally studied with large groups of teachers from varying subject areas and grade levels. This paper aims to focus primarily on mathematics teachers by studying four secondary mathematics teachers from diverse backgrounds about the stress placed on teachers of mathematics and how it effects their satisfaction with the teaching profession. This paper aims to answer the following questions:

1. From the viewpoint of the teachers, what are school administrators doing to help reduce stressors and increase retention?
2. What efforts are being made to retain mathematics teachers?
3. Do mathematics teachers believe they have the same needs as teachers in other disciplines?

Methodology

The teachers participating in this study were randomly chosen from a group of teachers used for a larger study on teacher stress and burnout. The teachers in the original study (n=385) were stratified into four groups based on their stress level and preventive coping skills and one mathematics teacher was randomly selected from each group. The groups were defined as low stress with low coping, low stress with high coping, high stress with low coping, and high stress with high coping.

Each teacher was interviewed using a semi-structured interview protocol. In order to determine the major commonalities and differences between the four participants, a thematic analysis will be used to analyze the data (Coffey & Atkinson, 1996). Outlying themes were determined through theory and the interview protocol was designed based on these themes. The themes chosen for the interview protocol were the following: Stress and Coping, Retention, Mathematics, and other miscellaneous questions. Topics addressed throughout the interviews also included their perceived needs of mathematics teachers, mathematics teachers' college preparatory courses and how it affected their preparation for teaching mathematics, professional development, collaboration with other teachers at their schools, high-stakes testing, salary, and the "perks" and "downfalls" of the profession. The participants' responses were entered into a meta data analysis table created in a spreadsheet to compare and contrast the results from each theme.

Group 1 (Low Stress, Low Coping) Representative: Candace

Candace was chosen as the participant from Group 1. She is a public school teacher that has been teaching for thirty years at the same school in a state in the southeast United States. Her school uses a traditional schedule. Candace holds a Master's degree and stated she is very satisfied with the teaching profession.

Group 2 (Low Stress, High Coping) Representative: Maggie

Maggie has only been teaching for four years and has taught at only one school. She teaches at a fine arts magnet school in a different state in the southeast United States. She holds a master's degree in Mathematics Education. At the time of the interview she was working towards an educational specialist degree and was contemplating entrance into a doctorate program in the future. Maggie's school also uses a traditional schedule. She stated she is only somewhat satisfied with the teaching profession.

Group 3 (High Stress, Low Coping) Representative: Becky

Becky has been a teacher for nineteen years and has taught at five different schools. At the time of the interview she taught at a public school in a state in the south-central United States. Her school operates on a 4x4 block schedule. In addition to teaching Becky had worked in other professions for seven years. She holds a bachelor's degree in Mathematics. Becky also stated to be only somewhat satisfied with the teaching profession.

Group 4 (High Stress, High Coping) Representative: Janel

Janel works at a private school in the northeast that uses a hybrid block schedule. She has been a teacher for only three years, but has taught at two different schools. The first school at which she taught was in the same state as Becky, but changes in her life required her to move out-of-state. She worked at another profession for one year between her two teaching assignments. Janel holds a bachelor's degree in Statistics and Mathematics Education. She indicated she is very satisfied with the teaching profession.

Stress and Coping

Different Schools, Different Stress

With the exception of Candace, who has taught at the same school for thirty-one years, the teachers all felt that the stress levels were different at each school where they have taught. Maggie has been teaching for only five years, all at the same school, but she could still remember vividly her student teaching experience and how the stress level at that school was not as high as the stress levels at her current school. At the time of the interview, she was teaching for a top-ranked magnet school in her state. While she stated that the pressure for teachers to produce top ranked results was significant, she also stated that it is acceptable as a tradeoff since they taught "great students."

Becky spent some time teaching at two middle schools before she settled into teaching high school. She described her time while teaching middle school as "doing time" because it was her first four years of teaching and she felt she was being suppressed because she did not have enough experience to find a job she was truly qualified for.

> *Middle school to me was just a far more stressful period. That is not my first love, that's not what my training was for, but it was where I could find a job. You know I always joke about that, I refer to those first four years of teaching as 'doing time.'*

Janel had more difficulty discussing the difference between the stress levels at the two schools where she had taught. She taught at the first school for only one year before getting married and moving out-of-state thus having to change jobs. She agreed that she was much more stressed during her first year of teaching, but she felt that her judgment might have been clouded by the fact that it was her first year of teaching and not necessarily the school where she taught. She feels the small age difference between her and her students played a role in her stress level that year, as well as the fact that she had to study the material she was teaching every night in order to stay current with her lesson plans.

Stress Relief: Other Teachers, Family, and a Nap

The teachers were asked what methods they use to relieve stress. The overwhelming response centered on the support of other teachers and that of their family. Becky, Janel, and Maggie all agreed that being around other teachers with whom they could decompress was the best form of stress relief for each of them.

Janel stated that a stressed teacher is detrimental to their students. So, she felt it was most important to take care of herself and calm herself down so she can be more supportive of her students. When asked about the most stressful days at work, both she and Maggie stated they went home and took naps to help them feel refreshed.

Becky was perhaps the most candid of all of the teachers about her methods of relieving stress. She passionately described how her husband, grandson, her dog, as well as her religion get her through difficult days.

> *I go home and I have a wonderful husband who will let me rant and rag, let me blow it out, and I know this will probably sound crazy to a lot of people, but there is nothing more relaxing than curling up with my dog…and just petting her and talking to her and stroking her. She's always happy to see me, animals are wonderful you know, and I go to see my grandson a lot, if I've had a really, really bad day I go to see my grandson on the way home and sit…and if he's sleeping I can just sit and look at him sleep and all is right with the world you know? And…I am very active in church, I prayed a lot this year and that's no lie, I have probably prayed more this year.*

Test Scores and Administration

All teachers interviewed were asked, "Do you feel the pressure of high stakes testing causes teachers to leave the profession?" None of the teachers hesitated before they said "Yes." During the conversations with these four teachers, the concern over test scores was a constant reference in their stories and feedback. Their responses ranged from a "laissez faire" attitude to one of being extremely nervous. The underlying concern, however, was the same: Administrators put tremendous pressure on teachers to produce high test scores.

Candace articulated the idea simply by stating that the pressure on testing forced teachers to teach how to take the test and, because of this, teachers lose autonomy in the classroom. Becky also expressed the idea that standardized testing costs a tremendous amount of money for school systems and too much time and money is "wasted on testing." Janel teaches in a private school where state standardized tests are not required. She has friends in the profession in public schools, however, and they have been told that their job status depended on their students' test scores. Maggie, the teacher in the top ranked school, almost yielded to her administrator's pressure for test scores. She spoke about how she almost quit teaching the previous year because of the tremendous pressure.

> *That's one reason I thought about leaving. I don't know why, but last year it really got to me. It was almost too much to handle, but again I think that might be a personality issue too…is that I take it too personal, that I have to just…I really have to know that if I try my best, I can't control whether someone sleeps through the test.*

While administrators can place enormous pressure on teachers to produce high test scores, there are other things that they can do to help alleviate stress on teachers. Teachers expressed the importance of discipline support from administrators. This support allowed teachers to teach without disruptions from those who did not wish to learn. Candace stated that in addition her administrator attempted to reduce the amount of paperwork and extra duties in which they must participate. Similarly, Becky appreciated that her administrators did not require them to attend every after-school event and meeting. This allowed her more time for doing her job completely.

Janel seemed to have one of the simplest suggestions for helping teachers alleviate stress from administrators: "I think they have to remember what it was like when they taught." She says that most administrators were previously teachers and many of them may have forgotten how demanding the profession can be. Maggie also added that simple encouragement also can increase the confidence level of teachers.

> *I think they could do a better job of encouraging. I think sometimes they get caught up with their people…breathing down their throat about how numbers have to be. Then they're breathing down our throat about how numbers have to be and they forget to say 'hey, we think you're doing a really great job.'*

Mathematics Teacher Retention

Needs of Mathematics Teachers

The teachers interviewed did not seem to feel that mathematics teachers had *more* needs than other teachers, but they have *different* needs. The most prevalent need mentioned by these teachers for mathematics teachers was the need for specialized professional development. Becky and Maggie both noted that mathematics courses undergo more curriculum changes than courses in other disciplines. Therefore, they stated, there is a greater need for professional development when these changes occur. Maggie also stated that mathematics teachers have to participate in after-school tutoring and remediation sessions more often than teachers in other disciplines. Teachers of other disciplines often used this extra time for their lesson planning and grading.

Mathematics Teacher Shortage

Only one teacher interviewed felt the need to increase the salary for teachers of mathematics. Others felt a differentiated salary increase was not fair to teachers in other disciplines who may work just as hard as a mathematics teacher. Janel proposed a novel concept for retaining mathematics teachers. She suggested letting mathematics teachers specialize in a specific area of mathematics. Many teachers of history and science specialize in one aspect of that discipline and she felt that mathematics teachers could benefit from the same process. She stated that if mathematics teachers teach only in one area such as Algebra, Geometry, Calculus, or Technical Math, then they would not feel they are "spread too thin" among their courses. This could result in a stronger ownership of courses and more successful teachers.

Candace discussed the need for more respect in the teaching profession by stating the following: "If your doctor says something, you would never think to refute them, but if a teacher does, a lot of time parents don't even think twice about saying 'I don't think that's right.'" This lack of respect from parents was a common issue among all of the interviewees. All teachers interviewed agreed that communication with parents is not a desirable part of the job. Becky and Janel both recounted instances within the last year where parents were excessively rude with them to the point at which they could not continue the conversation. In Becky's case, the parent wanted a conference with her, but she refused the conference unless an administrator was present. Maggie believed her personality causes her to dread this portion of her job as she did not like confrontation and admitted that communicating with parents usually involved some sort of confrontation. She dismissed this issue; however, and stated that "no job is perfect."

College Preparation

The participants in the study were asked about their college preparatory classes for becoming teachers. They were asked if they felt the mathematics courses they were required to take as undergraduate students were too difficult for potential teachers. They all agreed those courses were very difficult for them at the time, but none seemed to think they were not necessary. Candace and Maggie both stated that they felt the courses were relevant to them later in their careers because they helped build logical connections between mathematical topics and the courses prepared them to be better problem solvers. Maggie joked that she felt they "made us take those advanced mathematics courses in college in order to feel what it's like to not understand something so we can sympathize with our students."

Janel was the only participant who inferred that she did not think that her higher level mathematics courses were necessary. She never specifically stated she felt they were not needed. Instead, she expressed her disappointment in the lack of pedagogical courses that trained her how to *teach* mathematical concepts. She felt an exchange would have been optimal where they reduced the number of mathematics content courses and replaced them with mathematics pedagogical courses.

> *I feel like the downside to the math education programs in colleges, as much as I think they do a great job on the education side, we take all these advanced math classes and then we go teach like algebra one and no one's told us how to teach factoring. We did all these advanced things and then we go back and teach the most basic, and so I think that sometimes it is hard to remember how to go back and teach the way we can do it.*

Why They Stay

When asked what they feel are the "perks" of the teaching profession, a number of tangible results were discussed ranging from health and retirement benefits to the fact that teaching is "family-friendly" (except during emergencies such as sickness), they all expressed their love for teaching and the students. Becky stated that she could go home frustrated, aggravated, and ready to quit and her husband would ask "Tell me again why you teach?" She would respond:

> *I love the kids, they're not students to me, they're my children and ever since I've gotten back into high school they are my children…I go back the next day because I don't believe that anybody else will take as good of care of my children as I will.*

She also stated when asked about the perks that "I get to spend my day with kids everyday and I love it." She went on further to say:

> *I don't know, I think the perks of teaching is that if you love kids it's a way to spend time with kids and hopefully feel that you can make some kind of difference in their lives…I mean we spend a lot more time with them than their parents do at this high school age, a lot of them, not every one of them, but a lot of them, so it's the opportunity to give them a positive influence.*

Candace described the feelings she had when her students come back to the school to visit after they have graduated.

> *When they come back and talk to you, that's when you actually know that effect, that you have touched somebody's life, and then that gets passed on to the next generation, so even if I, if I do die, I know that a little part of me is all these people. That's how I think about it.*

Janel saw as a reward when students made good grades in her class. This reward, to her, outweighed the less desirable parts of the profession.

I just feel like the reward is greater…when that kid gets an A on the test or I just love being around the kids so…I can deal with all the rest of it because I feel like there's a reward there you know. I feel satisfied at the end of the school year…when I see the kids graduate…I feel like I accomplished something.

When asked why she remains in the teaching profession, Maggie jokingly responds "because I signed a contract." She then describes the reason as:

When it comes down to it, it's not about the test scores and when I really think about it, it's more about the students and I still enjoy interactions with students and making those interactions every day. When it comes down to it, that's what I always get down to, that's why I stay, because that's why I got into teaching. It's not because I have this great love for math, like I am still trying to decide if I even like math. It's just that I got into teaching for the students and so that's why I continue to stay.

Discussion

Kate Walsh, President of the National Council on Teacher Quality (NCTQ, 2008), states that the time period of the third through the fifth year of a teacher's career is "an opportunity lost for the health of the teaching profession" because of the high rates of teacher attrition within that valuable time of a teacher's career. Becky specifically stated that she seriously considered leaving the profession after her fifth year of teaching.

I had all of the bottom barrel weakest classes, had no bright spot in my day. That was the year I had three students that went to prison, you know, and it was just emotionally, it just tore me apart and I was frustrated, I felt like I was working myself to death and they weren't getting any better, like I was working really hard but the kids weren't working very hard…and no matter what I tried, I would try games, I would try you know different approaches…oh it was crazy and it was…just in a very difficult situation where I just thought, I do not know if I want to do this the rest of my life.

Maggie also discussed thoughts of leaving the profession. She seriously considered leaving at the end of her fourth year due to pressure from her administrators to produce high test scores. Studies, such as the "Teacher Follow-up Survey," also use the fifth year as a significant cut-off for years of experience (Cox, et al., 2007). The significance of using the fifth year as turning point is an interesting concept that could warrant future research.

Although the teachers in the study admitted that their administration has a tendency to increase their stress levels with added pressure on test scores, the teachers interviewed in this study were able to supply some specific items their administrators were currently doing to help alleviate their stress. Support with disciplining students, not requiring teachers to attend every afterschool meeting and event, and cutting down on paperwork and extra duties were all mentioned that help teachers do their job with ease. Previous studies have found student discipline problems and time management are problems for teachers and this support seems to parallel those findings. The lack of more significant responses could reveal that the efforts to retain teachers are not advertised enough to truly be effective, but it could also mean that there are no specific strategies in their school systems for retaining teachers and reducing stress.

Further investigations that pose the same question to administrators would be an interesting angle on this problem of teacher retention.

When questioned about how their schools retain mathematic teachers, none of the teachers could give specific, current, and relevant answers. Becky mentioned that a neighboring school system was offering signing bonuses for teachers of mathematics. More drastically, Maggie noted that her state was contemplating a rule that new teachers hired in “high needs areas”, such as mathematics, would be hired and begin their pay scale at the salary as those teachers with five years of experience in order to entice more teachers in those areas. Maggie did not condone this idea as she felt that was unfair to teachers in other disciplines who work just as hard as mathematics teachers. Once again, a study where this question was posed to school administrators may result in drastically different responses. Perhaps the participants’ lack of specific responses about their current school system’s tactics show that they are not enticed to stay in the profession by extravagant resources and outpouring of financial sources. They are more attracted to feeling “worthwhile” (Becky), having “great students” (Maggie), the reduction of menial tasks (Candace), and fewer meetings (Janel).

None of the teachers felt mathematics teachers were more privileged than teachers in other disciplines, but they did feel that some of the needs were different. The most common response was the need for specialized professional development. Becky and Maggie both stated that mathematics curriculum undergoes more changes in the standards more often than other disciplines. They were in agreement that mathematics teachers need to be better trained when these changes occur. On the contrary, Becky felt that all disciplines have varying needs, indicating that mathematics teachers were not more in need of items than teachers in other disciplines. Candace, being a science and mathematics teacher, felt that science teachers had more needs that teachers in other disciplines due to the “rather subjective” nature of the material and the time required to prepare for lessons. Surprisingly, none of the teachers mentioned specific tangible items that could be purchased with additional funding. An occasional mention of technology did occur, but only when the teachers were at a loss for a more descriptive answer. None of them seemed overly passionate that they were at a loss of technological resources that they needed in order to do their job effectively.

Teaching is a stressful career and few will refute the significance of the amount of stress involved in the career. Numerous studies exist researching stress and burnout. When Freudenberger (1974) initially began his research on burnout, he started by researching burnout on all professions, not specifically education. This led up to discussions on the teaching profession. When discussing who is prone to burnout, Freudenberger claims those most at risk are “the dedicated and the committed” who are “seeking to respond to the recognized needs of people.” When even more pressure is added from administrators, stress levels increase and burnout worsens resulting in more weary teachers who’d rather find other professions than teach children.

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REFLECTIONS ON TEACHING SPREADSHEET-BASED FUNCTIONS 'OUTSIDE THE CLASSROOM' AND TEACHERS' PRACTICAL RATIONALITY

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This report presents my reflections on teaching particular spreadsheet-based functions to teachers 'outside the classroom'. The reflections are framed in terms of interactions between two different practices, my own as a mathematics teacher educator and that of a teacher's practical rationality. The framework will enable us to pursue our thinking further in regards to the potential that spreadsheets offer as an instrument for optimising teacher understanding of functions and ways of engaging teachers' practical rationality. Findings have implications such as the need for aligning spreadsheet-based discourse, which is associated with functions, with official mathematics discourse in the classroom.

Introduction

Part of my practice entails teaching functions, by using spreadsheets, to teachers 'outside the classroom'. By 'outside the classroom' I refer to, and mean, the teaching and learning of mathematics that does not take place within the confines of a school classroom. In contrast, 'inside the classroom' refers to the teaching and learning of mathematics in the school classroom. Various studies point out the centrality of functions in school mathematics and teacher education (Even, 1990; Nyikahadzoyi, 2006). When teachers observe the interrelationship between quadratic and linear functions, as shown by means of spreadsheets, their responses to such observations reveal the conflict which tends to accompany the use of spreadsheets in the classroom. The integration of electronic spreadsheets (Pea, 1985) in classroom teaching, as is the case with any other technology, is marginal, according to Hoyles, Noss and Kent (2004). Observing teachers' responses outside the classroom to the use of spreadsheets, in relation to algebra teaching and learning with a focus on functions, therefore makes sense. The current paper, consequently, considers what we can learn from teachers' responses, in terms of their practical rationality, when they observe my teaching of spreadsheet-based functions outside the classroom.

Theoretical framework

The theoretical framework is driven by the research question, which draws on different, though overlapping, bodies of literature, such as those relating to spreadsheets as part of information communications technology (ICT), to school algebra reform, to varieties of mathematics and to teachers' practical rationality regarding mathematics teaching.

The body of literature on the potential for ICT, including the use of spreadsheets in school algebra teaching and learning with functions, is growing (Dettori, Garuti & Lemut, 2001; Haspekian, 2005; Sutherland & Rojano, 1993; Yerushalmy, 1999). Yerushalmy and Chazan's (2008) review of the prevailing discontinuities in school algebra assesses the role of various ICT components, such as spreadsheets, in the study of functions. In their review, they indicate the instructional benefits to be gained from the use of spreadsheets, one being the way in which multiple representations (whether graphic, tabular or symbolic) are made possible. With an appropriate habit of mind (Cuoco & Goldenberg, 1996), spreadsheets (in the form of *Excel*) can enable the user to adopt a problem posing (Brown & Walter, 1993) approach when it comes to the teaching and learning of functions. When such an approach is adopted within a learning environment, such as a school, the teaching of algebra can be rethought in such a way as to defy traditional curricular interpretations, including those which pertain to the separation of linear and quadratic functions. Pea (1985, p. 170) alludes to such reconceptualisation of traditional interpretations as an instance of the reorganisation metaphor, rather than of the amplification metaphor. Classically, the amplification metaphor is taken as referring to the adoption of a unidimensional focus, which retains the basic structure, meaning that straight lines and parabolas are treated as distinct objects. The reorganisation metaphor, in contrast, entails the emphasising of curricular continuities, such as exploring functions from an ‘interesting middle’ (Schwartz, 1995), in such a way as to show the interconnections existing between different functions.

The demonstrations of spreadsheet-based functions which I, as a teacher educator, have presented to groups of teachers are instances of where our mutual teaching practice connects. In part, my practice is informed by the mathematics of a mathematics education researcher (MER mathematics) and its particular form of discourse (Sfard, 1998) . According to Julie (2002), MER mathematics is the variety of mathematics which deals mostly with elementarised versions of mathematics, such as with the connections between, and the development of, mathematical ideas by means of spreadsheets, as in the current case. Julie further explains that MER mathematics is structured in terms of insights gleaned from learning theories, pedagogy, and the history and philosophy of mathematics. School mathematics teaching, to which he alludes as ‘school-teaching mathematics’, is what teachers do, being their practice. Watson (2008) expounds on such a variety of mathematics, which she calls ‘school mathematics’, arguing that it is a special kind of mathematics, which is subject to institutional constraints. An example of school mathematics is the official mathematical discourse of the classroom, which is the preferred term used in the current study to refer to such teaching. Unifying concepts such as a focus on the conceptual connections within and between functions may not be desirable, she notes, in school mathematics teaching. During my demonstrations, I use a form of MER mathematics discourse which is peculiar to spreadsheet use, about which I provide more detail later in the paper. As such discourse does not form part of the official mathematics discourse of the classroom (Hoyles et al., 2004), a discrepancy between the two varieties of mathematics and the practice of each is inevitable.

The French school on the instrumentation of spreadsheets in mathematics teaching (Haspekian, 2005) informs my teaching of MER mathematics. For instance, while working with spreadsheets, I use the Excel spreadsheet program and its capability, together with my knowledge of computer hardware and mathematics, to instrumentalise my classes. In this way, I can explore particular mathematical operations by means of such instrumented actions as ‘fill downs’. Figure 1 provides an example of such action, in which cell B5 contains =A5*A5 for the function $f(x) = x^2$.

The spreadsheet shown in Figure 1 is a numerical representation of a given phenomenon (Friedlander & Tabach, 2008), which, in the current case, is a parabola, $f(x) = x^2$, according to school mathematics and the teaching thereof. An acute awareness of the intimate connection between the calculation activated by the equals sign in cell B5 and the structure of x^2 is required for the learning of mathematics (Hoyles & Noss, 2009). The ‘opened up’ nature of cell B5 in Figure 1 reveals the connection of its contents to those of A5, indicating that a functional relationship exists between the two columns, A and B. The user and viewer of such a spreadsheet should, thus, be aware that they need to look *through* the spreadsheet, as well as *at* it (Artigue, 2002). Teachers generally adopt an approach to their mathematics teaching that is governed by practical rationality (Herbst & Chazan, 2003). For example, when they encounter the use of spreadsheets in relation to school algebra teaching and learning, such an encounter tends to impact on the ways in which they think about their own teaching practice. The teaching of mathematics at school level is shaped by various institutional, situational, epistemological, temporal and material conditions that prevail in the classroom (Watson, 2008), with such conditions serving to shape the use of spreadsheets in the reform of algebra teaching. Such reform will, inevitably, only take place over time. In order to impact on the approach taken by teachers of mathematics, according to Bourdieu (1998), they need to become sensitised to the use of spreadsheets outside the classroom in order that they might learn how most effectively to use them in the classroom.

	A	B
1		
2	X	X^2
3	-4	16
4	-3	9
5	-2	=A5*A5
6	-1	1
7	0	0
8	1	1
9	2	4

Figure 1. An example of instrumented action for the spreadsheet-based function f(x) = x^2

Outside the classroom		*Inside the classroom*
MER mathematics: Spreadsheet-based linear & quadratic functions through instrumented activity: Numerical (multiple) representations Technological mediation Reorganisation metaphor	Teachers' practical rationality ⟷ Framing/reframing (Schön, 1983)	School-teaching mathematics (Julie, 2002) / School mathematics (Watson, 2008) e.g. straight lines & parabolas

Figure 2. A diagrammatic representation of the theoretical framework of the current study

Figure 2 is a diagrammatic representation of a theoretical framework which I constructed to portray the alignment of the two practices. The primary setting for the current study is outside the classroom, comprising the holding of meetings with, and the giving of spreadsheet-based demonstrations to, the teachers concerned in the study. The nature of such a setting is revealed in the first two columns in the figure. The third column is presented as a broken line, due to the current absence of data on spreadsheet-based approaches to classroom-based functions. The information which belongs in the third column was inferred from the specific concerns and questions raised by, and sourced in the practical rationality of, those teachers who attended my spreadsheet-based demonstrations. The middle column indicates the positioning of the space in which I appealed to their practical rationality. The epistemological implications of Figure 2 lie in the way in which the figure captures ways of knowing and doing which are related to my practice as a mathematics teacher educator, and how such practice is interpreted by those whom I teach. In addition, Figure 2 has methodological implications, which will be elaborated later on in the course of the paper.

Methods

The methods of inquiry used in the current report are qualitative, stemming from constructs in the theoretical framework. Evidence of spreadsheet-based functions is provided by references to words and phrases in the first column in Figure 2. Evidence of teachers' practical rationality is present in data fragments, which were gleaned from teachers' responses to my teaching of the spreadsheet-based functions. In addition, the methodology employed made use of 'insider' research (Ball, 2000), in that it forms part of my own practice, so that the study is earmarked by both proximity and distance (Adler, Ball, Krainer, Lin F-L & Novotna, 2005). As findings must be treated with scepticism (in keeping with academic distancing) and subjected to a critical perspective, one way in which to conduct analysis is to reframe (Schön, 1983) data fragments by comparing them with forms of school mathematics, which is contained in the standard school mathematics curriculum. Reframing in such a sense is a type of comparison (Corbin & Strauss, 2008) between what happens outside the classroom with what happens inside it. For such a reason, I placed the notion of framing and reframing in the central column in Figure 2. In addition, I am aware that those findings which resemble MER mathematics should not be dismissed as solipsistic (Ball, 2000), due to their value in terms of illuminating the continual interplay between the varieties of mathematics resulting from instrumented activity conducted by means of spreadsheets.

Data sources, evidence, objects and materials

The data for the current report originates in a larger research project which was conducted into the role of spreadsheets in school mathematics reform relating to teacher education. In relation to the pre-service teacher education programme of a large university in the Western Cape, South Africa, we (Gierdien & Olivier, 2009) explored the possible use of spreadsheets with prospective teachers who were to be certified to teach mathematics up to Grade 10 in public schools in South Africa. Recently, both Olivier and I have started to work with practising teachers, who used to be marginalised under the apartheid regime, which officially ended in 1994. The involvement of officials (curriculum advisors) of the provincial Education Department and of the professional teachers' association in such work has made contact sessions with the teachers concerned more regular than they used to be.

The data fragments that will be presented for analysis in terms of the current study come from the related formal and informal meetings which were conducted with the teachers, as well as from a 50-minute presentation, titled *Drawing families of functions using spreadsheets*, which was attended by practising secondary school mathematics teachers and Education Department officials. In the demonstration, I used a data projector and spreadsheet software to construct the functions outlined in Figures 3 and 4 (see below). After observing my demonstration those teachers participating in the study commented on it, and also raised concerns and questions about the demonstration. It should be noted that the teachers did not have access to laptops or computerised aid during or after the demonstration. The venue was a normal classroom which could seat up to 40 students. The demonstration was undertaken mainly from a problem posing perspective, in terms of which questions were raised regarding what happens when we systematically vary linear functions and multiply them. Such pedagogical moves are in line with the notion of an ‘interesting middle’, which calls for a reorganisation, and not an amplification, perspective on linear and quadratic functions. Such moves should be viewed as examples (and evidence) of the practice of MER mathematics, with the goal of pointing out the connections between functions, for example, in terms of the approach which is taken by teachers in respect of practical rationality.

A more detailed description of the demonstration will reveal the instrumented actions involved. At the beginning of the demonstration, I drew the equivalent of a pencil and paper list of different linear functions on the board, after which I composed quadratic functions, based on the linear ones, as are shown in Figure 3 below.

Linear function	*Quadratic function*
$F_1(x) = x$	$F_1xF_1 = x^2$
$F_2(x) = x – 1$	$F_1xF_2= x(x – 1)$
$F_3(x) = x – 2$	$F_1xF_3= x(x – 2)$

Figure 3. A pencil-paper equivalent composition of linear and quadratic functions

	A	B	C	D	E	F
x	x	x - 1	x - 2	x^2	x(x - 1)	x(x - 2)
-5	-5	-6	-7	25	30	35
-4	-4	-5	-6	16	20	24
-3	-3	-4	-5	9	12	15
-2	-2	-3	-4	4	6	8
-1	-1	-2	-3	1	2	3
0	0	-1	-2	0	0	0
1	1	0	-1	1	0	-1
2	2	1	0	4	2	0
3	3	2	1	9	6	3
4	4	3	2	16	12	8
5	5	4	3	25	20	15
6	6	5	4	36	30	24
7	7	6	5	49	42	35
8	8	7	6	64	56	48

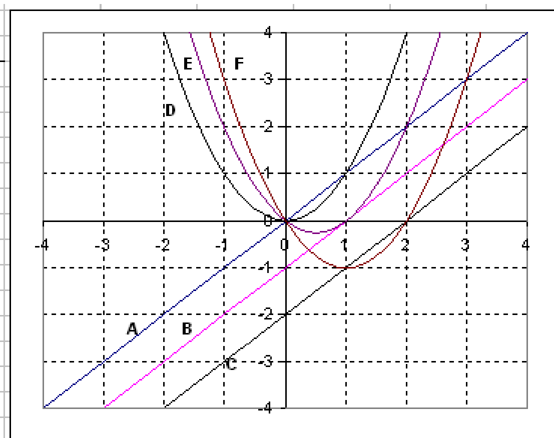

Figure 4. A spreadsheet-based multiple representation of linear and quadratic functions

I then proceeded to use the spreadsheets to construct the functions (see Figure 4) in multiple ways, with an emphasis on showing how to compose quadratic functions based on linear ones. In terms of a pencil and paper exercise, I considered the general case of the parabola $f(x) = ax^2 + bx + c$, where $a = 1$; and $c = 0$. My intention was to explore the parameter b, and to ask what happens if b = 0, -1, or -2. Such choices had been made prior to the demonstration, due to the time limits on such an exercise, as well as the epistemological benefits to be gained from starting in an ‘interesting middle’. Such a move amounted to a reorganisation of the traditional way of progressing from straight lines to parabolas which keeps the two conceptually separated from each other. The attention of the teachers concerned was then focused on the parameter ‘b’ in $f(x) = x^2 + bx$. Such decisions were in line with the habit of asking ‘what if?’, and with making an appeal to the practical rationality of the teachers concerned, in the hope that they might notice the conceptual connections within and between functions and how a particular use of spreadsheets enables such functions to occur in a different sequence, and at a different rate, compared to more traditional ways of using such functions. In the demonstration, I relied on the instructional affordances allowed by the spreadsheets, including the tabular, graphic and symbolic representations. Additional data sources for the exercise consisted of my own reflective notes and observations. Figure 4 shows the evidence provided in the form of spreadsheets data.

The above demonstration resulted from my instrumented actions and the degree to which I have made an instrument of spreadsheets, in terms of the French school of thought, and a tool for presenting and representing the various functions. During my construction of the various spreadsheet-based functions, I used a form of discourse which is peculiar to spreadsheet use, including use of the term ‘fill down’, and use of such concepts as parameters, cells, recursion, the selecting of columns in order to plot the various functions by means of the ‘scatter’ function, and multiple representations. I mention the use of such discourse to draw attention to the discrepancy between it and school mathematics teaching, or official classroom mathematical, discourse, which has to do with straight lines, parabolas, turning points, roots or zeroes, and the y-intercept.

In the current paper, I shall present an analysis of the following three incidents during the demonstration. The incidents serve as evidence of interactions between my practice and the practical rationality of teachers, which is associated with school mathematics teaching. The three incidents analysed bear witness to a comparison between the two practices. In addition, such incidents are significant, as they have implications for mathematics teacher education and the (lack of) learning opportunities to which they allude.

1. “What kind of mathematics is this?” was a question raised by a few teachers towards the end of the presentation.
2. A few teachers commented on the ‘mother function’ $f(x) = x^2 + c$ and its connections to the spreadsheet-based graphic representations in Figure 4.
3. Some teachers argued that the learners must first know the algebra concerned before they use the related spreadsheets.

My analysis is predicated on the notion framing/reframing, in terms of which I draw a comparison in order to keep the appropriate degree of academic distance.

Findings

My demonstration leading to the functions represented in Figure 4 evoked a lively discussion among the teachers concerned. One of the results of this discussion was that a few of the teachers asked the above-mentioned question.. By means of instrumented actions, I used the spreadsheets to reorganise the straight lines and parabolas with which teachers are familiar in novel ways, as indicated in Figure 4. The figure reflects an instance of the reorganisation metaphor, according to which familiar functions are conceptually connected by means of spreadsheets. The demonstration fell in the field of MER mathematics, with its peculiar discourse regarding spreadsheet-based linear and quadratic functions. Further evidence of operating within an MER mathematics framework lies in the title of the demonstration: families of functions. Such a title differs from that which would be assigned to such a demonstration in terms of a school mathematics perspective, as the denotation of such interrelationships would be highly unusual in school mathematics and the practice of school mathematics teaching, both of which tend to be highly fragmented. The teachers who participated in the study might have had difficulty in understanding the connections between the spreadsheet-based functions, especially as they had not only been required to look *at* the functions but also *through* them. The teachers were required to follow the peculiar discourse associated with the instrumented actions which I constructed for the various functions shown in Figure 4. The cells and tables used in the spreadsheet in Figure 4 lie on the arithmetic side, making it difficult for those who are new to the field to recognise the different algebraic functions when looking at the cells and tables for the first time.

In Figure 4, columns A through F can be seen to contain the numerical (integer-valued) representations of known objects, with straight lines and parabolas being reflected in the first column. In the given instance I relied on the instructional affordances, namely on the integer-valued linear and quadratic sequences in the tables and cells, in order to construct the spreadsheet-based functions. I inserted the headings 'x', 'x(x – 1)', and 'x(x – 2)' in Figure 4 in order to allow the teachers viewing the demonstration to relate the content to their conventional school mathematics teaching. My appeal to the teachers' practical rationality in the given instance was mediated by the computerisation of the school mathematics, using straight lines and parabolas, along with a reorganisation of familiar school mathematics content. Rather than giving a fact-oriented presentation of the various functions, my demonstration aimed at stressing the connections between the different functions, primarily by means of adopting a 'what if?' approach to the subject covered. The MER mathematics discourse associated with my instrumented actions might have resulted in the teachers asking "What kind of mathematics is this?" Such a question is significant, because it is evidence of the uniting of the two practices – my own as a mathematics teacher educator, and theirs as mathematics teachers. This type of reconceptualisation of the subject, which my demonstration required the mathematics teachers to undertake, demanded that they relinquish their traditional approach to the teaching of mathematics. Furthermore, I am aware that the content of Figure 4 is not covered in such assessment regimes as the Grade 12 final examinations, which is of pivotal importance to both teachers and learners alike.

'Mother function' $f(x) = x^2 + c$

The practical rationality of the teachers participating in the study was shown, in relation to their reference to the 'mother function', as a way of connecting with the type of MER mathematics which they witnessed during the demonstration. I intentionally started my demonstration with the linear function f(x) = x, with the goal of constructing quadratic functions by means of systematically varying the parameter b. Some of the teachers expressed the opinion that of $f(x) = x^2 + c$ could be considered as the 'mother function' and that it could be used to build other related quadratic functions, such as $f(x) = x^2 – 1$; $f(x) = x^2 – 2$; and $f(x) = x^2 – 3$. Evidently, they argued from within the framework of school mathematics in the given instance, because they explicitly referred to such a mother function as falling within the scope of the mathematics curriculum. Their use of the mother function framework for understanding the linear and quadratic functions which I constructed in terms of those instrumented actions which are shown in Figure 4 was apparent. However, the 'mother function' with which I started my demonstration was f(x) = x, instead of $f(x) = x^2 + c$, because I wished to stress the relations between the functions, as well as to begin at an 'interesting middle', where such relations become visible. In terms of the school mathematics framework, the two functions concerned are characterised by curricular discontinuity. A focus on epistemologically different mother functions might, therefore, be seen as a fruitful way in which to engage the practical rationality of teachers.

Learners must first know the algebra concerned before they can use the related spreadsheets

My agreement with "[l]earners must first know the algebra concerned" rests on my interpretation of such a statement as meaning that learners should have opportunities to see, and to explore, straight lines and parabolas as linear and quadratic integer-valued numerical sequences, respectively. However, I hold that they should, simultaneously, be offered opportunities of exploiting the affordances of the spreadsheets. In this way, they should learn how to represent, in multiple ways, such sequences as spreadsheet-based expressions, which can be found in the work of Friedlander and Tabach (2008), among others.

Lacking evidence of what the teachers concerned meant by ‘algebra’ in the given statement, taken from a school mathematics perspective, such algebra probably was not that which is explicitly conceptualised as numerical sequences per se. Instead, by ‘algebra’ the teachers might simply have meant straight lines and parabolas. Overall, the statement concerned highlights the challenge of engaging with teachers’ practical rationality, and especially their pedagogical beliefs, about the potential that spreadsheets offer in terms of the learning of algebra.

My demonstration was, in many ways, focused on the technological mediation of algebraic knowledge, namely functions. A necessary epistemological condition for the discovery of spreadsheets as a tool or instrument with which to learn algebra outside the classroom is that teachers have opportunities for nuancing the meaning of ‘know’ and ‘algebra’. As we saw earlier, some teachers attending the demonstration had difficulty in recognising the algebraic formulation shown in Figure 4. Whether the learners concerned actually understand the conceptual relationships between the different functions illustrated in the figure, which are numerical representations of known school mathematics, consisting of straight lines and parabolas, was explored for just under an hour with the teachers concerned. In the given case, such understanding related to coming to know what would happen if the integer values of ‘x’ and ‘b’ varied, and how ‘families of functions’ would be represented in multiple ways as a result of such variation. In terms of expert knowledge how changes in the values of ‘b’ affect $f(x) = x^2 + bx$ is ‘explicitly trivial’ (Yerushalmy, 1999). In contrast, however, such a concern is understandable when such relations are reframed from a school mathematics teaching perspective, in which context the potential for spreadsheets in algebra learning has yet to be embraced.

Conclusion

The findings reported on in the current paper have implications for mathematics teacher education. The interaction of my practice of MER mathematics and teachers’ practical rationality, with its roots in school mathematics teaching practice, has been demonstrated in this paper. Some of the teachers participating in the study had difficulty seeing familiar algebra in computerised form, with many conceiving of, and seeing, the connections between spreadsheet-based functions and the ‘mother function’ with which they are familiar in terms of teaching mathematics at school level. Other teachers disagreed with the role played by spreadsheets in the learning of algebra. The incomplete theoretical framework presented will, hopefully, help us to extend our thinking about how the two practices can benefit teacher, and, in turn, student, understanding of spreadsheet-based functions. The use of spreadsheets in teaching indubitably affects the teaching of mathematics at school level. The technological mediation which is provided by the use of spreadsheets allows for a new understanding of the subject in terms of generally recognisable mathematics, such as straight lines and parabolas. The three incidents explored in this paper indicate the need for a developmental approach to tool use in respect of spreadsheets in cases where spreadsheet-based discourse and the official classroom mathematical discourse align with each other. The attendant degree of skills and sophistication needs to be developed over time, with spreadsheets becoming an instrument for the learning of algebra. In the research cited earlier in the current paper, there is ample evidence of a place for spreadsheets in optimising the understanding of such an algebraic object as functions with numerical origins. As the three data-related incidents typify the kind of responses which have been encountered in the current researcher's ongoing meetings with teachers, much, clearly, still needs to be done in terms of making pedagogical moves to bridge the two practices concerned.

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TEACHER CHANGE FACILITATED BY INSTRUCTIONAL COACHES: A CUSTOMIZED APPROACH TO PROFESSIONAL DEVELOPMENT

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Although professional development can help teachers become familiar with new teaching practices, changing their instructional practices is still very difficult once removed from the professional development. We present results from a case study of teachers involved in a three-year professional development designed to provide teachers with an instructional coach during the school year. Results show that teachers are able to change certain instructional practices overtime and important interactions among instructional coaches and teachers are highlighted.

Introduction

Professional development is more effective at changing teachers' instructional practices when teachers engage in the professional development with their colleagues et al., 2002). What happens when teachers are either isolated as the only mathematics teacher at their school, or are the only teachers at their school teaching a particular course? The purpose of this paper is to report findings from one model of sustainable professional development centered on the use of an instructional coach who provided customized support for teachers as an extension of a summer program.

Theoretical Perspective

Implementing reform mathematics curricula represents a challenging transition for many teachers (Ziebarth, 2003), especially for those whose perceptions of mathematics education are grounded in traditional views of teaching mathematics. Although many view the textbook as the most important catalyst for changing what occurs in mathematics classrooms, the adoption of the curriculum alone is not likely to transform teachers' instructional practices (Wilson & Lloyd, 2000; Arbaugh et al., 2006). Further, research has shown convincingly that teachers are not likely to change their practices by attending isolated professional developments, as they need ongoing and sustainable support (Ball, 1996; Ball & Cohen, 1999; Guskey, 2002; Loucks-Horsley, Love, Stiles, Mundry, & Hewson, 2003; Putnam & Borko, 1997; Wilson & Berne, 1999). The *Principles an Standards for School Mathematics* (NCTM, 2000) recommend teachers to take part in "ongoing, sustained professional development" (p. 369), but research on how to provide continuous support for the types of collaborations necessary to facilitate changes in teacher practices is limited.

In our model instructional coaches facilitate interactions among teachers and provide them with the customized support they need to make changes to their classroom practices. Ball (1996) states, "the most effective professional development model is thought to involve follow-up activities, usually in the form of long-term support, coaching in teachers' classrooms, or ongoing interactions with colleagues" (pp. 501-502). Through an analysis of research on different types of teacher training, Joyce and Showers (2002) found that training programs which incorporate a coaching component are more effective in increasing teachers' knowledge, skills, and transfer to practice than trainings without such a component. Presented in this paper is an analysis of the changes teachers make through their interactions with an instructional coach and activities the instructional coach used to support these changes.

Context

The North Carolina Integrated Mathematics Project

The North Carolina Integrated Mathematics Project (NCIM) was developed to create and support a community of teachers using the *Core-Plus Mathematics* (CPMP) integrated curriculum materials particularly in high needs schools. Its aim was to educate teachers about the content and pedagogy of using integrated mathematics by creating a sustainable professional development model that could be replicated across the nation. Spread throughout the rural parts of the state, the seven partner schools were identified as low-performing (based on North Carolina accountability measures). The average student population for the project schools for the 2008-2009 school year ranged between 110 to 163 students and the ethnic make-up consisted of: 1% American Indian or Asian, 6% Hispanic, 16% White, and 77% Black. Approximately 72.8% of students at each school qualified for free and reduced lunch. The schools joined a collaboration called the New Schools Project and each had a focus on STEM (Science, Technology, Engineering and Mathematics) education. Twelve teachers were actively involved in the project, some who were the only mathematics teacher at their school. One aspect of the project was to assist rural schools in establishing statewide collaborations to address the challenges of isolation.

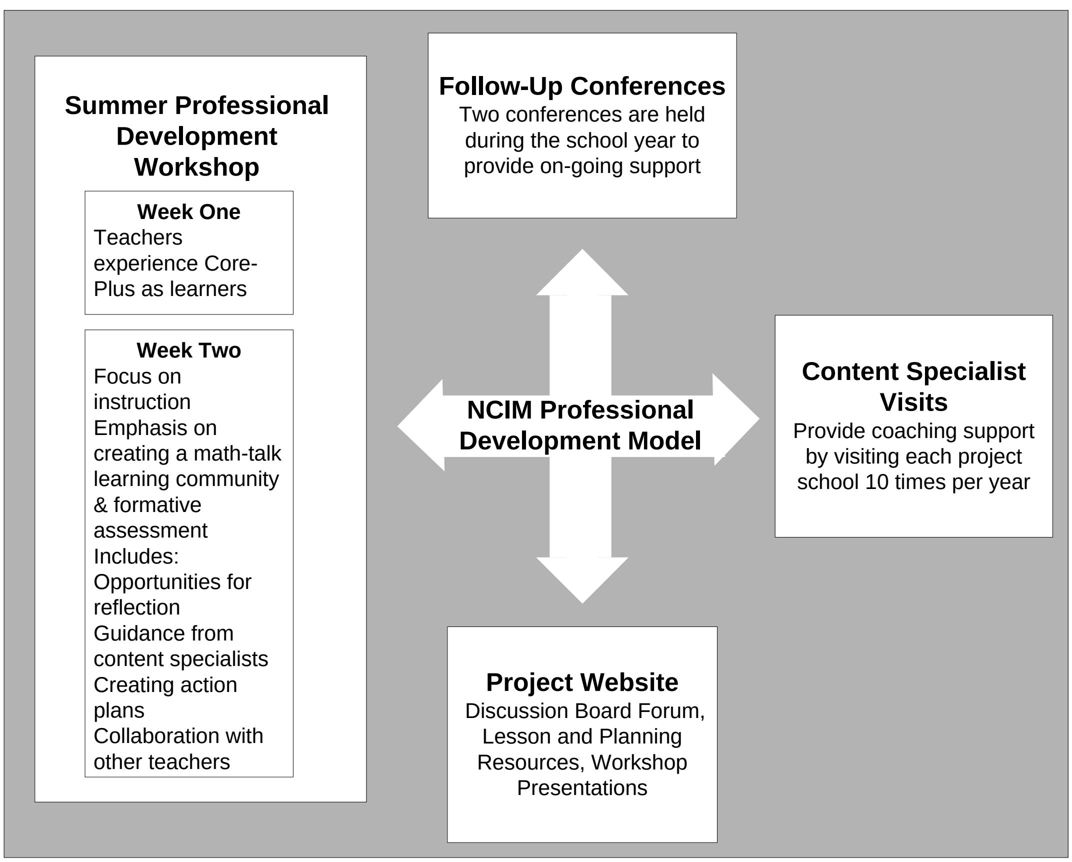

Figure 1. Four components of the NCIM professional development model

Four Components of the Professional Development Model

The coaching model was situated within a larger project, to prepare teachers to implement CPMP, in order to strengthen and invigorate STEM education at these schools. The goal of the overall professional development model was to improve and strengthen teachers' mathematical content and pedagogical knowledge (Loucks-Horsley, et al., 2003). The four main elements of the professional development model (see Figure 1) included a summer workshop providing in-depth education on use of curricular materials (one or two weeks), a web-based environment supporting information exchange, two face-to-face follow-up conferences, and instructional coaches who visited each site monthly. The project had two experienced teachers as instructional coaches, who had successfully used CPMP. This type of customized mentoring has been found to be responsive to individual teacher needs (Darling-Hammond, 1997).

Methodology

Participants

The participants in this study included the two instructional coaches, Linda and Margaret, and twelve NCIM project teachers from six of our partner schools. The project has grown since its inception in the spring of 2008 to include new teachers each year and has lost two due to teacher turnover. Included in this analysis are four teachers from the spring of 2008, 8 from the 2008-2009 school year, and 10 from the 2009-2010 school year. Of the twelve teachers involved in this study, five are mentored by Margaret and seven by Linda.

Data Collection

Data were collected from classroom observations of both teachers and of their interactions with their instructional coach, semi-structured interviews with both teachers and instructional coaches, a teacher content knowledge assessment, and instructional coach reports (see Table 1). Instructional coaches completed a report after each of their site visits to document their observations and activities with each teacher. The reports provided information on the teachers' use of time and pacing, instructional behaviors, collaborative learning, use of technology, formative assessment, and classroom management. Since the inception of the project in 2008, there have been approximately 140 reports. However, since some of the instructional coach visits occur on teacher workdays or during the summer, the majority of the results presented below are from the 89 reports in which our classroom observation protocol was used.

Table 1. Data Collection Matrix

	Interviews	Observation	Documents	Video Recording
Coaches	Yes	Yes	Yes	
Teachers	Yes	Yes	Yes	Yes
Students		Yes	Yes	Yes

Analysis

Since multiple sources of data were collected, detailing a "bounded system" over time, a case study was an appropriate method for this analysis (Creswell, 1998). Teachers were nested in different schools and assigned one of two different instructional coaches, thus an embedded analysis (Yin, 2008) was used to track changes in teachers' instructional practices over multiple units of analysis. Further, to determine which activities and behaviors seem to promote or detract from teacher growth overtime an analysis was also conducted across both instructional coaches.

Results

The Activities of the Instructional Coach

During the course of the project, the research and implementation team developed a listing of activities that mentors could engage in with teachers, helping them improve their instruction. The list was first brainstormed by the team and then revised in light of the first year instructional coach reports. The activity list then served as the basis for the reports to produce data on the relative frequency of the different activities. The broad categories for reporting are: (1) Curriculum and content assistance (2) Lesson planning, enactment and reflections (3) Assessment, feedback and grading, and (4) Professional community interactions.

Table 2. Percent of Total Activities of Instructional Coaches by Category

	Total	Linda	Margaret
I. Curriculum and Content Assistance:	18.57	16.57	20.29
II. Lesson Planning, Enactment and Reflection	52.08	44.10	58.94
III. Assessment, Feedback, and Grading	5.71	8.43	3.38
IV. Professional Community Interactions	23.64	30.90	17.39

Table 3. Percent of Activities Performed by Coaches Within Each Subcategory

	Total	Linda	Margaret
I. Curriculum and Content Assistance:			
Provide additional resources	32.87	47.46	22.62
Discuss or clarify content	30.07	32.20	28.57
Suggest methods for new vocabulary and reading investigations	16.08	5.08	23.81
Offer technical support for technology use	16.08	15.25	16.67
Communicate sequence with teachers and administrators	4.90	0.00	8.33
II. Lesson Planning, Enactment and Reflection			
Assist in daily lesson planning and structure	21.45	20.38	22.13
Offer classroom observation feedback	19.70	26.75	15.16
Fostering discourse and questioning	17.46	6.37	24.59
Model teaching	14.21	22.29	9.02
Develop long-term pacing guides	12.72	14.01	11.89
Provide strategies for collaborative learning	9.98	7.64	11.48
Define what a *Core-Plus* classroom looks like	4.49	2.55	5.74
III. Assessment, Feedback, and Grading			
Assist in developing appropriate assessments	27.27	13.33	57.14
Suggest appropriate homework	25.00	30.00	14.29
Develop strategies for providing feedback to students	25.00	26.67	21.43
Assist in developing appropriate rubrics for student work	22.73	30.00	7.14
IV. Professional Community Interactions			
Follow-up via email	47.25	46.36	48.61
Promote collaboration	23.63	27.27	18.06
Encourage web site use	19.23	18.18	20.83
Communicate needs of teacher	9.89	8.18	12.50

Once the categories and subcategories (see Table 3) were identified all reports of the content specialists were reviewed, and each time a behavior in the list was reported, a single instance was marked at the level of sub-categories. At the level of the categories, all tallies were totaled and the ratio of the tallies for that category to the total number of tallies for all categories was reported as a percentage (see Table 2). This yielded the results of the relative percent of instructional coach reports of distinct activities that fell into each of the four categories; percentages were also determined for each of the instructional coaches.

Then within each category, we report the ratio of the number of tallies per subcategory in relation to the total reports for that category as a percentage. This permits gaugeing the relative frequency of each of the subgroups within the category (Table 3).

Within the lesson planning category, Linda was more likely to model teach during her site visits. In an interview, Linda noted her favorite activities to engage in with teachers were model teaching and helping small groups, as she enjoyed student interactions. Her enjoyment of model teaching is evident from the activity list where she model taught about 22% of the time she was engaged in activities from the lesson planning category. Another activity that Linda performed more often than Margaret was promoting professional interactions (30.90%), and within this category she worked on developing collaboration with other STEM teachers (27.27%). Not only had Linda encouraged teachers to become model teachers for beginning teachers at their school, but Linda also took teachers that were isolated, as the only mathematics teacher at their schools, to observe teachers in other districts who had experience teaching CPMP.

Almost sixty percent of the activities Margaret engages in with the teachers were within the lesson planning, enactment, and reflection category. When Margaret was asked to explain her experiences as an instructional coach she replied, “It depends on the classroom I am in. Sometimes [I] interrupt group discussions and clarify content or go up to the board and help teach. Some teachers really just want me to sit and watch and give them feedback.”

A big focus Margaret had was customizing the professional development based on the needs of each individual teacher, while trying to improve their whole-class discourse strategies (24.59% of the lesson planning category). During the 2009-2010 school year she reported progress in four out of five teacher’s whole-class discourse. In one of her reports she noted, “I observed the teacher asking for more than the correct answers, asking students for reasoning. She allowed the students to do more of the explaining than last year.” And yet another she reported, “Students were engaged while working in groups and discussing things as a class…I have seen this teacher develop this over the last few years with these students.”

Time Spent With the Instructional Coach

On average instructional coaches spend approximately 120 minutes with each teacher during each visit, with a range from 15 to 240 minutes. The average amount time instructional coaches spent with each teacher is consistent for each year of participation in the project (see Table 4). Further, Linda and Margaret spend about the same amount of time at each site, 116 minutes and 127 minutes respectively.

Table 4. Time Spent With Coaches Based on Number of Years in the NCIM Project

Time Instructional Coaches Spend With Teachers				
Year in Project	First	Second	Third	Total
Average Amount of Time (minutes)	118.54	124.00	121.05	119.65

There is a wide discrepancy in the range of time instructional coaches spent with each teacher. Linda reported both the minimum and maximum times from her observations with Maria, who has received an instructional coach for two and a half years. Maria has made major changes to her teaching practices, which she attributes to the support of Linda and to her attendance at the workshops. One of her greatest accomplishments has been her transition to a facilitator in the classroom, and she points out that to get students to rely on one another she "goes around the asking them questions and every time they ask me a question, I ask them another question." At the beginning of the 2009-2010 school year Linda spent 240 minutes helping Maria prepare to teach a new course, the third course in the CPMP sequence. A week later, two new teachers joined the staff at Maria's school, both unfamiliar with reform mathematics and one with a language barrier. Linda consulted with the principal at the school and determined that she should spend more of her time with the two new teachers. While she still spends time in Maria's classroom and communicating with her via email, Linda is confident in Maria's growth that during one site visit she only devoted 15 minutes to observing Maria.

Teacher Change Overtime

Time management. Time management and pacing had become a tremendous concern for teachers and instructional coaches at the end of the 2008-2009 school year. Many teachers did not complete the final chapters of their textbooks before state assessments and others skipped crucial chapters in the middle of the book because they felt the sections were unimportant. During the 2009 Summer Workshop eight hours were reserved for long-term planning and there was a mini-afternoon session devoted towards effective time management strategies.

Using the observation protocol, instructional coaches report if time was used effectively for what the class needed. Instructional coaches respond on a 1 (ineffective or inappropriate) to 4 (effective and appropriate) scale, where the average effective time index for all project teachers is 2.87 (see Table 5). Further analysis showed that the longer teachers are involved in the NCIM project, the more effective they utilize class time.

Table 5. Average Effective Time Index

Year in Project	Effective Time Index
1st year	2.64
2nd year	3.02
3rd year	3.35
Total	2.87

Pacing. Instructional coaches also report if a teacher is on pace for the year. Separated by instructional coach, Table 6 indicates the percentage of teachers that were adhering to pacing guides they developed with their instructional coach. The differences between the pacing of teachers involved by instructional coach are surprising. The teachers Margaret coaches have a much higher percentage of being on pace for the year than Linda's teachers. There are two explanations for this discrepancy. First, based on a separate analysis of the instructional coach reports, Margaret reported in 28 of her 37 reports that she discussed daily lesson planning and in 29 of her 37 reports she discussed long-term pacing. Whereas Linda only devoted 24 of her 52 visits to daily lesson planning and 22 to long-term pacing. Second, Margaret made eight visits (to four teachers) before the beginning of school for planning sessions with teachers, whereas Linda made only three visits (to three teachers) before school was in session. During these visits Margaret helped teachers develop pacing guides for the entire year. The upfront time Margaret committed to long-term pacing resulted in having the higher percentage of teachers on pace.

