# Proceedings of the Thirty First Annual Meeting of the North American Chapter of the International Group for the

# **Psychology of Mathematics Education**

*Embracing Diverse Perspectives*

Atlanta, Georgia, USA

September 23-26, 2009

**Editors**
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*David W. Stinson*
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# History of PME

The International Group for the Psychology of Mathematics Education came into existence at the Third International Congress on Mathematical Education (ICME-3) in Karlsruhe, Germany in 1976. It is affiliated with the International Commission for Mathematical Instruction.

# Goals of PME-NA

The major goals of the North American Chapter of the International Group for the Psychology of Mathematics Education are:
1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education.
2. To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers.
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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# Preface

These Proceedings are the product of the 31st Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education held in Atlanta, Georgia, September 23-26, 2009. They are a written record of the research presented at the conference.

The theme for the conference was *Embracing Diverse Perspectives*, an appropriate one for Atlanta as a home to Dr. Martin Luther King, Jr. and the civil rights movement in the USA. This theme, which speaks to meeting the needs of all learners in mathematics education, is a worthy goal and researching issues around this goal is of critical importance.

*Lynn C. Hart*
*Conference Co-Chair*

We are pleased to present the Proceedings for the 31st Annual Meeting of PME-NA. The Proceedings CD includes 2 plenary papers, 67 research reports, 113 brief research reports, 42 posters, and 9 working group papers. Users can access these papers by selecting specific authors or particular topics. Additionally, users can view all conference papers in a book format. Editing the Proceedings has been a great opportunity, and I would like to thank my co-editors, David Stinson and Shonda Lemons-Smith, for their significant efforts in the editing process.

*Susan L. Swars*
*Lead Editor, Proceedings*

# Table of Contents

# ALGEBRA IS SYMBOLIC[1]

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*The symbolic nature of algebra is addressed from three angles: branch of mathematics, object of curriculum, and theme of research. As a branch of mathematics, algebra deals in symbols, and their manipulation poses sometimes daunting problems of learning and teaching. As an object of the mathematics curriculum, algebra symbolizes barrier and stepping-stone among other things. As a topic of research in mathematics education, algebra poses many of the problems of meaning and interpretation that mathematics itself poses, while allowing some special characteristics to be explored. Each angle has implications for researchers, and each touches on questions of diverse perspectives in our research.*

The title above as well as the paper that follows can be seen as a response to a doctoral student who a few months ago asked why algebra seems to be the only part of school mathematics that people are talking about these days. Certainly, algebra is much in the news.

Last year, for example, the California State Board of Education, by an 8 to 1 vote, approved a policy that as of 2011, all eighth graders in California public schools would be required to take the Algebra I course and the accompanying California Standards Test for Algebra I (Asimov, 2008). Algebra I has been a high school graduation requirement in California since 2004, but only about half of California students have been taking it in eighth grade. The others have been taking a test in general mathematics. Prompting the new policy was the ruling by the U.S. Department of Education in 2007 that the latter test was out of compliance with the No Child Left Behind Act because it measured no more than sixth- and seventh-grade mathematics content. Last December, the California School Boards Association and the Association of California School Administrators, joined by the state Superintendent of Public Instruction and the California Teachers Association, got an injunction to prevent the state board from implementing the Algebra I requirement (Garrett, 2009). Given California's continuing budget woes, it is difficult to see the policy being implemented anytime soon, but it certainly got people's attention.

As another example, how did the American Diploma Project (2004) decide on Algebra II to anchor its expectations for readiness in mathematics for college and the workplace? The Algebra II course was never intended to be a course for all students, let alone a requirement for high school graduation. Did research showing a correlation between completion of Algebra II and both college completion and employment in high-paying professional jobs (Adelman, 2006; Carnevale & Desrochers, 2003; Pelavin & Kane, 1990) lead people to make the causal inference that requiring Algebra II for high school graduation would send everyone on to college or into those jobs?

Finally, one has to wonder why the President's executive order of April 18, 2006, establishing the National Mathematics Advisory Panel (2008) contained the following, as the first item on which the panel was to make recommendations:

> "(a) The critical skills and skill progressions for students to acquire competence in algebra and readiness for higher levels of mathematics" (p. 71).

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Did George W. Bush, having decided that he needed to convene a panel to advise him on how the United States might implement the policy of fostering "greater knowledge of and improved performance in mathematics among American students" (p. 71), wake up one morning and say, "Competence in algebra ought to be the first order of business"? As Danny Martin (2008) notes, the president's choice of algebra was far from politically neutral, and one can raise questions about it: "Why algebra? Who decides? Algebra for whom and for what not-so-apparent purposes? Whose interests are served by these choices? Whose interests are not served?" (p. 393).

Algebra can be seen as symbolic in several ways. In this paper, I discuss three of those ways in an effort not simply to suggest why algebra seems to have taken center stage but also to relate that prominence to some of the diverse perspectives to be embraced in our research.

## Algebra as a Branch of Mathematics

Algebra is one of the main branches of mathematics, one that studies structure, relation, and quantity and in which symbols are used to represent numbers or members of a specified set. In the elementary algebra of school mathematics, the most prominent symbols are letters of the alphabet that are used, along with symbols for numbers, operations, and relations, to express relationships among known and unknown quantities. In fact, for many learners, algebra may appear to be entirely about symbolism. Because of its heavy use of alphabetic symbols, *x* in particular, school algebra has been facetiously defined as "the study of the 24th letter of the alphabet" (Mason, Graham, Pimm, & Gowar, 1985, p. 38).

The activities of school algebra have been characterized using various category schemes (e.g., Kilpatrick, Swafford, & Findell, 2001, pp. 294–295, note 4). Carolyn Kieran's (2004) recent formulation is as useful as any; she divides those activities into generational activity, transformational activity, and global/meta-level activity. Generational activity concerns the formation of the expressions and equations that are the objects of algebra; transformational activity involves rule-based manipulations of equivalent expressions and equations, and global/meta-level activity uses algebra as a tool for mathematical processes such as problem solving and reasoning that are not exclusive to algebra but that make it useful. Kieran devotes much of her discussion to the transformational activity of algebra, pointing out that technology is providing "a lens . . . for researching students' emergent conceptualisations of algebraic transformations" (p. 31). Two chapters in the 12th ICMI Study volume (Stacey, Chick, & Kendal, 2004, chs. 6 & 7) in which Kieran's paper appears deal with the learning of algebra in technological environments.

The increasing availability and use of computing technology in mathematics classes is complicating the role to be played by symbol manipulation in school algebra. The very term *symbol manipulation* seems pejorative, suggesting a procedure carried out by hand or machine and not involving thinking (for a sophisticated, generous analysis of manipulation in school mathematics, see Pimm, 1995). Because technology is now available to carry out algebraic symbol manipulation, questions are arising as to how much effort should be given to such manipulation when teaching algebra. The release in May 2009 of the Web site Wolfram|Alpha, an online service built on Mathematica that makes a supercharged computer algebra system (CAS) free and easily available to the general public, has touched off discussion, and some consternation, among high school and college mathematics teachers. They recognize that students will be able to use the system to solve any transformational problem of school or college algebra (Young, 2009). Given that potential, where is elementary algebra headed, or where should it be headed?

One perspective one might take is that of the technology enthusiast, arguing that CASs, and Wolfram|Alpha in particular, can be powerful teaching tools, removing the drudgery and mindlessness from much of algebra learning. Another perspective might be that of the technology skeptic, pointing out that we do not know very much about the potential cognitive benefits of symbol manipulation. Even Tony Ralston (2009), an unabashed proponent of doing away with pencil-and-paper algorithms in the arithmetic curriculum, argues that the question of what value there might be to teaching symbol manipulation in secondary school and college is not as simple as it is for elementary school arithmetic. It might also be pointed out that countries vary considerably in how much practice is given in symbol manipulation (Kendal & Stacey, 2004, p. 337) and that, even though the situation is changing, Asian countries that have been high-achieving in mathematics have long refrained from much technology use in mathematics instruction (p. 342). Perhaps there are benefits to manipulation by hand rather than by machine that are not so readily apparent.

Lest I be misunderstood, let me immediately add that I applaud efforts to incorporate CASs into curriculum materials in mathematics. But I also think it worthwhile to probe further and deeper into the effects of using CASs and other software. An example would be the work by French researchers (Artigue, 2002, 2005; Guin & Trouche, 2002; Laborde, 2001), who have been studying some of the ways in which technological tools can be integrated into teaching practice. Particularly relevant to the example of CASs is the work of Michèle Artigue and her colleagues, "who have developed the construct of *instrumental genesis:* the way in which users shape the artifacts they use, and the artifacts shape the users, and that yields instruments" (Artigue & Kilpatrick, 2008, p. 6). Artigue has noted that although one might expect a CAS to free students from the technical burden of symbol manipulation and thus allow them "to focus on conceptual thinking and understanding" (p. 8), she did not see that happening in the classrooms she observed in the early 1990s. She came to realize that mathematical techniques have "both an epistemic and a pragmatic value. A pragmatic value because they are operational; they produce results. And an epistemic value because they contribute to our understanding of the objects they involve" (p. 8). She could then see resistance to the use of technology in classrooms from a new perspective:

> The ordinary use of digital technology plays on the pragmatic power of technology, doing more things more quickly at the expense of its epistemic power. But what makes a technique legitimate at school cannot be its pragmatic power only, which is an essential difference between school and the outside world. Making technology legitimate and mathematically useful at school requires modes of integration that allow a reasonable balance between the pragmatic and the epistemic power of instrumented techniques. This balance . . . requires tasks and situations that cannot be reduced to simple adaptations of paper-and-pencil tasks. (p. 8)

The construct of instrumental genesis appears to have considerable promise for researchers looking into the ways in which technology is and is not being used in mathematics teaching.

My point in bringing technology into the discussion of symbol manipulation is to observe that every technology has both costs and benefits. Because researchers who are technology advocates are likely to be looking primarily for benefits, it is important to have some researchers with a different perspective looking at costs. This example illustrates the value for the research we do of promoting diversity in the perspectives we employ. Seen as a branch of mathematics, algebra may appear well defined, but research into its learning and teaching depends on the position from which the researcher views algebra and the symbols it uses.

## Algebra as an Object of Curriculum

Algebra is symbolic not merely in its use of symbols but also in the way, as a course or curriculum strand, it symbolizes academic success. School algebra is "higher" mathematics in a way that school arithmetic is not. A simplified metaphor for the school mathematics curriculum that I am fond of using portrays its justification as arising from the collision of two tectonic plates (Kilpatrick, 2003, pp. 319–322). The first plate developed within primary school education and treats mathematics as a tool for solving practical problems. The second plate developed within secondary and tertiary education; it arises from the liberal arts tradition and treats mathematics as a means of achieving intellectual growth. Over the centuries, the line separating the two curriculum plates has shifted considerably so that now it cuts across the grades, with mathematics in the primary grades taking on some of the intellectual character of higher mathematics, and secondary and tertiary mathematics becoming more oriented toward the solution of practical problems. Nevertheless, algebra has long been seen as justified primarily for its intellectual rather than its practical value. Recent efforts to teach algebra at all grades and to all students have run into stereotypic views of what the subject is and who should be learning it.

As a symbol, school algebra has many different interpretations. When Andrew Izsák and I wrote a history of algebra in the curriculum (Kilpatrick & Izsák, 2008), we took the following quotation as the epigraph because it captures so well how school algebra has been regarded for decades:

> If there is a heaven for school subjects, algebra will never go there. It is the one subject in the curriculum that has kept children from finishing high school, from developing their special interests and from enjoying much of their home study work. It has caused more family rows, more tears, more heartaches, and more sleepless nights than any other school subject. (Anonymous editorial writer quoted by Reeve, 1936, p. 2)

In this view, algebra is an evil force wreaking havoc across the land. In contrast, Fran Lebowitz (1981) advises, "Stand firm in your refusal to remain conscious during algebra. In real life, I assure you, there is no such thing as algebra" (p. 27).[2] Algebra does not exist outside school.

The title of the April 2000 issue of *Mathematics Education Dialogues*—"Algebra? A Gate! A Barrier! A Mystery!"—captures some of the varied symbolism attached to algebra. It has long been seen as a principal gatekeeper to educational and career opportunities, which is one reason the teaching and learning of algebra from kindergarten through Grade 12 was chosen by the RAND Mathematics Study Panel (2003) as one of three focus areas (along with teachers' mathematical knowledge for teaching and skills in teaching and learning mathematical thinking and problem solving) for a long-term research and development program, "three domains in which both proficiency and equity in proficiency present substantial challenges, and where past work would afford resources for some immediate progress" (p. xv). In recent years, Bob Moses (1994), through his Algebra Project, has been instrumental in shifting the dominant metaphor from gatekeeper to a new civil right for all:

> Algebra, once solely in place as the gatekeeper for higher math and the priesthood who gained access to it, now is the gatekeeper for citizenship; and people who don't have it are like the people who couldn't read and write in the industrial age. . . . [Lack of access to algebra] has become not a barrier to college entrance, but a barrier to citizenship.

That's the importance of algebra that has emerged with the new higher technology. (Moses & Cobb, 2001, p. 14)

Politicians, however, have not ordinarily looked on algebra as a civil right. For them, it is more likely to provide an opportunity for grandstanding. What better or cheaper way to "raise standards" and "promote reform" in school mathematics, and get applause while doing it, than to require students to take "algebra" regardless of what that might entail? The examples cited above from California, the American Diploma Project, and the executive order establishing the National Mathematics Advisory Panel all reflect political uses of algebra. George W. Bush's education advisor Tom Luce, who served on that panel while he was an Assistant Secretary of Education and currently heads up the National Math and Science Initiative, was one of the first to recommend that all students "take and pass algebra in eighth grade" (Luce & Thompson, 2005, p. 170), which may help explain how algebra landed at the top of the panel's agenda. Luce and his coauthor Lee Thompson characterized algebra as "the principal gatekeeper for more advanced math and science studies in high school" (p. 170)

In the public's eye, studies in algebra, along with "more advanced" mathematics, symbolize not just a hurdle but also accomplishment and validation. In L. Frank Baum's (1900/1999) book, *The Wonderful Wizard of Oz*, when the Scarecrow asks the Wizard for brains, he gets bran mixed with pins and needles ("a lot of bran-new brains"). In the movie version, in contrast, the Wizard hands out a diploma.[3] (The Tin Woodman gets a testimonial, and the Cowardly Lion gets a medal.) The writer Yip Harburg, who wrote the movie scene, said, "[I] devised the satiric and cynical idea of the Wizard handing out symbols because I was so aware of our lives being the images of things rather than the things themselves" (quoted in Harmetz, 1997, p. 58). In other words, symbols are our reality.

On receiving the diploma, the movie Scarecrow responds with a garbled version of the Pythagorean theorem: "The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side." Although the Scarecrow's "theorem" is more geometry than algebra, it makes my point: The Scarecrow does not get brains; instead, he gets a piece of paper. And rather than knowledge leading to certification, certification leads to knowledge. In popular culture, the important thing is not the algebra you have learned but the presence of "Algebra II" on your transcript.

As a curriculum object then, algebra, like the rest of school mathematics, has multiple symbolic meanings for learners, teachers, parents, politicians, and the public. Researchers should do more to recognize, explore, and attempt to understand those meanings. Research into the teaching and learning of mathematics that avoids attending to how the curriculum object is being interpreted by the participants in the educational encounter and the society in which they are situated is quite likely to yield useless or even invalid results.

## Algebra as a Topic of Research

Algebra is also symbolic as a topic of research into the teaching and learning of mathematics. As a presumably well-defined curriculum entity, algebra is studied by researchers around the world. Are they studying the same thing?

> Curriculum issues have apparently seemed relatively easy to discuss in international forums because they could be discussed in a manner that attended strongly to rather universal, mathematical characteristics and only weakly, if at all, to more local, sociopolitical characteristics. It is not surprising, therefore that the resulting conversations often bypassed subtle, yet important, curriculum issues related to variations

> in educational traditions and practices across countries. . . . To what extent can all important research questions be lifted out of the contextual details so that they can be considered within a broad international community of researchers? (Silver & Kilpatrick, 1994, pp. 249–250)

One characteristic that distinguishes researchers in the community of mathematics education from those outside that community who "use mathematics as a vehicle for their research into learning and instruction" (Kilpatrick, 1996, p. 33) is that the latter take mathematics as unproblematic, a black box whose performance characteristics can be studied without asking what is inside. In contrast, when mathematics education researchers study a topic like algebra, they do not—or should not—assume they know exactly what is being taught and how. An excellent example can be found in the chapter by Margaret Kendal and Kaye Stacey (2004) that ends the 12th ICMI Study volume. The chapter convincingly demonstrates that algebra is not the same school subject across educational jurisdictions. There are striking differences in who takes it; whether it is integrated or layered across years; how much emphasis is put on matters of generality and pattern; how much attention is given to symbolism, formalism, and abstraction; whether it is approached through functions and multiple representations; and what role is played by technology. Kendal and Stacey conclude, "Don't take your country's curriculum and approach to teaching algebra for granted and don't assume all other educational jurisdictions operate in a similar way—they conspicuously do not" (p. 345). Researchers interpreting national and international assessment studies involving algebra, therefore, should analyze how the term is being understood in the assessment items and how well that understanding applies to the issue in question (Kilpatrick, 2009; Kilpatrick, Mesa, & Sloane, 2007). Of course, that admonition applies equally well to other branches of school mathematics.

The National Mathematics Advisory Panel (2008) defined algebra in a conservative fashion with a list of what they deemed to be major topics of school algebra (Table 1, p. 16). They developed that list by reviewing state standards, current textbooks, 12th-grade objectives of the 2005 National Assessment of Educational Progress, the American Diploma Project's benchmarks, and the Singapore standards. Brian Greer (2008) characterizes the list of topics as

> a subset of what I learnt at grammar school in Northern Ireland nearly 50 years ago, with two exceptions, namely (a) fitting simple mathematical models to data, and (b) combinations and permutations, as applications of the binomial theorem and Pascal's Triangle. (p. 426)

Roschelle, Singleton, Sabelli, Pea, and Bransford (2008) point out:

> The Panel chose not to broadly and critically examine the relevance of (lowercase) algebra to modern economic life, its role in scientific activities, and its function in a highly technological world. Instead, the Panel defined Algebra I and Algebra II in the most conservative way (based on the intersection of the content of international curricula for such courses) and then further restricted its focus to "addressing the teaching and learning of mathematics from preschool to Grade 8 or so." (p. 610)

Pat Thompson (2008) observes that "the vision of algebra reflected in the Panel's content recommendations is a skills-based foundation for advanced symbolic manipulation and abstract algebra (especially the algebra of polynomial forms)" (p. 585).

The National Mathematics Advisory Panel (2008) notes in a footnote that their list of major topics "is meant as a catalog for coverage, not as a template for how courses should be sequenced or texts should be written" (p. 15). By cataloging topic coverage rather than suggesting approaches or offering interpretations, the panel assumed that school algebra is school algebra now and forever here and everywhere. Listing a topic such as "logarithmic functions" without considering how it might be introduced, understood, or used is equivalent to taking algebra itself as a black box. Also, as Greer (2008) noted, by ignoring the work of researchers such as Carolyn Kieran and Jim Kaput, the panel missed the opportunity to recommend "a more productive approach to the teaching of school algebra" (p. 424).

The draft mathematics standards from the Common Core Standards project of the National Governors Association and the Council of Chief State School Officers (Cavanagh & Gewertz, 2009) that were unexpectedly released in July suggest that, so far, that project is providing some context for the treatment of topics and not just a list. The argument can still be made, however, that the inclusion of more mathematics education researchers in the standards development process, as well as more attention to the work they do, would yield a better product. Anyone who has taken school algebra as a research topic should understand its problematic quality.

## Diversity, Balance, and Quality

By taking the symbolic nature of algebra as a focus, I have attempted to suggest how important it is to embrace diverse perspectives in conducting our research. Diversity in background, preparation, outlook, and approach enriches the research community. Diverse approaches, in particular, make our research useful for different purposes (National Council of Teachers of Mathematics Research Committee, 2009). But diversity is not all we need.

Diversity should be accompanied by balance. Enthusiastic ideologues are important; they are the ones who push us to try new things. Also important, however, are skeptical critics, and the latter always appear to be in short supply. Speaking for myself, I tend to avoid bandwagons, whether they concern problem solving (Kilpatrick, 1981), constructivism (Kilpatrick, 1996), or "algebra for all." It is fine if some of us want to hop on the current bandwagon, but it would not be good if all of us were there. To change the metaphor, there is little use in having a diverse set of passengers if all of them sit on the same side of the boat. We need "a more balanced perspective on research and the variety of ways it can be conducted" (p. 41).

Diversity and balance, of course, need to be accompanied by quality in what we do. As Heather Hill and Jeff Shih (2009) point out, "high-quality research" in education is a contentious construct, one that people continue to debate. Nonetheless, I think the community of mathematics education researchers has taken some important strides toward consensus regarding quality over the past several decades even if we continue to disagree regarding details.

A heartening sign that we are achieving diversity in research method comes from a recent study by Lynn Hart and her colleagues (Hart, Smith, Swars, & Smith, 2009). They examined 710 articles published from 1995 to 2005 that reported research in mathematics education and found that although, not surprisingly to anyone acquainted with the recent literature, qualitative studies were the most common, studies using mixed methods were the next most common, and a nontrivial number were quantitative studies. Hart et al. looked at six journals; Figure 1 shows the results for the *Journal for Research in Mathematics Education (JRME)*, which are not too different from those for the journals taken together, although the *JRME* had by far the most articles that used mixed methods with inferential statistics. The classification of research as qualitative, quantitative, or mixed turned out to present some serious problems of interpretation. In particular, studies using mixed methods were seldom identified as such in the report of the study. Although they did not find clear trends in the fraction of studies using mixed methods, Hart et al. expect that more high-quality mixed methods research will be published in the coming years, seeing mixed methods as possibly "an appropriate response to calls for greater generalizability of results while maintaining enough detail about the processes of teaching and learning to be valid and useful" (p. 39).

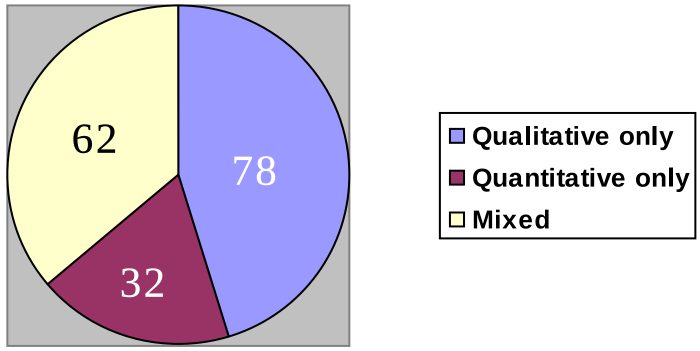


*Figure 1.* Frequencies of methods used in *Journal for Research in Mathematics Education* articles from 1995 to 2005. (*Source:* Hart et al., 2009, p. 34)

Before celebrating the diversity of methods to be found in the research literature in mathematics education, one should note that Hart et al. (2009) found much that could have been improved in the way researchers using mixed methods report their results. Hill and Shih (2009), who looked at statistical and mixed method reports in the *JRME* from 1997 to 2006 and judged the adequacy of the statistical treatment of data, also found considerable room for improvement in the quality of both analytic techniques and reporting. So we are not quite there yet.

School algebra is receiving more than its usual share of attention these days for reasons that are largely beyond the control of the North American community of researchers in mathematics education. As I have tried to suggest in this paper, they have some distinctive perspectives to contribute to the discussion. Like all research, theirs has its flaws and shortcomings. But to ignore both the research and the researchers is to limit the possibility of improving the teaching and learning of mathematics.

## Endnotes

[1] Plenary address at the annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Atlanta, 24 September 2009.

[2] Thanks to David Pimm (1995, p. 106) for this reference.

[3] Thanks to Eileen Donoghue for reminding me of this scene.

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# LITTLE BLACK BOYS AND LITTLE BLACK GIRLS: HOW DO MATHEMATICS EDUCATION RESEARCH AND POLICY EMBRACE THEM?

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Come round by my side and I'll sing you a song.
I'll sing it so softly, it'll do no one wrong.
On Birmingham Sunday the blood ran like wine,
And the choirs kept singing of Freedom.
That cold autumn morning no eyes saw the sun,
And Addie Mae Collins, her number was one.
At an old Baptist church there was no need to run.
And the choirs kept singing of Freedom,
The clouds they were grey and the autumn winds blew,
And Denise McNair brought the number to two.
The falcon of death was a creature they knew,
And the choirs kept singing of Freedom,
The church it was crowded, but no one could see
That Cynthia Wesley's dark number was three.
Her prayers and her feelings would shame you and me.
And the choirs kept singing of Freedom.
Young Carol Robertson entered the door
And the number her killers had given was four.
She asked for a blessing but asked for no more,
And the choirs kept singing of Freedom...[1]

## Preamble

The theme of this year's conference is *Embracing Diverse Perspectives*. This theme clearly represents an invitation for scholars in the field to consider and appreciate a wide range of theoretical and methodological perspectives on mathematics learning and participation, including those perspectives that diverge from what might be called conventional or mainstream thinking.

In my view, the conference theme also provides an opportunity to raise questions about how mathematics education research and policy have embraced and served the diversity of students who show up in mathematics classrooms, especially those students who must learn mathematics while simultaneously trying to negotiate the most difficult and oppressive life circumstances. These are often the same students who have been systematically and deliberately underserved in so many other societal and institutional contexts.

In this paper, and my accompanying plenary address, I take advantage of the conference theme to do two things that rarely occur in mainstream mathematics education contexts. First, I put Black children and their experiences at the center of the discussion. I share my perspective on how I believe mathematics education has and has not served these students. In many ways,

[1] Excerpted lyrics from the song *Birmingham Sunday* written by Richard Fariña and performed by Joan Baez.

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

Black children serve as canaries in the coal mine. If we have not, and cannot, do right by these children, it is extremely difficult for me to believe that we can accomplish the goals inherent in the conference theme.

My focus on Black children is not an exclusionary move; taking a *pro*-Black-child stance should not be interpreted as a stance *against* any other group of children given my sincere interest in insuring that all children experience mathematics learning and teaching in relevant and meaningful ways. However, I do believe that, in the context of discussing *diversity*, we should never lose sight of particularity. Similarly, when discussing particularity, we should never lose sight of diversity. Therefore, while it is important to discuss the needs of Black children as *children*, it is equally important to prioritize their needs as *Black* children (e.g., Hale, 2001; Lomotey, 1990; Martin, 2007; Perry, Steele, & Hilliard, 2003; Shujaa, 1994).

Regarding these last points, we should not lose sight of the fact that there is great diversity among Black children in the United States (e.g., Waters, 2001). There is no singular, essential characterization. They come from varied socioeconomic and family backgrounds and respond to schooling and education in multiple ways. Yet, there is a collective history and collective condition of Blacks in the United States that is clearly distinguished from other social groups. It is this history that gives partial meaning to what it has meant, and what it currently means, to be Black in America.

My focus on Black children in the United States does not deny that they are forever linked to other Blacks in the African *diaspora*, including Afro-Latins in the central and southern Americas, Afro-Carribeans in the West Indies, the Sidis in India, the Aboriginals in Australia, Afro-Arabs in the Middle East, and so on. These *diasporic* relations remind us that Black children in the United States are also children of the world. It is unfortunate that some policy makers and education researchers often lose sight of this fact by confining black children's existence to poverty-ridden communities, broken families, and low-quality schools and easily dismissing the historical and structural forces that create and maintain those conditions (D'Souza, 1991; McWhorter, 2001; Steele, 1990; S. Thernstrom & A. Thernstrom, 1997; A. Thernstrom & S. Thernstrom, 2004). Moreover, there is a disturbing trend in society that attempts to strip Black children of their childlike qualities altogether by using such labels as thugs, urban terrorists, predators, threats to society, and endangered species. Ignoring structural considerations, we are asked to believe that genetic, cultural, and intellectual inferiority account for these conditions (D'Souza, 1991; McWhorter, 2001; Steele, 1990; S. Thernstrom & A. Thernstrom, 1997; A. Thernstrom & S. Thernstrom, 2004).

While it is true that disproportionate numbers of Black children here and around the world continue to experience life conditions that not only limit their opportunities to learn but that also threaten their very lives, this is not the end of the story. It is equally true that, wherever they live and learn and no matter what their circumstances, Black children are also among the most resilient (Bowman & Howard, 1985; Gordon. 1995; McGee, 2009; Miller & MacIntosh, 1999; Sanders, 1997; Spencer, Cole, Dupree, Glymph, & Pierre, 1993). We need more studies of this resilience in mathematics education (e.g., Ellington, 2006; McGee, 2009).

Moreover, Black children in the U.S. are growing up in a time when geopolitical boundaries are being blurred by technology and globalization. Social media such as YouTube and MySpace are not only responsible for exporting and importing culture, ideology, protest, and revolution but also for exposing the human condition and helping Black children to contextualize their lives vis-à-vis the conditions in which other children live and learn. Black children can see that the struggle for a more humane existence is not confined to the boundaries of their own neighborhoods or cities.

Within this global perspective, the implication for Black children's mathematical education is clear:

> … meaningful mathematics education for African-American children should not only help them function in their local contexts in U.S. society but should also help them function as citizens of the globe, to function across boundaries of difference, and to recognize similarities in human conditions among people who wage the struggle against oppression" (Martin & McGee, 2009, p. 216).

This view on the aims and goals of mathematics education stands in sharp contrast to policy discussions that frame mathematics participation for Black children in terms of workforce participation and the preservation of U.S. international competitiveness (Committee on Science, Engineering, and Public Policy, 2007; Domestic Policy Council, 2006; National Research Council, 1989; National Science Board, 2003; U.S. Department of Education, 1997, 2008). While these may be worthy goals, they still reflect crude commodifications and self-serving concerns for Black learners, concerns that are typically couched in the easy-to-swallow language of equity and diversity (Gutstein, 2008; Martin, 2003; Martin 2009a, 2009c; Martin & McGee, 2009).

My own view is that even if larger numbers of Black Americans were to find themselves in the mathematics and engineering pipeline, they would only be absorbed into the workforce up to the point of not threatening the status and well-being of white workers. Examination of the public debate reveals the angst, resistance, and cries of racial preference that are often associated with the introduction of just one qualified African American into a given context even when that context has been historically all-white (Bonilla-Silva, 2003, 2005).

The second thing I do in this paper—in addition to centering the discussion on Black children—is to argue for racism and racialization (Miles, 1988) as central concerns in mathematics learning and participation and as lenses through which to critique mathematics education research and policy. I do so knowing that discussions of race[2] and racism are likely to produce knee-jerk negative reactions from those who have adopted a color-blind ideology and who believe that we now live in a post-racial society in which race and racism are no longer relevant, despite great evidence to the contrary (Bonilla-Silva, 2003, 2005; Macedo & Gounari, 2006; Omi & Winant, 1994; Winant, 2004). I do so also knowing that many discussions of race and racism are unproductive because they tend to aim for simplicity in framing and in solution.

In this discussion, I acknowledge the complexities of race and move well beyond the causal-factor approach utilized in mainstream research. For example, I agree with Essed (2002) who stated:

> "Race" is an *ideological construction*, and not just a social construction, because the idea of "race" has never existed outside a framework of group interest. As part of a nineteenth pseudoscientific theory, as well as in contemporary "popular" thinking, the notion of "race" is inherently part of a "model" of asymmetrically organized "races" in which Whites rank higher than "non-Whites." Furthermore, racism is a *structure* because racial

[2] Clearly, my focus on race does not diminish the importance of race, class, and gender intersections.

> and ethnic dominance exists in and is reproduced by the system through the formulation and applications of rules, laws, and regulations and through access to and the allocation of resources. Finally, racism is a *process* because structures and ideologies do not exist outside the everyday practices through which they are created and confirmed. (p. 185)

I draw on sociological theory in further characterizing racism as “the routinized outcome of practices that create or reproduce hierarchical social structures based on essentialized racial categories” (Winant, 2004, p. 126). As noted by Macedo and Gounari (2006), “Racism includes a set of ideologies, discourses, discursive practices, institutions, and vocabularies” (p. 4). These characterizations are important because they overcome the tendency to reduce racism to individual psychology (Omi & Winant, 1994). Instead, these characterizations acknowledge that racism operates at many levels—everyday, institutional, and structural—and involves all the actors, practices, and institutions in a given society.

Acknowledging sociological findings that race and racial categories are politically contested in any given sociohistorical and geopolitical context—through a process called *racial formation*[3] (Omi & Winant, 1994)—and also recognizing that racism is a global phenomenon (Macedo & Gounari, 2006), my references to race and racism in this paper is to their everyday, institutional, and structural instantiations in the United States (Bonilla-Silva, 1997; Essed, 2002). These peculiar and particular manifestations have ranged from (a) native American extermination, chattel slavery, Jim Crow apartheid, Chinese exclusion, and Japanese internment to (b) post-civil rights color-blindness to (c) a so-called post-racial context that allows for the passage of the Secure Fence Act which calls for 700 miles of physical and virtual fencing along the U.S.-Mexican border; a post-racial context that allows for the burning of Black churches and synagogues; a post-racial context that condones racial profiling of Arab Americans and Muslims; a post-racial context that allows a Republican activist to compare the First Lady of the United States to a gorilla and then issue a non-apology apology; a post-racial context that encourages a lunatic white supremacist to open fire in the Holocaust museum because of his hatred for Jews and Blacks (Macedo & Gounari, 2006).

The history and ubiquity of race offer some evidence for law professor Derrick Bell’s haunting claim that racism is permanent (Bell, 1993). This ubiquity also begs the question of how, not if, (mis)understandings of race and racism influence the ideologies and epistemologies found in mathematics education. I push this point further by asking how do race and racism structure the very nature of the mathematics education enterprise?

On one hand, there is the possibility that mathematics education is a race-neutral domain, free from racial contestation, stratification, and hierarchies, and different in character than all other racialized societal contexts. If so, how do we reconcile this neutral character with the racialized inequities faced outside of the domain by many of the students our work is intended to help?

On the other hand, I suggest that a structural analysis would show that mathematics education research and policy not only help to produce racial representations and meanings but also are themselves informed by societal meanings and representations of race. Not only do research and scholarly interpretations of children’s mathematical behavior serve to inform societal beliefs about race and racial categories, but race-based beliefs about children also serve

[3] Omi & Winant (2005, p. 16) define *racial formation* as the process by which social, economic, and political forces determine the content and importance of racial categories, and by which they are in turn shaped by racial meanings.

to inform mathematics education research and policy. Beliefs in so-called *racial* achievement gaps and attempts to close of such gaps by raising Black children to the level of white children exemplify these beliefs.

Moreover, a structural analysis would reveal that the pervasiveness whiteness—represented numerically, ideologically, epistemologically, and in material power—which characterizes mathematics education research and policy contexts bears a strong family resemblance to the manifestations of whiteness found in other societal contexts. In my view, the enterprise of mathematics education is no different than other racialized spaces and should be subjected to the same anti-racist scrutiny, especially as it pertains to the well-being of Black children.

It is in the ways just described that mathematics education research and policy can be implicated in New Right, conservative, liberal, and neoliberal *racial projects* (Omi & Winant, 1994; Winant, 2004) that shape larger racial dynamics. According to the sociological literature, a racial project is "simultaneously an interpretation, representation, or explanation of racial dynamics and an effort to reorganize or redistribute resources along particular racial lines. Racial projects connect what race means in a particular discursive practice and the ways in which both social structures and everyday experiences are racially organized, based upon that meaning" (Omi & Winant, 1994, p. 56). Consider this partial history of the *neoliberal* racial project:

> In order to win the [1992] election and reinvigorate the once-powerful Democratic coalition, Bill Clinton believed he needed to attract white working class voters—the "Reagan Democrats." His appeal was based on lessons learned from the right, lessons about race. Pragmatic liberals in the Democratic camp proposed a more activist social policy emphasizing greater state investment in job creation, education, and infrastructure development. But they conspicuously avoided discussing racial matters such as residential segregation or discrimination. The Democrats' approach, which harked back to Kennedy's remark that "A rising tide lifts all boats," aspired to "universalistic" rather than "group-specific" reforms. Thus the surprising shift in U.S. racial politics was not… the Republican analysis which placed blame on the racially defined minority poor and the welfare policies which has supposedly taught them irresponsibility and dependency. The "surprise" was rather the Democratic retreat from race and the party's limited but real adoption of Republican racial politics, with their support for "universalism" and their rejection of "race-specific" policies.…. This developing neoliberal project seeks to rearticulate the neoconservative and new right racial projects of the Reagan-Bush years in a centrist framework of moderate redistribution and cultural universalism. Neoliberals deliberately try to avoid racial themes, both because they fear the divisiveness and polarization which characterized the racial reaction, and because they mistrust the "identity politics" whose origins lie in the 1960s…. In its *signifying* or representational dimension, the neoliberal project avoids (as far as possible) framing issues or identities racially. Neoliberals argue that addressing social policy or political discourse overtly to matters of race simply serves to distract, or even hinder, the kinds of reforms which could most directly benefit racially defined minorities. To focus too much attention on race tends to fuel demagogy and separatism, and this exacerbates the very difficulties which much racial discourse has ostensibly been intended to solve. To speak of race is to enter a terrain where racism is hard to avoid. Better to address racism by ignoring race, at least publicly (Omi & Winant, 1994, pp. 146-148)

By way of example, recent reform movements and policy documents in mathematics education can be analyzed for their contributions to these racial projects. *Mathematics for All*, as one of the most egalitarian movements in the field, seeks to reorganize and redistribute access to mathematics by appealing to liberalism. In the liberal project, there is an underlying appeal to white middle- and upper-class consciousness to convince them that others must now share in the opportunities that they have long enjoyed (Winant, 2004). It also aligns well with the neoliberal racial project in that universal programs (i.e. *Algebra for All*) that work for all students are promoted in lieu of group-specific efforts and objectives (Winant, 2004). It is in this way that *Mathematics for All* rhetoric is about assimilation. In classical assimilation theory, assimilation is defined as “the decline, and at its endpoint the disappearance, of an ethnic/racial distinction and the cultural and social differences that express it” (Alba & Nee, 1997, p. 863).

Viewed more critically, *Mathematics for All* is also about nationalism because it appeals to U.S. international competitiveness in relation to real and perceived foreign threats (Gutstein, 2008; Martin, 2003, 2009c). Like assimilation, nationalism seeks to erase meaningful cultural differences among social groups and to silence internal racial identity politics in favor of collectivism.

So, while *Mathematics for All* has an equity-oriented veneer, it is clear to me that there are other ideologies at play that are not based on moral and humanistic concern for those who are marginalized in mathematics. In a paper titled *Hidden Assumptions and Unaddressed Questions in Mathematics for All Rhetoric* (Martin, 2003), I offer additional critique of this movement.

Similarly, a critical analysis of the *Final Report of the National Mathematics Advisory Panel* (U.S. Department of Education, 2008) report reveals how it, too, contributes to racial projects. In this report, the learning of mathematics in U.S. schools is linked directly to the preservation of national security. The third paragraph of the Panel’s *Executive Summary* is very clear in making this link:

> Much of the commentary on mathematics and science in the United States focuses on national economic competitiveness and the economic well-being of citizens and enterprises. There is reason enough for concern about these matters, but it is yet more fundamental to recognize that the safety of the nation and the quality of life—not just the prosperity of the nation—are at issue. (p. xi)

Considering the political origins of the National Math Panel, these security concerns can be linked to conservative Republican ideology, Islamophobia, anti-Muslim sentiments, and the globalization of U.S. racism and white privilege (Macedo & Gounari, 2006; Winant, 2004).

Beyond the policy arena, the frequent use of a race-comparative approach to examine mathematics achievement differences among U.S. students makes its own contribution to racial projects. This approach supports the normalization of whiteness and the subordination of poor, African American, Latino, and Native American students. Specifically, this approach has served to reify a racial hierarchy of mathematics ability that is now taken for granted by the general public and by many scholars and policy makers (Martin, 2009a, 2009c). Belief in this hierarchy contributes to the interpretation and representation of race and racial categories by supporting negative societal meanings for what it means to be poor, Black, Latino, and Native American. For example, in most of these studies, the resulting analyses often suggest that to *be Black* is to be mathematically illiterate and inferior relative to those who are identified as White and Asian.

## Researcher Identity

Having provided the extended preamble above, I do feel it is important to pause and provide a better sense of my motivation for raising these issues and where they fit into my life as a scholar. Readers who are familiar with my research and teaching know that my focus on Black children and issues of race and racialization is not a novelty for me. My efforts are not an attempt to jump on the equity and diversity bandwagons that have emerged in mathematics education over the last several years or an attempt to urbanize my research. Nor does my focus represent a sudden realization that it might be valuable to study the mathematical lives of Black children and to be explicit about attempts to construct them as less than ideal learners.

My research and teaching over the last twenty years have focused exclusively on the life and mathematical experiences of Black children and adults in school contexts ranging from middle school to community college. In my work, I have detailed aspects of their racial and mathematical socializations and characterized the identities they co-construct in light of their experiences. Moreover, rather than studying only underachievement and failure in mathematics, I have devoted a great deal of attention to documenting success and agency among African American children and adults. Up until a few years ago, little attention was given to this success and little was known about how students defined, achieved, and maintained it. My own studies have revealed a number of sociohistorical, community, school, and intrapersonal forces contributing to resilience and success in mathematics (Martin, 2000, 2006a, 2006b). This work has consistently highlighted issues of racism, racial identity, and racialization not because I impose these issues but because the participants in my research cite them over and over again as being both central and salient (Martin 2000, 2006a, 2006b, 2007, 2009a, 2009c).

So, although this paper has been composed to address the conference theme, it is clear that I also have a political agenda. This goes against the idea that research and scholarship should not drift towards advocacy. However, all research and scholarship are political. Moreover, the production of knowledge cannot be disconnected from who we are as people, what we have experienced, and what we believe. My multiple identities—racial, scholarly, mathematical, and otherwise—have informed, and continue to shape, my scholarly perspective. I am an African American through self and societal identification although these asserted and assigned identities do not always overlap. My own experiences with mathematics both mirror and diverge from those of other African Americans. Experiences with poverty and racism are not unfamiliar to me nor are experiences with academic and mathematics success.

I am also a scholar. I do not hesitate in identifying myself as an African American scholar in a field numerically dominated by white scholars. Identifying in this way does not limit or essentialize my perspective or discount the perspectives and experiences of others. Paraphrasing Supreme Court nominee Sonia Sotomayor:

> I would hope that a wise [African American man] with the richness of [his] experiences would more often than not reach a better conclusion than a white male who hasn't lived that life…. [However,] I… believe that we should not be so myopic as to believe that others of different experiences or backgrounds are incapable of understanding the values and needs of people from a different group. Many are so capable…. However, to understand takes time and effort, something that not all people are willing to give. For others, their experiences limit their ability to understand the experiences of others. Others simply do not care. Hence, one must accept the proposition that a difference there will be

by the presence of women and people of color. (http://feministlawprofessors.com/?p=10952)

In the remainder of this paper, I discuss Black children and issues of racism and racialization by structuring my comments around four inter-related topics which, admittedly, will be devoid of mathematics content[4] and may come across as sociological in nature, far afield of mathematics education.

First, I explore the meaning and significance of the title of this paper.

Second, I briefly discuss the representation of Black children in mainstream mathematics education research and policy so as to reveal the form and substance of these representations and to show how they have contributed to the construction of Black children as inferior to other children. Continued rhetoric around the so-called black-white or racial achievement gap is one example where Black children are told explicitly and matter of factly that they are inferior to white children.

Third, I briefly outline my own research theoretical perspective that conceptualizes mathematics learning and participation as racialized forms of experience, not just for African American children but also for all children. Within this perspective, I characterize mathematics education research and policy as instantiations of white institutional space, where pervasive myths and stereotypes about African American children have their genesis and are allowed to persist as common sense.

Finally, I present of a set of axioms for researching Black children and mathematics; these axioms have served as the foundation of my research and I believe they should inform all future work on Black children, helping to counter the masternarrative that has dominated discussions of these children.

## Little Black Boys and Little Black Girls?

My focus on little Black boys and little Black girls is simultaneously historical, present-day literal, and metaphorical. First, it recognizes the historical significance of this conference taking place in Atlanta, a key city in the United States civil rights movement as well as being the birthplace of reverend Dr. Martin Luther King, Jr. and the final resting place of Dr. King and his wife, Coretta Scott King. In his famous *I Have a Dream* speech, delivered on August 23, 1963, Dr. King envisioned a day when little Black boys and little Black girls would be able to experience full and humane lives, free from racism and subjugation and all that accompanies those oppressions.

Yet, on September 15, 1963, less than one month after that clarion call for social progress, four little Black girls—11-year-old Carole Denise McNair and 14-year-olds Addie Mae Collins, Cynthia Wesley, and Carole Robertson—were murdered by a bomb placed under the steps of the 16th Street Baptist Church located in Birmingham, Alabama. The ground floor of the church collapsed, killing the girls and injuring some twenty others. The lyrics that opened this paper are taken from the song *Birmingham Sunday,* which was performed by Joan Baez to mourn the girls' deaths.

[4] Although studies of Black children learning specific mathematics content are critically important, I do not dwell on this topic because I do not wish to suggest that there is something peculiar about these children's learning or that some content is especially problematic for them to learn. The fact is that normal, healthy Black children can learn whatever mathematics they are given the opportunity and necessary supports to learn.

Robert "Dynamite Bob" Chambliss, a member of the Ku Klux Klan, was identified by witnesses, arrested, and charged with murder and possession of dynamite without a permit. Other Klansmen were also identified but not initially charged. In his first trial on October 8, 1963, Chambliss was found not guilty of murder but received a small fine and sentenced to six months in jail for possessing dynamite. It was later revealed that FBI director J. Edgar Hoover interfered with prosecutions in the cases. In 1971, the case was re-opened by the Alabama attorney general. A grand jury indicted Chambliss for the murder of Denise McNair on September 24, 1977. In November 1977, Chambliss was retried, found guilty of murder, and sentenced to life in prison. It was not until 2001 and 2002 that two of the remaining suspects were convicted.

Although the murders of four little black girls punctuated September 15, 1963, two other murders of black children occurred in Birmingham on that day:

> James Robinson, a black 16-year-old, became involved in a rock-throwing incident with a gang of white teenagers. As he fled from the scene, Robinson ran down an alley near the Sixteenth St. Church and was promptly shot in the back and killed by a white City of Birmingham police officer. A few hours later, on the outskirts of the city, 13-year-old Virgil Ware was riding on the handlebars of a bicycle with his older brother. From the opposite direction, a red moped, decorated with the Confederate flag, quickly approached the two boys. Without warning, the operator of the motorbike, a white 16-year-old, pulled out a gun and shot Virgil twice in the chest, killing him instantly.
> http://www.trutv.com/library/crime/terrorists_spies/terrorists/birmingham_church/5.html

Why do I bring up civil rights history in a contemporary discussion of mathematics learning and participation? I do so because history reminds us that society has always had a high threshold for Black pain. Moreover, the lives of Denise McNair, Addie Mae Collins, Cynthia Wesley, Carole Robertson, James Robinson, and Virgil Ware were taken not because they were just any children. Their lives were taken because they were *Black* children. As I stated earlier in this paper, when discussing diversity, we should not lose sight of particularity. Any analysis of Black children's behavior in the world, including mathematics education, that fails to contextualize or appreciate what life was like, or is like, for these children is shortsighted and bound to be limited in its explanatory power.

There will be some who read this paper and say, "Get over it. Stop whining. Stop playing the race card. That's ancient history. Things have gotten better." and so on. However, these dismissals and resistance only amount to a desire to maintain the status quo and to avoid the work of understanding how society's laws, policies, and practices routinely continue to converge in subjugating Black children.[5]

### Representing and Constructing Black Children in Mathematics Education

My focus on little Black boys and little Black girls is present-day literal because I contend that even in a post-civil rights, color-blind era highlighted by the election of a President with biracial African heritage and the identification of mathematics literacy as a 21st century civil right, there is little reason to believe that the well-being of little Black boys and little Black girls

[5] As pointed out by educational anthropologist, studies of education for Black children should consider forces at many levels: societal, community, family, institutional, school, individual. I acknowledge that there are many internal, community- and family-based forces to consider. Those forces are not addressed in this paper. See Martin (2000).

is a priority in America or in mathematics education, in particular. We still live in a society where blackness and black life are denigrated. Just a few months ago, Bonnie Sweeten, a white woman from Philadelphia claimed that she and her 9-year-old daughter had been abducted by two Black men and thrown into the trunk of a Cadillac. In response to her 911 calls, massive local and national media attention was given to her abduction. Crisis intervention teams were sent to her daughter's school. Only after more careful police work was it revealed that Sweeten had faked the abductions and had flown to Disneyworld after withdrawing more than 12,000 dollars from her bank accounts. This is a repeat episode of earlier cases involving Susan Smith and Charles Stuart in which the villainous Black man was blamed for killing four white children and a white wife, respectively. In these two instances, Smith and Stuart were the guilty parties. Yet, in all these cases, society was quick to accept the accusations that were put forward. The media attention and concern for the well-being of white children, men, and women stands in stark contrast to the attention given to the alarming numbers of murders of Black children in my own city of Chicago. As of mid-May, 36 schoolchildren, most of them Black, had been killed, an average of more than one a week. National media attention was slow in coming.[6] In the eyes of many, each time a Black child's life is taken, it is just "another one gone."

A cursory examination of the ways Black children have been researched and represented in mainstream mathematics education research and policy further shows how Black children are devalued.

The dominant story line, or masternarrative, in research and policy contexts is one that normalizes failure, ignores success, and uses white children's mathematical behaviors and performance as the standard for all children. This masternarrative has helped to support negative social constructions of these children. Mathematics education policy reports dating back 25 years have explicitly labeled Black children as mathematically illiterate. More recently, African American 12th graders have been told, in a very public fashion, that they are only as skilled and demonstrate math abilities at the level of white 8th graders (Education Trust, 2003). After their comprehensive review of over 16,000 studies, the members of the National Mathematics Advisory Panel reduced their research recommendation for Black children to issues of motivation, task engagement, and self-efficacy. These areas are important but they focus attention on Black children as though they are unmotivated, inclined to disengagement, and lacking in agency. Institutional and structural barriers inside and outside of school, including racism, that affect student mathematics achievement, engagement, and motivation received little, if any, attention in the report (Martin, 2008). Resistance and disengagement among some students may, in fact, be rational responses to oppressive schooling practices.

In other research contexts, it has been claimed that poor (Black) children enter school with only *pre*-mathematical knowledge and lack the ability to mathematize their experiences, engage in abstraction and elaboration, and use mathematical ideas and symbols to create models of their everyday lives (e.g., Clements & Sarama, 2007). Left unanswered is whether researchers who report these findings understand, even partially, the "everyday lives" of Black children. As I have stated in other writings (Martin, 2009c):

> Because the tasks, assessments, and standards for competence used to draw these conclusions are typically not normed on African American children's cultural and life

[6] For an interesting mathematical analysis of media coverage on crimes against Black and White children see the Appendix to this paper or go to: http://www.thedefendersonline.com/2009/05/14/36-children-of-color-dead-in-chicago/

> experiences, one could also argue that … the preferred ways of abstracting, representing, an elaboration called for in these studies and reports are based on the normalized behavior of white, middle-class and upper-class children…. Very little consideration is giving to exploring patterns in the ways that [poor] African American children do engage in abstraction, representation, and elaboration to determine if these ways are mediated by their cultural experiences in out-of-school settings and whether the preferred ways of engaging in these processes serve useful functions relative to those experiences. (p. 15)

Moreover, despite these claims about Black children's mathematical knowledge, little seems to be known about their metacognitive and racial awareness during mathematical problem solving, particularly in contexts that are meaningful to them and where they are likely to demonstrate a range of mathematical behaviors. Research in these areas would not only provide insight into Black children's reasoning processes and strategy choices (e.g., Malloy & Jones, 1998) but also about their awareness of how they are socially constructed, and how they socially construct themselves, as mathematics learners.