Table 6. Average Effective Time Index

	Margaret	Linda
Not on Pace	20	85.71
On Pace	80	14.29

Content Knowledge. A content knowledge assessment is given at the beginning and end of each Summer Workshop to measure teachers' knowledge of mathematics. Initial results (see Table 7) are disheartening, however teachers show marked improvement at the conclusion of the workshop. On the observation reports when asked what assistance teachers need, instructional coaches most frequently respond that teachers need content support. Instructional coaches also report on the content delivery quality of teachers they observe.

Table 7. Teacher Content Knowledge Test Results

	Pre-Test	Post-Test	Percent Increase
2008-2009 NCIM Teachers	40.38%	57.63%	17.25
2009-2010 NCIM Teachers	49.73%	60.27%	10.55

Below is part of an interview transcript where Margaret discusses the difficulties she sees in teachers' content knowledge:

Margaret: She does really good stuff, but she has trouble with the content and when I'm there that's what we talk about. I very rarely talk about what happened in class that day. Except if it's affecting the content. We talked about how her warm-up took a really long time this time, but only because she isn't going to cover all her content if she sets her classroom up like that.

Interviewer: Is it strange for you to talk to teachers about content when they are math teachers?

Margaret: NO, because I see how weak they are! I've seen all of them make math mistakes.

Longitudinal results show that the longer a teacher was involved in the project, the more accurate they delivered content during (see Table 8). These results are encouraging for positive teacher change overtime, as inaccurate content delivery decreases and accurate content delivery increases with each additional year teachers are a part of the NCIM professional development.

Formative Assessment. A final topic examined based on the analysis of the instructional coach reports was how often teachers recognized when students were lost or misunderstood mathematics content (see Table 9). Formative assessment was an important component of the 2009 Summer Workshop based on an early evaluation of the instructional coach reports that highlighted a need for teachers to monitor and assess student progress. By their third year, 60% of project teachers began to frequently recognize when students had a misunderstanding about mathematics and another 20% occasionally noticed when students were confused or lost (the 20% that never recognized was one teacher who only attended week one of the workshop). These findings suggest teachers are more likely to consistently employ formative assessment measures based on the number of years they participate in the project.

Table 8. Percentage of Content Delivery for Teachers Based on Years in Project

Overtime	Accurate	Moderately Accurate	Inaccurate
1st year	44.44	22.22	33.33
2nd year	54.17	20.83	25.00
3rd year	80.00	10.00	10.00

Table 9. Percentage of Teachers Recognizing When Students Needed Assistance

	Frequently	Occasionally	Seldom	Never
1st year	22.86	60.00	14.29	2.86
2nd year	45.83	20.83	33.33	0.00
3rd year	60.00	20.00	0.00	20.00
Combined	34.29	42.86	18.57	4.29

Discussion

A model of professional development that has helped teachers in our project transition to teaching reform mathematics is to provide them with an instructional coach. Over time, by incorporating reports from instructional coaches and suggestions from teachers, project directors created a professional development in the context of teachers' classrooms that allowed them to make changes to their instructional practices. This type of coaching is aligned with Guskey's (1999) argument that "professional development is an ongoing activity [which should be] woven into the fabric of every educator's professional life" (p. 38) (our parenthetical added).

The embedded analysis presented throughout this paper is evidence that when provided with support, teachers are able to transfer the knowledge and skill that were learned from the collaborations with an instructional coach and during the workshops into practice. Though not all participating teachers experience the same level of success, findings show the long-term benefit of working one-on-one with an instructional coach. It is also important to realize the changes in instructional practices were gradual and the treatment with the instructional coaches took time and were influenced by the types of activities teachers engaged in with their coach.

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A FRAMEWORK FOR ANALYZING THE ROLE OF JUSTIFICATION IN PROFESSIONAL DEVELOPMENT

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NCTM standards specifically call for instructional programs that enable all students to make and investigate mathematical conjectures as well as develop and evaluate mathematical arguments and proofs (NCTM, 2000).

Hanna (2000) argued that school mathematics must reflect the range of functions proof serves in mathematics to include proof as verification, explanation, systemization, discovery, and communication. However, as Hanna noted these functions might take on different levels of importance in the classroom where the key role of proof is to promote student understanding. Similarly, we argue that a conceptualization of mathematical justification in PD must be situated in the work of teaching and the purposeful development of mathematical knowledge for teaching (Ball, Thames & Phelps, 2008). The teacher learning goals might include providing teachers with explicit knowledge of proof and its role in the mathematics discipline. PD might also target the development of a more unpacked understanding of justification (or specialized content knowledge) so that teachers might recognize the strategies students employ in justifying algebraic ideas or the assumptions that underlie a students' empirical argument.

In this poster, I present a framework for considering the purpose of justification in mathematics professional development that couples research on the role of justification and proof in the mathematics discipline and mathematics education with Ball and colleagues' elaboration of the mathematical knowledge needed for teaching (Hanna, 2000; Ball et. al, 2008). Examples from a current research and development project with math teacher leaders illustrate how this framework was elaborated to analyze leaders and teachers' understandings of justification and its use in teaching. This work thus begins to address the question of what teachers need to know or learn about justification in order to meet the current vision of reform and what this might then demand of PD leaders.

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ADVANCING LEADERS' CAPACITY TO SUPPORT TEACHER LEARNING WHILE DOING MATHEMATICS

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Researching Mathematics Leader Learning (RMLL) is a two-phase research and development project investigating leaders' understandings and practices associated with developing mathematically rich learning environments for teachers. We illustrate how our framework and design evolved to focus on developing teachers' specialized content knowledge (SCK) as a key purpose for engaging in mathematics during professional development (PD). The project began with a framework for leader learning based on sociomathematical norms (Yackel & Cobb, 1996) and practices for orchestrating discussions (Stein, et al., 2008). Analyses of data from Phase I focused on how leaders made sense of these constructs in seminars and how they took up these ideas in their own facilitation. These analyses led us to more fully explicate purposes for doing mathematics in PD and distinguish how mathematical work with teachers can and should be different from work with students (Kazemi et al., 2009). In particular, we found that the purpose for doing mathematics in PD was underspecified and needed to link to developing teachers' mathematical knowledge for teaching (MKT) as described in the work of Ball and colleagues (2008). Based on these findings, we made significant revisions to seminars in Phase II. Our current design and research frames mathematics tasks to orient leaders to SCK, considers the negotiation of sociomathematical norms for developing teachers' MKT, and outlines practices for facilitating mathematical work in PD.

We examine tasks and consider the facilitation work in explicating mathematics that unpacks key concepts and practices such as mathematical models for operations and mapping across representations to explore change. Given that a purpose of developing teachers' SCK is a relatively new idea in PD, it is important to consider what this mathematical work might entail and how to research this endeavor.

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INTRODUCING IMPULSIVE DISPOSITION VIA MATH PROBLEMS

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In solving mathematics problems, "doing whatever first comes to mind … or diving into the first approach that comes to mind" (Watson & Mason, 2007, p. 307) is commonly observed among students. Lim, Morera, and Tchoshanov (2009) use the term *impulsive disposition* to refer to students' proclivity to spontaneously proceed with an action that comes to mind without checking its relevance. We believe that math teachers should be aware of this undesirable habit of mind so as not to reinforce this habit in their classrooms. The information presented in this poster presentation was based on a 2.5-hour workshop conducted for a larger study that was aimed at testing a survey for assessing problem-solving disposition along the impulsive-analytic dimension. The objective of the workshop was to introduce impulsive and analytic dispositions to a group of 27 in-service and 10 pre-service teachers by allowing them to experience their own disposition via solving three math problems. The three problems were designed to elicit impulsive disposition. The first problem involves inferring the value of speed at a particular instant from a distance-time graph involving only linear segments. Participants with an impulsive disposition tended to apply the $r = d/t$ formula (d on the graph represents distance from home and t represents number of minutes passed). The second problem begins with "Suppose p kilometers is equal to q feet" and then asks which statement is correct: $p > q$, $p < q$, or $p = q$. The third problem is "Jorge is offered a raise if he can increase his weekly productivity by 15%. If Jorge works a five day week, how much does he need to increase his productivity each day?" Participants were asked to write their reasoning prior to voting their answers via clickers (personal units for a classroom response system).

Participants' answer choices were tabulated, written responses were analyzed, and solution strategies were identified. For example, 22 of the 37 respondents chose 3% each day for the third problem; their strategy of dividing 15% by 5 was considered an indicator of impulsive disposition. When polled about the lesson, about 96% of the participants scored a 4 (agree) or 5 (strongly agree) that the lesson was interesting, and was valuable. Participants' written comments about what they had learned from the workshop included "don't go for the first idea without attending to meaning," "read the problem carefully and understand the wording before we answer," and "mathematics cannot be solved by using formula but using logic."

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Chapter 18: Technology

Research Reports

Brief Research Reports

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Posters

EFFECTS OF TWO DIFFERENT MODELS OF PROFESSIONAL DEVELOPMENT ON STUDENTS' UNDERSTANDING OF ALGEBRAIC CONCEPTS

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This paper examines the effects of the first year of a two-year project to investigate formative assessment in a networked classroom. Participants were divided into two groups; one group receiving professional development on formative assessment with networked technology while the second group received professional development only on formative assessment. Data were gathered on participants' knowledge of formative assessment, teacher pedagogical content knowledge, mathematics background, and attitudes toward technology. Student data were collected and analyzed to examine the effects of teacher variables on student achievement.

Introduction

Past studies indicate professional development (PD) was the difference between teachers using technology to emphasize critical-thinking and problem-solving skills versus skill and drill (Wenglinsky, 1998; Brannigan, 2002). Thus, PD for integrating technology into the teaching of mathematics must include a strong focus on pedagogical approaches that have potential for impacting learning. This paper describes two models of PD for formative assessment and classroom-connected technology and reports on results from the first year of implementation. Building upon theory and practice that showed positive outcomes, Project FANC (Formative Assessment in a Networked Classroom)[1] was designed to address concerns raised about both the use of formative assessment and integration of technology. Teachers, recruited from 15 middle schools, were assigned to two models of PD. Each group participated in a five-day summer PD with five follow-up sessions during the school year along with in-school coaching. The design of both models took into consideration that one of the most salient characteristics of effective PD is providing teachers with opportunities to work with colleagues, both in their school building and beyond, giving them opportunities to learn from one another's successes and failures and to share ideas and knowledge (National Center for Research on Teacher Learning, 1995).

Theoretical Framework

Connected Classroom Technology

In *How People Learn* (NRC, 1999), one of the most promising technology-based innovations noted for transforming the classroom environment was the use of networks. In prior research on the implementation of a networking system, the use of TI-Navigator was found to support the development of a collaborative classroom environment by enhancing student interactions, focusing students' attention on multiple responses, and providing opportunities to peer- and self-assess student work. The ability to display class data or responses supports a problem-solving approach to developing skills and concepts (Dougherty, Akana, Cho, Fernandez, & Song, 2005;

Mackay, Olson & Slovin, 2006). One of the most promising uses of TI-Navigator is its potential to overcome a significant hurdle to improving formative classroom assessment: the collection, management and analysis of data (Roschelle, Penuel, & Abrahamson, 2004). While feedback loops in a regular classroom are slow, classroom networked technology can provide rapid cycles of feedback in real time. Beaty and Gerace (2009) remind us that technology is a tool and that pedagogical approaches aid or impact learning. Yet, although Owens, Demana, Abrahamson, Meagher & Herman (2004) found teachers in TI-Navigator classes perceived as more responsive to individual needs, more focused on knowledge building and assessment, and more community centered, they may not have changed their instruction based on information obtained.

There are four distinct, intertwined, functions of the TI-Navigator system particularly helpful for implementing formative assessment: 1) Quick Poll—to immediately collect and display all the students' answers to a question; 2) Screen Capture—to monitor individual students' work at anytime; 3) Learn Check—to administer quick, frequent formative assessments and provide timely feedback; and 4) Activity Center—allowing students to work collaboratively to contribute individual data to a class activity. While these tools are directly appropriate for formative assessment, teachers who may make significant changes in the use of technology in their classes do not necessarily make full use of the potential of the connected classroom for formative assessment (Owens, Pape, Irving, Sanalan, Boscardin, &Abrahamson, 2008).

Formative Assessment

Similarly, evidence has shown that appropriately implemented formative assessments can produce substantial learning gains at various ages and across subjects (Black, Harrison, Lee, Marshall, & Wiliam, 2004; Black & Wiliam, 1998b; Wiliam, Lee, Harrison, & Black, 2004). Black and Wiliam defined "formative assessment" as "all those activities undertaken by teachers, and/or by their students, which provide information to be used as feedback to modify the teaching and learning activities in which they are engaged" (p. 7). Formative assessment for learning includes activities such as questioning, discussion, seatwork, and student self-assessment. Teachers' preparation in assessment is often non-existent and teachers' content knowledge may be insufficient for deep understanding of concepts and principles they teach and assess (Heritage, 2007). Moreover, Heritage suggests that while there is no best way to carry out formative assessment, assessment and teaching are reciprocal activities, and need to be firmly situated in the practice of educators. Black and Wiliam (1998a) demonstrated that formative assessment improves student achievement if it guides changes in day-to-day practice. Shavelson, Yin, Furtak, Ruiz-Primo, Ayala, & Young (2006) classified formative assessment techniques into three categories on a continuum based on the amount of planning involved and the formality of technique used: 1) on-the-fly formative assessment, when teachable moments unexpectedly arise; 2) planned-for-interaction formative assessment, used during instruction but prepared deliberately before class to align with instructional goals; and 3) formal and-embedded-in-curriculum formative assessment designed to be implemented at the end of a unit of instruction to ensure students reach important goals before moving on. Despite their variety, when formative assessments are used, common steps are explicitly or implicitly involved: 1) determining achievement goals students are expected to reach—the expected level; 2) collecting information about what students know and can do—the actual level; 3) identifying the gap between the actual level and expected level; and 4) taking action to close the gap.

Many formative assessment strategies, however, are challenging because they take too much time to be used practically (Black & Wiliam, 1998b). It is time-consuming to count students' responses, and almost impossible to provide specific feedback on each student's work in a typical teaching load of four to six classes and 20 to 30 students per class. Even if teachers believe the time investment for formative assessment will yield reward in the future, Black and Wiliam (1998b) noted that teachers' work loads are often overwhelmed by agendas and dominated by district curriculum specifications and high-stake tests.

Methodology

Project FANC is a three-year research project funded by the National Science Foundation to investigate the use of formative assessment in a networked classroom as it affects 7^{th} grade student learning of algebra concepts. Thirty-two teachers from 15 schools were recruited and randomly assigned within schools to the Formative Sssessment (FA) or Navigator (NAV) group. This randomized-block design helps control for extraneous variables, such as student background and school context (e.g., teachers' work load, curriculum, class equipment, and community support). In the first year of the project, students' understanding of algebraic concepts taught by 7^{th} grade teachers participating in the NAV group were compared to students' understanding of algebraic concepts taught by 7^{th} grade teachers participating in the FA group.

While Project FANC provided PD to participants over a two-year period, only the first year of the PD is described since results are being reported only for that period. Input from the eight Advisory Board members along with numerous articles on PD, formative assessment, technology and related topics (Ayala & Brandon, 2008; Black & Wiliam, 1998a, 1998b, 2005; Gearhart, & Saxe, 2004; Guskey, 2007/2008; Wiliam, Lee, Harrison, & Black. 2004) were taken into consideration to design the two models of PD. The formative assessment model (Stiggins, Arter, Chappuis, & Chappuis, 2004, p.42), questioning strategies, and rich mathematics activities were essential components in both PD experiences, although the actual delivery and focus varied. The PD models delivered to two groups, FA and NAV, are briefly described below.

FA teachers participated in five days of PD on strategies for using formative assessment in their classrooms (not using TI-Navigator). In addition, five follow-up days and coaching were conducted throughout the academic year. The formative assessment model, questioning strategies, and mathematics activities focused on designing appropriate tasks where students could demonstrate thinking. In-depth discussions were held on techniques to analyze student work and interpret results in terms of revealing student understanding. FA participants were provided with opportunities to examine tasks and lessons and discussed how the tasks fit expected learning progressions. Participants worked on selecting and modifying tasks using processes of reversibility, flexibility, and generalization to deepen students' understanding of content. Ideas on creating a learning environment that included an assessment conversation with student-to-student interactions were shared with participants. At follow-up sessions, participants shared progress on implementation, examples of student work, and changes in the classrooms climate and were given opportunities involving classroom activities and questioning strategies.

NAV teachers participated in five days of PD on formative assessment and strategies for using TI-Navigator in their classrooms. In addition, five follow-up days and coaching were conducted throughout the academic year. The formative assessment model, questioning strategies, and activities were connected to the use of TI-Navigator. Significant time for hands-on experiences with TI-Navigator was provided. As participants practiced using handheld technology and TI-Navigator, emphasis was on creating and choosing good tasks and asking good questions. The PD focused on the 'added value' of handheld technology in a connected classroom. During follow-up, participants shared how they implemented formative assessment with TI-Navigator and discussed troubleshooting tips and changes in classroom climate.

At the beginning of the project all participants were given MacBook laptops, LCD projectors, Elmo visualizers (document cameras), and a classroom set of TI-73 graphing calculators. NAV participants were provided TI-Navigator Systems for their classrooms.

Data Collection

The results in this paper focus on student growth in achievement on a student assessment of algebraic concepts with a focus on patterns and relations, teacher's responses on questionnaires, and assessment of teachers content knowledge. A student assessment was developed to determine the extent to which participating students showed pre-post gains after implementation of the intervention. As much as possible, algebra items were drawn from existing sources such as state assessments and released NAEP, TIMSS, and SAT items. Items were reviewed for content, perceived alignment with the project's target content, perceived difficulty, and quality (e.g., clearness of item stems, etc.). Eighty-five items were selected from these sources. Once standards and knowledge levels were reconciled, items were selected for pilot testing. Two versions of a pilot-test were created from 48 items and piloted with over 1000 students in Grades 7 and 8. Item Response Theory was used to select items included on FANC Student Assessment.

To measure the growth of participants' content knowledge for teaching (CKT), the University of Michigan's *Learning Mathematics for Teaching* (LMT) instrument was administered at the beginning of the summer institute in year one, and then again after one year of participation. The LMT measures have been shown to be a significant predictor of student achievement (Hill, Rowan, & Ball, 2005). Teacher data were collected with three instruments that were developed for the study: a) the Teaching Practice and Perceptions Questionnaire includes two scales about teacher collaboration and four about teacher support; b) the Assessment Knowledge, Self-Efficacy, and Practice Survey includes four scales about assessment knowledge (assessment in general, student learning, subject content, and formative assessment) and one about teacher self-efficacy in using formative assessment; and c) Using Technology Questionnaire includes four scales about teachers' perceptions of using technology.

Results

Proposed Model

The analysis reported here is for 23 participants (FA (11), NAV (12)) for whom complete student data (1,629 students) was collected. A two-level hierarchical model (Luke, 2004) included one student covariate, the pretest score, and eight teacher variables. The six continuous variables included: a) teacher pretest score (labeled "TeaPre"); b) teachers' years of teaching experience ("TeaExp"); c) teachers' efficacy in formative assessment ("FAefficacy"); d) teachers' efficacy in technology ("TCHefficacy"); e) teachers' perception about their students' motivation ("StuMotiv"); and f) teachers' perception about overall support from school, colleagues, parents, and students ("Support"). The two discrete variables included a) the experiment group (FA vs. NAV) and b) teachers' mathematics or mathematics education major (major and minor vs. no math background in university, labeled "NoMath" and "Math").

The level 1 units were students taught by teachers who participated in FANC. The level 2 units were the teachers in FANC. Multilevel modeling was conducted using SAS PROCMIXED, Version 9. At the student level, the intercept was the only random component. A model in which the student pretest score was entered as a random variable failed to converge; therefore, the decision was made to treat the covariate as fixed effect. No interactions were hypothesized at the teacher level. Interactions between the level 2 predictors as well as between the level 1 and level 2 predictors were not significant. As anticipated, the interaction effect between the pretest score and the experiment group was not significant. This means students' pretest scores did not differ across the FA and NAV groups. Another interaction effect between the pretest score and teachers' mathematics education was also not significant, suggesting students' pretest scores did not differ by whether teachers had mathematics education in undergraduate and/or graduate schools. Based on these preliminary examinations, the following model is proposed:

Level 1:

$$Y_{ij} = \beta_{0j} + \beta_1(\text{StuPre}) + r_{ij},$$

where $r_{ij} \sim N(0, \sigma^2)$, β_{0j} a random intercept for a teacher j, β_1 a fixed slope of students' pretest scores, and r_{ij} random error associated with student i/teacher j.

Level 2:

$$\beta_{0j} = \gamma_{00} + \gamma_{01}(\text{TeaPre})_j + \gamma_{02}(\text{ExpGp})_j + \gamma_{02}(\text{MathBack})_j + \gamma_{03}(\text{TeaExp})_j$$
$$+ \gamma_{04}(\text{FAefficacy})_j + \gamma_{05}(\text{TCHefficacy})_j + \gamma_{06}(\text{StuMotiv})_j + \gamma_{07}(\text{Support})_j + \mu_{0j}$$

where $\mu_{0j} \sim N(0, \tau_{00})$, γ_{00} the grand mean of students' posttest scores, γ_{0k} the slope of the variable k where k=1,2,..8, and μ_{0j} the random deviation of students' mean posttest score of teacher *j* from the mean.

Preliminary Analysis

Table 1 presents descriptive statistics for student pretest and posttest scores. Table 2 presents descriptive statistics for six continuous variables in the model at student and teacher levels.

Table 1. Descriptive Statistics for Students' Pretest and Posttest Scores by Teacher Group

Group	Statistic	Student pretest	Student posttest
TI-Nav	*M*	13.720	16.130
(*n*=692)	*SD*	5.916	8.514
FA	*M*	14.040	16.850
(*n*=937)	*SD*	6.083	6.969
Total	*M*	13.900	16.550
(*n*=1,629)	*SD*	6.103	7.501

Table 2. Descriptive Statistics for Teacher Variables

Group	*Statistic*	*TeaPre*	*TeaExp*	*FAefficacy*	*TCHefficacy*	*StuMotiv*	*Support*
TI-Nav	*M*	20.100	10.455	7.475	7.674	3.643	3.227
(*n*=12)	*SD*	5.820	5.716	1.163	1.146	0.503	0.858
FA	*M*	19.900	7.364	7.143	7.843	3.250	3.300
(*n*=11)	*SD*	5.195	4.945	0.649	1.322	0.707	0.461
Total	*M*	20.000	8.909	7.292	7.763	3.456	3.262
(*n*=23)	*SD*	5.370	5.450	0.905	1.213	0.626	0.682

The initial step of the multilevel analysis was to examine the unconditional means model to ascertain whether the data warranted a multilevel model. Estimated variance at teacher level (τ_{00}) was found to be 18.330 ($p<.01$), and estimated variance at the student level (σ^2) was 42.624 ($p<.01$) in this null model. Hypothesis tests indicated that both variance components were significantly different from 0. These estimates suggest students' scores differ among teachers in average posttest scores, although there is more variation among students within the teachers than between teachers. The intra-class correlation, 0.301, shows that about 30% of the total variance occurs at the teacher level—a substantial proportion. This cluster of variability between different teachers' students' scores suggests that an OLS analysis of these data would likely yield misleading results.

Random Effects

This proposed model was significantly better than one in which the intercept was included at the initial step, $\chi^2(9)$ =10790.422-6891.167= 3899.255, $p< 0.01$. Table 3 presents random effects at level 1 and level 2. The conditional component representing variation among students' scores within the teacher level (σ^2) decreased from 42.624 in the unconditional means model to 20.484 ($p<.01$) after including the student level covariate, student pretest score. This indicated that inclusion of this student pretest score explained about 52 % ((42.624-20.484)/42.624 = 0.519) of variation within teachers. Even though the students' variation within the teacher level was substantially explained by student pretest score, the significant random variation of 20.284 shows possibilities to explain the remaining random variation using other student level variables.

Table 3. Random Effects at Level 1 and Level 2

Effect	Estimate	*SE*	Wald *Z*	*p* (1-sided)	95% confidence interval	
					Lower	Upper
Random effect at Level 1						
Residual	20.484	0.854	23.999	0.000	18.878	22.227
Random effect at Level 2						
Intercept	3.369	1.442	2.337	0.019	1.457	7.793

The variance component representing variation between the teachers (τ_{00}) drastically decreased from 18.330 in the unconditional model to 3.369 ($p<.05$) after controlling for the student and teacher level covariates. The predictors included in this proposed model explained substantive amount of variation in students' posttest scores at the teacher level. The intra-class correlation was 0.155, meaning that the proportion of between teacher variance to the total variance was 15.5 % after controlling for the predictors. The change of intra-class correlations from 0.301 in the unconditional means model to 0.155 in the proposed model also demonstrates that the teacher level predictors explained well the variation of students' posttest scores at the teacher level. Even though these results showed the random variations at the student and teacher levels were substantially explained by the predictors, the remaining clusters of variability at the teacher level as well as at the student level warrants further development of the multilevel model.

Fixed Effects

Table 4 shows results of hypothesis tests for fixed effects of student level and teacher level predictors. The fixed effect of students' pretest score was significant after controlling for the teacher level predictors. One of the variables of interest was Group, indicating whether a teacher belonged to the FA or NAV group. The student posttest mean in the FA group was 3.462 points higher than that of NAV group after controlling for the students' pretest scores and other teacher background variables. Teachers' knowledge in the subject content was a significant predictor. When a teacher scored 1 point higher in the pretest, their students achieved 0.448 higher points after the influence from the other variables was partialled out. Teachers' years of experience in teaching and mathematics education did not significantly influence students' achievement.

The variables asking the teachers' perceived efficacy in formative assessment and in technology use were not significant, nor was the teachers' perception about their students' motivation in class a significant predictor, after taking into account the other student and teacher level predictors. Finally, teachers' perception about overall support from school, colleagues, parents, and students showed a positive effect on students' achievement. Teachers who differ by 1 point in their perception about overall support differ by 3.041 points in students' achievement.

Table 4. Fixed Effects of the Student and Teacher Predictors

Effect	Estimate	*SE*	Approx *df*	*t*	*p* (2-sided)	95% confidence interval Lower	Upper
Intercept	6.084	7.101	13.938	0.857	0.406	-9.152	21.321
StuPre	0.797	0.023	1166.851	34.572	0.000	0.752	0.842
Group	3.462	1.465	13.179	2.363	0.034	0.301	6.623
MathBack	2.593	1.446	14.150	1.793	0.094	-0.506	5.691
TeaPre	0.448	0.163	13.281	2.745	0.016	0.096	0.800
TeaExp	-0.259	0.126	13.865	-2.060	0.059	-0.529	0.011
FAefficacy	-1.190	0.700	15.036	-1.699	0.110	-2.681	0.302
TCHefficacy	-0.917	0.588	13.854	-1.558	0.142	-2.180	0.346
StuMotiv	-1.096	1.010	13.977	-1.084	0.297	-3.263	1.072
Support	3.041	1.225	13.813	2.482	0.027	0.410	5.672

Conclusions and Discussions

The purpose of the study was to use Year 1 data to examine if students' achievement in the FA and NAV groups was different, after taking into account students' pre-test scores and teachers' initial statues such as their knowledge, teaching experiences, efficacy, and content knowledge for teaching. Student's understanding of patterns in the FA group was significantly better than students in the NAV group. The teacher variables that significantly affected the students' achievement on the post-test were the teacher's content knowledge and the support from the school, parents, and students. Data from nine of the 32 Project FANC participant's student data were not included in the analysis because they were involved in the pilot testing of the student assessment. However there were no significant differences on the teacher variables between the 9 teachers and the 23 teachers who were included in the analysis. While PD can be designed to increase teacher's content knowledge, support from the school, parents, and students is a much larger issue. During Year 2, the FA group will have PD experiences with using TI-Navigator for formative assessment and the NAV group will have a continuation of PD using the Navigator for formative assessment with more emphasis on formative assessment strategies. Extensive analysis of Project FANC will be completed at the end of Year 2.

Endnotes

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2. TI-Navigator™ is a networking system developed by Texas Instruments that wirelessly connects each student's graphing calculator to a classroom computer.

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IT'S NOT EASY BEING GREEN: EMBODIED ARTIFACTS AND THE GUIDED EMERGENCE OF MATHEMATICAL MEANING

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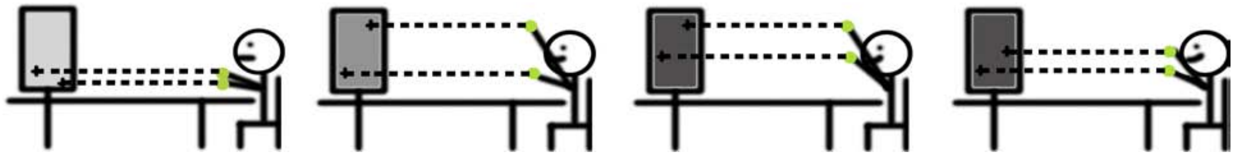
This on-going design-based research study focuses on Grade 4-6 students' guided task-based interaction with a novel computer-based hand-tracking system built to suggest the limitations of naïve additive schemes and create opportunities to develop core notions of proportionality as elaborations on these schemes, even before engaging numerical semiotic forms. Study participants struggled with canonical issues inherent to rational numbers. They formulated a string of insights leading up to a new type of equivalence class. Reported as a case study of Itamar, a 5th-grade middle-achieving student, our analyses reveal emergence of conceptually critical mathematical meanings in an activity that initially bears little mathematical significance.

Introduction

This paper draws on empirical data elicited in the context of an ongoing design-based research project that aims to foster and investigate Grade 4-6 student learning of the fundamental notions of ratio and proportion (Abrahamson & Howison, 2010). Drawing on grounded cognition theory (Barsalou, 2008), we conjectured that the enduring conceptual difficulty students experience with proportional reasoning (e.g., Lamon, 2007) may arise in part because everyday experience rarely affords opportunities to engage these quantitative notions sensomotorically; thus, students lack appropriate embodied experiences from which to construct the key concepts upon simulated dynamic imagery. Accordingly, we sought to engineer for students an embodied-proportion experience conductive to constructing these targeted concepts.

The particular experience designed for the study involved a non-routine activity—the student used two handheld devices, one in each hand, to remotely manipulate the vertical positions of two virtual objects displayed on a large monitor (see Figure 1). Unbeknownst to the student, the apparatus compared the ratio preset by the experimenter on a hidden monitor, e.g., a 1:2 ratio, to the ratio between the measured heights of each of the two handheld devices above the desk, e.g., 20 cm and 30 cm, respectively—a 2:3 ratio. The display's background color provided feedback to the student on how closely the performed ratio matched the unknown ratio, with red indicating "incorrect," yellow indicating "almost correct," and green indicating "correct." Students were tasked to "make the screen green" but were given no direction as to how to accomplish this task.

Students' guided inquiry into this mystery apparatus was designed to give rise to a succession of insights into the mathematical principles that covertly governed the automated feedback pattern they experienced as they manually explored the problem space. So doing, we expected, students would initially bring to bear naive "additive" reasoning and then cope with cognitive conflicts engendered by such contextually inappropriate reasoning. For example, students might expect the within-pair vertical distance to be constant across all "green" location pairs; therefore, once they found their first green location, they would tend to "lock" the vertical distance between their hands and raise this fixed-difference hand pair, only to see the green screen turn red; surprised, the students would then attempt to account for this anomaly by formulating and testing rules and strategies to enable the determination and enactment of further “green” locations. Thus, students were to reinvent core notions of proportionality by attempting to master an unknown computational function embodied in emergent features of an interactive apparatus.



a. b. c. d.

Figure 1. (a) Incorrect performance; (b) almost correct performance; (c) correct performance (1:2 ratio between hand heights); (d) another instance of correct performance (again, a 1:2 ratio)

Our rationale was that the color green, an arbitrary perceptual stimulus, would initially constitute for the student an objective, then serve as the student’s source of performance feedback while attempting to achieve that objective, and would ultimately give rise to an equivalence class—a collection of hand-location pairs which the student perceives as “the same” by virtue of their common effect on the screen. By subsequently introducing discursive prompts, measurement overlays, and representation templates, we hoped to guide students toward articulating mathematically principles governing this initially opaque yet gradually emerging class.

The above design-based research study serves in the current paper as an empirical context for examining the following pair of related questions on the nature of mathematical learning: (1) Can students develop mathematical meanings, signs, and concepts through engaging in a task that initially does not cue mathematical treatment as relevant to the solution of a problem, and if so, how?; (2) From the perspective of pedagogy and instructional design, what personal resources, technological artifacts, and discursive mechanisms could possibly give rise to mathematical meanings under circumstances of ostensibly *a*mathematical problems? These questions appear germane to the PME-NA 32 theme, Optimizing Student Understanding in Mathematics.

We sought to explore our research questions by analyzing videographed footage from a set of individual-student task-based clinical interviews, in which a protocol structured their interactions with the mystery device. As we report below, analyses of these sessions suggest that students constructed the targeted mathematical principles through engaging in the guided activity. In the conclusion we will advance a tentative assertion that, given appropriate materials, activities, and facilitation, students can tackle canonically difficult aspects of mathematical concepts while still engaged in *pre*symbolic quantitative reasoning. Specifically, we submit, students learning the subject matter of proportions can become aware of their own inclination to reason in terms of additive-only rather than multiplicative relations and, moreover, begin to juxtapose and reconcile these two basic types of quantitative reasoning even before inscribing mathematical expressions.

Background

The ongoing investigation reported in this paper draws on two theoretical perspectives of current interest in the mathematics-education research literature—embodied cognition and semiotic mediation—and seeks to explore relations between them. Our attention to the role of embodied action was inspired by grounded cognition theory (Barsalou, 2008), a rising cognitive-sciences paradigm that resonates well with genetic epistemology (Piaget, 1968), phenomenology (e.g., Husserl, 2000), and the construct of tacit knowledge (Polanyi, 1967; see also Varela et al., 1991, on enactive cognition). Grounded cognition rejects earlier notions of the mind as an information processing machine operating separately from the body's sensorimotor systems. Rather, perception, kinesthesia, and cognition are viewed as functionally linked, because reasoning consists of simulating fragments of embodied experiences. Quantitative reasoning, too, is thus necessarily grounded in sensorimotor experience (c.f. Lakoff & Núñez, 2000). It follows, we argue, that mathematical notions may differ with respect to the learning challenges they present to students as contingent on the everyday availability of embodied experiences that form the concepts' simulation substrate, so much so that in the absence of appropriate embodied experiences, students may be disadvantaged in developing grounded understandings of certain mathematical content. Specifically, we view the notion of ratio as "conceptually ambidextrous"; that is, the embodied substrate of a ratio, interpreted as isomorphism of measures (Vergnaud, 1983), could simulate the simultaneous enactment of two changes, such as the dynamic co-production of two hand gestures, a slower and a faster one. We conjecture that proportionality is challenging to students *conceptually* (see in Davis, 2003) in part because everyday experiences rarely if ever afford opportunities to perform and practice such actions *physically* as requisite of developing an embodied artifact that could then be leveraged to mathematize proportion.

Accordingly, the Kinemathics activity has been designed with the explicit goal that students construct what we call an *embodied artifact as an object-to-think-with*—an externally induced coordinated physical action that, similar to the embodied cultural legacy of martial arts, dance, or instrumented musical performance, students learn to perform even before, yet as a condition for, developing disciplinary meanings. Indeed, the semiotic potentials (Mariotti, 2009) of physical discursive action performed in the context of math instruction have been broadly demonstrated in gesture studies (e.g., Goldin–Meadow, Wagner Cook, & Mitchell, 2009). Our interest in such *action-before-concept* learning (ABC) is part of our larger inquiry into cultural precedence for pedagogical practice within explicitly embodied domains, wherein procedures are initially learned on trust yet subsequently—only toward perfecting the procedures toward mastery and further dissemination—are interpreted by experts as encoding emergent disciplinary knowledge.

The materials and protocol designed for this study, and in particular the initial absence of explicit mathematical signs in the core activity, appear to create an amenable empirical arena for studying artifacts and teachers' inter-connected roles in the semiotic mediation of mathematical knowledge (cf. Mariotti, 2009; Radford, 2009; Wertsch, 1979), because the gradual introduction of mathematical tools helps researchers disentangle these roles. That is, by initially refraining from enlisting mathematical signs to explain the task yet subsequently overlaying the situation with a set of graphical–numerical elements familiar to the students, we hoped to create data of students negotiating between naïve and scientific meanings for the variety of spatial magnitudes embedded in the situation. Specifically, we expected students to alternate between two types of mathematical reasoning that draw on different epistemological resources: presymbolic quantitative reasoning evoked by the hands-only activity and mathematical schemas evoked by symbolical mediation of this same activity (Abrahamson, 2009; Bamberger & diSessa, 2003).

We maintain that these two approaches—the embodied and the semiotic—can and should be modeled as pedagogically complementary resources for mathematics instruction aiming for deep understanding, precisely because they are co-instrumental in the learning–teaching dialectic (see diSessa, 2008, on dialectic theory; see Cole, 2009, on obuchenie). Researching student behavior around the Kinemathics task, wherein the embodied and semiotic are differentiable, could constitute a step toward further illuminating this dialectic. Accordingly, our study sought not so much to evaluate the effect of this brief intervention on students' understanding of the targeted content as much as to model the emergence of mathematical meaning in this non-routine design.

Methods

The device described above is a type of *Mathematical Image Trainer* (*MIT*, Abrahamson & Howison, 2010), which tracks the vertical positions of each of the student's hands. Infrared rays emanate from the device, reflect off special tape covering tennis balls held by the student, and are then sensed, interpreted, and visually represented on a display in the form of two crosshair symbols (trackers). The display is calibrated so as to continuously position the crosshairs at the actual physical height of each hand, in an attempt to enhance the embodied experience of virtual remote manipulation. Also, the researcher can control the error tolerance of students' hand positions. Finally, this MIT has a mode in which the trackers are controlled by inputting numbers into a ratio table rather than by remote handling. The color green, students' "correct" feedback, is a pedagogical scaffold for mathematizing proportional relations: this computationally fabricated *product* of measure indicates as equivalent a set of *isomorphism*-of-measure relations (cf. Vergnaud, 1983), thus supporting students' experience of embodied ratios as perceptual gestalts.

This paper draws on 20 interviews (total n = 24) conducted in an ongoing study with Grade 4-6 students from a private K-8 suburban school in the greater San Francisco Bay Area (33% on financial aid; 10% minority students). These participants were selected from a larger pool of student volunteers, all prior to formal instruction in ratio and proportion, in an attempt to achieve equal representation in terms of grade, gender, and teacher-ranked mathematical capability. Each student participated in a semi-structured interview (duration: mean 69 min.; SD 20.35 min.). Elsewhere, we report on results from pre/post-interview card-sorting tasks—one pictorial (a pair of hot-air balloons), the other numerical. In each of these tasks, students could assemble and narrate card sequences depicting either a "fixed-difference" or a "different difference" story.

The heart of the interview consisted of working with the MIT (see Figure 2a). At first, the condition for green was set as a 1:2 ratio, and no feedback other than the background color was given (see Figure 2b). Next, a grid was overlaid on the display monitor to help students plan, execute, and interpret their manipulations and, so doing, begin to articulate quantitative verbal assertions (see Figure 2c). In time, the numerical labels "1, 2, 3,…" were overlaid on the grid's vertical y-axis to help students construct further meanings by more readily recruiting arithmetic knowledge and skills and better distributing the problem-solving task (see Fig. 2d, partial view). Next, we asked students whether it was possible to keep the screen *continuously* green while moving the hands up and down. If students could not perform this task, the interviewer would do so by literally manipulating their hands up and down, enabling them gradually to assume agency.

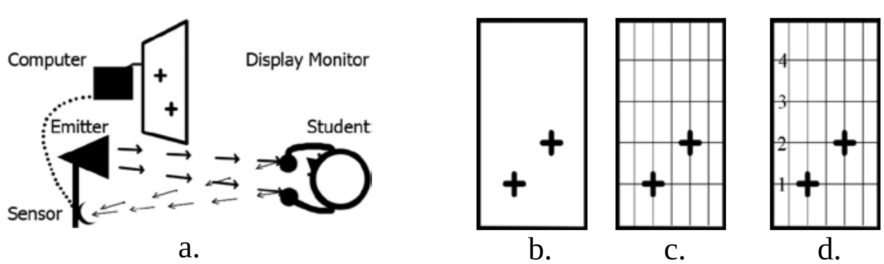


Figure 2. MIT: (a) system; the display in (b) free mode; (c) with grid overlay; (d) also numerals

After working with the 1:2 ratio, students were told that the "rule for green" would change, and ratios of 1:3 and then 2:3 were used. Finally, the technology was reset from manual to numerical control. The interviewer introduced students to the numerical mode by inputting and "running" a set of green ordered pairs the student had just discovered in the previous activity. The interviewer then cleared the ratio table, typed a new pair of numbers in the top row, and asked the student to fill into the table numbers that would generate a green screen throughout the run. While working in the numerical mode, students were allowed to revisit the manual mode.

Our research team has been analyzing the videographed sessions collaboratively by applying microgenetic analysis (Schoenfeld, Smith, & Arcavi, 1991). Whereas variability in students' mathematical competence was manifest in their performance, we discerned family resemblance among their trajectories. For this brief report we collapsed the variability and focused on the resemblance. General results will lead to a case study of Itamar, a G5 middle-level male student. An edited video of Itamar's interview accompanies the results (http://www.tinyurl.com/edrl-mit).

Results and Analyses

Recall that students were initially instructed to "make the screen green." As expected, after stumbling upon their first "green location," students attempted to create another by moving both hands while maintaining a fixed distance between them. They were surprised when the screen consequently turned red and moved their hands about until they found green again, whether in the same or a new location (see Figure 3; beams and sensors are three feet left of the monitor).

a. b. c.

Figure 3. Searching for green: (a) Itamar finds his first "green pair"; (b) his fixed-distance upward motion turns the screen red; (c) he lowers his left hand and finds a new green location

As students began to find more green pairs, they were inclined to determine principles underlying this emerging equivalence set of hand positions that each caused the screen to be green. Students tended to assert that as the hands rise, the greater the distance between the hands must be so as to sustain green. Students' gestures and utterances suggested they were surprised by this equivalence set, in which magnitudes are not fixed but, rather, grow across instances (different differences). Thus students shifted from viewing the within-pair relation as constant to viewing it as covariant with (vertical) location. Yet, prior to the Cartesian reticulation, they still conceptualized *individual* hand locations irrespective of their height above the desk.

The introduction of the grid appears to have imposed a pedagogically effective tradeoff en route to reinventing proportion, because students adopted a "snap-to-grid" discretization of the continuous Cartesian space. Namely, whereas students now confined their manual search for new green locations to integer-unit locations (upon the gridlines), so doing they discovered that as the left hand rose 1 grid unit up from a green location, the right hand must rise 2 grid units in order to re-green the screen, what we call the “per change” solution strategy. Thus the grid afforded or enhanced opportunities to engage in the mathematical activities of measurement, comparison, and counting, cueing students to objectify their hand locations and hand motions *numerically*.

Introducing numerals along the grid’s vertical axis further supported the emergence of multiplicative meanings for the activity. Initially, the numbers afforded succinct naming for the hand positions as a practical means of replicating these positions with precision and speed. Yet so doing, students became conscious of the particular number pairs and responded to them. Our selection of 1:2 as the protocol’s first unknown ratio proved supportive—it enabled students to leverage their familiarity with simple multiplicative relations (e.g., double, half). Such naming of multiplicative relations, in turn, was critical to the emergence of proportion, as we now elaborate.

Prior to introducing the grid, students had focused on the distance between the hands. Equipped with the grid, students viewed hand locations individually, then re-focused on relations between different hand-*pair* positions, namely on “per change” recursive productions, but no longer on the relation *within* each pair (distance), even though this relation remain proportionally invariant amid transitions between pairs. Introducing numbers as new tools in the working space appears to have refocused student attention upon within-pair properties of green pairs, contingent on their multiplicative fluency. For example, students were generally able to correctly assert that one hand magnitude should be double the other so as to recreate green. In sum, whereas the pre-grid portion of the activity sequence launched students’ construction of proportion by refuting their additive expectation and supporting a qualitative alternative, the grid and then numerals provided students with semiotic means of objectifying this embodied experience as additive, then multiplicative relations. We thus witness a compelling case where conceptual understanding emerges reflexively, contingent on the availability of notational systems supportive of inquiry.

Nevertheless, the meanings students built for proportion up to this point in the interview were still markedly discrete in nature. For example, asked if he could move his hands up the screen from bottom to top while keeping it continuously green, Itamar said he could not move his hands fast enough to prevent the screen from flashing red as he transitioned between two green pairs, in this case [2 4] and [3 6]. Itamar thus seems to have viewed his hands as transitioning between two discrete green states, rather than moving through a continuous set of green pairs. In other words, Itamar only attended to [2 4] and [3 6] while ignoring all hand location pairs in between.

With Itamar’s permission, Dor held his hands and gently demonstrated the ambidextrous feat of keeping the screen green while moving both hands up simultaneously and then down again.

Dor: So we start at you said 1 and 2. Here we go. [Moves hands up while maintaining green]
Itamar: Then 2 and 4.
Dor: Look, how am I doing it? [Refers to continuous production of green “despite” the grid]
Itamar: Oh, because it’s the same amount of space, and then this one’s slowly going higher.
Dor: Uh huh! This one’s slowly going higher, and that one’s going faster going higher?
Itamar: Oh! It’s like the air balloons! It’s like the hot-air balloons. This one’s going faster than the other one. They start at the same spot, and then one goes faster than the other.

Apparently, dynamic motion of two objects moving up at constant yet different speeds evoked for Itamar imagery of the balloons pictures used in the pre-interview card-sorting task, *despite his not having constructed a proportional story during that pretest*. He had only constructed a fixed-difference sequence, and when asked to compare the balloons’ speeds had insisted that they rose at the same speed, as evidenced in the following, earlier exchange during that pretest:

Dor: Now is one of them going faster than the other one, and if so, which one?
Itamar: Well if one *was* going faster it would be the red one [the higher balloon, on the right], but I don't think one is going faster, I think one just started before the other.

We now return to the interview. Whereas the previous meanings constructed for green involved discrete extensive quantities—either the distance between the hands or the per-change amounts between pairs—this portion of the protocol introduced rate-based intensive-quantity meanings, namely speed (distance/time). Introducing alternative meanings for one and the same situation is not just a matter of accumulating meaning. Rather, with a teacher's guidance, multiple meanings set up opportunities to struggle with and synthesize disparate notions into more complex conceptual structures. Importantly, understanding relationships between amount and rate is a key learning issue in both ratio-based and non-ratio-based contexts (Stroup, 2002).

Having manipulated the MIT manually, students then turned to operating the device numerically, using the virtual ratio table. In this phase, some students regressed from their multiplicative insight to additive fixed-difference reasoning. For instance, after inputting the sequence [2 3], [4 6], [6 9], [8 12] that he had found in the hands portion of the activity, Itamar correctly noted that the left side was counting by 2 and the right side by 3. Yet, prompted to begin from [3 4], he input [5 6], [7 8], [9 10] so as to "add 2 for each one." He had thus reverted to fixed-difference reasoning. We believe, though, that through further reflective interaction with the MIT, students could learn to articulate their nascent hands-on insights in numerical forms.

In sum, setting off from haphazard hand waving yet progressively discovering the MIT's systematicity, students offered the following sequence of observations that address many learning issues of the targeted notion of proportionality, with the more advanced students tackling and connecting more of these ideas: (a) the locations of *both* hands are necessary to achieve green; (b) the critical quality for achieving green is a type of *relation* between the hands' respective locations; (c) these locations should be reinterpreted as magnitudes—the objects' heights above the base line; (d) the difference between the heights of the hands in correct pairs should not be constant—it will change between correct pairs; (e) this difference should increase as the pair's height increases; (f) moving from one correct position to another can be achieved by increasing these heights differentially, for example the left hand should rise 2 ad hoc units for every 3 units the right hand rises—a constant coordinated-change principle that can be used iteratively; (g) the multiplicative relation within each pair, too, is a constant that characterizes that particular problem; and (h) one and the same number pair expresses three aspects of the interaction—the lowest integer-pair location on the grid, the differential additive change pair, and the multiplicative relation within each and every correct pair: for example 2 and 3 units are the lowest correct integer pair of heights, raising the left hand by 2 units for every 3 raised by the right results in another correct location, and 2/3 or 3/2 is the constant within-pair multiplicative relation. Along the way, students also realized that there are infinitely many location pairs, so that in fact one can *simultaneously* raise both hands, thus reinterpreting the solution as two hands moving at different speeds that could be characterized as, for example, 2 and 3 units per beat.

Conclusion

Students' successful modeling of a problem situation even before engaging normative mathematical media, notation systems, and formats, suggests promise in instructional designs affording embodied, presymbolic quantitative reasoning. Such experiences enable students to struggle with qualitative aspects of a mathematical domain and discover its quantitative principles before they are burdened with the supplementary cognitive load of the disciplinarily requisite inscription and calculation procedures. Namely, as students are guided through a sequence of insights into the properties of a mathematical phenomenon, they can perform core conceptual work even prior to symbolic articulation. Through the appropriate introduction of canonical semiotic means, students can then be guided to formulate these insights normatively: as Itamar summed up, referring to the within-pair differences of 1, 2, and 3 in the ordered-pair sequence [2 3] [4 6] [6 9], "the difference doesn't have to stay the same." That is, by virtue of mastering a computational device that engaged yet challenged their additive reasoning, students constructed a new type of equivalence class articulated as a proto-proportional elaboration on robust pre-multiplicative schemes. For just one hour, we propose, these results are promising.

Though this is pioneering work and further research is underway to validate and substantiate our arguments, the empirical data thus far tend to support our conjecture concerning the potential pedagogical role of embodied artifacts as presymbolic objects-to-think-with. Furthermore, our action-before-concept design appears to resonate with the view that learners often engage in social activities prior to, yet as a condition of, interpreting their own behaviors in accordance with adult definitions of the situation (see Wertsch, 1979). Building on these emergent insights, we can now better guide students to optimize their development of personal meanings, invoked by engaging with the artifact, into normative mathematical meanings. Moreover, through studying student learning, we, in turn, are better positioned to optimize our theory of learning.

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THE ROLE OF COMPUTER ALGEBRA SYSTEMS (CAS) AND A TASK ON THE SIMPLIFICATION OF RATIONAL EXPRESSIONS DESIGNED WITH A TECHNICAL-THEORETICAL APPROACH

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In this report we analyze and discuss the role of CAS with two 10th grade students on a task related to simplifying rational algebraic expressions. The theoretical elements adopted in this study are based on the instrumental approach. Results indicate that CAS and a technical-theoretical-oriented task provoked students to theorize on certain aspects of the simplification of rational expressions, thus illustrating the role of CAS in improving specific technical-theoretical components of algebra learning. However, the results also indicate that good task activity and CAS may not be enough for other related technical-theoretical understandings in this domain; the teacher's intervention may also be necessary.

Introduction

In the past few years, many research studies have reported the potential of calculators (e.g., Computer Algebra Systems, CAS) in facilitating symbolic manipulation in algebra learning (e.g., Kieran & Damboise, 2007; Kieran & Drijvers, 2006; Thomas, Monahan, & Pierce, 2004; among others). For instance, Kieran and Damboise (2007) point out how students who are weak in algebra can improve both technically and theoretically by means of a CAS experience involving the factoring of algebraic expressions such as $x^2 + px + q$, with p and q whole numbers. Taking into account Kieran and Drijvers (2006), it can be seen how the use of calculators and tasks that promote the interaction between CAS and paper-and-pencil environments leads to increasing the quality of students' algebraic thinking.

One concept that the algebra research community considers central to algebra learning is that of the equivalence of algebraic expressions. Kieran (2004), for example, views this concept to be a critical component of algebraic transformational activity. Not incidentally, it is well known (e.g., Davis, Jockusch & McKnight, 1978; Matz, 1980; among others) that students make frequent, common and persistent errors when they try to simplify rational expressions, appearing to neglect issues of equivalence. Some researchers have studied student thinking in this domain within CAS environments (e.g., Ball, Pierce & Stacey, 2003; Kieran & Drijvers, 2006). However, the majority of the reported research on algebraic syntax errors made by students has been carried out in paper-and-pencil environments (e.g., Booth, 1984; Davis, Jockusch & McKnight, 1978; Matz, 1980; Kirshner & Awtry, 2004; among others).

In the reports of these studies, different explanations are offered regarding the origins of many of these errors, as well as a variety of remedial teaching treatments (e.g., the use of different syntactical approaches) to help students overcome such errors. With respect to the use of technology-supported instructional treatments, among the few studies we note the research of, for example, Sutherland (1991, pp. 40-41) who points out that a computational environment influences students' algebraic conceptualization and helps them to overcome difficulties in the comprehension of algebraic symbolism. Similarly, Tall and Thomas (1991) argue that a computational environment helps students to overcome difficulties and errors due to their interpretation and notion of variable. Another related study on the influence of technology in the

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learning of algebraic concepts is by Guzmán and Martínez (2009), who point out that CAS is useful in the sense that it helps students to identify certain kinds of frequent and persistent algebraic errors. However, little is known about the influence of this kind of technology on students' thinking in relation to the simplification of rational expressions — a topic in which common algebraic syntax errors are made in a very persistent way due to a lack of conceptual comprehension on the part of students regarding the manipulation of this kind of expression.

Thus, the aim of this study is to answer the following research question: How do CAS and a task designed with a technical-theoretical approach influence students' thinking about the simplification of rational expressions?

Theoretical Framework

The instrumental approach to tool use has been recognized as a framework rich in theoretical elements for analyzing the processes of teaching and learning in a CAS context (e.g., Artigue, 2002; Lagrange 2003; Trouche, 2005; among others). The instrumental approach encompasses elements from both cognitive ergonomics (Vérillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). According to Monahan (2005), one can distinguish two directions within the instrumental approach: one in line with the cognitive ergonomics framework, and the other in line with the anthropological theory of didactics. In the former, the focus is the development of mental schemes within the instrumental genesis process. Within this approach, an essential point is the distinction between artifact and instrument (for more details see Drijvers & Trouche, 2008).

In line with the anthropological approach, researchers such as Artigue (2002) and Lagrange (2003, 2005) focus on the techniques that students develop while using technology (such as CAS). This approach is grounded in Chevallard's anthropological theory. Chevallard (1999) points out that mathematical objects emerge in a system of practices (praxeologies) that are characterized by four components: *task* (expressed in terms of verbs), in which the object is embedded; *technique*, used to solve the task; *technology*, the discourse that explains and justifies the technique; and *theory*, the discourse that provides the structural basis for the technology. Artigue (2002, p. 248) and her colleagues have reduced Chevallard's four components to three: *Task*, *Technique,* and *Theory*, where the term *Theory* combines Chevallard's *technology* and *theory* components.

Within this (Task-Technique-Theory) theoretical framework, not only does the term *theory* have a wider interpretation than is usual in the anthropological approach, so too does the term *technique* have a wider meaning than is usual in educational discourse. Here, a *technique* is a manner of carrying out a task; it is a complex assembly of reasoning and routine work and has both pragmatic and epistemic values (Artigue, 2002, p. 48). For Lagrange (2003, p. 271), technique is a mixture of routine work and reflection; it is a way of doing a task and it plays a pragmatic (in the sense of accomplishing the task) and epistemic role. With regard to the epistemic value of technique, Lagrange (2003, p. 271) has argued that: "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency". So, this epistemic value of techniques is crucial in studying students' conceptual reflections within a CAS environment. In our study, this T-T-T framework was taken into account in all aspects, including the designing of the tasks to be used, the conducting of the interviewer interventions, and the analyzing of the data that were collected.

The Study

This report is part of a wider research on common literal symbolic errors and the use of CAS. The analysis and results presented here are based on a pilot study that involved three sets of task activities, each one designed to take into account a different common error in algebraic syntax. Here, we analyze and discuss just a part of one of those activities.

Methodology

The Population

The participants were eight 10^{th} grade students (15 years old) of a Mexican public school. The selection of the students was made by their mathematics teacher, who believed that they were strong algebra students. It is also well known that students of this particular grade level make errors in trying to simplify rational expressions – the kinds of errors that we were interested in investigating. None of the students were accustomed to using CAS calculators; consequently, at the outset of the study, all the students received some basic training from the interviewer on how to use the calculator for basic symbol manipulation (use of the commands FACTOR, EXPAND, and SOLVE).

Task Design

The task proposed in this study concerns the simplification of rational expressions, both with paper and pencil and CAS. In this report, we use the term *activity* to refer to "the set of questions related to the task". For the design, theoretical elements from the instrumental approach were used. In other words, the activity was designed so that technical and theoretical questions were central to the task and, hence, that students would have the opportunity to reflect on both the technical and theoretical aspects in both paper-and-pencil and CAS environments. In the present report, only the following parts of the activity are reported: first, students' paper-and-pencil work (with technical and theoretical questions); second, their subsequent CAS work (technical question); and, finally, theoretical questions related to their work in both environments.

Implementation of the Study

The data collection was carried out by the interview method, led by the researcher. Students worked in pairs; each work session lasted between two and three hours. Each team of two students had a printed activity as well as a TI-Voyage 200 calculator. Every interview was audio and video-recorded to register the students' performance during the sessions.

Analysis and Discussion of the Data

In this report, because of lack of space, we analyze and discuss only one team's work on a subset of the actual task questions that were proposed. This team was chosen (we'll call each member of the team student A and student B) for this report because we consider that their work is typical and represents the role played by both the CAS and the designed task. The analysis, which is qualitative in nature, is based on the team's work sheets, as well as the video-recorded interview. The analysis and discussion of the data is detailed below.

The Role of the Proposed Task

We take the first part of the task as solved by the students A and B. As per the task design, the first section of the activity helped us to know how the students simplify, with paper and pencil, the given rational expressions. Figure 1a illustrates the students' techniques and their explanations related to the simplification of rational expressions. From this, we confirm that, in this environment, students made the expected errors: they eliminated the 'literal components' that were common to both numerator and denominator, without taking into account whether these 'literal components' were, in fact, a factor of both the numerator and the denominator.

Ia) Simplifica, usando papel y lápiz, las siguientes expresiones. Muestra todo tu trabajo. Completa la tabla comenzando con la primera fila.

Expresión	Explica tu procedimiento de simplificación
$\frac{x(3+x)}{x}$ $= \frac{3x+x^2}{x}$ $3+x^2$	Primero multiplicamos lo que esta antes del paréntesis por lo de adentro del mismo y después simplificamos x.
$\frac{4x+4y}{x+y}$ $= 4+4$ $= 8$	Al dividir letras iguales los exponentes se restan y como resultan elevadas las variables a la cero ya no se escriben.
$\frac{3x+4y}{x+y}$ $= 3+4$ $= 7$	//

Figure 1a. Simplification of expressions: Paper and pencil work

Ib) Verifica tus respuestas de Ia), para ello, utiliza la calculadora (usa la tecla *enter*). Escribe los resultados dados por la calculadora en la siguiente tabla.

Introduce en la calculadora	Respuesta dada por la calculadora
$\frac{x(3+x)}{x}$	$x+3$
$\frac{4x+4y}{x+y}$	4
$\frac{3x+4y}{x+y}$	$\frac{3x+4y}{x+y}$

Figure 1b. Simplification of expressions: CAS work

This part of the task allowed students to describe their paper-and-pencil technique for simplifying these expressions. We note that their explanation uses the terminology of *dividing* (see the second example of Figure 1a, where the students wrote, "we divide the same letters"). We also note that, whenever there are parentheses, the students first expand the expressions of the numerator and denominator before cancelling (see the first example of Figure 1a). The fact that the students spontaneously expanded expressions and didn't first observe them in terms of factors was something that hindered their theoretical reflection and seemed to lead them to make the kinds of errors that are reported in the literature.

Next, the students arrived at the part of the activity where they used the CAS calculator (Figure 1b). This allowed them to contrast their paper-and-pencil results with those obtained from the CAS. The differences between the two sets of results led them to wonder about their paper-and-pencil techniques and explanations. They began to question the theoretical underpinnings of their work. Hence, the design of the task (technical and theoretical questions in both environments) clearly led the students to confront their paper-and-pencil results (i.e., their initial techniques and theory) with the ones obtained from the calculator and promoted the search for other techniques in order to explain the CAS results.

Theoretical Reflection Promoted by the CAS in Students

From the analysis of the students' conversation following the surprising CAS results, it was clear that the results given by the CAS (Figure 1b) provoked in students a conceptual change (the epistemic role of the techniques, in this case the CAS technique). In other words, the use of the CAS in the context of the designed task led the students to rethink their techniques and explanations and provoked a theoretical reflection that could explain for them the results given by the CAS. For the expressions that involve just one term in the denominator (as in the first example of Figure 1a), the students could see that their paper-and-pencil technique was not correct, but could also see how to fix it. As the following extract suggests, they were able to make a quick adjustment to this technique so as to eliminate the discrepancy between the results:

[1] Student A: What is it? [*She asked for the result given by the calculator for the first expression of Figure 1b*]
[2] Student B: *x* plus 3 [*the CAS result for the first expression of Figure 1b*]
[3] Student A: And we wrote 3 plus *x* squared [*She refers to the result which they got by paper and pencil for the first expression of Figure 1a*]
[4] Student B: Yes. We must've taken off only one *x* [*Meaning that they had to eliminate another x*]. No matter. What's next?

It seemed a minor matter to student B to make this kind of adjustment to their technique for handling the simplification of rational expressions with a single term in the denominator – an adjustment that called for cancelling each occurrence of the given term in the numerator. But it is noted that no accompanying theoretical justification seemed forthcoming. However, for the second and third examples, the students could not easily come up with a simple adjustment to their paper-and-pencil technique for simplifying rational expressions containing a binomial as the denominator and two terms in the numerator – that is, an adjustment that would allow them to arrive at the same result as that produced by the CAS. The following extract illustrates their bewilderment at the CAS result for the second expression:

[5] Student B: Four [*She refers to the result obtained by the calculator for the second expression of Figure 1b*]
[6] Student A: Uh! [*Expressing surprise*] We are wrong too [*Because it doesn't match up with their paper-and-pencil result for the second expression of Figure 1a*]
[7] Student B: But I don't know why. Here I don't really know why.
[8] Student A: Neither do I.

For this expression, the students had cancelled the *x* in the numerator and the denominator, as well as the *y* in both, which had left them with 4 + 4, which they simplified to 8. Yet, the CAS result had been 4. They were similarly unable to explain why, for the third expression, the CAS did not simplify at all, but merely returned the given rational expression:

[9] Student B: Yes, here [*Referring to the first expression of the Figure 1b*], it makes sense [*the result given by the calculator*] because the *x's* were taken off, it first multiplied and we missed taking off the two *x's*. [*She states the multiplication procedure that she thinks the calculator did, just as they had expanded the numerator of the first expression of the Figure 1b*]. But in here, I'm not quite sure why it's 4, neither the result in here [*Referring to the last two results (Figure 1b) given by the calculator*]. Why it is the same [*referring to the 3rd result*], I don't have any idea.

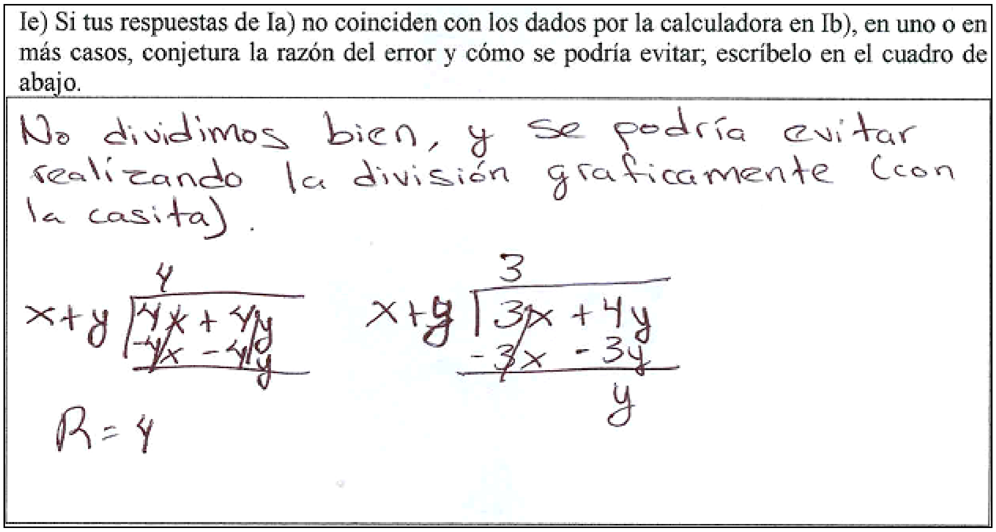
The two students continue to think about the discrepancies. While they could accommodate the result given by the CAS for the first example, the other two examples remained mysterious. They kept asking themselves if there were other ways to think about these simplifications. How might they justify the results offered by the CAS? The following extract underlines their dilemma, but then Student A suddenly had an idea:

[10] Student B: It's believed that in this case we should've taken off the *x* and the *y*, we take off both [*The repeated terms in the numerator and the denominator of the 2nd expression in Figure 1b*]. But why is it 4? [*The result given by CAS*]
[11] Student A: Let's see [*Pause*]. This is a division of polynomials!

In line [11], it is clear that the CAS result, for these two examples, has provoked a conceptual change in one of the students, which was understandable by the other. The theoretical reflection induced by the discrepant results moved the students from a technique involving eliminating literal symbols that are repeated in the numerator and the denominator to a technique involving division of polynomials. Thus, the numerator and denominator of the rational expression are now viewed as dividend and divisor respectively. But this new theoretical perspective, and its related technique of dividing the numerator by the denominator, is deemed necessary (by the students) only for rational expressions where the denominator is a binomial. For those cases where the denominator is a monomial, they continue to believe that the technique of cancelling the monomial of the denominator with all of its occurrences in the numerator is workable:

[12] Student A: In numbers 2 and 3 [*The last two expression of Figure 1a*], we didn't divide well.
[13] Interviewer: What do you mean by you didn't divide well?
[14] Student A: Ok, well, in here… we took off the *x's* and the *y's*. And [*Pause*].
[15] Interviewer: And can't we do that?
[16] Student A: Well [*Pause*] it would do if it were a division of a polynomial by a monomial.

The interviewer then asked the students to illustrate their new paper-and-pencil technique and to show (see Figure 2) how it helped them to avoid the errors they had made when simplifying the last two expressions of Figure 1a.


Ie) Si tus respuestas de Ia) no coinciden con los dados por la calculadora en Ib), en uno o en más casos, conjetura la razón del error y cómo se podría evitar; escríbelo en el cuadro de abajo.

No dividimos bien, y se podría evitar realizando la división graficamente (con la casita).

$x+y \,\overline{)\, 4x+4y}$ quotient 4; $-4x-4y$

$R=4$

$x+y \,\overline{)\, 3x+4y}$ quotient 3; $-3x-3y$; remainder y

Figure 2. The new technique to simplify rational expressions

Finally, based on their new technique (long division of polynomials) for simplifying rational expressions containing a binomial for the denominator, the students were able to explain the results given by the CAS. For them, to simplify such a rational expression came to mean that one should divide (numerator divided by denominator). They found, on their own, that if the quotient works out exactly, then the rational expression can be simplified -- the quotient of the division being the final simplification. But if the division is not exact, then the rational expression can't be simplified and the CAS calculator will give as the result the same expression. It's interesting to see how the students came to adapt their new technique and theory so as to make it also fit the case of rational expressions that could not be simplified. And so, through the technique of polynomial division, the students came explain the results given by the CAS. However, the connections between this technique and the one they used to simplify rational expressions where the denominator was a monomial were never made. For these connections, teacher intervention would clearly be necessary.

Conclusions

The present report shows that CAS and a task that promotes technical-theoretical thinking, within an environment involving the reconciling of paper-and-pencil and CAS results, can provoke in students a conceptual change regarding the simplification of rational expressions whose denominators are binomials. The change we observed involved moving from, on the one hand, simplifying expressions through the technique of cancelling out terms that were repetitive in the numerator and in the dominator to, on the other hand, the division of polynomials technique. The use of CAS to verify paper-and-pencil work, and the obtaining of surprising results by the CAS, led students to a theoretical reflection that provoked the use of a new technique and new theoretical explanations for both justifying the CAS results and rethinking their paper-and-pencil procedures of simplification. However, such theoretical reflection was not enough for them to understand the simplification of rational expressions in terms of cancelling out common factors of the numerator and denominator. These results thus suggest that, in spite of good tasks and the use of CAS, in order for students to more fully understand rational expressions and their simplification, including the relation between polynomial division and factored forms within rational expressions, the importance of teacher intervention is inescapable.

Endnotes

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THE USE OF DGS TO SUPPORT STUDENTS' CONCEPT IMAGE FORMATION OF LINEAR TRANSFORMATION

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"Vision, spatial sense and kinesthetic (motion) sense" is one of the six major facilities that significantly contribute to the development of mathematical thinking (Thurston, 1994, p.4). Drawing on embodied cognition theories, Núñez (2006) argues that mathematical ideas and concepts are ultimately embodied in the nature of human bodies, language and cognition. In this paper, I aim to examine the role of dynamic interactive representations of mathematical concepts in developing students' spatial and kinesthetic sense. In particular, I show how the spatial and kinaesthetic sense of thinking that students develop in their interactions with dynamic geometry software models greatly improves their previously operational understanding of linear transformation.

Introduction

Dynamic Geometry Software (DGS) offers the possibility for representing mathematical concepts symbolically and geometrically. The interactive feature of DGS enables students to perform multiple actions and generate a large number of examples effortlessly (Hollebrands, 2007; Mariotti, 2000), thus they can explore the relationships between the two representations. Research in this area has mostly focused on the design of tasks, students' use of technological tools, and the role of multiple representations in the learning of geometry (Hollebrands, Laborde & Straber, 2008). The use of DGS has been found to be effective in a variety of school mathematics subject areas, including geometry and algebra, and, also, the teaching of calculus, both at the high school and undergraduate levels (Hollebrands, 2003; Falcade, Laborde and Mariotti, 2007; Habre and Abboud, 2006). However, despite its appropriateness to other undergraduate subject areas, its potential in courses such as linear algebra has received less attention.

Prior research suggests that the use of DGS in linear algebra context has shown that DGS facilitate students' construction of their own mathematical objects and avoid the obstacle of formalism (Sierpinska, Dreyfus, and Hillel, 1999). This has motivated me to investigate the effect of the use of interactive dynamic representations of linear algebra concepts on students' concept formation. Given that representations are fundamentally dynamic and that time and motion play an important role in mathematical thinking (Lakoff & Núñez, 2000; Núñez, 2006; Thurston, 1994), I have become interested in investigating the effect of dynamic mathematical representations on students' modes of thinking. In this study, I explore the importance of dynamic representations of the concept of linear transformation in the development of "vision, spatial sense and kinesthetic (motion) sense," which Thurston (1994) includes as one of the six major facilities in the development of mathematical thinking (p.4).

Theoretical Background

Based on theories of embodied cognition, Núñez (2006) argues that mathematical ideas and concepts are ultimately embodied in the nature of human bodies, language and cognition. He has also shown that static objects can be unconsciously conceived in dynamic terms through imposing fictive motion; as he illustrates cases of the concepts of limits, curves and continuity. Recent research confirms Núñez's theory and shows evidence of time and motion in mathematicians' thinking about concepts such as eigenvectors (Sinclair and Gol Tabaghi, 2009). Research on students' modes of thinking has shown that visual ways of thinking include kinesthetic and dynamic imageries (Presmeg, 1986). Studies on the interrelationships between visual and analytic thinking also confirm that visual imagery sometimes is dynamic (Zazkis, Dubinsky, and Dautermann, 1996). These studies, as well as the emergence of embodied cognition theories, suggest we pay closer attention to the role of time and motion in mathematical thinking. It has been surmised that perceiving dynamic processes and objects requires the formation of more complex mental images than perceiving static objects (Zazkis, Dubinsky, and Dautermann, 1996). That could be due to extensive exposure to static representations of concepts. I hypothesise that the use of dynamic representations can help in the development of kinesthetic and dynamic imageries and thus, support concept image formation of mathematical ideas. This has motivated me to probe university students' modes of thinking in the presence of dynamic representations of concepts. More precisely, I wish to investigate whether dynamic representations of concepts affect students' ways of thinking.

In order to probe their ways of thinking, it is helpful to study students' linguistic expressions and bodily movements. Recent research has shown that speech and gesture are two facets of the same cognitive linguistic reality. In particular, research claims that gestures provide complementary content to speech content (Kendon, 2000) and that gestures are co-produced with abstract metaphorical thinking (McNeill, 1992). From a mathematics education perspective, gestures play an important role in cognition and can contribute to creating mathematical ideas (Arzarello et al., 2005, 2007; ESM special issue 2009). Therefore, gestures should be an effective way of providing evidence of kinesthetic and dynamic imageries (Núñez, 2006).

Sfard's theory on the dual nature of mathematical concepts (structural and operational) provides additional support in discerning students' ways of thinking. According to her, operational thinking involves conceiving a mathematical entity as a product of a certain process or to identify a mathematical entity with the process itself. In contrast, structural thinking, which usually evolves from operational thinking, involves conceiving a mathematical entity as an object. The two modes of thinking are not mutually exclusive, and are in fact complementary. One challenge in mathematics education is to help students make the transition from operational to structural thinking, a difficult transition when instruction is focused primarily on procedures and algorithms. Sfard notes that non-pictorial mental images (*inner representations)* are more pertinent to operational thinking, whereas pictorial mental images seem to support structural thinking (Sfard, 1991). She uses the terms *inner representation*, *mental image*, and *visual representation* interchangeably in describing ways that students envision mathematical concepts. Similarly, the term *concept image* is used to refer to "all the mental pictures and associated properties and processes" relating to the concept (Tall & Vinner, 1981, p.152). I shall use the terms *concept image* and *mental image* in my analysis.

In the context of linear algebra, studies have reported that linear algebra students mostly develop operational thinking and their level of thinking does not advance into structural level (Hillel and Sierpinska, 1994; Alves Dias and Artigue, 1995; Stewart, 2008). These studies have identified the lack of cognitive flexibility between different modes of thinking as one source of linear algebra students' difficulties. Because of its capacity to represent pictorially and dynamically a range of mathematical concepts, dynamic geometry software might enable students to enhance their concept images.

Research Context and Participants

I asked students to interact with sketches designed in *The Geometer's Sketchpad* (Jackiw, 1991) that involve a variety of linear algebra concepts such as vectors, spans, linear transformations and eigenvectors. I designed interviews using a set of tasks aimed at eliciting students' ways of thinking while they were interacting with sketches. I interviewed five students who were enrolled in a linear algebra course at the time of interviews, which has already covered the concepts mentioned above. Each interview lasted between 30 and 40 minutes. Interviews were videotaped and transcribed.

In this paper, I only present the analysis of two interviews focusing on the concept of linear transformation. More specifically, I report on students' understanding of transformation of a two-dimensional object on a plane. The participants, Julia and Mary, were second year university students who volunteered their time to participate in my study. Both verbal and gestural communication are used to provide evidence of dynamic and kinesthetic imageries.

Analysis of Study

I refer to Sfard's classification to discern students' ways of thinking about the concept of linear transformation. I also analyse the participants' linguistic and non-linguistic expressions to find out evidence of time and motion in their ways of thinking.

Interview Sketch and Tasks

Before seeing the designed dynamic sketch, participants were asked to describe linear transformation. Then, they were presented the linear transformation sketch (see Figure 1) that illustrates a linear transformation of a non-symmetrical object (blue-coloured F) under the matrix of transformation T. Users can drag either the pre-image or the image to any location on the screen. Additionally, the values of the matrix can be changed, which will result in a dynamic update of the image. The participants were asked to predict the image of F under different transformations: (a) $T = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, (b) $T = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, (c) $T = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$ and (d) $T = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.

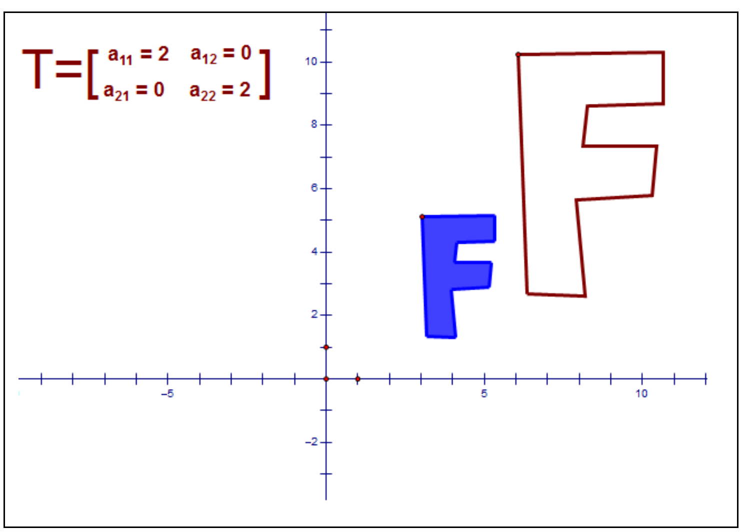

Figure1. The linear transformation sketch

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Analysis of Speech and Gestures: Julia

Julia initially described linear transformation as "changing vectors into something else so like mapping vectors to something else so it could be a matrix or anything else". It seemed that she recalled linear transformation in terms of matrix multiplication. She only mentioned transformation of vectors, which suggests she did not think of transforming other objects. This showed that she had developed an operational understanding of the concept of linear transformation as reported in the literature (see Stewart, 2008).

After her description, she was given the sketch (see Figure 1) that represented the transformation of an object under the transformation matrix $T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Looking at the sketch, I asked her to describe her observations of the image of F in regard to the given transformation. She said "expanded the original F by 2 in y-direction and x-direction, so there would be the four times of original size because it is 2 times 2." It was interesting that she compared the area of image and pre-image. She was then asked to predict the effect on pre-image F of the transformation (a). She predicted that transformation (a) would map the pre-image F into another F that would have three times expansion in its dimensions and nine times enlargement in its area. After her prediction, she was asked to change the values of the matrix into the transformation given in (a) to verify her predication.

Next, she was asked to predict the image of F under transformation (b), whereupon she whispered and used her right index figure (see Figure 2) to write up mental calculation on the desk. After a few seconds, the interviewer prompted her to explain her thought processes.

Figure 2. Julia's use of her right index figure to write up mental calculation

> Interviewer: Can you explain what are you calculating?
> Julia: I am just thinking of that as, um, so that would be the matrix and then say I have my x_1 and x_2 that would be my vectors so then if I have that vector times the matrix. So right now that would be two one zero two if I change a_{12} to one so then it would be, if you multiply them it would be two x_1 and then plus one x_2 [$2x_1 + 1x_2$] and the bottom would be zero x_1 plus two x_2 [$0x_1 + 2x_2$]. Then I do not know how that changes the figure, but in what way.

In order to predict the result of the transformation, Julia must rely on her operational knowledge to complete the matrix-vector multiplication. As she said her self, she did not know "how that changes the figure". Her gesture, using her index finger to write up the calculation, also revealed that her concept image of transformation was pertinent to operational thinking. I next asked her to predict the image of F under the transformation (c). She said “so now it is slanted that way, -1 would be slanted other way”. This showed that she referred to the sketch representation for the transformation (b) and compared the entries of two transformation (b) and (c) to predict the image of F. As she described what would happen, she held her right hand out and moved it from right to left, apparently also describing the slanting of the image of F (see Figure 3). This gesture was markedly different from her previous gesture using right index figure to write up the calculations since it communicated something about the transformation of the pre-image. This suggests that the dynamic representation enabled her to develop a pictorial mental image where she could manipulate the mental image “almost like real object” (Sfard, 1991, p.7).

Figure 3. Julia’s gesture as she describes transformation (c)

Next, she was asked to predict the image of F under transformation (d). She said “so two one zero two was like that before, um, it would be more fat, um, I do not know”. She recalled the entries of the transformation (b) and the sketch representation of it, then she used her right hand; slanted her hand to the right (see Figure 4) to illustrate the direction of the image of F under the transformation (b). Her gesture indicated that she evoked the representation of transformation (b) in her mind. That is evidence of kinesthetic and dynamic imageries that enabled her to think in terms of movement and direction so that she reconstructed the direction of the image of F in space and time using her hand.

Figure 4. Julia’s gesture as she describes the transformation (d)

Her prediction that the image “would be more fat” was incorrect. I asked her to use the sketch to show transformation (d). She was at first surprised when she saw that the image of F is a line segment, however, she immediately said “oh yeah because they are linearly dependent”. She realized the dependency between column vectors of the transformation (d) so that she justified the representation of the line segment on the sketch. This showed that she was able to coordinate the symbolic and geometric representations of the given linear transformation.

Analysis of Speech and Gestures: Mary

Mary initially described linear transformation as “like taking one vector and something, and transforming it using a set of like rules so then becomes something else.” She recalled linear transformation in terms of transformation of a vector into something else. Although the use of term “something else” gives a sense that she was presented with the idea of transformation being able to transform anything, she did not seem to have any example of what could actually be transformed. In response to the prompt, predict the image of F under transformation (a), she mentioned that transformation (a) maps the pre-image F into another F that is “wider, bigger, and three times the size”. After her prediction, she used the sketch to create the transformation (a). This showed that she had a sense of how the matrix actually affected the transformation of the pre-image.

Next, she was asked to predict the image of F under transformation (b). But, she was unable to articulate the shape of the image of F under transformation (b), so she immediately used the sketch to create this transformation. The sketch representation of the image seemed to prompt her memory, as she said “oh, it is a shear transformation”. It is worth mentioning that thus far no gesture accompanied her speech. In response to the next prompt, predict the image of F under transformation (c), she said the image would be “twice as large as now and sheared”. She also moved her right arm and hand upwards and to the right to depict the direction of the image of F (see Figure 5).

Figure 5. Mary’s hand represents the direction of the image of F under transformation (c)

Although, her arm and hand depicted a right-slanted image of F, the image would be left-slanted. Her gesture, movement of hand and arm, was a dynamic gesture that she used to illustrate the direction of the image of F under the transformation (c). This was evidence of kinesthetic and dynamic imageries that enabled her to reconstruct the image of F, to impose motion on it, and to position it in space. This showed that the use of dynamic interactive sketch had an impact on Mary’s ways of thinking. She then tried to create the transformation (c) using the sketch. While changing the values of the transformation matrix, the image of F became a line segment for $T=\begin{bmatrix} 0 & -2 \\ 0 & 3 \end{bmatrix}$. Mary, who was surprised by the sketch representation, tried to justify the representation and said that “oh, okay because a_{11} and a_{21} are zero”. She also gestured as she tried to justify the sketch representation: she moved her right hand that was slightly right-slanted from up to down to depict a line segment (Figure 6). Her gesture showed that she started to use kinesthetic and spatial senses to reason about the representation. I prompted her to explain more about the image, but she offered no further explanation.

Figure 6. Mary's hand represents the image of F under $T=\begin{bmatrix} 0 & -2 \\ 0 & 3 \end{bmatrix}$

Discussion

The two participants' initial descriptions of linear transformation at the beginning of the interviews confirmed research findings that students mostly develop operational understanding of linear algebra concepts—recall that these students had already completed the unit on linear transformations in their linear algebra course. Their concept image of linear transformation involved only operations (especially for Julia) and did not seem to include a pictorial mental image to support structural thinking.

After interacting with the transformation sketch, both participants developed a pictorial mental image of linear transformations, as evidenced by the descriptions they give of what will happen to the image (that it will slant, or that it will be fat, sheared, twice as large). Their descriptions indicate that they became focused on *how* the transformation changes the figure, and not simply on the calculations involved in the matrix multiplication. Their kinaesthetic conceptualisation is evidenced in their gestures, which describe the actual motion of the pre-image to the image. This is interesting in that the sketch does not actually show the continuous transformation of the pre-image to the image so that the participants actually impose this motion on the objects.

My findings suggest that the use of dynamic representations to illustrate the relationships between symbolic and geometric representations could support a pictorial concept image formation of advanced mathematical ideas such as linear transformation. Although the participants have not fully developed a structural understanding of linear transformation, they have developed pictorial mental images that support structural thinking. The structural understanding of a linear transformation involves understanding that transformation preserves linearity, i.e. the operations of vector addition and scalar multiplication. The dynamic and interactive features of DGS enable the designing of models appropriate for representing the linearity of transformation. Based on these results, I am pursuing further research to investigate students' understanding of the linearity of transformation.

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AN INTERACTIVE COMPUTATIONAL SYSTEM IN THE TEACHING OF RATIO AND PROPORTION NOTIONS

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In this paper we outline the design and results of the implementation of a computer system with interactive activities about the ratio and proportion topics used with 6th grade Mexican students (children under 11 years of age). The design of these activities was informed by studies in the computation and educational technology. The goal was to nurture psycho-pedagogical knowledge in mathematics and in the area of computing and educational technology.

Introduction

The call to help students enrolled in basic education programs develop the type mathematical competencies and cognitive structures needed for grasping more sophisticated ideas studies in school is currently one of the Mexico's major goals. Furthermore, attention has also been guided towards using technological environments as vehicles to engage children in such learning. Combined, these two issues, has raised the need for a careful examination of what and how research based teaching strategies might be utilized in designing a system of educational software that would support learning core mathematical concepts such as ratio and proportion. The goal here is to identify the theoretical foundations underlying the choice for inclusion of each of the instructional sections that make up this support tool for learning and teaching the concepts of ratio and proportion. In Mexico, the concepts of proportion and ratio are introduced in elementary grades and provide the basis for more study of more advanced mathematical topics including direct proportional change, linear function, among others.

This article focuses on describing our efforts at supporting the construction of the knowledge about concepts of ratio and proportion among children, for which we developed a computer system with interactive activities. The design of the proposed activities was based on observations and Ruiz's (2002) previous research with teachers and students in the sixth grade (primary education in Mexico) on the topics of ratio and proportion. The goal of the research project reported here was to identify whether the technology based activities designed for teaching of the concepts of ratio and proportion would support the development of proportional thinking among a group of adolescents (11 year olds).

Proportional thinking is based on qualitative surveys creating linguistic categories of comparison, as large or small. In qualitative terms also includes the intuitive, based on experience and empirical and the perceptual that relies on the senses. Quantitative proportional thinking refers to activities that allow students to count, measure and quantities used in the procedures.

Theoretical Background

Piaget (1978) noted that children between the ages of 11 and 12, begin to experience the notion of “proportion” in different areas and contexts, including spatial proportions (similar figures), the relations between weights and lengths arms in the balance of probabilities, and so on. Relying on the results of several experiments, Piaget (1988) indicated that the child acquires the identity qualitative rather than quantitative conservation and makes a distinction between qualitative comparisons and true quantification. Indeed, Piaget's notion of proportion begins in a qualitative manner before it is quantitatively structured. His theory emphasizes that in order for children to develop proportional and qualitative reasoning they must first begin to grasp notions of extension and reduction-- following the idea of the copier or drawing to scale. Furthermore, according to Piaget and Inhelder (1978), upon the development of perceptual receive of order children are then prepared to learn to compare quantities --which can be detected when they use phrases such as, "greater than...," and "less than ....”.

The transitional phase from qualitative to quantitative analysis is what Piaget referenced as intensive quantification. At a later phase, the student uses measurement to make comparisons among quantities--first confronting parts of the object and superimposing a figure in another, and then using measuring instruments, conventional or unconventional, to complete the action. Freudenthal (1983) described such comparisons as maintaining two forms: direct and indirect. The direct method of comparison is when an object overlaps another object, while the indirect method is when given two objects (A and B) and a third element to match (C).

Streefland (1991), regarding the emphasis to be given when teaching about ratio and proportion in early grades, suggested that the instructional approach must start from qualitative levels of recognition of these concepts and make use of educational resources that foster the development of perceptual patterns in support the corresponding process of quantification. It refers to the teaching of mathematics as a core activity, and highlights the importance of teaching tools developed by the designer and this is mentioned Freundenthal Teaching Phenomenology (1983), together with other background information considered realistic for the construction of mathematical knowledge. Similarly, Freudenthal posited that the understanding of the ratio may be guided and deepened by visualization and these can be illustrated using detailed constructions, where the pictures are different and show what items fit together in the original and the image. For example, in two contiguous figures extension or reduction of the other, in which the same linear rate can be established in each segment of the figure. The author further suggested that working the ratio of lengths of plane figures are used as means of representation, expressions of more global comparisons in the sense that the student is provided qualitative and quantitative understanding of magnitudes through visual perception.

In her doctoral thesis, Ruiz (2002) found various difficulties that students experienced as they worked through such structure, as described below.

> *There hasn’t been exploited to maximize the quality of students thinking about proportionality, which concentration was observed when expressed in one of the dimensions of the figures were asked to reduce or amplify.*
> *For some students, the qualitative is scarcely raised as a prelude to the quantitative, since within linguistic categories identified in them are the following: "is greater than ...", is smaller than ...", This reflects a certain understanding of proportion, but these same students did not find other categories by which showed greater understanding of the idea of proportion.*

> *Showed confusion in establishing relationships between quantities, so it was necessary to emphasize that to arrive at the notion of ratio.*

Technology and Learning

Some investigations have been done to determine if the use of learning environments based in web might help student learning. Galbraith and Haines (1998), for instance, showed that students who used the computers in the course of their mathematics learning, enjoyed mathematics. The authors reported that learners enjoyed the flexibility characteristics provided by the computer, they expended too much time using the computer to complete a task and liked to try new ideas. They also concluded that the web applications helped to elevate the level of trust, motivation and interaction among the learners.

Table 1. Purposes, indicators and measures of teaching the concepts of ratio and proportion

Concepts	Purposes	Indicator	Teaching Actions
Ratio: Relationship between two quantities through a quotient	Establish ratios intuitively	Compare	Overlap figures Use verbal categories like "one hand can be twice in one" or one side is the third part of the other " Count sides of squares (a grid)
	Establish explicit ratios	Express ratios as a fraction	Use a table Count sides of squares and write the ratio as a fraction.
Proportion: Ratio of ratios or equivalence of two or more ratios	Thought qualitative proportional	Show Using linguistic expressions	Select figures reduced or amplified. Amplify and reduce figures. Use verbal categories such as "greater than or smaller than"
	Transit proportional qualitative to quantitative thinking	Compare An indirect measure	Overlap figures Count sides of squares in a grid
	Thinking quantitatively proportional	Measure directly. Use the rule of three or excluded middle	Measured with a conventional instrument Use the table Solve numerical

Table 2. The relationship between the activities in Table 1 and actions on the computer

Teaching Actions	Actions in the computer
Overlap figures	drag the mouse
Use verbal categories	Use of Voice
Count sides of squares (in a grid)	using a pencil and in paint
Select figures reduced or amplified	clicking on the option
Amplify or reduce figures	using a pencil as in paint
Measured using a conventional instrument	Use slide rule
Use the chart	table to be completed by the student
Solve Numerical	Calculator

Other scholars (Nguyen & Kulm, 2005; Combs, 2004; Gourash, 2005; Engelbrecht & Harding, 2005_a, 2005_b), pointed out that the use of computer in an educative manner can help individual to seek and attach meaning to what they do. This results from the increased capacity for discovering relationships and concepts which in turn helps them to be more reflexives.

Drawing from the suggestions of research described above, Table I summarizes the referred indicators of understanding associated with the concepts of ratio and proportion as well instances of didactic actions associated to such indicators. Table 2 illustrates the relationship between the activities in Table 1 with the actions performed on the computer.

Methodology

In this section we describe the participants in the study, the elements of the computer system to perform the educational activities, and the educational activities themselves.

Participants

The participants in the study consisted of 29 Mexican students who were attending sixth grade in primary education (children under 11 years of age) at the time of data collection.

Sequence of activities and their presentation in the computer system

In designing and sequencing the activities both the didactic actions identified in the theoretical framework, and the theoretical elements associated with learning and teaching the concepts of ratio and proportion were utilized as described below.

Activity 1 Select the figure reduced, or amplified to that given by viewing

This activity is based on the ideas of "reduction" and "extension" type models supported by the experience of the scale drawing and photocopier, which is handling the situation of similarity, using the perceptual and observation. It assumes, as noted by Piaget, Streefland and Ruiz, that early teaching of ratio and proportion should be built on quality and level of recognition of this concept. Hence, we first used activities that did not require the use of numbers for arriving at solutions. We also took into consideration Freudenthal's suggestion that an understanding of the ratio maybe guided and deepened by visualization and these can be illustrated using detailed constructions, where the pictures are different and show what items fit together in the original and the image. For example, in two adjacent figures, an enlargement or reduction of the other, in which the same linear rate can be established in each segment of the figure.

In designing the figures used in the computer system we were also sensitive to what was meaningful to adolescent boys (11 years of age). We used objects to which they could related, for example: a boat, a bus, a star, a dog, but made with line segments.

Way it appears in the computer system

The environment shows 4 figures that are similar but with minor differences among them; is also one of these 4 figures but at double or triple or half or third of the size, linearly, and prompts the user to choose between the 4 figures, which is confined to the original (See Figure 11).

When any of the figures is selected by the user, we examine the choice made and immediately send a response including the test result to the individual taking the test. We ask if the individual would try again. Depending on whether the response is received and its content, the individual can restart the exercise or continue with the next activity.

Activity 2 Select the figure reduced, or amplified to that given by comparison

Case 1 Overlap a figure in another

The figure in another overlay allows students to recognize relationships of similarity between figures intuitively. The figures compare action is the beginning of the measurement without using a conventional instrument and is given by the superposition of them, as suggested by Freudenthal (1983).

Way it appears in the computer system

The user has the option of using the mouse to drag any of the four figures to overlay on the original and review by looking at it to see if the figure is enlarged or reduced on all sides by the same amount (see figure 1).

Case 2 using the grid

Using the grid gives the transition from qualitative to quantitative thinking and proportional reasoning (Ruiz, 2000). Count is used as the unit of measurement using the side of a grid box where the figures are. The result of counting the sides of the figures was made in order to get relations with the quantities obtained quotient, or ratios. This also allowed for recording of equivalence relations between two ratios or proportions.

Way it appears in the computer system

We show the user a figure on a grid. A grid is also shown where the user can draw a drawing of the figure. We asked the user to draw the figure at double or half, or third scales (see Figure 2). We support the figure that is in the grid and compare it to what the user draws. The user is notified if the answer is correct or not and you are if they wish to repeat the exercise. The users will respond if they wish to continue or repeat the exercise.

Figure 1. Source: Own creation

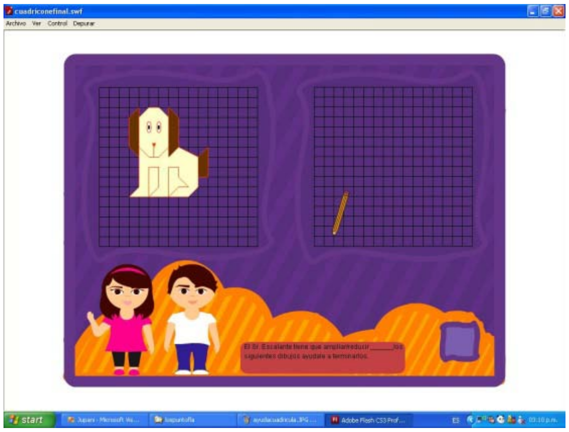


Figure 2. Source: Own creation

Activity 3. Using a Table. The table was used as a mode of representation for the determination of internal and external ratios (Freudenthal, 1983). They worked proportional change problems, where to obtain quantities was not only through the use of the operator, but establishing relationships among ratios. Finally, we worked the equivalence relation as a relation of proportionality. Figure 3 shows an example of how one student used the table and her ratios.

Results

Activity 1, select the figure by visualization

Eleven of 29 students chose the figure reduced by observing. The remaining 18 students chose to compare the figures by placing one over the other. To do this, they dragged the mouse and managed to see whether length or width were twice or half or a third part that I commented at the meeting. This result is consistent with what was proposed by Freudenthal (1983). If the figure was a circle, they compared the radius or diameter. Thus, all students managed to determine whether the figure was reduced by choice and they chose to drag and overlay a figure on the other to make the comparison. This form of student reaction is similar to that reported by Combs (2004).

Activity 2, Grid

Twenty three of the 29 students managed to draw correctly on the grid the similar figures that were requested. The remaining 6 students did about two to three attempts to achieve success on this task. The computer system was very useful because it presented the participants the option of working with different figures. Therefore, the activity did not become mechanical. Additionally, the student had the opportunity to discover what was happening. The recognition of ratios like a comparison, by a quotient of two magnitudes. It worked the notation as a fraction a/b with b different from (Is this what children did? Or what you had intended for the activity to support?)

Activity 3, using the table

Twenty of the 29 students filled the table correctly, they relied on numerical evidence to obtain the unit value if the result was wrong. Other students established ratios by reading from the table and wrote them as fractions. All participants used the calculator function of the environment. They did so to determine the operations to be realized. Figure 3 shows the case of Luis, as an example problem.

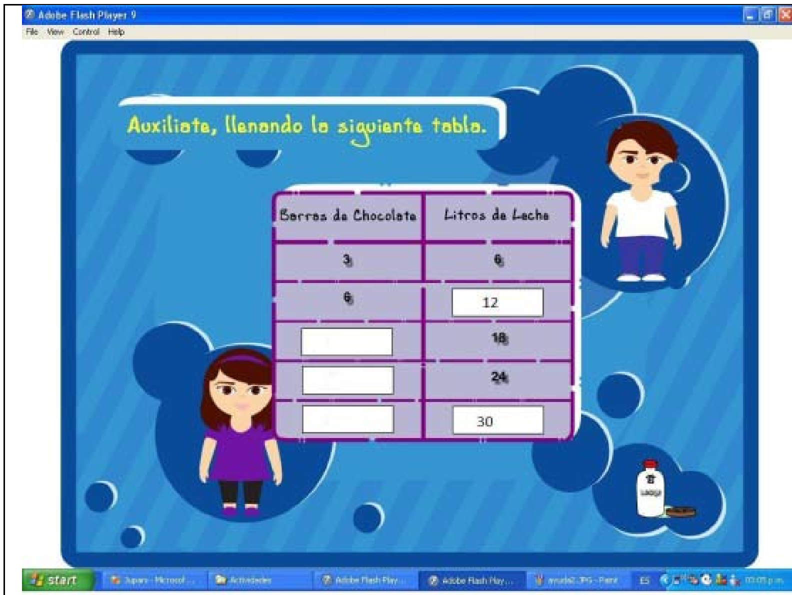
What Luis said in the activity was:
"I noticed that 3 should be 2 times in 6, and then 3 liters of milk will require 6 bars. Hmmm. That is, it takes twice chocolate bars. Ah, a liter then requires two bars, two-liter four 3 liters 6. Is better if I use the calculator and multiplying by two to fill the data they ask me "

Figure 3. Luis answers in the activity

In the case of Luis, it was observed that she established the relationships between two variables, milk and chocolate bars, as external rations, using Freudenthal's terms.
Figure 4 illustrates a second case, representing Manuel's work. Manuel was asked to give a conclusion. He stated:

> *The conclusion is that you ask me .., well, that to the chocolate bars have to use the multiplication table of 6, i.e., first hold 6 bars for 3 liters of milk, then 12 bars to 6 liters of milk, after 18 bars to 9 liters, so if I compare liter bar that is twice liter bar.*

As such, Manuel managed to establish relationships between the data given in two columns. This construct is what Freudenthal called internal rations.



What Manuel said to solve the activity was: See Figure 4 "I checked how changing the amounts in the column of chocolate bars, because they give us more numbers in that column. I saw that started with the 6, after there was a space to fill, then was 18 and soon 24. I used the calculator and divided 18 between 6, I got 3, then divided 24 between 6 and I gave it 4 then 6 multiplied by 2 and got 12, that is the value of the second and the last column is what I got 6 for 5 is 30."

Figure 4. Filling the table by Manuel, first column filled with gallons of milk

The use of the table allowed the students to establish relationships between magnitudes of figures, approaching towards constructing internal and external ratios (as defined by Freudenthal). Lastly, The participants managed to use different registers of representation (drawing, table and numeric) to solve problems of ratio and proportion. These findings suggest that working with interactive activities allowed the students to construct a conceptual understanding of the concepts of ratio and proportion, rather than just using the algorithm to work on these problems. These results are similar to those reported by other scholars (Galbraith & Haines, 1998; Nguyen & Kulm, 2005; Combs, 2004; Engelbrecht & Harding, 2005a,b).

Conclusions

Our findings suggest that the theoretically grounded, and research based activities designed to support children in consolidating the concepts of ratio and proportion were appropriate. The participants showed great interest in working with the computer system and were autonomous about how they resolved tasks. Although the teacher had to pose tasks (give them a figure and ask that they compare it with others to see which was smaller or larger), but the students developed that ability to respond to these questions through the proposed activities in the system. On basis of this finding we are compelled to infere that the participants developed both qualitative and quantitative proportional thinking. They showed a great deal of experimentation by freely dragging the mouse, using the grid, filling in tables, without being asked by the teacher to do so. Lastly, they developed visual and perceptual skills, that is, qualitatively proportional reasoning.

The presence of the table feature of the environment allowed the students to establish relationships between magnitudes of figures. This helped them to realize the internal and external ratios (as defined by Freudenthal). The different registers of representation (drawing, table and numeric) available to students when solving solve problems of ratio and proportion in interactive activities allowed the students to construct the concepts of ratio and proportion meaningfully, rather than using algorithms mechanically. This is consistent with the results reported by other scholars (Galbraith & Haines,1998; Nguyen & Kulm, 2005; Combs, 2004; Engelbrecht & Harding, 2005a,b).

The computer system was able to support the development of students' skills with the board down. However, this skill development did not occur for all participants.

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CONSTRUCTING COLLECTIVE ALGEBRAIC OBJECTS IN A CLASSROOM NETWORK

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This paper presents a novel learning environment designed to support Algebra teaching and learning using a classroom network. We analyze a class session first at the level of the whole class, and then through more detailed examination of simultaneous activity in three student pairs. Classroom activity in this environment was organized around successive task goals that emerged from dynamic interactions among teacher, students and tools. The interplay between these emergent objectives and other features of the learning environment presented students with resources that they were able to organize into emergent solution strategies.

Introduction

A rich body of prior research has explored ways of using classroom networks of handheld calculators or computers to support classroom interactions (Hegedus & Penuel, 2008; Stroup, Ares & Hurford, 2005; White, 2006). Designs for learning activities featuring classroom networks often emphasize the collective construction of a set of mathematical objects—each student in the classroom group uses his or her device to contribute a distinct member of a family of functions, a locus of points, or a class of equivalent expressions to an aggregation on a teacher's computer publicly displayed via LCD projection. In each case, the collective construct illustrates a map between the set and its elements that mirrors the relations between the classroom group and each individual student member, using the social organization and structure of the classroom as a resource for directing student attention to the relations among these objects and guiding classroom discussions about patterns within and generalizations across the array.

This paper reports on an ongoing design-based research project that seeks to build on these previous studies by exploring ways classroom networks might support not only mathematically rich collective activities at the level of the whole class, but also mathematical conversations and interactions among pairs and small groups of students. To this end, our design approach uses network links among student devices to align pairs or small groups of students with mathematical objects that participants in a pair or small group must alternately or jointly manipulate through their networked devices. Such objects are collective to the extent that they or their attributes appear—and change—simultaneously on multiple devices or in a shared display as a consequence of contributions from multiple students. Broadly, we aim to structure tasks around these collective mathematical objects in order to make the successful solving of problems dependent on contributions from and coordination between all participants in a small group. A central aim of our current work is to investigate the potential for these collaborative classroom network designs to reorganize or reshape conventional classroom activity structures—to study designs that blur the boundaries between forms of instruction oriented toward individual students, small groups, or the whole class. Below, we briefly discuss theoretical perspectives involving mathematical objects and introductory algebra, and then describe an activity design intended to support student learning about those concepts. We then present an analysis of the kinds of whole- and small-group classroom mathematical activity supported by this environment.

Collective Algebraic Objects

The meaning of algebraic expressions likely comes from a variety of sources, including the structure of those symbolic expressions themselves as well as their other representational forms, the problem contexts in which they are enacted, and other learner experiences and interactions exterior to the mathematical situation (Kieran 2007). For present purposes, we focus on the first (structural) and last (external) of these sources. Studies from a structural perspective emphasize the complexity of cultivating learners' recognition of such expressions as mathematical objects (Sfard & Linchevski, 1994). From a perspective more attuned to exterior sources of meaning, coming to understand algebraic expressions as mathematical objects is part of a broader social process of objectification, in which learners become aware of objects as stable elements of sociocultural practice through interaction and reflection (Radford, 2006). In the design and study of the learning environment and activities described below, we sought to attend to both these structural and interactional elements of algebraic learning.

The *Terms and Operations* Design

The learning environment described in this paper was created using the NetLogo modeling platform (Wilensky, 1999) and HubNet architecture (Wilensky & Stroup, 1999) in concert with a classroom set of Texas Instruments graphing calculators connected through a TI-Navigator™ network. In the activity design called *Terms and Operations*, student pairs share responsibility for constructing a polynomial expression. Each student can use the directional arrow keys on a calculator connected to a network server to move an icon in a whole-class shared display (Figure 1) populated with a variety of floating monomial terms (of order, sign, scale and number set by the teacher prior to each activity). The right side of this display also includes a field that monitors the current state of the collective polynomial expression for each pair.

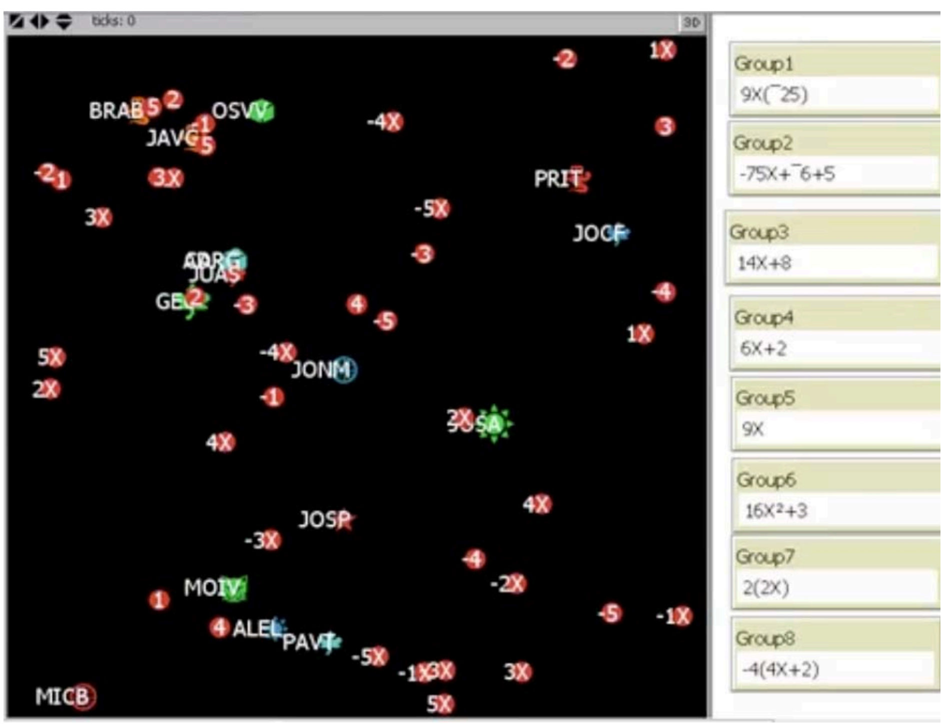
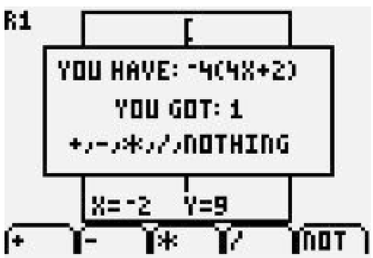


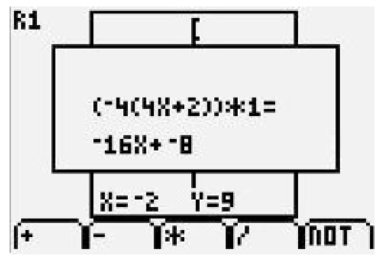
Figure 1. *Terms and Operations* shared display with floating terms and collective expressions

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

In the activities for the present study, the members of each pair took turns capturing and operating on a monomial term to construct a new collective polynomial expression. Each time a student captures a term, she chooses an operation (Figure 2a) and enters the result of combining these with the pair's previous expression (Figure 2b). If this combination is equivalent to the new entry, the collective expression updates accordingly. Typical activities in this environment involve either asking different groups to construct the same expression in different ways, or to construct different expressions that all share particular characteristics. Having student pairs construct these expressions by aggregating terms under different operations is intended to use both pair-level and whole-class interactions as contexts for engaging learners in dynamic construction activities which emphasize both (structural) equivalences among successively more complex objects, and the (operational) consequences of actions on those objects.



a) captured term and operation choices



b) a collective expression under the chosen term and operation and an equivalent student entry

Figures 2a and b. Student calculator screens featuring

Method

The *Terms and Operations* design was implemented in a classroom-based design experiment with two groups of 16 9th grade Algebra I students. Four days of *Terms and Operations* activities with each of these groups were part of a year-long project in which students participated in classroom network activities for a one-hour session each week as a supplement to their regular mathematics program. These sessions were taught by the first author; this arrangement reflects a researcher-teacher (Ball, 2000) approach in the larger design-based research project, and was intended to provide a context for nuanced investigation of forms of teaching practice supported by those designs. On the basis of informed consent, three student pairs in each class were selected as focus groups and videotaped during all activities. All screen states of the public computer display were recorded as a video file for each class session, and an additional camera with a wide zoom setting captured this projected display along with the whiteboard at the front of the room, as well as whole-class discussions and other teacher moves. Server logs recorded all terms and operations selected and expressions entered on student calculators. Below, we present data from the fourth session with one of these classes in order to examine the forms of whole and small-group classroom activity supported by the *Terms and Operations* environment.

Analysis

Table 1 summarizes the sequence of collective expressions constructed by each student pair during the fourth day of *Terms and Operations* activities. These successive expressions reflect an evolving set of goals that emerged over the course of the session, taking the shape of at least four distinct and overlapping tasks posed by the teacher and taken up to varying degrees by each student pair. In the next section, we describe the ways these emergent tasks unfolded through the interplay between students, teacher and tools in this learning environment.

Table 1. Summary of successive collective expressions constructed by each student pair

Group 1	Group 2	Group 3	Group 4	Group 5	Group 6	Group 7	Group 8
2	*5*	*-1*	*0*	*3*	*4*	*2*	*4x*
2(4x)	*5(5x)*	*5x*	*-3*	*3(3x)*	*4x*(4)*	*2(2x)*	*4x+(2)*
8x+1x	*25x+2*	*5x+4*	*6x*	*(3*3x)+5*	*4x*		*-4(4x+2)*
9x(5)	*-3(25x+2)*	*10x+8*	*6x+2*	*9x+9*	*4x*		*-16x+-8*
9x(-25)	*-75x+-6+5*	*14x+8*	*1.5+1/(2x)*	*9x+12*	*16x2*		*-15x+-8*
*(9x*2x)(-25)*	*-1x(-75x+-6+5)*	*14x+8+2+2*	*3+2/(2x)*	*9x+7*	*16x2+3*		*-14x+-8*
18x2(-25)	*75x2+4x*	*2x(14x+12)*	*6x+2*	*9x+2*	*16x2-1*		*-7x-4*
*(9x*2x)(-25)*	*x(75x+4)*	*28x2+24x+5*		*9x*	*(16x2-1)(-5x)*		
		28x2+24x+9		*4(9x)*			
				*4(9x*5)*			
				*4(9x*5/2)*			
				*4(9x*5/2)-3x*			

Whole Class-Level Activity: Emergent Tasks

After beginning the class with a brief review of the distributive property from the previous session, the teacher opened the *Terms & Operations* activity by asking each student group to make a collective expression that featured parentheses. When Group 2 completed this task by writing 5(5x), the teacher pointed out what they had done to the whole class, and then asked Group 2 pick up a new term, and to write their next expression without parentheses. As the other groups finished writing expressions with parentheses they were likewise given the same subsequent task. Continuing to build a succession of expressions with and without parentheses became an ongoing goal for each group throughout the remainder of the session.