Finally, those who choose to study Black children in high-poverty contexts must first acknowledge, and understand, that ghettos are not natural or normative contexts for Black children but, like slavery and Jim Crow, they are "race making institutions" (Wacquant, 2006, p. 103) designed to dehumanize and inflict material, structural, and symbolic violence (Bourdieu & Passeron, 1977) on those who are forced to live in them. As noted by Wacquant (2006):

> The ghetto, in short, operates as an ethnoracial prison: it encages a dishonoured category and severely curtails the life chances of its members in support of the 'monopolization of ideal and material goods or opportunities' by the dominant status groups dwelling on its outskirts. (p. 101)

Only recently have researchers begun to directly examine the mathematical *experiences* and *identities* of Black children versus a narrow focus on their achievement (Martin, 2007). Researchers doing this work have explored several important areas related to these students' mathematics learning and development: (1) their beliefs about their ability to participate in mathematical contexts, (2) their motivations to learn or do mathematics, (3) the ways in which they define the importance and value of mathematics knowledge and success in mathematics, (4) their mathematics socialization experiences in school and non-school contexts, and (5) the co-construction of mathematics identities and other social identities that are important to these students. Research in these areas supports the assertion made Theresa Perry, Claude Steele, and Asa Hilliard (2003) in their book *Young, Gifted, and Black: Promoting High Achievement Among African American Students:*

> African American students face challenges unique to them as students in American schools at all levels by virtue of their social identity as African Americans and of the way that identity can be a source of devaluation in contemporary American society…. Before we can theorize African-American school achievement, we need to have an understanding of what the nature of the task of achievement is for African Americans *as African Americans*. (pp. vi-9)

Readers are urged to consult the recent volumes *Mathematics Success and Failure Among African American Youth* (Martin, 2000), *Mathematics Teaching, Learning, and Liberation in the Lives of Black Children* (Martin, 2009b) and *Culturally Specific Pedagogy in the Mathematics Classroom: Strategies for Teachers and Students* (Leonard, 2008) for more details about some of this recent work.

### Racialized Forms of Experience and White Institutional Space

One of the most defining feature of the masternarrative on Black children and mathematics is that it frequently cites or implicates *race* as a causal variable in their achievement but just as frequently fails to define this concept or acknowledge that *racism*, not race, should be the key area of focus. I have argued elsewhere (Martin, 2009a) that:

> Within mathematics education, *race* remains undertheorized in relation to mathematics learning and participation. While race is characterized in the sociological and critical theory literatures as socially and politically constructed and with structural expressions, most studies of differential outcomes in mathematics education begin and end their analyses with static racial categories and group labels for the sole purpose of disaggregating data. One consequence is a widely accepted, and largely uncontested, racial hierarchy of mathematical ability. Rather than challenging and deconstructing this hierarchy, many math educators take it as their natural starting point. Disparities in achievement and persistence are then inadequately framed as reflecting race effects rather than as consequences of the *racialized* nature of students' mathematical experiences. (p. 295)

My own considerations of racism and racialization have led me to develop a conceptualization of mathematics learning and participation as *racialized forms of experience* for all children (Martin, 2006a, 2009a). I claim that these experiences are shaped and structured by the meanings and representations of race and racial groups that exist in the larger society. A summary of this perspective is provided in Figure 1. A more detailed discussion can be found in Martin (2009a).

I argue that this conceptualization of mathematics learning and participation may be more relevant to the mathematical experiences of African American learners than the dominant perspectives which typically characterize learning and participation as cultural, situated, or cognitive because this conceptualization situates the realities of racism and racialization at the center of these experiences (Martin & McGee, 2009).

| | **Conceptualizations of race** | **Conceptualizations of learners** | **Research, policy, and practice orientations to race** | **Aims and goals of mathematics education research, policy, and practice** |
|---|---|---|---|---|
| **Mainstream mathematics education research, policy, and practice** | Races as biologically determined. Race as a way to disaggregate data. Race as a causal variable for mathematics achievement. | Those who know mathematics. Those who do not. Those who are mathematically literate. Those who are | Resistance to realities of racism. Color-blindness. Racial apathy. Solution on demand. Interest convergence. | Close the racial achievement gap. Maintain white privilege and United States international |

| | | | | |
|---|---|---|---|---|
| | | mathematically illiterate. Students belong to a racial hierarchy of mathematical ability. | | competitiveness. |
| **Mathematics learning and participation as racialized forms of experience** | Race as a sociopolitical construction. Historically contingent nature of race. Consideration racism and racialization. | Consideration of the negotiated nature of identity with respect to mathematics. Asks what does it mean to be African American, Latino, Native American, white, and Asian American in the context of mathematics learning? | Consideration of everyday, institutional, and structural racism. Mainstream mathematics education research and policy contexts as instantiations of white institutional space. | Empowerment and liberation from oppression for marginalized learners. |

*Figure 1.* Contrasting approaches to race in mathematics education research, policy, and practice

I have utilized this race-critical perspective to address the production of knowledge about African American children and mathematics and to reframe the conversations about these children in several areas including mathematics teacher knowledge and teacher selection (Martin, 2007) and assessment (Davis & Martin, 2008). I have addressed questions such as: What is the study of African American children the study of? What should the study of African American children be the study of? Why should African American children learn mathematics? Who should teach mathematics to African American children? What does it mean to be African American in the context of mathematics learning? and What does it mean to be a learner of mathematics in an African American context?

*White Institutional Space*

Returning to the masternarrative on Black children, I contend that it is only within certain kinds of ideological and material spaces—contexts that sociologists have called *white institutional spaces*—that so-called *racial* achievement gaps and the mathematical illiteracy of Black children can assume common-sense status. The term *white institutional space* comes from the work of sociologists Joe Feagin (1996) and Wendy Moore, who, in her book *Reproducing Racism: White Space, Elite Law Schools, and Racial Inequality* (2008), examined the white space of law schools and how the ideologies and practices in these schools serve to privilege white perspectives, white ideological frames, white power, and white dominance all the while purporting to represent law as neutral and objective.

White institutional spaces are characterized by (1) numerical domination by whites and the exclusion of people of color from positions of power in institutional contexts, (2) the development of a white frame that organizes the logic of the institution or discipline, (3) the historical construction of curricular models based upon the thinking of white elites, and (4) the assertion of knowledge production as neutral and impartial unconnected to power relations.

In Martin (2008), I provide a more detailed discussion of how mathematics education research and policy contexts represent instantiations of white institutional space. For example, I offered a critique of the composition of the National Mathematics Advisory Panel as well as its failure to draw on the most insightful recent research about Black children and mathematics. My critique was not only directed at the Math Panel but also at scholars in the field who, from recognized positions of power, failed to object to the absence of African American math education researchers on the Panel. This kind of inaction, despite progressive rhetoric about equity and diversity, was noted by Macedo and Gounari (2006) as being characteristic of liberal approaches in white spaces:

> … many white liberals (and some black liberals as well) fail to understand how they can embody white supremacist values and beliefs, even though they may not embrace racism as prejudice or domination (especially domination that involves coercive control). They cannot recognize how their actions support and affirm the very structure of racist domination and oppression they profess to wish to see eradicated….By not understanding their complicity with white supremacist ideology, many white liberals reproduce a colonialist and assimilationist value system that gives rise to a form of tokenism parading under the rubric of diversity…. That is why many white liberals prefer to promote "diversity" to the extent that diversity as a cultural model not only fails to interrogate the white privilege extracted from a white supremacist ideology but also allows for white liberals to have blacks and other oppressed cultural groups as mascots in their Benetton color scheme of diversity. This form of diversity promoted through multicultural programs, for example, represents a mere reorganization of knowledge through which diversity is presented as a naturalization process whereby different ethnic and cultural groups (white groups are never associated with ethnicity, even though their ethnicity provides a yardstick against which all other groups are measured) are represented and their asymmetrical power relations with the dominant white group are never interrogated (p. 32)

These sentiments were echoed by Liz Appel (2003) in her focused critique of liberal white participants in the movement against the prison industrial complex:

> … many well-intentioned white folks wish to incorporate an anti-racist approach in their work. Seeking a quick resolve, the problem of racism is often superficially addressed, however. Focusing on tangible and visible solutions, they tokenize individual people of color, perhaps by bringing in a few nonwhite people into public spaces and circles of power (as board members, speakers, etc.), in an attempt to demonstrate the "diverse" nature of the struggle and those that make up the fight. This is not to say that every attempt to incorporate people of color is inherently racist and self-serving…. [But does] not the fact that whites are able to select people of color for inclusion… reaffirm our power and privilege? (p. 84)

It is through my analysis of mainstream mathematics education research and policy contexts as instantiations of white institutional space, and my understandings of other such spaces, that my focus on little Black boys and little Black girls in this paper becomes metaphorical. Sociologists tell us that when someone or something is socially blackened, it or they are relegated to marginalized status and thought of as inferior. Similarly, when something is whitened, it or they are elevated in social status or importance. In terms of racial dynamics of the United States, this has been documented in books with such provocative titles as *How the Irish Became White* (Ignatiev, 1996) and *The Price of Whiteness: Jews, Race, and American Identity* (Goldstein, 2006). This whitening has also been witnessed in the education arena where Asian Americans, collectively, have been given *model-minority* and *honorary white* status (Lee, 1996, 2005; Martin, 2009). Blackening, on the other hand, has most recently happened to Arab Americans and Muslims who are now are subject to racial profiling and other forms of subjugation. Blackening also explains how the diversity among those from the African diaspora is muted so as to create a singular perception and construction of these groups. Blacks from Caribbean, West Indian, and African backgrounds are all labeled by the dominant society as Black when they come to the U.S.

So, it is interesting to ask the following about the United States mathematics education enterprise: Who are the little Black boys and little Black girls in mathematics education and how are they, and their perspectives, embraced? Are they the scholars who take up race, racism, and power; issues that only occasionally find their way into mainstream mathematics education research and policy discussions? Are these scholars and their perspectives tolerated but also marginalized? Is it assumed that they are less-informed about mathematics content, teaching, learning, curriculum, and assessment to the degree that they are largely absent from key discussions in the field; called on only when issues of equity and diversity are considered?

In a field that purports to be committed to equity for all children, why are there no explicit discussions of the pervasive whiteness in mathematics education research and policy contexts or of the fact that the norms and values of these white institutional spaces are increasingly being applied to populations of other people's children? Why are there no discussions of how we continue to blacken some children by producing research that implies their inferiority? Is it that the characteristics of white institutional spaces are so strong that they lead us to believe this state of affairs is normal and acceptable?

**Where do we go From Here? Axioms for Researching Black Children and Mathematics**

In so far as Black children are concerned, I remain hopeful that mathematics education research and policy, if done right, can benefit these children. Clearly, what constitutes "right" is subject to much debate. Yet, little that constitutes right for these children will emerge from an enterprise that fails to understand its own complicity in these children's subjugation and negative social construction. Moving forward, I want to propose adherence to a set of *sociocritical* "axioms" for addressing Black children, in particular. An axiom is defined as a self-evident or universally recognized truth that is accepted without proof as the basis for argument. In mathematics, proofs of various conjectures and claims are essentially a function of the axioms upon which the system is organized. If you change the axioms, you change the system, and you also change what constitutes valid proof and what is regarded as true. My own research, as well as the comments and analysis in this paper, are premised on these axioms[7] and I believe they should undergird all future inquiry to the mathematical experiences of Black children:

- Axiom I: Black children are brilliant; researchers should not overly concern themselves with documenting how Black children differ from white children and reifying racial achievement gaps but with how black children can best attain and maintain excellence in mathematics;

- Axiom II: Black children possess the intellectual capacity to learn mathematics as well any other child; they do, however, often lack sufficient opportunities to engage in meaningful mathematical experiences;

[7] It is true that these are not axioms in the strict mathematical sense. I am appropriating the term to serve sociological and political purposes.

- Axiom III: *Race* is not a causal variable in determining mathematical achievement among Black children or any other group of children; research and policy purporting to cite *race* effects should be dismissed as scientifically invalid;

- Axiom IV: Racism, racial identity, and racialization are important considerations in mathematics learning and participation; Mathematics education research and policy are deeply involved in the production and reproduction of racial meanings;

- Axiom V: Mathematics education research and policy are simultaneously sites of oppression and liberation for Black children.

These statements are not meant to romanticize Black children nor do they ignore their struggles. However, they require attention to Black children's social realities and how forces, discourses, and projects in the larger society influence those realities. They also require reconsideration of the assumptions about the competencies and capacities of Black children in ways that move us beyond default characterizations of mathematical illiteracy and inferiority with respect to other learners.

As I stated earlier, Black children serve as canaries in the coal mine. If we cannot do right by these children, it is difficult to believe that we can accomplish the goal inherent in the theme of *Embracing Diverse Perspectives*.

## Conclusion

The discussion in this paper hints that a more thorough structural analysis of mathematics education would reveal that the discipline is no different than other racialized contexts in the larger society where issues of power and stratification are prominent. The analysis would confirm the racialized character of mathematics learning and participation not only for Black children but also for all children. I conjecture that a structural analysis would also show that mathematics education research and policy contribute to, and constitute, racial projects. Yet, the hopeful side of me continues to believe that mathematics education can simultaneously be a sight of, and means to, liberation for Black children, helping them to combat the negative consequences of these racial projects.

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## Appendix

Retrieved on July 12, 2009 from: http://www.thedefendersonline.com/2009/05/14/36-children-of-color-dead-in-chicago/

By Stacey Patton

Megan Kanka. Amber Hagerman. JonBenét Ramsey. Elizabeth Smart. Caylee Anthony. Sandra Cantu.

When these six cute, middle-class white girls, ranging from age 2 to 14, went missing or were horrifically murdered, national news outlets devoted hours, days and weeks of coverage to their cases. But when children of color are victimized in similar ways, the mainstream media often remains conspicuously silent or provides scant coverage at best.

A quick GOOGLE news archive search illustrates my point.

There are 3,670 articles on the 1994 murder of 7-year-old Megan Kanka, who was raped and abducted by a twice-convicted sex offender who lived next door. The 1996 murder and abduction of 9-year-old Amber Hagerman produced 2,570 headlines. An astonishing 13,500 news stories helped sensationalize the 1996 murder of JonBenét Ramsey, a 6-year-old beauty pageant contestant found bound and strangled in her home. Between June and November of 2002, 8,300 new stories were printed about the abduction and recovery of 14-year-old Elizabeth Smart. Since last October, 1,570 stories have discussed the murder of 2-year-old Caylee Anthony, whose skeletal remains were found a month later. And in one month, 424 articles have appeared on 8-year-old Cantu, who was raped, killed, stuffed in a suitcase and thrown in a pond in northern California on April 11.

Do the math. Six young white girls. One abducted and later returned. Five killed. 30, 134 news stories and nearly two million total web hits. And with the exception of the Ramsey case, suspects have been captured, indicted, tried, and even sentenced to death for the brutal crimes against these innocent children.

Each of these girls has her own Wikipedia entry, which discusses their lives, details of their investigation, and archives media references and external links to various websites, talk shows, and made-for-TV documentaries and movies as well as child and victims advocacy sites.

Now enter the names of the following children: Corey Hatter, Ordero Hillard, Marcus Washington, Andre Malcolm, Arthur Tyler, Sameer Conn, Shaun Brown, Shaun Bowens, Kiyanna Salter, Daniel Calderon, Ernest Williams, Julian King, Brian Murdock, Quentin Buckner, Devour Robinson, Dushawn Johnson, Isiah Stroud, Andre Stephens, Esteban Martinez, Itzel Fernandez, Johnel Ford, Rachael Beauchamp, Johnny Edwards, Kendrick Pitts, Raheem Washington, Carnell Pitts, Franco Avila, Gregory Robinson, Lee Ivory Miller, Rakeem Washington, Tommie Williams, Marquell Blake, Juan Cazares, Christina Campos, and Alex Arellano.

All 36 of these schoolchildren, mostly black and a few Latinos, were killed in the streets of Chicago during the past nine months. They were shot, stabbed, beaten with bats, kicked to death, burned and run over by cars.

GOOGLE their names and you won't get a return of hundreds of national news stories or thousands of web hits discussing their deaths. The only child of all these victims to gain a great deal of media attention was 7-year-old Julian King, the nephew of singer and actress Jennifer Hudson, killed last October by his mother's estranged husband.

For the rest of the children, there are no Wikipedia entries. No documentaries. No made for TV films. And there won't be. They'll be remembered in a few grainy YouTube video tributes posted by friends and family members. And if there are more shootings, all of these children will be lumped together and described as statistics and tragic victims of urban warfare, even though most were not high school dropouts, gang members, or criminals. They were killed during day-to-day activities: walking to the store, playing in a park, waiting for a bus, or riding in a car with a parent.

# DISCURSIVE ROUTINES AND ENDORSED NARRATIVES AS INSTANCES OF MATHEMATICAL COGNITION

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*The purpose of this research is to investigate the cognitive processes utilized by students when accessing mathematical knowledge while completing homework. In particular, we focus our attention on the ways in which students use distinctive features of mathematical discourse, as mathematical cognitive processes (i.e., "mathematical" mental functions based upon the function being performed), to support their own learning. Our findings suggest that there may be an important connection between discursive routines and endorsed narratives in student learning.*

## Purposes of the Study

Students' inability to complete mathematics homework has been linked to achievement (Cooper, Robinson, & Patall, 2006). In some instances, motivation and social circumstances may contribute to students' inabilities to complete mathematics homework. The curriculum and how it is experienced, understood, and subsequently accessed by students may also be important contributors to students' inability to complete homework. Nesher, Hershkovits, and Novotna (2003) contend that the ability to access mathematical knowledge, independently by a student, is significantly influenced by a student's *initial* ability to make sense of mathematical texts. Surprisingly, little research exists documenting the ways in which students access mathematical knowledge while completing homework, although research does exist in simulated settings (e.g., Berger, 2004).

Therefore, the purpose of this research is to investigate the cognitive processes utilized by students when accessing mathematical knowledge while completing homework. In particular, we focus our attention on the ways in which students use distinctive features of mathematical discourse, as mathematical cognitive processes (i.e., "mathematical" mental functions based upon the function being performed), to support their own learning. Four features of distinctive mathematical discourse are drawn from Ben-Yehuda, Lavy, Linchevski, and Sfard (2005) and thus include:

> (1) *uses of words* [authors' italics] that count as mathematical; (2) the use of uniquely mathematical *visual mediators* [authors' italics] in the form of symbolic artifacts that have been created specifically for the purpose of communicating about quantities; (3) special *discursive routines* [authors' italics] with which the participants implement well-defined types of task; and (4) *endorsed narratives*[authors' italics], such as definitions, postulates, and theorems, produced throughout the discursive activity. (p. 182)

We adopt for this analysis an elaborated view of mathematical discourse to include "all forms of language, including gesture, signs, artifacts, mimicking, and so on" (Lerman, 2001, p. 87).

For this research, six eighth grade children were invited to document their mathematical cognitive processes using at-home video diaries (i.e., "mathcams") while completing their homework to capture aspects of their mathematization that are causing problems, pause, and reflection. Students' verbalizing their thinking has been shown in multiple settings to improve mathematical performance (Mercer, Wegerif, & Dawes, 1999; Sfard & Kieran, 2001), and thus may be a real and tangible benefit for the students participating in this study. Beyond individual achievement, however, this research reporting data occurring in real-time has the potential to raise important insights and questions about classroom practices and student learning. Currently, "real-time analysis" of mathematical cognitive processes utilized by students during homework is scant amongst the literature.

## Research Questions

1. Which mathematical cognitive processes (i.e., mathematical words, visual mediators, discursive routines, and endorsed narratives) do students utilize when accessing mathematical knowledge while completing homework?
2. How does the interaction of mathematical cognitive processes suggest about the nature of student learning?
3. How can this knowledge inform teaching of mathematics?

## Theoretical Framework

In order to examine verbalized process as instances of mathematical cognition, we adopt the theoretical framework of Ericsson and Simon's (1993, 1998) work on talk- and think-aloud protocols. Ericson and Simon contend that verbal reports as data can, depending on the conditions of the verbal reports, be viewed as instances of cognition. This view is also supported by other theorists (Lerman, 2001; Radford, 2004; Sfard & Kieran, 2001). For example, Vygotsky (1962) describes talking aloud as the manifestation of inner thought or cognition (p. 149). Vygotsky theorizes that as children "solve practical tasks with the help of their speech, and action, which ultimately produces internalization of the visual field" (p. 26). Cognition "is not merely expressed in words; it comes into existence through them" (Vygotsky, 1962, p. 125). Ben-Yehuda et al. (2005) also suggest that "thinking can be regarded as a special case of the activity of communication [authors' italics]" (p. 181). Comparison between child and researcher-observer reports in a study by Wu et al. (2008) investigating verbalized reports of cognition showed high consistency (Kappa=.948). Wu et al.'s results suggest suggests that children can report on their cognitive processes accurately.

Particularly relevant to the present research are Ericsson and Simon's (1993; 1998) protocols associated with *Level 3 verbalizations*, which are verbalizations linked to instructions to explain or describe thinking, and thus employ intermediary cognitive processes by virtue of the instruction to "explain" or "elaborate." Ericsson and Simon also make the distinction between verbalizations that are given retrospectively or concurrently to task completion. In the present research, both applications apply. Students may be verbalizing their cognitive processes as they concurrently with homework completion or may retrospectively describe their cognitive processes associated with prior learning.

Faithful appropriations of Ericsson and Simon's (1993, 1998) talk- and think-aloud protocols occur exclusively in experimental settings. In later explications of their model for analyzing thinking and talk, Ericsson and Simon proposed that everyday situations can be reproduced in controlled laboratory settings. Indeed, many researchers simulate classroom learning with laboratory “training” (Anderson, Reder, & Simon, 2000, Summer). As Anderson, Reder, and Simon explain, learning as a complex skill is hierarchical in structure with multiple nested components that require both analyses in the laboratory and in real-world settings. Our research is poised to make important contributions in terms of real-time analysis of mathematical cognition and its relationship to teaching and learning, based upon what students tell us about their own thinking versus what a standardized test might show.

## Methods

Data for this paper were drawn from a year-long study investigating the home-school connection in mathematics learning in an eighth grade classroom. The teacher involved in the research (the third author), was a mentor mathematics teacher in his school board (Kotsopoulos & Heide, in press). At the time of this research, he had been teaching 12 years and had completed a master’s degree. He was approached to participate in this study and he agreed.

The school was located in an economically, socially and culturally diverse urban setting. For example, there were a reported 52 languages spoken amongst the student population, with 195 out of 960 students identified as English Language Learners (ELL). Only approximately 10% of the English Language Learners were born in Canada or resided in Canada for more than three years.

Duane’s class consisted of 28 students, 14 male, and 14 female. All students were either 13 or 14 years of age. From these 28 students, a purposive sample of six students (three males and three females) was selected and invited to participate in this study. In consultation with the grade-seven teacher, students were selected based upon the following considerations: (a) perceived ability of the student by the teacher to engage in thinking aloud, (b) perceived level of responsibility of the student to maintain continued engagement with the student and to care for the home equipment, (c) gender, and (d) ability. The goal in terms of ability and gender was to ensure mixed representation. In addition to the six students that were selected, two alternates were also selected in the event that a student had to withdraw from the study, which was the case with one of the initial student cohort.

Observational video data were gathered from the students and also daily from the classroom during mathematics instruction. The six students, their parents, the classroom teacher, and the research team, met to engage in a training session with the students and distribute the computers. Students at this time were trained on (a) how to video record their verbalizations using cameras built-into the laptop computers, (b) how to transfer their recordings using secure email or memory sticks, and (c) how to engage in the task of talking aloud about their mathematical thinking. We encouraged students to document everything they were thinking and doing in order to assist themselves with the understanding and completion of the mathematical homework. We anticipated one to five submissions from each student per week. Additionally, Amanda, the fourth author, video-taped the daily mathematics lesson for the duration of the school year.

All video data from the classroom and the students were transcribed by two different transcribers to ensure accuracy and then coded. There were four codes used. These codes were adopted from the four features of distinctive mathematical discourse from Ben-Yehuda et al. (2005) and thus include: identifying uses of words that count as mathematical, use of uniquely mathematical visual mediators, use of discursive routines, and use of endorsed narrative. Also appropriated from Ben-Yehuda et al. is the use of *high-resolution descriptive methods* to report our data. This method is intended to focus on detailed qualitative accounts of students’ cognitive process through transcription and, in our case, video analysis. The detailed qualitative accounts also aimed at ascertaining relationships between the four codes used. Thus, the codes were examined both individually and in relation to one another.

For this paper we analyzed 34 mathcam videos (mean length 4.79 minutes) from the six students (mean number of submissions 6 during the month of October 2008, in which the topic of study in the classroom was numeracy (e.g., exponents, factors, prime numbers, and square roots). In this paper we present specifically data emerging from one student—Kara who is 13 years old. We selected this case for its explanatory potential.

## Results

Our results across all six students suggested that the there were particular forms of interaction between the four mathematical cognitive processes under examination: uses of words use of uniquely mathematical visual mediators, discursive routines, and endorsed narratives. Use of words, specifically those mathematical words used in the homework sheet from which the students referred to, created some source of difficulty for each of the six students during one or more mathcam videos analyzed. In all cases, students indicated on the video that they would ask for assistance from their teacher. In one case, Kara, the student we will be discussing shortly, referred to the internet.

Mathematical visual mediators were used very infrequently by students (n=4) in an effort to support their understanding (see line 10). In all but one instance, the visual mediators used replicated those in the classroom by Duane during a lesson on integers (i.e., the use of counters to understand negative and positive numbers). Across all six students, an interesting interaction was observed between discursive routines and endorsed narratives. We illustrate this with two examples from mathcam videos submitted from Kara.

Instances where there were mathematical challenges exhibited by the students, discursive routines did not function in concert with endorsed narratives across all students. For example, Kara made inappropriate connections between her prior knowledge of finding the perimeter of a perfect square with square roots (line 2). She used a visual mediator to assist her but does not move forward in her trying to support her own understanding. Despite the fact that she is unsuccessful, she still viewed her approach as "cool" (line 4).

| | | |
|---|---|---|
| 1 | So today in math class we did the perfect square problem again, and while everybody else was doing the systematic trial system, I decided that I was gunna [SIC] try to figure out the perimeter, because if I could figure out the perimeter, I could divide it by four, and then I'd be able to find out the outside, then I could find out the width and the length for each side, and then from there I could multiply it and get the answer to the area. | Discursive routines |
| 2 | So, I was trying to figure it out, and I was trying to…first I made the rectangle, with the twenty square centimeters as the area, and then I cut off the four extra… um, squared centimeters that made it into a rectangle, and then I tried dividing that up. | Visual |
| 3 | And though it would have taken me a long time to figure it out, I'm pretty sure it would have eventually worked. But my um, other members of my group found it quickly by us—well not quickly, but they found it eventually by using the systematic trial system, but it took 'em [SIC] a while. Um, other people figured out a formula, and it turned out the entire thing was about roots, and I wish I would've thought about it because square root, it means the root of the square. If you think about it, it just kind a sounds like a big fancy word, but if you look at the word it makes sense to what it means. So if I thought about using the buttons on the calculator, maybe I could a figured it out. | Words<br>Endorsed narratives |

| | | |
|---|---|---|
| 4 | But, um, I definitely thought the way I decided to find the answer was kind of a cool way to try and do it. | Discursive routines |

A major source of misconception for Kara was from prior knowledge from previous classroom instruction linking the side length of a perfect square with an area of 16 units and the root of 16, which are both four. We see from her discourse that she was associating the number four for determining *all* perfect squares rather than observing that the square root is only four when the number is 16. Although she recognized that her method was related to perimeter (line 7), she continued with the strategy of dividing by four to find the other square roots. This method of estimation works relatively well for several of the questions (8–11), but failed her when trying to find the square root of 78. At this point, her method was not at issue but rather, according to Kara, the magnitude of the number under investigation (lines 12–13). The interesting point in this example is that the discursive routines that led her to incorrect mathematization are not simultaneously interacting with endorsed narratives.

| | | |
|---|---|---|
| 5 | And right now I just have to find the square root, and I know how to find the square root quick because of the calculator I'm using. So, I just have to type it in. If I can find the button. So [inaudible] the square root of thirty six is…six! | Discursive routines |
| 6 | Now, number four is estimate the value of each square root. Explain how you found one estimate. So, twenty-three, […]. Well, right now I'm just gunna [SIC] think of all the things that you can multiply twenty three by […]. I think I'm gunna try to figure out what…five times four is…five point seven five. I think it's going to be…five point seven five…because that is what it is divided by four, and there's four. Wait that's perimeter! | Discursive routines |
| 7 | I always forget that it's not the perimeter! It's the inside of the square. When I tried doing this before, I figured out the perimeter before I figured it out but it didn't work very well. | Endorsed narratives |
| 8 | I'm going to estimate. That it's going to be . . . I'm gunna [SIC] try it with thirty-six. Thirty-six divided by four is nine, the actual answer was three. So, if thirty-six is…if thirty-six, the perimeter for thirty-six is nine, but the square root is six, then maybe for twenty three it'll be three less too. | Discursive routines |
| 9 | So, if I do…twenty-three divided by four again and add five point seven, maybe it'll be…I'm gunna [SIC] go with the number four. Wow the answer was four point eighty, so I was close! | Discursive routines |
| 10 | It's definitely a good um, strategy for me to use if I just try and…if I just answer it that way then…I think I should try that for all my other questions too, and then I don't have to try and come up with some type of…I don't have to just guess a random number. | Discursive routines |
| 11 | So thirteen divided by four is three point two five, […]. I'm gunna [SIC] try three. The square root of three is my estimate. And the square root of thirteen is three point six one! See, my method definitely works, 'cause it might be over, but it's still close. | Discursive routines |
| 12 | Seventy-eight divided by four is 19.50, and I'm gunna [SIC] guess that it's gunna be a multiple of four and I'm not sure why but I'm going to subtract four from it and my answer's gunna be 15.50. So, my estimate is 15.50. And, the square root of 78 is eight point eighty three. Definitely didn't work out for me this time, but it's a good strategy. Probably because 78 is a larger number. | Discursive routines |
| 13 | I'm going to explain how I got the answer for thirteen. I got my estimate by dividing the digit, no, the number …by dividing the number by…dividing 13 by four, number of sides, and then subtracting one because of …because it's the area and not the perimeter. | Discursive routines |

In contrast to the preceding example, in the next set of transcriptions from another mathcam video, Kara described how she made sense of integers using the visual techniques in the classroom. She explained how Duane uses counters to demonstrate positive and negative integers and how this was proved to be very useful in assisting her understanding (line 14).

In this example related to integers, discursive routines are interacting with endorsed narratives (lines 14–17). This interaction has two important outcomes. First, Kara is able to move forward in her learning of integers. Second, she expands her understanding using the combined discursive routines and endorsed narratives to hypothesize about other relationships between integers. She calls this a “strategy” (line 15), and then proceeds to test and revisit her strategy to confirm her understanding (line 16). As she continued reading the homework sheet, she saw that the homework sheet ultimately outlined the “strategy” she had just developed (17). Her efforts are in contrast to those above where her strategy falls apart and there are no endorsed narratives engaged to assist her misconceptions or further her learning.

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| | | |
|---|---|---|
| 14 | So, today in math class, I really liked how Mr. Heidi explained integers—integers by using counters. He used white squares and red squares. And the red squares are positive and the white squares were negative, and what he did is he added and um, subtract them using integer numbers and it was really—it made it easier to understand when you could see that they canceled each other out. Now, I’m on question three, and I’ve—I kind of do the same thing with the counters, except I do it in my head, which makes it- because I’m really good at that kind of mental math, it just kind of works out. | Words, Discursive routines, Visual, Endorsed narratives |
| 15 | So four positives, three negatives, it’s three pairs, which makes it positive one. […] So now I have a new strategy—now, when there is a higher positive number than the negative number that’s getting added, from the negative number that’s getting added, I can just subtract the positive number as if it was just both positive numbers. So ten subtract six, which is four. And I know the answers gunna be four, ‘cause there’s four positives left. So I can just do that as a strategy now, so I don’t always have to… so I don’t always have to write it out. | Discursive routines, Endorsed narratives |
| 16 | I think the same thing might be for if there’s a higher negative than positive, except it’s the opposite- instead of having the sum of the equation a positive number, the sum of the equation would be a negative number. So, seven, my prediction that seven minus two is five…so I believe the answer’s gunna be negative five. Now, I’m just gunna prove my theory by doing one, two, three, four, five, six negatives and two positives. I’m circles the two positives, and that leaves one, two, three, four, five. That leaves five negatives left. That means that my hypothesis was correct. | Discursive routines, Endorsed narratives |
| 17 | This is a second part to question number four. It says: when you add a positive integer and a negative integer, the sum is positive…when the numerical larger integer is. Oh! Positive, when the numerical larger integer is positive, negative when the numerical larger integer is negative... just like my theory—my strategy that I was doing when I was figuring out! And, the sum is going to be zero when the integers are the same number. | Discursive routines, Endorsed narratives |

The preceding example from Kara is important. It illustrates what was seen throughout the students mathcam submissions; namely, that mathematical understanding as verbalized by the students was characterized by an interaction between discursive routines and endorsed narratives. Mathematical *mis*understanding was linked to inappropriately transfer knowledge from one context to another (i.e., Kara’s use of 4 as the divisor and potential root of many numbers) and consistently lacked an interface with the endorsed narratives.

Our results do not suggest that students improved their understanding as a result of verbalization (i.e., engaging in mathcams) (cf. Mercer, et al., 1999; Sfard & Kieran, 2001). There was minimal evidence of students engaging in self-correction based upon their verbalization (n = 6). However, there were more instances (n=12) where students verbalized incorrect calculations but proceeded with their work unaware of their errors.

As outlined by Nesher et al. (2003), a student's ability to access mathematical knowledge may be linked to their initial ability to make sense of mathematical texts. Our research adds to this finding to suggest that the inability to make sense of mathematical texts may be linked to the extent to which students are able to relate discursive practices to endorsed narratives. Our findings do not suggest the same sort of requisite interactions for mathematical visual mediators and words. Rather, the latter two were seen as having supporting roles in students' ability to complete homework independently following instruction.

## Conclusions

Real-time data collection in naturalistic settings is a strength of this research. However, a limitation of this design is the inability to track more than six students at one time due to technological constraints (i.e., only six laptops available for the project). The results presented in this paper represent a small sample drawn from the larger research project that is currently underway. We make no claims that the conclusions we have presented can be generalized or true of the remaining data to be analyzed.

Our results suggest that a students' inability complete homework accurately so that the task of completing homework is not reinforcing misconceptions may be related to the interaction between discursive routines and endorsed narratives. Our results show that misconceptions were largely discursive routines that were not paired with endorsed narratives and thus the mathematization was often incorrect.

More specific to the example we highlighted from Kara's mathcams, we see that also important in the learning of mathematics is the ability to transfer knowledge (or not), as a specific goal of instruction. Although understanding relationships in mathematics are widely viewed as significant in the learning of mathematics (National Council of Mathematics Teachers, 2000), it is also of critical importance to examine when certain relationships do not hold or fail to adhere to endorsed narratives of mathematics.

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# META REPRESENTATIONAL KNOWLEDGE, TRANSFER, AND MULTIPLE EMBODIMENTS IN LINEAR ALGEBRA

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*Previous research suggests spontaneous transfer between non-isomorphic problem settings to be rare in the absence of hints concerning the relationships between those settings (Holyoak & Koh, 1987). Two factorial experiments reinforce previous findings that transfer does not significantly occur between dissimilar settings, even when problems share abstract problem solving schemas and representations. A third experiment supports the Indirect Representational Transfer Hypothesis: meta-representational reflection on the meaning of a common abstract representation in relation to diverse mathematical settings induces schema transfer. Qualitative interviews uncover evidence of the role of meta-representational thought as part of a larger developmental process of actor-oriented transfer.*

## Introduction

A common theme in educational research on linear algebra concerns cognitive inflexibility associated with the appropriation of different linear algebra problem settings to similar matrix representations and methods, a consequence of the *multiple embodiments* of linear algebra concepts (Harel, 1989; Hillel & Mueller, 2006; DeVries & Arnon, 2004; Hillel & Sierpinska, 1994; Dias & Artique, 1995; Dorier, 2000). The study of multiple embodiments in linear algebra is important since students require an ability to establish meaningful links between representational forms in order to "understand the necessity for representing these situations by a general concept," a competence known as *representational fluency* (Harel, 1987, p.30-31). A common conclusion drawn from previous studies points to the general inadequacy for students to develop theoretical setting change competencies on their own, requiring didactic intervention on a meta-mathematical level. Informed by methods from experimental psychology, this study reconceives cognitive inflexibility and the role of meta-mathematical information as a study of the transfer of knowledge from familiar to unfamiliar problems settings sharing common abstract problem solving schemas.

## Theoretical Framework

*Meta Mathematical Knowledge as 'Sierpinska's Theoretical Thinking'*

Dorier (1995) defines meta-mathematical knowledge as:

> Information of that which constitutes mathematical knowledge concerning methods, structures, and (re)organization (of lower level competencies). Methods are defined as the procedures applicable to a set of similar problems within a given field: the methods designate that which is common to problem solving and not the technique itself (p.151).

Sierpinska (2000), defined the terms *practical* verses *theoretical* thinking, as similar to Vygotsky's notion of *everyday concepts* verses *scientific concepts*. Practical thinking is characterized by the propensity for students to view a subject, such as linear algebra, as an aggregate of "prototypical examples" interconnected and understood primarily through "goal-oriented, physical action(s)" (Sierpinska, 2000, p.212). Representations important at the practical level are the necessary symbols needed to perform computations; whereas, theoretical thinking produces formal systems in which "semiotic representation systems become themselves an object of reflection and analysis." In view of these definitions, the term *meta representational knowledge* is defined as meta-mathematical knowledge directed toward theoretical thinking upon the *meaning* of abstract representations in relation to mathematical contexts, settings, or schemas they are applicable to.

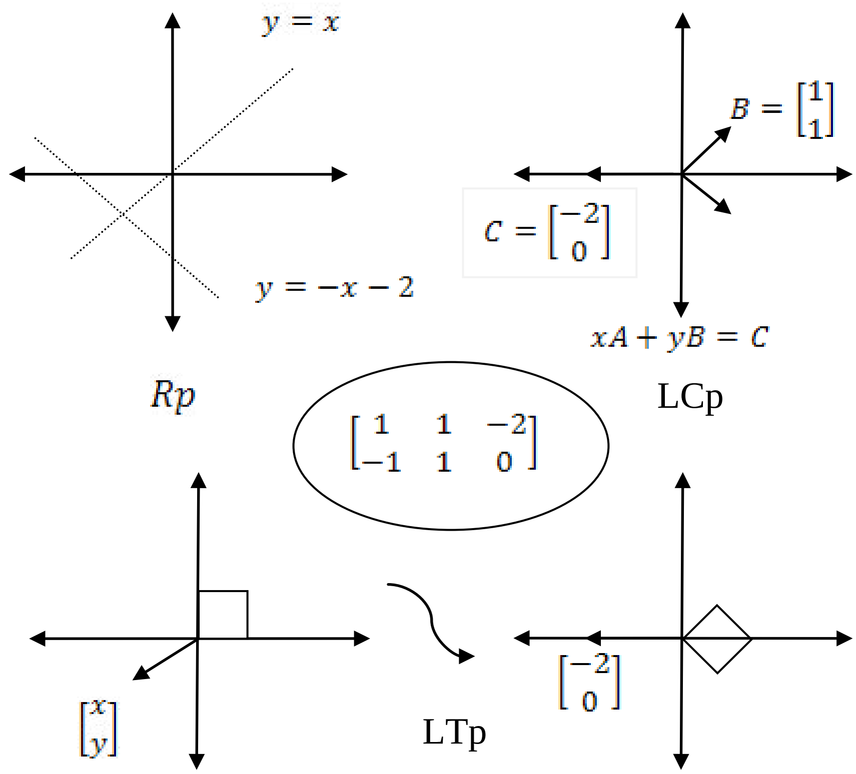


*Figure 1*. Row-picture (Rp), linear combination-picture (LCp), and linear transformation-picture (LTp) settings associated with central augmented matrix.

*Rp, LTp, and LCp Settings*

*Conceptually non-isomorphic settings* are defined as mathematical settings which are not mere re-labelings of each other, but consist of schemas containing different definitions, mathematical objects, concept-images, and structural relationships. The linear algebra settings of interest for this study include the Cartesian row setting (Rp), the linear combination setting (LCp), and the linear transformation setting (LTp). As an example, in figure 1 the matrix $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 1 & 0 \end{bmatrix}$ forms a representation having the following associated meanings: (i) in the Rp setting the matrix represents the system which solves for the intersection of two lines; (ii) in the LCp setting the matrix solves for the linear combination of vectors $A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which add to give the vector $C = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$; and (iii) in the LTp setting the matrix solves for the vector $\begin{bmatrix} x \\ y \end{bmatrix}$

which, upon clockwise rotation of 45 degrees, would result in the vector $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$. Prior to this study, a pilot experiment with 87 undergraduate linear algebra students was conducted which analyzed performance on an exam question involving all three of the above settings in figure 1. Based on the results of that experiment, and the preponderance of the Rp viewpoint in prerequisite coursework such as pre-calculus and calculus, the Rp setting is defined as a ‘familiar’ setting, while the LCp and LTp settings are defined as ‘unfamiliar.’

*Traditional and Actor-Oriented Transfer Perspectives*

The notion of traditional transfer derives from Thorndike’s *Theory of Identical Elements,* which states that transfer is more likely to occur when tasks share similar elements (Thorndike & Woodworth, 1901). When tasks have similar elemental structures, they are said to be *isomorphic*. Research in traditional transfer has often been criticized for its predominant emphasis on what MacKay terms, the *expert point of view,* as deciding criteria for task similarity or determination for whether or not transfer occurs (Marton, 2006, p.499; Lobato, 2006). Over the past several decades, a debate has ensued concerning the theoretical foundations of transfer theory. Rather than judging transfer according to normative, i.e., expert functionalist views, according to Lobato (2003), “actor-oriented transfer is defined as the personal construction of relations of similarity between the activities.” Both traditional and actor-oriented transfer perspectives comprise main elements of the theoretical framework for this mixed-methods study against the backdrop of an overall Piagetian developmental epistemology

## Methodology

Employing a *mixed methods* approach (Johnson, R. B., & Onwuegbuzie, 2004), three experiments and three interview case studies aimed to address the following questions:

Research Question 1 (Quantitative*)*

> Is there evidence, from the traditional transfer perspective, that transfer is facilitated between linear algebra problems from non-isomorphic settings which share similar (matrix) representations and solution procedures? (Experiments 1 and 2)

Research Question 2 (Qualitative)

> In what ways, from the theoretical perspective of actor-oriented transfer, do novice linear algebra students commonly have difficulty with conceptually non-isomorphic problem settings, even when sharing similar problem representations and solution procedures as familiar problem settings? (Interviews)

Research Question 3 (Mixed: Quantitative + Qualitative)

> Is there evidence that meta-representational intervention may facilitate traditional and/or actor-oriented transfer across conceptually non-isomorphic problem settings involving novel target problems sharing similar problem representations and solution procedures as more familiar problems? (Interviews and Experiment 3)

All of the subjects in this study were university sophomores and juniors, mostly engineering and science majors, in a university course. 260 participants took part in Experiments 1 and 2 (130 each), while 3 subjects participated in the interviews. 66 subjects participated in experiment 3. Subjects were randomly assigned to condition groups for the factorial experiments.

*Experiments 1 and 2*

To address research question 1, a multi-dimensional procedural problem-solving model, inspired by Chen & Mo (2004), was created for implementing factorial experiments employing a source problem – target problem design (see MSP model in figure 2). Problems used for the experiments all shared common goal structures and were matrix representable. Experiment 1 juxtaposed the familiar Rp setting with the Cp setting in the SETTING dimension of the MSP model, as well as manipulated different solution types to the matrix systems formed via the SOLUTION TYPE dimension. Experiment 2 was designed similarly to Experiment 1, except the LTp setting was used instead of the Cp setting in the manipulation of the SETTING factor. For both Experiments 1 and 2, preliminary pilot studies found significance and non-interaction for the BASIS and ROW-OPS dimensions, hence to avoid confounding variables, these dimensions of the MSP model were held fixed.

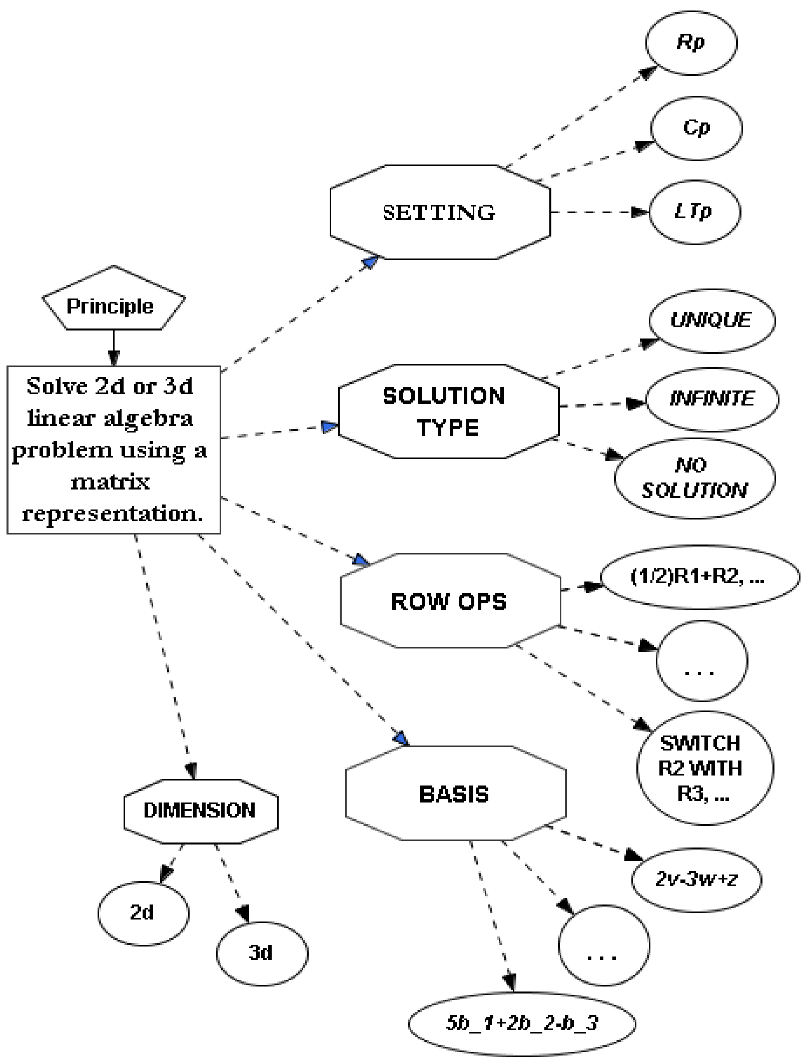


*Figure 2.* Multiple Setting Problem Model (MSP model).

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

*Interviews*

Using problems experimentally verified as belonging to novel non-isomorphic linear algebra settings (Experiments 1 and 2), it was the purpose of the interview portion of this study to investigate transfer difficulties from a non-expert, actor-oriented perspective, characterized by the subjects' "personal structuring" of phenomena, as well as "transfer distributed across social planes," hence; the interview format was largely semi-structured, with its adherence to the order and completion of the interview problems, and minimal interviewer interaction, unless subjects appeared stalled in problem solving (Lobato, 2003). The interview design consisted of two consecutive problems from the *Cp* setting, and one problem involving the *LTp* setting. As in the previous experiments, all of the interview problems could be solved through the successful use of the general goal structure:

1. Representation of the problem with an appropriate matrix.
2. Row reduction algorithm employed as solution procedure to solve matrix.
3. Interpretation of matrix system solution in the context of a novel problem setting.

*Experiment 3*

*Indirect representational transfer hypothesis.*

In order to coordinate the solution resulting from a process of row reduction on a matrix representative of a given problem embedded in a particular linear algebra setting, it is necessary and sufficient to understand the meaning of the initial matrix representation in relation to the setting.

Based on the results of Experiments 1 and 2 and the interviews, the *Indirect Representational Transfer Hypothesis* was conjectured, and a meta representational intervention was constructed for a treatment/non-treatment design. Treatment participants were given the two-fold interpretation of matrix multiplication as a system of row equations belonging to the row-picture (Rp) setting, or a linear combination of the columns of the matrix, belonging to the linear combination-picture (LCp) setting. Non-treatment participants were only given the target problem. Since the Rp setting was earlier categorized as a familiar setting for most undergraduate linear algebra students, Experiment 3 did not have a source problem. Target problem success was interpreted by the researcher as an indication of successful transfer from Rp to LCp settings.

## Results

*Experiments 1 and 2*

The results of Experiments 1 and 2 are summarized as follows:

1. Dissimilar SETTING conditions from the familiar Rp setting to the less familiar and conceptually non-isomorphic LCp and LTp settings created reliably significant obstacles in applying source solutions.

2. Dissimilar conditions in the SOLUTION TYPE dimension of the MSP model presented reliably significant difficulties in transferring source problem solutions.

*Interviews*

Upon initial coding analysis of the interview data, two stages of problem solving became evident in the form of (1) actor-oriented *representational transfer*, and (2) actor-oriented *solution-interpretational* transfer. From the perspective of actor-oriented transfer, the following six results were found:

1. Lack of understanding of the representational meaning of a matrix in relation to the problem setting, equation scalars, and equation variables (*scalar-variable conflict*), created obstacles in the transfer of correct matrix representations and solution interpretations.
2. Incomplete semantic-access to setting-specific information defining both the LTp and LCp settings, posed as an obstacle to successful representational and solution-setting transfer across conceptually non-isomorphic problem settings.
3. The beneficial effects of meta-cognitive intervention appear to consist in the formation of co-ordinations between non-isomorphic settings, matrix representations, and corresponding matrix solution interpretations.

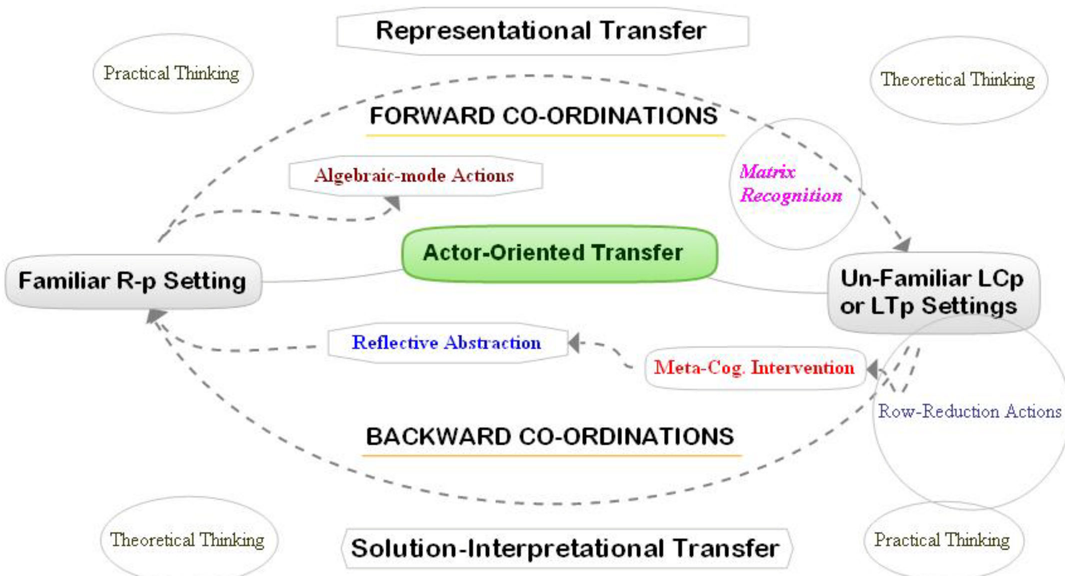


*Figure 3.* Result 4 illustration.

4. Algebraic mode computations characteristic of practical thought, in combination with meta representationally induced reflection characteristic of theoretical thinking, characterized a general pattern of *actor-oriented transfer* seen in the formation of co-ordinations between non-isomorphic settings, their representations, and corresponding solution interpretations. Furthermore: (a) Forward co-ordinations facilitating *representational transfer* were constructed from algebraic-mode transformations from definitions to familiar Rp-like systems of equations. (b) Backward co-ordinations facilitating setting-solution transfer involved the reversibility of reflectively interpreting the solution to a system of equations back through the representational co-ordinations connecting the matrix representation to the problem setting (see figure 3).

5. Upon successive exposure to problems from conceptually non-isomorphic settings sharing procedurality and matrix representations, the evidence indicated a reduction of the multiple constructive processes characteristic of previous actor-oriented transfer, leading to the conjecture that actor-oriented transfer become progressively streamlined in learning, encapsulating towards the spontaneous transfer of an abstract problem solving schema from familiar to conceptually non-isomorphic settings.

*Experiment 3*

The results of Experiment 3 indicated that the meta representational treatment was effective and reliable in promoting *target problem* success and traditional transfer from the assumed familiar knowledge in the Rp setting.

## Conclusions

The quantitative methods of this study produced three significant results in the area of traditional transfer experimentation. (i) Experiments 1 and 2 duplicated previous research by finding spontaneous transfer between *conceptually non-isomorphic* linear algebra problem settings to be rare in the absence of hints concerning the relationships between those settings. (ii) Experiments 1 and 2 extended research in the area of non-isomorphic transfer experimentation by finding that the combination of representational similarity and solution-procedural similarity between conceptually non-isomorphic linear algebra settings did not enhance transfer. And finally, (iii) Experiment 3 extended previous work by demonstrating transfer was indirectly facilitated with information of the meaning of a common representation (matrix) in the context of familiar and unfamiliar settings, not requiring meta information as to the relationships between the non-isomorphic Rp and LCp settings themselves (see figure 4).

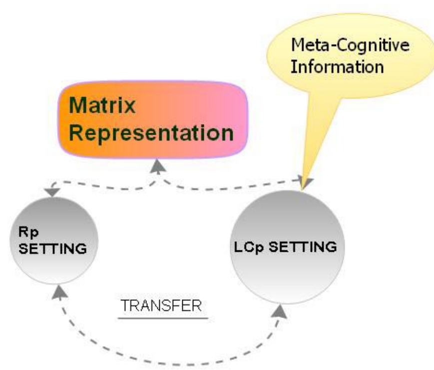


*Figure 4.* Indirect representational transfer.

From Result 5, as subjects repeatedly solved problems from conceptually non-isomorphic settings, the evidence showed a subtle decrease in the degree of personal constructions necessary in order to facilitate actor-oriented transfer. The author characterized this phenomenon as the occurrence of transfer, in the traditional sense, of prior problem solving knowledge from *within* the multiple constructive processes of actor-oriented transfer. From a Piagetian interpretation, the formation of forward and backward co-ordinations within the general process of actor-oriented transfer constitutes evidence of *reversibility* and potential *encapsulation,* marking the progress between *inter-operational* and *trans-operational* constructive phases, or in other terms, practical verses theoretical thinking (Piaget, 1983; Sierpinska, 2000).

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# THE EFFECTS OF THE FUNCTION MACHINE ON STUDENTS' UNDERSTANDING LEVELS AND THEIR IMAGE AND DEFINITION FOR THE CONCEPT OF FUNCTION

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*This study is on the effects of the function machine as a cognitive root on students' understandings of function concept. Data of the study was obtained from open-ended tests and clinical interviews. Results were analyzed according to the conception levels of the concept of function of APOS theory and concept image- concept definition framework of Tall and Vinner. It is concluded that function machine is a helpful cognitive root to raise the students' conceptual levels and to develop their concept images and concept definitions for the function concept.*

## Introduction

Mathematics education literature includes a lot of studies on function concept. Most of them agree with using multiple representation (table, graph, formula, procedure, verbal formulation, etc) environment in the teaching process of function concept (Janvier, 1987; Ferrini-Mundy and Graham, 1990; Confrey and Smith, 1991; Yerushalmy, 1997).

## Aim of the Study

In this study, our attention is to investigate the effects of the function machine on students' understanding of function concept.
We aimed to answer the following research questions:

- What is the effect of the function machine on students' understanding levels of the function concept?
- What is the effect of the function machine on the images and definitions that the advanced analysis course students have for the concept of function?

## Theoretical Framework

Because we aimed to investigate the effects of function machine on students' understandings of function concept, we evaluated their understanding levels according to APOS theoratical framework. Dubinsky and his colleagues' theory, APOS is a specific theoretical framework for research and curriculum development in collegiate mathematics education. The purpose of the theoretical analysis is to describe the specific mental constructions that a learner might make in order to develop her or his understanding of the concept. These mental constructions are called actions, processes, objects, and schemas. Dubinsky and his collagues (Breidenbach, Dubinsky, Hawks & Nichols, 1992; Dubinsky & Harel, 1992; Dubinsky, 1991) studied on conceptions of function concept. A subject who is at the level of an action conception of function is able to calculate the value of the function for a function formula and a point. To interpret a situation as a function unless a formula for computing values is given, inverses of functions, and the notion that the derivative of a function is a function are difficulties for a subject whose understanding of function concept is at the action level. When the action is repeated and reflected upon, it is interiorized to a process. The subject, who has the process conception of function thinks of a function as receiving one or more inputs that are independent variables, performing one or more

operations on the inputs and returning the results as outputs that are dependent variables. When a subject becomes aware of the process as a totally and is able to transform it by some actions or processes, it is said that the process has been encapsulated as an object. When a subject perceives manipulations of functions such as adding or multiplying, she or he encapsulates the process conception of function to an object. We should note that according to the Dubinsky and Harel (1992),
1. The three main restrictions students possess about what a function are:
(a)The *manipulation restriction* (you must be able to perform explict manipulations or you do not have a function)
(b) The *quantity restriction* (inputs and outputs must be numbers)
(c)The *continuity restriction (a graph representing a function must be continuous)*
*2.* Severity of the restriction. Some students feel, for example, that before they are willing to refer to a situation as a function, they personally have to know how to manipulate an explicit expression. (Dubinsky and Harel, 1992; pp. 86, 87)

Because other aim in this study is to investigate the effects of function machine as a cognitive root on students' concept images and concept definitions for the function concept, Tall and Vinner's framework of concept image-concept definition is also important for us. Tall and Vinner (1981) call the total cognitive structure that includes all mental pictures, associated properties and processes by concept image. According to Tall and Vinner, whether the concept definition is given to the student or constructed by himself, he may vary it from time to time, and in this way, a personal concept definition can differ from a formal concept definition. According to Vinner (1991) something which is evoked by the concept name in our memory is usually not the concept definition. Vinner (1991) takes the concept image and concept definition as two different "cells" in the cognitive structure. Vinner states the concept image cell is empty as long as some meaning is not associated with the concept name, and when the concept definition is memorized in a meaningless way, concept image cell is empty in many situations.

According to Vinner, after the teacher has given the definition of the concept to the student, who has a concept image, the concept image may be changed to include also the concept definition (satisfactory reconstruction or accommodation), the concept image may remain as it is and the concept definition will be forgotten or distorted after a short time (in this case the concept definition has not been assimilated), or both cells will remain as they are. Vinner states when the concept definition is first given, a similar process might occur. The concept image cell is empty at the beginning and it is gradually filled by examples and explanations. But, after the concept image cell has been filled, it does not always reflect all the aspects of the concept definition. According to Vinner, interplay between concept image and concept definition as seen in the Figure 1 refers to the long term processes of the concept formation.

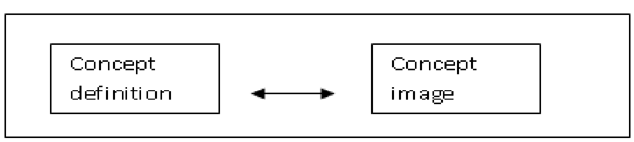


*Figure 1.* Interplay between concept image and concept definition.

Vinner adds that in addition to the process of the concept formation, there are also the processes of problem solving or task performance. When a cognitive task is posed to a student, different

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relationships between the cells and the cognitive task can be occur according to the student's concept image and concept definition cells.

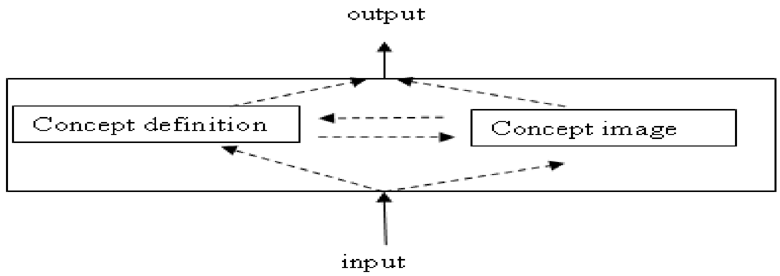


*Figure 2*. Relationships between the cells.

**Cognitive Root**

Tall, McGowen and DeMarois (2000) defined the notion of cognitive root as follows:
A *cognitive root* is a concept that:

- is a meaningful cognitive unit of core knowledge for the student at the beginning of the learning sequence,
- allows initial development through a strategy of cognitive expansion rather than significant cognitive reconstruction,
- contains the possibility of long-term meaning in later developments,
- is robust enough to remain useful as more sophisticated understanding develops.

(Tall, McGowen and DeMarois, 2000; p. 3)
Tall, McGoven and DeMarois (2000) suggests using the function machine (input-output box) as a cognitive root while teaching the function concept.

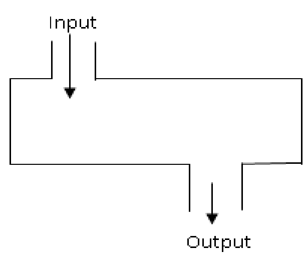


*Figure 3.* Function machine.

Tall and others stated the function machine (input-output box) as the cognitive box embodies both the process-object duality and also the multiple representations of the function concept as seen in following:

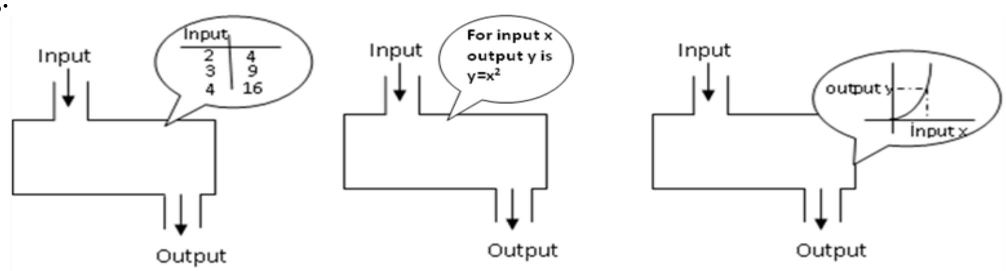


*Figure 4*. The function machine and representations of function.

Tall and others stated that the function box which is an embodied version of the more general function concept allows simple interpretations of profound ideas, for example two function boxes are “the same” if they have the same output for each input in the domain. They interpreted this perception of two function boxes being the same as occurring at the *process* level in the sense of Dubinsky.

## Methodology

This study was conducted to 23 students of one of the two groups of the advanced analysis course in the mathematics education program of a university in Turkey. Study is a part of a more comprehensive study in which understanding two variable function concept was aimed to investigate in the APOS theoretical framework (Asiala et al.,1996 ; Dubinsky, E. & Harel, G. , 1992). A lot of tests and clinic interviews were conducted in the advanced analysis course through an academic semester in the comprehensive study, whose method is teaching experiment (Cobb and Steffe, 1983). At the beginning, a test on single variable function concept was prepared and conducted to 23 students. After this test, single variable function concept was summarized by using function machine as a cognitive root. After the summary, another test on single variable functions was prepared in a similar manner to previous test. Then process of instruction of two variable function concept and preparation and conduction of tasks about this concept was started. After this process, clinic interviews were conducted with six of subject. These six subjects were chosen according to purposive sampling **(**Fraenkel and Wallen, 1996 ). Main criterion was to reach process conception for both single variable and two variable function concepts. Other criterion was accessibility. Because we aimed to report the effects of function machine as a cognitive root in this study, we will give the findings of first two tests and relevant tasks of interviews.

## Findings

According to the results of the first test, students had several difficulties and misconceptions about functions. Some graphs are given in the first question and it is asked if they are graph of a function whose independent variable is x. It is seen that most subjects did not have the notions of independent and dependent variables of a function. Several misunderstandings were seen in data. Most subjects (17 out of 23) have written an algebraic expression, which is familiar to them beside the graph. Generally (except three subjects), subjects focused on the notion of independent variable by overriding the concept definition.

In the aspect of algebraic (formula) representations of functions, students had similar misconceptions to the misconceptions students had in graph cases. Again, it is seen that the most effective reason of these misconceptions was that students did not have notion of independent variable of a function. Most students (16 out of 23) could perceive the functions only in table situations. That is, they could regard input-output and transference notions only in table situations. All students could find the input corresponding to the given output for the function, whose algebraic representation was given. Moreover, almost all students could complete the input-output table by using the given graph. The students who had faults in this task could not perceive the output corresponding the input in which the function was discontinuous. Almost all students could draw the graph of partial functions, while most students drew a whole circle instead of semi-circle by regardless of range of function. Contrary to the first test, most students (13 out of 23) could perceive the functions both graph and algebraic situations. Only four subjects had still focused on the independent variable notion instead of the function process.

For the definition of function, various types of definitions for the function concept were given in the pre-test. Most categories of the definitions are similar to the categories that Vinner and Dreyfus (1989) had. Subjects chose the class of item, which was defined, variously like equation, expression, or rule. Some of them also gave the uniqueness to the right condition of transfer. Three subjects defined the function correctly as a relation, while four subjects defined as a relation, but without uniqueness to the right condition. One subject defined as a correspondence with uniqueness to the right condition, while another subject defined as a correspondence without uniqueness to the right condition. When the function definitions in the pre-test and in the post-test were compared, it was seen that development was surprising. 18 subjects gave correct definition of function concept. Most of them defined the function concept as a special relation, and remaining defined as a correspondence correctly. While one of other subjects defined the function concept as a rule with uniqueness to the right condition, two of them defined as a machine. Only one subject defined as an expression.

Another task in which students had difficulties was that required the graph of an upper semi-circle in the sixth question of the first test. Similarly, the algebraic expression of lower semi-circle was given in the second test, and its graph was required. 17 out of 23 subjects gave correct graph in the second test. Only five subjects gave the graph of whole circle in the second test, while 13 out of 23 subjects gave the graph of whole circle in the first test. Remaining one subject gave completely wrong graph in the first test.

One of the relevant tasks to the aim of this study in the interview questionnaire was to find a function process on a set of functions. At the beginning of the interviews, the meaning of function was again questioned with a question like "what is a function?” or “according to you what does function mean?”. After following some other tasks, subject prompt to find a function process on a sets of functions with the questions as seen in following:

- You defined the function concept at the beginning of our interview. Is there any restriction on the domain of a function?
- Can the domain of a function be any set, which does not equal to the empty set?
- Do you know a function whose elements are also function?

If the subject could not see function as an element of a function, that is she/he could not see function as an object, she/he prompted as following:

- Can you imagine a function machine such that its inputs are functions?
- What can be the outputs of such a function machine?
- Do you know such a function machine on functions?

According to the responds of subjects to this task, four of six subjects could see function as an element of domain, and they gave the operations on a set of functions like composition, derivative, integral. Also one subject could see the taking inverse of a function as a function operation on the set of one to one and onto functions, but then it was seen that she could not have seen a function as an object. Other subject could not give any interpretation about this task.

## Results

According to the results of the first test, six of 23 students’ understanding of function concept was action level at the beginning of advanced analysis course. One of the remaining subjects’ conception level was transition from action to process. Except one subject, who had process conception, remaining 15 subjects had relatively weak process conceptions, which include at least one of the restrictions of severity, manipulation, and quantity. We should note that the conception levels were determined after qualitative analyze of data which were obtained from responds for open-ended tasks.

After brief instruction by using function machine as a cognitive root, two subjects' conception level rose from action to process, and four subjects' conception level rose from action to transference from action to process. Ten subject's conception levels were process even if some of them had restrictions, while their levels were relatively weak process conception before the brief instruction. Two subjects' conception levels were relatively weak process both before and after the brief instruction, but at least they abandoned the restrictions they had. Remaining subjects raised their conception levels from relatively weak process to weak process, because at least they developed to perceive functions in the table situations.

After the interviews which were conducted by six subjects whose conceptions levels were seen as at least process after brief instruction, it was concluded that conception levels of four of them were object level.

In the aspect of concept image and concept definitions of subjects for the function concept, most students' developments were well. Subjects' concept images, which included various misconceptions like familiarity, variable complexity at the beginning were disappeared after brief instruction. Moreover, most of them gained the concept definition and began to consult their concept definition and to relate their concept images and concept definitions, while they consulted only their concept images at the beginning.

## Conclusion

We concluded that function machine is very effective as a cognitive root on understanding the function concept. By instruction this cognitive root, students gain input-output and transference notions. Emphasizing with various examples that input of the machine, which is independent variable of function is an element of any set, which is not empty, this input transforms an output, which is dependent variable of function with the operation of the machine is gained the students the concept definition. Moreover, students develop the notions of domain and range by such an instruction. In the aspect of understanding levels of the students, it is helpful to rise the students' conceptual levels. Especially, by increasing in variety the inputs, students can begin to see a function as an object. For instance, students can make some operations on the set of functions, but generally they do not see neither these operations as functions nor the functions as an elements of a set. To give such operations on the sets of functions like arithmetical operations or composition as an example of function machine is gained the students seeing a function as an object. Moreover, such examples are helpful to be disappeared the restrictions like quantitative or manipulation. Such function machine examples can be given instruction of not only the single variable function concept but also two variable function concept.

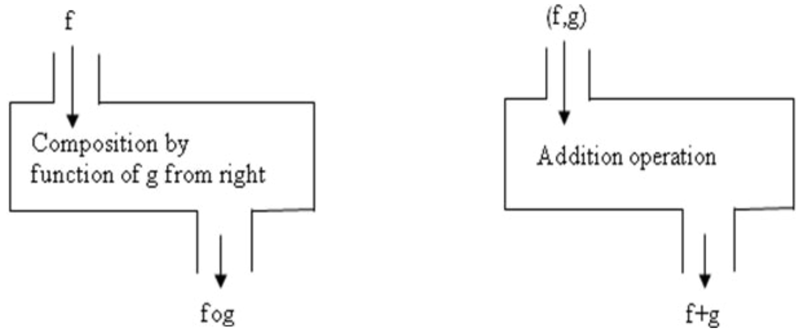


*Figure 5*. Function machine examples for single and two variables respectively.

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# ACCOMMODATING INFINITY: A LEAP OF IMAGINATION

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*This paper is the first installment of a study which seeks to identify the necessary and sufficient features of accommodating the idea of actual infinity. University mathematics majors' and graduates' engagement with the Ping-Pong Ball Conundrum is used as a means to this end. This paper focuses on one of the necessary features: the leap of imagination required to conceive of actual infinity, as well as its associated challenges.*

## Introduction

The concept of infinity has a distinctive quality which rouses the imagination, provoking controversy, and challenging fundamental ideas intuited as truth. In meeting these challenges and controversy an individual is invited to think in often new and complex ways—to engage in "advanced mathematical thinking."

The term 'advanced mathematical thinking' carries with it many descriptions. Although there is no agreement on the definition, many of the characteristics describing advanced mathematical thinking are exemplified in the concept of infinity.

One working description suggests advanced mathematical thinking (AMT) involves abstract, deductive thought (Tall 1991, 1992), and includes "proving in a logical manner based on definitions" (Tall, 1991, p. 20). Alternatively, ideas that exercise advanced mathematical thinking may be considered as ones that are not "entirely accessible to the five senses" (Edwards, Dubinsky, & McDonald, 2005, p. 18), and lack "an intuitive bases founded on experience" (Tall, 1992, p. 495). The abstract and intangible nature of actual infinity epitomises both of these descriptions.

This paper presents research from part of a broader investigation which aims to identify the necessary and sufficient features involved in accommodating the idea of actual infinity. It focuses on the 'leap of imagination' required to conceive of mathematical infinity, as well as on the challenges university mathematics students and graduates faced in making such a leap.

## Background

The concept of infinity carries with it a "surprisingly rich intuitive base that many students seem naturally to be endowed with" (Mamona-Downs, 2002, p. 49). Research into the nature of learners' intuitions of infinity has shown "that infinity appears intuitively as being equivalent with inexhaustible" (Fischbein, 2001, p. 324). Specifically, learners are naturally inclined to conceive of a potential or 'dynamic' infinity – a process for which every step is finite, but which continues endlessly. Intuitions of an 'endless infinite' have been observed in students of all levels, from middle school to university (e.g. Tirosh, 1991). In resonance with the general characteristics of intuitions, the idea of an endless infinite tends to be resilient: it is seen as self-evident, intrinsically certain, coercive, and resolute (Fischbein, 1987).

Current research suggests students' understanding of mathematical infinity tends to develop by reflecting on knowledge of related finite concepts and extending these familiar properties to the infinite case (Fischbein, 2001; Fischbein et al., 1979). As Fischbein (2001) observed, when learners attempt to establish an understanding of abstract concepts, their tacit mental

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representations in the reasoning process replace the abstract concepts by more accessible and familiar ones. For example, when analysing infinite sets, students may apply familiar methods for comparing sets that are acceptable in the case of finite sets, such as the inclusion (or part-whole) method, but which result in contradictions in the infinite case (e.g. Fischbein et al., 1979). With respect to infinite sets, only one method of comparison yields consistent solutions: one-to-one correspondence. Through the idea of 'coupling' elements, two (infinite) sets that can be put into one-to-one correspondence are accepted by the normative standard as having the same 'size', or cardinality.

Students' well-documented struggle to understand and appreciate aspects of infinite cardinality has motivated efforts to improve and refine pedagogical strategies. For instance, in set activities administered by Tsamir and Tirosh (1999), geometric figures were used to emphasize a correspondence between numerical sets, as well as to draw students' attention to the inconsistencies of comparing infinite sets with different methods. One of the goals of this study was to elicit cognitive conflict in the participants, thus a task which elicited an inclusion method of comparison was followed by one which elicited a one-to-one correspondence method. Tsamir and Tirosh concluded that this series of activities, which can elicit cognitive conflict "has the potential for raising students' awareness of incompatibilities in their own solutions to the same mathematical problem" (1999, p. 216). However, Tsamir (2003) warns that participants must come to appreciate one-to-one correspondence as the *only* appropriate method of infinite set comparison, and such was not the case in Tsamir and Tirosh (1999).

This study extends on prior research by examining conceptions of university mathematics students and graduates as they engaged with the well-known paradox 'The Ping-Pong Ball Conundrum'. Despite participants' sophisticated mathematical background and their experience and skill with abstract mathematics and mathematical thinking, participants faced challenges distancing themselves from realistic concerns and engaging with the mathematics presented in this novel problem-solving situation. These challenges are explored in this paper with the intent to shed light on the necessary and sufficient aspects required for a normative understanding of actual infinity, as well as on pedagogical strategies to guide learners toward such an understanding.

## Theoretical Framework

The concept of infinity relies on abstract, formal definitions of concepts for which intuition and the senses have no foundation. As such, one of the theoretical perspectives which guided this study is Hazzan's (1999) 'reducing levels of abstraction'. In Hazzan's (1999) perspective, when learners are confronted with a novel problem solving situation, they will attempt to make sense of unfamiliar or abstract concepts by reducing their level of abstraction. Hazzan describes different ways a learner may attempt to reduce abstraction. One such way she observed was in working with familiar entities when learners were faced with problems for which an understanding of the mathematical entities involved were not yet constructed. As an example, Hazzan noted that when learning abstract algebra, students would "often treat groups as if they were made only of numbers and of operations defined on numbers" (1999, p. 77). By basing arguments on familiar mathematical entities, such as numbers, in order to cope with unfamiliar concepts, such as groups, students lower the level of abstraction of those concepts.

In addition to relying on familiar entities to reduce the level of abstraction of novel ones, Hazzan interprets "students' personalization of formal expressions and logical arguments by using first-person language" as an attempt to reduce the level of abstraction of that expression.

For instance, language such as “I can find” or “I want to find” (Hazzan, 1999, p. 80), indicate, in Hazzan’s perspective, ways that a student may cope with unfamiliar terminology and concepts.

Hazzan (1999) relates her framework of reducing levels of abstraction to the APOS (Action, Process, Object, Schema) Theory of Dubinsky and McDonald (2001) through the observation that process conceptions of a mathematical entity may be considered on a lower level of abstraction than their corresponding conceptions as objects. She also argues that a learner’s attempt to reduce the level of abstraction of a mathematical entity through, for instance, the use of first-person language, indicate that the learner holds a process (rather than object) conception of that entity. Process and object conceptions are in the centre of the second framework considered in this study, that of the APOS Theory (Dubinsky & McDonald, 2001).

Dubinsky, Weller, McDonald, and Brown (2005a, 2005b) proposed an APOS analysis of infinity. They suggested that interiorising infinity to a process corresponds to an understanding of *potential* infinity, or the inexhaustible. As such, a process conception of infinity is imagined as performing an endless action, though without imagining the implementation of each step. Encapsulating this endless process to a completed object, in turn, corresponds to a conception of *actual* infinity, a quantity which describes the ‘size’, or cardinality, of a completed infinite set. As in the more general case, encapsulation of infinity is considered to have occurred once a learner is able to think of infinite quantities “as objects to which actions and processes (e.g., arithmetic operations, comparison of sets) could be applied” (Dubinsky et al., 2005a, p. 346). Dubinsky et al. also observed that “in the case of an infinite process, the object that results from encapsulation transcends the process, in the sense that it is not associated with nor is it produced by any step of the process” (2005a, p. 354).

## Methodology

### *Participants*

Data for this study were collected from five participants with advanced mathematical backgrounds. Each of the participants had prior experience with Cantor’s theory of transfinite numbers through formal instruction during upper level undergraduate mathematics courses. In particular, they were familiar with comparing infinite sets via one-to-one correspondence, such as corresponding the sets of natural numbers and rational numbers, and also with Cantor’s diagonal argument establishing the set of real numbers as having larger cardinality than the set of natural numbers. Participants also had substantial experience with infinity in calculus. The participants in this study included students enrolled in undergraduate degrees in mathematics, as well as participants who had completed at least a master’s degree in mathematics or mathematics education.

- Marc was a mathematics major in a south eastern state university in the USA. He was very bright and eager to engage with the paradoxes. In addition to his background with cardinal infinity, he had informally explored aspects of infinity in other contexts. Marc anticipated pursuing a graduate degree in mathematics.
- Maria was a classmate of Marc’s in the mathematics program. Her familiarity with Cantor’s theory included an awareness of the Continuum Hypothesis, as well as some properties of transfinite ordinal numbers.
- Joey was in his fourth year of an undergraduate degree in mathematics and physics at a university in eastern Canada. Joey had taken upper year courses in set theory and analysis, both of which touched on Cantor’s theory.

- Vince was a doctoral candidate in mathematics at a university in eastern Canada. Since his participation in this study, he had completed his degree, and began a professional research career in cryptography.
- Jenny was a doctoral candidate in mathematics education at a university in eastern Canada. Her area of research was didactiques des mathematics, and her scholarly background included an undergraduate degree in mathematics and physics.

*Data Collection*

Data from the five participants were collected through email correspondence, which was intended to offer participants the opportunity to put their thoughts in writing in order to contribute more precise and balanced responses than possible in a formal interview. The Ping-Pong Ball Conundrum (described below) was presented to participants, and they were asked to determine how many ping-pong balls remained in the barrel at the end of the 60-second experiment, and to explain their reasoning. The Ping-Pong Ball Conundrum was chosen because of its level of complexity, and because of the necessity to address a bound infinite set in the paradox resolution.

*The Ping-Pong Ball Conundrum*

> An infinite set of numbered ping-pong balls and a large barrel are instruments in the following experiment, which lasts 60 seconds. In 30 seconds, the task is to place the first 10 balls into the barrel and remove the ball numbered 1. In half of the remaining time, the next 10 balls are placed in the barrel and ball number 2 is removed. Again, in half the remaining time, balls numbered 21 to 30 are placed in the barrel, and ball number 3 is removed, and so on. At the end of the 60 seconds, how many ping-pong balls remain in the barrel?

The normative resolution to this paradox involves coordinating three infinite sets: the in-going ping-pong balls, the out-going ping-pong balls, and the intervals of time. Although there are more in-going than out-going ping-pong balls at each time interval, at the end of the experiment the barrel will be empty. A fundamental aspect in the resolution of this paradox is the one-to-one correspondence between any two of the three infinite sets in question. Given these equivalences, at the end of the experiment, the same amount of ping-pong balls went into the barrel as came out. Moreover, since the balls were removed in order, there is a specific time for which each of the in-going balls was removed. Thus the barrel is empty at the end of the 60 seconds.

## Results and Analysis

Surprisingly, despite the sophisticated mathematical knowledge of participants, only one participant, Marc, provided a resolution to the Ping-Pong Ball Conundrum that was consistent with the normative one. Indeed, as participants attempted to reconcile properties of actual infinity with the notions that were elicited by the Ping-Pong Ball Conundrum many were unwilling to take a leap of imagination beyond practical or realistic considerations and toward the 'realm of mathematics'.

The inability to leap toward the imaginative surfaced in participants' reluctance to distance themselves from concerns such as physical possibilities and constraints. For instance, Vince, a doctoral student in mathematics, objected to the feasibility of the experiment, and refuted the possibility of completing the experiment. He remarked that the "first thing that comes to mind is that the problem is not really that well-defined as the time left, 1/2n, never reaches zero." Vince went on to consider the processes of inserting and removing balls, and concluded that:

> You'll have lots of balls in the barrel when you reach 0 [end of the experiment], which you won't. And this is clearly a ridiculous answer if you consider the whole thing to be something that could take place. So my final answer is 136.

Vince's desire to consider the experiment as "something that could take place" suggests a reluctance to engage with the thought experiment in the Ping-Pong Ball Conundrum, which is by no means an experiment that could actually take place. Vince's resistance to let go of practical experience is also recognised in his comment that "lots of balls in the barrel" is "clearly a ridiculous answer", which suggest he was unable or unwilling to conceive of a barrel that could contain infinitely many balls. Also, Vince's notion of an endless experiment corresponds to a conception of potential infinity and is suggestive of a process conception, in terms of the APOS Theory. A conception of potential infinity, together with resilient practical concerns, seemed to prevent Vince from considering the infinite sets of balls or time intervals as completed objects.

Imagining the experiment being carried out also influenced Joey, an undergraduate student in mathematics. Joey's response began by describing the physical items in the paradox – such as the barrel and the balls – and speculating on the outcome if he were to actually perform the experiment. Joey wrote:

> Well, at first I'm thinking about a massive collection of white ping-pong balls. And an actual wooden barrel. Clearly thinking about actually performing the experiment and then realising there is no way I can actually move that fast in real life so I realise the final ping-pong ball count would be finite. However thought experiment… so... mathematically…

Joey's instinct was to consider the experiment in a 'realistic' way, and like Vince, he initially approached the paradox as though it were an experiment that he could perform. Once Joey realised that the physical constraints of reality restricted his solution to a finite "count", he distinguished between what was possible practically versus mathematically. This distinction suggests Joey was, to a degree, aware of a conflict between practical experiences and the realm of infinity. Nevertheless, Joey's realistic approach of "thinking about actually performing the experiment" seemed to influence his deliberations even as he addressed the thought experiment "mathematically." For instance, he continued to describe the experiment as though it were being carried out:

> Since I keep halving the time I add and remove ping-pong balls, I will never reach 60 seconds. So the experiment should never end, really. Meaning I have an infinite number of ping-pong balls, and yet there are more in the barrel. Since infinity is not an actual number, you can't say I have infinity here, but 9 times infinity there.

Joey maintained a personal connection to the experiment, and described his own involvement in terms of the actions he would take and the outcomes he would face. The use of personal language to cope with an abstract concept is, in Hazzan's (1999) perspective, an attempt to reduce the level of abstraction of that concept, and as such is indication of a process conception of infinity. Further, Joey's description of infinity as "more like a destination, an indication of an unlimited amount", and his remark that the "experiment should never end", are consistent with a process conception of infinity.

A use of personal language was also identified in Jenny's response, as she too resisted letting go of 'practical' concerns. Jenny, a doctoral candidate in mathematics education, also commented on physical limitations in the experiment. She found the issues of speed and time problematic, suggesting, "there is not enough time to work so fast." She also noted that "the fastest speed is light speed" and that if she could work at the speed of light then time would slow down and there would be "infinity time, so there will never be a last ball." She described herself as being stuck in an endless experiment, left to insert and remove ping-pong balls for eternity, lamenting "but I don't want to do that with my life."

The cognitive leap from the 'realistic' to the realm of mathematical infinity was a source of difficulty for Vince, Joey, and Jenny. Their resistance to engage with the realm of imagination and mathematics noticeably impacted each of their resolutions, and in the case of Vince prevented him from resolving the paradox beyond giving an arbitrary number as his solution. Another student attempted to bridge her realistic concerns with the surreal thought experiment by introducing assumptions. This student, Maria, attempted to reconcile reality and infinity by assuming the impossible was possible. For instance, Maria, an undergraduate student in mathematics, reasoned there would be an infinite number of ping-pong balls remaining in the barrel at the end of the experiment "assuming the barrel has an infinite volume and can house an infinite number of ping-pong balls." Maria imagined an infinite iterative process, noting that "per iteration, the barrel gains an additional 9 ping pong balls than it had previously… so to determine the number of ping pong balls in the barrel at the end of the experiment, we can simply determine the number of iterations and then multiply this by 9." Maria seemed to treat infinity as a very large number, and as such, needed to assume that the physical constructs could accommodate this 'infinite size'. Interestingly, Maria did not assume the existence of infinitely many ping-pong balls, only that they could be housed in the barrel. Recalling the normative solution, Maria's assumption is superfluous as the barrel at no moment contains infinitely many ping-pong balls.

In accordance with Mamolo and Zazkis (2008), participants who resisted distancing themselves from reality by clinging to practical concerns or by introducing capricious assumptions tended to approach the paradox in intuitive, process-oriented ways, such as focusing on infinite iterations. As Fischbein (1987) observed, properties of actual infinity contradict the finiteness of mental schemas and intuitions. In contrast, the ability to take a leap of imagination away from the realistic or the intuitive corresponds to an ability to engage effectively in advanced mathematical thinking—thinking which lacks "intuitive bases founded on experience" (Tall, 1992, p. 495). Clarifying the limitations that finite experience has on an understanding of infinity seems to be a fundamental aspect of accommodating properties of actual infinity—properties which by all means lack intuitive bases founded on practical experience.

Distinguishing between an intuitive and a formal understanding of infinity was an essential aspect in Marc's reasoning as he addressed the Ping-Pong Ball Conundrum. Marc, an undergraduate mathematics student was the only participants who came up with a solution that was consistent with the normative resolution to the paradox. Marc's approach was proof by contradiction. He wrote:

> We can assume that some ball does remain after the minute is up, and without loss of generality, let's say it's the nth ball. But we know that this ball is taken out during one of the steps 10n-9, 10n-8, .... ,10n-1, 10n, and all of these steps occur within one minute due to the fact that the series: (Sum from k=1 to k=infinity of (1/2^k)) converges to 1. But then the aforementioned ball is NOT in the barrel at the end of the minute, which contradicts our original assumption that it was. Therefore, there are no balls left in the barrel at the end of the minute.

Notable in Marc's response is his use of language. Marc seems to refer to a general, and perhaps impersonal, body of knowledge. For instance, Marc refers to "our original assumption", commenting on what "we know" in his solution. His depersonalized use of language is in sharp contrast to other participants who referred to what they as individuals could achieve (i.e. 'I add', 'I find', 'I want').

In connection to his removed stance from the experiment, Marc was also the only participant to clarify a separation between his realistic intuitions and his mathematical thinking. Marc's awareness of the limitations of intuition and realistic experience seemed to contribute significantly to his understanding of actual infinity. After discussing his solution, Marc reflected that "the intuition we've learned from the physical world fails us when it comes to the infinite." His willingness to distinguish between intuitive and formal understandings can be linked to his ability to take the leap of imagination necessary for accommodating actual infinity. Indeed, holding on to realistic, finite experiences and intuitions seemed to hinder the encapsulation of a process for which there is no final step, but for which a completed totality does exist.

## Conclusion

Conceiving of actual infinity exemplifies mathematical thinking that "extrapolates beyond the practical experience of the individual" (Tall, 1980, p. 1). As such, problems addressing infinity require a leap of imagination away from practical experience. In resonance with observations made in Mamolo and Zazkis (2008), participants resisted extrapolating beyond their practical or realistic experiences when addressing the ping-pong experiment. Letting go of realistic considerations was problematic for participants despite their considerable experience with advanced and abstract mathematics – mathematics that is inaccessible to the five senses (Edwards et al., 2005) and that lacks an intuitive basis (Tall, 1992). The inability to take the cognitive leap into the realm of mathematical infinity manifested in participants' responses in striking similarity to the reactions of participants in Mamolo and Zazkis (2008), despite much more sophisticated experience with 'advanced mathematical thinking'. There were those participants who were unable to 'leap', others who recognised the need to 'leap' but resisted, and some who could 'leap' to work within the realm of mathematics and clarify a separation between 'real' possibilities and mathematical truth.

This study draws attention to a pedagogical necessity to guide learners away from 'realistic' or intuitive approaches when addressing abstract mathematics, particularly with respect to concepts such as infinity. Continued research on effective pedagogical strategies to achieve this end, as well as on further necessary and sufficient features of accommodating the idea of actual infinity is underway.

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# STUDENTS' IDEAS ON FUNCTIONS OF TWO VARIABLES: DOMAIN, RANGE, AND REPRESENTATIONS

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*The study of student understanding of multivariable functions is of fundamental importance given their role in mathematics and its applications. The present study analyses students' understanding of these functions, focusing on recognition of domain and range of functions given in different representational registers, as well as on uniqueness of function value. APOS and semiotic representation theory are used as theoretical framework. The present study includes results of the analysis of interviews to 13 students. The analysis focuses on student' constructions after a multivariate calculus course, and on the difficulties they face when addressing tasks related with this concept.*

## Introduction and Purpose of the Study

The notion of a multivariable function is of fundamental importance in advanced mathematics and its applications. Even though its understanding is essential for mathematics, science and engineering, little is known about students' ideas and difficulties. There are very few research based studies that probe student understanding of the particularities of a multivariable function. This lack of research findings limits our understanding of how students learn the main ideas of the multivariable calculus.

The present study is a continuation of a previous study (Trigueros & Martínez-Planell, 2007) which reported on student understanding of graphs of functions of two variables. The focus of the present study rests on the following research questions: What are students' conceptions of domain and range of functions of two variables when they finish a Multivariate Calculus course? How are these conceptions related to their abstract, general notion of function?

## Theoretical Framework

Two conceptual frameworks inform the theoretical basis used in this study. Firstly APOS theory is used to model the development of the concept of two variable functions and, secondly, semiotic representation theory, provides the conceptual tools to analyze flexibility in the use of different representations and its role in the cognitive evolution of the mathematical ideas under consideration.

As APOS Theory is a well known theory, only its application for the purpose of this study is described. For more detail the reader may consult Asiala et al. (1996), and Dubinsky (1991, 1994).

The application of APOS theory to describe particular constructions by students requires researchers to develop a genetic decomposition—a description of specific mental constructions one may make in understanding mathematical concepts and their relationships. A portion of a preliminary genetic decomposition for the function of two variables concept given in Trigueros and Martínez-Planell (2007), is summarized below since it will be referred to throughout this paper:

The Cartesian plane, real numbers, and the intuitive notion of space schemata must be coordinated in order to construct the Cartesian space of dimension three, $R^3$, through the action

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of assigning a real number to a point in $R^2$, and the actions of representing the resulting object both as a 3-tuple and as a point in space and making conversions between them. These actions are interiorized into a process that considers all the possible 3-tuples and subsets of 3-tuples, and their representation in space, to construct a process that when coordinated with the respective verbal, analytic and geometric representations can be thematized as three dimensional space, $R^3$.

This space schema is coordinated with the schemata for function and set through the action of assigning one and only one specific height to each point in a given subset of $R^2$, either analytically or graphically. This action is interiorized into the process of assigning a height to each point on a subset of $R^2$ to construct a two variable function, and the process of conversion needed to relate its different representations. When the process of generalization of these actions to consider any possible function of two variables, as a specific relation between subsets of $R^2$ and R is encapsulated, it can be considered that the notion of two-variable functions has been constructed as an object.

Duval (1999, 2006), argued that thinking processes in mathematics require not only the use of representation systems, but also their cognitive coordination. In Duval's analysis, understanding and learning mathematics require the comparison of similar and different representations. According to this author, there are two different types of transformations of semiotic representations: treatments, which are transformations of representations that happen within the same representation register, and conversions which consist of changes of representation register without changing the object being denoted. He argues that these two types of transformations are the source of many difficulties in learning mathematics, and that overcoming these difficulties needs to take them both into account: to compare similar representations and treatments within the same register in order to discriminate relevant values of the mathematical object so that students notice the features that are mathematically relevant and cognitively significant, and to convert a representation from one register to another to dissociate the represented object and the content of the particular representation introduced so that the register does not remain compartmentalized.

## Method

An instrument was designed to conduct semi-structured interviews with students and test their understanding of the different components of a proposed genetic decomposition (Trigueros & Martínez-Planell, 2007). Nine students were interviewed. The students were chosen from a group of undergraduate students at a private university who had taken the equivalent of an introductory multivariable calculus course the previous semester. The instructor of the mathematics course they were currently taking chose what he judged three good, three average, and three weak students to be interviewed. On the basis of the results obtained, and in preparation for the present study, the researchers decided to conduct more interviews focusing on items in which it seemed more data would be useful. The instrument was once again revised to do this and five new interviews were conducted. These students were chosen from a group of undergraduate students at a public university who had just finished taking the equivalent of an introductory multivariable calculus course that same semester. The instructor of the multivariable course they took chose what he judged two slightly above average, and three average students to be interviewed. All interviews lasted for 45-60 minutes, were audio-recorded, and all the students' work on paper was kept as part of the data. The results obtained in 13 of these interviews were independently analyzed by two researchers, and the conclusions negotiated.

## Results

Results of the analysis of students' responses during the interview showed that none of these students was able to demonstrate all the constructions described in the genetic decomposition. Their description of what they consider a function seems to be more closely related to that of relation and associated mainly with its analytical representation. Students differed in the difficulties they faced during the interview but most of them struggled with the description of the domain of these functions in all the representations included in the tasks and in the conversion between representation registers.

Eight of 13 students had difficulty with the arbitrary nature of the relational correspondence. For example, when Emily was asked if the rule: "Input: weight in kilograms and height in centimeters. Output: name of person with that weight and height" defined a function, she responded:

Emily: *ok … [nervous laugh] for me, the fact that you have the weight, a weight and a height in centimeters [nervous laugher], the name is not there anywhere, … I don't know how one gets to …*

The same type of response is observed in Rodrigo:

Rodrigo: *…because it can't be that with only the weight and the height we can obtain the name of the person.*

When asked to define a function of two variables Gaddis responded:

Gaddis: *…, for me a function of two variables is a function of the form* $f(x,y)$ *is equal to x … a term in x, a term in y, and a constant or any number, then the domain would be, the two variables, that would be independent … and the range would be the result of those two variables evaluated in the function.*

In the genetic decomposition we assumed that the construction of the notion of function of two variables requires the coordination of a schema for $R^2$ with that of function through the action of assignation of one and only one value to each element in the domain of the function. Some students did not show to have made this coordination. Among them, there were 4 students who did not consider that a function is determined by the uniqueness of that assignation. Rodrigo is one such student:

Interviewer: *and if I were to give you a list with all students at this university with their weight and their height*
Rodrigo: *ok, here you could, but you'd get several results*
Interviewer: *and that, would be a function?*
Rodrigo: *yes, hmm, yes, yes*

Another student, Gaddis, commented to this same question:

Interviewer: *…is that a function?*
Gaddis: *…yes, I think so*
Interviewer: *So if a give you the weight, say 60 kg and height 2 m, what could be the output?*
Gaddis: *It would be the name of a person with that data*
Interviewer: *and, if there were more than one person with that data?*
Gaddis: *…the output would be the name of the person*

Later, while further exploring Gaddis's uniqueness of value notion, the interviewer presented the equation $x^2 + y^2 = 1$ to him, and asked:

*Interviewer: … y, is it a function of x?*
*Gaddis: … it could be, if we solve for the y*
Interviewer: *ok, I solve for the y and get* $\pm\sqrt{1 \;\; x^2}$ *, the y, is it a function of x?*

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Gaddis: *y, there it depends on x in that case*
Interviewer: *then, is it a function of x?*
Gaddis: *I think it is.*

Most of the students had difficulties when describing the domain of a two variable function. It was found that 3 students do not have a clear idea that the elements in the domain of this type of function are always ordered pairs. Even though they accept they need ordered pairs in order to find the value of a particular function, when they are asked how many elements are there in the domain of a function given in a table representation, they count the elements of each pair as different elements in the domain. For example:

Fernando: *… the elements, would be eight [he was using a 4 by 4 table]*
Interviewer: *give me an example of an element in the domain of*
Fernando: *0 comma 2*

Even though he uses "0 comma 2" as example of an element in the domain, he still counts the domain as a set of numbers. In terms of the genetic decomposition it seems these students have not interiorized the action of assigning one and only one specific height to each point in a given subset of $R^2$; rather than applying the action to elements of $R^2$, they seem to be acting on sets of two real numbers. In the case of María, she correctly listed the elements in the domain, but when asked to count them, she stated:

Maria: *ok, then, do I count them as pairs or separate?*

Another problem some students had in finding the domain of functions was that they were not able to restrict the domain to specific subsets. For example, when analyzing the function defined by $f(x,y)=1+x^2+y^2$, where the domain is restricted to the pairs $(x,y)$ that satisfy: $-1\le x\le 1$ and $-1\le y\le 1$, Paola could not understand the meaning of that restriction. With some help, she eventually succeeded in drawing the domain. However, later on, when trying to find the range she said:

Paola: *the thing is that I'm not sure what it means that it is restricted to the pair of numbers that satisfy this*

Of the 8 students having difficulty representing the domain of the above function, 5 tried to draw the graph of *f* first, without considering any restriction. Emily and Gracielle are examples of this difficulty:

Emily: *I'm trying to do the graph, to know more or less what would be the domain …*
Gracielle: *… [mumbles] …It says x goes from -1 to 1, and y goes from -1 to 1, a circle*
Interviewer: *and why is it a circle?*
Gracielle: *because the graph is a paraboloid …*

It was also found that students' ability to find the domain of a given function was related to the representation register used to present the information. The following was one of the questions of the interview: The following is the complete graph of a function *f:*

a. Find the domain of *f.*
b. Evaluate $f(0,0)$, $f(2,0)$, $f(2,2)$, $f(0,2)$.
c. Find the range of *f.*

In parts (a) and (c), the domain and range were found correctly by 10 students, and incorrectly by 3. As was already mentioned, some students tried to graph functions in order to find their domain, even when a restriction defining it was given, so it seems that students did not have much difficulty obtaining domain and range information from a graphical representation of a two-variable function. However, as was reported previously (Trigueros & Martínez-Planell, 2007), students do have great difficulty obtaining the graphical representation of functions of two variables, as well as obtaining other kinds of information from such surfaces. In this case Patricia, who treated domain incorrectly as a set of numbers when presented with functions in tabular or algebraic representations, was able to correctly find the domain in this case. This is one of the several examples found which confirm what was observed by Gagatsis, Christou, and Elia (2004): the cognitive demands for translating (converting) among representations are not the same, and so each one needs to be specifically attended.

It was also observed that all students who could get neither the domain nor the range of a function when its graphical representation was given also had difficulties when asked to find them for functions given in tabular or algebraic representations.

Some students showed some confusion between the domain of the function and the intersection of the graph of the function with the *xy* plane:

Patricia: *…yes, and if this (pointing to* $x^2+y^2+1=0$ *) …is zero, if z = 0, then it is a circle with radio equal 1…*

All the above mentioned difficulties seem to be related to the coordination of students' schema for space and that for function. Although they are able to assign values to specific points in $R^2$, either analytically or graphically, they cannot consider sets of points in the plane as the domain of the function or do not clearly understand the role of the domain of the function. When they are able to consider sets of points, they have difficulty considering the result of applying a function to the whole set.

Regarding the range of the function, most students' responses showed that their difficulties were mainly related to the lack of interiorization of the actions needed to find values of the function into a process. Some students showed that their idea of range of a function was not clearly differentiated from that of graph of the function, but this was not a prevalent difficulty. Most of them were able to calculate specific values in the range or to read it from a given graph. This was also observed through a task where most students used this strategy to match the algebraic representation of six functions, with their corresponding graphs. Students showed many difficulties doing this task. We expected them to use sections in the analysis of the graphs but only one of them was able to do this. Gaddis was the only student to consistently, and without prodding, use sections to analyze graphs of two-variable functions:

Interviewer: *say, start with the second formula* $g(x,y)=\sin(x)+y$

Gaddis: *ok, we take that* $g(x,y)$ *is the function in z, if we put that the y is 0 , we have that* $z=\sin x$*, which is the normal function, and if we put that the x is 0, the sine of 0 is 0 and we have that the, that* $z=y$

Interviewer: *and you, could you identify, with what you've done, the graph of that function?*

Gaddis: *wouldn't it be the graph on the upper right hand corner?*

Interviewer: *that's the one … and let's say that* $h(x,y)=\sin(xy)$

Gaddis: *every time that x or y is 0, it will be 0, z is substituted equal to … one would have to assign values to the x and the y and see the behavior of … to me it would be the, the second on the left hand column [correct]*

He was asked more questions that showed that he understood what he was doing and used sections in his graph analysis.

These results are also evidence that independently of the grades obtained by students during their course, they were not able to construct a process conception of function. They did not demonstrate having generalized the action of taking a point in the domain of the function and assigning it a height, or to have constructed the process of conversion needed to develop effective strategies to relate different representations of functions.

We found that students' definitions of two variable functions can be classified in essentially three groups. The first one contains definitions of the function machine sort: two inputs$\rightarrow$ one output, or input$\rightarrow$ output (5 students); the second one uses variable dependence, algebraic expression, or formula (4 students); and the third type of definition is given in terms of geometric images (4 students).