Emergent Task 1. As other groups continued to produce initial expressions that included parentheses, they tended to follow the format of Group 2's initial solution by writing a constant term in front of a set of parentheses containing a linear term (as in the second entries in Table 1 for Groups 1, 5 and 7). While several pairs were generating their next expressions without parentheses, the teacher rewrote their respective initial solutions (5(5x), 2(4x), 3(3x), 4x*(x), 2(2x), 4x+(2)) on the board and highlighted the similarity, and then encouraged these groups to build new expressions that included two terms and an operation within the parentheses.

Emergent Task 2. Eleven minutes into the activity, and as other groups were continuing to operate on new terms and write resulting expressions with and without parentheses, Group 6 created an expression with an x^2 term. Taking note of this expression as it appeared in the public display, the teacher pointed it out to the whole class and noted that there were no x^2's among the floating terms. The teacher then prompted Group 6 to write a new expression that again featured parentheses, and meanwhile challenged the other groups to see if they could likewise construct an expression with an x^2 term.

Emergent Task 3. Over the next twelve minutes, groups continued to work on some combination of the parentheses and quadratic term challenges as the teacher alternated between assisting individual groups and leading brief discussions with the whole group. Twenty-three minutes into the activity, the teacher noted the successes by Group 3 ($28x^2+24x+5$) and Group 2 ($75x^2+4x$) in constructing quadratic expressions, and presented a fourth task, prompting them to try factoring these expressions by rewriting with parentheses but no x^2 term. When both groups initially struggled with how to simultaneously factor their current expression and pick up a new term, the teacher told the whole class about a discovery made earlier in the session by Group 8 (reported below) that they could keep their expressions the same by choosing a 1 as their new term and multiplication as their operation. In the final remaining minutes of the session, Group 2 was able to use this approach to rewrite their expression as x(75x+4).

Discussion. The four main tasks (writing increasingly complex expressions alternately with and without parentheses, including two terms and an operator within the parentheses, writing an expressions with a quadratic term, rewriting to eliminate the square) undertaken by students during this session were all initiated by direction of the teacher. However, while the initial alternating parentheses task was the planned emphasis for the activity, both the revision of the task to include an operator within the parentheses, and the subsequent challenges involving quadratic terms and factoring, emerged as responses by the teacher to specific student constructions appearing in the public display. We take these emergent objectives as both a consequence and a characteristic of the kind of classroom activity supported by this designed learning environment—as resulting from the superposition of small-group engagement with dynamic objects and teacher-led whole class discussion about those group-level objects as they appeared in fairly rapid succession on the collective display. In the next section, we examine three student pairs during a portion of this session in order to consider the relations between whole-class and small-group level activity in the *Terms and Operations* environment.

Pair-Level Activity: Collective Objects and Emergent Solutions

Table 2 presents a set of transcripts that span the simultaneous dialogue of three student groups, as well as comments made by the teacher both to the whole class and through tableside conversations with one of the groups, over a three-minute segment of the Terms & Operations session described above. In particular, this segment begins with the teacher's observation that Group 6 had constructed an expression featuring a quadratic term, and follows the varying degrees to which these groups took up this new challenge or continued to work on other tasks. Below, we briefly examine the work of each pair during this segment.

Group 8. The start of this episode found Group 8 actively blurring the lines between small- and whole group activity, picking up on the teacher's efforts to call the class's attention to the public display by first looking around the room to figure out which of their classmates were in Group 6 (lines 1-5), and then finding Group 1's expression on the screen and calling across the room to ask how they made it (lines 8-10). As the teacher set other groups to work generating an x^2 term, Group 8 agreed that they were "lost" and should ask for help (lines 16-17). Having previously constructed -4(4x + 2), they had repeatedly attempted to pick another term and to remove the parentheses without success, each time failing to correctly distribute the -4 to rewrite their current expression before incorporating the new term. On arriving at their table, the teacher looked at Anna's calculator and noted that she had just picked up a 1 and selected multiply, but not yet entered a new expression (lines 21-22). The teacher asked them about the effect of multiplying by 1, and then encouraged them to use this circumstance as an opportunity to simplify their current expression without also having to incorporate a new term and operation (lines 22-35). As the teacher moved away, the students discussed how to multiply both 4x and 2 by -4, and soon correctly entered -16x-8. Thus while other groups were taking up the new task of constructing a squared term, the teacher and these students were using their coincidental choice of an identity term and operation to resolve some persistent confusion about distributing over parentheses. This emergent solution would later mark an opportunity (described above) for the teacher to showcase this pair being successful and to share this lesson about multiplying by one, and become an important resource for other groups as they worked to rewrite increasingly complex expressions.

Group 1. In the moments preceding this segment, Group 1 had been struggling to pick up and operate on a new term that would allow them to rewrite 9x(-25) with two terms and an operation inside parentheses. Just before the teacher interrupted to show the class Group 6's expression, they had chosen to add a 3. As they returned to pair work, Melissa suggested adding this 3 to the -25 would generate 9x(-22) (lines 12-14). Just as the calculator showed that this new expression was incorrect (lines 16-18), the teacher encouraged all groups to try to construct an expression with a squared term. Melissa appears to have taken this new direction as an opportunity to sidestep their difficulties in inserting another term to their current set of parentheses using addition or subtraction; instead, in that moment she sought out a 2x, selected multiplication, and wrote a new set of parentheses in which she combined the new term and operation with their current 9x (lines 19-23). In this case, then, a newly emergent task (multiplying linear terms to construct a quadratic) also became an emergent solution to the previous task.

Group 2. As the teacher called the class's attention to Group 6's quadratic expression in the public display, the students in Group 2 discussed how Group 6 managed to "get a squared" (lines 1-6). As the teacher directed Group 6 to rewrite their quadratic expression using parentheses (line 7), the students in Group 2 wondered what they should be doing next (lines 8-14). Overhearing this comment, the teacher invited Group 2 and others looking for a new task to make an expression featuring x^2 (line 15). Group 2 immediately took up this challenge, seeking out a -1x to pick up and multiply times their current expression (-75x+-6+5) in order to both "get a squared" and "change the negative" (lines 16-26). In doing so, however, Ben felt they should also continue with the alternating parentheses task (lines 27-8). Similarly, in arguing that they should keep both uncombined constant terms inside the parentheses along with the linear term (lines 34-37), Jorge appears to have been continuing to attend to the emergent task constraint of writing multiple terms and an operator inside the parentheses. They merged these various tasks over the course of writing their next two collective expressions, first building -1x(-75x+-6+5) in lines 27-39, and then distributing and adding 3x to form $75x^2+4x$ in the minutes following this segment. Thus they were able to construct successively more complex objects and to integrate new challenges by successfully layering these with a progression of previous task constraints.

Discussion. These simultaneous episodes found each group negotiating an important set of challenges: because the environment required them to pick up and operate on a new term even as they undertook newly emerging tasks related to the construction of equivalent or more complex expressions, pairs had to simultaneously deal with the previous expression as object to be rewritten in different but equivalent form, and the chosen operation as process of combining old and new terms. In the first case, pair 8's choice of an identity operation enabled them to focus on collaboratively working through and resolving some confusion about the process of rewriting an equivalent expression without worrying about a new transformation. Similarly, in abandoning their unsuccessful attempts to add a constant term and instead multiplying by a linear, Group 1 turned the occasion of the second emergent task into a means of accomplishing the previous objectives. And Group 2 carefully selected a new term and discussed steps that would allow them to simultaneously accomplish several construction tasks. In each case, features of the learning environment—the properties of an arbitrarily chosen term and operation, system feedback via the calculator about the equivalence of expressions, the public display of another group's construction—provided resources that pairs were able to incorporate into emergent solution strategies.

Table 2. Overlapping teacher comments and dialogue of 3 student groups during a three-minute segment of the class session.

Line	Teacher	Group 8	Group 1	Group 2
1	T: Whoa. Group 6, what happened	D: [turns around, looks at groups seated behind him] Who's Group 6?		J: How'd they get a squared?
2	there? Nice.	A: [Turns and points to Group 6 in the back of the room]		B: Cause they got two x's. Cause they probably
3	T: Group 6, everybody see what	D: Yeah.		picked up, like, x and... 4x and 4x again.
4	group 6 has made up here?			J: Oh, two x's.
5	T: You guys have got an x^2 term,	A: Un-hnn.		J and B: No.
6	are there any x^2's out there?			
7	T: No, so you must have done something clever to make that. So, group 6, now I want to see if you can rewrite this with parentheses…			
8		D: [Calls to G in Group 1, points up at the screen and asks how his group got their current expression,		B: What do we do?
9		9x(-25).]		J: Yeah, what do we do?
10		G: [holds his hands up and shrugs in response to John's question.]		
11		A: Where am I, where am I, where am I? Go straight down.		
12	T: That's a good challenge actually.	D: Where are you going? Oh, I see you.	M: I guess you can try, um.... Try	B: What are you doing?
13		A: [picks up a 1] Hmmm....	9x. And then, in the parentheses,	J: I don't know. Just moving around.
14			put negative 22.	
15	T: The rest of you who are looking for some ways to challenge yourselves, see if you can get an x^2 into your expression.			
16		A: I'm hecka lost now. Hecka hecka lost.	G: [enters expression, realizes it is	B: Yeah.
17		D: [laughs] I know. Let's ask him for help.	incorrect when the collective	J: There's probably a, negative 1x or something.
18	T: Group 8, can you guys get n x^2 into your expression?		display does not update] Nope.	B: Yeah.
19	D: No, we already tried it like three times to get off the parentheses, but we can't.		M: No? [Goes to work on picking	J: Or 1x. I see negative 1x. I want to change the
20	T: Oh, you can't make them go away. Ok.		up a 2x and multiplying times	negative. I don't like negatives.
21	D: No. Somehow, well, she just got a number.		current expression. Changes	B: [as J picks up a -1x] Alright, try to multiply it.
22	T: Let's see what you've got. [looks at A's calculator] Ok, so, here's a suggestion. What do you think that		collective expression to (9x*2x)(-	J: Yeah, multiply. So you get two 75's… So then
23	that [(-4(4x+2))*1] is equal to?		25)]	everything's the same?
24	[D looks and A, neither answers]		G: Now Manuel's following you.	B: Wait, did you multiply? 1x by negative one?
25	T: Well, this is really interesting actually, because you picked up a one, right, and you multiplied times		M: There we go. I finally changed	J: Yeah.
26	one? So, what happens when you multiply, do you know what happens when you multiply something by		it.	B: So it'd be, no, you want to put it back in
27	one?		G: [looks closely at screen and	parentheses. So it would be…
28	D: It stays the same.		laughs] Oh.	J: Oh. [typing] x…parentheses…-75…
29	T: It stays the same. [A nods] So, your expression won't change, right?		M: [laughs] I know, it looks hard.	B: x, plus…er, plus negative six…and then
30	D: Sure.			minus five outside the parentheses.
31	T: What it's equal to won't change, but you can write it as something else that's equivalent. So how could			J: It's plus five.
32	you rewrite that expression without parentheses?			B: Plus five outside the parentheses.
33	A: Ohhhh, ok, I think I get it. So negative 4 multiplied…			J: Why outside?
34	D: It could be negative four multiplied by 4x+2, right?			B: You put 'em inside?
35	T: Sounds like it's worth trying.	D: Negative four?		J: I thought you would, cause like,
36	T: Whoa, Group 1. Can you rewrite	A: Negative, negative four, so it would be negative 16?	M: (smiles, nods)	B: Oh yeah, put it inside.
37	this?	D: No.	G: I'm trying.	J: [Enters new expression of -1x(-75x+-6+5)]
38		A: Yes, because you're multiplying them together… Is it? I'm sorry!		Dude, I have too many numbers.
39	T: So Group 1 has, check this out.	D: No, it's ok, yeah, it's right, you're right. It's right.		
40	Cause this might, If you didn't know	A: Plus? D: Yeah.		
41	this already, Group 6, how did you	A: Negative 8. [Collective expression updates to -16x-8.] Oh! Oh my	G: Can I?	
42	make x^2?	god, I got another one. [She and John high-five].	M: Hold on, go back.	

Conclusion

We summarize two main themes from the analysis presented in this paper. Firstly, classroom activity in this learning environment was organized around successive and overlapping task goals that emerged from dynamic interactions among teacher, students and tools. In addition to those objectives initially set by the teacher, other tasks and constraints arose as both teacher and students attended to new expressions constructed by groups and publicly displayed via the network. Secondly, the interplay between these emergent objectives and other features of the learning environment presented students with a dynamic set of challenges reflective of the complexity of this classroom activity, and also with an emergent set of resources that they were able to organize into solutions to those challenges. We take the above episode to illustrate a novel form of classroom mathematics activity supported by this learning environment, one which blurs the boundaries between conventional instructional modes such as student-centered small group work and teacher-led whole class discussion. This hybrid activity structure is mediated by collective objects belonging to each group but publicly displayed for the whole class, thus providing emergent resources for the teacher to orchestrate, and for all students to actively and successfully participate in, simultaneous mathematical activity across multiple groups.

Endnotes

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DESIGN AS AN OBJECT-TO-THINK-WITH: SEMIOTIC POTENTIAL EMERGES THROUGH COLLABORATIVE REFLECTIVE CONVERSATION WITH MATERIAL

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We chart a historical analysis of a collaborative design-based research project investigating the emergence of mathematical meaning from embodied interaction with a technological tracking-system supporting the learning of proportionality. Recounting iterative cycles of a conceptually critical perceptual feedback element, we articulate three interconnected images of research-based designers: (a) Janus the two-headed keeper of passageways who sees artifacts alternately as a student would or as an expert would; (b) an investigator searching to explicate design decisions coherently in light of learning-sciences theory; and (c) a reflective practitioner who embraces tradeoffs and is open to constructive criticism and to implementing radical changes to design and theory. Ultimately, we posit, we as researchers are continuously developing professional vision for our own design even as the design changes.

Introduction

Design-based research (DBR), a fast evolving major approach to the investigation of human learning, is still nascent. Accordingly, whereas the approach has led to important ontological innovation and “humble theory” (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; diSessa & Cobb, 2004), it still receives formative criticism as methodologically volatile (Kelly, 2004), under-theorized (Abrahamson, 2009), and rhetorically inchoate (Puntambekar & Sandoval, 2009). In this paper, we—a DBR team investigating mathematics learning by developing and implementing innovative instructional technology—reflect on milestones in an evolving project in an attempt to respond productively to some of these constructive observations.

We view our team-based design processes through the same theoretical models of learning that have been applied to students and teachers engaged in collaborative problem solving, particularly in open-ended construction-based learning (e.g., Blikstein, 2008). Several philosophical, cognitive, and practical dispositions converge in our analysis: (1) an epistemological perspective on knowledge as emerging from learners’ explorative, highly divergent, yet possibly guided goal-based interaction with materials bearing semiotic potential (Bartolini Bussi & Mariotti, 2008; Lakatos, 1976); (2) a pedagogical commitment to constructionist learning in which bricolage antecedes hypothesis (Papert, 1996); (3) a methodological technique of collaborative qualitative microgenetic analysis (Schoenfeld, Smith, & Arcavi, 1991); and (4) a coherent view of designers as reflective learners (Schön, 1992). Engaging this reflexive analysis, we bear in mind the methodological limitations of introspection, moreover *group* introspection, and the exacerbating factor of professional capital at stake and the consequent vulnerability of practitioners revealing the meandering, backtracking, abductive processes underlying their products. Nonetheless, we are committed to a view of insight as emerging from previous dedication, creativity, hard work, and collaboration (Sawyer, 2007).

Thus, to the extent that we can contribute to design theory by studying design, we wish to shift the investigation focus from design *product* to design *process*.

In sum, designers are learners just as much as their product consumers, the students, are. Whereas students engage in problems supporting content learning, designers engage in projects where interesting problems relating to epistemology, cognition, pedagogy, and design emerge. For designers, these problems are the semiotic potential of the evolving designs. And yet this potential can be availed of only in a collaborative culture of discourse. Just as teachers identify "teachable moments" bearing potential learning gains for students, so designers ought to identify "researchable moments" within their own design process that appear to bear potential learning gains for other designers and theoreticians—ontological innovations for the practice of design-based research. Similarly, just as classroom discussion is critical for students to avail of the semiotic potential of pedagogical artifacts, we submit, so design-team discussion is critical for researchers to avail of the theoretical potential of design decisions as revealing issues of cognition and instruction as well as design principles (cf. Edelson, 2002).

This paper is built as a reflective recounting of our collaborative design process in building instructional materials for the Kinemathics project, which centers on tapping Grade 4-6 students' physical action and embodied reasoning as a means of scaffolding their mathematization of proportionality (Abrahamson & Howison, 2010). The process involved numerous iterations of innovative technology. Whereas the iterations implemented superficially similar interactive affordances, reflecting on implications of nuanced differences among the iterations drove us to reconsider our foundational conjecture pertaining to the function of embodied artifacts in mathematical learning.

Background and Theoretical Framework

Beyond Historical Revisionism: Toward Dynamic Coherence of Project, Process, and Product

Reflecting on our mathematics background, the authors note one characteristic shared by mathematical proofs and design reports: the disproportionate emphasis on the product, which is often positioned as if it appeared *ex nihilo*. Initial attempts at problem solving, however relevant to consequent refinement or reorientation of the study, are oftentimes relegated to "just pilot talk" or, in the worst-case scenario, ignored wholesale or even ignominiously cashiered as unprofessional. In short, *historical revisionism*—reading onto the beginning of a process that which emerged only at its end—serves, by and large, as the normative modus operandi.

Yet it is precisely those early forays and bold conjectures that unshackle innovative design of extant unsatisfactory precedents, mark the unique character of the project, and lend functional coherence to the evolving group ethos and orientation—a coherence that enables the researchers to marshal and contain numerous local product oscillations typical in iterated development cycles of design-based research. If this historical process remains opaque, we submit, then the grounding of new theory may also remain opaque. We further submit that rendering the design process more transparent, and thus allowing a broad range of DBR practitioners operating in diverse domains to learn from one another's design decisions, is a crucial step towards developing DBR as a rigorous pan-project methodology.

Design 'Seeing' as an Emergent Process: Conversations With Material and Semiotic Potential

In his analysis of design, Donald Schön posits that the activity of design is necessarily *knowing-in-action* (1992)—that is, design acumen is implicit or procedural—and that, therefore, designers may gain best access to their professional knowledge when put in the actual mode of designing. Furthermore, due to limited information-processing capacity, designers cannot, in advance of implementing a particular “move,” consider all the consequences and qualities that may eventually be considered relevant to its evaluation. The immediate corollary is that some design decisions emerge organically through the designer’s conversation with the material of a design situation; that is, the designer analyzes the design material, plans and executes the design move, then reflects on the (oftentimes unintended) consequences of this move—a process Schön refers to as *seeing-moving-seeing*. The term “seeing” can be interpreted as sense-making: “in all this ‘seeing,’ the designer not only visually registers information but also constructs its meaning—identifies patterns and gives them meaning beyond themselves” (p. 133).

The clarity of Schön’s construct of seeing-moving-seeing notwithstanding, we found the work of Maria Alessandra Mariotti and collaborators (Bartolini Bussi & Mariotti, 2008) particularly useful in helping us better understand what a researcher *sees* when engaged in reflective design (cf. Abrahamson, 2009). Mariotti approaches issues of knowledge construction by examining epistemological and cognitive virtues of goal-oriented guided interaction with pedagogical artifacts—the artifact’s *semiotic potential*. She distinguishes between *personal meanings*—constructed meanings arising in the individual from using the artifact as a means of accomplishing the prescribed task, and *mathematical meanings*—meanings that an expert recognizes as mathematical when observing the student’s use of the artifact. Thus, cultural pedagogical artifacts, such as a compass, an abacus, or a geometry module, may offer valuable semiotic potential with respect to particular educational goals: by taking advantage of its semiotic affordance, the teacher utilizes the artifact to occasion opportunities for students to develop personal meaning into mathematical meanings that constitute the didactical goals.

We believe that Mariotti et al.’s theoretical constructs are applicable in arenas beyond teacher–artifact interaction. That is, we posit that one thing designers strive to *see* through collaborative conversation with an evolving design artifact is precisely the artifact’s semiotic potential—both project-specific and practice-general—in terms of issues akin to foreseeable learning trajectories based on implementing this potential. We thus propose to imagine the design artifact as a gateway between personal and mathematical meanings. Like Janus—the two-headed Roman deity of gateways, passages, beginnings and endings (the mythological as well as etymological epistemic *jani*tor)—designers approach their work with one face turned towards personal meanings and the other turned towards mathematical meanings, constantly striving to reconcile the two; designers construct the gateway–artifact and constantly modify and prune its corridors so as to optimize its wanderers’ learning trajectories.

In the following sections, having introduced our study’s orientation, objectives, and thesis, we present a design narrative consisting of vignettes of researchable moments that we view as indicative of the design process as a conversation wherein the emerging artifacts’ semiotic-cum-theoretical potentials are elicited; we then interpret the narrative through our theoretical lenses.

Research Context: An Embodied-Design Study of Presymbolic Proportional Reasoning

The study we report on is a subpart of *Action-Before-Concept* (*ABC*), a cluster of cross-disciplinary studies centered around questions respecting relations between procedural and conceptual knowledge, ultimately focused on the procedural–conceptual relation in mathematics instruction. ABC, writ large, is an inquiry into cultural precedence for pedagogical practice within explicitly embodied domains, wherein procedures are initially learned on trust yet subsequently—only toward perfecting the procedures toward mastery and further dissemination—are interpreted by experts as embodying disciplinary knowledge.

The particular study we report on is *Kinemathics: Kinetically Induced Mathematics* (Abrahamson & Howison, 2010), an investigation into how students operating interactive mathematical artifacts develop the emerging tacit, embodied, sensori-motor coupling with the device into mathematical meanings. Specifically, Kinemathics puts forth and pursues the bold conjecture that some mathematical concepts are difficult because ordinary life does not offer *physical* opportunities to develop multi-modal dynamical images by which to simulate these concepts *mentally*. Our approach is to devise tools and activities geared to induce in students physical experiences that are initially of little if any mathematical context yet gradually are seen as mathematical by having students appropriate strategically available disciplinary instruments as epistemic–discursive means of enhancing their inquiry and communication (see Abrahamson, 2009, for an outline and demonstration of this DBR meta-rationale).

Ultimately, this paper will focus on the evolution of a particular feedback element. At this point, however, we will build context for the subsequent discussion by overviewing the design's history through the perspective of its key artifact, the *Mathematics Imagery Trainer* (MIT).

The original MIT, MIT1, was implemented as a mechanical device using pulley wheels with 4'' and 6'' diameters to effect a 2:3 ratio in movement of the attached ropes. The student holds handles attached at the ends of the ropes, and the researcher cranks a lever to raise and lower the ropes at a steady rate (see Figure 1a). MIT1 was limited, particularly insofar that its somewhat cumbersome mechanism allowed for the student to experience only a single ratio, a structural constraint liable to limit the ultimate semiotic potential of generalizing the phenomenon of proportion (Janus frowned). These perceived limitations impelled us to redesign MIT2 in an attempt to increase the design's mathematical flexibility, performance precision, electronic module interactivity, and dissemination potential.

a. b. c.

Figure 1. (a) MIT1: researcher cranking the device, student holding on; (b) MIT2: student's incorrect performance turns the screen red; (c) correct performance turns the screen green

In our second iteration, MIT2, we leveraged the high-resolution infrared camera available in the inexpensive Nintendo Wii remote to perform motion tracking of students' hands, similar to work by Johnny Lee (http://johnnylee.net/projects/wii/). The Wii remote is a standard Bluetooth device, with several open-source libraries available to access it through Java or C#. An array of 84 infrared (940nm) LEDs aligned with the camera provides out light (source), and 3M 3000X high-gain reflective tape attached to a tennis ball allows effective motion capture at distances as far as 12 feet. In use, infrared rays emanate from the MIT2, reflect off tape covering tennis balls held by the student, and are then sensed, interpreted, and visually represented on a large display in the form of two crosshair symbols (trackers). The display is calibrated so as to continuously position the crosshairs at the actual physical height of each hand in an attempt to enhance the embodied experience of virtual remote manipulation. Advanced MIT2 prototypes provide visual feedback of the student's performance on a green–red gradient, a design feature we discuss in the following section (see Figures 1b and 1c; source and sensors are three feet left of the monitor).

MIT2 is our current technological vanguard, being successfully utilized in a study with Grade 4-6 participants. Analyses of twenty-one videotaped task-based interview sessions, in which individual or paired students worked with MIT2, suggests that participants struggled productively with canonical issues inherent to rational numbers, as evidenced by a succession of insights grounded in the embodied nature of the artifact. Namely, students first brought to bear naïve additive reasoning yet eventually assimilated the system's unexpected behavior as a new type of equivalence class that builds multiplicative structures upon additive reasoning. In our analysis of students' reasoning, we view mathematics learning as making connections between complementary mathematizations rather than progressive decontextualization. In this view, the MIT-induced embodied artifact of ambidextrous motion *is* proportionality, yet tacit proportionality initially unconnected, undisciplined: it needs to be reflected upon, signified in standard mathematical inscriptions, and substantiated and elaborated through explicit solution procedures. We detail our preliminary findings elsewhere (Reinholz, Trninic, Howison, & Abrahamson, 2010, in these same PME-NA32 proceedings; also see http://www.tinyurl.com/edrl-mit for a video overview).

Researchable Moments: The Trajectory of "Green"

Our decision to forsake MIT1 and re-embody our design in MIT2, and in particular our shift to an electronic medium, introduced both a wealth of opportunities and, concomitantly, a slew of new design challenges. Namely, whereas in MIT1 students were initially passive participants hanging onto the pullies, MIT2 required students to immediately take agency, and so a question emerged as to how this agency should be framed, elicited, and guided. That is, what were students to *do* in MIT2? What would be their task, how would they accomplish it, and what form of feedback might they receive on the quality of this performance? Our initial response was to create electronic "targets"—two circular symbols that move up and down the monitor screen at preset rates; we would task participants with tracking these targets by remotely manipulating two virtual crosshair symbols (trackers), thus constituting a digital analogue of MIT1's rope tautness.

Once the development of the MIT2 program and user interface had progressed to a point that we could present it to a larger circle of researchers, we were prepared to reengage issues of feedback engineering and optimization, particularly with respect to orienting young students towards task completion. Critical to this collaborative reflection was the question of how to optimize the task such that students could best avail of the technological affordances of this innovative design through exploring its embodied problem space. We had already considered implementing on the monitor a symbol-based feedback mechanism in the format of "2:*x*". Namely, "2" would remain constant while *x* varied according to the height of the participant's right hand as detected in the instrument's sensory field. However, the discussion session strongly suggested to us critical limitations of the "2:*x*" solution as a performance feedback: in addition to electronic calibration challenges of determining and displaying *x*, a selection of alphanumerical feedback was perceived as prematurely mathematizing students' essentially embodied activity—the group opined that students should be given ample "free range" opportunity to explore and discover quantitative properties of the novel situation initially unfettered by potential arithmetic encumbrance and biases invoked by symbolical mediation. Having debated the plausibility of various feedback types, ranging from purely numerical outputs to graphical animations such as growing flowers, it was suggested that color-based feedback might offer an adequate solution. In particular, the color green was suggested due to its positive cultural signification. Thence, the color red was selected in binary contradistinction to indicate poor performance. Finally, a color gradient between green and red would indicate intermediate performance. Yet, still a question remained: How exactly would the color feedback be implemented? Herein lay the rub.

Having established the green-to-red gradient as our feedback sign, we discussed the plausibility of coloring the targets *themselves* as feedback rather than presenting the colors as standalone signs. A dissenting group member suggested the alternative option of flooding the entire background with the feedback color. An evaluation of this suggestion, which was ultimately endorsed, led us to look anew at the very utility and purpose of incorporating targets in our design, in light of our intention to use color feedback. We concluded that the targets were now redundant and ultimately unwieldy. Specifically, participants' exploration would be limited by the requirement to track moving targets. In contrast, we wanted to provide students with an opportunity to move the trackers freely, without the restrictions introduced by moving targets. Furthermore, to our surprise, we recognized that green, initially conceptualized as *feedback*, could in fact serve as a *goal* onto its own (analogously, urban drivers might deliberately attempt to effect a "25 mph" radar feedback). We thus decided to remove the trackers from the design entirely and implement the green/red background as the primary goal-cum-feedback element. Students' task thus became, somewhat enigmatically, to "make the screen green."

A routine of our design team is to simultaneously track changes implemented to the design and articulate these changes in terms both of their apparent improvement on earlier versions and their implications for subsequent iteration of development and implementation. So doing, we strive to cohere with the learning-sciences constructs guiding our research. However, attempting to make sense of an ostensibly simple question—"What is green?"—proved unexpectedly challenging. Our initial definition of green as a feedback indicating appropriate interface actions, namely "correct" hand-to-hand ratio, seemed incomplete. A subsequent two-week email flurry following the adoption of green feedback evidences dogged pursuit of one recurring question: "What *is* green?" Gradually we came to reason as follows: to a participant, green would initially constitute an objective; then green would serve as the performance feedback in attempting to achieve the same objective; and would ultimately give rise to an equivalence class—a collection of hand-location pairs that the student would perceive as "the same" by virtue of their common effect on the screen. Given appropriate guidance within this context, we surmised, the semiotic potential of green could be cultivated into notions of proportion and reformulated numerically.

The green quandary settled, we faced a dilemma: having abolished the targets, would our design still enable learning processes consistent with the ultra-radical-constructivist version of our action-before-concept conjecture? That is, we worried that by having designated green as the students' goal-cum-feedback, students would not occasion opportunity initially to engage "meaninglessly" with the MIT. We had reasoned that MIT1 provided students with such passive participation, and this passivity cohered with our embodiment thesis; in contrast, MIT2 rushed students to a highly prescribed activity. Ultimately, we resolved this dilemma by modifying the ABC theoretical framework: rather than strive for completely meaningless activity—a design objective at odds with the Vygotskian thesis that complements our constructivist commitments—we start from an activity whose *performance* objective is prescribed and demands immediate agency, yet whose disciplinary *semiotic* implication is initially covert.

Discussion and Implications for Design

The episode above captures much of what we mean by “recognizing semiotic potential through collaborative conversation with material”: by implementing green feedback and reflecting on it, our research team could recognize in the design hitherto latent potentials as well as constraints. In particular, identifying emergent incongruence between the design and our theoretical commitments drove us to consider, and then implement, modification in our theory—a form, if you will, of productive cognitive dissonance. This is a particular advantage of working in the DBR approach: the seeing-moving-seeing cycles provide ample opportunities for both illuminating the design’s semiotic potential and reflecting on underlying and emerging theory.

Furthermore, inasmuch as the discussion helps other designers understand and improve their own processes, the episode would indicate the importance of avoiding historical revisionism in reporting design decisions as well as the value of persevering in making sense of intuitive design decisions (Abrahamson, 2009). Note that it is *not* the case that we introduced green feedback in order to replace the targets—claiming this would constitute historical revisionism and obscure the dynamic interactive nature of our collaborative design process. Rather, deciding to use green feedback led us to reflect in depth on the targets’ semiotic potential within the protean context of the evolving artifact, which in turn led to their removal from the design and our consequent focus on making sense of “green.” Thus a seemingly minor design decision resulted in drastic changes in both the semiotic potential of the resulting artifact and, consequently, the emerging theory.

The analysis of our shared design history, presented in this paper, portrays DBR practice as a complex, emergent system, with small sparks gathering together over time, multiple dead ends, and the constant reinterpretation of previous ideas (Sawyer, 2007). Our navigation of this maze, in particular the moment-to-moment contextual shifts framing the evolving artifact, yielded three images of design-based researchers: (a) Janus the two-headed keeper of passageways who sees artifacts alternately as a student would or as an expert would and strives to facilitate cognitive entries into, and safe passages through, these learning corridors; (b) an investigator doggedly searching to explicate design decisions and elements coherently in light of learning-sciences theory; and (c) a reflective practitioner who embraces tradeoffs as rules of the game, logs all design decisions, shares deliberations with colleagues, and is open to constructive criticism and the possibility of implementing radical changes to both design and theory. We have found the above images, which emerged from this study, helpful in structuring our project and, so doing, allowing us to constantly revisit (and frequently revise) our theoretical framework in productive ways. Though design-based research may follow divergent paths, it is not a random walk.

In closing, we wish to underscore the critical role of vigorous reflection and respectful challenge within the collaborative progress of a design team. Expressed within a collegial atmosphere, genuine dissent encourages divergent thinking and thus creativity (Nemeth, 2009). This cognitive and epistemic plurality, coupled with dogged insistence on systemic coherence, enabled us to pursue divergent research directions, including emergent issues of apparently nugatory significance that consequently proved crucial in advancing our theoretical understanding of students’ reasoning. Moreover, by engaging these materials as a group, a wide range of the artifacts’ semiotic potentials were broadcast, simulating some of the design’s interpretive diversity as we entered our empirical site and engaged our study participants. Note, though, that we do not restrict collaboration to the oral medium: indeed, the very activity of co-authoring and revising this text served to push our collective underdeveloped intuitions beyond the realm of “stygian shades” (Vygotsky, 1978) and towards articulated discursive coherence.

Ultimately, we as researchers are recursively developing professional vision for our own evolving design.

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EIGHTH GRADERS' USE OF DISTRIBUTIVE REASONING FOR MULTIPLICATIVE TRANSFORMATION OF UNIT FRACTIONS

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As an extended investigation of children's construction of Rational Numbers of Arithmetic (Olive & Steffe, 2010), the present study explored how two 8th grade students constructed a scheme to multiplicatively transform a unit fraction into another unit fraction through a year-long teaching experiment. The results of this study indicate that distributive reasoning was essential and provided a base to explore the operations that produce the transformation.

Introduction

There have been a lot of efforts in mathematics education to investigate children's learning of fractions since Davis, Hunting, and Pearn (1993) commented that "the learning of fractions is not only very hard, it is, in the broader scheme of things, a dismal failure" (p. 63). One group of researchers, whom Pitkethly and Hunting (1996) mentioned as one of the two main schools in the research on children's initial fraction concepts, argued that the equidivision of a unit into parts, the recursive division of a part into subparts, and the reconstruction of the unit were essential for developing rational number meaning (Kieren, 1992; Mack, 1990; Steffe & Olive, 1990). Specially, Steffe and Olive (1990) conducted *The Fractions Project* to investigate children's construction of fraction schemes using operations that produced their number sequences (Biddlecomb, 1994; Olive, 1999; Steffe & Tzur, 1994; Steffe, 1992, 2010). From their teaching experiment, Steffe and Olive introduced a scheme, called the *rational numbers of arithmetic* (RNA), in which fractions have become abstracted operations. A child can be judged to have constructed the RNA when "the child is aware of the operations needed not only to reconstruct the unit whole from any one of its parts but also to produce any fraction of the unit whole from any other fraction" (Olive, 1999, p. 281). For the construction of RNA, Olive argued that a child integrates the operations of his or her *generalized number sequence* (GNS) with the operations that produce fractions as measurement units. He also argued that the process of children's construction of the RNA should be investigated because the operations that produce the RNA would be those operations that undergird the division of fractions. Therefore, the present study, as an extended investigation of children's construction of RNA, was conducted using a year-long teaching experiment with two 8th graders in order to explore how the two eight-grade students constructed fraction schemes based on their GNS.

Configuration of Research Questions

Olive and Steffe (2010) reported that students modified their existing counting schemes to form a *connected number sequence* (CNS) with their own fraction language. In their teaching experiment, Nathan, who had constructed a GNS, constructed fraction schemes during the first year of the teaching experiment that surpassed the fraction schemes the other students, who had only constructed the *explicitly nested number sequences* (ENS), produced throughout three years. He developed schemes of operations with fractions that allowed him to add fractions with unlike denominators, find a fraction of a fraction, rename fractions, and simplify fractions to lowest terms (Olive & Steffe, 2010). Further, they observed the harbinger for the construction of RNA in the construction process of Nathan's fraction schemes. Producing a co-measurement unit [one-sixth is a co-measurement unit of one-half and one-third because both are multiples of one-sixth] seemed essential for students' construction of RNA. Nathan was able to produce comeasurement units for fractions with which he could produce any fraction from any other fraction. For example, Nathan was able to make 1/9 of a unit stick using 1/12 of the stick by finding 1/36 as a co-measurement unit for both 1/12 and 1/9.

However, it was not reported whether he was explicitly aware of the multiplicative relationship between the two unit fractions while engaging in the transformation process. We would attribute a scheme to multiplicatively transform one unit fraction into another to a student when he or she is not only able to transform a unit fraction into another unit fraction, but also explicitly aware of the fraction to be used for the transformation. After Nathan transformed 1/9 of a stick into 1/12 of the stick, it is questionable whether he could find that 1/36 was a co-measurement unit of the two fractions because this would involve reflectively abstracting the sequence of operations in which he engaged into a program of operations. Further, the role of distributive reasoning for students' construction of RNA is yet to be investigated even though Olive and Steffe (2010) maintained that the construction of distributive reasoning enabled their students to establish a fraction composition scheme in relation to fraction multiplication. Reasoning with distribution likely requires having abstracted a three-levels-of-units multiplicative structure that can be brought to a situation prior to activity (Hackenberg, 2007).

Our study basically explored how two eighth-grade students constructed fraction schemes based on their GNS. Specially, in this paper, we will focus on the students' construction of a scheme the purpose of which is to *multiplicatively transform a unit fraction into another unit fraction* by addressing two main questions:

1. How did two eight graders construct distributive partitioning operations using their GNS?
2. How did construction of a scheme for multiplicative transformation between two unit fractions emerge as a modification of the students' prior knowledge (GNS or distributive reasoning)?

Method of Inquiry

The data for the present study were collected from a year-long *teaching experiment* (Steffe & Thompson, 2000), in which the first author participated as a teacher-researcher with a pair of eighth graders at a rural middle school from October 2008 to May 2009. The experiment is part of a larger, longitudinal study whose purpose is to bring forth and understand middle school students' algebraic reasoning. Rosa, one of the two participants, was chosen after an individual selection interview conducted on October of 2008. Carol, the other participant, had been chosen on October of 2007 and was paired with Rosa for her second year of teaching experiment. The criterion for selection of the two students was the ability to use composite units as iterable units, which was an indicator of the GNS. During the teaching experiment, the first author met once or twice a week in about 40-minute teaching episodes and the findings for this study were derived from seven episodes among them. All teaching episodes were videotaped with two cameras for on-going and retrospective analysis. One camera usually captured the whole picture of interactions among the pair of students and the teacher-researcher, and the other camera followed the students' written or computer work with the aid of two witness-researchers. The role of the witness-researchers was not only in assisting video recording but also in providing other perspectives during all three phases of the experiment: the actual teaching episodes, the on-going analysis between episodes during the experiment, and the retrospective analysis of the videotapes.

In terms of data analysis, the first type of analysis was ongoing analysis that occurred by watching videos of the teaching episodes and discussing and planning future episodes. For the most part, the resources from two cameras were mixed for a single, digitalized video file on the day of each teaching episode. In this way, we created a restored view of our teaching experiment. “A restored view is a wider view of activity than can typically be captured with an individual camera, but is still a selective view that reflects the researchers’ perspective of the recorded lesson” (Olive & Vomvoridi, 2006, p. 21). Then a sequence of summaries for the teaching episodes were created, each of which provided not only written descriptions of the participating students’ mathematical activities and interactions with the teacher-researcher, but also emerging key points in the students’ thinking and learning that were taken into account for the next teaching episode. The second type of analysis was a retrospective analysis. The purpose of the retrospective analysis of the sequence of teaching episodes was to make models of the students’ ways of operating mathematically through conceptual analysis of the students’ mathematical activities. We, first of all, attempted to understand what the students’ behaviors were and hypothesized why the students behaved in such ways. Then the attribution of the researchers’ construction of a scheme to the students was made at this stage.

Data Excerpts

Although finding a co-measurement unit for two given unit fractions [a fractional unit that can be used to exactly measure each unit fraction] would be essential for students to transform one of the unit fractions into the other (Olive & Steffe, 2010), the results of this study indicate that distributive reasoning was essential as well and provided a base for the students to become aware of the operations that produced the transformation. During the teaching experiment period, the two participating students demonstrated that they had constructed distributive partitioning operations in the context of a sharing situation, which might be considered as a partitive division problem involving fractions. However, the observation that they used distributive partitioning operations in their transforming a unit fraction into another was unexpected.

Even though a difference was revealed between Carol’s explicit use and Rosa’s implicit use of distributive partitioning operations, in this paper, we will follow the overall constructive itinerary for multiplicative transformation between two unit fractions based on distributive reasoning, mainly by focusing on Carol’s mathematical operations. The detailed discussion for elaborating the distinction between implicit and explicit use of distributive partitioning and significance on further mathematical activities is beyond the range of the paper.

Distributive Partitioning Operations in a Sharing Situation

On the 10th of February, Carol demonstrated that she had constructed distributive partitioning operations in a sharing situation. Upon the request to find one person’s share when six persons equally share five candies using JavaBars[1] (Olive, 2007), Carol made five bars for five candies on the screen, divided each bar into six pieces and pulled out one piece from each of five bars, which was cut into six pieces. This was an indication of distributive reasoning. That is, she formed a goal of a distributive partitioning scheme, say, to share five identical candy bars equally among six people. She then partitioned each candy bar into six parts and distributed one part from each of the five candy bars to each of the six persons with understanding that the share of one person could be replicated six times to produce the whole of the five candy bars. She also knew that five-sixths of one candy bar was identical to one-sixth of all of the candy bars. Rosa also similarly drew five candy bars on the screen, partitioned each bar into six pieces and took five pieces out of thirty little parts (see Figure 1). Her comment, "So each person gonna get one-sixth of the candy bar and since there is five total candy bars... basically one person's gonna get five-sixths of one whole candy bar," corroborated her construction of distributive partitioning operations in the context of the sharing situation.

Figure1. Rosa's drawing for sharing five candies among six people

Emergence of Distributive Partitioning Operations in Transforming a Unit Fraction into Another

Since a scheme for a multiplicative transformation between two unit fractions requires being explicitly aware of the fractional operator to be used in the transformation as well as the transformation activity itself, the teacher-researcher decided to pose transformation problems in the milieu of Geometer's Sketchpad (GSP) using a DILATION option. The DILATION option opened the way to allow the students to produce a multiplicative geometric transformation. In contrast to the program, JavaBars, where the students could produce their transformation activities step by step, the DILATION option in GSP directly asked to generate a scale factor for the transformations (cf. Figure 2). In that sense, transformation activities in GSP provided an occasion for the students to reflect on and abstract their mathematical activities for the transformations, which might lead to construction of a scale factor, that is, the number to be multiplied by [the multiplier]. On the 30th of March, the teacher-researcher posed a problem to find a multiplicative fractional operator to transform one-fourth of a meter into one-tenth of a meter using dilation in GSP (See Figure 2).

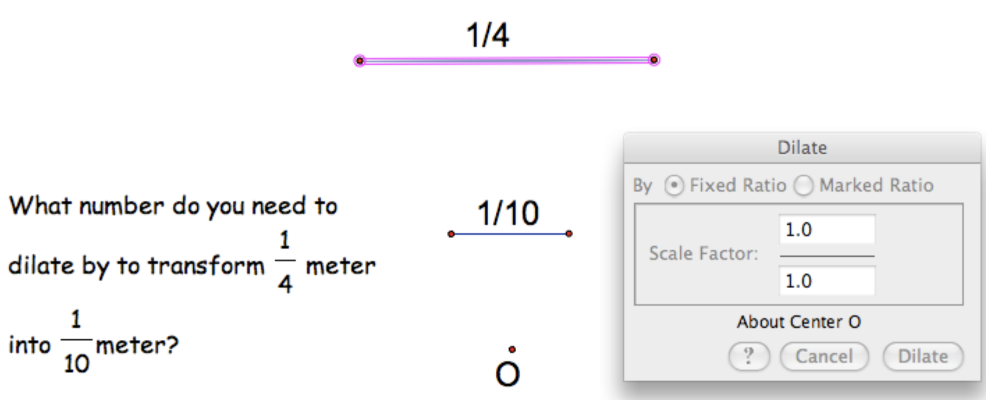

Figure 2. Multiplicative transformation using dilation in GSP

However, even though Carol and Rosa were familiar with the effect of dilation through several teaching episodes, say, transformation of a 2-meter segment into a 1/3-meter segment by dilating by one-sixth, they seemed stuck with finding one number that they could use to dilate 1/4-meter into 1/10-meter. Such a transformation activity in the GSP environment seemed too abstract for the students. Therefore, the teacher-researcher turned to using JavaBars in order for the students to possibly abstract their transforming activities. Carol and Rosa were asked to transform a 1/4-meter bar into a 1/10-meter bar using JavaBars. Both students started with the 1/4-meter bar and then copied it three times and joined the copies with the original 1/4-meter bar to make a 1-meter bar. Then Carol partitioned each 1/4-meter bar into ten parts and pulled out one part from each of the four 1/4-meter bars (see Figure 3). Her distributive reasoning was explicit in the sense that she anticipated one-tenth of the whole of the four parts could be found by taking one-tenth of each part and joining the four parts together, prior to engaging in the actual activity. When she was asked to reflect on her transformation processes in JavaBars in GSP environment and to find the number to be used in the dilation, she found four-tenths based on her construction in JavaBars. Although she already demonstrated the use of her distributive reasoning in a sharing situation, this transformation problem was a novel situation for her to use her distributive reasoning. In contrast, Rosa's distributive reasoning was not clear in that partitioning activities were usually initiated by Carol. Rosa then assimilated Carol's reasoning and imitated it. For Rosa, distributive reasoning did not emerge explicitly in such transformation situations during the teaching experiment.

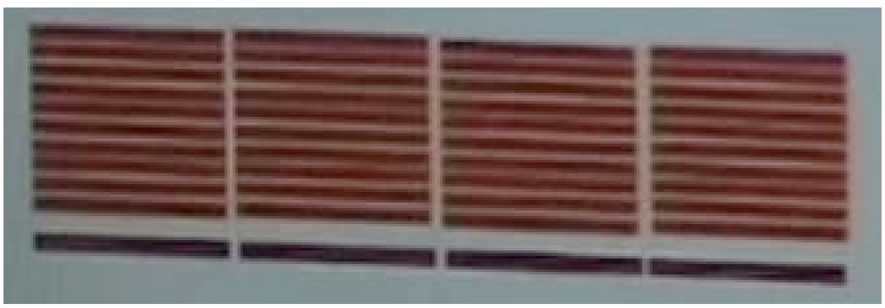

Figure 3. Carol's transformation from 1/4-meter to 1/10-meter

Multiplicative Transformation of Units Fractions based on Distributive Reasoning

On the 15th of April, Carol's behavior indicated construction of a scheme to multiplicatively transform one unit fraction into another based on her distributive reasoning. When the teacher-researcher asked the students to find a scale factor for transforming a 1/11-meter segment into a 1/13-meter segment using dilation in GSP, Carol immediately said "11/13" with an explanation that she had to multiply the 1/11-meter by 11 to get to one meter and then dilated the one meter by 1/13 to get 1/13, which led to the answer 11/13. The ability to take one-thirteenth of her re-presented 11/11-meter segment using her distributive partitioning operation seemed to enable her to find the fractional operator, 11/13, for the transformation by combining 11 with 1/13. On the other hand, Rosa could not independently establish the number to be used in dilation prior to activity throughout the teaching experiment even though she was able to enact transforming a unit fraction into any other unit fraction without such an explicit awareness. Rosa's implicit use of her distributive partitioning operations seemed to constrain her transformative activity.

Educational Importance of the Study

Only enacting the multiplicative transformation between two fractions is insufficient to construct the RNA because the RNA entail, "the construction of abstracted fractions as an ensemble of operations of which children are explicitly aware" (Steffe & Ulrich, 2010, p. 266). The RNA serve as an intermediate step in the constructive itinerary of students from fractional numbers to rational numbers of arithmetic as the latter are understood in conventional mathematical terms as an equivalent class of fractions. So, understanding the processes that are involved in students' construction of the operations that enable them to construct a multiplicative relation between any two fractions serves in understanding students' construction of the rational numbers of arithmetic in conventional mathematical terms. It also serves in understanding the steps that are involved in students' construction of fractional division and reciprocal reasoning. Further, this investigation of students' construction of multiplicative relations between two fractional quantities will contribute to the research in proportional reasoning, a form of a mathematical thinking that involves a sense of multiplicative co-variation.

Endnotes

1. JavaBars, a computer software, opens the possibility for students to create and enact a variety of different operations (i.e., partitioning, disembedding, iterating, etc.) on rectangles and squares. For instance, with JavaBars, students can create a rectangle, horizontally or vertically partition it into four parts and then pull out one of the four parts to construct three-fourths of the original referent whole by iterating it three times

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SECONDARY TEACHERS FIRST WAYS OF INCORPORATING MATHEMATICS TECHNOLOGY INTO CLASSROOM PRACTICE

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In this paper we present and analyze ways in which fifteen secondary teachers utilized technology in their classrooms having participated in an online technology training course for six months. The online course consisted of a series of activities designed for teachers to revise teaching some of the basic concepts relevant to high school mathematics, using technology and also HTML and JavaScript languages. Our training work hypothesis was that the uploaded activities would permit participating teachers to take up high school math innovatively through utilization of the conceptual or technological tools acquired by the mediation of the instrumental activity displayed along the mentioned course. Data consisted of the videos that the teachers submitted at the end of the OTC, demonstrating how they integrated technology in their teaching.

Introduction

The need to utilize new technologies throughout the educational system is now one of the prominent educational of one of current Mexico's official educational policies. To achieve this goal, Mexico's 2007-2012 National Development Plan (PND) states that adoption of new classroom technologies may be accelerated by offering teachers certified courses so that they become educated on how to use technology-mediated modalities in their instruction. As a consequence, efforts have been invested in designing learning environments for optimizing teacher development in that area.

Zbiek & Hollebrands (2008) argued that in order to assist develop technological competencies teachers must be considered as learners who move toward deeper understanding of what it means to use mathematics technology effectively with their students (p.288). In operationalizing this perspective, the authors endorse Beaudin and Bowers' (1997) PURIA model as a possible venue for PD design.

The PURIA model open some research-intervention issues that notably could facilitate or promote teachers development along a learning continuum. In fact, the model implies that the teachers experiment with different modes or development states to advance towards successful incorporation of technology in their instruction.

As Zbiek and Hollebrands (2008) stated, the growth during these modes includes the transition of the technology as the developer's tool into the teacher´s instrument for doing mathematics, a crucial aspect of learning to use technology captured by the notion of instrumental genesis. Similar ideas were suggested by other scholars (Verillon & Rabardel, 1995; Ruthven & Hennessy 2002).

Ruthven and Hennessy (2002) investigated the teachers' conceptions about what might constitute as successful practice using mathematics-focused technology. According to the authors, teachers' successful practice has to be considered as only tentative. Ruthven (2007) also argued that teachers would attain that ideal model only if they could develop knowledge through their own practice (craft knowledge). That is, starting from their craft knowledge teachers would be able to construct some more satisfactory ways for using new technologies into classroom.

Ruthven (2007) posited that craft knowledge is the largely reflex system of powerful heuristics which teachers bring to their classroom work. It draws on proceduralised and automated routines, tailored to the particular circumstances in which the teacher works. In particular, Ruthven (ibidem) said that much of the proposed innovation entails modification of this system (Ibidem, p. 6/16).

The PURIA Model

In a synthesis of the research regarding the use of technology into mathematics classroom practice, Zbiek and Hollebrands (2008) elaborated and extended what Beaudin and Bowers (1997) described as the PURIA Model (Play, Use, Recommend, Incorporate and Assess modes). According to Zbiek and Hollebrands (Ibidem), the extended PURIA Model considers the following five modes of using technology:

P, is the mode that involves playing with the technology. During this mode the use of technology does not have clear mathematical purpose.

U, is the mode that involves using technology as a personal tool. It is to say that the use of technology is for doing mathematics of one´s own design. May be using it in a classroom setting and not using it with students.

R, involves recommending technology to others. For instance, recommending use to a student, a peer, or a small group of students or peers. This likely is not in a formal classroom setting and it is not an integrated part of instruction.

I, involves incorporating technology into classroom instructions. In this mode, the teacher integrates the technology into classroom instruction. This mode occurs to varying degrees.

A, finally, this mode involves assessing students' use of technology. Here, the teacher examines how students use the technology and what they learn from using it (Zbiek & Hollebrands, 2008, p.295).

According to the authors, the resulting PURIA modes reflect the teacher becoming familiar with the technology as a tool for doing mathematics in the Play and Use modes. The growth during these modes includes the transition of the technology as the developer's tool into the teacher's instrument for doing mathematics, a crucial aspect of learning to use technology captured by the notion of instrumental genesis (Guin & Trouche, 1999; Trouche, 2000. Cited by Zbiek & Hollebrands, Ibidem).

In the Incorporate and the Assess modes, the teacher's attention turns, implicitly or consciously, toward the use of technology as a pedagogical tool, including the development of instructional orchestrations (Trouche, 2000) or elaborated plans regarding use of technology in the social dimensions of classrooms. The Recommends mode seems marked by a transition between mathematical and pedagogical aspects of the technology (Zbieck & Hollebrands, 2008, p.295).

The first two modes in the PURIA Model suggest that the teachers use the technology free of stress because they want to do so. If their initial experiences are successful, they will immediately proceed to work in a more pedagogically structured manner; in accordance with the Incorporate and Assess modes. Finally, the Recommend mode acts as a transitional phase from the teacher's initial experiences with technology to those pedagogically structured.

Zbieck & Hollebrands (2008) point out that the mathematical situations where the teachers find unexpected results, and the follow up on these tasks via opportunities for them to research, explain, and debate such situations, are experiences that may become the essence of powerful intervention when the teachers are using technology in according with the PURIA mode called Play or personal (p. 322). Finally, the authors mentiond that they found in this model a perspective that allowed for explicit consideration of teacher´s needs to learn to use technology, to learn to do mathematics with technology, to use the technology with students, and to attend to student learning as a guide for innovation.

The Instrumental Perspective and the Growth of Knowledge

According to Verillon and Rabardel (1995), artifact´s structure and function will foster certain knowledge or effect in cognitive development. That is, the introduction and use of instruments, whether material or psychological (language, computational means, symbols, diagrams, maps, and so on), leads to consumption of many structural and functional changes in learner's cognition. These authors perceive development to be the result of an extensive artificial process in which the acquisition of instruments plays a central role. Thus, it is not so much the instrument in itself which determines growth, but rather the functional reorganization and deployment that their acquisition and use impose onto innate mechanisms at distinct levels: sensory-motor, perceptive, mnemonic, representational, etc (Verillon & Rabardel, 1995, p.82).

Structure of Teaching Practice Using Technology

From our perspective, the activities implemented in the online course could and would allow participating teachers to learn to use technology and learn to do mathematics with technology. Moreover, asking teachers to incorporate technology into the classroom, with free choice of topic and technological tool, permitted us to study the participants' responses to the proposals for use of technology we had presented during the online course. It was important to examine how teachers responded to our technology teaching proposals implicitly included in the design of the activities uploaded for the course. According to Ruthven (2007): "If the technology must find a place in classroom practice, it must be examined in the context of class life, [just] as teachers life it." (Kerr, 1991, p. 121. Cited in Ruthven, 2007, p. 3). From Ruthven's point of view there are five key components of the structuring context of classroom practice: working environment, resource system, activity format, curriculum script and time economy. If researchers have these components on mind, they will draw on a range of more general work on classroom processes and teaching craft knowledge.

Methodology

Different Technological Resources Displayed Throughout the Online Training Course

We set up 24 weeks of activities uploaded in a *moodle* platform (link: http://upn.sems.gob.mx/main.php) that were designed for 15 high school mathematics teachers to help them learn how to appropriately use technological tools in their classrooms. Computer programming topics were addressed, particularly introduction to HTML and JavaScript programming, design of algorithms and their representations, algorithm development, flow charts, and codification, specifically with the purpose of teachers would experiment a change in the way they used to see or to approach to study mathematical algorithms.