Fernando's definition, for example, was given in terms of an input$\rightarrow$ output conception, which can be related to a process understanding of function:

Fernando: *so that for each, …, so that for my domain, for each element of my domain, I can only have one point in the range and no more points for each point in the domain*

Interviewer: *and what is it that makes the function be of two variables?*

Fernando: *of two variables? That I have two different … because my function, because my output depends on two inputs, precisely, and we have two variables*

Gaddis uses a formula; this can be related to an action conception of function:

Gaddis: *…, for me a function of two variables is a function of the form* $f(x,y)$ *is equal to x … a term in x, a term in y, and a constant or any number, then the domain would be, the two variables, that would be independent … and the range would be the result of those two variables evaluated in the function.*

Pablo, whose definition was very clear, seemed to be guided by a geometrical model:

Pablo: *…, a function of two variables means, is, that starting from a certain region in a plane one can, that is, of variables x y, a height function can be constructed and for each point in that x y region there exist only one defined height*

When considering only the definition for two variable functions given by students, it may seem that some of them have a process conception of these functions, however, when relating their definition with their responses to the other tasks it is shown that this is not the case: they have not interiorized their actions and they are not always able to do treatments and conversions on different representations.

## Discussion

All the interviewed students had successfully finished a course on multivariate calculus in their university, but in spite of this, showed a very shallow understanding of the concept of two variable functions. They were not able to coordinate the schema that were considered to be important in the construction of this concept.

If we review the historical development of the function concept as summarized in Kleiner (1989) and Sfard (1992), we find the development of the concept can be divided in three stages: The first formal definition of function was given by Johann Bernoulli in 1718: "one calls here Function of a variable a quantity composed in any manner whatever of this variable and constants." This definition is similar to that given by Euler in 1748: "a function of a variable quantity is an analytical expression composed in any manner from the variable quantity and numbers or constant quantities." Latter developments required that the definition of function could include cases where functions were not expressed by equations. Further developments spurred by the need for rigor, together with the great growth experienced in all fields of mathematics in the late 19$^{th}$ and early 20$^{th}$ centuries, led to the prevalent view of function as a mapping between arbitrary sets and Bourbaki's definition of a function as a set of ordered pairs.

Results show that students' idea of a function seems to be pre-Bourbaki, but this does not deter many of them from succeeding in an undergraduate multivariable calculus course. Objectives of teachers of these courses include helping students develop a deep understanding of the concept of function. In particular we consider teachers would like to help students construct an object conception of two-variable function, as a set of ordered pairs.

We can conclude that results of this study show that students' conception of domain of a two variable function is not clearly differentiated from that of real valued functions. Their difficulties finding or describing the domain of functions can be related to a lack of coordination between the schema of $R^2$ and that of function, and with conversions between representations. These difficulties underline the difficulty involved in the generalization that takes place in the transition between functions of a real variable and multivariable functions. It seems that the assumption that this generalization is straightforward for most of the students is not valid. Results on domain and range of functions show that these students were not able to interiorize the notion of two variable functions into a process even though they had already finished a calculus course on multivariable functions.

The relationship between students' notions of domain and range of a two variable function and their construction of the general concept of function is less clear from the data of this study. We found students who showed good understanding of domain and range of functions of two variables but whose general definition of function did not enable them to consider functions defined on arbitrary sets, and uniqueness of image. Other students' definition of two variable functions showed some aspects that could be related to interiorization of this concept, but demonstrated a poor understanding of domain and range of specific functions in different representations. We consider that more research is needed to gain a deeper understanding of this relationship.

The results of this study show that the description of how a student may construct the notion of a function of two variables has many subtleties that need to be addressed in instruction if students are to achieve at least a process conception of the modern function concept. One difficulty that had been reported has to do with the structure of students' schema for three-dimensional space, which includes building a subschema of subsets of $R^3$ to be able to analyze graphically functions of two variables (Trigueros & Martínez-Planell, 2007). The construction of the function concept as a process requires that students interiorize actions on sets. They also need to differentiate between functions of one and two variables and be able to consider subsets (in particular, restricted domains) of $R^2$ (and ordered pairs) as domains of functions. Further, the action of assigning a unique value to each point on a subset of $R^2$ needs to be interiorized into a process that includes the treatments and conversions needed to identify important elements in each representation and to relate different function representations. As seen repeatedly in the analysis of results, students frequently struggle when doing actions on representation registers and when converting between them. To address this, and agreeing with Duval's position on cognitive development (2006) and Gagatsis, et al (2004), we consider that much work was to be done with functions in different representations. Representations constitute different entities and, as such, require explicit instruction; actions to perform treatments and conversions and opportunities to interiorize them as processes must be part of the instructional process.

## Endnotes

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# THE ROLE OF REVERSIBILITY IN THE LEARNING OF THE CALCULUS DERIVATIVE AND ANTIDERIVATIVE GRAPHS

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*This study examines three calculus students' cognitive processes as they sketched antiderivative graphs when presented with derivative graphs. As we explored the students' analytic or visual strategies leading to different and sometimes divergent interpretations of derivative graphs, we provide insight into how students' understandings can be enriched by establishing reversible relationships between graphs of functions and their derivative or antiderivative graphs. Our results suggest that their work and thinking illustrate the importance of flexibility and reversibility of thinking in the complete understanding of differentiation and integration in calculus.*

## Background

The visual and analytic elements of mathematical thinking have been the subject of discussion and debate for decades (e.g., Zazkis, Dubinsky, & Dautermann, 1996). This study explores three calculus students' analytic or visual strategies as they sketched antiderivative graphs when presented with derivative graphs. Our descriptions of the students' thinking processes demonstrate that establishing reversible relations can greatly enhance their understanding of derivative and antiderivative graphs.

Visualization and visual thinking have been focal points of studies reforming the way calculus is taught. Zimmermann (1991) considers the understanding of differentiation from a graphical point of view as a fundamental aspect of visual thinking in calculus and suggests that students be able to sketch the graph of the derivative, the second derivative or an antiderivative of a function, given a graph of a function. Berry and Nyman (2003) conclude that students' abilities to draw graphs of functions from graphs of derivatives will enhance their conceptual understanding of the derivative and its connections to the concept of the integral.

Research literature (e.g., Presmeg, 2006) supports the assertion that understanding of mathematics is strongly related to the ability to use visual and analytic thinking. Similarly, Hughes-Hallett (2002) advocates a balance between graphical, numerical, analytical, and verbal expressions. Despite the considerable effort that has been spent exploring the relationship between visual and analytic thinking, the role of reversibility – switching from a direct to a reverse train of thought (Krutetskii, 1976) – in the understanding of the relationship between differentiation and integration, the inverse processes of calculus, has not been adequately described. Norman and Prichard (1994), whose work suggests that reversibility is closely related to the understanding of calculus, contend that when rules of integration are introduced, students who do not use reversibility to reason and understand differentiation and integration tend to view these processes as rather unrelated processes, each of which has its own rules to be memorized.

In this study, our goal was to gain understandings of three calculus students' thinking processes. We observed that these students' visual or analytic strategies led to different and sometimes divergent understandings of derivative graphs. In this paper, as the students attempted to relate the derivative graph to its antiderivative graphs, we illustrate how their understandings can be enriched by changing thinking processes and establishing reversible relations between

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graphs of functions and their derivative or antiderivative graphs. It was the thinking processes in these attempts that we analyzed based on their responses and sketches while solving the tasks during interviews. We describe our research findings about their cognitive processes and discuss pedagogical implications of these results.

## Theoretical Framework

The research of Krutetskii (1976) and Presmeg (2006) is influential in our analyses of students' cognitive processes in calculus and enables us to describe students' thinking processes and difficulties associated with the use of visualization. Krutetskii (1976) identifies analytic, visual, and harmonic thinkers according to their preferences for visual or analytic thinking. A student who has a predominance toward the analytic relies strongly on analytic thinking and relies little on visual thinking. Conversely, a student who has a predominance toward the visual relies strongly on visual thinking and relies less on analytic thinking. Harmonic students rely equally on analytic and visual thinking. Krutetskii found that analytic students' strong tendency to rely on analytic means results in a certain one-sidedness in their mathematical development. Visual students do need to interpret visually an expression of an abstract mathematical relationship or to operate with visual schemes and images even when a problem is easily solved by reasoning and the use of visualization is unnecessary or difficult. According to Krutetskii, to certain extent, it is a hindrance for visual students to be riveted to visual patterns. In his studies with gifted students, Krutetskii identified several mathematical abilities related to successful problem solving, including reversibility and flexibility. Reversibility refers to the ability of establishing two-way reversible relations as opposed to one-way relations which function only in one direction. According to Krutetskii, "In a reverse train of thought, the thought does not always have to travel over precisely the same route, but simply moves in reverse order" (p. 287). In the case of flexibility of cognitive processes, to find different or elegant solutions, the successful problem solvers switched from one cognitive process to another without any difficulty.

We take the position that visual imagery resides in a person's mind, and we have used the terms "image" and "imagery" interchangeably to mean a mental construct depicting visual and spatial information (Presmeg, 1986). Our work also is framed by Presmeg's (2006) study in which five kinds of visual imagery, associated with visual thinking, are identified: concrete imagery, memory images of formulae, pattern imagery, kinesthetic imagery, and dynamic imagery. During her interviews with visual students, rich and fully detailed concrete imagery was the most prevalent, and dynamic imagery, effective in depicting transformations and movements, was rarely used. In the literature, several difficulties verifying the limitations of imagery have been documented (e.g., Haciomeroglu & Aspinwall, 2007; Aspinwall, Shaw, & Presmeg, 1997). Presmeg observed that visual dynamic imagery was effective and visual students who were able to combine visual imagery and analytic means avoided the drawbacks associated with the use of imagery. Owen and Clements (1998) confirmed Presmeg's findings that dynamic visual imagery produces high levels of mathematical functioning.

## Methodology

In a semester-long study, we developed cases describing three calculus students, Amy, Bob, and Jack, who had completed an elementary calculus sequence. We conducted weekly task-based interviews, during which they were presented with derivative graphs of functions (see Figure 1) and asked to draw possible antiderivative graphs as we sought to gain understanding of their thinking processes. In this paper, the term antiderivative has been used to describe a graph of a function from which a given derivative graph can be drawn, and the terms differentiation and integration are used describe the two inverse processes of calculus. We prepared an initial set of graphical tasks to start the interviews, and the remaining tasks were developed based on analyses of these weekly interviews. Each interview lasted about 20 minutes and was video-and audio-taped. They were presented with one or two tasks during each interview and asked to think aloud while they were solving the tasks so that we could analyze their responses and strategies as well as describe and make inferences about their thinking processes. In the end, 16 graphical tasks were developed to describe the nature of the students' understandings and investigate their strategies to create meaning for derivative graphs. One graphical task that illustrates differences among these students' thinking and concomitant difficulties is discussed in this paper; other graphical tasks as well as the students' work can be found in Haciomeroglu's (2007) study.

## Data Collections and Results

We illustrate Amy's, Jack's, and Bob's thinking processes and show how analysis or visualization alone led to different interpretations of the derivative graph. In the next section, for the task in Figure 1, we discuss the case of Amy, who preferred analytic thinking, and later, in a second section, we discuss the cases of Bob and Jack for whom visualization was primary for the task. In a final section, we discuss the importance of reversibility and flexibility of thinking in the complete understanding of calculus and pedagogical implications for these results.

*Analytic Understanding*

The graphical task in Figure 1 with the following instructions was presented to Amy: *The graph of the derivative* $f'(x)$ *of a function f(x) is shown. Sketch a possible graph of f(x).*

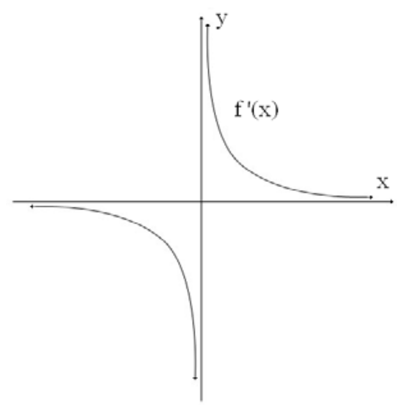


*Figure 1*. The task.

When we first presented the task, Amy made no mention of graphic interpretations of the task and instead translated from the graphic representation to the algebraic representation. That is, she estimated the equation of the derivative graph as $f'(x) = 1/x$. While integrating the equation, Amy, without using the absolute value of $x$, wrote $f(x) = \ln x$ as its integral and drew the graph in Figure 2. Amy knew that her graph could be shifted vertically; that is, shifting a graph of a function vertically does not change its derivative graph. Due to her methods of integration, Amy thought that her graph in Figure 2, the natural logarithm, was undefined on the interval $(-\infty, 0)$ and excluded this interval from the domain.

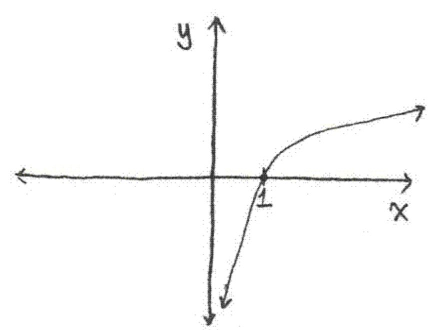


*Figure 2.* Amy's sketch.

At the end of the study, after discussing sixteen graphical tasks with Amy, Bob, and Jack, we decided to discuss the task in Figure 1 with Amy again to determine whether she would use the same strategies that led to her misunderstanding in integrating and interpreting equations in the previous interview. When presented with the task, Amy again estimated $f'(x) = 1/x$ as the equation of the derivative graph and computed its integral to draw the graph in Figure 3.

*Amy:* This graph looks a lot like the graph of *1/x*. It would be *1/2x* or *1/3x* but I just wrote it as *1/x* so the integral of *1/x* is *ln* |*x*| but I wrote it as absolute value to take into account the negative part of the graph too.

When asked if she could draw another graph from which the derivative graph in Figure 1 could be drawn, Amy said that since there were not any specific values for the derivative graph, her graph could be compressed or stretched horizontally due to different coefficients or it could be shifted up or down vertically but there were no other graphs that could produce the derivative graph in Figure 1. Consider this excerpt from the interview:

*Amy:* Another function [long pause] I don't think so.

*I:* Is this the only graph we can draw?

*Amy:* If that were *1/2x*, then we could have *1/2ln* |*x*|. This could be a little bit more gradual [draws a dotted line in Figure 3]. We could shift it up or down.

Since Amy assumed that the antiderivative graph had to be a logarithmic function, she said that her graph could not touch or cross the *y* axis because the graph in Figure 3 had an asymptote at $x = 0$. Consider the following passage:

*I:* It's approaching the *y* axis. Do you think it will touch or cross the *y* axis?

*Amy:* No, there is an asymptote at [$x =$]$0$. It'll keep going down.

*I:* The original function, *f*, can't be continuous?

*Amy:* Right. There is a discontinuity at [$x =$]$0$.

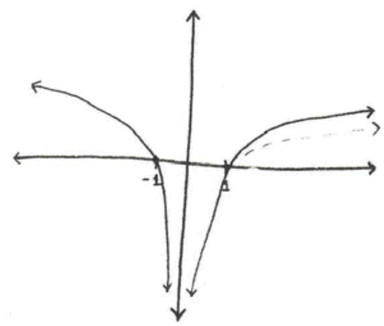


*Figure 3.* Amy's sketch.

*Analysis of Amy's Understanding*

Amy, without any apparent attempt to utilize visual means, estimated an equation of the derivative graph and integrated this equation to draw a possible antiderivative graph. Apparently, for Amy, the derivative graph represented an equation and this indicates her preference for analytic thinking for the graph in Figure 1. For the derivative task presented graphically, Amy's responses suggest that her thinking is Krutetskii's (1976) analytic type. We observed that Amy used analysis as the primary method in her work but her analytic approach without visual support hindered her thinking; that is, due to her methods of integration, initially she thought that her graph was not defined on the interval ($-\infty$, $0$). Later, she correctly integrated the equation but continued to assume that her graph had to be discontinuous. In addition, Amy's responses illustrate how her strong preference for analytic thinking and working in one-direction – from the derivative graph to its antiderivative graphs – instead of working in both directions exerts an influence on her thinking. Since Amy determined the equation as well as the graph of the antiderivative based on the equation extracted from the derivative graph, without considering whether other functions or graphs could create the same derivative graph, she said her graph did not touch the *y* axis and could not be continuous. If she had visualized the changing slopes of a possible antiderivative graph or considered the derivative of a graph with a cusp, perhaps she would have thought of other possible antiderivative graphs.

*Visual Understanding*

The graphical task in Figure 1 with the same instructions was presented to Bob and Jack. First, we consider the case of Bob. When we presented the task to Bob, he examined how the slopes changed on the derivative graph and transformed the slopes of tangent lines into the graph in Figure 4. Consider this excerpt from the interview:

*Bob:* Very little slope to big negative slope [approaches the *y* axis from the left in Figure 1] and it [slope] would be going to be a very high positive slope [approaches the *y* axis from the right in Figure 1].

Bob's descriptions of changing slopes suggest that he was employing Presmeg's (2006) dynamic imagery when he visualized the changing slopes of tangent lines to determine the shape of the antiderivative graph. Bob drew the graph in Figure 4 on the basis of his estimates for the slopes. Then, he tried to determine the continuity with the help of visual means as the following excerpts from the interview suggest:

*Bob:* This [the graph in Figure 4] can move up and down. Even though this [the graph in Figure 1] tells me the change in my parent function, in my integral, just because the derivative [in Figure 1] changes from positive to negative, doesn't mean your integral [in Figure 4] does. It just means slope goes from positive to negative.

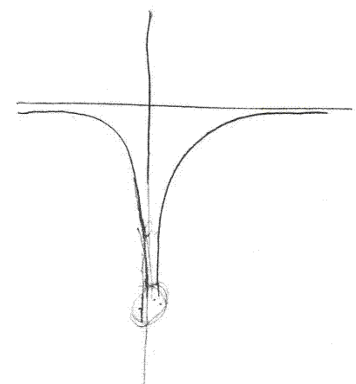

*Figure 4.* Bob's sketch.

Bob has just said that a discontinuity on a derivative graph indicated only a quick change of slopes on its antiderivative graph and not a discontinuity. Bob knew that his graph could be shifted vertically but was not sure whether it was continuous or not.

*Analysis of Bob's Understanding*

Bob described how the changing slopes (or *y* values of the derivative graph) determined the shape of the antiderivative graph without significant support from analytic thinking, and we consider this to be an example of the dynamic imagery described by Presmeg (2006). In this task, Bob's preference for visual thinking reinforces our belief that his thinking is representative of Krutetskii's (1976) visual type. For Bob, a derivative graph, representing the changing of slopes of its antiderivative graph, determined the antiderivative graph. Like Amy, Bob preferred to work in one direction – from the derivative graph to its antiderivative graphs – as he was working through the task. Without any attempt to estimate possible equations of the derivative graph or to consider derivative graphs of both continuous and discontinuous versions of his sketch, Bob determined the continuity of his graph based on his estimates for the changing slopes.

Now, we consider the case of Jack. For this task in Figure 1, Jack demonstrated a strong preference for visual thinking. That is, without any apparent attempt at translating the derivative graph into another representation, Jack visualized the changing slopes and transformed the derivative graph into the graph in Figure 5. Consider this excerpt from the interview:

*Jack:* This [the graph in Figure 1] approaches *0*. It [slope] gets to negative infinity [approaches the *y* axis from the left]. As we go to negative infinity, $f'(x)$ goes to *0* and likewise it happens to *f(x)*. We approach *y* axis from the right, it [slope] goes to positive infinity [in Figure 1]. We are going to positive infinity. It's [slope] approaching *0*.

For this task, Jack's thinking was predominantly visual, and he relied on his estimation of slopes to draw the graph in Figure 5. Jack employed dynamic images (Presmeg, 2006) while describing the changing slopes. When we asked whether his graph could be shifted vertically, Jack responded:

*Jack:* This [the graph in Figure 5] can go up or down. This can't go right or left. They can go up or down independently of each another. There is no value given for any specific point in *f(x)* here so you can change something there in the location along the y axis.

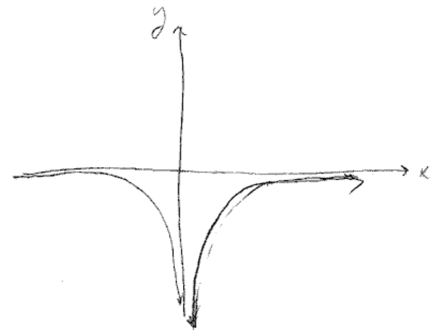

*Figure 5.* Jack's sketch

Jack's response illustrates how he could easily visualize different antiderivative graphs without analytic support. Jack said that the graphs on both sides of the *y* axis in Figure 5 could be shifted up or down independently or together but not horizontally. To further explore his thinking, we asked Jack whether his graph could be continuous. Consider the following passage.

*Jack:* Because $f'(x)$ [the graph in Figure 1] isn't continuous at the *y* axis, it's [the graph in Figure 5] not continuous at the *y* axis.

As seen in the excerpts, since the derivative graph in Figure 1 is not continuous at $x = 0$, Jack thought that its antiderivative graph had to be discontinuous at that point.

*Analysis of Jack's Understanding*

Although Jack, like Bob, used visualization as the primary method in their work, Jack had a different interpretation about the continuity of his graph; that is, Jack thought his graph had to be discontinuous at $x = 0$ where the derivative graph was discontinuous. Like Amy and Bob, Jack worked in only one direction – from the derivative graph to its antiderivative graphs – and did not demonstrate any attempts to reverse or change his thinking process. Instead, without apparent attempt to use analytic means, described how the slopes changed as he sketched a possible antiderivative graph. We considered this an act of Presmeg's (2006) dynamic imagery, as our inferences of his work supported our belief that Jack's thinking is representative of Krutetskii's (1976) visual type.

## Conclusions

For the calculus task presented graphically, Amy relied on analytic thinking, whereas Bob and Jack relied mainly on visual thinking. Their strong preferences for one mathematical thinking resulted in a one-sidedness in their understandings and presented different difficulties. For instance, since we presented a derivative graph, Amy, demonstrating a strong preference for analytic thinking, reversed the procedure by estimating and integrating the equation of the derivative graph and used that equation to draw a possible antiderivative graph. Due to the equation she estimated for the derivative graph, Amy thought that the antiderivative graph had to be a logarithmic function with a vertical asymptote. On the other hand, Bob's and Jack's visual approaches were quite similar in addressing the derivative graph. They employed dynamic images to transform the derivative graph into the antiderivative graph. From their responses, we inferred that in the absence of analytic support, dynamic images interfered with their cognition. Although, for them, a derivative graph represented the changing of slopes of its antiderivative graph, they interpreted the discontinuity on the derivative graph differently. For instance, at the point where the derivative graph was discontinuous, Jack claimed that the antiderivative graph had to be discontinuous while Bob said that the antiderivative graph was continuous with a drastic change of slopes.

We agree with Norman and Prichard (1994) that the Krutetskiian perspective and problem-solving processes – flexibility and reversibility – provide an insightful framework to analyze and describe students' understanding in calculus. From the students' descriptions of how they made their graphs, we conclude that their lack of flexibility in their cognitive processes or their reliance on one mathematical processing led to different and sometimes divergent interpretations of the same derivative graph. The idiosyncrasies in their work and thinking – estimating equations or visualizing slopes – illustrate the importance of flexibility and reversibility of thinking in the complete understanding of derivative and antiderivative graphs. The students did not demonstrate any apparent attempt to reverse their thought processes (or thinking in a reverse direction). Moreover, we consider their visual or analytic interpretations of the derivative graph to be an example of a one-way relationship (differentiation→ integration), described by Krutetskii (1976), not as a reversible two-way (differentiation↔ integration) relationship. Differentiation and integration, two fundamental concepts of calculus, are by their nature inverse processes, which suggest the use of reversibility of thinking when exploring the relationship between graphs of functions and their derivative or antiderivative graphs. We suggest that students' understanding of analytic and visual strategies and establishing reversible relationships between functions and their derivative or antiderivative graphs go together as students construct mathematical meaning for the concepts of differentiation and integration.

We conclude that failure to emphasize both analytic and visual aspects of these reversible processes can be an impediment to students' conceptual understanding of calculus. It is suggested that students be encouraged to associate a graph of a function with its derivative and a graph of a function with its antiderivatives as they examine graphs of various functions. For example, in this case, students can be asked to reverse their thinking to decide if any other antiderivative graph could have created the discontinuity in Figure 1. By encouraging students to utilize different mathematical thinking – for example, visual and analytic – and encouraging them to establish a two-way reversible relationship when interpreting data from derivative or antiderivative graphs, it is possible that students will overcome their difficulties and gain a wider and more robust perspective that cannot be provided without flexibility and reversibility of thinking.

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# IS SOCIAL ISOLATION NECESSARY IN DOCTORAL STUDY OF MATHEMATICS AND MATHEMATICAL RESEARCH?

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*Graduate school socialization describes the process in which graduate students become members of the community of an academic discipline. This is important because the primary purpose of graduate school is to prepare individuals for a career, typically as a professor and/or researcher, in a particular academic discipline. This paper uncovers a role of social interaction in the socialization of mathematics doctoral students into the community of mathematicians by examining one slice of mathematics doctoral student social interaction, namely peer interactions, to demonstrate how processes of affective socialization take place and their significance on students' views of mathematics and mathematics research.*

## Introduction

Over half of doctoral students in the United States leave prior to finishing their degrees (Lovitts & Nelson, 2000; Nettles & Millett, 2006). And for some disciplines, such as mathematics, the rate of departure is even greater (Council of Graduate School, 2008). The departure of students from doctoral study comes at a high cost for the student and the university (Gardner, 2008; Lovitts, 2001). And although many perceive that the high attrition is needed to "weed out" those who are not capable, prior research indicates that a good deal of attrition is unnecessary and preventable (Golde, 2005; Herzig, 2002).

In addition, few women and minorities participate in doctoral study of mathematics. In 2007-2008, of the 540 new mathematics doctoral recipients, who were U.S. citizens, only 166 (31%) were female and only 79 (15%) were minorities (Phipps et al., 2009). Broadening the focus of mathematics to increase the participation of women, minorities, and others who typically do not "fit in" may enrich the discipline of mathematics by expanding the range of mathematical thought and as a result, "help the profession flexibly meet the challenges posed by the growing quantitative sophistication of economic and political structures of the 21st century" (Herzig, 2004, p.173-174). Therefore, educational researchers need to determine what individual skills and knowledge and departmental systems and support help mathematics doctoral students (and women and minority mathematics doctoral students, in particular) succeed.

Many, including faculty, attribute student attrition to personal characteristics, such as interest in the field, lack of academic ability, and lack of drive, and conclude that raising admission requirements will increase student persistence. Yet, no difference in qualifications has been found between students who persist and students who leave (Lovitts & Nelson, 2000). Instead, both personal characteristics and departmental structures, including financial support, social and academic integration into the departmental community, family responsibilities, and impending requirements, have been shown to influence students' decisions not to persist (Earl-Novell, 2006; Herzig, 2004; Lovitts & Nelson, 2000).

It appears that many of the personal characteristics and departmental structures connected to attrition are aspects of the socialization of mathematics doctoral students into the mathematics community. However, no study of has looked at doctoral mathematics student socialization or examined attrition through the mathematics doctoral student socialization lens. This paper begins this inquiry by examining the role of social interaction in the socialization of mathematics doctoral students

## Defining Graduate Student Socialization

In general, socialization is the process through which a novices “acquire the values and attitudes, the interests, skills, and knowledge, in short the culture, current in the groups of which they are, or seek to become a member” (Merton, 1957, p.287). More specifically, and for the purposes of this paper, graduate school socialization is used to describe the process in which individuals in graduate school become members of the community of an academic discipline (Gardner, 2008; Golde, 1998). This is important because the primary purpose of graduate school is to prepare individuals for a career (i.e., participation), typically as a professor and/or researcher, in a particular academic discipline (Weidman & Stein, 2003). At the same time, however, graduate students must participate in the graduate school community. Therefore, the socialization of graduate school is a “double socialization”, in which students simultaneously learn to participate in graduate student life and prepare to participate in the community of their academic discipline (Golde, 1998).

Graduate student socialization has two dimensions: cognitive and affective. The cognitive dimension is the academic portion and refers to learning the knowledge and skills of the discipline. The affective dimension refers to learning how to participate in the social aspects of the discipline. Students become cognitively socialized as they take classes, learn to participate in research, take part in professional activities (including attending and presenting at conferences and writing and publishing professional papers). The means for affectively socializing graduate students is much less clear but occurs through personal interactions (both formal and informal) with faculty and peers in their department. And, since many of these social interactions occur during the activities in which students become cognitively socialized, both aspects of socialization typically occur simultaneously (Golde, 2000; Merton, 1957; Tinto, 1993; Weidman et al., 2001). Graduate school socialization has been closely linked to both students’ intellectual development and doctoral persistence (Golde, 2000, 2005; Tinto, 1993; Turner & Thompson, 1993).

### *Mathematics Graduate Student Affective Socialization*

In most fields, doctoral study is associated with isolation, and mathematics is no exception. Researchers continually discuss the virtually nonexistent social dimension of mathematics doctoral study (Herzig, 2002, 2004; Stage & Maple, 1996). And attrited mathematics doctoral students commonly cite isolation as a central reason in their decisions not to persist (Earl-Novell, 2006; Herzig, 2002, 2004; Hollenshead et al., 1994; Stage & Maple, 1996).

> Social interaction is important for affective socialization into the graduate school community. Social membership within one’s program becomes part and parcel of academic membership, and social interaction with one’s peers and faculty becomes closely linked not only to one’s intellectual development, but also to the development of important skills required for doctoral completion (Tinto, 1993, p.232).

It would also seem likely that social interaction is the primary means for affective socialization of graduate students into the community of mathematicians. This leads to questions of how doctoral students are being prepared to participate socially in the community of mathematicians and whether improving affective socialization of doctoral students would lower attrition. This paper begins to address these questions by examining one slice of mathematics doctoral student social interactions, namely peer interactions, to demonstrate how processes of affective socialization take place and the significance of these processes on students' views of mathematics and the mathematical community.

## Background: The Doctoral Study of Mathematics

Doctoral study of mathematics has two distinct stages. During the first stage, the coursework stage, students focus almost entirely on completing coursework requirements. This stage culminates in the successful completion of qualifying exams, which demonstrate to the department that the student has become an integral member of the course-taking community and is therefore, prepared to begin doctoral research. The second stage, the completion of a doctoral dissertation, marks the period between passing comprehensive exams and receiving the doctoral degree. During this stage, students complete a dissertation research proposal, go through the complete research process, and defend their final dissertation.

## Method

The analysis draws on the data collected from three female participants in a pilot study for a larger research project on women mathematics graduate students. The three female participants are doctoral students in the mathematics department at a university in the Southeastern United States. All participants engaged in semi-structured, individual interviews with the two authors. Interviews were open enough to allow participants to raise issues of their own that they deemed as significant to their experiences in K-12, undergraduate, and graduate mathematics. Each interview lasted approximately one hour. The interviews were audiotaped and then transcribed.

The data were analyzed using the constant comparative method, which has three phases in the data analysis: (a) intensive analysis, (b) developing categories, and (c) developing theory (Merriam, 1998). Contradictions were also examined and conjectures were made concerning the contradictory data. The data were analyzed for significant themes as well as differences in order to convey the complexity of individuals' responses (Fine & Weiss, 1998).

The primary themes identified in the data were the competitive nature of graduate study of mathematics and in the field of mathematics, in general; support for students' lives outside of the department; the challenging nature of mathematical research; the importance of peer support and peer interactions; conflicts between teaching and research; the role of advisors; the benefits of pursuing a Master's degree prior to enrolling in a doctoral program in mathematics; and independence and isolation in graduate study of mathematics. Through careful analysis of these categories and their relationships, *talking out problems*, a type of peer interaction, emerged as an important tool in successful study of graduate mathematics and mathematical research for these participants.

In the text that follows, particularly articulate quotes are used to present and highlight common themes discussed by the participants. Quotes from each of the participants have been included to ensure that all voices have been presented. To improve readability, quotes were minimally edited to remove stutters and distracting expressions such as 'uh' and to obscure references that might reveal participants' identities.

## Findings

### *Peer Interaction During Coursework*

The competitive nature of the department was a common theme across the interviews. Yet, the participants all indicated the importance of peer interaction in their success during the coursework stage. Specifically, *talking out problems* (i.e., a peer interactions in which a student talks to peers about a problem and the difficulties in solving it in order to better process, integrate, and understand the problem and how to approach it) was consistently described as the most valuable peer interaction for these students.

In graduate level mathematics, students face difficult problems with no clear path to a solution. In addition, professors often believe that students should struggle with problems on their own and therefore, hesitate to give students help on assignments. As a result, peers become a key resource on difficult assignments. Students indicated that they wanted to figure out the problems on their own; so, peers were primarily used as sounding boards to *talk out* problems.

> …I found out that I can solve problems by just talking with other people about them and somehow come up with solutions, even if they aren't really working on the problem yet or don't know the answer.

Thus, *talking out a problem* was a tool that helped students to solve complex mathematical problems that they had not been able to solve on their own and was seen as central to their success during the coursework phase of doctoral study.

*The Elimination of Peer Interaction During Dissertation Research*

However, upon entering the second stage of their doctoral program, participants were told that *talking out a problem* was not acceptable. One of the participant's advisors went as far as to forbid the student from discussing her dissertation research with other students.

> You're expected to do your work very independently and not collaborate with anybody… I was literally told by my dissertation advisor that I was not allowed to work with anybody on anything related to my dissertation.

And because the elimination of *talking out problems* was difficult for her; she attempted to explain its importance to her advisor.

> I try to tell him, "I need to talk these things out." For some reason he thinks that means that I don't have the ability to do it myself. So, I think it's a different perception.

The participants perceived that dissertation research was characteristic of mathematical research, in general. In other words, they indicated that mathematical research is done in isolation. This led one participant to question the necessity of social isolation in mathematical research.

> You have to ask yourself if it's always going to be that way. In an interdisciplinary field, like mine, I think I could be trained to be a team player, and I would be fine that way… And unfortunately, I think it will take 20 years for the community to catch up to that… I think there is a large section of old school professors who still believe that independent research is the only true research…

As these comments indicate, students discover that peer interactions (and *talking out problems*, in particular) can be useful tool in mathematical research. Yet, as they enter into their dissertation research, they are led to believe that it is unacceptable in the community of mathematicians.

## Conclusion

The primary purpose of doctoral study of mathematics is to prepare individuals for a career in mathematics, typically as a mathematics professor and/or researcher. So, during doctoral study, students should become socialized into the community of mathematicians and mathematical researchers. At the same time, however, graduate students also learn how to successfully participate in the mathematics graduate school community. This is sometimes problematic because socialization is not uniform across individuals or communities (Turner & Thompson, 1993).

Traditionally, mathematical researchers have been stereotyped as working alone on problems and only communicating with others after a solution has been found, as peers review the findings. Yet, this stereotype is often not accurate. Although some mathematicians do work in isolation; many mathematicians work collaboratively to solve problems. For example, in an interview study of 70 research mathematicians, Burton (1999) found that all but four of the participants used collaboration in their research.

Mathematics doctoral students typically have little experience with mathematical research prior to beginning the dissertation. And, the findings from this pilot study indicate that although mathematics doctoral students develop an appreciation for peer collaboration in mathematical research; they are led to believe that it is at odds with acceptable practice in mathematics research. This is in direct contradiction to what is actually happening in the community of mathematical researchers and leads to several questions, which can be addressed in future research.

The first, and foremost question is whether doctoral students are being adequately socialized into the mathematical community? In particular, if the community of mathematical researchers values collaboration, is forcing students to work in complete isolation on their dissertations a detriment to students' socialization into the community of mathematical researchers? Second, is *talking out a problem* a gender- or culture-specific strategy that is under-valued by the dominant culture. And, if this strategy becomes more accepted in mathematical research, would more women and minorities choose to participate in mathematics and mathematical research? Finally, this pilot studied uncovered one peer interaction that appears to be a valuable tool for solving mathematical problems. Are there others that can be uncovered in future research?

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# USING THE ONTO-SEMIOTIC APPROACH: DIFFERENT COORDINATE SYSTEMS AND DIMENSIONAL ANALOGY IN MULTIVARIATE CALCULUS

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*Dimensional analogy as a technique and different coordinate systems, apart from their intrinsic mathematical interest, are used in many types of applications in the sciences, engineering and art. As part of the process of the construction of an epistemic network for the subject, the identification of objects and dualities that emerge from this mathematical activity was carried out. The transformation of expressions to content through semiotic functions, and the identification of chains of signifiers and meanings, can be accomplished because of the rich layering and complexity of these mathematical concepts. Multivariate calculus students' responses to questions involving these notions were used for classification, according to the socio-epistemic network.*

## Dimensional Analogy and Different Coordinate Systems

The relationship between spatial dimension as a geometric concept, and the algebraic representation of dimension as lists of coordinates, has been recorded in textbooks as well as books and articles of general mathematical circulation, both pure and applied (Rucker, 1977; Banchoff, 1996; Tucker, 2007; Doren & Lasenby, 2007). The abstract relations are present from the beginning levels of algebra, but the present study privileges the multivariate calculus course where, as expressed in Montiel, Wilhelmi, Vidakovic and Elstak (in press):

> … it is in the multivariate calculus course where students, many for the first time, are expected to deal with space on a geometric and algebraic level after years of single variable functions and the Cartesian plane. They must define multivariable and vector functions, deal with hyperspace,… find that certain geometrical axioms for the plane do not hold over (lines cannot only intersect or be parallel, they can also be skew), and work with functions in different coordinate systems.

The issue of transiting between different coordinate systems, as well as the notion of dimension in its algebraic and geometric representations, are significant within undergraduate mathematics. Deep demands are made in both conceptual and application fields with respect to understanding and competence.

It is common to think of dimensional analogy as a method which permits the visualization of the fourth dimension (Weeks, 1985; Banchoff, 1996). To those interested in understanding the nature of multiple spatial dimensions, by taking a step back to the second dimension and trying to understand certain physical aspects**,** and then looking at the third dimension, an understanding of what the fourth spatial dimension would mean can be developed. For example, there are techniques that use generalizations, such as looking at the boundaries of one, two and three dimensional objects. On the other hand, there exist two ways of looking at dimension, that is, *extrinsically* or *intrinsically.* Briefly, the "extrinsic" point of view considers curves and surfaces, in particular lines and planes, as lying in a Euclidean space of higher dimension (for example a plane in an ambient space of three dimensions). According to the "intrinsic" point of view one

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cannot speak of moving 'outside' the geometric object because it is considered as self-contained. The inhabitants of Flatland consider their plane from an intrinsic point of view. For them, there is no space, no third-dimension.

The mathematical activity with lists of coordinates beyond the triple is usually introduced in the linear algebra context and, although mentioned in the multivariate calculus course, it is primarily the triple that displaces the ‘ordered pair’ (although, when working with gradients, the ordered pair returns, often in a way that may not be clear to the student). However, in triple integration the functions have a three dimensional domain (and one dimensional codomain), which means that the list indicating their geometrical representation would have four components. When dealing with functions whose three dimensional domains are expressed in the cylindrical or spherical coordinate system, it is not indicated, in any of the texts that were consulted, what that fourth component might look like. These are the types of mathematical issues that this study attempts to analyze when referring to dimensional analogy and different coordinate systems.

## Conceptual Framework

*Mathematical Objects*

A mathematical object, in this study, will be considered anything that can be used, suggested or pointed to when doing, communicating or learning mathematics. The onto-semiotic approach (Godino, Batanero & Roa, 2005; Font, Godino & D’Amore, 2007) considers six primary entities which are:

(1) *Language* (terms, expressions, notations, graphics);
(2) *Situations* (problems, extra or intra-mathematical applications, exercises, etc.);
(3) *Definitions* or descriptions of mathematical notions (number, point, straight line, mean, function, etc.);
(4) *Propositions*, properties or attributes, which usually are given as statements;
(5) *Procedures* or subjects’ actions when solving mathematical tasks (operations, algorithms, techniques, procedures);
(6) *Arguments* used to validate and explain the propositions or to contrast (justify or refute) subjects’ actions. (Godino et al., 2005; Font et al. 2007)

As important as the mathematical objects are: 1) the agents that move them and the meaning (straightforward or not) that is assigned to them; 2) the concrete appearance of these objects and the reference to ideal entities; and 3) their contextual and relational function with other mathematical objects. For these reasons, in the onto-semiotic framework, the following dual dimensions are also considered when analyzing mathematical objects: (Godino et al., 2005, 5):

(1) Personal / institutional;
(2) Ostensive / non-ostensive;
(3) Intensive / Extensive;
(4) Unitary / systemic;
(5) Expression / content.

These dual dimensions demonstrate how the primary entities must not be understood in an isolated manner, but according to their function and their relation in a contextualized mathematical activity.

*Systems of Practices, Emerging Objects and Epistemic Networks*

According to the onto-semiotic approach (Godino & Batanero, 1994; Wilhelmi et al., 2007a), it is necessary to determine the meanings (plural) associated with mathematical objects in different contexts and organize them (the meanings) as a complex and coherent whole. The *operative and discursive systems of practices*, and their subsystems, understood as depending on the institutional and personal contents that are associated to a mathematical object, and the objects that emerge within these systems, form epistemic and cognitive networks. This means that if the systems of practices are institutional, the emerging mathematical objects are considered to be institutional objects, and if the systems of practices correspond to an individual, then the objects are personal, according to the duality specified above. Also, following this duality, the objects that emerge can be ostensive (such as symbols and graphs) or non-ostensive, that is, conceptual or mental. The contextualized and functional use of these objects as elemental entities cannot be divorced from their essentially relational nature that, at the end, justifies their adaptation, whether in particular or general processes.

Whereas the *meaning* of a mathematical notion represents the structured complex of a system of practices in a context, the *holistic meaning* of a mathematical notion represents the expression of the different (partial) meanings associated with the notion as one system. The holistic meaning comes from the coordination of the partial meanings associated with a mathematical notion (Wilhelmi et al. 2007b). *Flexible mathematical thinking* (FMT) is what permits the passage between different partial meanings, and the coordination or partitioning of the different meanings when necessary.

Finally, in order to capture the semiotic complexity inherent in the communication of mathematics, it is important to identify the different objects that make up the mathematical practice in specific content areas and contexts. The resulting network is called a *socio-epistemic configuration*, and it captures the interplay of the objects and relations in a particular mathematical setting. This way, the personal meanings that are constructed when individuals carry out mathematical activity can be described by cognitive configurations. Evaluation of the learning and mathematical behavior of these individuals lies in the analysis of the relation between the socio-epistemic and cognitive configurations.

*Semiotic Functions and Representation*

The onto-semiotic approach places great value on the relation between mathematical objects by means of the *semiotic function* (Godino & Batanero, 1997), as a relation between an expression and a content established by ‘someone’, according to certain rules of correspondence. Not only language, but the other types of objects such as concepts, situations, actions, properties, or arguments, can be expressions or content of semiotic functions. Font et al. (2007) pointed out that to understand representation in terms of semiotic functions has the advantage of not segregating the object from its representation. Indeed, given an object and a representation, in general it is not possible to identify a unique semiotic function between them, and even the representation can constitute the content in another context. For example, Alson (1989, 1991) shows how a Cartesian graph can be given an algebraic structure before the introduction and analytic development of the theory of functions. This fact determines an objectification process of a representation (in its broadest sense) that is prototypical in mathematics.

When talking about semiotic functions, the dependence relations can be either *representational* (one object is put in the place of another), *instrumental* (an object is used as an instrument by another) or *structural* (two or more objects conform a system out of which new objects emerge) (Godino, Batanero, & Font, 2007). An example of a representative semiotic function, as opposed to structural or instrumental and related to other questions used in the interview that are not analyzed in this paper, could be a solid presented geometrically as the expression, and the formulation of a double integral 'setup' as the content. An instrumental semiotic function could have as expression the double integral, and as content the numerical answer, while the structural (or 'cooperative') semiotic function could take some region together with a double integral in terms of "$x$ and $y$" as the expression, and the set up of a double integral in terms of "$u$ and $v$" over a simpler region (using the Jacobian) as the content. It should also be clear that the expression in one semiotic function could be the content in another.

## Research Questions

(1) Do multivariate calculus students use dimensional analogy as a result of their transit from single variable to multivariable calculus? (2) In which primary entities is dimensional analogy used (if it is used)? (3) What primary entities and semiotic functions can be identified and classified, as students relate different coordinate reference systems to their previous study of calculus in the 2-dimensional Cartesian coordinate system?

## Context, Methodology, and Instrument

The context of the present study is multivariate calculus (calculus III) as the final course of a three course calculus sequence, taught at a large public research university in the southern United States. Seven students were interviewed once in two groups, the first consisting of four students and the second of three. The interviews were video-recorded. Each interview was approximately an hour long. The students were first given a questionnaire, which is included in this text, on which they wrote down their responses, and they were then asked to explain them. For each question, the students were chosen in a different order, but it was inevitable that who spoke first would influence, in some way, the others. They were asked to explain verbally on an individual basis, but group discussion was encouraged when it presented itself. This report focuses only on the first question from the interview (table 1).

Table 1. *Question 1*

| |
|---|
| In rectangular coordinates the coordinate surfaces: $x = x_0$, $y = y_0$, $z = z_0$ are three planes.<br>*(a)* In cylindrical coordinates, what are the three surfaces described by the equations: $r = r_0$, $\theta = \theta_0$, $z = z_0$? Sketch.<br>*(b)* In spherical coordinates, what are the three surfaces described by the equations: $\rho = \rho_0$, $\theta = \theta_0$, $\phi = \phi_0$? Sketch. |

The intention of this question was to detect the students' geometrical transition to 3D-space where, in the rectangular context, much emphasis was placed at the beginning of the course on the coordinate planes and the octants. These answers were important to begin to detect the process of dimensional analogy. The interview protocol included the question of why the equations represented planes, and not just points or lines. Although we have been unable to discover any literature on the subject, through informal discussions and comparisons it has been noted that the average student has difficulty with associating the algebraic equation, say, $y = a$, with a plane parallel to the $xz$-plane, or the actual $xz$-plane if $a = 0$. The protocol also indicated that, if the sketch was correct, to ask how the angle $\theta$ 'turns into' a plane.

## Analysis Using the Onto-Semiotic Approach

While analyzing dimensional analogy and different coordinate systems within the onto-semiotic approach, it is important to remember what was mentioned in the conceptual framework about the classification of the mathematical objects and the primary entities. There are aspects that characterize each of these entities, but by no means can there be a sharp separation between them. The first question from the interview will be analyzed; as there are seven students and two groups, S1, S2, S3 and S4 will represent the participants in the first group, and S5, S6, and S7 the participants in the second interview session. The two sessions will not be differentiated as emphasis will be placed on the question itself and the mathematical content. There are also written answers which will be referred to at times. Because of the natural limitations of space in a report such as this, emphasis will be placed on the participations that are directly related to dimensional analogy and the change of coordinate systems.

The interview question will be accompanied by a table that represents the socio-epistemic configuration relevant to its content and context. In the lecture sessions, the professor (one of the authors) had a well-defined system of objects and meanings that were meant to be developed within the context of the institutional mathematical practices.

In question 1, the socio-epistemic configuration of primary objects is structured in terms of the given situation (table 1), that is, the transition between algebraic and geometric representations of surfaces immersed in 3D, in different coordinate systems. The analysis and solutions detonate the use of different concepts, procedures, propositions and previous arguments, and open the possibility that new ones emerge. The activation of these emerging objects is brought about by the processes of definition, of creation of techniques (algorithmic or not), the determination of propositions and argumentation. All these processes are only possible through the use of language in different registers, that is, the use of languages that make the codification and transference of knowledge and meanings of the mathematical objects involved possible. (figure 1).

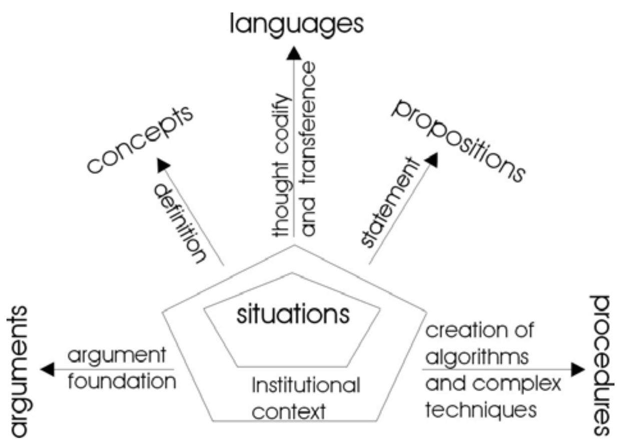


*Figure 1.* Mathematical' objects and process.

Question 1 had a marked geometrical emphasis, as students were asked to sketch what they understood by the algebraic equations representing surfaces immersed in 3D space, in the rectangular, cylindrical and spherical coordinate systems. The instructions were written in a combination of mathematical English (Wells, 2003) and symbols. The results of these instructions were sketches, a graphical language object. As institutional objects these sketches are considered language entities:

- *Verbal*: rectangular, cylindrical, spherical coordinates; dimension (3D);
- *Graphical*: graphs of the surfaces;
- *Symbolic*: $x = x_0$, $y = y_0$, $z = z_{0;}$ $r = r_0$, $\theta = \theta_0$, $z = z_{0;}$ $\rho = \rho_0$, $\theta = \theta_0$, $\phi = \phi_0$

We show in table 2 some of the previous and emergent concepts, procedures, propositions and arguments.

Table 2. *Previous and Emergent Concepts, Procedures, Propositions and Arguments*

| Objects | Previous | Emergent |
|---|---|---|
| Concepts-Definitions | • Cartesian coordinate system in 2D;<br>• Definitions of functions and relations in a single variable;<br>• Graphs of single variable functions and relations; | • Different coordinate systems in 3D;<br>• Graphs of planes;<br>• Definition of functions and relations in a multivariable context;<br>• Graphs of functions and relations in different coordinate systems in 3D;<br>• Extrinsic and intrinsic points of view |
| Procedures | • Graphing lines and curves in the plane;<br>• Evaluating and setting up single variable functions; | • Graphing planes and surfaces in 3D space;<br>• Evaluating and setting up multivariable functions in different coordinate systems; |
| Propositions | In rectangular coordinates:<br>From an intrinsical point of view:<br>• In 0D space, $x = x_0$ represents a point; in 1D space $x = x_0$, $y = y_0$ represent lines;<br>From an extrinsical point of view:<br>• In 1D space, $x = x_0$ represents a point on a line; in 2D space $x = x_0$, $y = y_0$ represent lines on a plane; $r = r_0$, $\theta = \theta_0$ represent curves on the polar plane. | From an intrinsical point of view<br>• In 2D space, $x = x_0$, $y = y_0$, $z = z_{0;}$ $r = r_0$, $\theta = \theta_0$, $z = z_{0;}$ $\rho = \rho_0$, $\theta = \theta_0$, $\phi = \phi_0$ represent surfaces.<br>From an extrinsical point of view:<br>• In 3D space, $x = x_0$, $y = y_0$, $z = z_{0;}$ $r = r_0$, $\theta = \theta_0$, $z = z_{0;}$ $\rho = \rho_0$, $\theta = \theta_0$, $\phi = \phi_0$ represent 2D surfaces immersed in 3D. |
| Arguments | Dimensions as degrees of freedom (intrinsic): a point has 0 degrees of freedom, a line has 1 degree of freedom. | Dimensions as coordinates (extrinsic in this case): a point has 1 coordinate, a line or curve in general has 2 coordinates, a plane or surface in general has 3 coordinates. |

Only S1 showed a complete grasp of the (institutional) meaning in his written work. The other students were able to translate the equations in rectangular coordinates, but their sketches of the surfaces in the cylindrical and spherical coordinate systems were, to different degrees, inaccurate.