The training course also included sequences of activities on the use of interactive software (eg. Logo, Geogebra, Aplusix, Excel, RecCon, FunDer); and exploration of a wide range of possibilities put in the Internet, for example, the library of virtual materials (eg. http://nlvm.usu.edu/en/nav/vlibrary.html) of Utah University (USA).

On the Mathematical and Pedagogical Tasks along the OTC

The mathematical topics covered in the activities included: Introductions to the fundamental arithmetic theorem, Goldbach's conjecture, graph theory, and calculating roots of polynomials (with the bisection method and the Regula Falsi method, the secant method, and Newton's method) purposed to review some of the algorithms important to these contexts, as well as the distinct possibilities of representing them in mathematics and computing. Although these topics are notably more advanced than what is included in the curriculum of Mexico's high schools, they were planned so as to challenge teachers' own existing knowledge by engaging them in solving problems that they may not have seen before or feel in control.

In relation to the pedagogical activities displayed throughout the OTC, we included four weeks of activities (two at the end of the first 10, and two at the end of the next 10 weeks). At the conclusion of OTC activities the teachers had to complete an assignment. The assignment asked that the teachers to first select one topic from high school mathematics curriculum, along with the software, tools, or digital materials they thought it would be useful to use on teaching the chosen topic. They then had to instrumentate a classroom work session with their students in a convenient way according to their chosen digital material. Following the video-recording of the work session, they had to upload a seven minutes long version of that recording to YouTube. Lastly, they had to upload their movie to the training platform a descriptive report of the video's content together with its URL. The results were based upon the analysis of the videos from five classrooms from the uploaded movies by teachers.

Results

The data illustrated that teachers displayed the PURIA modes of Play, Use, and Recommend. Additionally, the teachers also started to Incorporate mathematics technology into the classroom. More specifically, the analysis of the participants' videos revealed five different ways in which teachers' began to use technology in their instruction. These included use as: (1) A pattern of incorporation that derived from the classic approach to teaching, where the teacher use some technological tools mainly to explain to the students the topic they should learn (see Fig.1); (2) a modified version of this pattern that added teacher interaction with the students, basically by teacher questioning to the whole class (see Fig.2); (3) instrumentalization of the activity (Verillon & Rabardel, 1995; Assude et al., 2006) led by the use of a script (see Fig.3); (4) the teacher is orchestrating (Trouche, 2004) the activity using technological tools and leading comparisons between advantages and disadvantages of using paper and pencil techniques (see Fig.4); and finally, (5) organization of cooperative work centered on student appropriation of technology and project learning (see Fig.5).

An example of each mode of usage is presented below (detailed videos can be accessed at the listed URL address):

A. http://www.youtube.com/watch?v=PlLYsIO-Vh0&feature=related (See Fig.1).

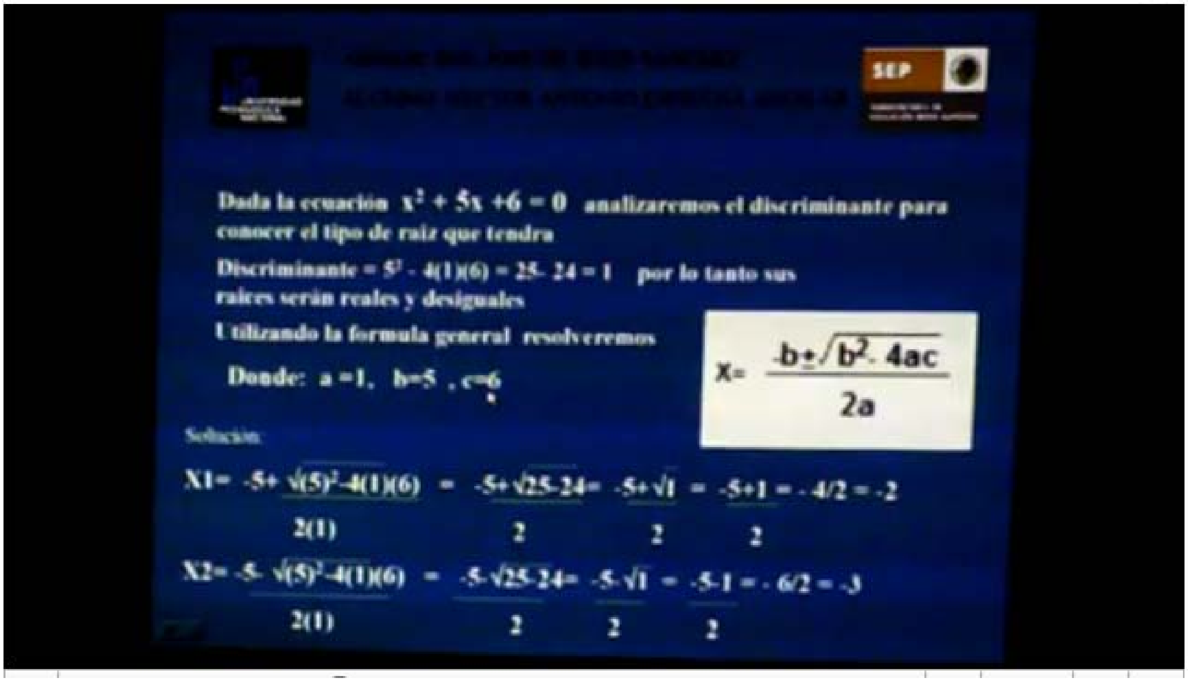

Figure 1. The teacher use some technological tools mainly to explain to the students a mathematical topic

B. http://mx.youtube.com/watch?v=N1FwbEo5KGI (As an example, see Fig.2).

Figure 2. Classic approach to teaching, adding teacher questioning to the whole class

C. http://es.youtube.com/watch?v=yhXs8BLMFlM (See Fig.3.)

Figure 3. The teacher is conducting an instrumentalization of the activity led by the use of a script

D. http://mx.youtube.com/watch?v=gwGcPtyXYbs (See Fig.4)

Figure 4. Orchestration of the activity using technological tools and comparing advantages and disadvantages of using paper and pencil technique

E. http://fcogurrola.blogspot.com (See Fig.5)

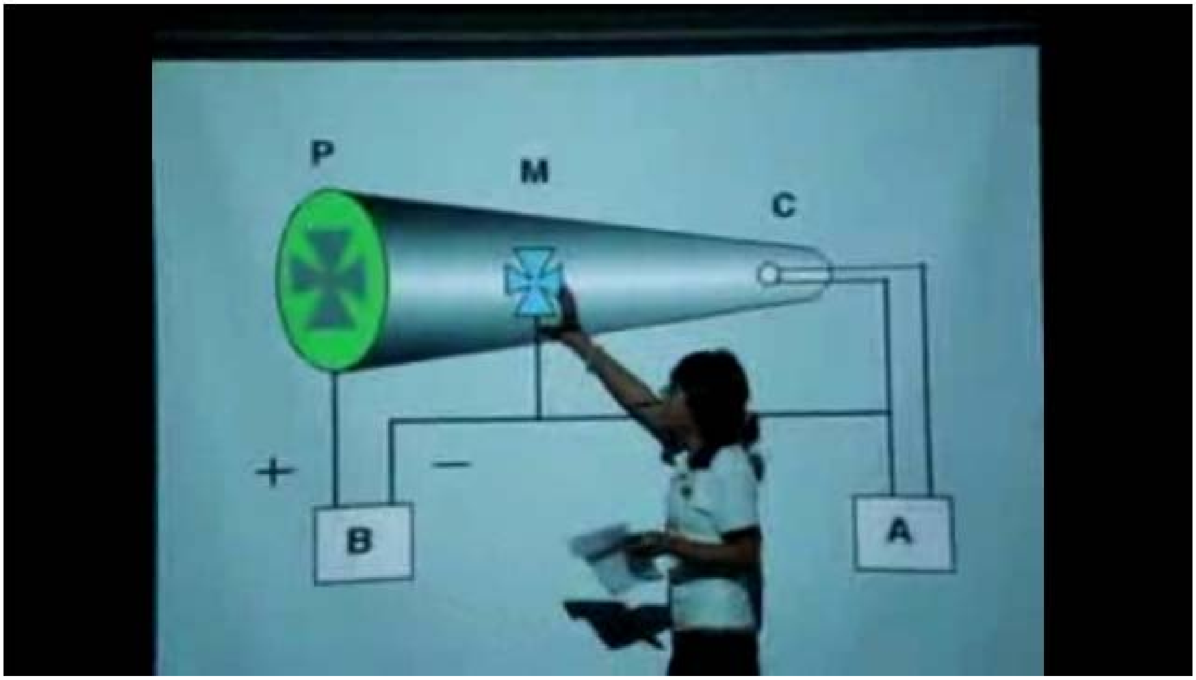

Figure 5. The teacher organize cooperative work centered on student appropriation of technology and project learning

Moreover, analysis of videos evidenced the presence of four of the five components Ruthven (2007) had distinguished as key elements to the structuring context of classroom practice with the goal of improving it. Specifically, these included the state of working environment, resource system, activity format and curriculum script, and teaching craft knowledge on the use of technology into classroom.

In relation with these components the evidences we obtained indicated that in a majority of the video-vignettes we reviewed the insufficient equipping of computers in the schools was patent, insufficient to providing one machine to each student and permit individual work. That is, in these cases the teachers only had one single computer in the room, (most likely their own,) and an LCD projector. Also, 37% of the teachers were able to complete their teaching practice in the computer lab that was fully equipped so that each student (or pair of students) had access to a machine. Nevertheless, the teachers showed competence selecting and mastering the technological tool to use into their classroom instruction, and choosing the topic to address. The teachers also showed different planning levels in relation with their interventions during their planned practice using technology. In particular, it was noticed that the teachers included more detail in their lesson planning and one more advanced mode they displayed during their practice, as it was the case of those that showed negotiation of meaning and organization of cooperative work between the students.

Finally, the different degrees teachers displayed to integrate mathematics technology into classroom instruction allow us to do a qualitative appraisal of the state of the development of their craft knowledge.

So that it becomes feasibly to predict progress in the process of learning to use the technology for teaching mathematics, in according to the extended PURIA Model, and when the participants *OTC* teachers were involved in assessing their students' learning, and when these teachers do their planning and their practice using the acquired technological and conceptual tools recently showed.

Conclusions

The advent of new technologies confronts teachers with great challenges as the one of having to learn to teach using new artifacts and technological tools into the mathematics classroom.

In this paper we focused on teachers' practice. Our specific interest was to consider how the teachers started to incorporate mathematics technology into their mathematics classrooms. We analyzed this process throughout the extended PURIA Model (Zbiek & Hollebrands, 2008), which is a development model based on different modes of use of the mathematics technology into the classroom. According to this model, the Assessment mode would be the one to reach, and it is reaching it when teachers would center their attention on the students as a guide for practice innovation.

Data was collected and analyzed using the video-vignettes of teachers' own practices. For some of the teachers (particularly those that included negotiation of meaning and organization of cooperative work between the students) developed competency to organize the instrumentation of the activity into their classroom. Therein underlies a latent potential to advance the thinking and development of the teachers towards the assessment of students' learning, the final stage in the PURIA Model of the learning process to teach with technology into mathematics classrooms.

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STUDENTS WITH LEARNING DIFFICULTIES ACCESSING MATHEMATICS THROUGH MOBILE LEARNING DEVICES

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Advances in technology are only useful to educators if they can be applied in a practical and appropriate way to daily instruction. A mobile learning device (MLD) is accessible technology that can be used to enhance mathematics instruction. Previously, students with learning difficulties found accessing the general education curriculum challenging. This study investigates the effectiveness of MLDs in mathematics instruction for typically performing students and students who have Individual Education Plans (IEPs). The findings of this study suggest that MLDs are beginning to have a positive effect on mathematics instruction for all students, including those with IEPs.

Introduction

Recently, the popular press and academic literature have published numerous articles concerning the use of cell phones in K-12 learning environments (Franklin & Peng, 2008; Liu, 2007; Traylor, 2009). While the topic has been debated in several publications, there is a paucity of research on the effectiveness of this technology on learning and student achievement (Dick, 2008). This study investigates the effectiveness of using mobile learning devices (MLDs) as a learning tool for all children in the general education classroom. We analyzed the results from standardized tests with special attention paid to students with individual education plans (IEPs) that included mathematics goals from a small school district in northwest Ohio who has been using MLDs for the past two years.

MLDs are educational technology that acts as a conduit between students who are digital natives and teachers who are digital immigrants (Prensky, 2006b). The National Council of Teachers of Mathematics (NCTM, 2000) suggests that "technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (p. 24). Technology is important, but how it is used and applied to the learning situation is even more critical. Some districts have specific goals and objectives when it comes to the use of technology, while others have no overt policy about the use of technology in the daily education of their students.

The technology mission statement for the participating school system is focused on creating life-long learners (Menchhofer, Sommer & Riepenhoff, 2009):

1. Technology creates a global learning environment.
2. The use of technology encompasses all learning styles.
3. Technology provides motivating and collaborative experiences.
4. Technology facilitates self-discovery for all learners.
5. Technology can transform the roles of teacher and learner.

The participating school system's primary technology goal is to sanction the use of technology as a transparent tool, assuming "Infusing technology into the classroom instruction will create the students who are academically competitive, technology literate, motivated and engaged in the learning process. The students will be proficient information users who have the ability to access, process and effectively communicate information in order to improve their learning and exceed in the national educational standards." (Menchhofer, Sommer & Riepenhoff, 2009, p. 4).

While many educators advocate and use certain school sanctioned technology (Smart boards, calculators, and laptops), many school systems across the United States do not advocate the use of mobile technology on school property. These policies ignore the learning potential of the relatively inexpensive mobile devices that are an integral part of the daily lives for a majority of students. These devices are often seen as a barrier to their students' learning while on school property. In order to capture students' interest, knowledge, and excitement, using MLDs, should be the goal of the 21st century education. These devices used daily can become a tool leading to success for typically performing students and students with special needs. Therefore, harnessing a technology students already successfully use should not be underestimated. We maintain that MLDs can be a crucial link that engages a typically marginalized group of students, children with learning difficulties. The purpose of research project reported here was to examine the effectiveness of MLDs in students' learning of mathematics, as well as to compare the effectiveness of MLDs in mathematics instruction for students with and without IEPs.

Theoretical Framework

Technology and 21st century students

The Pockets of Potential (Shuler, 2009) highlights five opportunities to seize mobile learning's unique attributes to improve education:

- Encourage "anywhere, anytime" learning;
- Reach underserved children;
- Improve 21st-century social interactions;
- Fit with learning environments; and
- Enable a personalized learning experience.

The first step in reaching students is to understand the role they play in the learning process. It seems logical to ask students what excites them about learning. However, educators often neglect or ignore the ideas of children who are the most affected by the educational decisions. It is not an effortless task to learn about what children think, especially for students with processing problems. By observing and discussing with students about their needs; their thoughts about learning are important data when determining the kinds of technology to be used in classrooms. Teachers and students need to be active partners in the design, as well as participate as informants in the testing and redesign of content (Druin, 2002). Engaging teachers and students from the beginning of the technology-content process allows for a greater chance to increase learning achievement, especially when the adults' and children's perspective are included in the redesigned content based learning modules (Lim, 2008; Prensky, 2008a). Teachers need to evolve in their roles as technologically advanced content facilitators if they wish to successfully teach today's students.

We maintain that in the 20th century education model, students attended school to learn the content that empowered them to use that knowledge outside the classroom. In the 21st century education model, students bring knowledge (i.e. technology) into the classroom, such as the ability to access information and the teachers are the facilitators for the processing, acquisition, accessing, analyzing, and translating the information into useful knowledge outside the classroom (Prensky, 2008b). Learning to implement the 21st century curriculum, which involves consistent use of technology, is a skill that teachers need to become proficient at and bring into the classroom (Freyvaud, 2008; Prensky, 2006a; Zbiek, Heid, Blume, & Dick, 2007).

MLDs technology levels the learning field, because they are an inexpensive, accessible technology for most households, including those households that lack laptops or desktop computers and internet connections. The project brings those students with limited or no access to technology at home into the “digital natives” generation (Prensky, 2006b). Students with learning difficulties generally have the same access as typically performing students. Even the students, who have not had constant access to technology at home, are still part of the “digital natives” generation, because of their consistent exposure to the vocabulary and information associated with almost instant access to information (Prensky, 2006b).

Students with Learning Difficulties

The Individuals with Disabilities Education Act (IDEA) and No Child Left Behind (NCLB) mandate that all students with special needs are expected to participate in general education classrooms. Motivating students to participate in instruction is often frustrating and a challenge for all educators. NCTM (2000) states that “technology offers teachers options for adapting instruction to special student needs” (p. 25) and that the computer or technology environment would be beneficial for students with attention and organizational issues. MLDs can provide a transition into the curriculum that is often a challenge for these students. Teachers must identify the content causing the struggle, such as computation, and then apply needed modifications (Peltenburg, van den Heuvel-Panhuizen & Doig, 2009). Also, teachers must examine their teaching behaviors and explore how teaching with technology opens up modification and accommodation options.

MLDs at the participating school were available to all students included in the general education mathematics classes. The special education teacher was a co-teacher and co-planner throughout the process and made sure the needs of any student on an IEP were being met. The students needed very little modification. Most students were very familiar with how MLDs work. Therefore, the training process was not any longer for students with IEPs than those without. A very small number of students needed a larger screen and were given a monitor to connect to their MLD, so that the print appeared on the monitor and, therefore, larger. If students were already have work modified in terms of less problems to solve or remember notes about formulas those modifications were easily built into the technology. The teachers were pleased at the progress that students with IEPs made. The device helped them to pay more attention to content that was more difficult for them than their typically performing peers. Learning how to use the MLD is part of a larger professional development process.

This process needs to include specific professional development that begins with technology facilitation training, rather than teachers’ random attempts to insert the technology into their classrooms (Kimmel, et. al, 1999). The short term goals of technology use included engagement in instruction and access to the curriculum for all students. The long term goal was to assist all students, especially students with learning difficulties, to realize their potential as active members of the community and workforce. This means these students also need exposure to Science, Technology, Engineering, and Mathematics (STEM) careers with emphasis on putting into place the skills they need to make this important transition (Lam, et. al, 2008).

Methodology

Data collection

This study investigated the effectiveness of MLDs in students' learning of mathematics, both typically performing students and students with IEPs, by analyzing students' performance on the SUCCESS test (https://reports.success-ode-state-oh-us.info/Login.aspx). The SUCCESS data analyzed in this study are provided by the Ohio Department of Education for dissemination of test data to the schools. Since student SUCCESS data is password protected via the educators' workroom link, the authors did not have a direct access to the data. The participating school district removed students' names and provided three years of SUCCESS data, 2007 - 2009 to the authors. The academic year 2008 was a pilot year, the first year for the participating schools using MLDs. Six teachers in grades third through fifth used MLDs in order to pilot test the effectiveness of the devices in education. In 2009, the second year, all the students used MLDs in grades third through sixth from two elementary schools and one middle school. More than 600 students used the MLDs daily, and their SUCCESS data were analyzed in the study.

Data analysis

The Spring SUCCESS data of 2007, 2008, and 2009 were analyzed using three variables:

1) The mathematics scale score,
2) The raw scores for the state five content standards: Measurement; Number Sense and Operations; Patterns, Functions, and Algebra; Data Analysis and Probability; and Geometry and Spatial Sense, and
3) The math total raw scores.

The descriptive analysis for the whole population was followed by the descriptive analysis comparing students with IEPs and the non-IEP student population. Due to the size and the nature of the sample, t-test for independent groups was used to compare students' scores for different years for IEP and Non-IEP student populations separately. Since students were at different grades each year taking different tests, they were treated as independent groups.

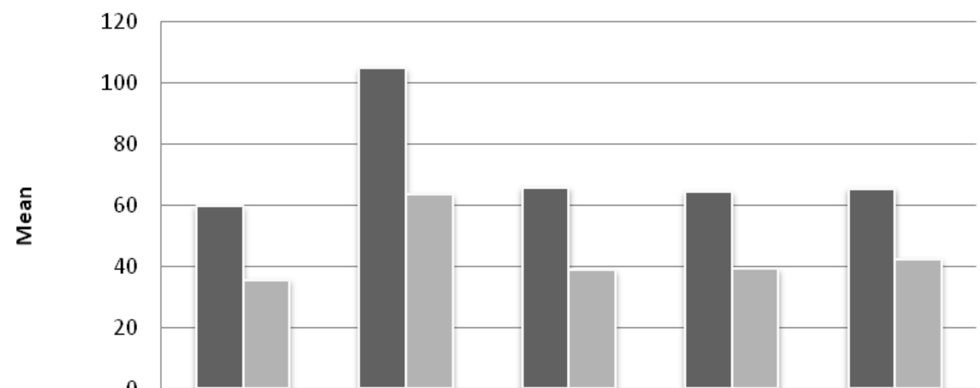
Results

Table 1 summarizes the data for the mathematics scale score, individual raw scores for five content standards: Measurement, Number Sense and Operations, Patterns, Functions, and Algebra, Data Analysis and Probability, and Geometry and Spatial Sense. The total mathematics scores for the whole population are also included. The average of total mathematics score for 2009 is higher than the other years. The mean score for the mathematics scale score in 2009 is also the highest. Mean scores for Measurement, Algebra, and Geometry gradually increased over the three years, but Data analysis and probability mean score does not show improvement. Mean score for Number and Number sense increased in 2009, but 2008 score was much lower than 2007. In contrary, mean score for Data analysis and probability show a noticeable improvement in 2008 but no increase in 2009.

Table 1. Descriptive statistics for the whole group across three years

	2007		2008		2009	
Whole Group	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
Math Scale Score	427.05	33.737	427.75	36.597	429.54	31.654
Measurement	55.38	21.581	56.21	22.612	56.48	19.946
Number	101.27	30.966	90.97	32.496	103.20	32.001
Algebra	58.26	22.380	63.04	25.940	64.14	22.266
Data & Probability	59.72	22.549	62.32	20.119	59.76	22.697
Geometry	59.76	20.126	60.80	21.772	65.66	16.878
Math Total Raw Score	334.39	98.170	333.34	101.132	349.25	91.738

Figure 1 provides a comparison of the data for students with and without IEPs for the mathematics scale and the total mathematics scores for 2007, 2008, and 2009 combined.



	Measurement	Number	Algebra	Data & Probability	Geometry
Non-IEP students	59.73	104.88	65.92	64.47	65.58
IEP students	35.61	63.68	38.78	39.26	42.47

Figure 1. Comparison of IEP and Non-IEP populations

Approximately 16 percent of the whole population is comprised of students on IEPs. Their scores compared to Non-IEP students are lower in all areas, which was predictable. However, this result can inform teachers with an overall picture and help diagnose where the biggest gap exists between students with and without IEPs. The ratio of students with IEPs and students without IEPs is different for each content standard. The biggest gap between students with and without IEPs appears in Algebra and the least difference in Geometry. Considering the new nation's emphasis on Algebra in high school graduation requirement (Achieve, 2009), early intervention in Algebra needs to be discussed between mathematics and special education teachers, especially ways to use the MLDs in Algebra instruction.

The t-test results for Non-IEP and IEP population are summarized below:

a) There was a significant difference between years 2007 and 2009, $t(1091) = -2.259$, $p < .05$, with 2009 mathematics total raw score for the Non-IEP students being higher than 2007.

b) There was a significant difference between years 2008 and 2009, $t(1058) = -2.013$, $p < .05$, with 2009 mathematics total raw score for the Non-IEP students being higher than 2008.
c) There was a significant difference between years 2008 and 2009, $t(185) = -2.060$, $p < .05$, with 2009 mathematics scale score for the students with IEPs being higher than 2008.
d) There was a significant difference between years 2008 and 2009, $t(180) = -3.147$, $p < .05$, with 2009 mathematics total raw score for the students with IEPs being higher than 2008.

Table 2. Comparison of IEP and Non-IEP populations across the years

	2007				2008				2009			
	Non-IEP Students		IEP Students		Non-IEP Students		IEP Students		Non-IEP Students		IEP Students	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
Math Scale Score	434.46	28.985	389.13	30.841	435.47	32.336	385.54	29.071	435.51	28.212	394.08	27.633
Measurement	59.18	19.848	35.96	19.632	60.60	20.244	32.16	19.644	59.44	18.361	38.89	20.025
Number	107.83	26.081	67.71	32.336	97.63	28.879	54.54	26.576	109.01	28.310	68.67	31.020
Algebra	62.62	20.178	35.96	19.726	67.95	23.683	36.19	20.889	67.36	20.398	45.00	23.380
Data & Probability	63.71	20.268	39.27	22.637	66.42	17.266	39.90	19.921	63.33	20.981	38.56	20.963
Geometry	63.37	17.596	41.28	22.115	64.99	19.244	37.84	20.526	68.49	14.590	48.89	19.685
Math Total Raw Score	356.70	81.372	220.18	97.695	357.59	84.151	200.62	81.571	367.63	78.515	240.00	88.989

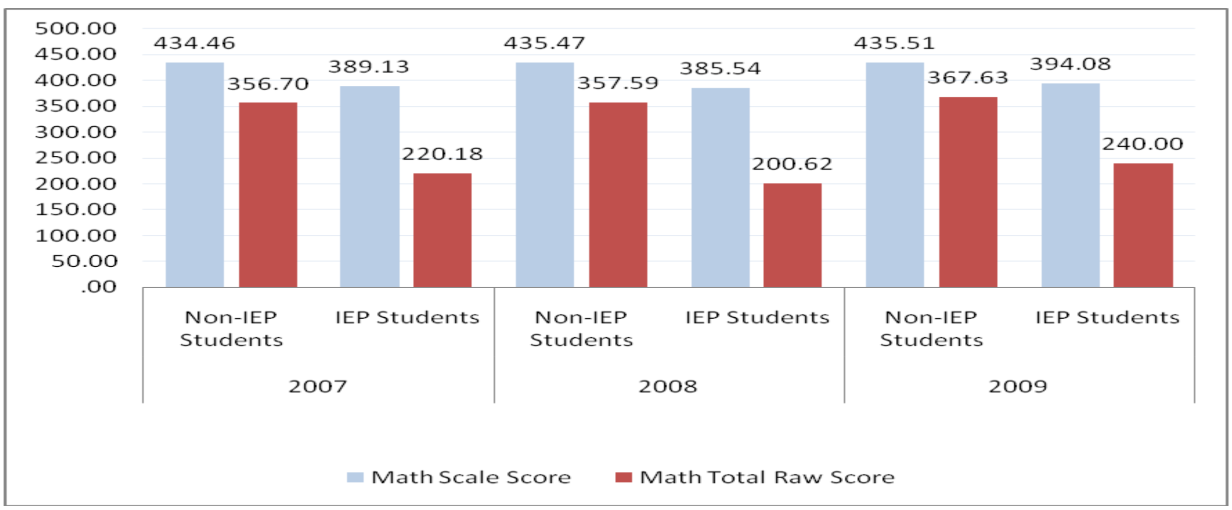


Figure 2. Comparison of IEP and non-IEP population for Math scale and total math scores

According to a) and b), Non-IEP students' mathematics total raw score shows statistically significant improvement in 2009. Findings summarized in c) and d) indicate that students with IEPs' mathematics scale score and total raw score show statistically significant increase in 2009 compared to 2008. Both students with and without IEPs had to learn new instructional technology, MLDs, in 2008. Students both with and without IEPs performance did not show a significant increase from 2007 to 2008, and it is possibly due to their slow learning pace compared to Non-IEP students. In other words, it is possible that students with IEPs took longer time to get familiar with and adopt MLDs into their learning style, compared to Non-IEP students. Fortunately, their scores in 2009 show significant improvement.

Discussion

The findings of this study suggest that MLDs are beginning to have a positive effect on mathematics instruction for all students, including those with IEPs. Students with IEPs show an increased math score in 2009 versus 2007 when they had no instruction with MLDs. While the students without IEPs showed a slightly higher total math score in year two, the total math score did increase from the start year. The increase appears to indicate that the teaching with MLDs is becoming more effective and leading to increased mathematics knowledge. Each school year is unique, as are the students, so the fact that scores are rising for students with IEPs is promising. In addition, comparison data between students with IEPSs and Non-IEP students' performance in each mathematics content standard informs mathematics teachers in various aspects: students with IEPs need more help with Algebra and Number and Operations; or Geometry is the area IEP students can enhance their understanding the most.

Future implications include extending the use of MLDs to other districts in other states and providing well structured teacher training on how to use MLDs and how to develop effective activities for MLDs. To accomplish this, targeted professional development is needed to create comprehensive teaching modules. While the goal of many technology-based games encourages basic review and drill; instructional modules should engage students at higher thinking levels. The content in the MLDs should be what draws the student in, not the device itself. Students already have a great understanding of the device, but teachers will continue to learn to exploit that in a way that increases content knowledge and readiness for STEM careers. It is the hope that with this added dimension, achievement scores will continue to increase and curriculum goals would be met for all students.

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AN INVESTIGATION OF DEVELOPING REPRESENTATIONS OF LINEAR FUNCTIONS IN THE PRESENCE OF CONNECTED CLASSROOM TECHNOLOGY

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This poster presents an investigation of the use of representations of linear functions in the classroom in the presence of connected classroom technology and its impact on student ability to transfer among different representations of linear functions using data from the Classroom Connectivity in Promoting Mathematics and Science Achievement (CCMS) project. CCMS is a national research study examining the impact of connected classroom technology on student achievement, self-regulated learning and dispositions of students toward mathematics and science. The research design is a randomized crossover field trial in 118 Algebra 1 classrooms (Owens et al., 2008). The intervention of the CCMS project consisted of providing teachers with Texas Instruments TI-Navigator™, a wireless classroom device that networks students' handhelds to the teacher's computer and pedagogical training to support implementation of this technology. Two major components of this training focused on the development of formative assessment and classroom discourse. The combination of formative assessment, classroom discourse and TI-Navigator's™ representational capabilities should provide a rich environment to develop representational thinking, since representation is inherently a social activity comprised of the social construction of shared meaning of mathematical representations (Pape & Tchoshanov, 2001).

The data from the CCMS project being used in this investigation consists of an algebra pretest, algebra posttest, teacher interviews, a survey on student perceptions of instruction and video data of classroom lessons. Teachers were observed two consecutive days each year of the project. For the investigation, fifteen teachers were chosen because their videotaped lessons were on linear topics. The algebra posttest was analyzed using Cunningham's (2005) guidelines for items requiring transfer of representations and 12 items were found. A qualitative analysis of the video tapes will be conducted to see the relationship of the connected classroom technology use and the use of linear representations. Statistical analysis will be conducted to determine if there is a difference between students being taught with a navigator™ and without on their ability to transfer among linear representations. Results of both analyses will be presented.

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APPLETS THAT PUT THE 'INTERACTIVITY' IN INTERACTIVE WHITE BOARDS

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The proliferation of interactive white boards (IWBs) in schools across the world represents a bold attempt to improve education via significant financial investment. Unfortunately, classroom-based research indicates that because teachers are only given scant amounts of training and support, they tend to use only the superficial affordances of these technologies without changing overall interaction patterns or resulting test scores (Lerman & Zevenbergen, 2007). This study examines how a teacher used various features of an applet that was designed to leverage the features of an IWB to enhance students' efforts to solve challenging tasks. Our work is guided by a commognitive view of learning which suggests that interpersonal communication and individual thinking are two facets of the same phenomenon. This socially-oriented perspective highlights the central role of tools as mediators of communication and thinking. Our goal as tool developers is to support teachers' efforts to engage students in thought/communication activities that leverage the unique aspects of whiteboard to mediate these discussions.

This work involved iterative cycles of design and research. We began by imagining how an applet might have affected the class discussion that Henningsen and Stein (1997) describe in which students offered innovative strategies for representing portions of grids. We then designed an initial interactive tool and videotaped groups of high school students using the applet to solve similar tasks. A second round of design and research involved comparing results of groups with and without an interactive whiteboard.

Preliminary results indicate that while the groups of students who used the applet projected onto a screen without an IWB were able to communicate their ideas, they were hindered by the fact that they had to describe moves such as "Drag the third red dot down two places". These indexical statements separated their activities from the underlying mathematical ideas that supported them. In contrast, when students were using the IWB and sharing the interactive pen, their efforts to think, gesture, and debate were fused as they interacted with the mathematical representations. In particular, their words and gestures ignited shifts in activity that resulted in new ways of conceptualizing the task that made use of the various features built into the activity for use with IWBs. We argue that these features enabled the students to communicate ideas in an interactive public space that could not have been replicated using pencil and paper or the projector alone.

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CONCEPTUAL FRAMEWORK FOR DESIGNING MATH COMPUTER GAMES: ELEMENTARY GAME THEORY DIMENSIONS FOR EDUCATORS

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Our study addresses the research problem of constructing an interdisciplinary conceptual framework for analyzing math games through a series of design decisions. While we focus on computer games, many of the principles apply to physical space games. Definitions of decisions come from game theory research and gaming studies (Gee, 2007; Myerson, 1997). The gameplay consequences of each decision are analyzed based on existing games viewed through the lens of these definitions. The mathematics education consequences of each decision are then analyzed based on the pedagogy embodied in the gameplay, and viewed through the lens of learning theories (Piaget, 1970; Pirie & Kieren, 1994; Vygotskii & Kozulin, 1986). A series of parallels between gaming concepts and pedagogical notions helps mathematics educators make sense of game theory concepts, and apply these concepts to teaching. The resulting structure makes it clear that some types of math games are overused, and other promising types are rarely employed by mathematics education game developers.

The decisions, as well as their mathematics and math education parallels, are made along these dimensions that provide dichotomies, gradients or levels:

- *Abstraction dichotomy:* narrative-based vs. abstract; situated vs. formalized
- *Revelation gradient:* full disclosure to hidden information; open-book to closed-book
- *Strategic gradient:* strategic to typed; problem-solving to exercises
- *Resource levels:* bounded rationality gameplay or not; level or stage learning theories
- *Agency and autonomy gradient:* high to none; open-ended to closed-ended tasks
- *Planning levels:* interactions, tasks, tactics, strategies; order of math tasks
- *Depth gradient:* expert to superficial knowledge; deep learning to expository learning
- *Goal gradient:* sandbox play to clear goals; conceptual learning to procedural fluency

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FIFTH GRADERS' CONCEPTION OF FRACTIONS ON NUMBERLINE REPRESENTATIONS

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While students' experiences related to representations, such as pie charts (Cramer, Wyberg, & Leavitt, 2008) and rectangular objects, and how they conceive fractions in relation to those representations are well investigated (Cramer, et al., 2008), students fraction conceptions related to number line needs further investigations. In addition, comparing students' conceptions of fractions in different external representations and which representations might support students conception of fractions (Taber, 2001) is also a research focus that deserves attention.

In this study, our purpose was to investigate the ways in which eight 5th graders conceived fractional numbers with different representations such as pie chart, rectangles, and number line. The eight students were interviewed for 45 minutes long sessions on 16 fraction problems as part of data collection for a bigger study. The problems had different representations and aimed to investigate different fractional knowledge such as partitioning, iterating operations as well as multiplicative reasoning. The data and analysis is part of the bigger research with which students had a computer tutor intervention that had hint and feedback features. The analysis of interview data revealed that while students are much more familiar with fractional numbers with pie chart representations, their conceptions of where fractional numbers are located on the number line is usually (unlike their interpretations of fractions with pie charts) independent from the whole/unit on the number line and the activities that produce fractional numbers discussed in the literature (Steffe, 2002; Norton & Wilkins, 2009).

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THE NATURE OF CLASSROOM DYNAMICS IN HIGH SCHOOL MATHEMATICS CLASSES WHERE GRAPHING CALCULATORS ARE USED

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Several studies have shown mixed findings about the roles of teachers in technology-enriched classrooms. Doerr and Zangor (2000), Farrell (1996), and Slavit (1996) found that teachers assume the roles of fellow investigator, facilitator, and consultant when they teach with graphing calculators while Goos, Galbraith, Renshaw, and Geiger (2003) reported mixed results as did Tharp et al. (1997).

This study investigated the nature of classroom dynamics in high school mathematics classes where graphing calculators were used. Participants were six high school mathematics teachers from a midsized urban school district in the north eastern United States. Data were collected through classroom observations. Analysis of the data revealed that the nature of classroom discourse tended to influence the role for which the graphing calculators were used. I found four of the five roles of graphing calculator use discussed by Doerr and Zangor (2000). These findings can be summarized in two major results:

(1) Teacher directed lessons were characterized by teachers demonstrating specific calculator functions, making decisions about particular calculator settings, and correcting student errors and/or confirming solutions. On the other hand, lessons involving student exploration were characterized by teachers involving students in decision making regarding calculator use, guiding students in refining their thinking with regard to calculator use, and challenging students to interpret calculator results in the context of the problem situation and communicating this understanding to the whole class.

(2) During student explorations the graphing calculator was mainly used as a visualization tool, a checking tool, or a data collection and analysis tool. During teacher directed sessions the graphing calculator was mainly used as a computational tool.

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USING TECHNOLOGY TO ENAHANCE THE STUDY OF MATHEMATICS

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Technology plays an important role in teaching and learning mathematics. The effective use of technology often depends on the teacher. As students engage in a variety of mathematical activities in the classroom, the teacher must decide when and how best to apply technology to assist in the learning of mathematical concepts. Cognitive technological tools such as computer software and handheld graphing technology, if used appropriately in the mathematics classroom, have the potential to enhance teaching and learning of mathematics by empowering students to concentrate on more advanced problems. Cognitive technological tools have the potential to transcend the limitations of the mind in "thinking, learning, and problem-solving activities" (Pea, 1987, p.91).

Zbeik, Heid, and Blume (2007) identified two types of mathematical activity: technical and conceptual. The technical refers to the procedural actions and performance on mathematical objects; actions such as numerical computation, algebraic manipulations, graphical transformation, and solving equations. Conceptual mathematical activity, on the other hand, involves understanding, communicating, using and making mathematical connections, conjecturing, representing, and proving. The distinction between the two dimensions of activity can play an important role in understanding how best to integrate technology throughout the curriculum in a way that fosters student understanding. When engaging in cognitively demanding mathematical inquiry activities, carrying out procedures can be performed by technology and reflecting on those procedures, interpreting results, describing connections between representations, making predictions, generalizing, and making conjectures can be emphasized. This can lead students to deeper mathematical thinking and understanding. When learning mathematical algorithms and doing complicated computations are not the goal of the problem solving process, the computer software "does the math" which allows students to focus on the deeper ideas of the problem.

In this poster presentation, an examination of how cognitive technological tools such as the computer algebra systems (CAS) handheld graphing calculator, Excel, and TI-Interactive! are used as students engaged in different types of mathematical activity in an undergraduate mathematics and mathematics methods course.

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WHERE IS MATH 2.0? ALARMING TRENDS AND NEW HOPES

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The Math 2.0 Interest Group (Math 2.0 Interest Group, 2010a, 2010b) is an international network of mathematics educators, online developers, teachers and families working on social media and computational mathematics projects. Directions of Group's work include analysis of math-rich social objects, studies of mathematical behavior online, development of courses and curricula, and organization of meetings and conferences for researchers and practitioners.

The trend that alarms us most is the lag of mathematics behind other subjects in class-centered online communities, and a larger lag in other communities. Mathematics remains confined to classes, homework, and tests. Most class-centered communities are not sustainable, dissolving after the class ends. The existing sustainable math communities can be intellectually elitist and demographically exclusive, and isolated from one another.

We identified (Deubel, 2010; Droujkova, 2009) four directions for support, development and promotion in the immediate future, and as tracks for events and publications we host:

- *Community mathematics:* networks, content sharing, clubs, grassroots curriculum
- *Executable mathematics:* software, manipulatives, programming platforms, social objects
- *Humanistic mathematics:* visual arts, digital storytelling, ethnomathematics
- *Psychology of mathematics:* meta-cognition, attitudes, learning theories, well-being

Mathematical authoring, including content creation and sharing by students, and development of activities, communities, curricula and software by educators, is at the heart of all four directions.

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Chapter 19: Working Group Paper

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

ADDRESSING EQUITY AND DIVERSITY ISSUES IN MATHEMATICS EDUCATION

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As the title suggests, this Working Group has a dual focus on issues of mathematics teaching and learning and issues of equity and diversity. This includes topics identified at the Working Group in 2009 such as pre-service mathematics teacher education for social justice, culturally relevant and responsive mathematics education, and supporting mathematical discourse in the linguistically diverse classroom. In addition, we focus on questions about systematic inequities and their impact on the teaching and learning of mathematics, and on methodologies that allow researchers to examine the complexity of the socio-political contexts of mathematics teaching and learning This work attempts to take a multifaceted approach, aimed at multiple levels from the classroom to broader social structures, within a variety of contexts both in and out of school, and at a broad span of relationships, including researcher to study participants, teachers to schools, schools to districts, and districts to national policy.

Brief History

This Working Group builds on and extends the work of the Diversity in Mathematics Education (DiME) Group, one of the Centers for Learning and Teaching (CLT) funded by the National Science Foundation (NSF). DiME is a group of emerging scholars (new faculty and graduate students) who graduated from, or in some cases are still studying at, three major universities (University of Wisconsin-Madison, University of California-Berkeley, and UCLA). The Center is dedicated to creating a community of scholars poised to address some critical problems facing mathematics education, specifically with respect to issues of equity (or, more accurately, issues of inequity).

The DiME Group (as well as subsets of that group) has already engaged in important scholarly activities. After two years of a cross-campus collaboration dedicated to studying issues framed by the question of why particular groups of students (i.e. poor students, students of color, English learners) fail in school mathematics in comparison to their white (and sometimes Asian) peers, we presented a symposium at AERA 2005 (DiME Group, 2005). This was followed by the writing of a chapter in the recently published *Handbook of Research on Mathematics Teaching and Learning* which examined issues of culture, race, and power in mathematics education (DiME Group, 2007). Further, in an effort to bring together and expand the community of scholars interested in this work, DiME, at AERA in 2008, sponsored a one-day Professional Development session examining equity and diversity issues in Mathematics Education. In addition DiME members have joined with other scholars in joint presentations and conferences. A book on research of professional development that attends to both equity and mathematics issues is currently in press (Foote, in press). Many DiME members as well as other scholars contributed to this volume.

Moreover, the Center has historically held DiME conferences each summer. These conferences provide a place for fellows and faculty to discuss their current work as well as to hear from leaders in the emerging field of equity and diversity issues in mathematics education. Since the summer of 2008, the DiME Conference opened to non-DiME graduate students with similar research interests from other CLTs such as the Center for the Mathematics Education of Latinos/as (CEMELA), as well as graduate students not affiliated with an NSF CLT. This was an initial attempt to bring together a larger group of emerging scholars whose research focuses on issues of equity and diversity in mathematics education. In addition, DiME graduates, as they have moved to other universities, have begun to work with scholars and graduate students with connections to other NSF CLTs such as MetroMath and the Urban Case Studies Project in MAC-MTL whose projects also incorporate issues of equity and diversity in mathematics education.

It is important to acknowledge some of the people whose work in the field of diversity and equity in mathematics education has been important to our work. Theoretically we have been building on the work of such scholars as Marta Civil (Civil & Bernier, 2006; González, Andrade, Civil, & Moll, 2001), Megan Franke (Franke, Kazemi, & Battey, 2007), Eric Gutstein (Gutstein, 2006), Danny Martin (Martin, 2000), Judit Moschkovitch (Moschkovich, 2002), and Na'ilah Nasir (Nasir, 2002). We have as well been building on the work of our advisors, Tom Carpenter (Carpenter, Fennema, & Franke, 1996), Geoff Saxe (Saxe, 2002), Alan Schoenfeld (Schoenfeld, 2002), and again Megan Franke (Kazemi & Franke, 2004), as well as many others outside of the field of mathematics education.

A significant strand of the work of the DiME Center for Learning and Teaching included implementing professional development programs grounded in teachers' practice and focusing on equity at each site. The research and professional development efforts of DiME scholars are deeply intertwined, and much of the research thus far produced by members of the DiME Group addresses issues of equity within Professional Development. Additionally, since the majority of the DiME graduates, as new professors, along with a number of current Fellows, are engaged in teaching Mathematics Methods courses, the integration of issues of equity with issues of mathematics teaching and learning in Math Methods has become a site of interest for research. These scholars have learned through their work that collaboration is a critical component to our work. We were pleased for the opportunity offered by the first year of being a Working Group at PME 2009 to continue working together as well as to expand the group to include other interested scholars with similar research interests. We were encouraged that our efforts were well received; more than 40 scholars from a wide variety of universities and other educational organizations took part in the Working Group.

Focal Issues

Under the umbrella of attending to equity and diversity issues in mathematics education, researchers are currently focusing on such issues as teaching and classroom interactions, Professional Development, pre-service teacher education (primarily in Math Methods classes), student learning (including the learning of particular sub-groups of students such as African-American students or English learners), and parent perspectives. Much of the work attempts to contextualize the teaching and learning of mathematics within the local contexts in which it happens, as well as to examine the structures within which this teaching and learning occurs (e.g. large urban, suburban, or rural districts; under-resourced or well-resourced schools; and high-stakes testing environments). How the greater contexts and policies at the national, state, and district level impact the teaching and learning of mathematics at specific local sites is an important issue, as is how issues of culture, race, and power intersect with issues of student achievement and learning in mathematics.

Existing research tends either to focus on professional development in mathematics (e.g., Barnett, 1998; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Kazemi & Franke, 2004; Lewis, 2000; Saxe, Gearhart, & Nasir, 2001; Schifter, 1998; Schifter & Fosnot, 1993; Sherin & vanEs, 2003), or professional development for equity (e.g., Sleeter, 1992, 1997; Lawrence & Tatum, 1997a). Little research exists, however, which examines professional development or mathematics methods courses that integrate both. The effects of these separate bodies of work, one based on mathematics and one based on equity, limits the impact that teachers can have in actual classrooms. The former can help us uncover the complexities of children's mathematical thinking as well as the ways in which curriculum can support mathematical understanding in a number of domains. The latter has produced a body of literature that has helped to reveal educational inequities as well as demonstrated ways in which inequities in the educational enterprise could be overcome.

To bridge these separate bodies of work, the Working Group has begun and will continue to focus on analyzing what counts as mathematics learning, in whose eyes, and how these culturally bound distinctions afford and constrain opportunities for students of color to have access to mathematical trajectories in school and beyond. Further, asking questions about systematic inequities leads to methodologies that allow the researcher to look at multiple levels simultaneously. This research begins to take a multifaceted approach, aimed at multiple levels from the classroom to broader social structures, within a variety of contexts both in and out of school, and at a broad span of relationships including researcher to study participants, teachers to schools, schools to districts, and districts to national policy.

Some of the research questions that the Working Group will consider are:

Teachers and Teaching

- What are the characteristics, dispositions, etc. of successful mathematics teachers for *all* students across a variety of local contexts and schools? How do they convey a sense of purpose for learning mathematical content through their instruction?
- How do beginning mathematics teachers perceive and negotiate the multiple challenges of the school context? How do they talk about the challenges and supports for their work in achieving and equitable mathematics pedagogy?
- What impediments do teachers face in teaching mathematics for understanding?
- How can mathematics teachers learn to teach mathematics with a culturally relevant approach?
- What does teaching mathematics for social justice look like in a variety of local contexts?
- What are the complexities inherent in teacher learning about equity pedagogy when students come from a variety of cultural and/or linguistic backgrounds all of which may differ from the teacher's background?
- What are dominant discourses of mathematics teachers?

Students and learning

- What is the role of student academic and mathematics identity in achievement?
- How do students' out-of-school experiences influence their learning of school mathematics?

- What is the role of perceived/historical opportunity on student participation in mathematics?

Policy

- How does an environment of high-stakes standardized testing affect whether and how teachers teach mathematics for understanding? How does this play out across a variety of local contexts? How can we support teachers to teach mathematics for understanding in that environment?
- How do we address issues of tracking/ability grouping and in particular the grouping of students by test designation?

Plan for Working Group

The overarching goal of the group continues to be to facilitate collaboration within the growing community of scholars and practitioners concerned with understanding and addressing the challenges of attending to issues of equity and diversity in mathematics education. The PME Working Group provides an important forum for these scholars to come together with other interested researchers who identify their work as attending to equity and diversity issues within mathematics education in order to develop plans for future research. Our main goal for this year, then, is to continue a sustained collaboration around key issues (theoretical and methodological) related to research design and analysis in studies attending to issues of equity and diversity in mathematics education.

In order to support this collaborative research, smaller research groups were formed from participants in the large Working Group. The plans and goals for these sub-groups are detailed in the final section: Previous Work of the Group and Anticipated Follow-up Activities.

Much of our work is qualitative in nature and we recognize that one way to increase the number of participants is to conduct research across several sites. In order to do this, we need to use the same protocols for data gathering. One of our sub-groups has taken as its charge to develop an observation protocol. We intend that, during what we hope to be the long life of this Working Group, this and other research protocols may be developed and used across a variety of research projects. Although this was not a topic in 2009, it is anticipated that one aspect of the sub-group meetings will be to discuss potential funding opportunities. Here we may identify and begin to draft grant proposals to fund research across contexts. More specifically, for PME 2010 we will proceed as follows.

SESSION 1:

- Presentation and discussion of goals of Working Group.
- Introduction of participants.
- Reports of the work of sub-groups during 2009-2010.
- Adjustment (addition or collapsing) of current sub-groups as appropriate.

SESSION 2:

- Sub-group meetings to discuss and delineate further plans to address and expand research questions developed at PME 2009.

SESSION 3:

- Sharing and discussion of work from Session 2.
- Planning for further collaboration.
- Developing a tentative agenda for future Working Group meetings.

Previous Work of the Group and Anticipated Follow-up Activities

The Working Group met for three productive sessions at PME 2009. In the first session, we identified areas of interest to the participants within the broad area of equity and diversity issues in mathematics education. Much fruitful discussion was had as areas were identified and examined. During the second session, participants met in an identified sub-group of their choice. Within these sub-groups, rich conversations ensued regarding theoretical and practical considerations of the topics. Sub-groups began to identify plans and goals for the 2009-2010 academic year. During the third session discussions continued, sub-groups finalized their plans, and then reported to the large Working Group. What follows are the plans and goals of the sub-groups formed at PME 2009. The activities stated in these plans are currently in progress.

Pre-service Teacher Education that Frames Mathematics Education as a Social and Political Activity

This sub-group acknowledges tensions in our work focusing on equity and social justice in relationship to reform mathematics. Frameworks are needed to understand these issues. These can build on work in culturally relevant pedagogy (Ladson-Billings, 1995), teaching for social justice (Gutstein, 2003), funds-of-knowledge (González et al., 2001) as well as more general issues of equity, diversity, social analysis, and critical pedagogy. We need to begin by defining what we mean by these terms (e.g. reform mathematics, social activity, political activity); and how we recognize them in the classroom. This group would like to develop a collection of materials, resources, and activities that are being used in Math Methods courses. Another aspect of this work may be studying our own professional development as mathematics teacher educators oriented toward incorporating equity issues centrally in our Math Methods courses.

Goal for 2009-2010. We intend to design a common lesson and teach it in varied contexts (different universities, different grade levels of methods, alternative certification teachers that are also pre-service teachers).

Culturally Relevant and Responsive Mathematics Education (CRRME)

The title reflects the fact that the responsive piece is a bit different from the relevant piece and that we want the mathematics piece included. We are interested in language, discourse, ethnicity, ways of interacting, family, community, experiences, generational issues, expectations (not high and low, but individual or community’s expectations). We want to examine what we mean by social justice. Issues can be examined, such as teaching (a) about social justice (the context), (b) with social justice (status and participation), and (c) for social justice (power and question). Various aspects of CRRME include (a) local contexts, (b) local associations, (c) using cultural referents, (d) ethnomathematics, (e) critical pedagogy, and (f) teaching “classical” math. We are concerned as well with how literature on culturally relevant pedagogy is grounded in existing theory and research on culture and social constructivism.

Goal for 2009-2010. Since many in the mathematics education community are not familiar with what social scientists say about culture, we intend to review that literature with a focus on how it relates to research in mathematics education.

Creating Observation Protocols around Instructional Practices

This group hopes to develop a protocol that can measure instructional practice AND be a tool to help teachers improve their instructional practice. Our focus is on the importance of improving instruction for students of color; this is our goal. We recognize that protocols have limits (and dangers!). For example, protocols do not necessarily look at microgenesis, teacher change, structural issues, dispositions.

Goal for 2009-2010. We plan to have at least three conversations together as a group over the next year. During these conversations, we will (a) discuss existing protocols posted on google groups to stimulate rich discussions around questions such as: What do we like? What are they missing? How might we combine, extend, and so forth; (b) move conversation to issues of teacher change/microgenesis; (c) discuss what it means to develop this protocol and train people in its use as we don't want people to take any tool we develop and usurp it, using it in ways detrimental to teachers; and (d) discuss the importance of theoretical grounding of this work. Our longer-term goal is to produce a paper or report that is the theoretical grounding of an observation tool that does not ignore issues of race, class, and so forth.

Feminist Theory Literature Group:

This will be a reading group. Currently feminist theory is under-utilized in equity research.

Goal for 2009-2010. We plan to create a bibliography and report next year to this group about how feminist theory can be used to inform equity research.

Language and Discourse Group: Issues around Supporting Mathematical Discourse in Linguistically Diverse Classrooms

This will begin as a group reading widely in the area of language and discourse.

Goal for 2009-2010. We intend to begin reading widely in the area of language and discourse and create a bibliography that can serve as a resource.

Plans for 2009-2010

It is anticipated that Working Group participants will leave the conference ready to (a) continue to (and in some cases begin to) work on developing particular aspects of their research (such as particular data gathering methods) and/or (b) continue (and again in some cases to begin) a collaboration with other Working Group participants. It is our hope that the work of this group that began at the PME Conference in Atlanta in 2009 will continue in Columbus, Ohio in 2010 and for many years after.

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BUILDING ON STUDENTS' CURRENT WAYS OF REASONING TO DEVELOP MORE FORMAL MATHEMATICS IN LINEAR EQUATIONS

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This working group will bring together researchers interested in investigating learning and teaching mathematics that seeks to build on students' current ways of reasoning to develop more formal and robust ways of thinking in the content domain of linear algebra. Linear algebra represents a content domain that is a critical juncture for students as they transition to more abstract courses and that is of interest to the broader community. The main activities of the working group will be (a) reflection on students' current ways of reasoning and how these ways of reasoning progress; (b) theoretical elaboration on what such progress entails, from both psychological and sociological points of view; and (c) explicit attention to issues of instructional design. In broad terms, the sessions will allow time for participants to work through some of the mathematical ideas for themselves, to discuss core theoretical underpinnings on cognition and instructional design, to examine instructional tasks, and to discuss video of student work on these tasks.

A prominent problem in the teaching and learning of K-16 mathematics is how to build on students' current ways of reasoning to develop more conventional, generalizable, and abstract ways of reasoning. This problem is particularly pressing in undergraduate courses that often serve as a transitional point for students as they attempt to progress from more computationally based courses to more abstract courses that feature reasoning with formal definitions and proof construction. Such transitional courses can block students from continuing to pursue a science, technology, engineering or mathematics (STEM) major. It is well documented that the number of students in STEM majors in general, and mathematics majors in particular, is decreasing, despite the fact that a greater percentage of the population is attending college (Bressoud, 2009; Holton, 2001). This problematic trend is not unique to the United States.

This working group will bring together researchers interested in investigating learning and teaching that seeks to build on students' current ways of reasoning to develop more formal and robust mathematics. The main activities of the working group will be (a) reflection on students' current ways of reasoning and how these ways of reasoning progress, (b) theoretical elaboration on what such progress entails, from both psychological and sociological points of view, and (c) explicit attention to issues of instructional design. Progress on these three fronts requires a content domain that is of interest to the broader community and that is a critical juncture for students as they transition to more abstract courses. In the following section we explicate why linear algebra is an appropriate content domain for the working group. This is followed by a discussion of key theoretical perspectives on learning and instructional design that will likely inform the working group. The final three sections detail the plan for the active engagement of participants, anticipated follow-up activities, and ways in which the working group builds on and extends prior collaborations.

Why Linear Algebra?