As a reply to the question of why the equations in rectangular coordinates represented planes and not lines, S3 said “For each plane there’s only a restriction in one dimension, so that dimension is throughout the whole plane. Each other dimension can be anything, that’s how you get just one infinite plane”. The deictic signs that accompanied the verbal expression can be used to classify the relations that were established as a representational semiotic function, going from the equations, as the expression, to the sketches and gestures as content (in particular, the communication of the ‘infinite plane’ was clearly done by spatial gestures). In this same vein, an interchange with the two interviewers (I1 and I2) and S5, when asked why the equations represented planes, will be presented.

S5: I don't think I know how to verbalize why they're planes and not points.

I1: But why do you see $x = x_0$ as parallel to the $yz$ -plane?

I2: For the record, you're looking at the room.

S5: I'm looking at $zy$ $x$ and .

I2: You're pointing, what exactly are you pointing at?

S5: The corners, I'm pointing at the corners to get my head around it in space.

On the other hand, S7 used dimensional analogy to explain:

S7: When we have 2 dimensional, say $x = 2$ . We fix $x$ at 2 and $y$ could be anything. Now we have 3D, another variable which is $z$ . Instead of say, $x = 3$ and $y$ going on forever, it would be $z$ going up and down forever as well.

In the context of the cylindrical coordinate system, as was mentioned, only S1 drew all three expected answers. However, S3, once hearing and observing S1's geometrical description and representation of the three surfaces, realized he had misunderstood the question, but "it was asking the same thing as the previous one" (the three surfaces in rectangular coordinates), and offered his interpretation:

S3: $r = r_0$ would be a cylinder of infinite height, $\theta = \theta_0$, would be a slice of the cylinder and $z = z_0$ is an infinite plane, no, not a plane, it's an infinite disk at $z_0$.

I1: What is the difference between an infinite disk as opposed to a plane?

S3: It's just the coordinate system that you use, it's not rectangular coordinates any more, it's polar coordinates.

I2: But why think of it as a disk? I could think of it as an oval or a rectangular thing.

S3: It has no restriction on $\theta$, it turns into a plane with an angle of $2\pi$ , the radius has no restriction, so it goes on and on.

S1: It has to do with how you build the plane in your head. If you take one $r$ value at a time, it's just concentric disks. It's just the shape of the space.

As was established in the conceptual framework, the objects that have emerged, such as language, situations, procedures, definitions, reflect their duality. In particular, the duality ostensive (symbols, graphs, gestures) or non-ostensive, that is, conceptual or mental, can be detected in the chosen fragments related to question 1, through the semiotic functions with their expression and content.

## Synthesis, Conclusions, and Prospective

Dimensional analogy as a technique and different coordinate systems, apart from their intrinsic mathematical interest, are used in many types of applications in the sciences, engineering and art. The generic notion of representation is central in the cognitive and instructional processes involved in communicating these notions. The focus on changes of registers and on individual processes of objectification, conceptualization and meaning contributes to a coherent view of mathematical knowledge and the means of its construction and communication. Based on the onto-semiotic approach, it can be added that it is also important to emphasize the anthropological and socio-cultural character of this knowledge, indicating the tensions between the personal and institutional meanings. The primary entities and their dualities, together with the semiotic functions, allow the description of this personal-institutional tension, related to the notion of meaning and mathematical objects that are relevant, in this case, to dimension, dimensional analogy and different coordinate systems.

The onto-semiotic complexity that was identified is an empirical indicator that should guide the search for ways to improve and control the didactic systems related to the learning and teaching of notions, methods and meanings associated with dimension and different coordinate systems in the multivariate context. Dimensional analogy, as was seen in the previous section, is used by the students, especially when dealing with the primary entities of language, situations and concepts. However, this concept needs to be formalized and consciously incorporated as a technique in the communication of this mathematical subject, and others similar to it.

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# PROJECT M$^2$: A RESEARCH PROJECT DEVELOPING ADVANCED GEOMETRY AND MEASUREMENT CURRICULUM FOR *ALL* PRIMARY STUDENTS

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Project M$^2$: Mentoring Young Mathematicians is a 5-year National Science Foundation research grant aimed to develop advanced geometry and measurement units for K-2 students and measure the impact on student achievement. Mathematical performance has been unimpressive for U.S. students on both national and international measures, particularly diverse and poor students (Perie, Grigg, & Dion, 2005; TIMSS, 2004). Measurement and geometry are areas of particular concern. U.S. fourth graders ranked only 17$^{th}$ in geometry and 13$^{th}$ in measurement out of the 25 countries in the 2003 TIMSS (Mullis et al., 2004); similarly, geometric shapes and measures was the lowest averaging cognitive domain in the 2007 TIMSS (Gonzales et al., 2008). This performance is reflective of insufficient time devoted to geometry instruction (Clements, 2004) and weak measurement instruction in the primary grades (Clements & Stephan, 2004).

In response, Project M$^2$ is undertaking the development and research on the efficacy of geometry and measurement units in each grade, K-2. The units are focused on higher-level mathematics using research-based practices and standards in mathematics and early childhood education and augmented by exemplary gifted education practices. To address the needs of *all* students, the Project M$^2$ units are being piloted in urban, suburban, and rural classes. Following, they are being field-tested in classrooms nationwide that include underrepresented students (12 classes and $n$ ~ 240 in each the intervention and control group). Teachers from the same schools were randomly selected to teach either the units or regular curriculum. Performance-based tasks, criterion-referenced unit tests, and standardized tests are being used to determine the impact of the Project M$^2$ units. This session will report on the research endeavors and curricular innovations as well as lessons learned from the first two years of the grant.

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# CHANGING STUDENTS' MINDS ABOUT MATHEMATICS: EXAMINING SHORT-TERM CHANGES IN THE BELIEFS OF MIDDLE-SCHOOL STUDENTS

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*This study was designed to determine if a 5-week summer school program could change both beliefs about mathematics and intelligence and, if so, to examine how these changes coincided with changes in students' effort and achievement. A curriculum was designed to expose students to difficult mathematics via rich, open-ended problems that necessitated daily group work, presentation, and discussion and that focused on teachers providing specific, qualitative feedback in lieu of scores or grades. This "learning-focused" curriculum was delivered to 23 seventh- and eighth-grade students, all of whom were required to complete a summer make-up course to pass into the next grade. Results suggest in part that specific curricula and instruction can change the beliefs and increase the performance of low-performing students, even in a short amount of time.*

## Introduction and Significance

The question of why beliefs[1] matter in education research is a complicated one to answer. If we care most about helping students be successful in mathematics, which for now at a minimum means passing math classes and achieving proficient scores on state math tests, perhaps what students believe or don't believe is irrelevant. Furthermore, for many studies positing that beliefs do impact achievement there are others arguing beliefs actually follow from successes or failures in school. What is clear, however, is that there are indeed certain beliefs and theories held by secondary math students that affect both their motivation and achievement. We know that a junior high math student's theory of intelligence as unchanging or evolving affects his/her classroom motivation and is a strong predictor of achievement in mathematics (Blackwell, Trzesniewski, & Dweck, 2007), for example, and children's beliefs about mathematics and mathematics learning, especially when the beliefs have remain unchanged for many years, affect their future effort (Haladyna, Shaughnessy, & Shaughnessy, 1983) and achievement (Mason, 2003). Furthermore, research suggests that "negative experiences have lasting negative effects [on achievement] *primarily* when they affect an individual's beliefs" (Blackwell, et al., 2007).

The majority of beliefs research seems to confirm that *strong* beliefs about oneself and one's learning are resistant to change and do impact achievement, including affecting both cognition and behavior (Krosnick, 2007). In a study by Peter Kloosterman and Frances Stage, six categories of "strong" math beliefs were identified as being salient across student populations, then used to design a set of beliefs scales (Kloosterman & Stage, 1992). Part of the impetus for the study was from prior research suggesting that there are certain beliefs students hold about mathematics that exist across different secondary math classrooms (Schoenfeld, 1989). From a national assessment in 1985 for example, 83% of eighth-grade students reported to agree or strongly agree with the statement, "There is always a rule to follow in mathematics" (Dossey, Mullis, Lindquist, & Chambers, 1988). The six beliefs measured by the Indiana Mathematics Beliefs Scales (IMBS) are:

- Belief 1: I can solve time-consuming mathematics problems.

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

- Belief 2: There are word problems that cannot be solved with simple, step-by-step procedures.
- Belief 3: Understanding concepts is important in mathematics.
- Belief 4: Word problems are important in mathematics.
- Belief 5: Effort can increase mathematical ability.
- Belief 6: Mathematics is useful and relevant to my life.

Another set of “strong” beliefs that have recently been found to affect achievement in mathematics is students’ implicit theories of intelligence. While some students believe that intelligence is fixed (“entity theorists”) others believe it is malleable (“incremental theorists”), viewpoints which shape students’ responses to academic challenges quite differently (Dweck, 2000). Incremental theorists, for example, tend to display mastery-oriented responses to failure more than entity theorists because they believe in the utility of effort given a roadblock or setback. In a study by Elliott and Dweck, researchers manipulated the value of a goal (as either learning-focused or performance-focused) and students’ perceived ability (as either high or low) to examine affects on problem-solving strategies and persistence. The students in the performance-goal, low-perceived-ability group manifested rapid strategy deterioration, increased failure attribution and increased negative affect, including learned helplessness (Elliott & Dweck, 1988). The findings suggest that learning-focused environments for learning mathematics may be most effective for maximizing student self-efficacy, persistence and achievement.

To our knowledge, no research has examined both beliefs about mathematics and beliefs about intelligence in the same study, despite that it is reasonable to expect these two sets of beliefs to be related. Recently however, some curricular and programmatic efforts have been implemented to expose students to learning-focused mathematics education that both bolsters students’ achievement in mathematics and changes their beliefs about mathematics and mathematics learning. In 2005 for example, Boaler and colleagues wrote and taught a summer math curriculum in which students were encouraged to perform daily “meta-cognitive acts,” including mathematical summarizing, questioning, clarifying and predicting (Boaler, et al., under review). Prior to Boaler’s study, much research had shown that engaging in metacognitive acts positively affects achievement and engagement in mathematics (see White & Frederikson, 1998; Palincasar & Brown, 1984; Adey & Shaher, 1994). In just five weeks the students in the summer math program showed both math achievement gains and positive attitudinal shifts. Boaler says that “supporting and knowledgeable teachers, open and interesting tasks, and classroom culture of peer collaboration tasks, all of which characterized our summer school intervention, made it very likely that virtually all students would find intellectual stimulation and feel valued as mathematical thinkers.”

In the study described here we intend to replicate and expand on the model designed by Boaler and colleagues in the summer of 2005, in order to measure if changes in students’ beliefs about mathematics (if they exist) coincide with changes in students’ beliefs about the malleability of intelligence. The research questions addressed are:

1. Do students’ beliefs about mathematics and mathematics learning change over five weeks of a learning-focused math program? If so, which ones?
2. Are students’ beliefs about mathematics and mathematics learning correlated with their belief about the nature of intelligence?
3. Does a change in a student’s beliefs about the nature of mathematics and/or intelligence coincide with a change in achievement or effort?

## Methods

*Subjects*

The subjects were one class of 23 seventh- and eighth-grade mathematics students in a summer school program in the Bay Area. Most students were allocated to classes randomly by the registrar, but some were switched by the principal for behavioral and motivational reasons.[2] The experimental class was a heterogeneous mix of students from several different middle schools. Though technically students can elect to attend summer school program to "get ahead" the next year, 90% of students this particular summer had been mandated to attend and pass summer courses to move on to the next grade, and all students in the experimental class were there because they needed enrichment in mathematics and/or had earned a D or F in Math the previous year (Stoffal, principal, personal communication 2008). Specifically, according to an introductory survey, 22 of 23 students in the course reported being there because they had failed mathematics the previous school year.

*The Curriculum*

The "learning-focused" curriculum used in the experimental class was designed to necessitate collaborative problem-solving and emphasized both the debriefing of final answers and the sharing and discussing of mistakes and roadblocks. To motivate students to persevere, frequent, qualitative feedback from the teacher on students' progress toward the standards for the course was given, and final grades for the summer were *pass* or *fail*, where a pass was achieved by participating meaningfully in group work, presentations and whole-class discussions. For consistent formative assessment and to develop personal relationships with the students, the teacher wrote individualized responses to students' journal entries completed at the end of each day. Finally, the activities and debriefs focused on students developing and justifying their own conclusions about mathematics, and not necessarily on right answers or complete work.

The curriculum had four middle-school math content area emphases, four computational emphases and four process emphases. The content emphases, probability and number theory, proportional reasoning, patterns and generalizability and the four representations of data, were drawn from both state and national standards for 7th and 8th grade mathematics and represent topics middle-school students tend to struggle with and that are important for success in Algebra. The computational emphases, chosen for similar reasons, included students getting practice with and deepening conceptual understanding of fractions, decimals and percents, exponents and roots, ratios and proportions and order of operations. Finally, the process emphases, organization, communication, justification and questioning, came from the National Council of Teachers of Mathematics standards for mathematical thinking.

*Belief Measures*

In order to measure students' beliefs about mathematics and intelligence, on the first day of class all students were given a survey consisting of randomly-ordered questions from the Indiana Mathematics Beliefs Scales (IMBS) (Kloosterman & Cougan, 1994) and from the Implicit Theories of Intelligence Scale for Children (ITIS) (Dweck, 2000). Items assessing one of the IMBS factors, "*Word problems are important in mathematics*," were omitted because the researcher worried the phrase "word problems" might have been unfamiliar to some students or defined in a variety of ways. For this reason, items measuring the belief "*There are word problems that cannot be solved with simple, step-by-step procedures,*" used the phrase "*math problems*" instead of "*word problems*."

For the purposes of this paper, scores from both measures will be referred to as "beliefs scores." The final survey consisted of 36 items: six items measuring each of six belief factors (5 beliefs about mathematics factors and 1 belief about intelligence factor). Both scales were tested in prior research by their designers for stability and test-rest reliability and have been used in prior empirical research. The students' scores on the survey items were directionally-adjusted so that a score of 1 consistently correlated with a more negative belief and a 5 correlated with a more positive belief.

*Achievement and Effort*

In order to measure general achievement, on the second day of class all students solved two problems from the 2000 Math Assessment Collaborative (MAC) for eighth grade, consisting of an open-ended patterns and proportional reasoning problem. These particular problems were chosen because they were unrelated directly to content from the course, but were two, different measures of general problem-solving ability. First, each assessment was scored using a rubric and given a numerical score for correctness ("Paths" problem out of 4 points; "Lockers" problem out of 6 points). These scores and rubrics were derived from the existing scoring methods from the designers of the problems. As a measure of effort, each assessment was also given a binary score for completeness, based on whether or not the subject tried all parts of the problem. Two scorers scored all assessments, compared scores, then discussed disparities and came to agreements on final scores before analyses began. Because of absenteeism only 17 of 23 students took the post-assessment.

## Results

*Changes in Achievement and Effort*

The change from pre-test to post-test in achievement scores was significant ($t$=1.80 $p$=0.046), as well as the change in effort scores ($t$=2.7, $p$=0.007) (Tables 1.1, 1.2).

*Table 1.1*

Comparison of Pre- and Post-test Achievement Scores (N=17)

| | **Pre-Test** | | **Post-Test** | | ***Change*** |
|---|---|---|---|---|---|
| | M | Sd | M | Sd | |
| **Overall Change** | ***1.74*** | ***1.38*** | ***2.32*** | ***1.71*** | ***0.59**** |
| **Problem #1: Paths** | 0.74 | 0.66 | 1.03 | 0.76 | 0.29* |
| **Problem #2: Lockers** | 1 | 0.98 | 1.29 | 1.26 | 0.29 |

*Note*. *$p$ < .05

*Table 1.2*

Comparison of Pre- and Post-test Effort Scores (N=17)

| | **% who tried all parts of test** | | ***Change*** |
|---|---|---|---|
| | Pre | Post | |
| | | | |
| **Problem #1: Paths** | 47.1% | 70.6% | 23.5%* |
| **Problem #1: Lockers** | 41.2% | 70.6% | 29%** |

*Note*. **p* < 0.025, ***p* < 0.05

*Changes in Beliefs Scores*

All of the mean beliefs' scores became more positive from pre- to post-survey, and four of these changes were statistically significant (Table 2). Belief 4 did not change from pre-survey to post.

*Table 2*

Paired T-test Results for Pre-/Post-scores on Beliefs Survey, Overall and by Belief (N=23)

| | | **Pre-Survey** | | **Post-Survey** | | ***Change*** | **t-value** |
|---|---|---|---|---|---|---|---|
| | | M | Sd | M | Sd | | |
| ***Overall Change*** | | ***3.24*** | *1.01* | ***3.37*** | *0.99* | *0.13* | ***3.4**** |
| | | | | | | | |
| **Belief #1** | Understanding concepts is important in mathematics. | 3.21 | 0.9 | 3.5 | 0.89 | 0.29 | **3.0***** |
| **Belief #2** | Effort can increase mathematical ability. | 3.85 | 0.7 | 3.93 | 0.75 | 0.08 | **1.5*** |
| **Belief #4** | There are problems that cannot be solved with simple, step-by-step procedures. | 2.71 | 0.9 | 2.71 | 0.86 | 0.00 | 0.15 |
| **Belief #5** | Mathematics is useful and relevant to my life. | 3.37 | 1.1 | 3.5 | 0.91 | 0.13 | **1.4*** |
| **Belief #6** | I can solve time-consuming mathematics problems. | 2.97 | 1.14 | 3.17 | 1.07 | 0.20 | **1.7**** |
| | | | | | | | |
| **Belief #3** | Intelligence is malleable. | 3.39 | 1.0 | 3.42 | 1.04 | 0.03 | 0.89 |

*Note*. **p* < 0.025, ***p* < 0.05, ****p* < 0.01

*Correlations between Beliefs and Achievement*

The correlation between change in beliefs and change in achievement was (R=0.12, NS). It is clear from the scatterplot that the four greatest changes in achievement scores were correlated with large changes in beliefs scores. There was a more positive correlation between changes in effort scores and changes in achievement scores (R=0.42).

## Discussion

The main findings of this study are that students' beliefs about mathematics and mathematics learning can change over five weeks of a learning-focused math program. The largest change was found in students' beliefs about the importance of understanding concepts in learning mathematics. There were also significant, positive changes in students' achievement and effort scores. Despite these findings, there was no significant change in students' beliefs about the malleability of intelligence, nor significant correlations between changes among any of the beliefs. Overall, changes in students' achievement scores correlated only weakly with changes in students' beliefs, but much more strongly with changes in effort scores.

The fact that a large number of the analyses shown above resulted in non-significant results is not surprising, considering the small sample size and short time frame for implementing the learning-focused curriculum. This makes the significant results, however, much more interesting and important for future research. Clearly students' beliefs can be changed in a short amount of time, as evidenced by the four small but significant changes and overall significant change in scores. These beliefs were:

- "Effort can increase mathematical ability."
- "I can solve time-consuming mathematics problems."
- "Understanding concepts is important in mathematics."
- "Mathematics is useful and relevant to my life."

The fact that these beliefs changed and not the other beliefs ("*Intelligence is malleable*" and "*There are problems that cannot be solved with simple, step-by-step procedures*") can be qualified in part by certain characteristics of the curriculum used. Specifically, the curriculum was focused primarily on large, open-ended problems that we both asked students to try their best on (as opposed to solve completely) and persevere on as long as possible. Students spent a great deal of time trying things out and were never discouraged from attempting a particular method even if it was clearly wrong. In class debriefs and discussions the students then came together to share processes and answers and often went back to work afterward. It was an important part of the design of the curriculum for students to both grapple with difficult mathematics but also have enough access so they wouldn't shut down. It is not surprising that after five weeks of doing mathematics in this way students would have different beliefs about effort's role in what defines "mathematical ability" and more confidence solving time-consuming math problems. The former is a pleasing result considering that Americans tend to be one of few groups internationally who believe you are either born with math ability or not (TIMSS, 1999). Additionally, the focus on explanation, justification and questioning in the curriculum could have contributed to the positive changes in students' beliefs about the importance of conceptual understanding in mathematics as well.

These same curricular characteristics, however, make the non-significant change in students' beliefs about the quality of math problems curious. It would be expected that students feeling more confident in their ability to solve tough problems and believing that effort has more to do with ability than not would also believe that solving math problems has less to do with following one, correct procedure than it does with hypothesizing, testing, manipulating, etc. Furthermore, students' beliefs about the nature of intelligence generally did not change over the course of the summer. This could be explained perhaps by the fact that these questions were blended in with the other 30 questions focused on mathematics, and thus could have been misinterpreted. Perhaps some students' engrained beliefs that they *personally* are not "intelligent" mathematically overrode the fact that the questions were intended to measure beliefs about intelligence *in general*. Or perhaps some students assumed the questions were referring to their beliefs about themselves and not their beliefs about general intelligence. This kind of measurement error would need to be addressed in future studies.

Another explanation for any of the changes in beliefs could come from the social desirability of certain answers. By the end of the five weeks, it is possible that some students may have circled certain responses in order to impress or satisfy the teacher. But there are two reasons why this is probably not the case. First, both achievement and effort scores significantly increased form pre- to post-program. This provides evidence that even if students had grown to like the teacher enough to falsify answers on their survey, it is unlikely they all, as well, “decided” to get a higher score on the post-assessment. Second, in a study where both entity and incremental options were given to students on a survey, children showed “a tendency to endorse incremental statements,” despite actually believing different things about the malleability of intelligence (Erdley & Dweck, 1993). Though this suggests that students may generally have ideas about what they should or should not believe about intelligence, on this survey the only two non-significant changes *included* beliefs about intelligence.

Clearly the small sample size of the students is problematic, and could partly explain why not all changes in beliefs scores were significant or that those that were small in magnitude. Perhaps if these latter changes had been larger, we would have seen a significant change in beliefs 3 and 4 as well. If this were true, it would imply that certain changes in certain beliefs about mathematics are correlated with changes in students’ beliefs about intelligence.

It is not surprising that the change in achievement was correlated to some degree with change in effort. This could explain the change in achievement entirely, in fact. Perhaps if students had actually grown more confident in some way over the course of the five weeks in trying problems, even tough ones, their willingness to try the post-assessment problems had increased, and not necessarily their problem-solving ability. The correlation between changes in beliefs and achievement, though weak, provides some support for the argument that it was not effort entirely that might explain achievement gains. Though no causal claims of any kind can be made from this data, this finding provides impetus to examine the connection between changes in these beliefs, due to this kind of curriculum, and changes in achievement. Looking at the scatterplots from Diagram 4, for example, it may be true that above a certain level of achievement there is an interaction between achievement and beliefs such that a high level of achievement is associated with a high change in belief (the 4 plots in the upper right hand corner). It would be interesting to manipulate belief in a math class without doing any problems (e.g. work to convey to the class they are special, good problem-solvers, intelligent, etc. in other ways) and see if problem-solving still improves.

## Endnotes

1. The concept *beliefs* is defined here as “the personal assumptions from which individuals make decisions about the actions they will undertake” (Kloosterman, Raymond, & Emenaker, 1996).

2. For example, one student had failed math the year before in a class taught by one of the summer school teachers. The registrar had randomly placed him in her class for the summer program, so he was manually switched into another teachers’ class with hopes that exposure to a different teacher might help motivate him to do better.

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# STUDYING UNDERGRADUATE MATHEMATICS: EXPLORING STUDENTS' BELIEFS, EXPERIENCES AND GAINS

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*This study explores undergraduate students' self-reported gains in college mathematics courses and how these relate to the inquiry-based learning (IBL) methods used in their courses. These experiences were studied against the qualities and development of students' beliefs, motivation and strategies for learning and solving mathematical problems. Pre- and post survey data during one semester-long course showed that students found the instructional practices beneficial and reported cognitive, affective and social gains due to the course. Clear positive correlations appeared between students' gains and their experiences of instructional practices as well as between the gains and their beliefs, motivation and strategies. Moreover, positive changes between pre- and post-surveys in beliefs, motivation and strategies indicate the positive impact of the classes on students' perceptions and practices in studying college mathematics.*

## Background

The important role of beliefs, affect, and motivation in learning mathematics is well acknowledged in mathematics education research. Students' mathematical beliefs and attitudes have powerful impacts on their engagement and achievement, especially on problem solving. Students' beliefs about the nature of mathematical knowledge and skills, about mathematical problem-solving, and about their own mathematical capability, often determine their level of attendance and learning. Negative attitudes and emotions, together with inadequate self-regulatory behaviors, are often connected with students' preventive beliefs and perceptions in mathematics learning situations (DeBellis & Goldin, 2006; Malmivuori, 2001, 2007; McLeod, 1992; Schoenfeld, 1992). Such beliefs and behaviors derive from students' previous classroom experiences, both positive and negative; they are highly stable and difficult to change (e.g., Bishop, 2001; Cobb, Yackel & McCain, 2000).

Students who choose to study college mathematics differ from those studying secondary and high school mathematics, as does their learning context. College students who study mathematics as their major or minor subject usually show positive attitudes towards mathematics and high motivation, but nonetheless have varying goals for their study of mathematics and varying beliefs about mathematics and mathematical problem solving, for example in their beliefs about mathematical proofs (Selden & Selden, 2007; Sowder & Harel, 2003). Other students, required to take mathematics courses for another major, may exhibit less positive attitudes. Pre-service teachers represent a third group of college students with distinct beliefs, goals and attitudes.

All these various personal, social and instructional aspects determine the context in which students develop knowledge, beliefs and attitudes. Active teaching approaches all share goals of engaging students in their own learning processes and activating their responsibility for their own learning (Prince, 2004). Inquiry-based learning (IBL) is one such approach. Closely related to discovery learning or guided discovery (Bruner, 1961; Dewey, 1938) and problem-based learning (e.g., Savin-Baden & Major, 2004), IBL provides opportunities for students to engage in knowledge creation and argumentation (Rasmussen & Kwon, 2007) and promotes problem solving skills, independent thinking and intellectual growth (Buch & Wolff, 2000; Duch, Gron &

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Allen, 2001). In addition, pedagogical practices that emphasize cooperative learning are seen to foster student dialogue, build positive interdependence within groups, and promote higher-order thinking (Gillies, 2007; King, 2002). Such instructional approaches offer a context that clearly differs from the lecturing, exams and transmittance model of providing content knowledge that are traditional in college mathematics courses. This study explores undergraduate students' beliefs, experiences and learning gains in this kind of an active instructional context.

## Objectives of the Study

This study reports preliminary findings on undergraduate mathematics students' beliefs, motivation, strategies and gains with respect to their experiences of instructional practices during one semester of college mathematics applying IBL instructional methods. The main focus is on students' beliefs, goals and experiences students while studying college mathematics, and how these are related to their achievement, measured in the form of self-reported learning gains. The results also show changes in students' beliefs, motivation, and strategies during their IBL mathematics course, and, further, how these changes are related to their learning gains.

## Methodology

*Subjects of the Study*

Survey data was gathered from undergraduate students studying mathematics in four different US research universities during one term. The students represent mostly general math major and minor students in advanced mathematics courses; three sections of elementary and secondary pre-service teachers were also included. A structured, paper-and-pencil questionnaire was administered in the beginning and end of each course. The data derive from 13 different IBL sections with 192 students responding to the pre- and post-surveys. Four additional IBL sections were included in the analysis of post-survey data, for a total of 233 students.

*Instrument*

The first part of the survey was constructed on the basis of theory about mathematical beliefs, affect, goals and strategies of learning and problem solving. It was tested and revised using item analysis. The seven sections measured students' interest in and enjoyment of mathematics, goals in studying mathematics, learning and problem solving actions taken while doing mathematics, and beliefs about learning mathematics, problem solving, and proofs. Responses varied on a seven-point Likert-scale (e.g., from "not at all important" to "extremely important").

The post-survey included the same items, plus six additional sections. Four sections asked about students' experiences of instructional practices: how much various practices helped their learning, on a five-point scale from "no help" to "a great help." Two sections measured students' learning gains in understanding, attitudes, confidence and capabilities, on a five-point scale from "no gain" to "great gain". These sections were based on the SALG instrument (Student Assessment of their Learning Gains; SALG, 2008), which was developed to enable faculty and program evaluators to gather formative and summative data on classroom practices. Both the pre- and post-surveys gathered information on students' personal and mathematical background.

*Variables and Data Analysis*

New composite variables were constructed on the basis of the designed scales and exploratory factor analyses: 17 measures of beliefs, motivation, affect and strategies; 5 measures of instructional practices; and 4 measures of gains (see Table 1). Scores varied between 1 and 7 (where 4 points to a neutral or average view) or between 1 and 5 (where 3 refers to moderate help or gain). Reliability scores for these scales varied between 0.64 and 0.96. Statistical analysis included descriptive statistics, correlation analysis, and parametric tests (T-tests, ANOVA).

Table 1. *Composite Variables Measuring Student Beliefs, Experiences and Learning Gains*

| **Variable** | **items** | **scale** | **Emphasis of items** |
|---|---|---|---|
| Motivation | | | |
| *Interest* | 3 | 7 | Interest in learning and discussing mathematics |
| *Math major* | 1 | 7 | Desire to graduate with a math major |
| *Math future* | 2 | 7 | Desire to pursue math in future work or education |
| *Teaching* | 1 | 7 | Desire to teach math |
| Enjoyment | 7 | 7 | Pleasure in doing and discovering mathematics |
| Goals for studying math | | | |
| *Intrinsic* | 4 | 7 | Learning new ways to think & to apply math |
| *Extrinsic* | 4 | 7 | Meeting requirements; degree, good grades |
| *Communicating* | 2 | 7 | Communication of mathematical ideas to others |
| Beliefs about learning | | | |
| *Instructor-driven* | 5 | 7 | Exams, lectures, instructor activities |
| *Group work* | 3 | 7 | Whole-class or small group work |
| *Exchange of ideas* | 3 | 7 | Active verbal interaction with other students |
| Beliefs about problem-solving | | | |
| *Practice* | 2 | 7 | Repeated practice, remembering |
| *Reasoning* | 5 | 7 | Rigorous reasoning, flexibility in solving |
| Beliefs about proofs (Yoo & Smith, 2007) | | | |
| *Constructive* | 4 | 7 | Process view; revealing mathematical ideas |
| *Confirming* | 4 | 7 | Product view; recall and confirming conjectures |
| Strategies | | | |
| *Independent* | 4 | 7 | Finding one's own way to think & solve problems |
| *Collaborative* | 4 | 7 | Seeking help, actively sharing with others |
| *Self-regulatory* | 6 | 7 | Planning, organizing, reviewing one's own work |
| Experience of classroom practices (what helped me learn) | | | |
| *Overall* | 7 | 5 | Teaching approach, atmosphere, pace, workload |
| *Active participation* | 5 | 5 | Personal engagement in discussion & group work |
| *Individual work* | 4 | 5 | Studying & problem-solving on one's own |
| *Assignments* | 8 | 5 | Nature of tests, homework, other assigned tasks |
| *Personal interactions* | 6 | | Interaction with peers & instructor, in/out of class |
| Learning gains | | | |
| *Mathematical thinking* | 4 | 5 | Understanding concepts, how mathematicians think |
| *Application* | 3 | 5 | Applying ideas elsewhere, understanding others' ideas |
| *Empowerment* | 10 | 5 | Confidence to do math, appreciation, persistence |
| *Working with others* | 3 | 5 | Working with others, seeking help |

## Results

*Experiences of Instructional Practices and Gains*

Table 2 shows the average ratings of students' experiences of the helpfulness to them of various classroom practices and of their self-reported gains due to participating in a college IBL mathematics course. These are reported separately for each campus with more than one section and advanced mathematics students (CM1-CM3) are distinguished from pre-service teachers at one campus (CT4).

Table 2. *Average Ratings of Experiences of Classroom Practices and Self-reported Learning Gains*

| | | CAMPUS | | | |
|---|---|---|---|---|---|
| | | CM1 | CM2 | CM3 | CT4 |
| Classroom practices (5-point scale) | *overall* | 3.99 | 3.43 | 3.56 | 2.56 |
| | *active participation* | 4.18 | 3.84 | 4.05 | 3.65 |
| | *individual work* | 4.09 | 3.94 | 3.77 | 3.32 |
| | *assignments* | 3.87 | 3.36 | 3.42 | 2.63 |
| | *interactions* | 4.14 | 3.89 | 3.95 | 3.57 |
| Learning gains (5-point scale) | *mathematical thinking* | 4.37 | 3.89 | 4.01 | 2.72 |
| | *application* | 3.19 | 3.03 | 3.45 | 2.26 |
| | *empowerment* | 3.72 | 3.45 | 3.59 | 2.41 |
| | *working with others* | 3.66 | 3.39 | 3.92 | 2.86 |

** N = 34-77 for each campus.*

The averages in Table 2 reflect that students on each campus found IBL instructional practices helpful to their learning. Averages near or above 4.0 further indicate that many students experienced great help due to their participation in the IBL class. Students reported the most help to their learning from active participation and interaction during class work, while the particular assignments were least helpful.

Table 2 further shows that students' experiences slightly varied between the campuses. Pre-service teachers (CT4) found less benefit in the instructional practices than other undergraduate mathematics students. Lower ratings on the overall approach to teaching and learning and class atmosphere may reflect what students reported (in separate interviews) to be high workloads. This was valid especially for pre-service teachers, but also appeared among the advanced mathematics students. Campus 1 students (CM1) seemed to find their classroom experiences slightly more helpful than students at other campuses, but the differences were not large and probably reflected real variation in the actual instructional practices among classes.

Averages for the four learning gain variables show that students reported moderate or good gains due to their course work. Again, averages near and above 4.0 indicate that many students felt they made great gains in their IBL class. Among all groups, the highest gains were reported in understanding mathematical thinking and concepts, with lower gains in understanding how the course ideas were applied outside mathematics or how to make mathematics understandable for other people. Again, pre-service teachers reported weaker gains than did advanced mathematics students; relatively speaking, they reported stronger gains in learning to work well with others.

Campus 2 students (CM2) reported lower gains than other students, consistent with their slightly less positive reports of benefits from the instructional practices in their courses. On the other hand, students at Campus 3 (CM3) had the highest gains in working with others. Unlike Campus 2 students, they found interaction and active participation more helpful than individual work. These variations among student groups may thus reflect the differences in actual instructional practices, consistent with our (separate) observations of classroom sessions. For example, individual work may have been more emphasized at Campus 2 than elsewhere.

Analysis of the correlations among the instructional practices and learning gains for all students revealed statistically significant positive correlations between all the variables (from 0.297 to 0.749). That is, those students who experienced various class practices and interactions as clearly helpful also reported higher gains from their class, and vice versa. These correlations indicate that students found IBL instructional approaches beneficial to their learning.

*Beliefs, Motivation, and Strategies*

Table 3 lists the averages on scales for mathematical beliefs, motivation and strategies for learning and problem solving. The results concern advanced mathematics students (not pre-service teachers) in 10 IBL sections at four campuses who took both pre- and post-surveys.

Table 3. *Averages for Measures of Beliefs, Motivation, and Strategies for Pre- and Post Surveys*

| | | AVERAGES (7-point scale) | | |
|---|---|---|---|---|
| VARIABLE | | Pre survey | Post survey | Sig. level |
| Motivation | *interest* | 4.68 | 5.08 | ** |
| | *math major* | 4.49 | 5.12 | * |
| | *math future* | 6.20 | 6.19 | |
| | *teaching* | 3.70 | 4.27 | * |
| Enjoyment | | 5.34 | 5.56 | |
| Goals | *intrinsic* | 5.78 | 5.79 | |
| | *extrinsic* | 5.28 | 5.36 | |
| | *communicating* | 5.18 | 5.50 | * |
| Beliefs about learning | *instructor-centered* | 5.14 | 4.90 | * |
| | *group work* | 4.67 | 4.98 | * |
| | *interaction* | 5.26 | 5.35 | |
| Beliefs about problem solving | *practice* | 4.88 | 4.83 | |
| | *reasoning* | 5.27 | 5.45 | * |
| Beliefs about proofs | *constructive* | 5.65 | 5.85 | * |
| | *confirming* | 4.96 | 4.77 | |
| Strategies | *independent* | 5.23 | 5.55 | ** |
| | *collaborative* | 4.46 | 4.97 | ** |
| | *self-regulatory* | 5.10 | 5.20 | |

** N =112-184; p < .05*, p < .01**.*

Table 3 indicates that these students had rather strong motivation and promotive beliefs that emphasized rigorous reasoning, flexibility and construction in problem solving. They stressed the value of interaction in learning and were motivated both by intrinsic and extrinsic goals and an ability to communicate about mathematics. In their preferred strategies for learning and problem solving, students emphasized both individual work and collaboration and reported a high level of self-regulatory activities.

Despite these rather high initial averages, some general changes between the pre- and post-surveys could nonetheless be observed in students' beliefs, motivation and strategies. Most of these involved increases in the strength of students' beliefs about the importance of collaboration and group work in studying mathematics, and in their motivation and their use of effective problem-solving strategies. Females showed increases from pre- to post-surveys more often than males. Students reported more use of both individual and collaborative ways to learn after the IBL course. They also reported higher interest in mathematics in general, in graduating with a college math major, and a slightly higher likelihood that they would teach mathematics in the future. Moreover, the observed changes showed some growth in a constructive view of proving and in seeing the importance of rigorous reasoning and multiple approaches in solving math problems—views more consistent with mathematicians' views. Finally, students showed declines in a transmittal view of learning that emphasized instructor explanation and seeing similar examples to their homework. All these changes suggest a positive impact of IBL classes on students' perceptions of mathematics and on their practices in studying college mathematics.

*Connections Between Beliefs, Motivation, Strategies and Gains*

Correlations were computed between the constructed post-survey variables and gains for all four campuses. Table 4 displays the strongest correlations between the four gain scales and the post-survey scales on beliefs, motivation, and strategies for learning mathematics.

Table 4. *The Strongest Correlations between Learning Gains and Post-survey Beliefs, Motivation, and Strategies, among all Students*

| VARIABLE | | GAINS | | | |
|---|---|---|---|---|---|
| | | Mathematical thinking | Application | Empowerment | Working with others |
| Motivation | *interest* | 0.515 | 0.338 | 0.466 | 0.220 |
| Goals | *intrinsic* | 0.505 | 0.371 | 0.469 | 0.261 |
| Enjoyment | | 0.564 | 0.346 | 0.489 | 0.182 |
| Beliefs about learning | *interactions* | 0.272 | 0.307 | 0.343 | 0.296 |
| Beliefs about problem solving | *reasoning* | 0.421 | 0.319 | 0.327 | 0.247 |
| Beliefs about proofs | *constructive* | 0.412 | 0.289 | 0.289 | 0.222 |
| Strategies | *independence* | 0.452 | 0.249 | 0.351 | 0.226 |
| | *self-regulation* | 0.359 | 0.320 | 0.415 | 0.355 |

* *All correlations are significant at the level $p < .01$* ** *$N = 197$-$222$*

Table 4 shows clear positive connections between students' cognitive, affective, and social gains and their beliefs, motivation and learning strategies. The strongest correlations related to students' gains in understanding mathematical concepts and thinking, and to their gains in empowerment: confidence, positive attitude, persistence, and ability to improve their own mathematical capacity. High interest, enjoyment and intrinsic goals for learning college mathematics were the most positively related to learning gains. Interestingly, these motivational connections were clearest among pre-service teachers—among whom interest and enjoyment were initially lower. Students' preference for independent thinking strategies was most strongly linked to higher gains in mathematical thinking, while strategies of active self-regulation in solving problems were most clearly linked to higher empowerment.

Students' gains in mathematical thinking were linked to beliefs about the importance of rigorous reasoning and flexibility in solving problems, and to constructive views of mathematical proof. The belief that it is important to have active interaction with other students when learning mathematics was also clearly related to cognitive, affective and social gains. Unsurprisingly, this relation was strongest for students' gains in willingness and ability to work with others.

Comparisons between Tables 3 and 4 indicate that the strongest correlations to gains were represented by those variables that also showed increases from pre- to post-surveys. This applied in particular to students' interest and independence of learning but also to the importance they attached to rigorous reasoning and flexibility in solving mathematical problems and to constructive views of proofs. Positive changes in these views and approaches may have a critical role in affecting students' learning gains, not only among these students in IBL classes but also among college mathematics students more generally.

## Discussion

The role of beliefs, affect, and motivation has been widely studied in secondary school contexts, but less so among college mathematics students. Teaching methods applying inquiry-based learning with active collaboration create a context different from traditional college mathematics instruction. Active engagement of students in their own learning processes, with responsibility, collaboration, and creative use of personal resources, is seen to enhance growth of thinking and problem solving (Prince & Felder, 2007) and social skills (Duch et al., 2001; Jordan & Metais, 1997). Promotion of cognitive, affective and social skills in such learning contexts may then be reflected in students' beliefs, experiences, and activities (e.g., Kwon, Rasmussen, & Allen, 2005; Smith, 2006). Our preliminary findings point to such positive impacts.

The advanced math students in this study began with high motivation and adequate beliefs about mathematics learning and problem solving. However, participation in an IBL mathematics course seemed to further promote these views and approaches. Students gained interest in and motivation to study mathematics and exhibited less belief in rote learning methods. They attached greater importance to group work and active collaboration, and to communicating mathematical ideas. Their choice of problem-solving strategies emphasized both independent thinking and collaboration. After taking an IBL course, they saw rigorous reasoning in solving problems as more important and reflected a more process-based view of mathematical proofs.

Students reported rather high cognitive, affective and social gains due to their participation in an IBL course and reported IBL classroom practices as helpful, especially their own active participation and interaction during the class work. Pre-service teachers reported lower gains and less benefit from IBL learning approaches than advanced math students.

The results also showed important, direct connections between these cognitive, affective, and social gains and students' beliefs, motivation, and use of particular learning and problem solving strategies. Higher interest, enjoyment, and intrinsic goals for learning college mathematics were most clearly connected to higher gains. The strongest connections to gains appeared in the beliefs, motivation and strategies that also showed increases during an IBL course.

Limitations to the study include a rather low sample size and large variation in the instructional practices, nature of courses, and in student backgrounds within and between the campus sections. More data are needed to confirm these preliminary findings. However, the results indicate that use of active instructional methods with inquiry and collaboration represents a learning context that may have powerful effects on students' learning and positive attitudes toward college mathematics. Future study will show how these gains and experiences vary between different student groups, especially with respect to gender and in comparison with students experiencing more traditional college mathematics teaching.

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# STRATEGIZING DEVELOPMENTAL MATHEMATICS AT A LARGE AND DIVERSE STATE UNIVERSITY

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*This study investigated the current status of developmental mathematics at a large and diverse state university, as it existed within national practices and recommendations. Findings suggest that students' attitudes toward mathematics and learning, the time they are willing to spend, knowledge of how to learn mathematics, and learning experiences within remedial courses all play a critical role in their academic success. Strategies for improving the quality of student learning in developmental mathematics are proposed.*

## Objectives and Research Perspective

Statistics indicate that almost sixty percent of students who enroll in community colleges must take developmental mathematics before entering college-level coursework (Schwartz, 2007). This statistic is also alarming at the university level, primarily at universities intended for non-traditional and local student populations. For example, the largest proportion (80%) of 1st-year college students taking a developmental course at public, 4-year institutions in 2000 took developmental mathematics (Duranczyk & Higbee, 2006). When students face remediation, it means they have a longer road to completing their mathematics degree requirements. Many give up before they finish the sequence of courses. Additionally, taking remedial courses the entire first year is costly, as well as delays graduation for those needing mathematics courses. Time for remediation can also dissuade students from seeking majors requiring mathematics. Alternative pathways are needed to help facilitate students' expeditious and successful completion of remediation, as well as help students arrive on university campuses ready for undergraduate mathematics. One common theme in this literature on instruction in developmental mathematics courses is that no single set of practices will be effective with every student (Biswas, 2007; Schwartz & Jenkins, 2007). There is a broad consensus in the literature that educators ought to take a holistic approach to developmental education.

Our university has a highly diverse student population of nearly 38,000. Each year approximately one-third of the first-time freshmen need to take and pass pre-baccalaureate developmental mathematics courses before moving on to the next level (general education). Some of the students taking developmental mathematics struggle to successfully complete it in one year; approximately 30% of them do not return for a second year. In order to understand the current situation of developmental mathematics within our Department (Mathematics and Statistics), as well as propose strategies for improving the experience and completion rates of our developmental students, we embarked on a research study. Our Department and the larger University were eager to carefully examine the current situation of the program after recent changes to both placement and course structure, and strategize future possible changes, including specific pilot studies.

Two primary research questions guided our work:

1. What is the current situation of developmental mathematics at our university?
2. What practically should be implemented over the next two years to improve the existing program?

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## Methodology

As neither of us had previously conducted research in the area of remedial mathematics, we began by casting a large net across both the existing literature and other programs. We sought to accomplish the following: gain knowledge of what other institutions have tried and are currently trying, generate a portrait of what existing literature tells relative to the design and implementation of effective developmental mathematics programs, compile other existing data and completed analyses from our Department and the campus community, examine students' experiences, expectations, attitudes and success in developmental mathematics courses, and gather insights from key individuals and other departments on campus on how to handle the difficult process of developmental mathematics.

Our primary data sources included:

- Relevant literature in developmental mathematics and mathematics education;
- Website searches (California State University campuses, similar universities in California and across nation, etc.);
- Email communication with colleagues at universities and community colleges across the nation;
- Existing data (Department, University);
- Surveys administered to developmental mathematics students at the beginning and end of the Fall 2008 semester (380 responses), a subset of students who had just finished developmental mathematics in Spring 2008 (28 responses), and developmental mathematics instructors (6 responses);
- Interviews with individuals (10) within our Department, Department of English, and University administrators;
- Town meetings (3) with key individuals across campus (e.g., higher administration, academic advisors).

Student surveys contained both open-ended and Likert-type questions (1-6 scale) and asked about their expectations for and experiences within the developmental mathematics courses, their anxiety and attitudes toward mathematics and mathematics learning, their level of skill preparation (pre- and post-course), and impressions of course and instructor. Instructor surveys included only open-ended questions regarding their experiences teaching developmental courses, observations of their students' learning and needs, and recommendations for our Department and future iterations of all courses they had taught. All interviews were semi-structured and informal in nature.

## Results

*Key Findings from Relevant Literature*

Two fundamental factors have proven to be crucial to student success in mathematics remediation: (1) the existence of an early alarm system for remediation needs, and (2) a strong alignment between college mathematics placement tests and high school mathematics curriculum and assessments (Brown & Niemi, 2007; California Partnership for Achieving Student Success, 2008; Hill, 2008). However, students often do not perceive the need for intervention, and a student's pathway within a given high school context is dependent upon his/her success at the middle school level.

Once in a college or university environment, traditional approaches to instruction (e.g., lecture) in developmental mathematics courses do not engage students (e.g.,McGlynn, 2008). A focus on rote memorization of formulas and rules is a hindrance to students at the developmental level. Developmental mathematics courses need to focus on critical thinking and how to be an effective student of mathematics, and spend time and attention to enhancing the students' work, study habits and concentration skills. Non-traditional instructional methods in developmental mathematics education can work; for example, cooperative learning can lesson students' "fear of mathematics" and the "fear of failing" (e.g., NADE Mathematics Special Professional Interest Network, 2002, 2003). Guided instruction appears to be more effective for low performing students than structured instruction (e.g., Kroesbergen & van Luit, 2002). Students' negative attitudes and anxiety toward mathematics must be overtly addressed (Tobias, 1993).

Technology could be integrated where appropriate to encourage active learning and help students explore concepts and visualize real-world data. Computer-based instruction is most successful when it is used as a supplement to regular classroom activities in developmental courses. In order for technology to be effective, the student-teacher interaction must be the main form of formal instruction with technology "acting as a conduit" for this to take place effectively (Robinson, 1995).

Students in developmental courses need to be motivated and inspired, hence the role of teacher is critical. Teachers must listen carefully to students and meet them at their level or place of cognition, and attack students' mathematics anxiety–one good strategy is to instill a participatory, non-threatening classroom environment (e.g., Ironsmith, Marva, Harju, & Eppler, 2003). Teachers must have training in and appropriate attitudes toward working with this special population of students. Research suggests that teachers' attitudes impact instruction and student achievement (Richardson, 1996; Thompson, 1992).

*Prevalent Models and Strategies*

Across higher education institutions in the United States, we have isolated three models for developmental mathematics courses that are most prevalent:

*(1) The Supplement Model*. This model retains the basic structure of the traditional course, supplements lectures and textbooks with technology-based, out-of-class activities, or small-group or individual tutoring.

*(2) The Replacement Model*. This model reduces the number of in-class meetings and replaces with out-of-class, online, interactive learning activities in computer-labs. The in-class meetings could be combined into large sections and team-taught by multiple instructors.

*(3) The Emporium* (*or Fully-online*) *Model*. This model replaces all class meetings with a learning resource center featuring online materials, multi-media resources, commercial software, on-demand personalized assistance, automatically evaluated assessments with guided feedback and alternative staffing. Course material is often organized into modules, which students will complete at individualized paces.

Within any model, there are recommended considerations for the most valuable developmental mathematics experience. These include: (1) Proper placement of students through the administering of placement tests into appropriate courses is essential to student success; (2) A clearly defined philosophy for the program, accompanied by clearly specified goals and objectives; (3) On-going communication among those who teach developmental courses with centralized supervision; (4) A strong counseling component; counseling components should be integrated into the overall structure of the developmental program and be carried out by counselors specifically trained to work with developmental students; and (5) Tutoring for students; tutoring is most effective when the tutors are trained to work with students at developmental levels.

*Major Findings from Student Surveys*

In September 2008, we conducted surveys with two groups of CSULB students: (1) Those students who had just finished taking an intermediate algebra course in Spring 2008 and were currently enrolled in a general education course (on our campus intermediate algebra is the second level of developmental mathematics course), and (2) Those students who had just begun intermediate algebra. In December 2008, an end-of-course follow-up survey was conducted with the students in the second group. Some common themes were identified across the three groups of responses:

- Over three-quarters of the students took two semesters of mathematics in the senior year in high school;
- On a scale of 1 to 6 (6 highest), 70% of the students entering intermediate algebra chose 3 or 4 in rating the levels of their own mathematics skills and enjoyment. At the end of the course, such percentage increased to 80%;
- Almost all of the students thought the state-level placement exam was important or necessary, but did no or very little preparation for it;
- The majority of the students felt the developmental mathematics placement was fair or necessary, with a few expressing surprise over the placement;
- Students did not believe they were informed of various opportunities and resources available;
- Besides the in-class time, 80% of the students spent only 2 – 4 hours per week on studying for the course. More than half of the students believed they should have spent more time for better understanding;
- As a result of taking the course, the majority of the students expressed that they were "somewhat" or "a little" better in mathematics. Many felt that this class reviewed or "refreshed" what they had studied in high school;
- Over 90% of the students believed the class met their expectations, and the instructor and instruction were helpful over the semester;
- On the December 2008 end-of-course survey, around 90% of the students felt "somewhat" or "very" confident in passing the class, which showed increased confidence levels than in September when they just started.

*Lessons Learned from Interviews with Key Individuals*

Interviews revealed that many aspects of existing practice are critical to students' success in developmental mathematics. First, effort matters. For every additional hour that students study in our Department's remedial courses, the effect on their performances is significant. The passing rate for those who spend three hours per week in developmental courses is 61%. For those who spend four hours per week, the rate is 67%. And, for those students who spend five hours per week, the passing rate jumps to 78%. Students who are most successful in developmental mathematics courses understand the importance of attending class, studying, and completing developmental mathematics early.