Linear algebra is an especially relevant site for the working group for several reasons. First, it is a conceptually rich domain in which concepts and procedures are intimately connected. The unifying power of linear algebra is particularly elegant and beautiful, and offers students a glimpse into what often excites and motivates mathematicians. Second, there are many opportunities for working with formal definitions and abstractions. These opportunities also present challenges to students as they attempt to relate their current ways of thinking to more formal and abstract mathematics. Third, there is a plethora of applications of linear algebra to science, engineering, and economics. As such there are many opportunities to use these applications to motivate and develop mathematical ideas. Fourth, linear algebra is a complex domain for interpreting and creating various symbolic and graphical representations. Hence there are many opportunities to study how students mathematize their current ways of reasoning. Finally, linear algebra is situated between courses with more of a problem solving or computational emphasis, such as calculus, and courses with a primarily proof-based emphasis such as abstract algebra and real analysis. All of these reasons point to the fact that linear algebra is a critical and often difficult juncture in students' university studies.

Given the complex and difficult nature of this content domain, there has been and continues to be fairly broad interest, both nationally and internationally, in improving the teaching and learning of linear algebra. A brief review of this prior work is in order because it lays the foundation for new work that builds on contemporary learning theories and instructional design perspectives.

In the late 1990's, research in linear algebra conducted by Harel and Sowder (1998) in the United States resulted in the articulation of a trajectory that characterizes the evolution of how students understand and use proofs. Harel (2000) also proposed the following three principles for learning: the concreteness principle, the necessity principle, and the generalizability principle. These principles emphasize the need for curriculum that (among other things) begins with situations that are sensible to students and can be generalized to the important abstract or more formal constructs of linear algebra.

In Canada, Sierpinska (2000) noted the difference in what she calls theoretical thinking and practical thinking. This difference in thinking is related to the distinction between students working from their concept image (practical thinking) versus working from a concept definition (theoretical thinking). This distinction is central to students' transition from their current ways to reasoning to more formal or abstract ways of reasoning. In related work, Hillel (2000) described three languages (representations or modes) in linear algebra that students must coordinate: the abstract language of the general theory, the algebraic language of R^n, and the geometric language of two and three-dimensional spaces. The necessary coordination of all three modes of reasoning is one reason why linear algebra is often such a challenging course for students.

In France, Dorier, Robert, Robinet and Rogalski (2000) have noted that there appears to be an insurmountable *obstacle to formalism* inherent in the learning of linear algebra. Dorier (2002) summarized this obstacle as follows:

> *Students' difficulties with the formal aspect of the theory of vector spaces are not just a general problem with formalism but mostly a difficulty of understanding the specific use of formalism within the theory of vector spaces and the interpretation of the formal concepts in relation with more intuitive contexts like geometry or systems of linear equations, in which they historically emerged. (p. 877)*

There has been considerable activity on the part of French researchers in developing a program for the linear algebra course that occurs during the first year at French universities for science majors. In the United States and Mexico this course occurs later in the university program and these students do not have the same secondary school experience with proof that French students do. In Mexico there is also a research project to understand students' construction of Linear Algebra concepts (Trigueros & Oktaç, 2005; Kú, Trigueros & Oktaç, 2008).

In addition to research that either categorizes what students struggle with doing or explains what an expert thinks is necessary to understand an idea, more recent work has examined the productive and creative ways that students are able to interact with the ideas of linear algebra. In Mexico, Possani, Trigueros, Preciado, and Lozano (2010) analyzed the use of a teaching sequence that started with a real life problem. The authors reported on student progress as they advanced through different solution strategies. For instance, they found that students learn what they are supposed to learn and they can even do more than what is normally expected from them when given the opportunity. In a similar spirit, Larson, Zandieh, and Rasmussen (2008) reported on ways that students were using their current ways of reasoning about vectors and matrices to develop algebraic methods to determine the eigenvectors and eigenvalues of a given matrix. In particular, the authors present a key idea that emerged as a central and powerful way in which students came to reason and eventually develop the formal ideas and procedures for eigenvalues and eigenvectors. These two lines of more recent work in linear algebra are particularly relevant to the charge of the working group because rather than detailing student difficulties, they detail progress in building on students' current ways of reasoning to develop more formal and conventional mathematics. In the next section the theoretical foundations for these two lines of work are detailed.

Theoretical Foundations

This section describes several different ways to characterize student progression from their current ways of reasoning to more formal ways of reasoning as well as the perspectives underlying these characterizations. The way in which researchers operationalize this transition is essential because these underlying perspectives bring forth possibly different aspects of the same phenomenon. Part of the goal of the working group is to clarify and contrast these underlying perspectives in order to sort out points of real difference and real similarity (although different terminology might be used). The following paragraphs identify a few of the more prominent approaches to characterizing the progression from students' current ways of reasoning to more formal ways of reasoning.

The term “concept definition” refers to the formal definition whereas “concept image” refers to the “set of all mental pictures associated in the students’ mind with the concept name, together with all the properties characterizing them” (Vinner & Dreyfus, 1989, p. 356). A student’s concept image is typically the result of his or her experiences with examples and nonexamples. One strand of studies within this theme details how students often reason from their concept image even though they “know” the definition. Paradigmatic of this type of study is the analysis reported by Vinner and Dreyfus (1989) in which students stated essentially correct definitions for function but failed to use the definition in their analysis of whether a given graph/equation/table is or is not an example of a function. Instead, students used their concept image, often relying on a prototypical example in order to decide whether the given example is or is not an example of a function. Similarly, Edwards (1997) found that although real analysis students knew the formal definition of continuity, many tended to reason from their concept image, and Moore (1994) found that even though undergraduate mathematics majors in a transition to proof course knew a formal definition for a one-to-one function, they had difficulty using the definition to structure or organize a proof that a given function is one-to-one.

These studies provide examples of a framework that was developed from an individual cognitive point of view and is typically used to compare student current concepts relative to more formal mathematical definitions. The definition game, described next, can be thought of in terms of individuals, but is oriented more toward students’ mathematical activity than their internal cognitive structures. In relation to the theme of the working group, the definition game characterizes a type of interplay between student current ways of reasoning and their introduction to new, formal mathematical definitions or settings. The definition game takes a variety of forms, including using an *unfamiliar* definition in a *familiar* mathematical setting and using a *familiar* definition in an *unfamiliar* mathematical setting. More generally, playing the definition game consists of using a ________ definition in a ________ mathematical setting, where either blank can be filled in with adjective along the unfamiliar to familiar continuum (Zandieh & Rasmussen, in press).

For example, Dahlberg and Housman (1997) investigated the strategies senior and junior undergraduate mathematics majors used when presented with a new definition in a relatively familiar setting. In particular, students were presented with the definition of a fine function (a function with a root at each integer), something that none of the students had previously encountered, and were asked to generate examples of fine functions, to decide whether given functions were fine, and to determine the validity of different statements about fine functions.

In contrast to Dahlberg and Housman’s study in which students used an unfamiliar definition in a relatively familiar mathematical setting, Borasi (1991) described a lesson with two secondary school students where they used a familiar definition of circle in an unfamiliar mathematical setting. The unfamiliar mathematical setting was that of taxicab geometry, which is an idealization of an urban area with a regular grid pattern of streets. Although street grids were familiar to these students, this environment was *mathematically* unfamiliar to students due to the way distances are measured.

The Action-Process-Object-Schema (APOS) theory (Asiala et al., 1996; Dubinsky, 1991) offers a particular cognitive framing for how cognitive structures evolve as students progress from their current conceptions to more formal conceptions. In this perspective, an action conception is a transformation of a mathematical object by individuals, according to an explicit algorithm that is conceived as externally driven. As individuals reflect on their actions, they can *interiorize* them into a process. Each step of a transformation may be described or reflected upon without actually performing it. An object conception is constructed when a person reflects on actions applied to a particular process and becomes aware of the process as a totality, or *encapsulates* it. A mathematical schema is considered to be a collection of action, process and object conceptions, and other previously constructed schemas, which are synthesized to form mathematical structures utilized in problem situations (Baker, Cooley, & Trigueros, 2000). These schemas evolve as relations between new and previous action, process, and object conceptions and other schemas are constructed and reconstructed. Their evolution may be described by three stages that Piaget and García (1983) refer to as the “triad.” At the general *intra*- stage some operational actions are possible, but there is an absence of relationships between properties. At the *inter*- stage, the identification of relations between different processes and objects, and transformations are starting to form but remain isolated. The *trans*- stage is defined in terms of the construction of a synthesis between relations to form a structure (Cooley, Trigueros, & Baker, 2007). Also, although it might be thought that in APOS theory there is a linear progression from action to process to object and then to having different actions, processes, and objects organized in schemas, this often appears more like a dialectical progression where there can be partial developments, passages and returns from one to other conception (Czarnocha, Dubinsky, Prabhu, & Vidakovic, 1999). What the theory states is that the way a student works with diverse mathematical tasks related to the concept is different depending on his or her conception.

The application of APOS theory to describe particular constructions by students requires researchers to develop a genetic decomposition—a description of specific mental constructions one may make in understanding mathematical concepts and their relationships. Some clarifications are pertinent. A genetic decomposition for a concept is not unique: it is a general model about how a concept may be constructed. Different researchers can develop diverse genetic decompositions of how students in general construct that particular concept, but, once one is proposed, it needs to be supported by research data from students. Frequently, gathered data reveal overlooked constructions, which then give rise to a revised genetic decomposition.

In comparison to perspectives that mainly account for an individual’s cognitive structures or activity, some work has focused on constructs that allow one to track collective discipline specific practices that begin with students’ current reasoning and move toward more formal reasoning. For example, Rasmussen, Zandieh, King and Teppo (2005) tracks the progression of the practices of symbolizing, algorithmatizing and defining across such a continuum by elaborating and extending of the framework of horizontal and vertical mathematizing (Treffers, 1987). Zandieh and Rasmussen (in press) detail more thoroughly the case of defining, framing the analysis within the theory of Realistic Mathematics Education (RME). In particular, these researchers adapted the RME heuristic of emergent models (Gravemeijer, 1999) as a means to organize and reflect on a significant learning experience that involved creating and using definitions.

The intention of the emergent model heuristic is to create a sequence of tasks in which students first develop *models-of* their mathematical activity, which later become *models-for* more sophisticated mathematical reasoning (Gravemeijer, 1999). In a similar way, Zandieh and Rasmussen (in press) demonstrate that definitions can first come to the fore as a *definition-of* students' previous activity and later these definitions serve as tools for (that is, *definitions-for*) further mathematical reasoning. The shift from model-of to model-for (or definitions-of to definitions-for) is compatible with Sfard’s (1991) process of reification. While the literature offers several examples of emergent models, the precise meaning of the term “model” is left implicit. To add clarity to the field, Zandieh and Rasmussen (in press) define models as student generated ways of organizing their activity with observable and mental tools. By observable tools they mean things in their environment, such as graphs, diagrams, explicitly stated definitions, physical objects, etc. By mental tools they mean the ways in which students think and reason as they solve problems – their mental organizing activity.

As described by Gravemeijer, Bowers and Stephan (2003), the model-of / model-for transition is reserved for a significant conceptual transition in student activity, where activity includes mental activity. Otherwise, each act of symbolizing would be labeled as a model-of / model-for transition and hence lose analytic power. The model-of / model-for transition is therefore concurrent with the creation of a new mathematical reality.

The models and modeling perspective, on the other hand, focuses on the development of conceptual tools which are useful in decision making. Researchers working from this perspective (Kelly & Lesh, 2001; Lesh & Doerr, 2003, Lesh & English, 2005) have developed criteria that the problems to be posed to the students must satisfy in order to be successfully applied in the classroom to contribute to the learning process of students. Here, the main idea in modeling consists of introducing realistic complex situations where students engage in mathematical thinking and generating complex products and conceptual tools to accomplish the intended goal. These products are constructed during cycles of work and reflection and can be, in each cycle, self-evaluated by students.

Plan for Active Engagement of Participants

In this section we describe the specific tasks we expect to accomplish during the three working group sessions. In broad terms, the sessions will allow time for participants to work through some of the mathematical ideas for themselves, to discuss core theoretical underpinnings on cognition and instructional design, to examine instructional tasks, and to discuss video of student work on these tasks.

Session One

A brief presentation of the goals of the working group will be presented. Two different linear algebra problems will be presented together with the following questions to be discussed:

1. What linear algebra concepts do you believe can be introduced with the problem?
2. What strategies do you anticipate will motivate students to start working on these problems?
3. What kind of activities or questions would you introduce to help students move their strategies forward?

Participants will be split into small groups, and one of these two problems will be assigned to each group so that half of the participants work on one problem and the other half work on the other problem. After a period of discussion in small groups, the whole group will discuss the answers to the questions. As a closure to this session we will talk briefly about what will follow in the next sessions.

Session Two

After a brief introduction to the session we will go back to the problems and present the results of the discussion from the previous session and present the activity for the present session. Results from actual student work with the two problems introduced in the first session will be presented to the participants; these can be parts of video recording or written work. Participants will split again to work in groups and work with the data to answer the following questions:

1. What did students do and how does it compare with what we expected them to do?
2. How would you analyze these data?
3. Are there specific results you can describe from your analysis?
4. Write a summary of your findings.

It is important to note that work from students used in this part will be bilingual. The group will reunite to discuss the answers to these questions. Groups that worked with one problem will present their conclusions to those working with the other problem and then the same will be done by the groups that worked with the other problem. As we anticipate answers to the posed questions will depend on the selected theoretical framework chosen by different groups, we will close the session by talking about the focus of the third session, theoretical frameworks.

Session Three

Organizers will start the session by summarizing work from the previous sessions.
The organizers will present and discuss some results they found for the same problems using the theoretical lens each group used. The group will split into small groups to discuss the possibilities for the application of different theoretical frameworks to research on models, in particular models related to linear algebra. Discussion can be focused by the questions:

1. What restrictions on the way problems are posed are imposed by different theoretical frameworks?
2. What specific contributions to the analysis of student modeling activities can different theoretical approaches make?
3. What limitations can be expected from the use of different theoretical frameworks?

Each group will write down their answers to the questions to be discussed by the whole group. The group will reunite to discuss the answers to the questions and possibilities for future work.

Anticipated Follow-up Activities

We anticipate three follow-up activities: a wiki for further interaction throughout the year, a special issue in a journal on the group's theme, and a second working group in 2011. First, we will create a wiki for the purposes of the dissemination of ideas and further communication amongst members of the working group. The wiki will be an interactive community centered on the activities of the working group, housing tasks that are presented at the working group and possibly other tasks from interested parties. Accompanying these tasks will be teaching notes and materials designed to help teachers and researchers to better implement the tasks in their classrooms. Those who use the tasks will have the opportunity to post their thoughts and reactions to the tasks. The intention is to provide a forum for participants to refine their understanding of student thinking about linear algebra, how the tasks are able to elicit and refine student thinking, and how student' transition from informal, intuitive ways of thinking to formal understandings.

Second, we will seek to create a special journal issue focusing on the theme of transitioning from students' current ways of thinking about mathematics and mathematical situations towards more formal ways of reasoning in mathematics. While the issue will be primarily centered on instructional design and student thinking in linear algebra, there will also be opportunities for other authors to include work in other content areas, given that the work is consistent with the theme of the working group. In conjunction with the international makeup of this working group's organizers, the journal issue will be intended for an international audience and hence could be written in Spanish as well as English in order to reach a wider community of mathematics education professionals.

Third, we plan to have another PME-NA working group in 2011 in order to continue the work started in 2010 and capitalize on the activity that the wiki instigates over the course of the year. A new working group in 2011 will allow members of the 2010 working group to congregate and continue their work on the themes of the original group. Furthermore, we anticipate that new issues will arise as a result of conversations and work done on the wiki. The 2011 group will allow the participants to address these issues and discuss how to further connecting students' current ways of reasoning to the more conventional, formal ways of reasoning in mathematics. Finally, the 2011 working group will be an opportunity for people not already a part of the group to take part in our ongoing activities. As this group matures, we would make our work available via the Linear Algebra Education section of the International Linear Algebra Society website.

Prior Work and Collaborations

The proposed working group builds on and extends the working group proposers' previous collaborative efforts. The collaborative work of the USA researchers stems from their involvement in an NSF-funded project that investigates the transition from informal to formal reasoning of students within the context of undergraduate linear algebra. Under the direction of Chris Rasmussen and Michelle Zandieh, this project is composed of two main research strands, one of which is a linear algebra stand. Within this strand, there are three primary goals: (a) to detail student conceptions of fundamental ideas, (b) to trace students' intellectual growth from informal to more formal ways of reasoning, and (c) to create instructional sequences that support students' growth.

The Mexican research team, funded by the Consejo Nacional de Ciencia y Tecnología, is also investigating the teaching and learning of linear algebra. The goal of their project is the design of modeling activities to teach linear algebra concepts in an application context. The activities are designed so that students can discuss and reflect not only on the potential of application of the concepts involved in each activity, but also their properties and the relation with other concepts. The intention of the project is to promote significant learning of the abstract concepts included in a Linear Algebra course. The modeling activities are being used with different groups of students and research is being conducted on the way students use mathematics they already know, the strategies they follow when working with the activities, and how this leads to more formal and conventional mathematics.

In 2008, Maria Trigueros and Asuman Oktaç organized a symposium focused on linear algebra education at the 15th Conference of the International Linear Algebra Society in Cancun, Mexico. They alerted the US team to the conference and encouraged them to apply to present their work at the symposium. As a result, members from both the Mexican and US team presented their work at this conference (Possani, Preciado, & Lozano, 2008; Trigueros, Ku, &

Oktaç, 2008; Zandieh, Larson, & Rasmussen, 2008). It was at this meeting that the two research teams became aware of their similar interests in the use of modeling problems in order to investigate ways to assist students in transitioning from their current ways of reasoning to more formal or conventional ways of reasoning in linear algebra. In particular, the American researchers have been designing, implementing, and studying a curriculum for an undergraduate linear algebra course that is based on student inquiry and conceptual development of key ideas, and the researchers at the Instituto Tecnológico Autónomo de México have had a similar ongoing project.

Since meeting in Cancun, the researchers have conducted regular meetings via Internet video and audio chats during which they share ideas, results, and innovations for task development and implementation in their respective linear algebra classrooms. They have convened a face-to-face meeting at the Instituto Tecnológico Autónomo de México in Mexico City on November 9-11, 2009. At this meeting, the two research teams worked on various modeling tasks that have been used in their respective linear algebra classrooms, shared results of how these tasks functioned in the classrooms, and collaborated towards refinement of these problems. The bringing together and the merging of two research teams that are working from different yet compatible theoretical perspectives have proven invaluable to both in the past. Building upon this foundation at the working group, where additional experienced researchers in mathematics education will have the opportunity to participate and contribute, will be beneficial for all involved.

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CONTINUING DISCUSSION OF MATHEMATICAL HABITS OF MIND

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The idea of "mathematical habits of mind" has been introduced to emphasize the need to help students think about mathematics "the way mathematicians do." There seems to be considerable interest among mathematics educators and mathematicians in helping students develop mathematical habits of mind. The objectives of this working group are: (a) to continue the discussion of various views and aspects of mathematical habits of mind begun at PME-NA 31, (b) to explore avenues for research, (c) to encourage research collaborations, and (d) to interest doctoral students in this topic. In the Proceedings of PME-NA 31, we provided an overview of mathematical habits of mind, including concepts that are closely related to habits of mind—ways of thinking, mathematical practices, knowing-to act in the moment, cognitive disposition, and behavioral schemas. Below we provide a summary of the discussions held at PME-NA 31. We invite returning participants, as well as other mathematics educators who are interested in mathematical habits of mind, especially those who have conducted research related to habits of mind, to our discussions.

An Overview of Mathematical Habits of Mind

The term *habits of mind* was introduced by Cuoco, Goldenberg, and Mark (1996) as an organizing principle for mathematics curricula in which high-school students and college students think about mathematics the way mathematicians do. Lim and Selden (2009) highlighted two key attributes of habits of mind: the *habitual* characteristic and the *thinking* characteristic. The habitual nature of habits of mind was underscored in Goldenberg's (1996) description of habits of mind, which "one acquires so well, makes so natural, and incorporates so fully into one's repertoire, that they become mental habits—one not only can draw upon them easily, but one is likely to do so" (p. 13). Mason and Spence's (1999) notion of *knowing-to act in the moment* accentuates this habitual character. They have differentiated between two types of knowledge. The first type, referred to as *knowing-about*, consists of Ryle's (1949, cited in Mason & Spence) three classes of knowledge: *knowing-that* (factual knowledge), *knowing-how* (procedural skills), and *knowing-why* (personal stories to account for phenomena). The second type, referred to as *knowing-to*, is tacit knowledge that is context/situation dependent and becomes present in the moment when it is required. This distinction is important because "knowing to act when the moment comes requires more than having accumulated knowledge-about . . ." (Mason & Spence, 1999, p. 135).

> *Knowing-about … forms the heart of institutionalized education: students can learn and be tested on it. But success in examinations gives little indication of whether that knowledge can be used or called upon when required, which is the essence of knowing-to* (Mason & Spence, p. 138).

Mason and Spence advocate the practice of reflection as a means to help students improve their knowing-to act in the moment. Students should be encouraged to reflect on (a) what they have done after an action, and (b) what they are doing while enacting it, which were termed by

Schön (1983) *reflection-on-action* and *reflection-in-action* respectively. With respect to reflection-in-action, students should routinely ask themselves “What do I know?” and “What do I want?” (Mason & Spence, p. 154).

The thinking characteristic differentiates habits of mind from behavioral habits such as knuckle cracking and nail biting. Costa and Kallick (2000) identified sixteen habits of mind that can and should be cultivated in schools. Habits of minds in Costa and Kallick’s list that are related to mathematical thinking and learning include persisting, managing impulsivity, thinking flexibly, metacognition, striving for accuracy, and thinking and communicating with clarity and precision. Cuoco (1996) differentiates mathematical habits of mind, such as talking big thinking small, talking small thinking big, thinking in terms of functions, and mixing deduction and experiment, from more general ones like pattern-sniffing, experimenting, formulating, tinkering, inventing, visualizing, and conjecturing. Harel’s (2008) distinction between ways of thinking and ways of understanding highlights two complementary subsets of mathematics: the former refers to conceptual tools or mathematical habits of mind that are used for creating the latter, which refers to collections of institutionalized definitions, theorems, proofs, problems, and solutions. These two aspects of mathematics are analogous to the two types of standards—process and content—outlined in the NCTM’s (2000) *Principles and Standards for School Mathematics*.

According to Levasseur and Cuoco (2003), mathematical habits of mind should not “be the explicit objects of our teaching, rather, each student should internalize them as they do math” (p. 34). Mathematical habits of mind can be fostered by providing students opportunities to engage in authentic mathematical activities such as modeling and realistic problem solving (see Lesh & Doerr, 2003; Schoenfeld, 1985). The National Council of Teachers of Mathematics published two books on *Teaching Mathematics through Problem Solving*; one for Grades PreK-6 and the other for Grades 6-12. There is a chapter on mathematical habits of mind in each book: (a) Goldenberg, Shteingold, and Feurzeig (2003) discuss five habits of mind that are particularly relevant to elementary grade levels; they include thinking about word meaning, justifying claims and proving conjectures, distinguishing between agreement and logical necessity, analyzing answers, problems, and methods, and seeking and using heuristics to solve problems; and (b) Levasseur and Cuoco (2003) discuss how secondary school students can acquire habits of mind such as guessing, challenging solutions, looking for patterns, conserving memory, using alternative representations, thinking algebraically, and classifying carefully. Driscoll and colleagues (1999, 2007) have created professional development materials to foster algebraic thinking and geometric thinking. Their goal is to help teachers and their students develop algebraic habits of mind such as doing/undoing, building rules to represent functions, abstracting from computation, reasoning with relationships, generalizing geometric ideas, investigating invariants, and sustaining reasoned exploration by trying different approaches and stepping back to reflect. Lewis (2008, January) commented that their professional development project was designed to help teachers develop “habits of mind of a mathematical thinker” and in turn foster these habits in their students. Rasmussen (2009, January) emphasized the need for teachers to be deliberate about initiating and sustaining particular classroom norms so as to promote certain desirable habits of mind and effect students’ beliefs and values.

The *Standards for Mathematical Practice* in the Common Core State Standards in Mathematics (CCSSI, 2010) highlight the following habits of minds that mathematics educators at all levels should seek to develop in their students:

- Make sense of problems and persevere in solving them
- Reason abstractly and quantitatively
- Construct viable arguments and critique the reasoning of others
- Model with mathematics
- Use appropriate tools strategically
- Attend to precision
- Look for and make use of structure
- Look for and express regularity in repeated reasoning

A collection of lists of habits of mind from various sources, including CCSSI (2010), can be downloaded from http://works.bepress.com/kien_lim/19/. More information on mathematical habits of mind can be found in the Mathematical Habits of Mind Working Group paper in the *Proceedings of PME-NA 31* (Lim & Selden, 2009) and on the website: http://www.math.utep.edu/Faculty/kienlim/mhom.

The working group on mathematical habits of mind met for the first time in September 2009 at PME-NA 31. There were three meetings. We began with individual presentations on research related to habits of mind. We then had an open forum to discuss theoretical and pedagogical issues related to this topic followed by small subgroup breakout sessions. We concluded the meeting by having reports from the various subgroups. To facilitate the continuation of this working group at PME-NA 32, we summarize below the information gathered during the previous working group sessions.

Individual Presentations at the PME-NA 31 Working Group

The working group began with six 10-minute presentations. Below are the title, name of the presenter(s), and a short summary for each presentation, listed in the order they were presented.

Mathematical Habits of Mind for Preservice Elementary Teachers
Richard S. Millman, Georgia Institute of Technology

The concept of mathematical habits of mind urges preservice teachers to use, in their teaching, ideas such as posing questions (e.g. “Is there a different way to think about this problem?” and “What is there that I am not seeing?”), seeking possible generalizations, considering the necessity and use of careful definitions, and dealing with the vagueness of open-ended questions. These ideas are aimed at having preservice elementary teachers understand how mathematicians might think. In the presentation, Millman described how mathematical habits of mind could be introduced in a mathematics content course for future elementary teachers through their inclusion in the textbook, *Mathematical Reasoning for Elementary Teachers,5th Edition* (Long, DeTemple, & Millman, 2009). He provided some reactions of pre-service teachers, as well as those of eleven reviewers of the text (six of whom used the textbook and five of whom didn’t), concerning the inclusion and emphasis of mathematical habits of mind in the textbook.

Undesirable Habits of Mind of Preservice Teachers
Kien H. Lim, University of Texas at El Paso

Many preservice K-8 teachers enter college with undesirable habits of mind such as (a) spontaneously proceeding with the first action that comes to mind without analyzing the problem situation, and (b) not attending to meaning of numbers and symbols. Such habits of mind can negatively impact what and how they learn mathematics. For example, students often tend to focus on procedures for solving problems rather than on underlying mathematical structures. In the presentation, Lim offered several strategies to help prospective teachers address their undesirable habits of mind. One strategy is to pose problems for which a recently learned idea will not work, and thereby present students with a need to investigate the principles that underlie that idea. Another strategy is to emphasize the need for understanding the quantities embedded in a problem situation and how these quantities are related. A study was conducted to investigate the viability of using nonproportional missing-value problems to address students' tendency to overgeneralize proportional approaches (see Lim & Morera, 2010). An instrument, called the likelihood-to-act survey, has been developed to assess students' problem-solving disposition along the impulsive-analytic disposition (see Lim, Morera, & Tchoshanov, 2009).

Transforming pre-service teachers' dispositions towards mathematics through reflection in-activity and post-activity

Dionne I. Cross, Indiana University-Bloomington

Mathematics education is still undergoing a transition from a transmission view of instruction to one that involves students actively engaging in "doing" mathematics. One way to address this problem is by working with pre-service teachers to begin transforming their ideas about mathematics and mathematics learning. Using examples, the transformation of the teachers' approaches to non-traditional problems were described—*from* initially being unable to solve problems that do not have a singular solution or problems that prioritized thinking and reasoning *to* making generalizations and aligning algebraic notation with specific aspects of a problem. Students' development of these habits of mind (Driscoll, 1999) were attributed to focused efforts to teach students (a) how to engage in reflection both during an activity and following the activity (Schön, 1983), and (b) how to articulate their thoughts both verbally and in writing.

Habits of Mind in the Proving Process

Annie Selden & John Selden, New Mexico State University

The Seldens view the proving process as a sequence of mental or physical actions that cannot be fully reconstructed from the final written proof. Such actions often appear to be due to the enactment of small, automated situation-action pairs that they call *behavioral schemas* (Selden & Selden, 2008; Selden, McKee, & Selden, 2010). A common beneficial behavioral schema consists of a situation where one has to prove a universally quantified statement like, "For all real numbers x, P(x)" and the action is writing into the proof something like, "Let x be a real number," meaning x is arbitrary but fixed. Focusing on such behavioral schemas, that is., small habits of mind, has two advantages. First, the uses, interactions, and origins of behavioral schemas are relatively easy to examine. Second, this perspective is not only explanatory but also suggests concrete teaching actions, such as the use of practice to encourage the formation of beneficial schemas and the elimination of detrimental ones.

Mathematics Immersion and Educators' Habits of Mind: Preliminary Results from Two Programs

Karen Graham & Todd Abel, University of New Hampshire

Mathematics immersion is a form professional development where educators are encouraged to work through unfamiliar mathematical content in ways that simulate the activities and practices of mathematicians. Graham and Abel briefly described two such programs that they have examined. The first program was a one-week summer institute, participated in by 18 faculty members and 7 graduate students. Analysis of participant journals and follow-up surveys uncovered the following themes: (a) freedom to experiment, conjecture, and guess, (b) value in using multiple points of view, and (c) joy in doing mathematics. The second was a two-summer professional development program participated in by 50 high-school and middle-school teachers. Analysis of pre-and-post interviews with six participants revealed the following themes: (a) the motivational role of pattern-sniffing, (b) the importance of mixing deduction and experiment, and (c) the importance of classroom practice (e.g., the usefulness of activities for classroom work). Mathematics immersion increased participants' awareness of, but did seem to impact participants' use of, habits of mind.

Richard Lesh, Indiana University-Bloomington

Do students develop rigid and unchanging profiles of habits, dispositions, and attitudes? Or, do productive problem solvers manipulate their own profiles to suit circumstances? Evidence was presented to show that (a) productive-but-implicitly-functioning habits of mind can be developed using reflection activities similar to those used by athletes and performing artists; (b) students can develop more powerful ways of seeing (or interpreting) their own problem solving experiences; (c) both learning and application of ideas and processes develop synchronously during mathematical model-development activities; and (d) the productivity of relevant processes, beliefs, dispositions, and habits of mind vary across time. Productive students can learn to manipulate their own profiles to suite circumstances. This research is based on models and modeling perspectives of mathematical problem solving, learning, and teaching

PowerPoint slides or write-ups for the above presentations can be downloaded at http://www.math.utep.edu/Faculty/kienlim/mhom. In addition, other presentations on this topic at the 2008 and the 2009 Joint Mathematics Meetings are posted at http://www.math.utep.edu/Faculty/kienlim/hom. Presenters at those sessions included Hyman Bass, Al Cuoco, Paul Goldenberg, Guershon Harel, Kien Lim, Chris Rasmussen, Annie Selden and John Selden.

Questions Raised During the Brainstorming Session

A number of ideas and questions emerged during a brainstorming session by the working group last year. These ideas and questions can be grouped into four broad categories: epistemology, cognition, pedagogy, and research.

Epistemology-related questions. What do we, or should we, mean by mathematical habits of mind? What is meant by a mathematical disposition? Can we get to a common understanding of what we mean by mathematical habits of mind? What does it mean to think mathematically? How do mathematicians' (whether pure, applied, or statisticians) views of mathematical habits of mind differ from those of mathematics teachers? What are engineers' views of mathematical

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

habits of mind? With the introduction of technology (e.g., computers and calculators), are there new mathematical habits of mind that it would be beneficial to acquire? Are there special mathematical habits of mind that are useful when using internet resources?

Cognition-related questions. How are mathematical habits of mind different from beliefs, metacognitive strategies, or problem-solving strategies? Is curiosity a habit of mind? Is being meticulous a habit of mind? What is not a habit of mind? What are some tacitly functioning mathematical habits of mind? How have mathematicians acquired mathematical habits of mind? Is it adaptive or useful for students to think like mathematicians? Are there negative mathematical habits of mind? What are they? What impact do positive (or negative) mathematical habits of mind have on students' problem solving ability?

Pedagogy-related questions. Are there progressive stages to developing mathematical habits of mind? What are some impediments or obstacles to acquiring positive mathematical habits of mind? Can you teach positive mathematical habits of mind to students? If so, how? Is there teacher-to-student transfer or professor-to-teacher-to-student transfer of mathematical habits of mind? How do teachers' mathematical habits of mind affect their teaching? What are some ways of assessing the acquisition of positive mathematical habits of mind?

Research-related questions. What sorts of theoretical frameworks could one use in researching mathematical habits of mind? Are mathematical habits of mind culturally influenced? Is their acquisition a matter of enculturation into the larger mathematical culture? What is the role of language in acquiring mathematical habits of mind? In general, how would one conduct research on mathematical habits of mind?

PME-NA 31 Working Subgroup Discussions

The larger working group at PME-NA 31 broke-out into four subgroups to discuss the following topics: (a) Defining habits of mind and the role of language in developing mathematical habits of mind; (b) How does one help teachers or students to develop mathematical habits of mind? (c) Small mathematical habits of mind (behavioral schemas) in proving and problem-solving, including tacit habits of mind; and (d) Negative mathematical habits of mind.

Defining mathematical habits of mind. The first subgroup came up with the following definitions and characteristics of habits of mind which they summarized for the entire working group. Habits of mind are automatic mental processes in response to stimuli that can produce behavior. There is a hierarchy, or spectrum, of habits of mind from low level to high level. An example of a low-level of habit of mind is helplessness (e.g., the response of "show me how to do it") often exhibited by novices. A middle-level habit of mind might be the ability to go through several problem-solving strategies, often exhibited by those developing expert habits of mind. A high-level of habit of mind includes reflecting, monitoring, and questioning, as is often exhibited by experts.

Encouraging mathematical habits of mind in students. The second subgroup took as their starting point Cuoco, Goldenberg, and Mark's (1996) view that "Much more important than specific mathematical results are the habits of mind used by the people who created those results." The subgroup also considered mathematical habits of mind to be "the methods by which mathematics is created, and the techniques used by researchers." They considered the following aspects of, and questions related to, mathematical habits of mind: Justification and defense of results to others in the community. Are there mathematical versus non-mathematical habits of mind? Are mathematical habits of mind dependent on personal experience? Is there some aspect of enculturation? How do the habits of mind used by researchers in the creation of new mathematical results develop? For possible research questions, they suggested: (a) By what processes does a person develop the habits of mind that are typical of mathematicians? (b) How does a person become enculturated into mathematical practices? (c) How do young children develop early mathematical habits of mind?

Small, possibly tacit, mathematical habits of mind. The third subgroup considered the question: Why does it matter to have a small, possibly tacit, habit of mind? They viewed a small, possibly tacit, habit of mind as having two components: (a) the interpretation component, of which one is aware, and (b) the execution, or doing, component, which is often automatic and of which one is usually not aware. The interpretation of the situation is the key, or important, part of the habit. Having an automated habit doesn't take up much working memory, so one can concentrate on other things.

This subgroup made some basic assumptions: (a) Habits of mind develop over time (as there must be a time when one did not have a particular habit); and (b) People have profiles, or constellations of related habits of mind. The following research question was proposed: How do such small, possibly tacit, habits of mind, or other habits of mind, develop?

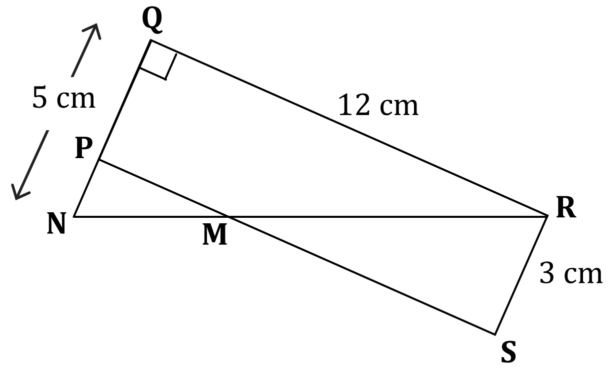
The subgroup also considered whether there were habits of mind with, and without, understanding the underlying habit. Is doing so a mathematical habit of mind? It was observed by one of the group members that he had not found a habit of mind that was not sometimes counterproductive.

The group made the following observations: (a) The tacit part of some small habits of mind are absolutely critical; (b) People can have habits of mind, and other habits, and not know they have them. For example, some people walk in a certain way. Other people know this and can recognize them from their walk, but they aren't aware of the way they walk; and (c) From problem solving studies, it has been observed, for example, that the observer can say a student is drawing a picture, but that's what the observer thinks is going on. However, the student may say that's not what he/she was doing. If one questions the student, he/she may say, "I was trying to figure out what was going on [not drawing a picture]."

Dick Lesh reported a recent study conducted with two groups of students. One group watched a PBS program, *Cyberquest*, about problem-solving teams. Both groups of students were given two problems to solve at the start and two problems to solve at the end. The researchers were given a list of things to notice, such as the role of individuals (the leader, etc.); group functioning, data gathering and data processing. Most of what they observed could be considered habits of mind. At the start, both groups of children had the same number of habits of mind. At the end, both groups invoked the same habits of mind, but the students who watched the program got the problems correct. Further, there was no correlation between the number of processes (habits) and whether the students got the problems right. So what was the difference between the two groups? The group that got the problems correct did them (the habits of mind) at the "right time" and for the "right reasons."

Negative habits of mind. This subgroup considered mainly negative instances of spontaneous, or impulsive, disposition or tendency to act. For example, when a student is asked to solve $(x+3)(x-4)=0$ and automatically multiplies the left hand side. It seems that students are doing what is familiar to them, that they are programmed to react to certain patterns in certain manners, and that this might be the result of certain instructional environments. Such spontaneous dispositions may be limiting to, or block, cognitive growth.

Another example considered was geometrical. The figure below shows a right triangle NQR and a rectangle PQRS; NQ = 5 cm, QR = 12 cm, and RS = 3 cm. When asked to find the length PM, a student may begin by using the Pythagorean theorem to find the length NR. Such a student is said to be impulsive because he or she spontaneously applied the first idea that came to mind without checking its appropriateness.



This subgroup generated the following research questions. (a) How can we detect spontaneous dispositions in situations when the responses are correct responses? (b) What situations/tasks elicit a spontaneous disposition? (c) Is a spontaneous disposition visually influenced? What is the case for blind people? And (d) Can eye tracking collect useful information that can help us understand a student's impulsive disposition?

Plans for the Working Group at PME-NA 32

At the request of the 2009 participants of the Working Group on Mathematical Habits of Mind, we do not intend to have formal presentations. Instead, during the first session we will determine the interests of the participants and formulate an agenda for subsequent sessions. We hope to make progress in this second working group by accomplishing the following:

- Come up with an operational definition that is useful for research purposes.
- Identify important issues/questions and develop research agendas to address/answer them.
- Form collaborations to conduct research.
- Discuss the usefulness and viability of having a support group for those who are conducting research on mathematical habits of mind, and explore the interest in and feasibility of having a special issue of a research journal dedicated to mathematical habits of mind.

Participants with similar interests will team up to discuss theoretical, pedagogical, and/or research issues related to mathematical habits of mind. These teams will report back their discussions and findings to the entire working group at the final working group session. We will conclude the working group with a discussion of next steps.

To facilitate communication and discussion among individuals who are interested in mathematical habits of mind, we have created a site at http://habitsofmind.ning.com/. Anyone can register and be a member of this professional network.

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DEVELOPING MODELS FOR LOCALIZED CROSS-INSTITUTIONAL MATHEMATICS EDUCATION RESEARCH GROUPS

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Mathematics education researchers face a set of professional challenges that are unique to their discipline. The primary purpose of this PME-NA working group is to discuss, develop and ultimately document effective models for localized professional support in research, writing, and publication, as well as opportunities for cross-institutional collaboration. The objectives of this working group are: (a) to discuss the needs of the mathematics education research community regarding professional support and collaboration, (b) to explore models for successful localized mathematics education research working groups, (c) to facilitate mathematics educators to form localized working groups, and (d) to establish a plan of assessment for evolving localized working groups. This working group will be appropriate for anyone interested in forming their own localized mathematics education research group.

Background

National organizations focusing on mathematics education and mathematics education research such as the National Council of Teachers of Mathematics (NCTM), the School Science and Mathematics Association (SSMA), Psychology of Mathematics Education-North American Chapter (PMENA), Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME), and the Association of Mathematics Teacher Educators (AMTE) provide rigorous outlets for reporting research studies completed or in progress, action research, and for disseminating research frameworks, practitioner knowledge, and other ideas. However, support for on-going development of researchers and for connecting local communities of researchers must also come from local or web-based formats. Formation of local, cross-institution, research groups attends to the ongoing local mentoring, collaboration, and research needs of individual researchers and also alleviates possible obstacles in connecting to the greater research community such as issues of time for travel, funding for travel, and other acute regional needs.

Recognizing a need to facilitate the development of a professionally supportive network of mathematics education researchers across Texas, the mathematics education group at Sam Houston State University in Huntsville, Texas organized the Mathematics Education Research in Texas (MERiT) conference, which has been held three times since 2008. The majority of time at MERiT is devoted to small-group discussions among participants from different institutions. The informal and collegial atmosphere at the MERiT meetings fosters professional interactions that often focus on the beginning stages and design aspects of research in addition to the later stages of summarizing and presenting findings. Some conversations provide opportunities for feedback on individual research efforts, while others focus on collaborations for grant proposals, manuscripts, research projects, and presentations. Thirty-five participants representing 16 different institutions have attended one or more of these MERiT conferences.

At the third MERiT conference, this supportive environment for mathematics education researchers served as a catalyst for the formation of two smaller ongoing working groups. One group formed as the result of common research interests, and the second group grew out of geographic proximity. The first group is a state-wide group of mathematics educators with a research interest in mathematics teacher efficacy. This group is currently focused on the development of secondary mathematics teachers and is in the process of developing a teacher efficacy instrument with Algebra as the focus. The second group is made up of mathematics educators at universities in North Texas, who decided to form a local working group that could meet monthly for support in research, writing, and publication. The group members represent research and teaching universities as well as departments of mathematics and education. The participants represent all stages of academic careers, from newly graduated Ph.D.s, to full professors.

Developing a model for a mathematics education working group

Not having a roadmap for how to proceed, this latter group, aptly named the Mathematics Education Research Group in North Texas (MERGiNT), started meeting and began to establish goals. Several themes emerged regarding the unique opportunities such a group afforded. First, many members of MERGiNT experience some level of institutional or departmental isolation in that they were either the only or one of very few mathematics educators in their department or university. This isolation directly hampers the pursuit of research and publication due to both a paucity of formative feedback and a lack of creative discourse needed from colleagues pursuing similar goals. In addition, expanding the number of collaborators on research studies in progress or in planning stages arose as a common need. For example, some researchers at smaller institutions need access to larger populations of pre-service teachers. MERGiNT participants' proximity to each other facilitates regular meetings focused on providing peer review and feedback as well as discussions on appropriate theoretical frameworks. Also, it seems likely that opportunities for collaboration will increase at the local level both because of the logistical feasibility of meeting and the common school systems in which various local universities have investment and influence.

The second theme that emerged was the issue of presence and influence within local school districts. Much of MERGiNT participants' research interests connects to schools and inservice teachers in the region and educational frameworks and structures in the state. The benefits from collaboration and coordinated regional efforts may assist in building stronger networks for maintaining awareness and effective feedback mechanisms for local, regional, and state levels efforts. An added strength may be that a cross-institutional working group streamlines communication with school administrators, staff, and teachers by presenting common goals, rather than disparate competitors asking for collaboration on competing projects. The potential of MERGiNT becoming more visible to the school districts provides the group greater opportunity for cooperation from districts and collaboration with each other.

The need for professional support and regional presence with school systems is not at all unique to the mathematics education researchers of North Texas. The MERiT conference, out of which our organization grew, similarly addresses these needs at the statewide scale. This PME-NA working group seeks to explore how the national mathematics education research community in other regions could benefit from developing similar systems of support.

This working group brings forward the need for local cross-institutional collaborations among mathematics education researchers to help establish more frequent opportunities for mentoring, refining research agendas, and forming local initiatives that present common regional goals versus individualized institutional goals to local school districts and other institutions. We learn from the research literature on collaborative research groups to inform this work and aim to define a process for which the goals of helping researchers grow professionally and of fostering further near-peer collaborations.

Effective models for professional support and collaboration: structures and characteristics

The two primary goals of the MERGiNT group are to support its members' full individual professional development as mathematics education researchers and to foster collaborations within the working group. To develop a structure that supports these goals, we turn to the research literature on collaborative working groups.

One of the challenges for university faculty is balancing multiple responsibilities including teaching, scholarship, and service. In addition to navigating multiple roles, new faculty also express feelings of isolation (Savage, Karp, & Logue, 2004). One means of helping junior faculty acclimate to the community of higher education is mentoring. Savage et al. (2004) describe six goals of mentoring: support new faculty in their professional growth, advance faculty fulfillment by building community, recruit and keep faculty by specifically outlining institutional expectations, encourage relations between new and experienced faculty, guide the transition of new faculty into the new culture, and help new faculty in balancing their duties. However, if a new faculty member is the only mathematics educator on campus, the mentor can contribute cultural knowledge and professional support but cannot provide insider knowledge of the field of mathematics education. Therefore, despite the aforementioned benefits of mentorship programs, there are potentially some limitations.

Another means of inducting new faculty into the university community is support groups. The idea of support groups among teaching professionals is not a novel concept. Educators at all ends of the spectrum, such as inservice teachers, principals, preservice teachers, and college professors, have formed groups to support one another in their endeavors. For instance, Balach and Szymanski (2003) found that through their diverse group of educators that progress was made in the form of "dialogic skills, a shared understanding of how teachers must lead a life of the mind, and realization of how to create a context supportive of change" (p.25). Sterrett and Haus (2009), two principals from the same school district, formed a collaborative group to help improve their schools from the top down. They met with certain goals in mind and helped each other develop plans for school enhancements. Even specific groups like Science, Technology, Engineering, and Math (STEM) programs (Hamos, et al., 2009) have developed across the country to bring individuals from different disciplines together. Through all these various types of meetings, many educators have benefited from support and encouragement, as well as higher productivity rates.

The idea of higher productivity also applies to master's and doctoral students in collaborative support groups. Similar to new faculty members, graduate students, at the thesis or dissertation writing stage, often feel a sense of isolation which is one of the factors contributing to delays or abandonment in completing the degree (Conrad & Phillips, 1995). In these support groups, graduate students discuss their progress and research findings with each other. Members of a group of three master's students, who were all writing their thesis and met monthly with their supervisor, outlined the benefits of their collaboration: "(1) Support; (2) help in gathering materials; (3) seeding of ideas, stimulation; (4) the alternative perspectives that the others provided; (5) a timetable and pressure; (6) experience in 'thinking on your feet'; (7) elimination of 'blocks' [to writing]; (8) encouragement to write; (9) and help in clarifying and communicating ideas" (Conrad & Phillips, 1995, p. 315). University faculty members could also gain from this type of collaborative mentoring from a working group made up of fellow faculty members, especially those in their field, with respect to writing manuscripts for publication.

Whether supporting each other's individual endeavors (first goal) or working together on projects (second goal), the key idea is collaboration. Collaboration occurs when members work together on a shared project or toward a shared goal. Each member contributes toward the effort. All members have a shared vision and communicate with one another to provide progress toward fulfilling that vision (Leonard, 2002). As teacher educators, there is further imperative for collaboration: "If collaboration is a valued skill for teachers, then it is essential that teacher educators find ways to make collaboration a more integral part of the university context" (Ross, et. al. 2005, p. 278). This last quote is corroborated by several associations including the Association of Teacher Educators (ATE), the National Association for the Accreditation of Teacher Education (NCATE), Interstate New Teacher Assessment and Support Consortium (INTASC), and the National Board for Professional Teaching Standards (NBPTS), which all incorporate collaboration in part of their standards, and the Association of Mathematics Teacher Educators (AMTE), which includes collaboration in its mission and goals.

In an attempt to formulate a model of cross-institutional collaboration, the characteristics of other types of collaborations such as professional learning communities and partnerships between universities and schools will be discussed. A professional learning community (PLC) is a group of teachers, often within a school, who work together to improve student learning. The definition of a PLC can be gathered from the three words in the term. The teachers are *professionals* who collaborate in fulfilling their responsibilities of their *learning* and the *learning* of their students. *Community* is, "individuals coming together in a group in order to interact in meaningful activities to learn deeply with colleagues about an identified topic, to develop shared meaning, and identify shared purposes related to the topic" (Hord, 2009, p.41). Hord (2009) identifies conditions for success of PLC's, several of which can be extended to a cross-institutional working group. The members of the PLC must decide how they are going to structure their experiences. There must be a dedicated time and space for meetings including the idea that the location of the meetings be rotated. Parr and Ward (2006) found that collaborations among K through 12 teachers via the internet may not be productive, resulting in limited sharing of materials and website use, supporting the importance of face-to-face meeting time. Finally, the power and the authority must be shared, also termed distributed leadership.

Distributed (or distributive) leadership, a concept primarily used in K through 12 schools, has a bottom-up versus top-down orientation. Whitby (2006) states that, "Distributive leadership involves the leadership functions of a school being shared by many people in ways that strengthen the whole school community, intensifying a sense of engagement and shared responsibility while making the workload more manageable" (p. 2). Brown and Littrich (2008) used and assessed the distributive leadership model for a cross-institutional collaboration which planned a conference roundtable on assessment. The project participants identified eight characteristics of distributive leadership: generates engagement; acknowledges and recognizes leadership irrespective of position; focuses on people's strengths; is different things in different contexts; is enduring; requires the development of strong relationships and networks; is about capacity building and development; and assists and informs succession planning. Using surveys and reflective discussions to evaluate the framework based of these eight characteristics, it was found that the collaboration, on the whole, aligned with the principles. However, the authors also noted that although the project opened up the possibility of further collaborations (intra- and cross-institutional), there was more success with collaborations within institutions.

There is also documentation about collaborations between institutions of higher education and elementary, middle, and secondary schools. For example, Dolly (1998) discussed numerous research articles and how professional development schools work as environments where new collegiate faculty can learn about working at the university level, as well as partner with others on university issues. Frost, Coomes, and Lindeblad (2009) researched the characteristics and outcomes of a collaborative professional development project between secondary school mathematics teachers and postsecondary mathematics faculty. They were focused on the difficulties that students face when they transition from secondary mathematics courses to postsecondary mathematics courses. Their activities included readings, discussions, and comparison of student work. They also used activities such as team goal setting and analysis of school curricula and in-school teamwork with the aim of creating a sense of belonging to smaller PLCs within the larger group. DuFour and Eaker (1998) referred to this arrangement as a nested PLC. Participants in Frost, Coomes, and Lindeblad's (2009) study developed a sense of safety, trust, and mutual respect for each other. Postsecondary faculty also valued the opportunity to work with high school teachers.

Robbins and Cooper (2003), working with schools and at a university, respectively, described their collaborative endeavors over a nine-year period. They identified several "rules" by which they worked together including maintaining communication, establishing mutual respect, upholding confidentiality, and regularly identifying and assessing shared goals. The majority of the piece discusses the various phases through which their collaboration progressed highlighting the evolutionary nature of the collaboration. Since they were working in two different contexts they needed to find a place in the middle to meet; that is, they sought to, "'construct' their own spaces for shared professionalism" (Robbins & Cooper, 2003, p. 225).

In terms of external collaborations in higher education, the majority of the research is about why the collaboration exists such as sharing resources or enhancing authority, effectiveness, and/or product (Kezar, 2006). In addition to sharing the reasons for forming our working group, we plan to identify the characteristics that support cross-institutional collaboration by drawing on the literature of other types of collaborations. The result will be a framework of collaboration that applies to our situation of building a "space of shared professionalism" among mathematics education researchers from multiple institutions in a localized area.

Purpose of this PME-NA working group

This PME-NA working group builds on the development of three localized cross-institutional working groups for mathematics education researchers in different regions of Texas. The primary purpose of the working group is to discuss, develop and ultimately document effective models for localized professional support and collaboration that are transferrable to other regions. We expect interest from other mathematics educators who face the same assortment of unique professional challenges.

The objectives of this working group are:

- To discuss needs of the mathematics education research community regarding professional support and collaboration
- To explore models for successful working groups
- To facilitate mathematics educators to form localized working groups
- To establish a plan of assessment for evolving localized working groups

Proposed Activities for this PME-NA working group

Session 1: Goals & Structure of Mathematics Education Research (MER) Working Groups

- Introduction of participants
- History and development of the MERGiNT and MERiT groups
- Presentation and discussion of goals and structure of a MER working group
    - Why create a MER working group?
    - Needs of the community and major needs of the MER working group
    - Characteristics needed to be a successful evolving working group
- Individual presentations, if any, on research related to developing models for localized cross-institutional mathematics education research groups

Session 2: How to Get Started

- Review needs of working groups, as identified in Session 1
- Discussion of:
    - How to start a working group?
    - How to identify other people to participate?
    - Role of community colleges
    - Building interdisciplinary connections
        - Education Faculty + Mathematics Faculty + K-12 Teacher

Session 3: Models of MER working groups

- Discussion of:
    - Various models of working groups
        - State working groups & subgroups

        - Cross institutional working groups
        - Working groups within institutions
    - How to collaborate and share resources?
        - Websites, Blackboard, Skype, Wikis, etc.
    - Developing a framework
    - Assessing your framework
    - How is the working group meeting the needs of the educational community?
- Planning for further collaboration and dissemination
- Developing tentative timeline for creation of future Working Groups

Anticipated follow-up

We anticipate follow-up activities on two levels. We expect that this PME-NA working group will be a platform for the formation of new mathematics education research working groups. We plan to document the evolution of these groups, and on a local level, we plan to use the insights gained in this PME-NA working group to develop a framework for assessing the growth and success of the MERGiNT group.

- Identify the new groups formed from this working group
    - Determine nature of group (similar research interest, geographic proximity, other)
    - Have new groups establish tentative goals and meeting format/times
    - Each new group submit a progress report to MERGiNT prior to PMENA 2011
- MERGiNT group:
    - Develop a theoretical framework to support the formation of MER working groups
    - Is the theoretical framework of this type of group unique to mathematics education or are there universal traits to such a group?
        - Theories are like toothbrushes…everyone has their own and no one wants to use anyone else's. (Campbell,2006)
    - Develop possible ways to assess the effectiveness of a MER working group

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INVESTIGATING MATHEMATICALLY IMPORTANT PEDAGOGICAL OPPORTUNITIES

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Mathematically Important Pedagogical Opportunities (MIPOs) are instances in a classroom lesson in which the teacher has an opportunity to move the class forward in their development of significant mathematics. Although this construct is widely recognized in the literature as important to mathematics teaching and learning, it is neither well defined nor clearly identified as a construct that can be studied. This working group will build on the efforts of two research groups, represented by the organizers, to define, identify, and characterize MIPOs. Specifically, Session 1 will focus on identifying MIPOs, including questioning and critiquing working definitions and preliminary dimensions of MIPOs. Session 2 will explore sub-constructs of MIPOs and the potential of sub-constructs to provide leverage in studying the broader construct. The first two sessions will include examining instances of classroom practice (written/video) that have been identified as containing MIPOs. Session 3 will focus on issues around developing a research agenda for investigating MIPOs and generating plans for continuing work on MIPOs.

History of Working Group

We propose a new working group focused on investigating mathematically important pedagogical opportunities. The two research groups represented by the organizers have been interacting over the past year, including meeting informally at PME-NA 2009, around their overlapping work on defining, identifying, and characterizing mathematically important pedagogical opportunities. The working group will be an opportunity to expand this work to include a broader range of researchers interested in this topic.

Focal Issues

Although skilled teachers and teacher educators often intuitively “know” when mathematically important pedagogical opportunities occur during a lesson and can readily produce ideas about how to capitalize on such opportunities, the literature reveals a construct that is not well-defined. Ideas related to these opportunities are mentioned in many different ways. For example, Jaworski (1994) refers to such opportunities as “critical moments in the classroom when students created a moment of choice or opportunity” (p. 527). Davies and Walker (2005) use the term “significant mathematical instances” (p. 275) and Davis (1997) calls them “potentially powerful learning opportunities” (p. 360). Schoenfeld (2008) refers to such moments as “the fodder for a content-related conversation” (p. 57), as “an issue that the teacher judges to be a candidate for classroom discussion” (p. 65) and as the “grist for later discussion or reflection” (p. 70). Schifter (1996) spoke of “novel student idea[s] that prompt teachers to reflect on and rethink their instruction” (p. 130). In the context of teacher professional development, Remillard and Geist (2002) described *openings in the curriculum* as moments in which teachers’ questions, observations, or challenges require the facilitator to make a decision about how to incorporate into the discussion the mathematical or pedagogical issues that are raised. The nature of the facilitator's decision determines the extent to which the teachers' ideas advance the learning of the group. Similarly, when a MIPO occurs in a classroom lesson, a teacher first needs to recognize it as such, and then make a decision about how to respond. Depending on the action taken by the teacher, the MIPO may or may not support the development of students' mathematical understanding.

Thus, there is clear recognition that such opportunities, whatever they are called, are important to mathematics teaching and learning. These opportunities, however, frequently either go unnoticed or are not acted upon by many teachers, particularly novices (Peterson & Leatham, 2010). This raises the question of how teacher educators can help teachers recognize MIPOs during their instruction and use them to support student learning. The purpose of this working group is to engage with the construct of MIPOs in order to consider ways that researchers can identify, explore, and study these opportunities.

In the following, we first give a brief overview of the work around MIPOs done by each of the two research groups. We then provide a description of how we will engage participants with the construct of MIPOs during the three working group sessions, and end by outlining potential follow-up activities.

A Broad View of Mathematically Important Pedagogical Opportunities
Leatham & Peterson

For us, a MIPO is when students' observed mathematical thinking provides the teacher with an opportunity to move the class forward in their development of significant mathematics. MIPOs are driven by significant mathematics and observed student thinking. When students articulate mathematical ideas, expert mathematics teachers use their professional knowledge to discern whether there is evidence that students are ready, at that moment, to engage with important underlying mathematics. Many teachers, however, do not recognize MIPOs in their classrooms. From our observations of novice teachers, MIPOs often go unrecognized because of the wide variation of circumstances under which they occur. We describe four types of variation in MIPOs: two related to the nature of mathematics within a MIPO (focus and problematization) and two related to the way in which a MIPO surfaces (visibility and predictability). Our belief is that an awareness of these variations is a first step to helping teachers recognize MIPOs more often in their classrooms.

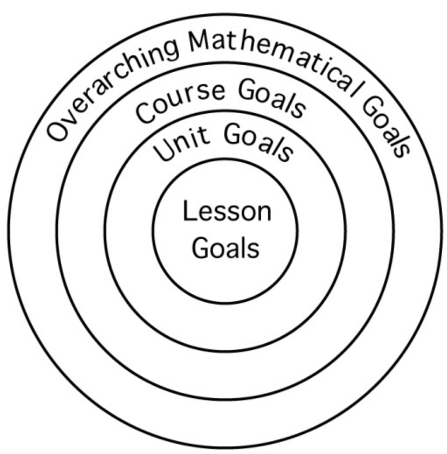

Figure 1. Layers of mathematical goals one might consider in recognizing MIPOs

The nature of the mathematics in MIPOs varies in at least two important ways: focus and problematization. First, MIPOs vary in the degree to which they support the mathematics a given lesson or discussion was designed to elicit. Thus, although MIPOs could lead to discussion about the mathematics within the content of the day's lesson, they could also lead to discussion about mathematics related to broader goals of the unit, course or mathematics as a whole (see Figure 1). To be prepared for such variation in mathematical focus, Leinhardt and Steele (2005) suggested that teachers need "both a primary mathematical agenda… and sensible, valued, predefined classes of situations that suggest deviating from the agenda" (p. 110).

Second we consider variation with respect to problematization. For us MIPOs are always an opportunity to help students to clarify important mathematical ideas. There is variation, however, in the degree to which students initially see the need for this clarification—that is, already see the mathematics of the situation as problematic. On one extreme, students may raise the problematic nature of the mathematics themselves (e.g., "But yesterday you said…"). On the other extreme students may share their thinking, yet not be aware, as far as the teacher has evidence, that there is something problematic or noteworthy in the mathematics at hand. In such situations teachers have the opportunity to help students to problematize the situation.

In addition to variation related to mathematics, the third and fourth dimensions relate to the ways in which teachers observe students' mathematical thinking in a MIPO: visibility and predictability. The third dimension is the variation in how MIPOs are made visible to teachers. On one extreme, students may simply share their thinking with little teacher elicitation (e.g., student raises their hand and asks a question). On the other hand, the teacher may have observed students' mathematical thinking as they worked in small groups or individually. In these instances, the teacher may choose to share the thinking (e.g., "I saw several students using this strategy…") or ask the student to do so. Thus MIPOs initially may become visible to the teacher in the context of whole class, small group, or individual work, and may or may not be made public by the students themselves.

Finally, MIPOs vary in the degree to which they are predictable. For example, teachers often engage students in tasks that are meant to elicit particular mathematical thinking; thus teachers often anticipate certain solution strategies and the mathematics to which those strategies give rise. In addition, over time teachers begin to recognize common student conceptions and misconceptions. Because of this anticipation and knowledge of students' thinking, opportunities often arise that are predictable. Even when the thinking is predictable, however, the details often are not. For example, we may anticipate *that* students will respond in certain ways, but just *when* they do so and *how* they articulate their thinking are often less predictable. Some MIPOs are even less predictable. There are a multitude of pathways down which students' thinking can take them, particularly when they are engaged in classrooms that encourage them to try to actively make sense of the mathematics at hand.

In our experience, teachers most easily recognize MIPOs with these characteristics: (1) the mathematical focus aligns well with the day's lesson, (2) there is evidence that students see the mathematics at hand as problematic, (3) the thinking is made visible in the context of a whole class discussion, or (4) the thinking was anticipated. An understanding of the variations we have just described, however, can help teachers to recognize less obvious variations of MIPOs, particularly when the mathematical focus is beyond the day's lesson, students do not yet perceive the situation as problematic, the thinking is made visible in small group or individual conversation, or the thinking is unexpected. Although we find these variations useful, they do not help one to identify when occurrences of student mathematics thinking are MIPOs. To aid in their identification, we are in the process of describing common types of MIPOs. Two common types are (1) incorrect or incomplete responses and (2) multiple responses or solutions. First, incorrect responses are often MIPOs because giving students opportunities to discuss the thinking behind their responses leads to important mathematics related both to why the response was incorrect as well as the mathematics underlying a correct response. Similarly, vague or incomplete thinking is also a potential MIPO. Encouraging students to elaborate and clarify often reveals confusion and questions related to important underlying mathematical ideas.

Another common type of MIPO occurs when there exist multiple student responses or solutions to a given question. These situations give rise to MIPOs for a couple of reasons: (1) the solutions are inconsistent or (2) there are commonalities between different methods. When solutions are inconsistent by one being correct and the other incorrect, students can evaluate the thinking that supports these solutions to identify the critical elements of the underlying concepts. If two correct solutions have relied on different methods, a comparison of these methods can highlight the underlying mathematics that is common to both. An awareness of these common types of MIPOs can help teachers learn how to identify them. We expect there are other common types worth exploring.

Pivotal Teaching Moments as a Sub-construct of MIPOs
Stockero & Van Zoest

We have narrowed our work on MIPOs to focus on a sub-construct that we call *pivotal teaching moments* (PTM). We define a PTM as *an instance in a classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend, or change the nature of, students' mathematical understanding*. We ground our work in classroom observations of beginning teachers' practice based on the hypothesis that the decision-making process involved with PTMs may be more obvious with this group than with skilled teachers. That is, skilled teachers may recognize a PTM and make the decision to act so quickly and smoothly that it wouldn't be clear that they had modified their instruction.

We believe that an important first step in capitalizing on PTMs is recognizing that such moments exist. Without this awareness, teachers may experience *inattentional blindness* (Simons, 2000)—a phenomenon described in the psychology literature as a failure to focus attention on unexpected events. This is also related to the idea of *framing* (Levin, Hammer & Coffey, 2009)—the way in which a teacher makes sense of a classroom situation. From this perspective, whether a teacher notices the value in an unexpected event depends on how he or she frames what is taking place during instruction. If, for example, a teacher views a student error as something that needs to be corrected, he or she is unlikely to consider the mathematical thinking behind the error or whether the error could be used to highlight a specific mathematical idea. On the other hand, a teacher who views an error as a site for learning is more likely to consider both the mathematics underlying the error and how it could be used to develop mathematical understanding. Thus, we began our research by investigating the circumstances in which PTMs seem likely to occur.

In general, PTMs seem most likely to occur when students are actively engaged in and contributing to the mathematics lesson in some way—either by the design of the lesson or by their own initiative. In our work to date, we have identified five circumstances that are fertile grounds for PTMs. These are circumstances in which: (1) student comments or questions go beyond the mathematics that the teacher had planned to discuss; (2) students are trying to make sense of the mathematics in the lesson; (3) incorrect mathematical thinking or an incorrect solution is made public; (4) a mathematical contradiction occurs; and (5) confusion is expressed about specific mathematical ideas.

Drawing on Stein, Smith, Henningsen, and Silver's (2000) work on cognitive demand of tasks, we have conceptualized PTMs as triples that include the PTM, the teacher decision and the likely impact on student learning. We further characterize the first component of the triple, the PTM, by the circumstance that led to it and the potential of the moment to improve students' opportunities to learn mathematics. The second component, teacher decision, is characterized by the action the teacher takes and how effectively they implement that action. Teacher actions that we have identified include: (1) ignoring or dismissing the interruption; (2) incorporating the interruption into the plan; (3) pursuing student thinking; (4) emphasizing the meaning of the mathematics; and (5) extending the mathematics or making connections to other topics. The final component of the triple, likely impact on student learning, is identified as negative impact, neutral impact, or low, medium, or high positive impact.

This PTM work provides an instance of studying a sub-construct of MIPOs that can be used to inform the working group's thinking about developing frameworks and methodologies for studying the broader MIPO construct.

Plan for Working Group

Session 1: Identifying mathematically important pedagogical opportunities

Session 1 will begin with introductions of the working group organizers and participants to gain a sense of participants' contexts and the reasons they are interested in working on or thinking about the MIPO construct. In order to allow participants to more easily become engaged in discussions about MIPOs, particularly those for whom the idea is new, a short presentation will be given by the organizers related to defining, characterizing, and identifying MIPOs. This will be followed by small and large group discussions that will provide particpants an opportunity to question and critique the ideas that have been presented. The group discussions will center on questions and ideas about MIPOs with which the organizers have been wrestling, including:

- How does one know "mathematically important" when one sees it?
- To what extent does the working definition capture the idea of a MIPO? What is not being captured by this definition?
- Do the dimensions described in the presentation help one think about MIPOs? Are there other dimensions that should be considered? What is the relative importance of these dimensions?
- Can MIPOs be both planned and unplanned?

Classroom video or written clip(s) that have been identified as containing MIPOs will then be shared both as a means of further engaging participants with the idea of a MIPO and of introducing the types of activities in which participants will engage in Session 2. Session 1 will conclude with a short discussion related to issues that will be discussed further in Sessions 2 and 3, including how the MIPO construct might be narrowed in order to gain some traction in understanding the construct and issues of practicality and interest. In particular, Sessions 2 and 3 will be responsive to the interests of participants through the selection of the instances of classroom practice and the implementation issues on which we will focus.

The goals of Session 1 are to:

- Develop an understanding of participants' contexts and interests in MIPOs in order to structure the working group sessions in a way that is responsive to the participants.

- Engage participants in thinking about the idea of a MIPO by presenting an example of how they have been defined, characterized, and identified.
- Begin to refine definitions and ideas related to MIPOs through small and whole group discussions.

Session 2: *Exploring mathematically important pedagogical opportunities*

To stimulate a conversation about the possibility of sub-constructs that could be used to gain leverage in making sense of the broader MIPO construct, Session 2 will begin by introducing pivotal teaching moments (PTMs) as an example of a sub-construct. The PTM work was chosen because it provides an example of how narrowing the scope of the MIPOs can lead to the development of frameworks and methodologies that may be applicable to studying the broader MIPO construct.

After giving a brief overview of work on PTMs, we will engage working group members in examining instances of practice (written/video) that have been identified as containing MIPOs. This will provide an opportunity to begin to test out and refine existing definitions and frameworks related to both PTMs and the broader MIPO construct, as well as raise additional issues about MIPOs.

The goals of Session 2 are to:

- Test out and refine existing MIPO definitions and frameworks.
- Consider sub-constructs of MIPOs that might provide leverage in studying the broader construct.

Session 3: Setting a research agenda for investigating mathematically important pedagogical opportunities

Session 3 will focus on issues around developing a research agenda for investigating MIPOs. We will begin by considering what the field needs to know about MIPOs in order to use the construct to improve the practice of both mathematics teacher educators and classroom teachers. For example, what knowledge or dispositions might teachers need to capitalize on MIPOs? Specifically, we will discuss:

- What questions need to be addressed?
- What methodologies might be appropriate?
- What contexts should be considered?

Based on these discussions, small groups will be formed around the working group participants' expressed interests. Collaborative endeavors will be encouraged and venues for continuing MIPO work will be discussed.

The goals of Session 3 are to:

- Outline a research agenda for studying MIPOs.
- Generate plans for continuing MIPO work.

Anticipated Follow-up Activities

The work of the group will be continued by the core research teams through ongoing collaboration. If there is sufficient interest among working group participants, the organizers will facilitate broader research collaborations and additional working group sessions through venues such as PME-NA conferences, AMTE pre-sessions, and web-conferencing.

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MODELS & MODELING WORKING GROUP

Subgroup Leaders: Dick Lesh (Indiana), Tamara Moore (Minnesota), Helen Doerr (Syracuse), Miriam Amit (Israel), Lyn English (Australia), and Bharath Sriraman (Montana).

The Models & Modeling Working Group has been one of PMENA's most active working groups – beginning with the very first PMEMA that Dick Lesh held at Northwestern University. And, it has a long history of encouraging collaborations among group members – and of including junior colleagues in a variety of research and development activities. Consequently, since the founding of PMENA, models & modeling perspectives (MMP) have provides a useful framework a large number of books, journal publications, funded projects, and conference presentations that featured collaborations among multiple working group members.

For the 2010 meeting of PMENA, the MMP working group will focus on four topic areas:

(a) modeling activities for primary grade children,
(b) modeling activities focusing on integrated approaches to algebra, calculus, statistics, probability, and geometry,
(c) modeling-based activities for teacher development,
(d) modeling-focused design research methodologies. …

Newcomers are very welcome. Please bring 1-2 page handouts for any projects you would like to discuss.

PRESERVICE ELEMENTARY SCHOOL TEACHERS' CONTENT KNOWLEDGE IN MATHEMATICS

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This working group continues its focus on preservice elementary teachers' content knowledge in mathematics for teaching. Participants continue the discussion of (a) a synthesis of the current research on the knowledge on preservice teacher content knowledge, (b) pedagogical principles they developed for working with preservice teachers, and (c) ways to support the development of such knowledge, for example how video and other artifacts of children's mathematical thinking can be used to address PSTs specialized content knowledge for teaching. Dialogues (on- and off line), resource sharing, as well as collaboration among members of the study group will be continued, encouraged, and facilitated during and after the conference.

Foci and Aims of the Working Group

The goal of this working group is to examine the various types of knowledge needed for teachers of mathematics focusing in particular on describing specialized content knowledge for teaching (Hill, Ball, & Schilling, 2008) and the development thereof. Building on previous work by this group at various meetings (for example: PME-NA 2007, PMENA 2009, AMTE 2008, AMTE 2009, AMTE 2010) we will continue discussions to gain a better understanding of what preservice elementary school teachers should know and how they come to know in order to become effective teachers and to explore how mathematics educators can assist preservice and in-service elementary school teachers in developing such knowledge. In particular we will continue our discussions on (a) the synthesis of the research on preservice elementary teachers' content knowledge, (b) a common pedagogy developed by this group, c) ways to help PSTs develop more sophisticated conceptions, for example, using artifacts of children's mathematical thinking , and (d) possible collaborations. Below we outline what we have accomplished thus far and what we intend to accomplish at PME-NA 2010.

Synthesizing the Research on Preservice Elementary Teachers' Content Knowledge and Identifying Areas of Further Study.

This working group has begun to synthesize the research on PSTs' content knowledge and to identify areas of further study. Various members of the group took the lead to examine the current state of knowledge in a particular content area. The content areas examined by this group (with a select number of citations) are: Whole Numbers and Place Value (Southwell & Penglase, 2005; Thanheiser, 2009a, in preparation-b), Decimals (Putt, 1995; Stacey, et al., 2001; Zazkis & Khoury, 1993), Fractions (Deborah Lowenberg Ball, 1990; Graeber, 1989; Newton, 2008) , Algebra (Deborah Loewenberg Ball, 1990; Bednarz, Kieran, Lee, & International research, 1996; Doerr, 2004), Geometry & Measurement (Fujita & Jones, 2006; Jones, Mooney, & Harries, 2002; Olkun & Toluk, 2004), Statistics (Groth, 2007; Groth & Bergner, 2006; Leavy & O'Loughlin, 2006), and Probability (Canada, 2006).

We are working on synthesizing the research findings relative to PSTs content knowledge in each of these areas and plan to create a combined synthesis across all content areas. A rough draft of the synthesis is expected to be finished by October 2010. Working group participants are encouraged to contact the first author to receive a copy of the current document prior to the conference.

The synthesis of the current research is essential for us to know for the following reasons:

1. We need to understand the conceptions with which the PSTs enter our classrooms so we can build on those conceptions (Bransford, Brown, & Cocking, 1999). To help teachers develop the content knowledge needed to teach mathematics, educators need to understand the PSTs' currently held conceptions. As the authors of *The Mathematical Education of Teachers* suggest, "The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they *do* know" (CMBS, 2001, p. 17).
2. To identify research needs we need to know *what* has been established in terms of PSTs' content knowledge and *how* that knowledge has been developed and *what* we still do not know or understand.

Mathematics teacher educators need more empirical evidence of what learning opportunities contribute most to more knowledgeable and confident teachers in order to make more informed changes to their programs (Mewborn, 2000). Once the synthesis paper is completed we envision this to be a starting point to conduct further research on PSTs' content knowledge and the development of that knowledge. In addition we will examine similarities and differences in PSTs' content knowledge and the development thereof across the various content areas.

Further developing a common pedagogy developed by this group.

Our pedagogy is based on the framework for mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008; Hill, et al., 2008; Hill, Rowan, & Ball, 2005). Hill, Ball, & Shilling (2008) introduced a framework for distinguishing between different types of knowledge included in the construct of *mathematical knowledge for teaching*. This framework distinguishes between subject matter knowledge and pedagogical content knowledge. Subject matter knowledge is subdivided into common content knowledge, specialized content knowledge, and knowledge on the mathematical horizon. Pedagogical content knowledge is subdivided into knowledge of content and students, knowledge of content and teaching and knowledge of curricula. Hill, Rowan & Ball (2005) provide empirical support linking teachers' mathematical knowledge for teaching to student achievement gains.

The authors of this working group find the mathematical knowledge for teaching framework to be useful when discussing different types of knowledge they want their preservice teachers to develop. However, much work remains in determining what kinds of learning opportunities effectively help preservice teachers to develop such knowledge. Therefore our attention is focused on the question of how we can use our current understanding of mathematical knowledge for teaching as a framework to address content preparation of PSTs' content knowledge in content and method courses.

Through examination of their collective work, several participants of this working group found they shared a common framework of design principles:

1. Mathematical ideas are built on preservice teachers' currently held conceptions.
2. Classes for preservice teachers should model teaching for understanding (i.e. we teach our PST the same way we want them to teach their children)
3. We focus on developing connections between content knowledge and:
    - Knowledge of teaching;
    - Knowledge of children's mathematics;
    - Knowledge of curriculum. .

Further collaborative work will allow the group to continue to refine this framework and build on it.

Ways to support the development of specialized content knowledge

Building on the pedagogy developed by this group we consider the uses of artifacts of children's mathematical thinking as a concrete example of ways to address the PSTs' specialized content knowledge. This group has begun to examine this question and our plan is to continue to do so at this working group session. Using the pedagogy outlined above we developed a framework for various uses of artifacts of children's mathematical thinking (see Figure 1). We will address all three elements of the pedagogy but our focus for this working group lies on the first element, being informed by the other two.

Example: Use of a sequence of two artifacts of children's mathematical thinking

For example, to address the PSTs' currently held conceptions, artifacts are carefully chosen to match/question PSTs' currently held conceptions. In the context of multidigit whole numbers, for example, we know that many PSTs think of the digits in a whole number in terms of ones, rather than in terms of their values (Thanheiser, 2009b, in preparation-b). These PSTs may think of the regrouped digits in the context of subtraction (see Figure 2) in terms of ones rather than their values. One PST, for example, explained "You put a 1 over next to the number and that gives you ten … I don't get how the 1 can become a 10. One and 10 are two different numbers. How can you subtract 1 from here and then add 10 over here? Where did the other 9 come from?"

To address the question of how children think about algorithms, research has shown that children, if allowed, will invent their own algorithms (which make sense to them) . This strengthens their place value understanding in terms of connecting digits to their values and relating between place values (Ambrose, 1998; Hiebert & Wearne, 1996; Constance Kamii, 1994; Constance; Kamii, Lewis, & Livingston, 1993; Sowder & Schappelle, 1994). However, children who are not developing their own algorithms often view digits in terms of ones rather than their values (just as the PSTs). Based on this information we can now choose artifacts of children's mathematical thinking which will: (a) address the PSTs' incorrect interpretation of the regrouped digit as 1, (b) build on that incorrect interpretation to develop a more meaningful interpretation, (c) model how we integrate such thinking into our classroom, (d) connect to knowledge of content and children (how do children think about this?), (e) connect to knowledge of content and teaching (address the teaching of this topic), and (f) connect to knowledge of content and curriculum (address curricular issues of this topic). One such artifact is a video clip of a 2nd grader adding 274+368 (San Diego State Foundation, Philipp, & Cabral). This child initially adds 200+300 =500, then 70+60=130 and then 4+8=12. He then adds those partial sums together to get his answer (see Figure 2).

PEDAGOGY		USING ARTIFACTS
1. Mathematical ideas are built on preservice teachers' currently held conceptions.	⇦	Use of artifacts of children's mathematical thinking to address and build on PSTs' currently held conceptions.
2. Classes for preservice teachers should model teaching for understanding (i.e. we teach our PST the same way we want them to teach their children)	⇦	Use of artifacts of children's mathematical thinking allows us to 'model' how such thinking could be addressed in a classroom setting. It also allows us to 'model' how thinking could be addressed at all levels.
3. We focus on developing connections between content knowledge and: • Knowledge of teaching; • Knowledge of children's mathematics; • Knowledge of curriculum	⇦	Using artifacts of children's thinking allows us to draw connections to other areas.

Figure 1. Using artifacts of children's mathematical thinking within our pedagogy

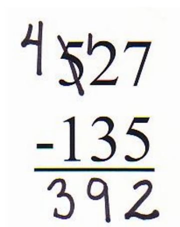

Figure 2. One PSTs' application of the standard algorithm for regrouping in 527-135

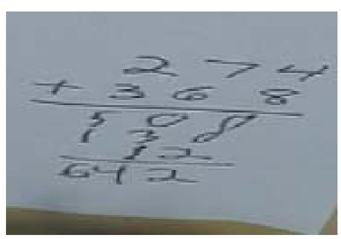

Figure 3. 2nd grader's strategy to add 274 + 368 (San Diego State Foundation, et al.)

PSTs' reactions to this video clip vary widely and thus offer a good starting point for discussion. Sample PSTs responses to this video are:

- "[the child] thinks too quickly without understanding the rules of place value."
- "It seems as though [the child] is thinking in a manner that helps him add large numbers in an easier way than having to carry numbers over. His thought process was a little confusing at first, but once I caught on to what he was doing I realized that it was a very smart way to go about adding such large numbers at a young age. He was thinking in a mathematically developed manner which actually made quite a lot of sense"
- "He added the hundreds and then added the tens and then added the ones. To find the final answer, he added all three numbers together."

At the beginning of a discussion of an artifact like this not all PSTs are in a position to understand the child's thinking. PSTs who think of the digits in terms of ones may not consider breaking a number like 274 up into 200 and 70 and 4. Reacting to a child doing this requires the PSTs to decide whether it makes sense and if it does why it works. Using this particular artifact could raise the issue of values of digits in a classroom. In addition it could be used to examine regrouping. How is it that the child adds 70 + 60, what exactly happens in his process? While this strategy may seem obvious to the reader it is not to all PSTs. When PSTs were asked how this child might add 389 + 475 (using the same strategy), only 12 of 22 PSTs (Thanheiser, in preparation-a) were able to do so. Thus, this artifact could be used to address the three aspects of the pedagogy outlined above (see Figure 4).

PEDAGOGY	**CHILD'S ALTERNATIVE ALGORITM VIDEO**
1. Mathematical ideas are built on preservice teachers' currently held conceptions.	• relating digits in a number to their values • explicating regrouping and the value of regrouped digits • relating various place values to each other • Discussing alternative/invented strategies.
2. Classes for preservice teachers should model teaching for understanding (i.e. we teach our PST the same way we want them to teach their children)	• Integrating/explicating/discussing alternative/invented strategies. • Modeling respectful treatment of alternative strategies • Modeling sense making of alternative ways of thinking
3. We focus on developing connections between content knowledge and: • Knowledge of teaching; • Knowledge of children's mathematics • Knowledge of curriculum	• Explicating the fact that children can and do develop their own algorithms • Explicate that developing own algorithms deepens number sense/place value understanding • Show various alternative algorithms

Figure 4. Relation of a partial sums artifact to pedagogy

Brosnan, P., Erchick, D. B., & Flevares, L. (Eds.). (2010). Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Columbus, OH: The Ohio State University.

A second video clip of children's mathematical thinking is used in combination with the one described above. While the previous one is an alternate correct strategy that raises several mathematical issues, some of these issues may still be dismissed by the PSTs as different from their own. Some PSTs may not be ready to listen to alternate viable strategies until they see a limitation of their own (Thanheiser, 2009c). An artifact of children's mathematical thinking addressing this limitation shows a child who has successfully used the strategy described above with 2-digit numbers but struggles with 3-digit numbers. She solves the problem by adding single digits (see Figure 5)

The goals of this video clip are to address the insufficiency of a conception of seeing all digits in terms of ones and emphasizing the need to relate digits to their values. It also emphasizes the need for making sense of alternate strategies. The coupling of a correct and an incorrect solution allows PSTs to examine alternate correct mathematical thinking and examine why it makes sense and address their own conceptions and issues with those they may not have been aware of.

Figure 5. One student's correct solution to 38+45 and incorrect solution to 638 + 456 (San Diego State Foundation, et al.)

Collaboration

This group began its collaborations in 2007. A small subgroup of the current group began to meet at PME-NA, NCTM and AMTE meetings on a regular basis since then and has presented at several of those meetings (NCTM, 2007, PME-NA 2009, AMTE 2009, AMTE 2010). Our research is similar in its focus pertaining to preservice elementary teachers' content knowledge but varies in specific content, use of prior work (conceptual analysis of content, prior work with children, prior work with adults), and theoretical framework.

At PME-NA 2007 we agreed on the need for the construction of a research base for the study of preservice teacher content knowledge. This includes a need to synthesize and summarize existing (completed and current) research and develop a research agenda. At PME-NA 2009 we began this work by dividing content areas between organizers and synthesizing the existing literature. At AMTE 2010, we met to refine the guidelines for creating these individual syntheses and at PME-NA 2010 we plan to refine a rough draft of the combined syntheses (we encourage interested participants to contact the first author for a copy of this draft before the conference). The aims of this working group are to continue established collaborations (synthesis paper, developing our pedagogy, addressing the pedagogical issues with artifacts of children's mathematical thinking), develop new collaborations, and support each other in our work.

Outline of Working Group Sessions

In our first session, we will start with a brief introduction and overview of the working group. This will be followed by a brief summary of the last working groups (PME-NA 2007, 2009) and other activities since (working groups at AMTE 2008, 2009, 2010; symposium at NCTM 2007).

We will then share the current version of the individual sections of the synthesis paper. Using a common framework each content area leader will present a 5-10 min presentation on what the findings are in their area.

After individual presentations we will discuss commonalities and differences among the various content areas and discuss the synthesis paper as a whole (drafts will be available to participants).

In our second session, we will address the use of children's mathematical thinking artifacts to address/develop PSTs specialized content knowledge. We will pay particular attention to the constructs of noticing and authentic listening. Noticing requires PSTs to (a) attend to the child's strategy, (b) interpret the child's mathematical understanding, and (c) deciding how to respond on the basis of the child's understanding (Jacobs, Clement, & Philipp, 2010). Authentic listening requires support of the talker through the listener's attention (Weissglass, 1990) and a change in the views of both the talker and the listener as a result of the interaction (D'Ambrosio, 2004; Davis, 1997). Tyminski, Kastberg, Richardson, & Winarski (2010) have noted phases of listening within the development of authentic listening: readiness, hearing, understanding, and connecting. PSTs need to be able to develop authentic listening skills as well as ask questions of children that elicit their mathematical talk for listening.

Questions we will consider: Are noticing and listening skills necessary components of PSTs specialized content knowledge? What protocols can be developed to help PSTs develop such skills?

In our third session we will synthesize the current status of the working group and discuss modes of communication to sustain collaboration to investigate the research questions emerged from the first and the second sections throughout the year. A calendar of discussion chats will be established and published through an online forum.

Follow-Up Activities

Participants will establish an on-line forum for open discussion of topics of interest. In addition monthly online chat meetings will be established. Possible collaborations may include joint research projects, mini-conferences, and a book proposal to the Mathematics Teacher Education series at Springer, and the publication of joint research articles.

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RESEARCH AND PRACTICE ON LESSON STUDY: WORK TO DATE AND FUTURE DIRECTIONS

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The Lesson Study Working Group met four previous times: Merida, Mexico (2006), Lake Tahoe (2007) and Morelia, Mexico, (2008) and Atlanta (2009). The outcome of the work of over 30 researchers at these meetings is a book under contract with Springer Publishers. At this the final meeting of the working group, chapter authors will share their research and practice as documented in the chapters of the book. Following is a summary of the work that will be discussed. Newcomers are welcome to one or all of the working group sessions. This will be followed by a discussion of future directions for research on lesson study.

Lesson Study Overview

Structures, History, and Variation

Lesson study incorporates characteristics of effective professional development programs identified in prior research: it is site-based, practice-oriented, focused on student learning, collaboration-based, and research-oriented (Bell and Gilbert, 2004; Borko, 2004; Cochran-Smith and Lytle, 1999, 2001; Darling-Hammond, 1994; Wang and O'Dell, 2002; Little, 2001; Hawley and Valli, 1999; Wilson and Berne, 1999). What separates lesson study from other instructional improvement approaches is that it places teachers at the center of the professional activity, with their interests and desire to better understand student learning based on their own teaching experiences. The idea is simple: teachers organically come together with a shared question regarding their students' learning, plan a lesson to make student learning visible, and examine and discuss what they observe. Through multiple iterations of the process, teachers have many opportunities to discuss student learning and how their teaching affects it.

After identifying a lesson goal, teachers plan a lesson. The goals can be general at first (e.g., how students understand equivalent fractions), and are increasingly refined and focused throughout the lesson study process to become specific research questions at the end (e.g., strategies students use to compare 2/4 and 3/6). Teachers choose and/or design a teaching approach to make student learning visible, keeping their lesson goal in mind. The main purpose of this step is not to plan a perfect lesson but to test a teaching approach (or investigate a question about teaching) in a live context to study how students learn. As they plan, they anticipate students' possible responses and craft the details of the lesson. Teachers come to know the key aspects of the lesson, to anticipate how students may respond to these aspects, and to explore different thinking and reasoning that may lie behind the possible responses. During planning, teachers also have an opportunity to study curricular materials, which can help teachers' content knowledge development. During the lesson, teachers attend to student thinking and take notes on different student approaches. During the debriefing after the lesson, teachers discuss the data they have collected during the observation.

There are other professional development programs that incorporate many of the characteristics of lesson study (e.g., action research, teacher research). However, what sets lesson study apart is the *live* research lesson. The live research lesson creates a unique learning opportunity for teachers. Shared classroom experiences expose teachers' professional knowledge that may otherwise not be shared: teachers notice certain aspects of teaching and learning, and this implicit and organic noticing does not happen in artificially replicated professional development settings.

In Japan, lesson study has been widely used for over a century. Many Japanese educators attribute success in changing their teaching practice to participation in lesson study (Lewis, Perry, and Murata, 2006; Murata and Takahashi, 2002: Shimizu, et. al., 2005). As a foundational mechanism to support the improvement of teaching, lesson study is used to examine and better understand new educational approaches, curricular content, and instructional sequences introduced in Japan. In many cases, teachers play the central role in making new approaches adoptable and content accessible. Lesson study makes teaching approaches more practical and understandable to teachers through developing deeper understanding of content and student thinking. In this manner, lesson study works effectively to connect theory and practice.

While lesson study is known in the United States (and other parts of the world) as a small, school-based collaboration, typically in the subject area of mathematics, lesson study comes in many different shapes and sizes in Japan. There is small and school-based lesson study as well as large-scale, national-level lesson study (Murata and Takahashi, 2002; Lewis and Tsuchida, 1998; Shimizu, et. al., 2005). Different formats for lesson study meet different needs and interests of the teachers. A typical Japanese teacher has multiple opportunities to participate in lesson study throughout his/her professional career.

Introducing Lesson Study to the World

Lesson study came to attract the attention of an international audience in the past decade, and in 2002 it was one of the foci for the Ninth Conference of the International Congress on Mathematics Education (ICME). It subsequently spread to many other countries and more than a dozen international conferences and workshops were held around the world in which people shared their experiences and progress with lesson study as they adopted this new form of professional development in their unique cultural contexts (e.g., Conference on Learning Study, 2006; Fujita, et. al., 2004; Lo, 2003; National College for Educational Leadership, 2004; Shimizu, et. al., 2005).

Issues of fidelity of implementation

There are several issues and concerns around *implementing* lesson study in diverse settings. For example, there are unique issues of implementation with preservice or inservice teachers. Other issues include: content knowledge competency, variations in curricula, time and availability to meet, administrative support and cultural differences. As a result of these difficulties in maintaining fidelity of implementation with the Japanese model, lesson study is being *adapted* to meet unique needs in a variety of settings.

The limited depth of mathematical knowledge of some teacher groups attempting to implement lesson study, particularly at the elementary-level, has raised the question of whether lesson study work can be completely teacher-driven. Related to this issue is the role the outside coach or expert should or could play in such a lesson study community. A second issue is implementation within existing, traditional and/or rigidly structured curricula. Unlike the Japanese mathematics curriculum which provides a loosely defined framework for teachers to build off of, many curricula used in other countries are quite structured or scripted and not conducive to the planning cycles used in lesson study. A third issue is a lack of administrative support necessary to alter existing curricula, provide financial support, and schedule opportunities to meet and plan. The daily schedule of most *elementary* teachers prevents regular meetings and opportunities to collaborate. A related issue is the limited extrinsic reward available. Lesson study presupposes blocks of time for teachers to work together. This frequently must be outside school hours and districts are often constrained by contracts that demand stipends and/or release time. Finally there may be fundamental differences in the cultures of teachers from different communities and countries. It was suggested that the fiercely independent nature of some cultures may limit success in building collaborative groups.

Lesson Study: Research and Practice

The body of knowledge about lesson study is growing, but remains somewhat elusive and composed of discrete research endeavors. While the literature suggests that lesson study can facilitate greater reflection and more focused conversations about teaching and learning than are often realized with other types of professional development (Lewis, 2002), as well as specific and authentic conversations about management, student learning, and impact of significant and subtle changes in lesson design (Marble, 2006), there is still much to be learned. Following are summaries of the work that will be shared.

Inservice Teachers

Three reports will be given within the group looking at lesson study with Inservice Elementary Teachers. Rachelle Meyer and TrenaWilkerson (Baylor University, US) conducted a multiple case study which examined the effects lesson study had on middle school mathematics teachers' content knowledge. Participants for this study consisted of 26 middle school mathematics teachers, from a large urban school district, who formed eight lesson study groups. The researchers sought to examine the experiences and impact lesson study had on the participating teachers' content knowledge in mathematics from the eight case studies. More specifically, the researchers focused on (1) the need for teachers' content understanding while planning the research lesson and (2) the participating teachers' growth in content knowledge. This qualitative research used seven measures to gather data which consisted of the following: two baseline surveys; transcripts from planning and reflection sessions; observation notes; lesson plans; and a reflective questionnaire. Analysis of the data consisted of both a within and across case comparison. For the with-in case analysis, each case was first treated as a comprehensive case in and of itself. Once the analysis of each case was completed, a cross-case analysis began in order to develop more sophisticated descriptions and more powerful explanations. Data revealed lesson study did improve teachers' content knowledge for three of eight case studies as a result of teacher collaboration.

Lynn C. Hart (Georgia State University, US) and Jane Carriere (City Schools of Decatur, US) describe implementation of a lesson study project with third grade teachers in a small school district to study the development of the critical lenses (habits of mind) necessary for meaningful lesson study work. Adapting the lesson study process to meet school system needs, two outside facilitators stimulated development of the critical lenses through mathematics explorations and probing/what if questioning. Using a qualitative methodology and the group as the unit of analysis, data were coded for evidence of and change in the lenses. After one year, the 8 participating teachers showed a qualitative difference in two of the three lenses: the student lens and the curriculum developer lens. No change was seen in the researcher lens.

Jo Clay Olson (Washington State University, US), Paul White (Australian Catholic University), and Len Sparrow (Curtin University, Australia) suggest that while lesson study is an effective model for teacher professional development and growth in Japan, there is less evidence that such a model is viable in international settings. This chapter reports on the experiences of five elementary teachers with a lesson study approach to professional development over a year. Two groups were formed, but the results of their experiences were different. One group accepted the challenges highlighted by the lesson study process. They reflected on and changed their classroom practice in fundamental ways. The other team rejected the challenges and maintained their traditional pedagogy. System-wide requirements, for example state testing, constrained the development of one team while the ability to personalize insights from the lesson study process and critical reflection became a catalyst to personal professional growth for the other.

Diane Tepylo and Joan Moss (University of Toronto) examine the Mathematical Knowledge for Teaching (MKT) of four grades 5 and 6 teachers in a small-town elementary school as they participated in three cycles of lesson study on teaching fractions. The school, in a rural school board, is a dual-track school (English and French Immersion) resulting in classes in the English stream with 25- 48% of students working on Individualized Educational Plans as compared to the 18% provincial average. The four teachers, Jeri, Brenda, Leslie and Francis, ranged in age from 23 to 49 with three months to eight years of teaching experience. Before this study, the teachers had little exposure to mathematics professional development and none had any experience with lesson study.

Preservice Teachers: The next section present work with preservice teachers. Maria Lorelei Fernandez (Florida International University, US) and Joseph Zilliox (University of Hawai'I, US) present the work of two mathematics educators, each using a lesson study approach with prospective teachers of mathematics. One educator worked with prospective secondary teachers in a Microteaching Lesson Study context and the other worked with prospective elementary teachers during initial field experiences in K-6 schools. The lesson study experiences in both contexts incorporated important features of Japanese lesson study including operationalizing an overarching learning goal driving recursive cycles of collaborative planning, lesson observation by colleagues and other knowledgeable advisors, analytic reflection, and ongoing revision. The prospective teachers exposed their knowledge, beliefs and practices to the scrutiny of peers and other experts, developing and reconsidering their thinking and practices through collaboration on shared teaching experiences. Similarities and differences in the secondary and elementary prospective teachers' experiences and learning in relation to elements comprising the lesson study approaches are discussed. Similarities included trajectories of their lesson plans toward more student-centered teaching, importance of negotiation for their learning, and value of the cooperative nature of the experiences for sharing varying ideas and perspectives. Differences included development of mathematics knowledge, extent of focus on classroom processes and management, use of videotaped lessons, conduct of oral reports of their group lesson study, and participation of knowledgeable advisors.

Aki Murata and Bindu Pothen (Stanford University, US) outline how lesson study is used in preservice elementary mathematics methods courses to support preservice teachers' connections between their emerging practice and understanding. The course structure is described, and week-by-week course activities and assignments are summarized. Lesson study in preservice teacher education program has a potential to support on-going teacher learning by connecting the course experiences with field-based assignments. By continuously focusing on student learning of mathematics, research lesson teaching ties together the various experiences in the course to help preservice teachers develop new understanding of their practice. Pedagogical content knowledge is meaningfully developed in the collaborative learning settings. Short summary of research findings on teacher learning is presented based on quantitative and qualitative data.

Paul W. Yu (Grand Valley State University, US) presents a theoretical framework based on a review of the literature across two different areas of research in the mathematics education of pre-service teachers: field based experiences (Zeichner, 1981) and Japanese lesson study (Stigler and Hiebert, 1999; Takahashi and Yoshida, 2004). Zeichner (1981) discusses two contrasting issues related to pre-service teachers’ field based experiences. First, these field-based experiences are perceived to be a necessary component for teacher preparation. In contrast, some scholars question the significance of the experience other than an enculturation into the existing socio-cultural norms of the teaching profession. An emerging framework for in-service teacher improvement is lesson study. The chapter reflects on the use of lesson study as a framework for pre-service teachers' field-based experiences that takes place early in their collegiate coursework. The goal was to use lesson study to give students a different model for professional development, that is (1) collaborative, (2) focused on children’s understanding of mathematics, and (3) exposes the pre-service teachers to the nature of mathematics instruction. The chapter describes how lesson study was modified to accommodate the difference between pre-service and in-service teachers’ experiences, and reflects on the enactment of these modifications in the collegiate course.

Beyond K-12: The third group looks at lesson study in university settings. Alice Alston, Lou Pedrick, Kim Morris, and Roya Bassu, (Robert B. Davis Institute for Learning in the Graduate School of Education, Rutgers University, US) were engaged with teachers and administrators of partnering districts in implementing school-based professional development using a modified form of lesson study as part of a graduate course. During a semester a group of ten teachers resourced by two university researchers worked together to develop a series of mathematical tasks intended to embody concepts that are basic to their district’s curriculum and address specific mathematical goals that they had identified as important for their students. The series of tasks were implemented in six classes including grades 5 through 8 that were taught by members of the group during several weeks at the end of the term. The teacher-researchers, studying the videotapes, observer notes and student work from their session, select, transcribe and analyze critical events from each class that provide evocative examples of the mathematical strategies and representations of their students, These analyses are shared in follow-up discussions and compiled to produce an overall analysis of the development of the mathematical ideas as evidenced in the mathematical activity of the students across the four grades involved in the lesson study project. The research of the university educators is based on data that includes notes from the earlier sessions when goals were set and the tasks developed as well as the videotapes of the implementations, debriefing discussions, and the subsequent analyses and group discussions of the teachers. This analysis focuses on the teachers’ reflections and actions for evidence of a shift from surface characteristics of the classroom activity toward a closer attention to students’ thinking and subsequent implications for instruction.

Andrea Knapp (University of Georgia, US), Megan Bomer (Illinois Central College, US), and Cynthia Moore (Illinois State University, US) focus on the professional development of two Coaches (graduate students) of mathematics teachers and one classroom Teacher as they engaged in the lesson study process. The Coaches progressively designed, taught, and refined standards-based lessons which they co-taught with the classroom Teacher. Participants developed three aspects of mathematical knowledge for teaching (MKT): knowledge of content and teaching (KCT), knowledge of content and students (KCS), and specialized content knowledge (SCK)

(Hill, Rowan & Ball, 2005). KCT developed as the Coaches and Teacher collaborated during lesson study to place a stronger emphasis on inquiry in lessons. In particular, Coaches investigated reform curricula and research which enhanced the Teacher's KCT. In addition, the group developed KCS as they listened to students and observed them on videotape. Furthermore, Coaches developed KCS from the Teacher as the Teacher shared with Coaches his knowledge of student difficulties. Finally, the Coaches and Teacher developed SCK by considering mathematical perturbations from the lesson with the Teacher. Thus, lesson study mutually enhanced the teaching abilities of both Coaches and the Teacher whom they supported.

The next chapter describes how Michael Kamen and Stephen Marble (Southwestern University, Georgetown, Texas) became interested in how to adapt lesson study to the university level. A pilot project with colleagues from Southwestern University was implemented that used a collaborative professional development model based on lesson study for structuring peer visitation. The project culminated in the facilitation of lesson study, with four research teams consisting of professors from five academic departments. The second and third projects involved two lesson study teams with instructors from other universities (Debra L. Junk, Texas Regional Collaboratives at The University of Texas-Austin; Sandi Cooper, Baylor University; Colleen Eddy, University of North Texas; Trena Wilkerson, Baylor University and Cami Sawyer, Awatapu College, Palmerston North, New Zealand). One of the teams developed a research lesson on teaching preservice teachers about assessment issues during hands-on science instruction, and the other was a research lesson designed to help preservice elementary teachers understand the importance of supporting children's learning with invented mathematics problem-solving strategies. In this chapter the authors focus on the last of these three projects, an elementary mathematics methods research lesson.

The Importance of the Task within the Curriculum

Part Four of the book looks at the importance of the task in lesson study. Brian Doig and Susie Groves (Deakin University, Australia) and Toshiakira Fujii, (Tokyo Gakugei University, Japan) present an argument for focusing lesson study approaches to teacher professional development firmly on the type and rôle of the mathematical task (*hatsumon*) used in the mathematics classroom. Drawing on research conducted in Australia and Japan, the authors argue that not all elements of lesson study, or particularly the research lesson, are equal in the impact that they have on children's learning. Further, it is demonstrated how Japanese educators place a strong emphasis on task selection, and that this effort is largely ignored by non-Japanese adapters of Lesson Study. Finally, the authors suggest that in order to use lesson study effectively in non-Japanese mathematics classrooms, it is necessary to build on the current practices of teachers that are commensurate with the elements of lesson study. Examples, from Japanese and Australian classrooms, are presented as illustrations of how the selection of the task is critical to the outcomes of the lesson.

Jacqueline Sack (University of Texas, Houston, US) and Irma Vazquez (University of St. Thomas) share how two teacher-researchers and a teacher apply the principles of lesson study in their research process for developing a 3-dimensional visualization program for elementary children. They share a common belief that children learn best through social constructivist approaches (Cobb et al., 2001) with explicit opportunities for differentiated instruction (Tomlinson and McTighe, 2006). While many lesson-study experiences offer teachers opportunities for personal professional development for deepening their pedagogical content knowledge, their focus is to develop and investigate new curricular materials that enable students to move among various visual and verbal representations (van Niekerk, 1997). They utilize a dynamic computer interface, Geocadabra (Lecluse, 2005) that simultaneously integrates several of these representations. The goal is to extend the body of knowledge on how children think and learn about geometric space, ultimately to publish instructional materials to support children's development of 3-dimensional and 2-dimensional spatial reasoning skills. The study takes place in a linguistically- and academically-diverse inner-city school, during its after-school program, with third- and fourth-grade students, for one hour each week for each grade level. They illustrate our adaptation of lesson study as our process for lesson design, enactment, reflection, and iterative re-enactment.

Penina Kamina (SUNY College at Oneonta, US) and Patricia Tinto (Syracuse University New York, US) argue that practicing teachers are often at the heart of reform initiatives and often with little professional development or support. Teachers wrestle with shedding their old pedagogical beliefs, understanding mathematical content, and learning how to use curricular materials such as *Investigations* (Putnam, 2003). The discrepancy between the implementers' prior experiences, National Council of Teachers of Mathematics (NCTM) principles, and *Investigations'* objectives presented an important problem for study. A qualitative case study research design was used to explore teachers' implementation of *Investigations'* mathematics in fifth-grade classrooms. Data were collected in the form of lesson plans and audiotape and videotape of lesson study meetings. Results of this study showed that teachers that collaborated with each other in lesson-study meetings were quickly able to establish new classroom instructional approaches and implement new curriculum. Their enhanced content knowledge, pedagogical knowledge, and reformed pedagogical beliefs that emerged from participating in lesson study enabled these teachers to be versatile in implementing the *Investigations* curriculum.

Unpacking Lesson Study: More than three easy steps

Dolores Corcoran (St Patrick's College, Dublin City University, IRE) presents results from a study to shed light on aspects of lesson study that emerged during its use as part of a primary teacher education programme in Dublin, Ireland. She describes five aspects to her role as broker and varying degrees of power associated with each of them. In an effort to make these more visible, she examines the junctures at which they appeared to interact with student teachers' roles, by first listing the different roles she was aware of assuming throughout the program. She was a college lecturer /researcher/data manager. She was also lesson study co-coordinator/audio-visual technician and manager. She was Knowledgeable Other. She sought to influence students' attitudes to mathematics and mathematics teaching by offering them opportunities to air and discuss their stories about learning mathematics. This aspect of her role focused on 'affect' in relation to learning mathematics and was welcomed by the group as an opportunity to strengthen the *identity of participation* of the members (Wenger, 1998, p. 215). Student teachers in the study showed marked changes in how they approached planning for, and teaching of, mathematics.

Mary Pat Sjostrom (Chaminade University of Honolulu, US) and Melfried Olson (University of Hawai'i at Mānoa, US) describe the experiences of one group of elementary school teachers as they engaged in a three-year professional development experience culminating in a one-year lesson study program. The partners in this project, university professors, school administrators and teachers, worked together to modify the professional development plan to serve the needs of the teachers and students. Although lesson study was not part of the original plan, it became the focus of year three. This case study illustrates the difficulties encountered in introducing lesson study, and examines the way in which the components of the first two years, notably the Reflective Teaching Model, collaborative problem solving and analysis of student work, helped pave the way for success in lesson study in year three.

Final Thoughts

The research and practice on Lesson Study shared in this working group only begins to uncover the complexity of the process and the myriad of opportunities to learn more about teacher development in mathematics education. We have identified many salient themes and issues that will require in-depth study to tease out their impact on lesson study communities. We hope this will take us to the next phase on our work on lesson study.

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REPRESENTATIONS OF MATHEMATICS TEACHING AND THEIR USE IN TEACHER EDUCATION: WHAT DO WE NEED IN A PEDAGOGY FOR THE 21ST CENTURY?

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"Facilitating sessions where teachers interact with and discuss representations of teaching"[1]

Brief history of the Working Group

This is the first meeting at PMENA of this RMT working group. The idea of this working group emerged during a series of three-day conferences on representations of mathematics teaching held in Ann Arbor, Michigan in August 2009 and June 2010. These conferences were organized by project ThEMaT (Thought Experiments in Mathematics Teaching), an NSF-funded research and development project directed by Herbst and Chazan. ThEMaT originally created animated representations of teaching using cartoon characters to be used for research, specifically to prompt experienced teachers to relay the rationality they draw upon to justify or indict actions in teaching. The project also aimed to disseminate those animations to be used in teacher development and for that purpose held summer workshops in 2007 and 2008. The workshops evolved into the RMT conferences in 2009 and 2010, whose purpose was to gather developers and users of all kinds of representations of teaching to present their work and discuss issues that might be common to them. RMT conference participants included users of video, written cases, dialogues, photographs, comic strips, and animations. An outcome of the 2009 RMT conference was a special double issue of the journal *ZDM--The International Journal of Mathematics Education*, guest edited by Herbst and Chazan; the issue will appear in 2011 and contains 15 papers, currently at various stages of post-review production. Discussions at the 2010 conference created agendas for working groups and sessions in other, more prominent conferences. One of those agendas, based on a discussion about the facilitation of sessions with representations of teaching and the tools available to facilitate those sessions online and face-to-face, and for different clienteles[6] (practicing teachers, preservice teachers, teacher leaders, others), stimulated this working group.

Issues in the psychology of mathematics education that will be the focus of the work

The working group is focused on elaborating a pedagogy of mathematics teacher education assisted by representations of teaching. Like other technological innovations, representations of teaching not only offer opportunities for teachers' learning but also call for specialized pedagogical practices from teacher developers. The working group will engage in such elaboration in two ways. First, the work will consist of using some conceptual and technological tools (described below) to design some teacher development experiences. Second, reflections on such design work are expected to call attention for more and better tools.

For some years now, teacher educators worldwide have used classroom video records, samples of student work, narrative cases, and other artifacts to engage teachers in discussions about teaching (Fishman, 2003; Lampert & Ball, 1998; Merseth, 2003; Sherin & Han, 2004; Smith, Silver, & Stein, 2004; Tochon, 1999). These artifacts afford opportunities for teachers to learn from practice, whether this learning focuses on pedagogical or mathematical aspects of the work of teaching or on understanding students and their thinking. More recently, animations and comic books using cartoon characters have been created and used for similar purposes (Herbst, Chazan, Chen, Chieu, and Weiss, in press). In parallel, advances in information technologies have made it easy to create and manipulate rich media objects (graphics, photo, video) and share them in the Internet where they can be tagged, commented, and repurposed. This technology enables collaborative work across geographic boundaries. More importantly, it enables a different kind of work with records of teaching, particularly work that, by enabling more detailed and active experiences with the media, has the potential to increase learning opportunities for clients of teacher education.

Materials exist describing how to use some of these artifacts in teacher learning contexts (e.g., Merseth, 2003; Seago, Mumme, & Branca, 2004). The scholarly literature has also addressed facilitation in the context of describing teacher learning from professional development (e.g., Borko, 2004) and even the learning of the facilitators (e.g., Stein, Smith, & Silver, 1999). We choose not to review this literature here.

Our present purpose is to stimulate the development of pedagogical practices attuned to the possibilities that novel media--particularly cartoon-based representations of teaching--and new technologies offer for teacher development. Herbst, Chazan, Chen, Chieu, and Weiss (in press) have argued that cartoon-based representations of teaching can have virtues similar to video (e.g., the possibility of an animation to immerse the viewer in a timeline and cadence of events comparable to that of real action) as well as some of the virtues of written cases (e.g., the possibility of cartoons to represent selected facets of the individuality of people and settings rather than show by default as many of those facets as the recording technology allows). These characteristics add to available technologies that permit to create, annotate, and reuse computer graphics and interact with others about them. These tools make cartoon-based representations of teaching a malleable medium for learning in, from, and for practice (Lampert, 2010). In calling for the development of a pedagogy adapted to the use of representations of teaching we operate on the assumption that representations of teaching can do more than support usual teacher development activities. They can also create new spaces for the development of professional knowledge and skills. In a way this is analogous to how the availability of new technological artifacts (calculators, computers) not only permits the emergence of new ways of knowing but also requires novel pedagogical practices to fulfill their promise.

In this document we contribute to a discussion on a pedagogy of mathematics teacher education assisted by representations of teaching by proposing some basic categories for such pedagogy. We also provide examples that can get the working group started in the work of fleshing out such pedagogy. We start this discussion, however, with a more basic conceptualization of representations of teaching that can underscore the important role that cartoon-based representations can play in teacher development practice.

What is a Representation of Teaching

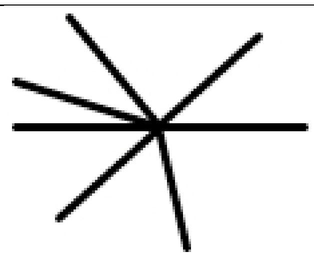
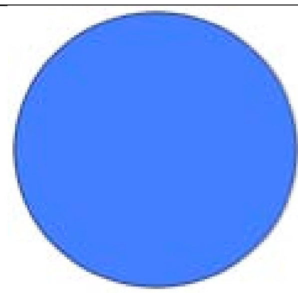
The expression “representation of mathematics teaching” suggests a semiotic mediation—a *sign*, or *representamen* (Peirce, 1955) pointing to mathematics teaching as the *referent*, the *object* represented. That expression might trigger associations with the notion of representation in the teaching of mathematics. At first blush those associations could be dispelled just by noting that the preposition “of” refers not to the role that representations may play in teaching mathematics but to the representation of the practice of teaching mathematics itself and the role that these can play in the learning of teaching. But on second thought, the extant literature on mathematical representations might be of use in understanding what a representation is and what role it could play in the learning of teaching. We come back to it below, after we consider the object or referent in representations of teaching.