Second, the most at-risk students need to be appropriately mentored; they have low or unclear personal expectations and drift easily. As part of this process, resources that help them study can be available, and they can be pushed to meet deadlines and encouraged in all aspects of their learning. As part of this process, mathematics information should be disseminated in the spring prior to students' arrival at the University. In general, we need to push these students to make preparations for learning and inform them of resources and supports.

Third, more tutoring should be made available to developmental students, with tutors who are experienced in working with this special student population.

Fourth, relative to instruction, students need to be: motivated through alternative approaches, evaluated through less formal assessments, and taught through a well-structured curriculum. Intermediate algebra instructors interviewed had varied views on the nature and goals of the course; goals should be aligned. One example existed within the English Department; their remedial sections have a common exit requirement (portfolio) to hold instructors accountable; instructors meet regularly and work collaboratively toward common goals.

Fifth and finally, the Department has provided ongoing, formal professional development for graduate teaching assistants; this training should include specific attention to this special population of students.

## Proposed Strategies

Based on the reality of mathematics instruction and practice in our Department and the aforementioned findings, we proposed the following strategies at town meetings across the University for improving students' success in our developmental mathematics program:

- Organize mathematics placement exam workshops for those graduating high school in the spring;
- Offer mathematics courses to selected incoming freshmen in the summer, in order to reduce the number of students needing remediation in their freshman year;
- Revise the developmental mathematics curriculum, particularly in intermediate algebra;
- Implement common structures, assessments, and final exams across all sections of a given course, particularly intermediate algebra;
- Increase course time each week in intermediate algebra by one hour for individual or group tutoring or Supplemental Instruction sessions;
- Assign experienced graduate teaching assistants (as opposed to part-time instructors) to teach developmental mathematics courses, thereby allowing for a more aligned program;
- Provide ongoing professional development to teaching assistants and tutors (with respect to mathematics content, pedagogy, student needs, etc.);
- Identify and provide additional support to students with special needs, such as double pre-baccalaureate students (those needing remediation in both mathematics and English) and student athletes.

Based on feedback relative to our proposed strategies, we suggested to the Department and University a number of pilot studies that could examine changes to our existing Developmental Mathematics Program. Highlights include:

(1) Provide 1-2 daylong placement exam workshops for the State-level exam to high school seniors on campus in Spring on the Saturday prior to the testing dates. Garner information (pre-test, survey) during first hour in order to sort students by ability. The objectives are to help participants prepare for the exam, gain insight into participants' expectations for and attitudes toward the exam and mathematics at the University, and decrease the number of students needing developmental mathematics in fall. Data would include pre-post student tests, pre-post student surveys, student interviews, and placement exam scores.

(2) Revise the intermediate algebra course curriculum to better reflect the nature of the course and these students' needs. As part of this process, generate common assessments (e.g., final exam). The goal is to develop a curriculum that is more appropriate to developmental level students, as well as assessments that are required by all instructors,

with improvement in students' passing rates. Data would include artifacts (e.g., assessments), detailed record of curriculum development, instructor surveys, and passing rates of future course sections.

(3) Expand the number of developmental courses offered in summer and actively recruit from surrounding districts to target students who have just missed the placement exam cut-off score to bypass remediation. The goals are to decrease number of students needing developmental mathematics in fall and better understand the background and attitudes of students selecting this option. Data would include pre-post student surveys, passing rates of these summer sections, and numbers of students completing remediation as a result of this offering prior to freshman year.

(4) Have graduate students teach as many sections of intermediate algebra as possible to provide structured and on-going support led by an assigned coordinator that addresses pedagogy, content, and special needs of students. The goals are to improve the students' passing rate and enhance the learning experience for all students. Data would include pre-post student surveys, students' passing rate, graduate student instructor surveys and interviews, and interviews with the graduate student coordinator.

(5) Increase the weekly course length of intermediate algebra by one hour for individual or group tutoring or Supplemental Instruction sessions (computer-based software, online system, human tutors or SI leaders). The expected outcomes include improved student passing rate and increased time to address affective and support the needs of students in this developmental course. Data would include pre-post student surveys, students' passing and completion rates, and instructor interviews.

## Discussion

Within this paper we carefully describe our methodology, as well as results from both of our research questions. Presenting our work in this manner allows it to serve as a model for how other institutions can examine their own developmental mathematics programs and make the changes/revisions necessary to address this nation-wide problem. While popular perception is that most students needing remediation either did not take mathematics in their last year of high school or did not successfully complete prerequisite courses, we have found, and the literature supports, that this perception is not accurate. Students' attitudes toward mathematics and learning, the time they are willing to spend, knowledge of how to learn mathematics, and learning experiences within remedial courses all play a critical role in their academic success. Developmental programs, particularly those serving diverse student populations, must examine both the content and pedagogy within courses and the structure by which students' learning is supported. More research of a long-term nature that qualitatively studies student success and learning experiences in developmental mathematics programs is needed.

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# A CASE STUDY OF RESILIENCE BASED ON MATHEMATICS SELF-EFFICACY AND SOCIAL IDENTITY

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*Despite statistics showing African-American students' less than average mathematics performance), there exists a population of African-American students with records of mathematics achievement. This paper highlights one such student enrolled at a two-year residential high school in the Southeast. Her academic strategies and ideology were uncovered to determine their influence on her mathematical success. This analysis is part of a larger study highlighting the academic and social strategies of mathematically successful African-American students attending Caldwell Academy and factors contributing to that success and social balance, and how both contribute to resilience. The study also investigated the lack of enrollment in higher-level mathematics classes among eligible African-American students.*

## Purpose

African-American students have historically not performed as well mathematically as their White and Asian counterparts (http://nces.ed.gov, 2007). However, at specialized schools like the one in this study, students generally enter on a much more level academic playing field than in traditional high schools based largely on the acceptance criteria. At Caldwell Academy, a specialized residential high school, African-Americans were not enrolling in higher level advanced mathematics course for which they were eligible.

The study developed as the result of concerns that African-American students at Caldwell academy were not enrolling in higher-level mathematics courses, despite their academic eligibility. To uncover the reasons why students were not taking courses, it became necessary to determine how the students felt about their mathematical abilities, then identify the academic and social strategies they employed to be mathematically successful. Reasons for not taking the courses were shared by students during final stages of the study. Shared here is the portrait of one African-American student who attributed her mathematics success to her work ethic; her ideology – one indicative of inclusion and success – is also perceived to be directly related to her mathematics success.

## Theoretical Framework

Resilience theory serves as the primary framework for the study. While resilience theory outlines the interventions, protective mechanisms, and coping strategies employed by students to surmount obstacles (Nettles, 1993), academic resilience focuses on the latter two (Floyd, 1996; Lee, 1991; Steward, 1996). Both high-levels of mathematics self-efficacy and positive social identity can be factors contributing to the academic resilience of African-American students. Resilient students have been shown both to possess high levels of self-perceived ability (efficacy) and to rely on others in their social circles for guidance and support (Lee, 1991; Wade & Oskeola, 2002). Without both strategies, students are less able to employ the coping strategies necessary for resilience.

Self-efficacy and racial and social identity development are secondary frameworks. Race can be used to predict mathematics self-efficacy (O'Brien, Martinez-Ponz, & Kopala, 1999). That

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self-efficacy, in turn, is strongly related to interest (and performance) in mathematics courses and STEM-based career fields. A high level of mathematics self-efficacy is the result of a student's belief in their own mathematics ability. It is generally perceived as task-specific (Hackett & Betz, 1989) and for the purposes of this study was addressed from a more general perspective in terms of mathematics courses which study participants believed they could take and successfully complete with a passing grade.

Positive social identity for African-American adolescents generally develops as they progress though the stages of *nigrescence,* as defined by Cross (1991). Cross' five stages include: pre-encounter, encounter, immersion/emersion, internalization, and commitment. The significance of race and how central it is to the identity of African-Americans can change during each stage of nigrescence. Race is not as significant during pre-encounter, but becomes much more so after on experiences some type of encounter that causes and African-American to become more aware of their race and how they are perceived by others based on race.

Though there is disagreement about the age at which encounter occurs, Tatum argues that it can occur as early as middle school (1997). During adolescence, African-Americans are thought to become more aware of their race as it pertains to mainstream society and in comparison (or contrast) to their peers. Encounter precipitates immersion/emersion where many adolescents become focused on race and associate strongly with Black culture to define them and prove their "Blackness." They also begin to negotiate between their sometimes contrasting school and home environments. In those situations, African-American adolescents may have one set of friends or peers in the classroom while they choose a social circle comprised of more African-American peers with whom they share experiences that are non-academic. In a residential high school, like Caldwell Academy, the students' school and home environments overlap and there may not be any distinction between those with whom they share academic and social experiences.

Internalization and commitment occur after one matures to allow incorporating the values and ideas of mainstream society without jeopardizing their racial identity. This occurs for many students in college after they are exposed to more opportunities to take African-American history classes and join Black organizations.

## Methodology

A qualitative study using phenomenological methods was used to capture the experiences of the study participants in their own words. Phenomenology allowed the researcher to use the students' experiences while inserting one's own understanding of the phenomenon to convey their experiences (Tesch, 1987).

The larger study was designed to determine what factors contributed to low African-American enrollment in higher-level mathematics courses at a specialized, residential, high school. The following research questions were posed to identify those factors:

1. How confident do African-American students feel about their ability to successfully complete mathematics courses?
2. What academic and social strategies do mathematically successful African-American students employ?
3. What type of shift occurs in the development of racial and social identity of African-American adolescents at a specialized, residential high school?

Caldwell Academy, the two-year residential school that served as the setting for the study, is a state-funded high school in the Southeastern United States. A member of the National Consortium of Specialized Secondary Schools of Science, Mathematics, and Technology, Caldwell Academy required a strong academic history and records of achievement from the applicants chosen to attend. Application to Caldwell is open to all state residents and students are chosen during the spring of their sophomore year to attend the school during their junior and senior years. The admissions process was based on academic records, teacher recommendations, standardized tests scores, student activities, and an essay. Student enrollment at the time of the study was over 600 students. At a state-funded institution Caldwell Academy maintains an enrollment goal that reflects that composition of the state population. At the time of the study, African-American population in the state was 21.6% while 71 of the students at Caldwell Academy were African-American (less than 12%).

Participants in the study were chosen by criteria sampling based on their eligibility to enroll in mathematics courses beyond Calculus. At Caldwell Academy, a mathematically successful student was one who earned a B- or higher in their Precalculus their junior year, as well as those who entered the school enrolled in Calculus or higher. This allowed the student met the prerequisites for mathematics courses that were considered more advanced and required a stronger mathematics foundation.

Each study participant completed a mathematics autobiography, racial identity assessment (developed by Sellers, etc.), and a semi-structured individual interview. Questions from the autobiography and select items from the interview addressed mathematics efficacy and academic strategies. Other interview items addressed the social strategies the students employed to remain successful at Caldwell Academy, including their social circles and support systems. The racial identity assessment was used as an added dimension to interpret students' reported strategies and responses to interview questions.

The mathematics autobiography was designed to help students identify when and how their confidence in their individual mathematical abilities developed. By responding to items such as "When did you first realize you were good at mathematics?" and "Who helped you value mathematics?" students reflected on their relationships with mathematics and the events contributing to their levels of mathematical self-efficacy. The mathematics autobiography also asked that they share a strong (positive or negative) mathematics memory and the mathematics teacher they considered to be their favorite.

The racial assessment used for the study was the Multidimensional Inventory of Black Identity (MIBI) developed by Sellers, Smith, Shelton, Rowley, and Chavous (1998). The MIBI is based on their Multidimensional Model of Racial Identity (MMRI) which creates a composite model of racial identity combining the universal properties of African-Americans with the specific qualitative nuances of individual African-Americans. Sellers, Smith, Shelton, Rowley, & Chavous identified four dimensions that collectively define African-Americans while acknowledging individual experiences. The four dimensions of the MMRI are salience, centrality, regard, and ideology. Salience, how aware a person is of their own racial identity, can be situation-specific and is considered less stable than the other three dimensions. The remaining dimensions, centrality, regard, and ideology serve as the subscales of the MIBI. Centrality measures how much race comes before other aspects in regard to an individual's sense of identity and self. How an individual feels about Black people and how they believe others perceive Black people, are personal and private regard, respectively. Ideology has four different categories to identify in which one would characterize their perspectives on African-American in America and where their priorities lie in relating to other African-Americans, minorities, American, and humans. The ideology subscales are not exclusive and many African-Americans demonstrate some form of each. The items from each subscale are presented in a 7-point Likert scale and are averaged to produce a unique profile of each participant.

The individual interviews were conducted after study participants completed the mathematics autobiography and the racial identity assessment. The items from the interview were a combination of those created during a pilot to address the students' academic and social strategies and other questions that were developed specifically for each participant as a result of the previous two instruments. Each student was asked to describe academic and social experiences at their home schools prior to attending Caldwell Academy, and then describe the same experiences after enrollment. The students were asked to about their perceived mathematics ability, effort, and performance at their home schools and at Caldwell Academy. They were also asked to describe the social circles they created at both schools.

## Findings

Jackie was one of 71 African-American students enrolled at the time of the study, with approximately 20 in the senior class. Her initial mathematics placement exam placed her in Precalculus. Because she completed Honors Precalculus at her home school Jackie requested to be re-tested. Based on her second score, she was enrolled in Calculus as a junior at Caldwell Academy. Enrolling in Calculus during her junior year meant Jackie could meet the mathematics graduation requirement by taking *any* mathematics courses during her senior year. She was the only student in the study enrolled in higher level mathematics courses.

Jackie's individual portrait highlighted her strong self-concept and diverse social circle. She credited her mother for her developed confidence in her mathematical abilities. According to Jackie, her mother helped her realize her mathematical talent, while her teachers were surprised. She was constantly and consistently encouraged by her mother, despite the fact that others doubted her academic ability. The constant encouragement helped her feel strongly about being re-tested. In describing her own perception of her mathematics ability in her autobiography, Jackie stated:

> I liked to learn anything when I was younger, math was an easier thing to learn because I soon realized that there were formulas and tricks. I never considered myself a "math person" but I knew I was good at it. I liked the cut and dry, right/wrong angle to math.

The MIBI revealed Jackie's unique experience as an African-American and could be used to add dimension to her responses to other assessment instruments. Her centrality was a bit higher than average for study participants, suggesting that being Black was important to her, but not the only thing that defined her. Jackie's private regard was higher than average, while her public regard was lower than average. Of the ideology subscales, Jackie scored highest as an assimilationist, suggesting that she believed in working within the mainstream American system to be most effective in changing it. Her ideology also explained her perspective on taking advantage of opportunities available.

Jackie revealed a multi-faceted definition of success during her interview. Although she perceived the majority of the students at Caldwell Academy to be focused primarily on academic endeavors, her interests varied and she believed that a student with few or no other activities should always be academically successful. In contrast to the academics-only model, Jackie embraced a definition of success based on involvement in activities outside of the classroom as well. She had interests that included athletics, music, and performing arts and chose to surround herself with others possessing similar interests.

Jackie perceived her academic ability to be as strong after attending Caldwell Academy as she did prior to enrollment. However, she acknowledged that she had to spend more time working to understand mathematics at Caldwell Academy and that her performance was not as great. She described spending down time during basketball practice working on Calculus and going to see her instructor much more than was necessary at her previous school. Jackie believed that work ethic could contribute to mathematical success just as much as natural ability. Although she was the only African-American student enrolled in her any higher-level mathematics class, not just her own, Jackie did not place emphasis on that. She was accustomed to that scenario and focused more on her work. She also shared that mathematics was easier at Caldwell Academy after she discarded the idea that mathematics came from formulas and began to demonstrate a more conceptual understanding of mathematics, as opposed to the procedural understanding to which she was accustomed.

Social identity was the aspect that Jackie believed suffered more from being African-American at Caldwell Academy than her academic identity. She believed it was because she didn't feel compelled to spend time and create relationships with other Black students based on race alone; she needed more core values and interests in common with those in her social circle. As a result, her social circle was very diverse, both prior to enrollment at Caldwell Academy and while enrolled. The foundations on which she based her relationships allowed her to rely on her friends to help her through both academic and social challenges. Academically, Jackie thought students at Caldwell Academy could take advantage of the available resources to further develop their academic and personal interests.

## Discussion

Jackie's resilience appeared to be the result of a high level of mathematics self-efficacy and support systems – both familial and social. Jackie was the only student in the study enrolled in a higher level mathematics course although she could have taken less advanced courses to meet her graduation requirements. She enrolled in the mathematics courses for the academic challenge and relied on her work ethic to help her complete the courses with a passing grade.

Jackie possessed a sense of her mathematics ability of which she became aware by a series of mathematics successes. It was also consistently fostered by her mother's consistent belief in her. Jackie's previous mathematics experiences, support systems and ideology contributed to her strong work ethic and ability to create diverse social circles. She was accustomed to being the minority and was not as affected by that as others in the study. She attributed her success to her work ethic and was determined to be successful by taking advantage of academic opportunities.

As the US population becomes more diverse and the workforce becomes more mathematically driven, mathematic success is necessary to enter students in the pipeline for Science, Technology, Engineering, and Mathematics (STEM) fields. It then becomes necessary to identify the factors contributing to academic resilience. Once identified, these factors can be used to create nurturing learning environments that recognize and incorporate the unique challenges and needs of minority students, particularly those that vary from mainstream culture. Using Jackie's experiences help create a template for improving African-American participation in advanced mathematics.

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# THE INFLUENCE OF REACTANCE ON ELEMENTARY SCHOOL TEACHERS' WILLINGNESS TO CHANGE THEIR CLASSROOM PRACTICE

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*The gap between teachers' perceptions of how they teach mathematics and their actual classroom practice has been well-documented. To understand this disparity, data was collected for three years on pre-service and in-service elementary school teachers. The Dowd Therapeutic Reactance Scale (TRS) was used to identify psychological reactance that may predispose teachers' willingness to change, thus influencing their efforts to incorporate reform strategies into their practice. Data analysis shows that teachers tend to be low in reactance, implying a greater desire to impress and to be socially appropriate. Scores indicate a low locus of self-control and a preference for high levels of professional advice and structured support.*

## Background and Perspectives

In 2000, the National Council of Teachers of Mathematics (NCTM) revised their standards for teaching mathematics. Over 95 percent of teachers in the U.S. claim to be aware of the these standards and 70 percent indicated, when questioned, that researchers would find proof of standards-based teaching in their classrooms. However, very little evidence of reform instruction has actually been found when teachers were observed (Stigler & Hiebert, 1999). Ma found that most practicing elementary teachers in the U.S. believed mathematics was "an arbitrary collection of facts and rules in which doing mathematics means following set procedures step-by-step to arrive at answers" (1999, p. 123). Thus when teachers engage in pre-service methods courses at the university or in professional development courses later in their career that focus on "understanding concepts," they often face uncomfortable confrontations with their existing attitudes as well as their own knowledge of mathematics.

*Factors Involved in Changing Teachers' Beliefs, Attitudes, and Classroom Practice*

Reconstructing beliefs is more complex than providing teachers with standards-based curricula and workshops on implementation of those standards. As Schifter and Fosnot (1993) point out, "... significant and enduring change in the way teachers teach cannot be induced by a course of lectures, a handful of workshops, or even books...no matter how informative or persuasive." Instead, teachers change more readily "in ecologically embedded settings of real classroom practices, real students, and real curricula - elements that teachers define as central to their profession" (Confrey, 2000, p. 100). Learning occurs when teachers are given the opportunity to reflect on and communicate about the mathematical thinking of their students (Franke et al., 2001; Margolinas et al., 2005). They must examine their beliefs about how children come to know mathematics and discover ways to teach given that information.

These issues are compounded by tensions that exist in school settings that actually sustain traditional teaching rather than supporting reform (Gregg, 1995). Furthermore, Remillard and Bryans (2004) found that a teacher's orientation towards a curriculum influences how he or she engages those materials in the classroom as much as the curriculum itself. As mathematics educators have already discovered, supplying standards-based materials does not necessarily translate into any discernable change in classroom practice. Since teachers' beliefs can either support or constrain their students' learning, mathematics teacher educators need to attend

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carefully to teacher beliefs (Warfield et al, 2005). If the gap between teachers' perceptions of how they teach mathematics and their actual classroom practice is ever to be rectified, all of these obstacles must be overcome.

*The Influence of Reactance*

Any change in classroom practice is further complicated by a teacher's individual character traits. Research shows that personality factors such as psychological reactance, extroversion, anxiety, independence, and self-control affect an individual's ability to change (Bartram, 1995). The Theory of Psychological Reactance (Brehm, 1966; Brehm & Brehm, 1981) postulates that:

> Individuals possess "free behaviors" that can be engaged in at a moment or at some future time and that the motivational state of psychological reactance will be aroused whenever any of these free behaviors are eliminated or threatened with elimination. This motivational state will be directed toward the restoration of the eliminated or threatened behavior and will result in behavior known as reactance effects (Dowd, Milne, & Wise, 1991).

Psychological reactance theory offers a helpful framework for understanding oppositional behavior in individuals. For instance, therapy clients who are high in reactance - a "state of mind aroused by a threat to one's perceived legitimate freedom" (Brehm, 1966) - have low expectations for change and low therapy outcomes (Dowd & Wallbrown, 1993). Counselors who are high in reactance prefer unstructured supervision and greater degrees of professional freedom. In contrast, those counselors who are low in reactance were most extreme in their preference for structured supervision (Tracey, Ellickson, & Sherry, 1989). In another study, low-reactant patients reduced cigarette consumption more when they were provided with high amounts of physician advice. However, high-reactant patients perceived high amounts of physician advice as a threat, especially if that advice was negatively toned. For these patients, the best interaction was a low amount of negatively toned advice (Graybar et al., 1989).

Reactance potential, like other personality traits, may be higher in some individuals than it is in others (Brehm, 1966) and is believed to remain relatively stable over an individual's lifetime. And, as is often the case with other personality factors, there may be a correlation between reactance, career choice, and eventual success at an individual's chosen occupation.

Extrapolating from the psychotherapy literature to education, then, it is probable that teachers who are high in reactance might engage in oppositional behavior when confronted with requests to adapt reform strategies. They would view standards-based curricula and workshops as a threat to their autonomy and exhibit a decreased willingness to change classroom practice. Teachers who are low in reactance would feel less threatened by outside attempts to reform their teaching practice and be more willing to change. However, they would desire higher levels of advice and structured support from perceived authority. Although "research on the relationship between teachers' characteristics and teacher effectiveness as been underway for over a century" (Rockoff et al., 2008), little empirical research exists in the area of reactance, so this study explores teaching dynamics from the unique perspective of considering the social and emotional dimensions of teachers.

## Research Question

How does the psychological reactance of elementary school teacher influence their willingness to incorporate mathematics reform strategies in classroom practice?

## Participants, Methods, and Data Sources

From 2004 - 2007, data was collected on 191 pre-service and 65 in-service teachers. The pre-service teachers attended the same university and were in eleven cohort groups spread out over six semesters. The in-service teachers were employed at four different local elementary schools during this time frame. In one instance during the study, two of the pre-service teachers were hired and became in-service teachers at one of the schools.

Participants completed a personality measure called the Therapeutic Reactance Scale (TRS; Dowd, Milne, & Wise, 1991) which gives scores on psychological reactance. The TRS is a one-page, 28-item test using a 4-point Likert scale. Several of the items are reverse-keyed to eliminate any effect from acquiescence response sets. The instrument is based on normative data and has a very slightly positively skewed normal distribution. The usual mean is 67.78 and the median 67.75. The minimum and maximum attainable scores are 28 and 112, respectively. The TRS has an internal consistency ranging from .75 to .84. This clinical measure of reactance has been used in psychometric applications, such as counseling and human resource management, for more than a decade.

## Results

Descriptive statistics were calculated for each of the eleven pre-service and four in-service groups separately. ANOVA: Single Factor was performed on all possible combinations, taken two at a time, to yield a simple analysis of variance on data for the samples. This analysis tested the hypothesis that each sample was drawn from the same underlying probability distribution against the alternative hypothesis that the underlying probability distributions were not the same for all samples. In each instance, the $p > 0.90$, which did not lend sufficient support to reject the null hypothesis. In other words, the groups were sufficiently similar that they likely represented the same population.

The data from all teachers was then combined into “composite” pre-service and in-service scores, yielding means of 62.43 and 61.99 respectively (see Table 1.) Another ANOVA was performed on the composite scores, producing a $p = 0.909$. This confirmed that the null hypothesis should not be rejected: teachers, regardless of teaching experience, had relatively similar scores and thus represented similar populations. Interestingly, performing ANOVA against the “normal” TRS scores yielded a $p < 0.05$, showing that elementary teachers’ scores did differ significantly from usual scores used to normalize the TRS. In this case, the alternate hypothesis would be supported: Teachers had significantly lower scores in psychological reactance than the general population.

A t-Test (two-sample assuming unequal variances) was also performed (see Table 1). This t-test form assumed that the two data sets came from distributions with unequal variances (a heteroscedastic t-test) since descriptive statistics had exhibited this disparity. Results from this analysis supported the ANOVA results. They indicated that the two samples were likely to have come from distributions with equal population means.

Table 1. *T-test: Two-sample Assuming Unequal Variances*

| | Pre-service | In-Service |
|---|---|---|
| Mean | 62.43455497 | 61.99230769 |
| Variance | 45.2338523 | 62.23040865 |
| Observations | 191 | 65 |
| Hypoth Mean Difference | 0.5 | |
| Df | 98 | |
| t Stat | -0.052848266 | |
| P(T<=t) one-tail | 0.478980192 | |
| t Critical one-tail | 1.660551218 | |
| P(T<=t) two-tail | 0.957960383 | |
| t Critical two-tail | 1.984467404 | |

Graphing the data from the two groups provided some interesting results (see Figures 1 and 2). A class width of 3.4 was chosen to display TRS scores, which ranged from 44.5 to 78. In both instances, the data was bi-modal. One mode was near TRS's normative central tendency measures (mean 67.78 and median 67.75). However, there was a second mode at a lower reactance score (54.8 to 58) for both sets of teachers. This less-reactant mode accounted for the shift to an overall lower mean and median reactance score.

**Figure 2: In-Service Teacher Reactance**

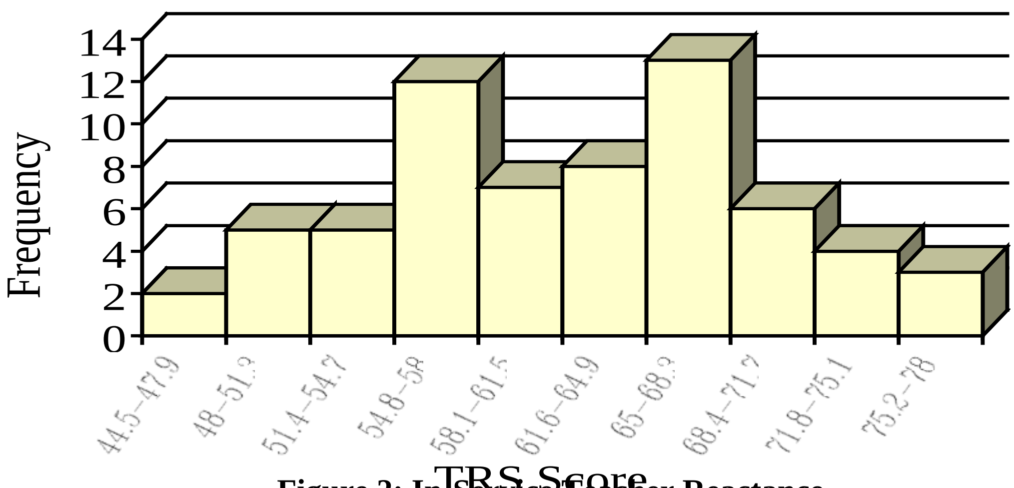
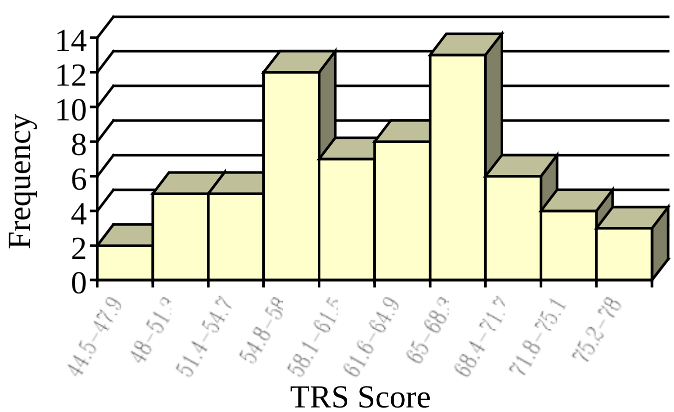



**Figure 2: In-Service Teacher Reactance**



## Discussion

The gap between teachers' perceptions of how they teach and their actual classroom practice has been studied from a variety of perspectives. This mismatch of belief and behavior appears to exist at several levels. For instance, Schussler, Bercaw, and Stocksberry (2008) found that an inverse relationship exists between teacher awareness and assumptions. In research focusing on the moral disposition of teachers, Johnson (2008) found an inconsistency between teacher candidates' quantitative assessments (tests) and qualitative data (contextual written assignments), suggesting that teachers had an inaccurate picture of their own beliefs and how they put those beliefs into action. When Singh and Stoloff (2008) examined teacher dispositions, they discovered that teachers gave lip service to research based instructional strategies, but needed to reshape their behavior in order to actually engage in those practices. Perhaps matching perception to actual behavior is a common human dilemma; certainly teachers are not exempt from exhibiting this disparity.

The analysis of TRS data yielded some interesting findings. First, the mean reactance value for teachers is low, whatever their career stage. In fact, as a group, elementary school teachers are significantly lower in reactance than the general population. It is interesting to speculate that certain personality types may be drawn (and perhaps better suited) to particular careers. Decades of research in vocational studies would certainly give both theoretical and empirical support for that hypothesis, as the goal of vocational counseling is to match the client with a vocation in which he will be both satisfied and satisfactory (Botterbusch, 1978; Ehrhart & Makransky, 2007; Whitehead, 2005).

The bimodal nature of TRS scores suggest that one group of teachers is about average in reactance, but many more individuals than expected from the general population tend to fall below the normative level. In fact, the second mode indicates a large proportion of teachers are grouped around an unusually low reactance value. Teachers with these low TRS scores are unlikely to feel threatened by reform strategies. Because reactance is a personality trait that remains fairly constant over time, it is not surprising to note that pre-service and in-service teachers exhibit the same distributions. (After all, the ANOVA and t-Test indicated that the two samples were likely to have come from distributions with equal population means.)

Research evidence has suggested that reactance potential may mediate compliance with behavioral tasks. Brehm and Brehm (1966) found that the tendency to resist overt influence is positively correlated with internal locus of control. Thus, individuals with Type A personality may possess a lower threshold of threat for arousal of reactance than Type B personalities (Dowd, Milne, & Wise, 1991). What does this mean in terms of low reactant elementary school teachers? It means that teachers have a lesser tendency to resist influence - correlated with a low locus of control - and demonstrate more expectation of changing their behavior (Rohrbaugh et al., 1981). Also, low-reactant people become significantly less anxious than high-reactant individuals when confronted with pressures from the "outside" (Dowd, Milne, & Wise, 1991). Low reactance could be a distinct advantage in a profession that requires its workers to incorporate an ever-changing landscape of external demands levied by administrators, parents, politicians, and mathematics educators.

Low reactance is also related to a greater desire to impress others and to be socially appropriate (Dowd, Milne, & Wise, 1991). As teachers work in the public arena (regardless of whether they are employed in a public or private school), this desire might actually increase their willingness to change teaching practice if modifications are perceived as socially acceptable. Teacher preparation and development courses would therefore be more effective at changing classroom practice if teachers worked in communities of respected peers, and if they experienced support, acceptance, and positive reinforcement from administrators and persons of recognized social authority.

A final implication of low reactance is closely related to the desire to be socially acceptable. Research has shown low-reactant individuals change their behavior more readily with high levels of professional advice, and they are more extreme in their preference for structured support (Dowd, Milne, & Wise, 1991; Tracey, Ellickson, & Sherry, 1989). Elementary school teachers' low reactance would thus indicate the desire for perceived "experts" to design highly structured methods and professional development courses. Mathematics educators, mathematics specialists, and others with professional expertise must be available to answer questions and provide supportive counsel on a regular basis. Contrary to oppositional behavior, low-reactant individuals like elementary school teachers want suggestions from more experienced leaders because they value such guidance when trying to implement a change in their own behavior.

Any discussion of the use of psychological measures (such as reactance testing) in educational settings should include some attention to ethical concerns. Without question, tests that have been used successfully for clinical diagnosis and to reveal psychopathology are best administered by a clinician with specialized training in their interpretation. One must also consider the potential implications of using clinical tests and the relevance of personality constructs and their validity if the results of these tests are directed at pre-employment screening or post-employment evaluation. Camara and Merenda (2000) suggest that the fear of litigation resulting from testing that is used in an exclusionary manner (such as for pre-employment screening) may be the primary disincentive for using personality tests inappropriately. Even "psychologists are increasingly on 'thin ice' in using clinical instruments to make high stakes decisions in employment settings" (Camara & Merenda, 2000, p. 1183). However, these ethical constraints play little or no role in this study. A licensed clinical psychologist is involved in the research. The TRS was only used to identify psychological reactance that may predispose teachers' willingness to change in order to suggest more effective strategies for delivering mathematics education courses to teachers.

However, reactance scores by themselves contribute only one small piece to the complex jigsaw puzzle of designing effective teacher education courses. Additional data associated with this study is currently under analysis to examine the interplay of reactance with other personality traits, while tracking the change in classroom practice that is anticipated from standards-based instruction. Participants took a second personality measure called the 16PF Questionnaire (Cattell, Cattell, & Cattell, 1993). The 16PF is a 185 item self-report which measures normal adult personality dimensions (i.e. extraversion, anxiety, tough-mindedness, independence). In addition, teachers in this study have been involved in either a university mathematics methods course or a school-wide two-year long biweekly professional development (PD) course (or both, as was the case for two of the pre-service teachers who were hired by the participating school districts). These courses have been aimed at instruction in NCTM reform beliefs and practice.

To measure beliefs and attitudes about teaching mathematics, each participant took the Integrating Mathematics and Pedagogy Beliefs Survey (IMAP, Philipp, 2002). This belief-assessment instrument is a web-based, in-context survey using video clips that allow respondents to interpret and react to student actions in well-defined situations. Participants took the IMAP twice - at the beginning and end of their coursework – thus data has been collected to examine any change or beliefs or attitudes that might likely have occurred due to the reform-based instruction.

To document change in classroom practice (perceived or actual), in-service teachers completed self-assessments to rate their own progress at reforming their teaching practices. In addition, an outside evaluator made formal observations of each teacher's classroom practice during regular mathematics lessons twice a year. The collaborative lessons associated with both the methods and the PD courses were formally observed. Pre-service and in-service teachers' reflective journals were collected. Qualitative data from self-assessments, observations, and teacher writings is being coded for emergent themes and trends. The findings and correlates from these analyses will be reported and should give a more accurate picture of the dynamics of personality and willingness/ability to change classroom practice.

## Use of Results Related to PME-NA Goals

Because teachers are individuals, they respond to training differently. Research needs to identify these differences and make suggestions for adapting methods and professional development courses to meet teacher needs. This study hopes to gain a deeper understanding of the influence of psychological reactance that may predispose teachers' willingness to change, thus influencing their efforts to incorporate reform strategies into classroom practice. These findings meet the goal of embracing the diverse perspectives of teachers in order to benefit the children they teach.

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# TAKING A CLOSER LOOK AT PRACTICE AND PARTICIPATION: TEACHER INTERACTIONS AND ENGAGEMENT IN AN URBAN MIDDLE SCHOOL MATHEMATICS CLASSROOM

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*From the detailed analysis of videotapes in an urban middle school classroom taken as part of a larger study, we provide interpretation of the notion of an "emotionally safe environment." This "teacher research" project looks at proposed ways of describing a complex behavioral, social, and affective environment that can enhance or hinder students' motivation to engage mathematically. The analysis presented here focuses on three categories: reframing frustration and impasse; reframing the mathematical problem solving process; and maintaining high cognitive demand and motivational potential in the selection and implementation of mathematical problems.*

## Background and Theoretical Framework

This analysis investigates features of a classroom that are perceived to promote an emotionally safe environment for student exploration of conceptually challenging mathematics. The research reported here is part of a larger study investigating the occurrence and development of powerful affect around conceptually challenging mathematics. Its focus is on urban middle school classrooms serving low-income students primarily of African American, Latino, and Caribbean descent. The class is one of three urban, middle school mathematics classes (in two different districts), that were studied in depth over the course of a school year, starting in September and ending in late May. The teachers were selected based on evidence of strong mathematics teaching skill, research interest, and their likely ability to elicit powerful affect around conceptually challenging mathematics. Data were collected during five cycles of twenty visits. For each cycle, data included videotapes of two consecutive lessons, pre- and post-interviews with the teacher, and videotaped stimulated-recall interviews with four focus students. Three cameras were used for each class session: two following the focus students, and the third stationary camera capturing an overall view of the class. Additional data included students' written work, observers' field notes and earlier analysis (Alston *et al*., 2007; Alston *et al*., 2008). The classroom teacher joined the research team subsequent to the school year and is participating in the analysis; he is the first author of the present report.

Teachers face the general problem of how to create effective learning environments in mathematics, and the specific problem of how to interact most effectively with students engaged in conceptually challenging mathematical activity. Current literature suggests the value of attending to issues of affect, context, social interactions, race, and culture in helping students gain confidence, motivation, and improving performance (Ball & Bass, 2003; Cobb & Yackel, 1998, Goldin, Epstein, & Schorr, 2007, Goldin, 2000a, 2000b; Martin, 2000; Moschkovich, 2002; Stigler & Hiebert, 1998). As students and teachers engage in conceptually challenging mathematics, a variety of emotional feelings occur that may influence instruction, motivation, and learning. These issues have immediate implications for learners in the mathematics classroom. Existing mathematics education research highlights the reciprocal relationship

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

between teacher actions and student behaviors--actions and behaviors that, in turn, appear to contribute to an interactive classroom environment where students are engaging with conceptually challenging mathematics (e.g., Carpenter, & Lehrer, 1999; Kaput, 1999; Martino, & Maher, 1999; Schorr, Warner, Gearhart, & Samuels, 2007; Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A., 2000; Warner, Schorr, Arias & Sanchez, in press). In taking this stance the role of the teacher needs to be carefully considered when it comes to students engaging in exploring conceptually challenging mathematics and valuing their mathematical thinking as they construct, reason, and justify their solutions (Hiebert *et. al.,* 1997).

Conceptually challenging mathematics is defined as mathematical content that requires some development of new concepts or changes in existing ones (Schorr, & Goldin, 2008), thus making it cognitively demanding. This frequently involves figuring something out within a problem situation in a "relational" manner (Skemp, 1976). Classroom social interactions around such mathematics may include students presenting ideas that are challenged publicly by their peers. Problem-solving efforts are likely to evoke discussions, explorations, and challenges to individuals' thinking. Some students' conjectures may turn out to be incorrect while the teacher or the class accepts others. Valid conjectures may at times be rejected as well. Students may lose track of underlying mathematical concepts as they bring personal details to the situation and experience impasse (Schoenfeld, 1992).

An emotionally safe environment for exploring conceptually challenging mathematics is defined as one where students' ideas and solutions are valued. Students are encouraged to share ideas. Mistakes are not criticized but transformed into opportunities for learning. Argumentation, reasoning, and proof are seen as part of the process in working out mathematical ideas (Ecceles, & Midgley, 1989; Goldin, Epstein, & Schorr, 2007; Maher, 1998; Maher, Davis, & Alston, 1991a; Schorr & Goldin, 2008, Yackel & Cobb, 1996).

The affective domain refers to emotional feelings, attitudes, beliefs, and values in relation to mathematics (DeBellis & Goldin, 2006; Evans, 2000; Evans, Morgan, & Tsatsaroni, 2006; Malmivuori, 2001; McLeod, 1994; Phillip, 2007). Powerful affect refers to those patterns of affect and behavior that lead to interest, engagement, persistence, and mathematical success. It is not restricted to positive emotions, such as curiosity, pleasure, and satisfaction, but includes the effective management and uses of feelings such as bewilderment and frustration (Epstein, Goldin, & Schorr, 2007; Goldin, 2000a,b; Gomez-Chacon, 2000a,b; Hannula, 2002; Malmivuori, 2006; Schorr & Goldin, 2008). The conjecture of several of the researchers cited above is that powerful affect in relation to conceptually challenging mathematics is fundamental to developing mathematical ability and essential to present and future mathematical achievement.

## Research Questions and Methods

This report will focus on the following qualitative research question: What teacher interactions foster engagement during exploration of conceptually challenging mathematics?

This analysis focuses on data from class sessions with students during the second cycle. The lesson was based on the unit project from "Stretching and Shrinking," a unit of *Connected Mathematics 2* (Lappan et. al., 2006). The activity began the night before when for homework students were instructed to bring in either a hand drawn picture or print out of their favorite cartoon character or television/music celebrity of their choice. The students were to enlarge their figures after applying them to a grid, plotting the points that would recreate their character, and apply a rule to the coordinates to enlarge them by a scale factor of at least four. This investigation continues for three more class periods of which analysis is ongoing.

With input from colleagues (Epstein et. al., 2008) involved with the larger study emerged some examples for exploring features of a classroom that promotes an emotionally safe environment for student exploration of conceptually challenging mathematics. The following abbreviated list builds from this collaboration with a focus on teacher strategies that foster engagement: 1) Reframing frustration and impasse (RFI); 2) Reframing the mathematical problem solving process (RP); and 3) Maintaining high cognitive demand and motivational potential in the selection and implementation of mathematical problems (HCD). At this early stage in analysis, we make no claim regarding the reliability of coding. The categories are preliminary and conjectural, though intended to lay the groundwork for future work.

## Results

The students were working in small groups attempting to take their figure from its original state to a scale factor at least four times larger than the original. The initial goal was a bit confusing and the teacher reframed impasse (RFI) by having some students simplify their character to a boxier version to convert to the coordinate grid system and to enlarge their figure by applying a rule to the coordinates. He also had students trace their figure on a transparency to enlarge using an overhead projector and compare a sampling of measurements from the original to the enlargement to determine the scale factor (RP). The teacher also encourages another student to stay course with the original plan because he believed the student would have success (HCD). In the tables that follow are examples of reframing frustration and impasse (RFI), reframing the mathematical problem solving process (RP), and maintaining high cognitive demand and motivational potential in the selection and implementation of mathematical problems (HCD).

Mari (M) wanted to work with a television wrestler named Batista. The teacher suggested she make a championship belt with the wrester’s name in the middle. This provided an opportunity for the student to stay engaged while she plotted points for her belt that kept in line with the mathematical goals of the lesson.

| | | | |
|---|---|---|---|
| 22:43 | Mr. P | Just to stay simple I would make little lines but I would do like a um like that. You could write the name in between and go, some thing like that, you happy with that? | RP |
| 23:09 | M | Uh huh (smiling) | |
| 23:10 | Mr. P | So something like that, so put it on there | |
| 23:11 | M | So can I do that and then put it on the other | |
| 23:12 | Mr. P | And then you would make sure that you’re able to put his name, you’re going to have to make block letters like this, making them curve might be rough but you’re if you want to make a B it might look like this (teacher draws boxy B) | |

Mari comes back at a later point to get assistance with the writing of the letters in block form to put on the coordinate grid.

| | | | |
|---|---|---|---|
| 41:52 | M | Can I just write them in? | |
| 41:53 | Mr. P | Yes, but you want them to know um can I see your pencil? Okay so B, like you did okay, A that one’s actually pretty easy A right? ‘cause you just need, you might need 3 points here to make that curve. You might need to start back on the belt and across, you understand? And the S you’re gonna have to | RFI |

| | | | |
|---|---|---|---|
| | | make one of these kind of S's. I is just right? | |
| 42:52 | M | Yeah and then there's a T and another A. | |
| 42:56 | Mr. P | Okay so i'll just do that part. You're gonna need to do the points in the middle right? You're gonna have to tell me where these points are but you know you got your lines here, that's fine, yeah I think this will work out good. | |
| 43:21 | M | And now I gotta put it on the bigger paper? | |
| 43:23 | Mr. P | Well now the goal, yeah, now the goal is to tell me where all these coordinates are. Do you know where all these points are? | HCD |
| 43:31 | M | Oh like if it's 1 and 15 and something like that | |
| 43:34 | Mr. P | Yes and then, and then we want to apply a formula to it or a rule | HCD |

Petri (P) had spent a good portion of the period trying to find a character and apply it to the grid. Her group was trying to explain to her that she could make Stewie from the television show "Family Guy" into a diamond to be able to graph it easier. Petri didn't understand and became frustrated. Tyanna (T) also comes to the teacher to try and get some help. Mr. P suggests to her to put her character on a transparency and use the overhead projector to enlarge the figure. He also sees Petri is frustrated and suggest she do the same.

| | | | |
|---|---|---|---|
| 48:49 | E | Mines was easier. | |
| 48:52 | P | You could just make a box. | |
| 48:56 | R | I should make a football. | |
| 49:00 | E | Make a big diamond. | |
| 49:03 | R | Yeah a big diamond head | |
| 49:04 | P | What character is a big diamond head? | |
| 49:06 | R | Stewie, it's a football head but it's diamond | |
| 49:10 | G | Oh yeah you just do it like that, yeah look | |
| 49:16 | P | What ya'll talking about? | |
| 49:17 | R | You see this look, um cube forms like that. It's a diamond. | |
| 49:23 | P | You guys are strange. You're saying make his head a diamond. It's gonna be ugly (laughs) | |
| 49:33 | G | You don't even know how to draw a diamond. Oh my god. | |
| 49:38 | P | Ah ya'll aggravating. | |
| 49:52 | E | My dog has small hands. This dog. (Q comes to table) | |
| 49:57 | R | What? | |
| 49:59 | Q | Where? | |
| 50:00 | R | Diamond head. | |
| 50:03 | P | I don't got no diamond head | |
| 50:04 | R | Oh yeah that's the same one as yours. | |
| 50:10 | P | I'm gonna make a box with some ears. | |
| 50:17 | Q | She in my seat, I don't know why she in my seat. Are you the one they supposed to record? | |
| 50:22 | P | Shut up Q, you're retarded | |
| 50:25 | Q | No I'm not. get up. | |
| 50:49 | E | Ha ha look at my dog. | |
| 50:53 | R | He's hungry | |
| 50:55 | Mr. P | Are you getting frustrated? | RFI |
| 50:57 | T | Uh ha | |

| | | | |
|---|---|---|---|
| 50:58 | Mr. P | Draw him on the plastic paper and then you can enlarge him and you can find the scale factor from using the ruler instead of all the coordinates. Does that sound like something you'd rather do? | RP, HCD |
| 51:08 | T | Um hm. I need a marker | |
| 51:10 | Mr. P | Take one, use a permanent marker. I don't…I don't have the other ones | |
| 51:19 | P | Mr. P, I'm fed up! | |
| 51:21 | Mr. P | Yes? You're fed up too | RFI |
| 51:23 | P | Yes I'm just gonna draw something | |
| 51:24 | Mr. P | Okay you can either, the option is to use either boxes. Glenda did you make that printout at home? Good job. | |
| 51:32 | P | Okay this right here. | |
| 51:36 | Mr. P | What? | |
| 51:38 | E | I made a new one but it's smaller | |
| 51:40 | Mr. P | Okay | |
| 51:43 | P | Can I make a crown or something? | |
| 51:45 | Mr. P | You could do a crown or who was your original person? Stewie? | RFI |
| 51:48 | P | Yes | |
| 51:50 | Mr. P | Some of you - some of you that are getting frustrated | RFI |
| 51:52 | R | I don't k now how to do this square thing. It's not that, I can do it but it's too many dots. Stewie's easy. Not the square thing. You know how to do the square? | |
| 52:09 | Mr. P | After you get it on a large piece of paper then we'll have to use measurement to find the scale factor | RP, HCD |
| 52:11 | P | Okay | |
| 52:14 | Mr. P | 'Cause I don't want you to get too frustrated here. | RFI |

Ryan (R) who is an excellent artist was plugging along and trying to apply his talent to this mathematical situation. He approaches the teacher to see how he is progressing.

| | | | |
|---|---|---|---|
| 43:52 | R | What number do I got to go up to? | |
| 43:56 | Mr. P | Yours is very detailed. You're gonna need a lot of points. Um that's fine if you stop there. | HCD |
| 44:01 | M | Who is that - Sponge Bob? | |
| 44:03 | R | Yeah, that's Sponge Bob. | |
| 44:05 | Mr. P | Um the only thing, this is the only thing, I don't want you to start over Bryant but one of the things, and I didn't do a good job explaining it. If you're gonna make Sponge Bob think about him more | RFI, HCD |
| 44:25 | R | Cubed? More easier? | |
| 44:26 | Mr. P | I'm not saying you have to do it over but um it might make your enlargement easier too. Here I'm having trouble drawing it, let me see him again how is he? He's wide on, so yeah so you need right? Is that it? Something like this and then for his eyes you might want to make them uh | RFI, HCD |
| 45:04 | R | Hexagon | |
| 45:06 | Mr. P | Yeah because see how the sharp edges you can put on the lines easier. I think yours is very detailed and beautiful but it might be hard to make a bigger one. All right? Because these are a lot of points to identify so either you can go ahead with this one or try and just make him, and then if you want to do your X's in the middle like you have, you can be able to do you know what I'm saying like if you want to keep with that theme with the | RFI, HCD |

| | | | |
|---|---|---|---|
| | | Jamaican flag is that what it is or Dominican? | |
| 45:37 | R | Dominican | |
| 45:38 | Mr. P | Dominican flag in the middle then you can still do that and then when you make his nose, you're gonna make it square like this | |
| 45:53 | R | 'Cause if I stick with this I don't know what number to go up to. | |
| 45:56 | Mr. P | You would have to go up really far, you would need a lot more points right? Well you don't really need that many, you can go as high as you want but you can keep going with this theme, I don't see any problem with it, you can tell who it is. 'Cause you've done, you've done some boxy shapes on the sides, you just now need to identify the points. So like where is that point? | RFI |
| 46:22 | R | Like at 8, 3? | |

A bit later after a few students were encouraged to use another method due to being frustrated by the first method but Mr. P encourages Ryan to continue using his original method.

| | | | |
|---|---|---|---|
| 53:04 | Mr. P | No no you don't have to do it on other paper. If you're doing that method, there's a somewhat different method and if I'm I'm pushing first for you to graph the points and make your thing because we want a library of good characters for the next group of kids so that we don't just have to use the wumps. Right? | HCD |
| 53:26 | R | Yeah so this is a good character? | |
| 53:27 | Mr. P | Yeah so if you're not too flustered, just keep pushing on. Okay? | HCD |
| 53:32 | R | I'm a do the coordinates in this picture | |
| 53:33 | Mr. P | Yeah I think you're doing fine Ryan, you can handle it. | HCD |

## Conclusions and Implications

This analysis documents the value of students' bringing personal interests to mathematical concepts. The task itself brings interest to scaling figures larger or smaller with some reframing by the teacher. During this lesson the mathematical goal of scale factor seemed to move to the background while students focused on creating the likeness of their character. Despite the students' struggles transforming their characters during the lesson, the agency and personalization that students exhibited are valuable and need to be navigated with care as to not disengage them. Furthermore these findings hope to shed light on the time, consideration, and decisions it takes to engage learners from a perspective that involves motivation and affect during the course of a lesson. Reframing problems, frustration, and impasse while maintaining high cognitive demand are relevant issues in mathematics education and need to be highly considered when choosing tasks and engaging learners. Teachers try to understand the mathematical strategies and representations a student might use in solving a particular problem. If we can understand and represent teacher interactions and student behaviors that enhance or impede emotional safety and mathematical engagement, it could be useful for creating other environments that encourage students to make mathematical meaning for themselves. Being the voice of the teacher in the classroom where the data were collected enhances the qualitative research paradigm by allowing for exploration of the unique features surrounding these particular cases and perhaps arriving at what Paul Ernest describes as the, "truth derived from identification with, and living through, a story with the richness and complex interrelationships of social, and human life" (Ernest 1994, p. 34). By adopting this perspective the present research affords an opportunity to understand the situation in a way that may be valuable to other practitioners facing similar challenges.