The question of what is mathematics teaching has been vastly addressed in the literature. The teacher’s work used to be seen as administering an unproblematic body of subject matter to children and, or alternatively, cultivating a personal relationship with children. But successive improvements in conceiving the subject of studies, students’ cognition and learning, and eventually the work of teaching itself, have contributed to portray a rather complex profession (see Doyle, 2006; Fenstermacher, 1994; Lampert, 2001). The conception of teaching proposed by Cohen (in press) asserting that teaching is a practice that deliberately attends to students’ learning of disciplinary subject matter by attending to the representations of disciplinary knowledge, the cognitions of students, and the instructional medium in which teacher and student interact, seems useful as a starting point in describing the object or referent in representations of teaching.

Representations of teaching could help the work of teacher developers by pointing to the many tasks of teaching that derive from the three domains of teacher work that Cohen describes. The notion that the work of the teacher includes attending to the representation of disciplinary knowledge includes tasks of teaching such as selecting or designing embodiments of mathematical ideas, formulating mathematical statements that are true, crafting mathematically compelling explanations, identifying errors, choosing problems for students that give opportunity to use target mathematical ideas, etc. The notion that the work of the teacher includes attending to the cognitions of students includes tasks of teaching such as eliciting students’ thinking, interpreting students’ conceptions, creating and issuing specific challenges to students’ conceptions, etc. The notion that the work of the teacher includes attending to the instructional medium includes a number of diverse tasks of teaching associated with shepherding interpersonal dynamics, communication through personal relationships, and the affordances and constraints of the institution where the work is done (e.g., using well the time allotted for class or the space allotted for public displays; see also Ball, Thames, and Phelps, 2008; Lampert, 2001). The work of learning to teach includes learning to attend to those three commonplaces, eventually acting in such a way as to exercise attentiveness to those different foci simultaneously and over relatively long expanses of time. From such a brief, initial description of mathematics teaching as the object of learning it should be apparent that we are talking about an object of study that potentially makes high demands on intellectual and performance capacity. Experiences that scaffold that learning, enabling learners of teaching to engage with and learn about some features of that complex practice while keeping others simple, may succeed in creating those capacities incrementally. Representations of teaching can display particular enactments of those tasks; cartoon-based representations of teaching, particularly those realized with non-descript characters, can help focus on the enactments and the tasks enacted rather than on the actors themselves (McCloud, 1994). Along those lines an analogy with representations of mathematical ideas can be quite productive.

Mathematical representations can help make some aspects of mathematical ideas salient by embodying the structure or function of those ideas in the structure and function of the sign system that performs the representation (see also Goldin, 2003; Resnick and Omanson, 1986; Schoenfeld, 1987). Peirce (1955) addresses this with the notion of the *interpretant*, a decodification of the sign into another sign (what some have called an internal representation). A representation of a mathematical object by a sign is thus realized through the creation of an interpretant that basically establishes how the sign points to the object (and what aspects of the sign predicate about the object).

Various kinds of mathematical signs (such as those in figure 2) can perform a representation of a mathematical idea (e.g., circle); they do so by creating an interpretant that makes the mapping between sign and object operational, by noting how the sign calls forth the object. Cartoon-based animations and comic strips (and for that matter also video clips of images of real people and written words and sentences) can do the same with respect to teaching. The role of any of those representations of teaching is to build the interpretant–to enable ways of thinking about teaching called forth by the sign that might progressively grasp the complexities of the practice itself. While the things represented, mathematical ideas on the one hand and the practice of mathematics teaching on the other, are quite dissimilar, we might learn something about what representations do by working through a metaphor between them. Like any metaphor one could push this one to limits where it would stop making sense, but like any metaphor, this metaphor can, within boundaries, help us understand the power of some representations of teaching in regard to the development of the interpretant.

		$\{(x,y) \in \Re^2 / (x-a)^2 + (y-b)^2 = r^2\}$ The intersection of a sphere with a plane
Figure 1. Several positions of a spinner	**Figure 2. Other representations of circle realized with different semiotic resources (diagram, symbols, language)**	

We'd like to propose an analogy between a particular representation of circle (the spinner alluded to in Figure 1) and the representation of the practice of mathematics teaching made possible through comic strips or animations of cartoons characters. Of course there are other representations of circle, such as those shown in Figure 2. The spinner in Figure 1 contrasts with all of those in Figure 2 in its dynamic nature. One could think of the spinner as creating a message over time that says, "this is a circle." At any one moment in time the spinner only points at one point, as if writing one letter of that message. One has to look at it over some length of time to gather what the aggregate is doing. The spinner also contrasts with the other representations in regard to how it shows not just what the circle is but also how the points in the circle are obtained. Finally, this spinner also represents the circle without actually depicting any of its points—we see the circle by putting into the image a response of sorts to the stimulus provided by the rotating segment, but neither the segment is part of the circle nor the circle is really drawn there. In all these regards cartoon-based representations are like the spinner.

The spinner representation of the circle reminds us of Walt Whitman's[7] words "all music is what awakes from you when you are reminded by the instruments." In Peirce's terms, the representamen (or sign) is the collection of strokes at different locations, sorted over time; the referent or object is the mathematical notion of circle, say, the set of points on a plane at a fixed, given distance from a given point on that plane; the interpretant notes that the strokes are the same length and start from one common point at each moment while the other point makes over time a familiar figure. Indeed, through the interpretant, the moving strokes remind us where the circle's points should be, and over time they remind us where the complete circle should be; finally the interpretant elides the moving strokes–they are not the circle. The moving strokes are like the instruments in Whitman's quote, the notion of circle could be the orchestral arrangement handled by the orchestra director, while the points becoming the circle are the music we hear. It can also be noted that we see the circle through the work of the spinner partly because we have an idea of what the circle should be—the interpretant has some prior associations such as for example one between the sign in Figure 2a and a synthetic idea of the circle (e.g., an equivalence class of all the diagrams that look like that one).

Just as the spinner shows over time how the circle is made to someone who has a prior idea of what a circle is supposed to be like and without actually creating a physical sign for the circle itself but by evoking it from the viewer, a representation of teaching can point us to how teaching is done even if the signs themselves don't show each of its actions but just point to them. A comic strip including a sequence of photographs of the teacher writing at the board can portray an explanation or the setting up of a task (see also Crespo, 2010). So can a comic strip containing snapshots of action in response to a problem such as those shown in Figure 3. The interpretant associates those signs with the actual events that could happen in practice, say when a student solves a problem about linear functions, and can not only fill the gaps in between frames but also predict the actions that might come after.

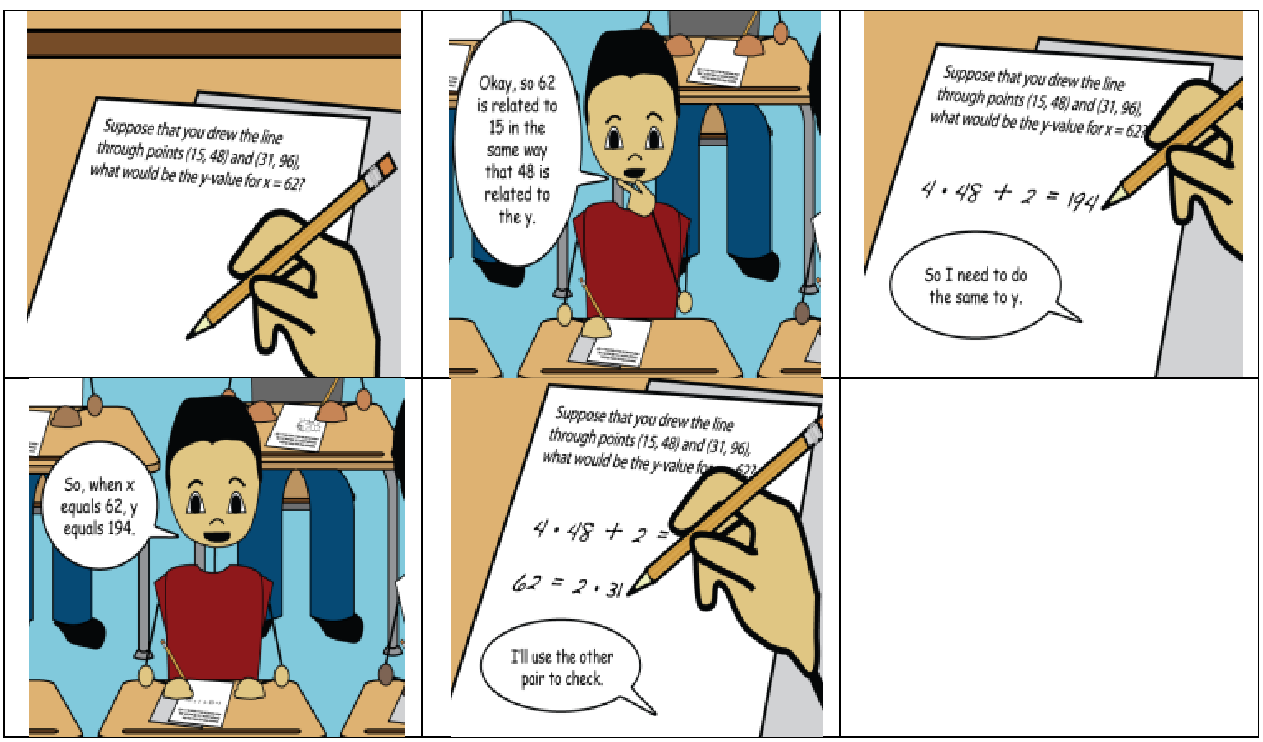


Figure 3. A solution to a linear function problem that reveals a misconception at play

Building the Interpretant

All communication about ideas, be that in mathematics instruction or in teacher education, happens through transacting representations; at the very least because language is also a semiotic system. In regard to transacting about the practice of teaching, we'd say that language can be quite an abstract semiotic system. Words such as *constructivist* or *traditional* may be useful to name chunks of action in the manner that the English word "circle" names the mathematical object; but those words are not like the spinner that tells you how a circle comes to be. For teacher education we need representations that help connect *theory* of what teaching is with *practice* on how to do it. Note, along those lines, that the spinner allows one to interact with the notion of circle, for example by permitting the construction of a different, static representation of the circle (such as the one on Figure 2a) by plotting a point at the mobile end of the spinner at various moments in time and then by using that representation to make conjectures about properties of the circle other than the constant distance from a center that defines it. Clearly a symbolic representation of circle, such as the one on Figure 2b also enables one to do things with the circle, and notice different properties. A diagrammatic representation of circle has been a preferred one for beginners for ages. In the case of teaching we ask what kind of sign can produce a representation of teaching that can be so generative of knowledge about teaching as the spinner is for the circle?

Why are these semiotic considerations of any use in thinking about a representations-based pedagogy of mathematics teacher education? Grossman et al. (2009) note the role that representations of practice play in professional education; while they distinguish between *representations* as those artifacts that display practice (e.g., video records), *decompositions* as those artifacts that unpack and elaborate on aspects of practice (e.g., a rubric that outlines the characteristics of an instructional explanation), and *approximations* as those artifacts that create opportunities to engage in practice (e.g., a simulation software that invites teacher candidates to act the teacher in front of virtual students), it seems to us that all of those are to some extent based on a representation of teaching. We prefer the words in their gerund form as *representing*, *decomposing*, and *approximating* practice and use them to designate the activity systems in which the interpretant employs the signs to transact and think about the object of consideration.

The work of mathematics teacher education, the pedagogy of teaching a practice assisted by representations of practice thus involves building the interpretant through activities of representing, decomposing, and approximating practice with different kinds of semiotic resources. It includes, in particular, developing teacher capacity for using signs to describe practice, for 'reading' a practice from signs, and for creating enactments of that practice within the grammar of a semiotic system. Such activities include exercises like "Ms. Shackleforth's lesson," (see Ghousseini, 2008) where teacher candidates are asked to supply the lines spoken by the teacher in a written dialogue where only student lines are provided. Clearly such approximation of practice does not preserve all the complexities of practice; but rather than discarding such approximations for what they miss, we could see them as building some opportunity for practice. In generating the teacher's lines in a hypothetical classroom dialogue, clients have the chance to design the precise words that a teacher would say (rather than just the informational content of their response), offering an opportunity for them to consider the informational and relational entailments of teacher talk in the context of a specific mathematical discussion. While engaging with representations of teaching (doing things with signs) does not substitute engagement with actual teaching, the former scaffolds the latter much in the same way as engaging diagrammatically with a representation of a circle (such as drawing a stroke between a point on the circle and the center) enables the user to think about aspects of the circle (e.g., the notion of radius) without loss of generality.

What kinds of tasks can elaborate the three activities noted above (representing, decomposing, and approximating)? How can facilitation with such tasks engage teachers in grappling with important aspects of practice that they need to learn? This is where we want the working group to get started. A preliminary framework to guide the work of the group is provided below.

A Pedagogical Framework for Facilitation

The framework we propose includes four kinds of elements. One of those can be described at face value as *open-ended expressions* or boundary objects that can be used in transactions between teacher developers and their clients without needing to be completely defined. One of those expressions is "mathematical action" which we observed being used in a geometry class for future teachers and in the context of having the students watch an animation of geometry instruction. The college instructor's goal was to have her clients flag moments that had mathematical significance hoping to get them to bring up things like "making a claim," "offering a counterexample," "extending a claim to a larger set of objects," etc. Clearly, it would have defeated the purpose of the activity to give such a list (and its vocabulary) before watching the movie; and the work could have been stifled or miscued if the instructor had used better understood words (e.g. if she had said something like "flag moments where they are using a mathematical concept," the students might have restricted themselves to uses of known mathematical vocabulary). Other open-ended expressions that can serve comparable purposes are "student thinking" and "teaching move." Their usefulness in mobilizing the work with representations depends on their ambiguous nature—the work set up will then involve discovering or inventing their meaning.

Figure 4. A student's response to a problem inviting the teacher to make a move

The first proposal of this framework is that a pedagogy for mathematics teacher education assisted by representations of teaching needs a set of usable open-ended expressions or boundary objects that the teacher developer can use to formulate, in rather non-descript ways (i.e., not calling attention to the words), tasks that the learner of teaching can do with representations. For example, a client could be offered the student solution to a problem sketched in Figure 4 and then asked to represent, using the cartoon character set, an answer to the question "What would be your next move?" This question is ambiguous in comparison with the more concrete "What would you say next?" and for that reason it might allow the client to make the cartoon teacher do other things than talking. A client could have the teacher erase the board, while another might have the teacher correct the construction, and another might have the teacher ask Alpha what they know about the relationship between radii and tangents. A discussion of those options and the possible justification for these moves could ensue without ever dwelling on the word *move*.

A second element of this framework is relatively expected and documented in earlier literature about facilitation. It consists of having a taxonomy of activity structures or activity types for mathematics teacher education. Clearly these activity types could have elements in common with those found in K-12 classrooms, such as triadic dialogue, homework review, etc. (see Lemke, 1990). But there are other activity types that are particular to the work of mathematics teacher education assisted by representations of teaching. A quite common activity type could be described as "working on the math." Quite often, mathematics teacher developers who intend to show a video that displays students working on a mathematical task will first have their clients work on the mathematics that will be featured in the video. The goal of this activity is not necessarily to teach the mathematics at stake in the problem to the clients, and for that reason the clients may or may not be expected to complete the problem. Yet this activity is done for the clients to develop some familiarity with the mathematics at play in the representation so that they can attend to other things, such as student thinking, when they interact with the representation.

> You are going to teach a lesson on tangents to circles. The lesson includes teaching the procedure for constructing a tangent to a given circle. In your plan you have sketched the procedure as follows.
>
> (1) Given, a circle with center O, and a point P
> (2) Draw segment OP, find its midpoint D
> (3) With compass centered at D draw a circle of radius DO, intersect the circle at points A and B
> (4) With straightedge, draw lines PA and PB, which are the required tangents
>
> Your field instructor indicates that the procedure is okay but that it is not clear how your students will know that the lines are indeed tangents to the given circle. Script a few lines where you explain to the students why is it that the lines they constructed are the tangents to the given circle.

Figure 5. A homework problem for clients to practice a task of teaching

Another activity type we have used in the context of teacher education is a form of review of homework in which clients enact scripts of action that they conceive outside of class in response to practical problems, such as shown in Figure 5, in the context of learning to explain procedures. Usually those enactments give clients some practice in delivering a teaching move that had been planned; they also get other clients involved in giving feedback, and help raise more substantive questions about the task of teaching being learned. In the context of the teacher study groups organized by ThEMaT, an activity structure involved asking participants to watch animations and tap the table to stop the animation so they could make a comment. Stopping the video not only enabled the person who tapped the table to make a comment, but it also opened the floor to other participants to chime in. The video would only continue when the participants were no longer interested in following up. We bring these examples to illustrate the more general contention that a pedagogy for mathematics teacher education that uses representations of teaching needs to include a taxonomy of activity structures. These activity structures could specify in particular how the clients might interact with (manipulate, annotate, etc.) the representations being used.

A tird element of the framework consist of problem types. By this we mean specific intellectual work that participants do within an activity type involving representations. An initial list of types derives from the three words used by Grossman et al. (2009): representing (the client might view), decomposing (the client might study), and approximating (the client might do). Clearly more is hinted—one could expect that participants would not just view a representation and be impressed by it; they might also describe a segment of it or point to special moments. In fact the wording of the tasks do matter (Morris, 2006). Likewise, a representation of teaching useful for decomposing teaching could look like a rubric that unpacks the characteristics of a task of teaching (e.g., how to explain a concept). The study of a decomposition of teaching could involve work such as judging a performance on that task of teaching using the decomposition. As noted above, completing dialogues or comic strips could be used as activity types for approximating teaching. Approximating teaching might include not only problems for clients to conceive the actions they might do by writing lines of dialogue but also rehearsing those actions by working on aspects of professional performance such as voice, use of interjections, posture, board writing, etc. Table 1 provides an initial list of problem types based on what Herbst & Chazan (2006) identified as attributes of cartoon-based representations of teaching.

Table 1. Problem types to engage participants with representations

Problem Type Name	Sample Problem Statement
Alternativity problem	What else could one do at this moment?
Generality problem	What would you call that teaching move?
Normativity problem	What should be done in these circumstances?
Projectiveness problem	How would you have felt if those were your students?
Reflectiveness problem	What do you think about this episode?
Temporality problem	At what moment would you say [such thing] happened?

All of the elements in Table 1 unpack the work that the viewer could do when confronting what Grossman et al. (2009) called a representation of teaching (and what we would call a representation used in representing and viewing teaching). But some of them can also be used in the context of decomposing and studying teaching—for example a normativity problem could be used along with a video record or animation to give learners a chance to learn a rubric about an activity of teaching. For example, they could be given a rubric that describes the components of an instructional explanation, and a video clip that purportedly contains one such explanation (not necessarily a good one); and they could be asked a normativity problem so that they apply the rubric. The problem types listed in Table 1 are only some examples. More generally, the question to the members of the working group is what are other types of problems that could be used along with representations of teaching and in activities such as representing, decomposing, or approximating teaching? Our contention in this paper is that a pedagogy of mathematics teacher education assisted by representations of teaching needs a taxonomy of problem types.

The final element of this emerging framework addresses the technological affordances needed to realize this pedagogy of mathematics teacher education. Clearly one could do many of these activities having only a video projector and playing media off a single computer. But there are important pedagogical considerations associated with more technology-intensive environments. For example, if every client could play the media at their leisure but contribute comments to a common conversation, in face-to-face, chat, or forum, this could have the beneficial effect of diversifying the discussions, particularly in regard to temporality problems. Along those lines a recent study by Chieu, Herbst, and Weiss (in press) shows evidence that clients' comments in forum or chat benefitted from having an embedded screen for the animation being discussed, which they could access at the same time as they interacted with peers in a forum or chat. This media-enabled-forum is one of several functionalities available in ThEMaT Online, a resource for teacher developers to use with their clients[2]. Another functionality present in ThEMaT Online is a lesson sketching software that permits users to develop or contribute to a classroom episode using cartoon characters (Herbst et al., in press). Such lesson sketching software is the first step of what a virtual setting for teacher education could look like; Chieu and Herbst (in review) offer a longer term prospect with their design of a teaching simulator. An authoring tool, which is part of ThEMaT Online, allows teacher developers to create lessons and assessments by juxtaposing different functionalities like those in a sequence of activities. We expect the working group will be able to explore these functionalities, and the potential combinations that could be made with them; we also expect the working group to have suggestions of new functionalities to add.

More generally we think that the development of a pedagogy assisted by representations of teaching requires a menu of tools that facilitates the creation and delivery of experiences for clients as well as the collection of their work. It requires more than a course management system because the representations are not atomic objects. In order to pose temporality problems (see Table 1) for example, it is important to be able to find, tag, and communicate specific moments in the timeline of a representation of teaching. In general, we believe that the continuous development of the pedagogical functionalities of the ThEMaT Online system can be a scaffold in the process of developing a pedagogy of teacher development assisted by representations of teaching.

Plan for active engagement of participants in productive reflection on the issues

The plan includes starting with a brief exposition by the authors of the contents of the paper followed by a collective discussion framed around the following questions:

- On the *facilitation* of discussions or investigations assisted by representations of teaching: What are the possibilities and demands of a pedagogy assisted by representations? What considerations are needed to make when framing the encounter of an audience with an artifact, depending on who the audience is and the specifics of the artifact? What else can be done beyond the general "watch and we'll talk"? How does the multidimensionality of representations (the fact that they involve representations of mathematics and of students' thinking as much as those of teacher action) feature in the organization of encounters with representations? In particular, does it help to organize an earlier encounter with the mathematics of the representation?
- On *learning technologies* to navigate representations of teaching: What is available and what is needed? Web 2.0 technologies have brought up the possibility for users to do more than read or view media artifacts—they can record, share, tag, comment, rate, index, and mix media artifacts. Are those capabilities useful in enabling the work with representations of teaching? What other capabilities are desirable?

We estimate that the first meeting of the working group will be consumed by the exposition and the discussion. The second meeting will be organized around pairs of participants involved in the creation of exemplar sessions for a chosen clientele and around a particular representation of teaching. Participants are invited to bring a representation of teaching they would want to work with, and accompanying notes about facilitation. The organizers will provide some scaffolds in the form of virtual index cards (realized as Power point slides) that contain some of the elements of the taxonomy. People will divide in groups and design a lesson, session, tutorial, or assessment that implements elements of the taxonomy with the representation chosen. The purpose of this task is to push for the development of the taxonomies in a concrete context. Inasmuch as possible these materials will be shared among the members of the group. The third, and final, meeting will provide a forum for subgroups to share these emerging products and the extensions of the taxonomies that these products require. Participants will also discuss opportunities for follow-up activities.

Anticipated follow-up activities

We have been allotted a pre session slot at the AMTE Annual Meeting in 2011. We plan to use that slot to display the exemplars developed at the PMENA meeting and to engage in further work on (1) improving the exemplars and (2) using the exemplars to improve the taxonomies. We hope we will be able to use those products to propose a second iteration of this working group at next year's PMENA.

The characteristics of the working group are such that people may expect collaboration and sharing of materials. We will strive to acknowledge all authors in these materials as well as in the development of the ideas.

Endnotes

1. The introductory content of this paper is taken from the first author's Pattishall Award Lecture at the UM School of Education in May 2010. The work of project ThEMaT has been done with the support of NSF grants ESI-0353285 and DRL- 0918425. All opinions are those of the authors and do not necessarily represent the views of the foundation. We thank Emina Alibegovic for important comments.
2. We use the word "client" to refer to the students of teaching. Among other things, this choice prevents ambiguity with our references to students in the representations of teaching that those clients might be exposed in their teacher development opportunities.
3. A song for occupations, 4. In Leaves of Grass.
4. ThEMaT Online is a set of representations of teaching and tools to manipulate them. The system is currently being designed for teacher educators to use with their clients. It can be previewed at http://grip.umich.edu/themat.

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SUPPORTING TEACHERS TO INCREASE RETENTION

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National reports have identified the need to increase the pool of highly qualified mathematics teachers as a way to improve mathematics education. However, providing high quality mathematics education for all students goes beyond the recruitment of knowledgeable teachers. This working group is designed to offer an opportunity to examine the role that professional development and support play in the work and retention of mathematics teachers. Retention focuses on new teachers, especially those in urban area and mathematics teachers in hard-to-hire settings. Discussions concentrate on the study of interventions through professional development and support models. Efforts to deepen our understanding of the complex and multifaceted picture of why teachers leave and why they stay, and how efforts to retain teachers impact their work in the classroom and their decisions to stay or leave are developed through the sharing of research designs, data collection, and on-going results. This working group is appropriate for anyone who has work to share or who is thinking about supporting a retention project. Throughout, we address this very complex task both in terms of the opportunities and challenges for the mathematics education researchers to provide quantitative and qualitative input on a major political issue. It is hoped that this working group will enrich the dialogue about a national crisis in mathematics education.

Brief Overview of the STIR Working Group

Although teacher retention is a topic being included in many conferences and position papers, the Supporting Teachers to Increase Retention Working Group met for the first time in 2009 at PME-NA 31 in Atlanta, GA to investigate the relationship between Professional Development/Support and the retention of mathematics teachers. A total of 11 participants attended the three working group sessions focused on identifying the absences in the research on mathematics teacher retention in order to move forward in tackling this complex national issue. The emerging dialogue was based on the results of research studies and ideas of the participants. Participants shared their backgrounds and interests in retention issues, and discussed potential research directions with the ultimate goal of identifying key research issues that would constitute a research agenda for the group. Participants were interested in examining the on-going preparation, support and retention of grade 7-12 mathematics teachers from a variety of angles, including: (1) Impact of Professional Development on Teacher Retention ; (2) Relationship between Content Knowledge and Retention ; (3) Obtaining Research that examines the Dimensions of Professional Development and Support that impact Retention. The overarching issue of Mathematics Teacher Retention is often overlooked under the assumption that effective professional development would in essence lead to increased retention. However a closer look at what type of support helps teachers stay in their school, let alone their profession is necessary.

At PME-NA 32 in Columbus, OH, the group will elaborate on the research agenda that was outlined at PME-NA 31 in Atlanta, GA. To stimulate discussions, members of the group will be invited to provide summaries of on-going projects to be discussed during the group meeting sessions. To date these projects fall under the following categories: (1) Role of technology in support/retention; (2) Professional support community that reflect the building of networks and contacts to support work, decisions, challenges, and opportunities that arise in the teaching of mathematics, including lesson study and electronically based communities of practice; (3) Role of leadership and/or career enhancement in retaining mathematics teachers, including the PD directed towards new teachers entering the field through alternative certifications, those coming from other careers and shifts in PD needed as new teachers move through the challenges of their first 5 years of teaching; (4) Content based Professional development with emphases on conceptual linking and problem solving; and (5) Research issues that arise in examining teacher retention.

Issues of Psychology of Mathematics Education to be focused on

The study of the relationship between Professional Development and Support Models on the Work and Retention of Mathematics Teachers in grades 7 – 12 merits careful examination. Several national reports have pointed to the need to increase the pool of highly qualified mathematics teachers as a way to improve mathematics education and maintain the United States' economic competitiveness (National Academy of Sciences, 2007; Glenn Commission, 2000). However, providing high quality mathematics education for all students goes beyond the recruitment of mathematically knowledgeable teachers to encompass issues of teacher support, professional development, and retention. Over the past two decades, analyses of teacher employment patterns reveal that new recruits leave their school and teaching a short time after they enter. Ingersoll, using data from the School and Staffing Survey concluded that in 1999-2000, 27% of first year teachers left their schools. Of those, 11 percent left teaching and 16 percent transferred to new schools (Smith and Ingersoll, 2003). Earlier research revealed that teachers who leave first are likely to be those with the highest qualifications (Murnane and other 1991; Schlecty and Vance 1981). This "revolving door" is even higher in large urban districts; for example, 25% of the teachers new to Philadelphia in 1999-2000 left after their first year and more than half left within four years (Neild and other 2003). In Chicago, an analysis of turnover rates in 64 high-poverty, high-minority schools revealed that 23.3 percent of new teachers left in 2001-2002.

Reasons for the lack of retention of new teachers and teacher in high-poverty schools are often related to "working conditions" and lack of support (Ingersoll, 2001; Smith & Ingersoll, 2004; Johnson et al., 2004), though pay also plays a role (Hanushek, Kain, & Rivkin, 2001). This support includes professional and collegial support such as working collaboratively with colleagues, coherent, job-embedded assistance, professional development, having input on key issues and progressively expanding influence and increasing opportunities (Johnson 2006). Preparation, support, and working conditions are important, because they are essential to teachers' effectiveness on the job and their ability to realize the intrinsic rewards that attract many to teaching and keep them in the profession despite the profession's relatively low pay (Johnson & Birkeland, 2003; Liu, Johnson, & Peske, 2004).

Recently, a status report on teacher development focusing on professional development and support of teachers (Darling-Hammond et al, 2009) summarized findings and put forth recommendations for effective professional development. The basis for the paper included national surveys with self reported data, a meta-analysis of 1,300 research studies and specific studies. The conclusion is that "well designed" professional development can influence teacher practice and student performance. The paper focuses on what is or could be considered as well defined. One strand of the paper is that of effective support for new teachers. Although half of the states require support for new teachers (Education Week, 2008) it was found that rates of participation in teacher induction programs varied by school types with highest rates in schools with least poverty and lowest in schools with high levels of poverty. Beyond the rates of participation and availability of support, there is the question of what is effective support. An on-going large-scale research project was sited which is currently underway that aims to measure impacts in terms of classroom practices, student achievement and teacher mobility. Initial results seem to reflect the difficulty in identifying the impact of support.

Another study presently in its fourth year, Supporting Teachers to Increase Retention (STIR) is studying the relationship between retention and support of mathematics teachers across the State of California. This five year study is looking through the lens of 10 support models to relate retention to content knowledge, classroom practices, professional communities of support, leadership and needed support. Initial results are complex but are showing relationships between sustained professional development and support and teacher retention. Data collected to establish a base line for retention across a 5 years period preceding STIR showed that yearly attrition averaged 20% across all 10 sites and for the five year period the attrition average was 54% with sites reporting an attrition of mathematic teachers as high as 73%. In first year the attrition dropped from 20% to 14%. Additional data will be available in October 2010.

But, what is the relationship between the support and the retention? One of the 10 sites from this study observed that success of a retention initiative takes root in a variety of needs: the need to know your District and its teachers – a necessity that often relies on established, long-term relationships between the university and district leaders; the need to offer sustained support as opposed to punctual interventions in order to break the isolation of beginning teachers and create a sustainable community; the need to establish relevance in the professional development activities proposed by engaging participants in deep introspection of their own knowledge gaps; the need to involve all actors of the community to prevent miscommunication from annihilating attempts made towards change; the need to nurture the community created by moving its members forward into roles and responsibilities they are ready to take on; and last but not least, the need to refine even successful models to keep the momentum (Felter & Faughn, 2009).

As is indicated in the comments above, support comes from multiple sources. Another recent study from Peabody College, Vanderbilt University, finds that principals play a critical role in the support of new mathematics teachers (McGraner, 2009). But how do we successfully involve principals in supporting professional development that sustains retention?

An additional aspect of the issue at stake is the retention of mathematics teachers entering the profession through alternative certification as brought up by one working group participant. In a presentation to the group at PME-NA 2009, Brian Evans emphasized retention issues within the New York City Teaching Fellows program and provided us with the following literature review: "Teachers leave teaching in New York City for three reasons (Stein, 2002): retirement, leaving the profession, and transferring to a school outside New York City. […] A concern with alternative certification is lack of retention, especially in large urban areas such as New York City (Darling-Hammond, Holtzman, Gatlin, & Heilig, 2005). Sipe and D'Angelo (2006) found when surveying Fellows that about 70% of them intended to stay in education. NYCTF reports that 89 percent of Fellows begin a second year of teaching after the completion of their first year (NYCTF, 2008). Boyd, Grossman, Lankford, Michelli, Loeb, and Wyckoff (2006) reported that about 46% of Teaching Fellows stay in teaching after four years as compared to 55% to 63% of traditionally prepared teachers. Kane, Rockoff, and Staiger (2006) found that Teaching Fellows and traditionally prepared teachers have similar retention rates. Further, Tai, Liu, and Fan (2006) claim that alternative certification teachers, in general, have comparable commitment to the teaching profession as do traditionally trained teachers. In a survey of 31 Teaching Fellows, 90 percent said they were considering leaving their high needs schools for better schools in or outside of New York City, or leaving the teaching profession altogether (Stein, 2002). […] Similar to results found in other studies (Costigan, 2004; Cruickshank, Jenkins, & Metcalf, 2006; Evans, 2009), teachers were very concerned with student behavioral problems and unsupportive administration." (Evans, 2009)

This working group is designed to offer a comprehensive, multifaceted examination of the on-going preparation, support and retention of grade 7-12 mathematics teachers based on the results of research studies and ideas of the participants. It is hoped that this working group will enrich the dialogue relating the "support gap" and the work and retention of teachers of mathematics. It is also expected that this working group will propose areas ripe for further research. In light of the National efforts to close "poor performing schools" this work to identify ways to improve retention of mathematics teachers becomes especially critical.

Year 1 & 2 of the working group: Proposals and Summary

A Summary Report from the first year of the working group is included below. This year's proposal builds upon the proposal from last year by integrating participants' research interests and focusing the discussions around 4 major themes relating to teacher retention. A reference list produced by collapsing the references from the proposal and work of the first year is found at the end of this proposal.

Summary Report: Activities of Working Group PME-NA meeting 2009

<u>Day 1:</u> *Candice Ridlon (UMES, MD); Brian Evans (Pace University, NYC); Christine D. Thomas (Georgia State University, GA); Ellen Clay (Drexel University, PA); Allyson Hallman (UGA, GA) ; Michael Meagher (CUNY, NYC); Barbara Pence (SJSU, CA); Axelle Faughn (WCU, NC).*

After an Introduction to the working group, participants introduced themselves and shared individual interests & concerns regarding the "support gap" and the work and retention of teachers of mathematics. Goals and Key questions from the proposal were presented, followed by a Review of the literature. Participants were then asked to consider the question "Can support impact teacher retention?", and more specifically "What are the different aspects of support? How is impact measured? What are opportunities and challenges encountered when researching teacher retention?" A major task for our first session consisted in coming up with directions to work and identifying missing foci in literature. The list of themes and interests resulting from this initial brainstorming is provided later in this report.

<u>Day 2:</u> *Douglas Owens (Ohio State University, OH); Drew Polly (UNC Charlotte, NC); Candice Ridlon ; Christine D. Thomas ; Barbara Pence ; Axelle Faughn.*

The second meeting session of this working group had two major foci: Sharing Models of Support and Identifying categories from Day 1 list of questions.

A. Models of support

Although sharing models was a recurring activity throughout the three meeting session of the working group, the list below synthesizes the models that were shared over the three days:

- Brian Evans – NYCTF literature review
- Drew Polly – UNC Charlotte - Researcher-beginning teacher onsite mentorship

This presentation shares the findings of a study in which the researcher supported beginning teachers' mathematics instruction through mentorship activities. The researcher *provided opportunities to co-plan, co-teach and feedback on lessons to four teachers who were either new educators or new to their grade level. Analyses of field notes and interview data indicates that the teachers who sought more in class support during their mathematics teaching used more standards-based pedagogies and reported a greater satisfaction in teaching mathematics in a standards-based approach.*

- Candice Ridlon – PD through reform curricula in Utah
- Axelle Faughn – Addressing content knowledge through MATM

This presentation offers an examination of the role that professional development plays in the work and retention of new teachers and/or teachers in hard-to-staff settings. Based in California and including 10 sites and more than 250 teachers, first and second year results from data collected through large group surveys, online logs and site level focus groups helps to explain why the attrition rate over the first year of the study dropped from 20% to 14%.

- Barbara Pence – CMP STIR 10 sites, 10 models
- Christine Thomas – Developing an online community of support

Four professors from Georgia State University, located in an urban environment in the United States, share their conceptualization, operationalization and implementation of a study of mathematics teachers in an online learning community in Second Life. This is a virtual professional learning community (PLC) that is located in Five Points at Georgia State University. Second Life is a multiple user virtual environment. It allows users to create and control a representation of themselves known as an avatar. They can use these avatars to explore and interact with a simulated reality. This simulated reality is limited only by imagination.

We use Second Life as a virtual learning environment for learning about and engaging in situated contexts designed to support induction year teachers in developing as exemplary teachers of mathematics. The teachers are able to experience and examine a broad range of classroom situations within Second Life. More specifically, teachers are able to convene and as a community interacting both with audio and with non-verbal cues within the Second Life environment. They are able to model best practices and reflect on their experiences within a virtual community of their peers

B. Categories from Day 1 along with the related list of questions

Participants organized the questions and interests identified during the first meeting into 4 main categories that emerged from the initial brainstorming: Impact of PD on Teacher Retention; Content Knowledge and Retention ; Research issues and Retention ; Equity and Retention. The revisited list of questions below provides an arrangement of interests under the 4 major themes, although it was clear from our discussions that these themes overlapped and that many questions could be considered from more than one angle and crossed over categories.

1. Can Professional Development impact Teacher Retention?

What is professional Development? What models are successful?
What type of teacher professional learning encourages transformations in teaching practices and rewards resulting in improved retention?
Is Coaching/Mentoring (induction) part of PD?
How does pre-service fit into the issue of retention?
Looking at induction as PD for first five years;
Retention and alternative certification (NYCTF, TFA);
Retention of beginning teachers;
Can we establish a continuum from pre-service to in-service experiences (Medical School

model)?
Support and retention through online learning communities for Secondary Teachers beyond induction;
Is there a difference between rural/urban/hard-to-hire settings?
Who are the participants? (In-service/Pre-service, Teacher Educators, Administrators…)
What are the goals/activities of the online community?
Can a community be exclusively online? Or is Face-to-face needed?
Retention and Leadership;
What is leadership?
How long should people stay in the classroom? And/or move into more influential positions in education?
Role of administrators in retention and professional development;
Guidelines for support - technology, school and state policies driving support, job-embedded professional development, or necessary time, content and opportunities for support
Role of videotaping in PD;

2. Content
Seeing success through Math Content to improve retention. Does knowing more mathematics give a better sense of success?
How do we define "Highly Qualified" (State/Federal/Teacher perceived)?
Would relaxing admissions standards into university teacher training graduate programs and increasing standards for completion help address teacher shortage? Programs can serve as filters…
Role of technology and retention; (gaining experience/competence/confidence with Instrumentation while learning and teaching mathematics)
Getting National Board Certified.

3. Research
Research issues related to data collection and interpretation;
Self-reported data;
Perception on leavers/movers;
Beginning teacher preparation and professional development: when to start "counting" for retention;
What are opportunities and challenges encountered when researching teacher retention?
How do we measure impact?
How many leave for a few years and then come back?
Question for participants: How do you keep track of retention?
Do we start counting pre-service? Do we count somebody who moved to administration?

4. *Retention and Equity (maybe considered as a sub-piece of each category above)*
How to attract, recruit and maintain minorities into teaching?
What retention issues are similar/different when looking through the lenses of equity?
Effect of retention on schools/students
Day 3: *Nancy Schoolcraft (IN); Christine D. Thomas ; Douglas Owens ; Barbara Pence ; Axelle Faughn*

In order to support discussions on the third meeting session of the working group, participants were invited to read Darling-Hammond's report on teacher professional development and support. A reflection followed on the categories identified above (Professional Development ; Content ; Research ; Equity) in the light of the report. Additional focus and questions emerged as we noted that PD must be sustained, long-term, and involve a community of learners: Mentoring necessitates careful pairing, could be done through videotapes and reflections; Decreasing the number of preps for beginning teachers could provide more time for planning and reflecting; We need to connect retention to student learning: Is seeing students succeed part of teacher perceived success that could help with retention? i.e., would evidence of increased students' performance help build confidence & a sense of competence? Finally, advantages of "whole school" reforms and developing leadership in a single individual to bring PD back to their site were emphasized to increase onsite presence through lesson study, lesson planning, online community, coaching, and/or videotapes.

A final activity on day 3 consisted in sharing ideas on further work and dissemination. Participants indicated several directions we could take such as co-developing research and papers, co-presenting at future conferences (New Teacher Center; NCTM Research Pre-session; AERA; T^3; others?) establishing a website/webbase on teacher retention, and collecting papers for a Monograph on Mathematics Teacher Retention.

Plans for the second year of the working group

As can be seen in the summary of the working group, the core question of the relationship between Professional Development/Support and retention was a significant and central to all discussions but due to lack of directly related research, formed the springboard for lists of questions. This year, it is hoped that the work from last year along with an additional year of work and significant political actions will help to build a more focused and active research base. Specifically, it is anticipated that we will plan for active engagement of participants in productive reflection on the issues across projects. We hope to begin with the review of previous work for those who are new to the group and then move to breakout groups that will report out and collectively lead to the beginning of a collaborative publishable work. In advance of the conference, last year's participants will be surveyed to identify any questions or shifts in foci that occurred this year. Additional people will also be contacted. Once the topics for the breakout sessions are identified, specific people will be contacted to chair breakout sessions. Reports will be anticipated from each of the breakout sessions. The general outline of the three days of the working group will include:

Day 1 • Introductions
• Review of last years work and update of work by those who are attending.
• Overview of each of the breakout groups. It is expected that we will form 4 groups.
• A 15 minute breakout discussion where participants can take part in one of two discussion and then switch to the next group.

Day 2 • Introductions of new people
• Continue with basic ground work for the remaining two breakout groups and discussions in each breakout group.
• Brief discussion of issues addressed in each group

• Small group work that focuses on challenges and knowledge base for each group

Day 3 • Breakout group reports

• Discussion of where we go from here

Productivity of the working group will be a function of the advanced organization. But at this point, we anticipate that the breakout groups will address the questions generated from the year 1 discussions and listed initially in this proposal. That is:

(1) Role of technology in support/retention;

(2) Professional support community that reflect the building of networks and contacts to support work, decisions, challenges, and opportunities that arise in the teaching of mathematics, including lesson study and electronically based communities of practice;

(3) Role of early career leadership and/or career enhancement in retaining mathematics teachers, including the PD directed towards new teachers entering the field through alternative certifications, those coming from other careers, and shifts in PD needed as new teachers move through the challenges of their first 5 years of teaching;

(4) Content based Professional development with emphases on conceptual linking and problem solving.

The core questions of relating retention and professional development/support, equity and that of research issues that arise in examining teacher retention will be integrated across each of these breakout topics.

Description of anticipated working group breakout sessions listed by category

(1) Category I: Role of technology in support/retention.

This break-out session will be facilitated by Axelle Faughn (CMP-STIR Bakersfield). We will discuss issues pertaining to the role of technology in empowering teachers both in the classroom and the larger mathematics education community. Questions to be addressed include: (1) What model of PD has been provided that engaged participants in Technology use? (2) What trends are you noticing in technology use by teachers? (3) Are you able to relate technology to Retention or Leadership? (4) What constitutes technology in the work of mathematics teachers? (5) What are issues related to equity when working with instructional technologies? The session's facilitator will provide data and material for participants to examine from several projects that make use of technology in their professional development. Throughout participants will consider how to measure impact by collecting carefully designed research evidence.

(2) Category II: Professional support community that reflect the building of networks and contacts to support work, decisions, challenges, and opportunities that arise in the teaching of mathematics, including lesson study and electronically based communities of practice.

Discussion I. This break-out session will address implementation strategies and techniques for implementing lesson study. Questions to be discussed include: (1) What is the timeline for planning, teaching and re-teaching? (2) How do teachers define/identify misconceptions by students? (3) How and what do teachers decide in changing their lesson? (4) Who is involved in the Lesson Study planning and implementation?

Discussion II. Online professional learning community: A platform for professional development and research on mathematics teacher retention in an urban environment – facilitated by Christine D. Thomas (Georgia State University)

In this working group session, we will share our evaluation and synthesis of the literature on Second Life, professional development within a PLC, and distance learning used in our participatory action research study of this Second Life project. The Second Life project that spans August 2009-May 2011 aims at: (1) sustaining mathematics teachers who are attempting to improve their teaching and students' learning and (2) conducting research for the dissemination of knowledge on retention of secondary mathematics teachers in these schools. We use this as an opportunity to share our understandings, interpretations, and analysis of the literature on the topic as it relates to the goals, and needs of the urban community and the mathematics teachers we serve. In addition, we share our beginning analysis, and initial findings of the study based on the baseline data collected.

(3) Category III: Role of early career leadership and/or career enhancement in retaining mathematics teachers, including the shifts in PD needed as new teachers move through the challenges of their first 5 years of teaching. Discussions under this category will be facilitated by Barbara Pence (CMP-STIR, San Jose State University). It is hoped that a representative from the New York City Teaching Fellows will co-chair this break-out session with Pence.

(4) Category IV: Content based Professional development with emphases on conceptual linking and problem solving and **establishing** collaboration with mathematicians – Facilitated by Imre Tuba (CMP STIR, San Diego State University)

Anticipated Results of the Working Group

To begin addressing the particular issues of retention and technology, retention and community, retention and leadership, and/or retention and content-based professional development, each participant will share their project by including a description of the work to date, the stage of development the project is in, research design and instrumentation, and a summary of initial findings. The organizers of the working group plan to solicit several papers emerging from this collaborative work, and possibly the development of a monograph synthesizing our work on mathematics teacher retention and support. The network will also be included in the development of a conference on Supporting Teachers to Increase Retention to take place in two years.

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USING TECHNOLOGY FOR TEACHING AND LEARNING PROBABILITY

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The working group focuses on the technological approaches to learning and teaching probability. We address several issues in the discussion such as the importance of learning probability, frameworks describing criteria for understanding probability, technology use in teaching and learning probability in conjunction with the visual characteristics of new learning form of probability. Among various representations used in mathematics and mathematics education, visual representations of probability concepts, the effects of the implementation of visual techniques, and more importantly the importance of visual exploration of probability and its contribution to learning probability will also be discussed.

Background

The focus of this discussion group will be situated at the intersection of the technology use in mathematics education, visual learning in mathematics education, and developing a conceptual understanding of probability. An enormous corpus of literature has been accumulated on technology use in mathematics education and visual learning of mathematics, and many researchers discussed the topics in the various national and international meetings such as PME, PMENA, and ICMI (Arcavi, 1999; Duval, 1999; Hitt, 1999; Hoyles, 2008; Kaput, 1999; Kaput & Hegedus, 2000; Leatham, & McGehee, 2004; McDougall, 1999; Moreno-Armella, 1999; Presmeg, 1999; Radford, 1999; Santos-Trigo, 1999; Thompson, 1999).

Moreover, a specific discussion group focused on the visual learning during the 2009 PME-NA meeting and the discussion was facilitated by two authors of this proposal. The discussion group in that meeting started by identifying the current situation and contemporary perspectives in technology use and visual learning mathematics education and looked for the possible collaboration opportunities. The group brainstormed on the use of technology in mathematics education, on the various forms of representations used in mathematics, and on the meanings of visualization. The proposed discussion will be a continuation of that discussion with the focus on probability.

The rest of the proposal includes a conceptual framework for the importance of learning probability, probabilistic thinking and reasoning, as well as some examples of software used in learning and teaching probability. We finalize the proposal by suggesting an outline for the meeting by addressing the topics to be discussed in each day.

The Importance of Learning Probability

Gal (2005) categorizes motivations for learning probability into two categories, namely internal and external reasons. Internal reasons are connected with the importance of probability within the broader discipline of mathematics. According to this view, learning probability is important because it serves as the foundation for other mathematical disciplines such as statistics and decision theory. In addition, since probability is important because of its connection to other mathematical concepts (rational numbers, equations, integrals, and sets), solving problems in probability provides opportunity for further mastery of those concepts.

External reasons are connected with the fact that probability could be used to explain many natural and social phenomena. Probability models are at the core of many theories and models including the quantum-theoretic model of the atom, kinetic gas theory, and genetics. Many concepts in social science also use sophisticated probabilistic models. They include voting, choosing an insurance plan, crime, and racial discrimination.

Probabilistic Thinking and Reasoning

There are several frameworks and building blocks ("lists of criteria") that describe probabilistic reasoning. The major frameworks mentioned in the literature are: core domain of probability concepts (Moore, 1990), probability thinking framework (Jones et al., 1997), and building blocks of probability literacy (Gal, 2005). These frameworks can be divided into two groups: prescriptive and descriptive.

Moore's (1990) probability thinking framework is primarily prescriptive because it describes the pieces of knowledge that a person needs to have in order to be able to reason probabilistically. Moore (1990) describes the conditions that students need to satisfy in order to be able to move towards more difficult concepts including conditional probability. These conditions are: (1) learning to discern the overall pattern of events and not attempt a causal explanation of each outcome; (2) recognizing the stability of long run frequencies; (3) assigning probabilities to finite sets of outcomes and compare observed proportions to these probabilities; (4) overcoming the tendency to believe that the regularity described by probability applies to short sequences of random outcomes; and (5) applying an understanding of proportions to construct a math model of probability and develop an understanding of some "basic laws or axioms that include the addition rules for disjoint sets." (p. 120). The first condition deals with the understanding of randomness and is relevant to this thesis.

Gal's (2005) model consists of two parts which the author calls "the building blocks." (p. 46). The first set of "building blocks" consists of the knowledge elements. The knowledge elements are prescriptive because, similar to the Moore's model, it lists pieces of knowledge that students should have in order to master probability The second part of the model consists of dispositions that play "a key role in how people think about probabilistic information or act in situations that involve chance and uncertainty, whether in the real world or in the classroom." (Gal, 2005, p. 45). This part of the model is descriptive because it does not exclude dispositions that are detrimental for understanding probability. In Gal's model, the knowledge elements are: (1) big ideas (variation, randomness, independence, predictability, and uncertainty); (2) figuring probabilities (ways to find or estimate the probability of events); (3) language (the terms and methods used to communicate about chance); (4) context (understanding the role and implications of probabilistic issues and messages in various contexts and in personal and public discourse); and (5) critical questions (issues to reflect upon when dealing with probabilities). Dispositional elements in Gal's model are: (1) critical stance; (2) beliefs and attitudes; and (3) personal sentiments regarding uncertainty and risk (e.g., risk aversion).

One of the most comprehensive frameworks describing probabilistic reasoning is provided by Jones et al (1997). The framework is divided into four constructs: sample space, probability of an event, probability comparisons, and conditional probability. Furthermore, the framework recognizes four levels of reasoning: level 1- subjective, level 2 - transitional between subjective and naive quantitative, level 3 - informal quantitative, and level 4- numerical reasoning. Jones' framework is descriptive, because, similar to Gal's dispositions, it goes beyond the description of the knowledge of probability that students should possess.

All above-mentioned frameworks imply the following criteria for understanding of probability: (1) acknowledgment that not all phenomena can be viewed causally and deterministically; (2) understanding that random events might have a pattern that is stable in the long run, but that this pattern cannot be used to predict the next outcome; and (3) acknowledgment that randomness is a concept that permeates all aspect of life.

Examples of Software for Teaching and Learning Probability

Probability Explorer (Figure 1) is a research-based application developed by Stohl (2002) which enables students to explore variety of situations involving randomness and probability.

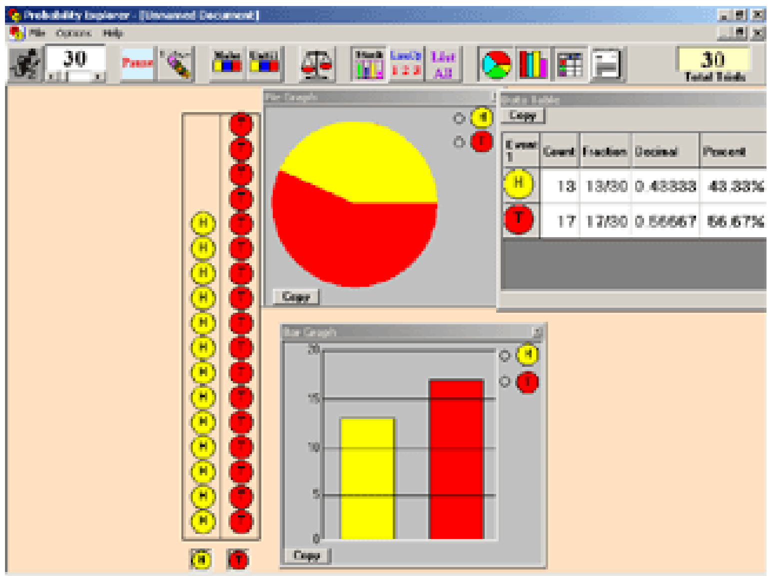

Figure 1

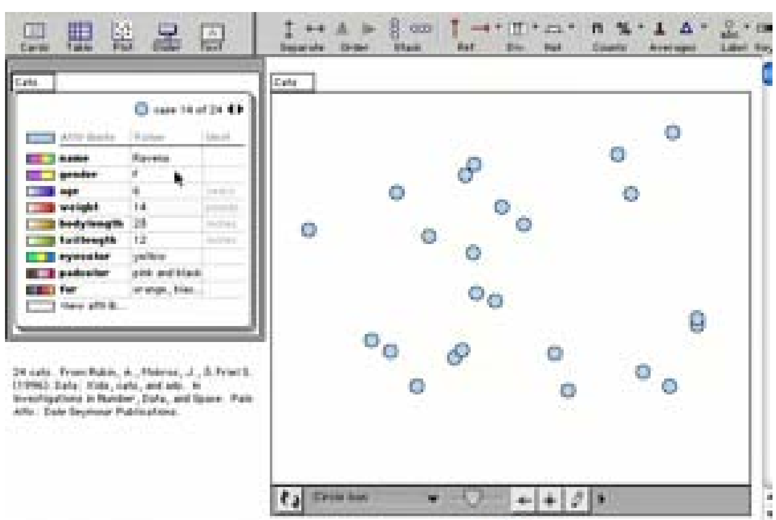

Figure 2

Probability Explorer is useful for designing the second aspect of probability understanding. By repeating experiments arbitrary many number of times and by various types of representations (graphs, pictures, pie charts) , students can explore the idea that the frequencies are stable across many trials but that this information cannot be used to predict the next outcome. Tinker Plots (Figure 2) is a software developed by Konold and Miller (2005) that allows students to explore various aspects of randomness and probability including the link between theoretical and experimental probability, as well as exploring randomness in variety of contexts.

The Rationale and Goals of the Discussion Group

The goals of this discussion group are to explore and discuss the technological and visual approaches in learning and teaching probability, to improve awareness on the technology use in conceptual understanding of probabilistic concepts, and to set up a research agenda on the study of technology use in learning probability.

The discussion group will review theoretical discussions in learning probability, the past and current use of technology in learning probability and seek for opportunities for possible use of technology in the future. The discussion, then, will seek possible ways to deepen our understanding of our basic concerns in understanding and teaching probability and the contribution of technology to cope with these concerns. In addition, we will seek opportunities to implement open source technology in classrooms to learn and teach probability.

Objectives:

1) To identify building blocks for understanding probability in elementary and high school (focus on one or the other depends on the participants)
2) To determine contexts in which probability software is appropriate
3) To offer comprehensive analysis of existing software and how it could be used to learn the building blocks
4) To give suggestions for software developers

Questions To Format the Discussion:

1) What is special about learning probability? How is learning probability different from learning other subjects (e.g. algebra and geometry)?
2) For what features of probability is software suitable?
3) For what features is not suitable?
4) Can software be used to learn probability in different contexts or does the software limit the context?
5) If we agree that students should understand probability and randomness in real world, how can computers be used to understand the "real world"?
6) What is the role of new internet technologies in learning probability?

Format for the Discussion Group Meeting

Day 1:

- *Theoretical discussion about learning probability*
- *Past research on learning probability*
- *Basic challenges in understanding probability concepts*

Day2:

- *Possible solutions to the problems addressed in the first day*
- *Technology use learning probability*
- *Comparing real life and technology use in understanding probability*

- *Could simulation be a solution*

Day3:

- *Wrap up the discussion and themes emerged in the first two days*
- *Suggestion for the future*
- *Setting a possible agenda for the future*

Possible Future Agenda Items

- To set up a team for working on an open source software to learn probability

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