## Endnote

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## Background and Theoretical Framework

This analysis investigates features of a classroom that are perceived to promote an emotionally safe environment for student exploration of conceptually challenging mathematics. The research reported here is part of a larger study investigating the occurrence and development of powerful affect around conceptually challenging mathematics. Its focus is on urban middle school classrooms serving low-income students primarily of African American, Latino, and Caribbean descent. The class is one of three urban, middle school mathematics classes (in two different districts), that were studied in depth over the course of a school year, starting in September and ending in late May. The teachers were selected based on evidence of strong mathematics teaching skill, research interest, and their likely ability to elicit powerful affect around conceptually challenging mathematics. Data were collected during five cycles of twenty visits. For each cycle, data included videotapes of two consecutive lessons, pre- and post-interviews with the teacher, and videotaped stimulated-recall interviews with four focus students. Three cameras were used for each class session: two following the focus students, and the third stationary camera capturing an overall view of the class. Additional data included students' written work, observers' field notes and earlier analysis (Alston *et al*., 2007; Alston *et al*., 2008). The classroom teacher joined the research team subsequent to the school year and is participating in the analysis; he is the first author of the present report.

Teachers face the general problem of how to create effective learning environments in mathematics, and the specific problem of how to interact most effectively with students engaged in conceptually challenging mathematical activity. Current literature suggests the value of attending to issues of affect, context, social interactions, race, and culture in helping students gain confidence, motivation, and improving performance (Ball & Bass, 2003; Cobb & Yackel, 1998, Goldin, Epstein, & Schorr, 2007, Goldin, 2000a, 2000b; Martin, 2000; Moschkovich, 2002; Stigler & Hiebert, 1998). As students and teachers engage in conceptually challenging mathematics, a variety of emotional feelings occur that may influence instruction, motivation, and learning. These issues have immediate implications for learners in the mathematics classroom. Existing mathematics education research highlights the reciprocal relationship between teacher actions and student behaviors--actions and behaviors that, in turn, appear to contribute to an interactive classroom environment where students are engaging with conceptually challenging mathematics (e.g., Carpenter, & Lehrer, 1999; Kaput, 1999; Martino, & Maher, 1999; Schorr, Warner, Gearhart, & Samuels, 2007; Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A., 2000; Warner, Schorr, Arias & Sanchez, in press). In taking this stance the role of the teacher needs to be carefully considered when it comes to students engaging in exploring conceptually challenging mathematics and valuing their mathematical thinking as they construct, reason, and justify their solutions (Hiebert *et. al.,* 1997).

Conceptually challenging mathematics is defined as mathematical content that requires some development of new concepts or changes in existing ones (Schorr, & Goldin, 2008), thus making it cognitively demanding. This frequently involves figuring something out within a problem situation in a "relational" manner (Skemp, 1976). Classroom social interactions around such mathematics may include students presenting ideas that are challenged publicly by their peers. Problem-solving efforts are likely to evoke discussions, explorations, and challenges to individuals' thinking. Some students' conjectures may turn out to be incorrect while the teacher or the class accepts others. Valid conjectures may at times be rejected as well. Students may lose track of underlying mathematical concepts as they bring personal details to the situation and experience impasse (Schoenfeld, 1992).

An emotionally safe environment for exploring conceptually challenging mathematics is defined as one where students' ideas and solutions are valued. Students are encouraged to share ideas. Mistakes are not criticized but transformed into opportunities for learning. Argumentation, reasoning, and proof are seen as part of the process in working out mathematical ideas (Ecceles, & Midgley, 1989; Goldin, Epstein, & Schorr, 2007; Maher, 1998; Maher, Davis, & Alston, 1991a; Schorr & Goldin, 2008, Yackel & Cobb, 1996).

The affective domain refers to emotional feelings, attitudes, beliefs, and values in relation to mathematics (DeBellis & Goldin, 2006; Evans, 2000; Evans, Morgan, & Tsatsaroni, 2006; Malmivuori, 2001; McLeod, 1994; Phillip, 2007). Powerful affect refers to those patterns of affect and behavior that lead to interest, engagement, persistence, and mathematical success. It is not restricted to positive emotions, such as curiosity, pleasure, and satisfaction, but includes the effective management and uses of feelings such as bewilderment and frustration (Epstein, Goldin, & Schorr, 2007; Goldin, 2000a,b; Gomez-Chacon, 2000a,b; Hannula, 2002; Malmivuori, 2006; Schorr & Goldin, 2008). The conjecture of several of the researchers cited above is that powerful affect in relation to conceptually challenging mathematics is fundamental to developing mathematical ability and essential to present and future mathematical achievement.

## Research Questions and Methods

This report will focus on the following qualitative research question: What teacher interactions foster engagement during exploration of conceptually challenging mathematics?

This analysis focuses on data from class sessions with students during the second cycle. The lesson was based on the unit project from "Stretching and Shrinking," a unit of *Connected Mathematics 2* (Lappan et. al., 2006). The activity began the night before when for homework students were instructed to bring in either a hand drawn picture or print out of their favorite cartoon character or television/music celebrity of their choice. The students were to enlarge their figures after applying them to a grid, plotting the points that would recreate their character, and apply a rule to the coordinates to enlarge them by a scale factor of at least four. This investigation continues for three more class periods of which analysis is ongoing.

With input from colleagues (Epstein et. al., 2008) involved with the larger study emerged some examples for exploring features of a classroom that promotes an emotionally safe environment for student exploration of conceptually challenging mathematics. The following abbreviated list builds from this collaboration with a focus on teacher strategies that foster engagement: 1) Reframing frustration and impasse (RFI); 2) Reframing the mathematical problem solving process (RP); and 3) Maintaining high cognitive demand and motivational potential in the selection and implementation of mathematical problems (HCD). At this early stage in analysis, we make no claim regarding the reliability of coding. The categories are preliminary and conjectural, though intended to lay the groundwork for future work.

## Results

The students were working in small groups attempting to take their figure from its original state to a scale factor at least four times larger than the original. The initial goal was a bit confusing and the teacher reframed impasse (RFI) by having some students simplify their character to a boxier version to convert to the coordinate grid system and to enlarge their figure by applying a rule to the coordinates. He also had students trace their figure on a transparency to enlarge using an overhead projector and compare a sampling of measurements from the original to the enlargement to determine the scale factor (RP). The teacher also encourages another student to stay course with the original plan because he believed the student would have success (HCD). In the tables that follow are examples of reframing frustration and impasse (RFI), reframing the mathematical problem solving process (RP), and maintaining high cognitive demand and motivational potential in the selection and implementation of mathematical problems (HCD).

Mari (M) wanted to work with a television wrestler named Batista. The teacher suggested she make a championship belt with the wrester’s name in the middle. This provided an opportunity for the student to stay engaged while she plotted points for her belt that kept in line with the mathematical goals of the lesson.

| | | | |
|---|---|---|---|
| 22:43 | Mr. P | Just to stay simple I would make little lines but I would do like a um like that. You could write the name in between and go, some thing like that, you happy with that? | RP |
| 23:09 | M | Uh huh (smiling) | |
| 23:10 | Mr. P | So something like that, so put it on there | |
| 23:11 | M | So can I do that and then put it on the other | |
| 23:12 | Mr. P | And then you would make sure that you’re able to put his name, you’re going to have to make block letters like this, making them curve might be rough but you’re if you want to make a B it might look like this (teacher draws boxy B) | |

Mari comes back at a later point to get assistance with the writing of the letters in block form to put on the coordinate grid.

| | | | |
|---|---|---|---|
| 41:52 | M | Can I just write them in? | |
| 41:53 | Mr. P | Yes, but you want them to know um can I see your pencil? Okay so B, like you did okay, A that one’s actually pretty easy A right? ‘cause you just need, you might need 3 points here to make that curve. You might need to start back on the belt and across, you understand? And the S you’re gonna have to | RFI |

| | | | |
|---|---|---|---|
| | | make one of these kind of S's. I is just right? | |
| 42:52 | M | Yeah and then there's a T and another A. | |
| 42:56 | Mr. P | Okay so i'll just do that part. You're gonna need to do the points in the middle right? You're gonna have to tell me where these points are but you know you got your lines here, that's fine, yeah I think this will work out good. | |
| 43:21 | M | And now I gotta put it on the bigger paper? | |
| 43:23 | Mr. P | Well now the goal, yeah, now the goal is to tell me where all these coordinates are. Do you know where all these points are? | HCD |
| 43:31 | M | Oh like if it's 1 and 15 and something like that | |
| 43:34 | Mr. P | Yes and then, and then we want to apply a formula to it or a rule | HCD |

Petri (P) had spent a good portion of the period trying to find a character and apply it to the grid. Her group was trying to explain to her that she could make Stewie from the television show "Family Guy" into a diamond to be able to graph it easier. Petri didn't understand and became frustrated. Tyanna (T) also comes to the teacher to try and get some help. Mr. P suggests to her to put her character on a transparency and use the overhead projector to enlarge the figure. He also sees Petri is frustrated and suggest she do the same.

| | | | |
|---|---|---|---|
| 48:49 | E | Mines was easier. | |
| 48:52 | P | You could just make a box. | |
| 48:56 | R | I should make a football. | |
| 49:00 | E | Make a big diamond. | |
| 49:03 | R | Yeah a big diamond head | |
| 49:04 | P | What character is a big diamond head? | |
| 49:06 | R | Stewie, it's a football head but it's diamond | |
| 49:10 | G | Oh yeah you just do it like that, yeah look | |
| 49:16 | P | What ya'll talking about? | |
| 49:17 | R | You see this look, um cube forms like that. It's a diamond. | |
| 49:23 | P | You guys are strange. You're saying make his head a diamond. It's gonna be ugly (laughs) | |
| 49:33 | G | You don't even know how to draw a diamond. Oh my god. | |
| 49:38 | P | Ah ya'll aggravating. | |
| 49:52 | E | My dog has small hands. This dog. (Q comes to table) | |
| 49:57 | R | What? | |
| 49:59 | Q | Where? | |
| 50:00 | R | Diamond head. | |
| 50:03 | P | I don't got no diamond head | |
| 50:04 | R | Oh yeah that's the same one as yours. | |
| 50:10 | P | I'm gonna make a box with some ears. | |
| 50:17 | Q | She in my seat, I don't know why she in my seat. Are you the one they supposed to record? | |
| 50:22 | P | Shut up Q, you're retarded | |
| 50:25 | Q | No I'm not. get up. | |
| 50:49 | E | Ha ha look at my dog. | |
| 50:53 | R | He's hungry | |
| 50:55 | Mr. P | Are you getting frustrated? | RFI |
| 50:57 | T | Uh ha | |

| | | | |
|---|---|---|---|
| 50:58 | Mr. P | Draw him on the plastic paper and then you can enlarge him and you can find the scale factor from using the ruler instead of all the coordinates. Does that sound like something you'd rather do? | RP, HCD |
| 51:08 | T | Um hm. I need a marker | |
| 51:10 | Mr. P | Take one, use a permanent marker. I don't…I don't have the other ones | |
| 51:19 | P | Mr. P, I'm fed up! | |
| 51:21 | Mr. P | Yes? You're fed up too | RFI |
| 51:23 | P | Yes I'm just gonna draw something | |
| 51:24 | Mr. P | Okay you can either, the option is to use either boxes. Glenda did you make that printout at home? Good job. | |
| 51:32 | P | Okay this right here. | |
| 51:36 | Mr. P | What? | |
| 51:38 | E | I made a new one but it's smaller | |
| 51:40 | Mr. P | Okay | |
| 51:43 | P | Can I make a crown or something? | |
| 51:45 | Mr. P | You could do a crown or who was your original person? Stewie? | RFI |
| 51:48 | P | Yes | |
| 51:50 | Mr. P | Some of you - some of you that are getting frustrated | RFI |
| 51:52 | R | I don't k now how to do this square thing. It's not that, I can do it but it's too many dots. Stewie's easy. Not the square thing. You know how to do the square? | |
| 52:09 | Mr. P | After you get it on a large piece of paper then we'll have to use measurement to find the scale factor | RP, HCD |
| 52:11 | P | Okay | |
| 52:14 | Mr. P | 'Cause I don't want you to get too frustrated here. | RFI |

Ryan (R) who is an excellent artist was plugging along and trying to apply his talent to this mathematical situation. He approaches the teacher to see how he is progressing.

| | | | |
|---|---|---|---|
| 43:52 | R | What number do I got to go up to? | |
| 43:56 | Mr. P | Yours is very detailed. You're gonna need a lot of points. Um that's fine if you stop there. | HCD |
| 44:01 | M | Who is that - Sponge Bob? | |
| 44:03 | R | Yeah, that's Sponge Bob. | |
| 44:05 | Mr. P | Um the only thing, this is the only thing, I don't want you to start over Bryant but one of the things, and I didn't do a good job explaining it. If you're gonna make Sponge Bob think about him more | RFI, HCD |
| 44:25 | R | Cubed? More easier? | |
| 44:26 | Mr. P | I'm not saying you have to do it over but um it might make your enlargement easier too. Here I'm having trouble drawing it, let me see him again how is he? He's wide on, so yeah so you need right? Is that it? Something like this and then for his eyes you might want to make them uh | RFI, HCD |
| 45:04 | R | Hexagon | |
| 45:06 | Mr. P | Yeah because see how the sharp edges you can put on the lines easier. I think yours is very detailed and beautiful but it might be hard to make a bigger one. All right? Because these are a lot of points to identify so either you can go ahead with this one or try and just make him, and then if you want to do your X's in the middle like you have, you can be able to do you know what I'm saying like if you want to keep with that theme with the | RFI, HCD |

| | | | |
|---|---|---|---|
| | | Jamaican flag is that what it is or Dominican? | |
| 45:37 | R | Dominican | |
| 45:38 | Mr. P | Dominican flag in the middle then you can still do that and then when you make his nose, you’re gonna make it square like this | |
| 45:53 | R | ‘Cause if I stick with this I don’t know what number to go up to. | |
| 45:56 | Mr. P | You would have to go up really far, you would need a lot more points right? Well you don’t really need that many, you can go as high as you want but you can keep going with this theme, I don’t see any problem with it, you can tell who it is. ‘Cause you’ve done, you’ve done some boxy shapes on the sides, you just now need to identify the points. So like where is that point? | RFI |
| 46:22 | R | Like at 8, 3? | |

A bit later after a few students were encouraged to use another method due to being frustrated by the first method but Mr. P encourages Ryan to continue using his original method.

| | | | |
|---|---|---|---|
| 53:04 | Mr. P | No no you don’t have to do it on other paper. If you’re doing that method, there’s a somewhat different method and if I’m I’m pushing first for you to graph the points and make your thing because we want a library of good characters for the next group of kids so that we don’t just have to use the wumps. Right? | HCD |
| 53:26 | R | Yeah so this is a good character? | |
| 53:27 | Mr. P | Yeah so if you’re not too flustered, just keep pushing on. Okay? | HCD |
| 53:32 | R | I’m a do the coordinates in this picture | |
| 53:33 | Mr. P | Yeah I think you’re doing fine Ryan, you can handle it. | HCD |

## Conclusions and Implications

This analysis documents the value of students’ bringing personal interests to mathematical concepts. The task itself brings interest to scaling figures larger or smaller with some reframing by the teacher. During this lesson the mathematical goal of scale factor seemed to move to the background while students focused on creating the likeness of their character. Despite the students’ struggles transforming their characters during the lesson, the agency and personalization that students exhibited are valuable and need to be navigated with care as to not disengage them. Furthermore these findings hope to shed light on the time, consideration, and decisions it takes to engage learners from a perspective that involves motivation and affect during the course of a lesson. Reframing problems, frustration, and impasse while maintaining high cognitive demand are relevant issues in mathematics education and need to be highly considered when choosing tasks and engaging learners. Teachers try to understand the mathematical strategies and representations a student might use in solving a particular problem. If we can understand and represent teacher interactions and student behaviors that enhance or impede emotional safety and mathematical engagement, it could be useful for creating other environments that encourage students to make mathematical meaning for themselves. Being the voice of the teacher in the classroom where the data were collected enhances the qualitative research paradigm by allowing for exploration of the unique features surrounding these particular cases and perhaps arriving at what Paul Ernest describes as the, “truth derived from identification with, and living through, a story with the richness and complex interrelationships of social, and human life” (Ernest 1994, p. 34). By adopting this perspective the present research affords an opportunity to understand the situation in a way that may be valuable to other practitioners facing similar challenges.

**Endnote**

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# CONCEPTUALIZING PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' AFFECTIVE, META-COGNITIVE, AND MATHEMATICAL BEHAVIOR DURING PROBLEM-SOLVING

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Early studies in problem-solving were concerned with the aspects of a problem that made solving the problem difficult (Geiger and Galbraith, 1998). The concern focused on the cognitive aspects of problem-solving without taking into account the background or characteristics of the problem-solver. Geiger and Galbraith (1998) suggested that the relationship between the learner and a problem is what is of significance and not so much the difficulty level of the problem. Schoenfeld (1992) explained that purely cognitive behavior is rare, and that learners perform most mathematics task within the context established by their perspectives on the nature of those tasks. Only a few studies give attention to understanding the interrelationship between affect and cognitive processes during mathematics learning and problem-solving.

The purpose of this grounded theory study was to investigate the interplay of prospective secondary mathematics teachers' affect, metacognition, and mathematical cognition in a problem-solving context. Constructivist grounded theory methodology (Charmaz, 2005) was used to answer the question: What is the characterization of the interplay among prospective teachers' mathematical beliefs, metacognition, and mathematical knowledge in the context of solving mathematics problems? Over a ten-week period, I conducted four interviews with four prospective secondary mathematics teachers enrolled in an undergraduate mathematics problem-solving course. One of the interviews included a think-aloud (Ericsson & Simon, 1993) problem-solving episode. Participant artifacts, observations, and researcher reflections were regularly recorded and included as part of the data collection.

In this poster, I present the results of the study in the form of a model. The interpretive model that emerged from the study describes the interplay among affect, metacognition, and mathematical cognition during problem-solving as meta-affect, persistence and autonomy, and meta-strategic knowledge. The analysis of data showed that, for the participants, "Knowing How and Knowing Why" mathematics procedures work, having the ability to justify their reasoning and problem solutions, and persisting in difficult problem-solving situations represented mathematics knowledge and understanding that could empower them to become productive problem-solvers and effective secondary mathematics teachers.

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# MOTIVATION IN COLLEGE MATHEMATICS: DOES IT EVEN EXIST?

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One belief that impacts both attitudes towards mathematics (including motivation) and mathematics achievement is self-efficacy. Self-efficacy influences attitudes and behavior in several considerable ways. Choice, effort and persistence are just a few of the ways that self-efficacy influences behavior (Schunk & Pajares, 2005). One of the ways that self-efficacy influences attitude is that the amount of stress and anxiety that a student may or may not experience during a task (Pajares, 2002).

It has been found that self-ratings of overall academic ability, self-efficacy in the domain of mathematics, and expectancy of success in college mathematics are all significant predictors of achievement in college mathematics (House, 2001). However, it has also been found that there is a significant difference in the level of mathematics self-efficacy between freshman college students enrolled in an intermediate or developmental algebra course and freshman college students enrolled in a calculus course (Hall & Ponton, 2005). This in-progress study seeks to explore these differences in more detail.

Data will be collected from freshman college students enrolled in calculus classes and in college algebra classes. Students' self-beliefs about mathematics will be measured using four scales from the Fennema-Sherman Mathematics Attitudes Scales: Attitude Towards Success in Mathematics scale, Confidence in Learning Mathematics scale, Mathematics Anxiety scale, and Effectance Motivation in Mathematics scale. In addition to measuring students' self-beliefs and motivation in mathematics, students' mathematics achievement will also be measured throughout the semester through students' exam scores.

The plan for analysis of this study is to make use of simple t-tests to determine whether or not there is a significant difference in self-beliefs and motivation between the two groups of students. Also, multiple regression will be used in the analysis of this study to determine the effect of each of the FSMAS domains on mathematics achievement.

Because it is becoming increasingly common for colleges and universities to require all students to successfully complete a quantitative course during their undergraduate studies, the results of the study could be useful for college mathematics professors. Possible implications of the study are for the teaching and learning of college mathematics, particularly when it comes to the design and structure of developmental college algebra courses.

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# THE NATURE AND DEVELOPMENT OF EXPERTS' STRATEGY FLEXIBILITY FOR SOLVING EQUATIONS

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*Absent from the literature on flexibility is a consideration of experts' flexibility. Do experts exhibit strategy flexibility, as one might assume? If so, how do experts perceive that this capacity developed? We describe results from interviews with eight content experts to explore strategy flexibility for solving equations. Our analysis indicates that experts were capable of making subtle judgments about the most appropriate strategy for a given problem, based on factors including mental and rapid testing of strategies, the problem solver's goals, and familiarity with a given problem type. Implications for future research on flexibility and on mathematics instruction are discussed.*

## Introduction

Success in algebra has been and continues to be a concern among educators and policy makers because of its important role as a gateway to college (National Mathematics Advisory Panel, 2008). Increasingly, proficiency with algebra and mathematics in general is considered to involve more than just skill; it involves an integration of skill and understanding that allows for flexible, adaptive, and appropriate use of algorithms, all of which contribute to efficiency, problem solving, and transfer of ideas to new situations (Baroody & Dowker, 2003; National Research Council, 2001)). Yet, research on flexibility and how it develops is only emerging, particularly in the post-elementary years. Existing research suggests that flexibility can be enhanced through appropriate instruction (Blöte, Van der Burg, & Klein, 2001; Rittle-Johnson & Star, 2007; Star & Rittle-Johnson, in press; Star & Seifert, 2006), but much more research is needed to fully understand the development of flexibility.

Largely absent from the emerging literature on flexibility is a consideration of experts' flexibility. If researchers are to fully understand the trajectory of flexible problem solving in school mathematics, then it seems important to examine experts' approaches to advanced school topics such as algebra. Do experts exhibit strategy flexibility with algebra, as one might assume? If so, how do experts perceive that this capacity developed in themselves? Do experts feel that flexibility is an important instructional outcome in school mathematics? In this paper, we describe results from several interviews with experts to explore strategy flexibility for solving equations.

### *Strategy Flexibility*

Flexibility, particularly in terms of flexible use of strategies, is not a construct that has been consistently defined by researchers. Some distinguish flexibility from adaptability, while others equate the two terms. In this paper, we take the latter perspective by defining *flexibility* as knowledge of multiple solutions as well as the ability and tendency to adaptively select the most appropriate ones for a given problem and a particular goal (Star & Rittle-Johnson, 2008; Star & Seifert, 2006).

Several recent studies have examined the development of flexibility among school-aged learners, and this work has identified promising instructional interventions that appear to improve students' flexibility (e.g., Blöte et al., 2001; Rittle-Johnson & Star, 2007; Star & Seifert,

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2006). However, if flexibility is an important component of students' proficiency in mathematics (Kilpatrick et al., 2001), it also seems critical to have more in-depth knowledge of what this capacity looks like in experts. Do experts exhibit flexibility? How and when do experts become flexible? As discussed below, the literature on experts' flexibility is quite limited and yields inconsistent results.

*Experts and Flexibility*

Although many studies have examined differences between experts and novices with regard to complex academic tasks (particularly in physics; Chi, Feltovich, & Glaser, 1981; Larkin, McDermott, Simon, & Simon, 1980), few have focused specifically on strategy flexibility in mathematics. The limited research suggests that experts do tend to have multiple, efficient strategies for solving problems, but findings are inconsistent about the extent to which experts employ these strategies.

For example, Dowker (1992) found that mathematicians not only exhibited a high level of accuracy, but they used a variety of strategies, and the strategies tended to illustrate their knowledge of number properties and relationships. Similarly, Cortés (2003) found that experts were not only efficient, but they rapidly analyzed the task and decided on an approach based on the characteristics of the problem. In contrast, Carry and colleagues (Carry, Lewis, & Bernard, 1979) reported that the primary difference between more and less able solvers was *not* that experts knew and used a greater number of strategies but rather that more able solvers tended to make fewer errors. These and other studies provide a useful starting point for understanding experts' flexibility. However, there are several key weaknesses to this literature that suggest the need for additional studies.

First, existing research does not explore the *development* of flexibility. Do experts attribute their flexibility to prior instruction? If not, how did they become flexible problem solvers? Do they believe instruction has a role in developing flexibility? Second, existing research does not sufficiently explore the *nature* of experts' flexibility. In particular, research does not clearly distinguish between experts' use of algorithms, knowledge of alternate methods, and preference for particular strategies. Finally, existing studies have provided limited opportunities for experts to demonstrate flexibility. Although some problems that allow for efficient or elegant solutions have been used in past research, the primary purpose of most studies was to understand experts' strategies and accuracy in general, not to explore flexibility per se. As a result, they were given few problems that were explicitly designed to test strategy flexibility.

*Current Study*

The current study attempts to address the above weaknesses in the literature on experts' flexibility. Specifically, we designed tasks that provided opportunities for experts to demonstrate flexibility; we probed experts about the approaches they noticed and preferred; and we asked experts to reflect on the emergence of this capacity in their own learning. Similar to Cortés (2003), we explored flexibility among experts from several different fields, including mathematics, mathematics education, engineering, and secondary mathematics instruction. Our analysis of experts' interviews and problem solving focuses on the following issues.

First, we are interested in the *nature* of experts' flexibility for solving algebra problems. Prior research has shown that, while experts are less likely to exhibit errors in problem solving than novices, the degree to which experts show knowledge of multiple strategies and the ability to adaptively select the most appropriate strategy has varied across studies. Our investigation of experts' flexibility will focus on several aspects of strategy choice, including use of multiple strategies, knowledge of multiple strategies, and preferences for certain strategies. A second focus of the current work is *development* of experts' flexibility developed. We believe that this developmental perspective on flexibility among experts has not been well-explored in prior research. Related issues include whether experts believe that flexibility is an important outcome and how experts believe flexibility can or should be fostered in school mathematics. As such, several research questions guided our study. To what extent do experts use multiple approaches and adaptively select the most appropriate strategies for a given problem? How do experts select problem solving strategies? Do experts hold preferences for particular strategies for certain problems, and, if so, what are these preferences based on? How do experts become flexible, and what are their views about the role of instruction in developing flexibility?

## Method

### *Participants*

Participants for the current study included eight experts in school algebra. Specifically, a convenience sample of two mathematicians, two mathematics educators, two secondary mathematics teachers, and two engineers were included in the study. One of the mathematicians, (Mark; pseudonyms are used for all experts), held a doctorate in mathematics and had worked for years in a university mathematics department. The other mathematician (Matthew) was finishing his doctorate in mathematics but had also taught high school mathematics and worked with teachers for many years. Both mathematics educators held masters degrees in mathematics; one also held a doctorate in educational studies with a concentration in mathematics education (Evelyn), and the other was finishing her doctorate in mathematics education at the time of data collection (Ellen). One of the mathematics teachers was a veteran secondary teacher with a bachelors degree in physics and a masters in education with a mathematics emphasis (Tara). The other teacher held degrees in psychology and sociology (Timothy) but had been a professional tutor for many years and often taught high school mathematics in the summer. One of the engineers held bachelors degrees in mathematics and in aerospace engineering (Nathan). His job included writing flight simulations for rockets and satellites, which involved extensive use of algebra, calculus, and other areas of mathematics. The other was a mechanical engineer, with both a bachelors and a masters degree in the field. His job involved designing and testing electrical tools (Nicholas). These types of experts were chosen in order to provide a range of perspectives on flexibility and potentially different approaches to solving problems. Participants were from five different states, all within in the eastern United States. Five of the experts were male and three were female.

### *Measures*

Opportunities to assess flexibility were embedded in a researcher-designed algebra test. The 55-item test was originally designed to be used as a final examination for a three-week summer course for high school students that reviewed more advanced topics from a first year of school algebra. Test items were symbolic mathematics problems taken or adapted from a standard algebra text on solving and graphing both linear and quadratic equations, as well as simplifying expressions with exponents and square roots. The exam was edited to ensure opportunities to demonstrate flexibility. For example, one problem in the form $a(\mathrm{x} + b) = c$ was altered such that $c$ was divisible by $a$. Whereas students are often taught to distribute first for equations that include parentheses, dividing both sides of the equation by $a$ is an alternative first step to solving this equation.

Semi-structured interviews were conducted to probe experts' thinking about their strategies for solving algebraic problems. Experts were asked to explain how they solved certain problems, why they chose the strategies, whether they knew of other ways to solve the problems, and which strategies they preferred. Problems were selected based on their possible relevance for exploring issues of flexibility. After explaining their thinking on these problems, experts were asked about own experiences that led to becoming flexible and whether they thought flexibility should be or could be taught in schools.

*Procedure*

The algebra test was administered in a one-on-one setting to the experts at a time and place convenient to them. At the beginning of the test administration, experts were told that the focus would be on the methods they used to solve the problems. As such, showing work was encouraged. They were informed that an interview would follow the test in which they would be asked about selected problems, but they were not told in advance which particular problems would be selected. The test was not timed, but most of the experts completed it within 20 minutes. The interviews were conducted immediately following completion of the test. Interview times varied, ranging from about 10 to 30 minutes. Interviews were audiotaped and subsequently transcribed.

## Results

We begin by reporting results on the nature of experts' flexibility, followed by an analysis of experts' views on how their flexibility developed.

*The Nature of Experts' Flexibility*

It is important to first note that the eight experts interviewed in this study were quite successful on the mathematics tasks that they were asked to complete. Consistent with prior work on experts in mathematics and other domains, our experts rarely made errors. When errors were made, experts quickly noticed and corrected them. Our interest, however, was more on the strategies that experts used and their flexibility. We found that the experts interviewed were quite flexible. They exhibited knowledge of and use of multiple strategies for solving a range of problems, and they generally used and/or expressed a preference for the most adaptive strategies for a given problem.

*Choice of strategies.* We first consider the criteria that experts used to select their chosen strategy (which was often the optimal strategy) for a given problem. Overwhelmingly, experts indicated a preference for strategies that they deemed to be easiest. In general, the easiest strategy was the one that was "faster, quicker, less steps" (Tara). However, fewer steps was not the only consideration for choosing an easy or efficient strategy; reducing effort was also important. As one expert suggested, "It's not about extra steps. I don't mind putting in extra steps if extra steps makes it easier." (Nathan). Referring to the problem, $7(n + 13) = 42$, one expert noted, "I wanted to minimize the effort. So another way to do it is distribute, but I didn't want to do that" (Matthew).

The experts also referred to the "neatness" of a strategy in explaining their strategy choices. For example, one expert noted that to solve the above problem, he felt it was best to use division by seven as the first step because "That was evenly divisible. If it wasn't divisible it wouldn't get a nice, clean answer" (Nicholas). Similarly, another expert also noted that, "Distributing, I would have had to deal with fractions and finding common denominators and things wouldn't have been as nice" (Tara). Avoiding fractions was of particular interest to many of the experts. One expert explained that she did not like to work with fractions because they are "slower to operate with" (Ellen). In general, reducing the arithmetic complexity of the problem was important for both speed and accuracy. On expert explained, "I'm less likely to make a calculation error"

(Ellen). Another expert concurred, stating that "I am not real quick at arithmetic, so I like to keep the arithmetic as simple as possible" (Evelyn).

In addition to selecting a strategy based on its perceived ease of execution, experts also considered the specific characteristics (e.g., structure and coefficients) of problems in selecting a strategy. This criteria for selecting strategies was particularly noticeable when the experts solved systems of linear equations. For example, when solving the system which included the lines $3y + 4x = 0$ and $y = x - 7$, one expert explained "I used substitution because it was set-up that way" (Timothy). Presumably Timothy is referring to the fact that the second line is already written in the form $y$ = ?, which makes it particularly amenable to substitution in the other equation. Similarly, when another expert solved the system containing the lines $2x + y = 1$ and $x + y = 3$, he explained that he used the addition/subtraction method because "you have to get rid of something and in this case you just observed that $y$ is the same, so subtraction will get rid of it" (Mark).

*Failure to choose optimal strategy.* Consistent with prior research (Carry et al., 1979), experts did not always choose the most efficient strategy for a given problem. Typically, when experts failed to choose the optimal strategy and were subsequently asked about their strategy choice, they tended to note that they "weren't thinking." For example, one expert solved $7(n + 13) = 42$ by distributing the seven as his first step. When he later noticed that it would have been more efficient to divide both sides by seven first, he stated, "I just didn't think about it at the time. I just blew through it" (Nathan). In addition to "not thinking," other experts indicated that their initial choice of non-optimal strategies came because of well-practiced, automatized approaches that they initiated very quickly after seeing a problem. For example, another expert, who also used distribution as the first step to solve this same problem, explained his choice as, "For some reason I just went straight to the formula. I guess [I was] in that mindset" (Timothy).

However, experts' rationales for their strategy choices clearly indicated their strong tendency to prefer elegant, efficient strategies even when these strategies were not used for solving a given problem. For the problem $1/3(x + 5) + 2/3(x + 5) = 7$, both Ellen and Nathan multiplied both sides of the equation by 3 in an attempt to avoid calculating with fractions. However, when asked if they knew of another way to solve the problem, both said a better first step would be to combine "like" terms, obtaining $x + 5 = 7$, and both expressed a preference for this method. When asked why she preferred the second method, Ellen suggested, "There's something about recognizing that those two things equal one that I, I don't know, that I like….There is something pretty about it." After noticing this alternate approach, Nathan described his original strategy as "incredibly convoluted," adding that, "In this case, my fixation on getting rid of the denominators kind of obscured the problem there."

*Development of Experts' Flexibility*

Toward the end of the interview, experts were introduced to the construct of flexibility (as we conceptualize it here) and were asked to reflect on their flexibility. After learning about flexibility, not surprisingly experts uniformly believed that they themselves were flexible. Furthermore, when considering how their own flexibility developed, experts did not believe that flexibility was an overt instructional goal for their K-12 or university mathematics instructors. A typical response to being asked if they were taught flexibility was, "No, never!" (Evelyn). Instead, experts offered two explanations for how they developed strategy flexibility.

First, several experts felt that their own flexibility had emerged as a natural consequence of exposure to seeing similar kinds of problems over and over again, combined with a desire to solve problems as quickly as possible. One expert stated, "My best guess is just a lot of repetition and when you to do small equations over and over and over you're going to want to find the quickest way of doing things to get done faster" (Tara). This desire to complete problems as quickly as possible could have been present because of disinterest (e.g., wanting to complete math homework as quickly as possible), but among those we interviewed, it seemed more common that experts found the search for the quickest and easiest strategies for given problems to be intellectually challenging and interesting. Experts did not report that their mathematics instructors had pushed them to search for optimal strategies, but rather that this was a desire that they themselves brought to problem solving. As one expert stated, "It's a challenge to solve them in different ways, but then you start to learn which methods are quickest or easiest for certain problems, and you notice certain patterns" (Nathan).

Another explanation that several experts favored was that their flexibility was the result of their teaching. As one expert explained, "I've taught this stuff. I guess when you teach kids you get to know ten different ways of doing it based on what kids like" (Matthew). When describing how he developed flexibility, Timothy suggested, "I tutor students mainly who struggle, so I have just learned that if they can't see it one way, often trying another way helps them see something they didn't see before." It appears that, through interactions with students -- both having to explain problems in multiple ways to struggling students and also by exposure to the idiosyncratic, original, or even erroneous strategies that students produce -- experts developed more robust knowledge of multiple strategies for solving algebra problems.

Overall, the experts appeared to value flexibility and felt that it was an integral part of doing mathematics. As one expert described, "Problem solving is a general skill, and you have to be adaptable and flexible to the context" (Nathan). However, and consistent with experts' views on how their own flexibility developed, there was variation as to whether experts felt it was a good idea to teach students to be flexible. Some experts thought that teaching flexibility was a good way to teach students to notice the mathematical structure in problems and that it would help "the students understand the mathematics more deeply" (Evelyn). Yet others thought that teaching for flexibility would confuse students and that students should learn from trial and error. One expert stated, "You can learn some tricks from your teachers, but eventually it all comes down to doing it yourself" (Mark).

## Discussion

The purpose of this study was to explore the strategy flexibility of experts. Despite the recent emergence of a literature on students' flexibility, relatively little research exists on experts' flexibility. Our results indicated that our experts did exhibit strategy flexibility on the tasks that they were asked to complete: Experts showed knowledge of multiple strategies and the ability to select appropriate strategies for given problems. Experts expressed a strong preference for strategies that they deemed to be easiest, where the easiest strategies were those which resulted in the fewest steps, the least effort to execute, and/or the reduction of arithmetic complexity. Experts also considered the specific characteristics of problems (including a problem's structure and its coefficients) when selecting a strategy. Consistent with prior research, we found that experts did not always select the optimal strategy for a given problem, despite their knowledge of and preference for the most efficient strategies. With respect to the development of their own flexibility, experts did not believe that flexibility had been an overt instructional goal for their prior mathematics instructors. Rather, the experts interviewed here believed that their flexibility emerged from their own initiative and/or as a consequence of their teaching experiences.

Below, we discuss several implications of our results, for research on flexibility and for mathematics instruction more generally.

*Implications for Mathematics Instruction*

The experts interviewed here were in agreement that flexibility was *not* an explicit focus of their K-12 and university mathematics experience (as Evelyn noted, “No, never!”). One interpretation of experts’ views would be that flexibility should not be an instructional target in elementary and secondary schools – that flexibility is best developed implicitly and individually by the repeated problem solving experiences of learners. According to this interpretation, developing flexibility requires a significant amount of personal initiative and thus may be considered as an advanced instructional goal – one that was only available to those with a great deal of talent in mathematics and an exceptional drive to develop this competency in themselves, largely without the aid of teachers.

This point of view runs counter to the current emphasis in the US on the importance of flexibility for *all* students in K-12 mathematics instruction (e.g., National Mathematics Advisory Council, 2008; Kilpatrick et al., 2001). A proponent of this current emphasis on flexibility for all might point out that much has changed in the years since these experts attended elementary and secondary school, including a greater variety of curricula, greater diversity in instructional methods, and more generally increased attention to providing quality mathematics instruction to all students. Improvements such as these suggest that, despite some experts’ views to the contrary, it may be possible to consider flexibility as an instructional goal for all students, rather than as an outcome that is only available to future experts who pursue it themselves.

*Future Research on Flexibility*

This study underscores the importance in future research of using tasks that are specifically designed to elucidate flexibility. Prior work with students has shown that knowledge of innovative strategies for problem solving often precedes the ability to implement these strategies (e.g., Star & Rittle-Johnson, 2008). Similarly, this study and others (e.g., Carry et al., 1980) suggests that even experts do not always use the most efficient strategy for solving a given problem. As a result, merely giving students a list of problems to solve may not be a good indicator of flexibility; students may choose to use the same strategy for all problems, even when it is not the most efficient choice, and despite their knowledge of alternative approaches.

In this study and in our prior work, we have used two kinds of tasks to more effectively assess students’ flexibility. First, we have conducted interviews to accompany problem solving, asking participants to explain and justify their choices of strategies. And second, we have incorporated different kinds of problems into our assessments to better tap participants’ flexibility. In some cases, we have merely changed problem coefficients, to accentuate the benefits of using an alternative strategy. For example, instead of using a problem such as $3(x + 1) = 22$, we might alter the problem to $3(x + 1) = 21$, to perhaps increase the likelihood that participants who were aware that they could divide both sides by 3 first would actually implement this strategy. In addition, we also devised other kinds of tasks to tap flexibility, including asking participants to solve the same problem in more than one way and to identify which strategy was optimal.

## Conclusion

Strategy flexibility is an important instructional goal in mathematics instruction at all levels. An emerging research base on students’ flexibility is beginning to provide helpful guidance on the development of flexibility and instructional tasks that reliably lead to greater flexibility. However, experts’ flexibility has been relatively unexplored. This paper provides initial evidence about experts’ views on the nature, development of and importance of flexibility. Despite expert agreement that flexibility was not emphasized in their own learning of mathematics, experts in this study had a natural and pervasive tendency to value and use efficient, elegant strategies to solve algebra problems. This tendency seems to be deeply related to their knowledge of the subject, underscoring the need to include flexibility as a goal for instruction at the secondary level.

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# SEVENTH GRADERS' USE OF ADDITIVE AND MULTIPLICATIVE REASONING FOR ENUMERATIVE COMBINATORIAL PROBLEMS

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*In a year-long teaching experiment with two 7th grade students, this study investigated how those students dealt with enumerative combinatorial problems based on their additive and multiplicative reasoning. The results show that three distinctive levels of enumeration appeared in the students' mathematical behavior: additive enumeration, multiplicative enumeration, and recursive multiplicative enumeration. Permutation problems of more than five elements seem to involve conceptual constructs beyond recursive multiplicative enumeration.*

## Background

Research on combinatorics in mathematics education has not been studied extensively when compared with other topics such as whole number arithmetic and fractions. Further, ever since Piaget and his colleagues conducted a number of studies investigating the development of children's combinatoric operations (e.g. Piaget, 1975), combinatorial reasoning as a research topic has been given little attention although combinatorial problems have been adapted for the research on students' development of proof justification or verification strategies (Eizenberg & Zaslavsky, 2004; Maher, 2005; Maher & Martino, 1996). However, as Piaget and Inhelder (1975) pointed out, children's combinatorial reasoning is a very fundamental mathematical idea, whose basis is in additive and multiplicative reasoning.

> The general characteristic is to start from the additive idea of juxtaposition and not from the intersection or interference, that is, from multiplicative association. … His ability to presuppose this multiplicative association would be necessary to construct a complete operative system. (Piaget & Inhelder, 1975, p.169)

Therefore, this study was conducted using a year-long teaching experiment with two 7th graders in order to investigate how those students dealt with combinatorial problems based on their additive and multiplicative reasoning and whether distinctive features, which can be regarded as combinatorial reasoning, would be drawn from their mathematical operations in those problem situations. For this report, students' mathematical activities in the problems related to enumerative combinatorics will be the focus.

## Theoretical Perspectives

Enumerative combinatorial problems are basically counting problems. That is, children are asked to count the number of ways that certain patterns can be formed. However, when compared with a simple counting problem such as "How many apples are on the table?" there is a huge difference between the two counting situations because enumerative combinatorial problems involve more than children's basic counting schemes. In other words, in a combinatorial problem, children attend, first of all, to units of the indefinite quantity to be counted in the problem situation and, further, to monitoring their counting activity to decide when to stop counting. For example, consider the outfit problem, "If you have two shirts and three pairs of pants, how many outfits can you make?" For this problem, children must construct the units to be counted as the combination of one shirt and a pair of pants. The countable unit is a pair of units rather than a singleton unit as in most counting situations children encounter.

Piaget and Inhelder (1975) suggested three stages of children's combinatorial operations in enumerative combination, permutation, and arrangement problems through clinical interviews with children from six to twelve years old. At the first stage of empirical operations, children construct countable units for the problem without establishing a systematic way to make pairs of units as countable. That is, for a two card arrangement problem with three different cards, they take any card, combine with any other one, and simply look around next to see if that pair is already on the board. At the second stage, children have a sense of regularity, but it always remains empirical at the beginning. In other words, they quickly understand that the combinations can be ordered according to the first numbers (1, 2, or 3) in each arrangement, but this is only an empirical discovery. Finally, the third stage of understanding of the system, children can construct the law $n^2$ for two card arrangements with n cards. However, this is only empirical generalization, which does not bring about a reflective and anticipatory scheme. Later, based on Piaget's works, English (1991) addressed the *odometer strategy*, which holds one item constant while systematically varying each of the other items. She showed that children seven years of age demonstrated such systemic operations and argued that the combinatorial domain should be considered as a topic of investigation in the elementary school curriculum. However, there remains an unexplained aspect in her research results in that she did not provide operative explanations based on children' additive and multiplicative reasoning even though Piaget and Inhelder (1975) already mentioned that combinatorial operations should be rooted in additive juxtaposition and multiplicative association.

Steffe (1992) observed children's construction of such mathematical operations and referred to them as *lexicographic units-coordinating operations*, which have similar observable features to English's (1991) odometer strategy. A *units-coordination* means "to distribute a composite unit over the elements of another composite unit" (p.279). The first multiplicative concept occurred as a recursive counting scheme, which was used to produce the first units-coordination. It follows that children's construction of units-coordinating operations is crucial in their transition from an additive to a multiplicative world and researchers must understand these operations when investigating the status of children's additive and multiplicative reasoning. Specifically, children's units-coordinating operations that are involved when making possible pairs, for instance, with three numeral cards and seven letter cards are called lexicographical operations because of the dictionary ordering of the pairs. Thus, it is possible that the construction of lexicographic units-coordinating operations and symbolizing actions for those operations open pathways for children's construction of combinatorial reasoning. So the main concern in the analysis of the teaching experiment data was to investigate how children's construction of unit-coordinating operations emerged and was transformed through their mathematical activities in combinatorial problem situations.

## Methods of Inquiry

The data for this report were collected from a year-long constructivist teaching experiment (Steffe & Thompson, 2000), in which a pair of seventh-grade students were taught at a rural middle school in north Georgia from October 2007 to May 2008. The experiment is a part of the larger, longitudinal study called the Ontogenesis of Algebraic Knowing (OAK), whose purpose is to bring forth and understand middle school students' algebraic reasoning. Two students for the research were chosen after individual selection interviews conducted during September and

October of 2007. The criteria for selection of the students were that they should be able to use three levels of units for solving multiplication and division problems. During the teaching experiment, the two students were met twice a week in 40-minute teaching episodes. All teaching episodes were videotaped with two cameras for on-going and retrospective analysis. One camera usually captured the whole picture of interactions among the pair of students and the teacher-researcher, and the other camera followed the students' written or computer work with the aid of two witness-researchers. The role of the witness-researcher was not only assisting in video recording but also in providing other perspectives during all three phases of the experiment: the actual teaching episodes, the on-going analysis between episodes during the experiment, and the retrospective analysis of the videotapes. In terms of data analysis, the first type of analysis was ongoing analysis that occurred by watching videos of the teaching episodes and debating and planning future episodes. The second type of analysis, which was conducted after the data had been collected, was retrospective analysis. The purpose of the retrospective analysis was to make models of the students' ways of operating mathematically.

## Data Excerpts

*Additive Enumeration and Multiplicative Enumeration*

The two students, Carol and Damon, demonstrated that they were able to execute *additive enumeration,* in which they attended to a unit to be counted and executed counting additively. For example, on February 4, Carol and Damon solved the following coloring-window problem based on their additive counting, "Find how many ways to paint four windows with two colors." Carol and Damon were given a picture of a window consisting of four sub-windows. Carol tried to draw pictorial symbols for all possible windows and counted them whereas Damon repeated writing and erasing the letter 'r' for red and 'b' for black on a given window while adding tally marks above the window to keep track of his 'r' and 'b' entries (see Figure 1 and 2). Finally, Damon found sixteen cases although he wrote fifteen tally marks down, but Carol got fourteen at first, missing two cases.

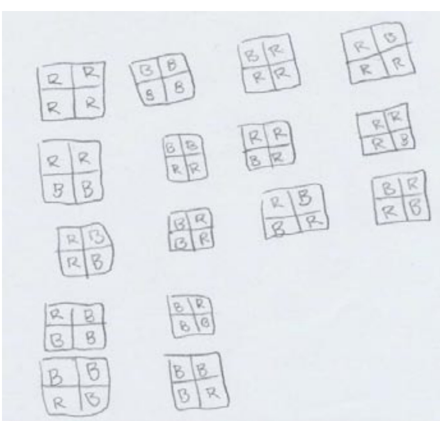

*Figure 1.* Carol's final pictorial representation.

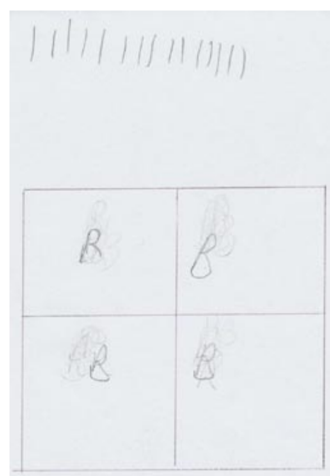

*Figure 2.* Damon's final representation using using tally marks.

Carol represented units of her unit-coordinating operations by drawing identical-looking windows like the given window, whereas Damon symbolized his results of units-coordinating

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operations using tally marks with an aid of writing possible initial letters for each window. Carol's missing two cases indicated that she engaged in additive enumeration because she proceeded sequentially and lost track of where she was at in enumerating the windows. Further, the continuing activities of the two students revealed a crucial difference in their reasoning about the problem situation that was important in their development of combinatorial reasoning. Right after checking out his answer with the teacher, Damon exclaimed, "Look at it. There is an easier way!" His explanation indicated that he constructed a multiplicative structure for counting his windows. He argued that the answer could be easily found as two to the fourth because we could use two colors for four squares (sub-windows). Damon's reflective way of counting might be considered as multiplicative enumeration in that he counted the countable units by producing two to the fourth. Carol did not even seem to assimilate Damon's explanation because she did not organize her activity as multiplicative although she finally drew her missing two windows and got sixteen as an answer. In Damon's case, even though he said, "two to the fourth", the structure seemed to be a product of his operating rather than as a given in recursively operating further because he could not provide a satisfactory justification for why two to the fourth worked. This conjecture was corroborated on February 11 while solving a two-digit number problem: "How many two-digit numbers can you make?"

For the two-digit number problem, Damon started to write two-digit numerals from ten to ninety in a column and kept writing eleven, twenty-one, thirty-one, …, ninety-one making another column and then stopped (cf. Figure 3). After murmuring something, he wrote '81' a little away from two columns as an answer, but resumed writing two-digit numbers for making sure of his answer because the teacher commented, "go ahead (keep writing) and check your answer." Then he finally corrected it as '90'. As an explanation, he claimed that zero in one-digit place could go with nine numbers in ten-digit place and so did the other numbers in one-digit place. However, he could not find the right answer until he finished writing down tabular forms of numerals for all possible two-digit numbers.

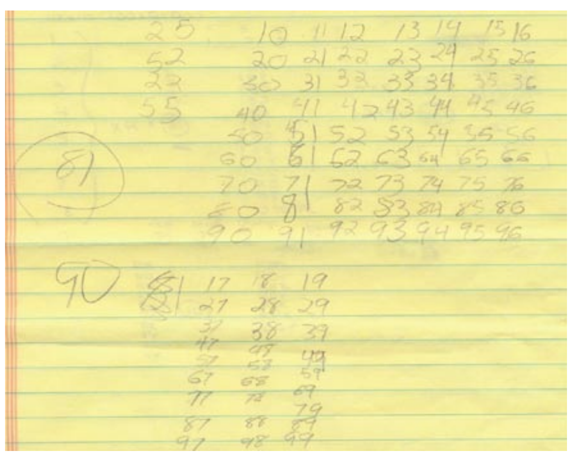


*Figure 3*. Damon's answer for two-digit number problem.

His comment that he thought his table was going to be a nine by nine table in the middle of his solving process when he said "eighty-one" and the fact that he had to transform all possible units-coordinating operations into written notation for him to be able to complete his counting activity indicate that his construction of a multiplicative structure for this combinatorial problem was a product of operating and not available for analysis prior to operating. For example, he could have transformed his counting activity into a compressed form of symbols like '9x10 = 90' just as he exclaimed $2^4$ in the coloring-window problem. Similarly, Carol made a table for arranging all possible two-digit numbers and after writing all possible two-digit numbers, she wrote '9x10 = 90' under the table (see Figure 4).

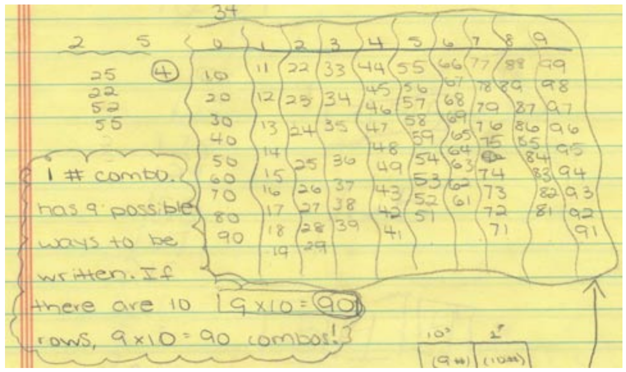
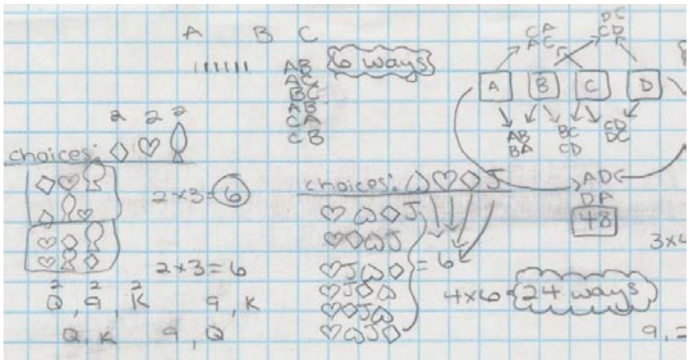



*Figure 4.* Carol's answer for two-digit number problem.

After getting the solution, she exclaimed "I know a faster way. I could have done that. Oh~" and she explained "nine different combinations (pointing out columns) in each way. There are ten rows (actually ten columns) and ten times nine is ninety… If I figured out that each row has nine, I could have just done that." Nevertheless, an inference that her numerical symbols and her reflective explanation concerning how she could organize her work were available to her prior to actually engaging in operating was not warranted. That is, an inference that Carol's construction of the problem situation and her mathematical operations for the solution were transformed into the numerical symbols, '9x10 = 90' prior to operating would be too strong. Rather, as Skemp (1987) illustrated as one of the functions of symbols, Carol's numeral table for her solution made possible her reflective thinking and reorganization of the situation. Her retrospective analysis using her table led her to represent her construction of the situation and operations for the answer by writing a compressed form using numerical notation. Thus, at this moment, Carol did not seem to produce a multiplicative structure as a given for the solution prior to operating. To construct a multiplicative structure and use it to find an indefinite quantity is an important mathematical modification because such a structure permits a child to curtail her counting activity, and further helps her interiorize conducted actions and operations so that she could use them a priori for the construction of more abstracted combinatorial reasoning, that is, recursive multiplicative reasoning.

*Recursive Multiplicative Enumeration*

The teaching episode on February 13 was focused on card arrangement problems not allowing a repeated use of any card. For instance, "There are five different cards. How many different ways can you arrange those cards?" Carol and Damon began with a two-card arrangement problem and went through a five-card arrangement problem. While solving those problems, they used similar representations in that they symbolized by writing all possible units-coordinating operations of card arrangements (see Figure 5 and 6).



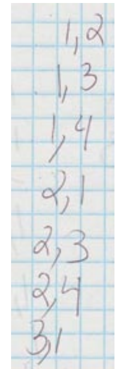




*Figure 5*. Carol's card arrangement. *Figure 6*. Damon's card arrangement.

However, when they were faced with the five-card arrangement problem, a distinct feature from the previous card arrangement problems was observed. Damon tried to write down all possible arrangements of five cards by fixing '1' for the first card, but seemed to lose track of his systematic way of proceeding. After two minutes passed, he suddenly wrote down '120' on his worksheet. After the teacher's requested an explanation for his answer, he said he fixed the first two cards as '1' and '2' and counted all possible five-card arrangements with the fixed two numbers, which were six cases and then he got one hundred twenty by multiplying by four and five in order. However, he could not provide a satisfactory justification for why he multiplied by four and five. On the other hand, Carol represented those possible cases more systemically and provided a clearer explanation than Damon did (see Figure 7).

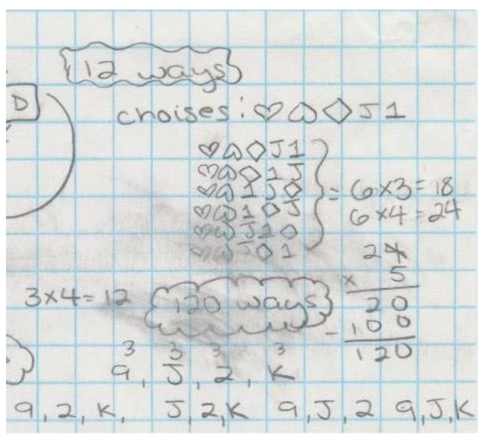



*Figure 7*. Carol's solution for five-card arrangement problem.

Nonetheless, whether Carol, let alone Damon, was explicitly aware of a multiplicative structure is still ambiguous even though she worked symbolically. The five-card arrangement problem seemed to reveal a limitation in their multiplicative enumeration. Writing down all possible five-card arrangements seemed to be very hard for them. They both could hold the two initial marks constant and systematically and exhaustively vary the elements to place in the three remaining places. But this was nothing other than a modification in the lexicographic orderings that they initially demonstrated. They still needed to "run through" this way of ordering in the case of the last three places. Once that was completed, they could again use their lexicographic method of ordering, but this time they anticipated the number of choices—four and then five. However, they seemed yet to construct the concept of a "slot" [abstracted unit] that anticipation of the number of choices of four and five might have indicated. A slot is an abstraction from the mental process of running through a deck of 52 cards, say, and choosing each one in order when finding how many pairs could be made using the 52 cards. The abstraction follows from unitizing the mental process. The construction of this concept of variable seemingly is critical in recursive multiplicative enumeration. For students' multiplicative enumeration to be recursive means that the students are able to externalize the results of their units-coordinating scheme and operate on these results with operations external or internal to the scheme (Tillema, 2007). When they fixed the first and the second card, counted all possible cases with the two fixed cards and arrived at six, both students simply multiplied six by four and that result by five, which indicated recursive operations. However, because they did not a priori operate in this way, the inference that the students had constructed the concept of a slot that stood in for making five choices, then four choices, etc. was not warranted. Still, both students distributed the six possibilities they generated by holding the two initial marks constant over each element of four possibilities of the second card. Furthermore, the two students took the results of the multiplicative enumeration up to the second card as given and operated on another multiplicative enumeration with the first card position. They could engage in recursive multiplicative enumeration in the process of operating, but they were yet to do so prior to operating. The two students' distinct solving activities with five cards as well as less than five cards leads to the hypothesis that arrangement problems of more than five elements require students to have constructed a program of operations that involves the multiplicative coordination of the operations that are recorded in a sequence of slots if the students are to mentally solve them without engaging in actual coordinating activity.

## Conclusion

This study suggests that (a) the students' enumerative combinatorial counting constructs can be based on their units-coordinating operations, (b) three distinctive levels of enumeration appeared in the students' mathematical behavior: additive enumeration, multiplicative enumeration, and recursive multiplicative enumeration, and (c) permutation problems of more than five elements seem to involve conceptual constructs beyond recursive multiplicative enumeration, that is, the concept of a program of multiplicative operations. These explanations open the way to study students' construction of combinatorial reasoning beyond the formal-operational period in Piaget's works.

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# COLLEGE STUDENTS' UNDERSTANDING OF RATIONAL EXPONENTS: A TEACHING EXPERIMENT

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*Five college students' understanding of exponents was investigated. The students then participated in a teaching experiment to test an alternative route for studying rational and negative exponents. The focus of the study was the role of the laws of exponents in the process of students' understanding of exponents that are not natural numbers. What is the impact of the teaching experiment on the students' understanding of such exponents? Results suggest that students do not base their understanding of non-natural exponents on the laws of exponents and that the teaching method can improve understanding of exponents, depending on their earlier experiences.*

## Introduction

Exponents are basic tools in the arsenal of college students. Functions that model population growth, absorption of light, decay of radioactive material, mortgage rates and interest on capital, all require forms of exponents. Learning exponents starts early in school mathematics. Counting how many times a number is multiplied by itself is probably the main notion in learning exponents. When students move up the mathematical ladder, exponents can be negative, fractions or even irrational numbers. How students construct their notion of such exponents has not been extensively studied. Do students reconcile their early notion of exponents as counting numbers with the new exponents they learn? Do they develop meaningful explanations for rational exponents that make sense to them? What do they think of the zero exponents?

## Research Questions

1. What is the role of the laws of exponents in the students' construction and understanding of rational and negative exponents?
2. How do these college students understand and justify their notions of rational and negative exponents?
3. What is the impact of an alternative route in teaching rational and negative exponents on these students' understanding of exponents?

## The Teaching Conjecture

Postponing formal definitions the idea of rational exponents was built around the notion of relative rates of change. The context was population growth. Rates of change were transformed into factors of multiplication per "time" unit: if a population grows 10 % each year, and the present population is 5000, then the next year the population will be 5000 * 1.1. The factor 1.1 is the factor of multiplication over one year. What could be the population after two years? After three years? After half a year? Each unit of time has its own factor of multiplication.

The emphasis shifted from just following the formal definition to the calculation of factors of multiplication per unit of time or parts of the unit. These factors of multiplication were taken as the intuitive basis for rational and negative exponents. The link between the early definition of

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exponents and the new factors of multiplication using roots and powers of roots was actively pursued.

A major part of the conjecture was a procedure to solve exponential equations by finding the decimal digits of the exponents. The conjecture was that through this procedure students would create, discuss, and get familiar with decimal exponents and their relations to the laws of exponents for rational numbers, the place value system for decimal exponents and an alternative interpretation of the zero exponent.

## Empirical and Theoretical Background

Many textbooks define $A^N$ as A*A*A*….*A with N factors of the number A. $A^1$ is defined as A and $A^0$ as one (1). Researchers have commented that these definitions together with those for rational and negative exponents represent an inconsistency in terms of presentation, language, and /or meaning of exponents compared with the early definition, in this paper the common definition, of exponents (Lockhead, 1991).

*Empirical Studies on Multiplication and Exponential Forms*

Many problem areas in learning and understanding multiplication have been identified in empirical studies on multiplication and exponential forms. Studies by Bell, Fischbein, & Greer (1984), Ekenstam & Greger (1983), Tirosh & Graeber (1986), have identified misconceptions about multiplications and divisions with decimals and fractions. Students' struggles with word problems involving multiplicative contexts with or without fractions were documented by DeCorte, Verschaffel & Van Collie (1988), Taber (1991). Others, like Boulet (1998), Mulligan & Mitchellmore (1997), have studied ways to conceptualize multiplication and to link it to models of repeated multiplication of smaller and smaller units. An important step was made when Confrey and her colleagues created the splitting model of multiplication to propose an intuitive model for exponential multiplication. Teaching experiments testing the model suggested a strong learning potential of the model for visualizing exponents and multiplication (Confrey & Smith, 1995).

*Students' Understanding of Definitions*

Rational and negative exponents are defined in textbooks by relying completely on a definition. Natural exponents refer to actions that are clear and meaningful to the student. Definitions that have a strong familiar side and tend to be immediately clear to students are classified as "logical" definitions (i.e., logical for the learner!). Definitions that rely on features and characteristics of a concept and rely solely on these properties are classified as "lexical." Students, even in mathematics, seem to have problems with lexical definitions (Edwards, 1997; Edwards & Ward, 2004). What they develop over time is a concept-definition and a concept-image. These are cognitive structures associated with the concept or the definition and constructed based on experience mostly (Tall & Vinner, 1981). Students can also show considerable skill in computations with a concept without being able to explain what the steps mean or how to attach conceptual meaning to the symbols involved in the computation (Owens & Super, 1993).

*Theoretical Framework of Understanding of Mathematical Concepts*

Herscovics, Bergeron and Goldin developed a framework for the concept of mathematical understanding with three components. The first component deals with intuitive (visual, perceptual) understanding, logical-physical and procedural understanding, and logico-physical abstraction. The second component describes the emergence of mathematical concepts through logical and procedural understanding, mathematical abstraction and formalization. At each stage the learner constructs conceptual schemes that involve layers of knowledge for problem solving. The construction of notions of invariance of operations, generalizations, and more abstraction are part of the second component. The more advanced the mathematical concepts, the less physical pre-concepts and more prior mathematical concepts are involved (Goldin & Herscovics, 1991). The third component of the framework is the theory of internal and external representations. External representations are symbols, diagrams, notations, graphs, equations, in general tools for thinking. Internal representations are inferred mental constructions, and cognitive conceptual schemes. Language, including definitions, visual, kinesthetic systems etc., are all part of internal representations. External and internal representations form one system that also includes affect (Goldin, 1998; Goldin & Herscovics, 1991). External systems are sometimes expressions of internal systems, while external systems can be internalized. The learner constructs her/his internal representation through acts interacting with tools and systems. Internal representations can also interact and mediate the working of other internal systems (Goldin, 1998).

*Encapsulation of processes.* In the Goldin-Herscovics-Bergeron model of mathematical understanding (Herscovics & Bergeron, 1988; Goldin & Herscovics, 1991) it is not clear how learners' knowledge evolves from first actions to mature mathematical concepts. APOS theory by Dubinsky (1994) provides a framework. Actions (A) on physical or mental (mathematical) objects are internalized by the learner and (can) become processes (P). When the internalized process has evolved to the point that the learner can reverse or transform such processes and actions mentally, it takes on the form of a mathematical object (O) for the learner. This is the stage of encapsulation. The learner thinks or talks about a process or an operation from a global, structural, point of view. She/He can perform actions on the mental image of the process as if it was an independent object. The whole mental construction becomes part of a new scheme (S).

## Methodology

Five undergraduate students at a large Midwestern university were first selected based on their willingness to be interviewed and to participate afterwards in a four week teaching experiment. One student was a freshman, two were in their second year and had completed pre-calculus classes with the researcher, and two were in the education program in their third year and had completed calculus classes successfully. The students are given fictitious names and are called Ann, Chandra (both female), Bob, Dennis, and Eddy.

The research started with the pre-interview: a semi-structured interview, lasting a little over one hour, conducted by the researcher with each student separately. The interviews were audio taped and then transcribed. The purpose of the pre-interview was to document the knowledge and understanding of rational and negative exponents of each student at the beginning of the teaching experiment.

The five students then participated in a videotaped teaching experiment of four weeks to test the effectiveness and impact of an alternative route to teaching and justifying exponents and improving the understanding of decimal exponents. In the teaching experiment the zero exponent was treated separately in two different contexts. Once as related to smaller and smaller fractional exponents that tended to a factor of multiplication of one. And later, as related to its role as a place holder in the decimal representation of exponents when solving exponential equations. The students were again interviewed separately after the second, the third and the last week of the teaching episodes. These interviews were semi-structured and audio taped. After the conclusion of the teaching experiment the post-interview was conducted separately with each student, using the same questionnaire as the pre-interview.

The video tapes made during the teaching experiment were studied in various cycles to extract as much as possible relevant material for transcription and analysis. The pre- and post-interviews together with the data from the interviews conducted during the teaching experiment provided a basis for studying changes in the students' ideas and their understanding of exponents

## Results

*The Role of the Laws of Exponents*

The Laws of Exponents (LOE) operate for all the students as basic properties for computational purposes. Each student in the study was able to work with positive integer exponents using the LOE "spontaneously." Only one student (Dennis) was aware of the actual name of LOE. All the others just focused on knowing the specific properties that were needed to work with exponents. The first conclusion is that the Laws of Exponents functioned at the computational level for the students.

The notion that the LOE would act as a collection of properties for understanding the invariant patterns or properties for all exponents and thus would provide a strong basis for creating rational and negative exponents was not confirmed. The most powerful concept image of all the students turned out to be the Common Definition of Exponents (CDE). This well established network of ideas states that an exponent represents the number of factors in a repeated multiplication of a certain unit or base number. The concept implies the existence of a base, or a unit, a stage of multiplication, and the presence of a sufficient number of factors to be multiplied.

*Students' Understanding of Rational and Negative Exponents*

The CDE is more powerful than the LOE in forming the notion of students in their efforts to understand what rational exponents or negative exponents are. All the students reported serious conflicts in their learning activity when trying to explain what these rational or negative exponents were. From a learning perspective, the most important activity was the effort of the students for overcoming the notions of the CDE as a basic but conflicting set of ideas for expanding the concept of exponents.

The zero exponent exposed the limitations of the traditional approach to exponents further. All students reported a conflict in trying to apply the CDE to the zero exponent. When asked to explain the zero exponent they all expressed reservations as to the justification for the value one assigned to powers of zero. Only Dennis, the freshman offered an (incorrect) explanation why the value should be one. All students mentioned the authority of the teacher as a basis for accepting the value of one (1) for zero exponents. None invoked the laws of exponents as a basis for their beliefs. Each student expressed the feeling that the value one was either "weird" or that it should be zero, because there were "no factors."

All the students seemed to show attempts to explain their images of exponents of various kinds by trying to fit these new notions of exponents into the definition that they knew and understood fully, the Common Definition of Exponents (CDE). The zero exponents were conceived as finding a multiplication when zero factors are used or as a special case with an explanation that seemed plausible, but avoided answering the question how the CDE is related to the value of 1. The negative exponents also presented challenges, voiced by students as "weird" or "unacceptable," or "you can't have negative exponents," or they referred to the "school teachers" to justify the meaning of negative exponents. The fact that there was no apparent conceptual connection for the students between the CDE and the traditional definition of negative exponents did not stop them from applying the working definition of negative exponents and solving problems. The authority of the teachers, the school and the existence of "answers" probably overruled their concerns. With the rational exponents the basic picture was similar in the sense that the CDE seemed to function as the point of departure for interpreting the given definition. No reference was made to rational actions to be applied to the underlying properties of roots and powers. This notion that all you needed was the value of the exponent seemed to be connected also to the unambiguous world of the CDE where one given exponent fixes the value of the expression.

Each of the five students showed some hesitancy or uncertainty about how one knew if one given number is a power of another given number. The common operations with exponents seemed to go one way only: given a base and a number, find the value when the base had that number as an exponent. The reverse action seemed quite unfamiliar to the students. This problem would re-appear when the chain of multiplications was proposed in the teaching experiment.

The CDE seemed to be the only concept that had a clear operational process and a meaningful interpretation for the concept of exponents for the students. The lack of a meaningful extension and development of the CDE into a broader notion that reconciled the CDE with the new definitions for the zero exponent, the rational exponent and the negative exponent made it unlikely that students like those interviewed would develop on their own, an overarching concept of exponents that was applicable to all types of exponents, from integers, to zero, to rational and negative numbers.

When asked to explain what $5^{1/3}$ meant to them they were able, after some hints, to explain what that symbol stood for. Chandra and Dennis commented that $5^{1/3}$ did not mean 1/3 times the factor 5. Chandra saw the definition as problematic: "How do you multiply the number five 1/3 times? That's impossible." The equality of forms like $5^{1/3}$ and $5^{2/6}$ were not connected to radicals and their properties, but to the fact that 1/3 = 2/6. Only Anna and Chandra were able to explain after some hints, how the decimal digit in $3^{2.1}$ could increase the value of $3^2$ from 9 to 10.045…

The conclusion is that the laws of exponents did not provide the students with sufficient support to build an understanding of rational, negative or the zero exponents. The definition based on counting factors (common definition) was still the lens through which students tried to understand exponents of all kinds. Much more cognitive support was needed to build strong understanding of rational and negative forms of exponents for these students.

*Impact of the Teaching Experiment*

The biggest hurdle for the students was to understand that the model for population growth was not linear in its mechanics. Their initial ideas involved using linear and additive thinking models to find rates of change over multiple periods or over parts of a period. If a population grows by 10% each year then why is it not growing 20% over two years? If a bacterial population grows by 100% each three hours why does it not grow 33% per hour? The central role of factors of multiplication and the multiplicative nature of the process required extensive work and discussion.

A procedure for numerically solving (without the use of logarithms) exponential equations like: find A if $5^{A} = 10$, made students create their own versions of the procedure. This part was internalized by the students with minor problems. The problems were associated with how to handle and explain digits of zero in exponents like $3^{2.0959} = 10$.

Students struggled with the requirement in the teaching to justify the rational properties of fractional exponents from the properties of radicals and powers of radicals. This was most pronounced for Bob, Dennis, and Eddy. They stated never to have been exposed to such knowledge before. All roots had to be checked with numerical examples and calculators by these students to convince them of certain properties of radicals.

Negative exponents were introduced using three perspectives. The first focus was on the graph along the x-axis to find previous population numbers by dividing by the factor of multiplication. This was explained as a “backward” movement of exponents associated with directed numbers in ordinary subtraction. Then the functions $F(x) = a^x$ and $G(x) = (1/a)^x$ were compared and found to be symmetrical with intersections at X = 0. Finally the symmetry was used as a way to solve exponential equations like, find X if $(0.8)^X = 10$. Finding Y in $(1.25)^Y = 10$ first, then using the symmetry to understand that X = -Y provided a solution for the first equation.

All the students accepted the metaphor of backward movement for the exponents. Bob did not fully understand the use of symmetry for solving equations, but accepted the “backward movement” idea associated with the division. He even proposed the term “factor of division” for the number used. Ann and Chandra repeatedly used the metaphor themselves, while Eddy created his own symbolic representation to describe his understanding of negative, positive and zero exponents by what he called “forward, backward and no movement at all in the graph.” Dennis vacillated between using the metaphor and using his high school method of replacing negative exponents by finding the inverse of the base.

The post-interviews revealed that Ann and Chandra were able to formulate a general, single definition of exponents that applied to rational, decimal, zero and negative numbers for exponents. The other students were not able to do that, but none mentioned weirdness or inconsistency again. Their definitions were still the common definition with special cases for rational and negative exponents. Dennis still stuck to his definition of zero from high school, but all the others linked the zero exponents to the fact that higher roots of a number tend to produce numbers like 1.0000… with expanding rows of zeros. This was associated to a final value of 1 for exponent zero.

## Discussion

The teaching experiment did not result in all students being able to voice a generalized notion of exponents that applied to all possible numbers. Two students did reach that stage however. They were the students who had studied calculus and seemed to be more mathematically mature compared to the other students.

There is still more research to be done to find out what is needed beyond what was offered in the teaching experiment that can guide students to understand fully what exponents stand for and how the original laws can be extended to fractions, decimals, negative and even irrational exponents. The method to show how exponential equations can be solved numerically can possibly be used to help students understand intuitively what logarithms are.

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# RELATIONSHIPS BETWEEN MOTIVATION AND STUDENT PERFORMANCE IN A TECHNOLOGY-RICH CLASSROOM ENVIRONMENT

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*We report results from a larger study using a mixed-methods approach to examine the relationship between changes in attitude and achievement using the connected SimCalc MathWorlds® environment. Aspects of students' anxiety linked to sharing work publicly declined during our curriculum intervention. This was significantly correlated with increases in student knowledge and interpretation of multiple representations of functions. Similarly, positive change in student attitude towards technology was related to increases in knowledge of algebra concepts related to linear functions. After analyzing video data, we believe positive student outcomes result from increased motivation through active and mathematically-meaningful participation in the classroom.*

## Background

SimCalc MathWorlds® software (herein referred to as SimCalc) allows students to create mathematical objects on graphing calculators and see dynamic representations of these functions through the animations of characters whose motion is driven by the defined functions. Students are then able to send their work to a teacher's computer. Calculators are connected to hubs that wirelessly communicate to the teacher's computer via a local access point. The flow of data around a classroom can be very fast allowing large iterations of activities to be executed during one class.

In our intervention, we included activities that allow students to create functions in SimCalc on the TI-83 Plus or TI-84 Plus graphing calculator which can then be collected (or "aggregated") by a teacher into the SimCalc software running in parallel on a computer using TI's Navigator Wireless network. The activities are part of a curriculum, developed and refined over many years, that focuses on core high school Algebra ideas such as linear functions, simultaneity, covariation, and slope-as-rate — rather than slope as $m$ in the equations, $y=mx+b$. The activities utilize Classroom Connectivity (CC) in new ways to supplement or replace existing traditional algebra curriculum (Hegedus & Kaput, 2004).

We report some specific findings from a larger quasi-experimental study where we investigated the implementation of SimCalc—a dynamic software and curriculum package—in regular U.S. high school classrooms. In particular, we focus on the impact of our resources on student motivation and attitude, and its correlation with mathematical performance. Significant learning gains by high school Algebra 1 students were measured across a 3-6 week quasi-experimental intervention conducted in several ninth grade Algebra 1 classrooms across two medium- to low-achieving districts in Massachusetts, U.S., with teachers of varying experience

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(t(322)=2.711, p=0.007).

## Theoretical Perspectives

Our connected approach to classroom learning is reiterated in seminal works that highlight the potential of classroom response systems to achieve a transformation of the classroom-learning environment (Bransford, Brown & Cocking, 1999). Similarly other investigators have expanded their approaches to include devices that allow aggregation of mathematical objects submitted by students (Wilensky & Stroup, 2000). Linking private work in a mathematically meaningful way through networks, and displaying the aggregations of whole class work, potentially enhances students' metacognitive ability to reflect upon their own work in reference to others (Huffaker & Calvert, 2003). These activities create an intrinsic motivation context with a socio-cultural view to "motivation in context" (Hickey, 2003) that is also intrinsically mathematical, accomplishing a much more intimate intertwining of motivation and mathematics that can be typically accomplished in existing classrooms.

## Research Questions

In this paper, we focus on one particular SimCalc class, taught by an experienced SimCalc teacher, with respect to changes in attitude and learning gain. We explore the following questions:

- What relationship(s) exists between student gains in performance on pre- and post content tests and measures of attitude?
- In what ways might attitude influence performance gains?
    - Which classroom behaviors and/or interactions suggest an attitude–performance connection?
- How might the SimCalc environment impact relationships between student performance gains and attitude?

## Methodology

*Sample*

Five teachers in a total of eight classes in two school districts participated in the larger SimCalc study. The remaining Algebra 1 classes in each district, eight in total, were used as comparison classes. Algebra 1 is typically taught in ninth grade where students are 14-15 years old. The teachers involved in the study were not randomly chosen; rather they agreed to be a part of the project for various reasons. We chose the single SimCalc classroom on which to focus our paper based upon the robustness of the relationship between the variables we wished to investigate, i.e., performance gain and attitude.

*Data Collection*

Motivation was measured using pre- and post-intervention attitude surveys. The student survey used to measure attitudes and beliefs about mathematics, school, and technology was comprised of 27 items that participants responded to on a Likert-type scale ranging from "0 – Strongly Disagree" to "4 – Strongly Agree." An example item was, "I think mathematics is important in life."

Video data was collected for each class in the SimCalc group and one class in the comparison group during the intervention. Each class was recorded with two digital cameras. One of the cameras was focused on the teacher and the whiteboard space where connected SimCalc was projected. The other was positioned at the front of the class and was focused on the students using a wide-angled lens to pan out and observe whole class dynamics and small group interactions. Both cameras were used as roaming cameras when the class was involved in small group work. Video data was collected for twenty-six classes, each class lasting approximately fifty-five minutes for the SimCalc class in this case study. A second researcher took detailed field notes of the classes. Selected students were interviewed at the end of the intervention. Classroom video episodes were used as a qualitative component that re-enforced our quantitative findings because surveys and/or structured interview analyses do not always accurately reflect the attitude/affect of a student (Goldin, 2008; Ma & Kishor, 1997; Schorr & Goldin, 2008).

The instrument used to measure learning gains was a mathematics content test compiled from various state high-stakes state tests used to determine school success for No Child Left Behind (NCLB) accountability. The 22 item pre- and post-test included twenty multiple choice items worth 1 point, one short answer question worth a maximum of two points, and one long answer question worth a maximum of four points.

## Results

*Quantitative*

Our initial analysis focused on student gain in specific content areas and on changes in attitude. Our goal was to identify patterns of change and potential relationships between various dimensions of the data.

The content test was broken down into concept categories: graphical interpretation items (41% of the test), proportion and rate items (23%), recognizing and determining a pattern items (9%), and multiple representation items (27%). These groups did not include the long answer open response item, which was omitted due to a low response rate by both treatment and comparison groups. The SimCalc group showed gains on each category of the content test. In particular, the SimCalc students had a significantly greater gain than the comparison group on multiple representation items $t(322)=3.069$, $p<0.01$. These items dealt with generalizing relationships across representations. Conceptual transfer across multiple representations of a mathematical concept or object is an important theme in mathematics education and one of the National Council of Teachers of Mathematics process standards (NCTM, 2000).

We conducted a principle components analysis on items from the student attitude survey, which produced a four-component model that accounted for 48% of the total variance. We categorized these into four broad components that coincided with our theoretical model of attitude: *Deep Affect/Beliefs not subject to casual change (20.8%)*, *Anxiety (11.4%)*, *Preference to work alone (8.5%)*, and the *Perception and Use of Technology (7.3%)*. Based on this analysis we computed weighted composite scores for each dimension on the survey to allow us to measure change in attitude over time. Table 1 below displays descriptive statistics for each subscale used in the analysis from our sample population.

*The SimCalc case study class.* The case study class demonstrated gains in both performance and attitude when compared with the aggregated treatment and comparison groups. This class had a lower score mean score on the pre-content test than the aggregated treatment or comparison groups. However, the case study class demonstrated greater content gains (2.42) from pre to posttest than either the treatment (1.99) or comparison (.96) groups. They also had higher mean scores on the pre-survey for the *Deep Affect* subscale (11.3, compared to 10.6 for treatment and 10.4 for control), and the *Anxiety* subscale (9.0, compared to 8.4 for treatment and 8.3 for control), and recorded a greater gain (.40 and -1.2 respectively) than the other two groups (-.19 and -.27 for treatment and -.48 and -.07 for control).

Table 1
*Descriptive Statistics for Attitude and Content Test Sub-scales on the Pre Survey and Pre Content Test for all Classrooms in the Study*

| | | N | Mean Initial Score | SD | Min | Max |
|---|---|---|---|---|---|---|
| Content Test | Content Test Score (0-26)* | 396 | 10.2 | 3.6 | 1 | 19 |
| | Graphical Interpretation (0-9)* | 396 | 5.1 | 1.6 | 1 | 9 |
| | Proportion and Rate (0-6)* | 380 | 2.9 | 1.8 | 0 | 6 |
| | Recognizing and Determining a Pattern (0-2)* | 396 | .36 | .48 | 0 | 1 |
| | Multiple Representations (0-6)* | 396 | 1.8 | 1.4 | 0 | 6 |
| Student Survey | Deep Affect | 301 | 10.5 | 3.6 | 0 | 18.2 |
| | Anxiety | 296 | 8.3 | 3.4 | 1.1 | 18.7 |
| | Preference to Work Alone | 304 | 3.2 | 2.5 | 0 | 9.3 |
| | Perception and Use of Technology | 301 | 5.7 | 2.1 | 0 | 10.9 |

*The range of scores for the subscale

A larger positive gain on the *Deep Affect* subscale implied that students chose higher agreement responses on items describing math as interesting and important. This gain also suggested a more positive overall attitude toward school at the end of the intervention than at the beginning. Similarly, the case study class had a larger negative gain for the *Anxiety* subscale, which suggests students were less anxious at the end of the intervention then they were at the beginning. The case study class had a lower mean for the *Perception and Use of Technology* subscale on the pre-survey (5.3) but had the highest positive gain among classes (.36). The gain for the aggregated treatment group was negative (-.20) indicating that they enjoyed technology slightly less than they had at the beginning of the intervention.

We explored the relationships between changes in attitude and gain in content knowledge by analyzing a correlation matrix of these dimensions (Table 2). Most notably, changes in anxiety and preference to work alone were correlated to gain on the multiple representations subscale. A regression analysis confirmed that *Anxiety* significantly predicted gain on this subscale, $ß=-.39$, $t(16)=-2.82$, $p=.012$. *Anxiety* also explained a significant portion of variance in gain scores on this factor, $R^2=.33$, $F(1, 16)=7.96$, $p=.012$.

Table 2
*Correlations between Gain on the Content Test Subscales and Changes in Attitude Subscales for the Specific SimCalc Class*

| | Mult. Rep. Gain | Gain in Graph Int. | Change in Deep Affect | Change in Anxiety | Change in Work Preferences | Change in Perception and Use of Technology |
|---|---|---|---|---|---|---|
| Total Gain | .711** | .764** | .327 | -.422$^{\dagger}$ | .370 | .460$^{\dagger}$ |
| Mult. Rep. Gain | | .261 | .214 | -.576* | .515* | -.047 |
| Gain in Graph Int. | | | .336 | .004 | .350 | .535* |
| Change in Deep Affect | | | | -.389 | -.007 | .633** |
| Change in Anxiety | | | | | .065 | -.016 |
| Change in Work Preferences | | | | | | -.004 |

*p<.05, **p<.01, $^{\dagger}$ p<.1

*Qualitative*

In the video data collected in our comparison classrooms, and at the beginning of the SimCalc intervention for this particular class, students were seated in rows, answered teacher questions when called upon, and the Initiation-Reply-Evaluation sequence was the primary discourse method of the class (Wertsch & Toma, 1995).

Towards the end of the SimCalc intervention, the classroom discourse in this specific SimCalc class was quite different. Dialogic function was evident as students actively questioned, reacted to, and transformed the ideas and utterances of their peers, their peers' work and their own work (Wertsch & Toma, 1995). We propose that these illustrate the significant learning and, more importantly, the correlation with attitudinal changes that are evident in our measurements presented above. The particular classroom episode on which we have focused is not an outlier. This episode is a representative sample of the class from the latter portion of the intervention that exemplified the discursive and pedagogical practices that may be attributed to significant changes in our content and motivation sub-scales. In this episode students worked on an activity called *Coming Together*. In this activity students created a motion for a SimCalc actor, B, such that B started at 2 times their group number of feet (each group was assigned a different group number) and ended in a tie with Actor A. Actor A was defined by the function: y=2x on a domain of [0,6]. Students built a function expression to fit the goal of the activity. Once students developed their functions, the teacher collected them. Before any student work was shown the teacher conducted a class discussion. A discussion about the motion began when the teacher asked what would happen when she ran the animation.

(1) S5: We're all gonna go different speeds but we're all gonna end at the same position cause that uh end at
(2) Teacher: Where is everyone going to be at the end of the motion?
S4: Not all different speeds.
S6: 12.
(5) S1: 12.
Teacher: 12 what?
S1: feet.
Teacher: Feet. Right okay... So at the end
S4: They're gonna be at point (6,12).
(10) S5: Group 6 isn't gonna move. {S5 is in group 6}
S8: yes we move. We go like this. {In the air with her pointer finger outstretched, she gestures a horizontal line}
S4: They go sideways.
S1: No they don't move, time goes on.
S9: Cause they start at 12.
(15) S1: Yeah.
S3: It's like they already won.
S4: It's like they're at rest.
S5: Yeah but on the world we don't move.
S4: They're resting for 6 seconds.
(20) Teacher: S5 says in the world Group 6 is not gonna move.
S1: Yeah.
S3: No, they're not gonna move at all.

S3: That's right. They just stay there.
Teacher: Do we agree with that?
(25) *{Multiple students reply with yeah}*
S8: Yeah but time is moving.
Teacher: Time is moving but they aren't.
S1: Yeah cause you'll like see everyone else move but them.

In this excerpt, the teacher repeated what a student said twice, and facilitated the conversation three times: once to ask a question, once to clarify the units, and once to see if the class has come to an agreement. Student agency was evident as they debated their work and the work of their peers (lines 9-19). The students in Group 6 started at 12 feet and must end at 12 feet, therefore their character would not be moving. The students were building their understanding of how time and position co-vary.

In line (9), a student responded with a multiple representational answer. The class was discussing the motion, which is measured in feet, but his answer was in terms of the graph, the line segments would "end" at the point (6,12). Time was implicit in the motion.

Line (2) showed the teacher accepting the first response and then attempting to ask a different question. S4 did not agree with S5 and he initiated a discussion. While his comment was not addressed in this except, S4 followed up a little later in the class again saying, "not all different speeds." S4 then explained that some groups would have the same speed, but different velocities, because they were traveling in a different direction. He also conceptualized the symmetry that was created in the class' set of motions and graphs as evidenced by his discussion of the slopes of the line segments that followed later in the class:

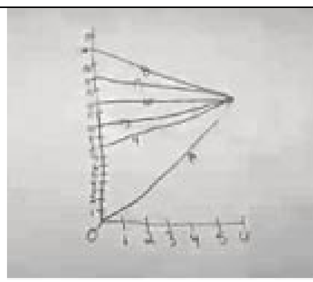
S4: We were the opposite of them. We were their opposite.
S6: And so then the next two...
S4: Cause they were 2 and we were negative 2
Teacher: So the slope. So you think group...you were saying group 4…
S6: 5 and 7 I think
Teacher: So group 5 and group 7 had opposite slopes?
S6: Yeah
S4: And group A and group 4

The SimCalc activities allowed discussions to emerge that were not present prior to the intervention. The students talked to one another about the mathematical objects they created on their calculators and that had been aggregated by the teacher. The networked classroom appeared to have enhanced a rich set of communication events where analysis of mathematical variation was brought to a social plane and where students could understand the core mathematical ideas in focus from a collaborative perspective. (Hegedus, Dalton, Cambridge et al., 2006).

The rich collaborative communication continued as the teacher asked students what the class set of graphs would look like when she displayed them. The students began using metaphors to describe this. A few students mentioned the set of graphs would look like a fan or a rake with the graph of Group 6 as the *handle of the rake* because it was perpendicular to the y-axis. The teacher followed up students' initial response with her own metaphor.

Teacher: How about if I see it as a hand?
S1: Yeah I can see it as a hand.
S2: Kinda, skinny hand.
S: No.
S1: Yeah huh cause look.

S3: A hand with six fingers on it?
S9: Doesn't make sense.
S1: Yeah, a hand.
S4: Yeah a hand with 6 fingers {laughs}.
Teacher: Think about it though. What would be your y-intercept? Where would your y-intercept be?
S9: Your pinky.
S4: At the webbings.
S9: That doesn't make sense.
Teacher: Pretend there's an extra one [an extra finger]
S3: That would be where they end though {See image 2 where S3 is referring to the "webbings" or palm of his hand}
Teacher: That would be where they end—I'm sorry. You're right. So what would my palm be? My palm would be where they?
S3: End. {Referring to the point (6,12)}

| | |
|---|---|
|  |  |
| *Figure 1.* A student's drawing on the board of the class set of graphs. | *Figure 2.* The gestures representing the class set of graphs shown by the teacher and student, S3. |

Within one activity, students relied on three major representations SimCalc offered them to make deductions on the behavior of the family of functions. They used this knowledge to help them derive a function rule that can be generalized for any group in the class. The group relied heavily on the animation space and graphs to understand which groups would have a negative and positive slope, which was at the heart of their debate.

Throughout this entire classroom episode, the teacher acted as a mediator. She prompted or guided students if they were stuck but provided students an opportunity to make their own discoveries. The students debated and argued about the underlying mathematics. They challenged each other and made conjectures, correct or not, with the goal to derive a general expression and understand its meaning.

## Conclusion

Several aspects of learning in the SimCalc environment may contribute to lower anxiety measures. While this research is in its early stages, we speculate that three effects may be at work: (i) SimCalc provides a malleable environment with which to explore concepts in personally meaningful ways, (ii) students can make numerous conjectures, some of which may be false, before coming to a final answer, and (iii) the use of multiple representations in the curriculum and software provide for the deliberate generalization of concepts.

At this point, there are many potential explanations for motivation/learning performance relationships in SimCalc classrooms. As longitudinal data accumulates, we will investigate the connection between attitudinal changes and gains in performance using larger sample sizes and more refined analyses. We will also explore whether our hypothesized factors—the richness of the SimCalc context, the reduced emphasis on “one right answer”, and the explicit transfer of concepts to a variety of mathematical representations—contribute to such changes.

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# A CONCEPTUAL CHANGE LENS ON THE EMERGENCE OF A NOVEL STRATEGY DURING MATHEMATICAL PROBLEM SOLVING

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*This paper reports on an analytic case study of a pre-algebra student who makes a surprising and significant mathematical discovery over the course of several episodes of problem solving. The research reported in this paper is motivated by the goal of understanding how and why the student's strategies shifted from a simple, yet purposeful, guessing and checking approach to a sophisticated approach based on linear interpolation. The paper illustrates how a conceptual change framework developed in the science education literature can provide useful analytic tools for understanding shifts in problem solving strategies in terms of underlying conceptual refinements and reorganization.*

## Introduction

The phenomenon of interest in this paper is how new strategies emerge during mathematical activities, such as problem solving. Microgenetic analyses of strategy change (See Siegler, 2006 for a review) have focused on developing techniques for tracking shifts in strategy usage at a fine grain level of detail. While we share this attention to fine-grained analyses, our focus in the line of research reported in this paper will ultimately be on the processes by which an individual constructs a novel strategy from existing conceptual resources as opposed to the processes by which individuals come to reliably activate and use one strategy over another competitor strategy. In other words, the approach proposed by this research project is to re-frame analyses of strategy change in terms of underlying conceptual change. We will illustrate how an analytical approach known as "knowledge analysis" (diSessa, 1993; Sherin, 2001) for studying growth and change of conceptual structures can provide useful analytic tools for understanding the shifts in problem solving strategies that come about due to underlying conceptual refinements and reorganization.

To illustrate the potential of this approach, we will explore a case study of a pre-algebra student, Liam, who largely independently re-invents a deterministic and essentially algebraic problem solving strategy, known as linear interpolation, through the activity of solving algebra word problems using a purposeful guessing and checking strategy. Previous research has documented that students use informal problem solving approaches such as guessing and checking prior to instruction with algebraic solving techniques (Johanning, 2004, 2007; Kieran, Boileau & Garançon, 1996; Nathan & Koedinger, 2000; Stacey & MacGregor, 2000). However, the conceptual nature of students' guessing strategies and what kind of mathematical ideas can potentially be developed as a consequence had not previously been objects of extensive study. One reason for this is that the prior studies of students' pre-algebraic problem solving approaches were based primarily on written records and hence did not offer access to the richness and learning potential of students' informal strategies.

The case shared in this paper offers a surprisingly clear demonstration of how important algebraic ideas such as function, co-variation and rate of change can emerge and be developed through the successive refinement of informal problem solving strategies. Such potential for the development of algebraic reasoning is discussed in Levin, 2008. An important contribution of

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the current research is the explicit identification of key knowledge resources that are activated and used as the student constructs the linear interpolation strategy. Thus, the specifics of the case of Liam are new to the literature on the development of algebraic thinking, but beyond that, the case of Liam is an interesting site to begin elaborating theoretical and analytical tools for studying the growth and change of knowledge (i.e., conceptual change) in mathematics, and is thus of more general interest.

## Theoretical Framework

In this paper, the "knowledge in pieces" epistemological perspective proposed by diSessa, (diSessa, 1993) is adapted to analyze the conceptual underpinnings of an observed strategy shift in the case of Liam. In this theoretical perspective, an important underlying assumption is that individual knowledge can be thought of as a complex system comprised of many knowledge elements of diverse types. As individuals learn and gain in expertise, activation of relevant knowledge elements becomes more appropriately context sensitive and coordinated as ensembles of elements. DiSessa, 1993 and Sherin, 2001 argue that it is fruitful analytically to engage with the complexity of individual knowledge systems by defining a base vocabulary of sub-conceptual primitive knowledge elements. One reason this is argued to be useful analytically is because important features of expertise may manifest themselves only at the level of primitives.

## Data Collection and Methodology

The data corpus for the Liam case study includes video and written work collected over the course of six individual semi-structured tutorial sessions with a researcher, each approximately one hour in length. Liam was one of six pre-algebra students participating in this study aimed at analyzing students' emergent understanding of variable and letter-notation using a curricular tool, a Guess and Check chart, as suggested by a widely-used algebra curriculum (Sallee, Kysh, Kasimatis & Hoey, 2002). The Liam data corpus was selected for extended analysis because of the unexpectedly rich conceptual development that occurred during the sessions. The analysis of video and transcripts of problem solving in this study allowed access to students' real-time reasoning as they solved problems. Video data was transcribed for analysis, annotated with relevant details such as students' gestures, and coordinated with written work artifacts.

## Research Questions

1. How can we characterize Liam's conceptual understanding in a way that will make tracking moment-by-moment shifts in understanding analytically feasible?
2. What conceptual understandings did Liam develop between two contrasting episodes that may have allowed the observed change in problem solving strategy to occur?

## Background and Context

Liam is a pre-algebra student who initially approached solving word problems using a purposeful guessing and checking approach, which he devised (as research shows that many students do) without any previous instruction about how to solve such problems. Over the course of several individual sessions, he refined his purposeful guessing and checking approach, organized in tabular form, to an essentially algebraic algorithm (linear interpolation) for solving word problems. The linear interpolation strategy we will examine in this paper emerged naturally over the course of the sessions with Liam, and was not something that was explicitly designed to be part of the sessions with the tutor/researcher.

Data excerpts illustrating the approach taken by Liam at the beginning and ending of the sessions are given below. One can see that while Liam's initial approach was based on purposeful guessing and checking, his later approach is deterministic, building off the linear structure underlying the problem contexts, and in fact no longer involves "guessing" at all.

**Episode One**: **Systematic and Purposeful Guessing and Checking**

In this first focal episode, one observes that Liam is using a purposeful, systematic guessing and checking approach to solve the given word problem. This problem was the first in the series of sessions where the tutor had suggested that Liam organize his guessing and checking strategy in a Guess and Check chart. Previously, Liam had used the invented strategy of "guessing and checking," (though not arranged in a chart).

*The base of a rectangle is three more than twice the height. If the perimeter of the rectangle is sixty inches, find the height and the base of the rectangle.*

Below is a reproduction of the chart Liam constructed, along with excerpts from the transcript coordinated with his activity with the chart.

| Height | Base | Perimeter | 60 Check |
|---|---|---|---|
| 18 | 2(18)+3=39 | 36+(39+39)=114 | ↑ |
| 10 | 2(10)+3=23 | 20+46=66 | ↑ |
| 8 | 2(8)+3=19 | 8+8+19+19=54 | ↓ |
| 9 | 2(9)+3=21 | 9+9+21+21=60 | = |

"Oh. So it's actually a lot lower. I just realized that."

"This is way too much." … "Almost twice as much. So I'll try with 10."

A little too high…

"It's probably nine."

"Well, it was actually definitely 9 if this [result for guess of 8] was too low and this [result of guess of 10] was too high. Unless it was a decimal number."

*Figure 1.* Transcript from Liam's problem solving approach in episode one in which Liam used "successive approximation" to find the solution to the word problem. The chart is a typed reproduction of Liam's work.

Already in episode one, Liam is already making very purposeful choices about the sequence of trial values he constructs. Certainly, his choices of guesses are far from "random." In fact, he already appears to have an approximate sense for how the input/output pairs he generates co-vary linearly. One can also notice that he is making inferences in terms of both "scalar" judgments ("a little too high") and also "proportional" judgments ("it's a little less than twice" the target value).

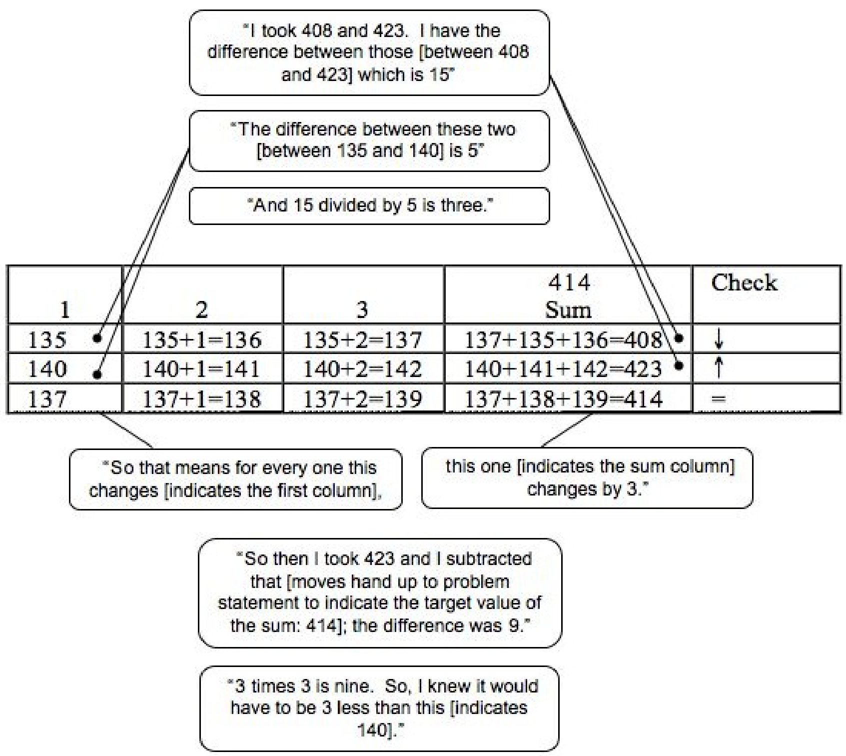
### Episode Two: Leveraging Linearity to Solve Problems

Later in the series of sessions (session 5 out of 6), Liam had refined his strategy from merely "purposeful guessing and checking" to "linear interpolation." In this data excerpt, one can see Liam deploy his newly constructed linear interpolation strategy to solve a problem of a similar underlying (linear) form as in focal episode one. The problem he was working on this episode was:

*The sum of three consecutive integers is 414. Find the three integers.*

In solving this problem, Liam continues to organize his work in a "guess and check" chart (a typed reproduction is pictured below). After having solved the problem and when asked to explain his solution strategy in this later, contrasting episode, Liam says

*"I took 408 and 423 [see chart below]. I have the difference between those [between 408 and 423] which is 15. The difference between these two [between 135 and 140] is 5. And 15 divided by 5 is three. So that means that for every one this changes [indicates the first column], this one [indicates the sum column] changes by 3. So, then I took 423 and I subtracted that [moved hand up to problem statement to indicate the target value of the sum: 414]; the difference was 9. 3 times 3 is nine. So, I knew that it would have to be three less than this [indicates 140]."*




| 1 | 2 | 3 | 414 Sum | Check |
|---|---|---|---|---|
| 135 | 135+1=136 | 135+2=137 | 137+135+136=408 | ↓ |
| 140 | 140+1=141 | 140+2=142 | 140+141+142=423 | ↑ |
| 137 | 137+1=138 | 137+2=139 | 137+138+139=414 | = |

*Figure 2.* Transcript from episode two in which Liam used the "linear interpolation" method he constructed. The chart is a typed reproduction of Liam's work. Though quotations are presented separately (to highlight the multiple steps involved in Liam's strategy), this constitutes one uninterrupted utterance by Liam.

To give a quick recap of Liam's activity in the second episode, we see that after Liam has finished the computations with two trial input values, Liam forms the ratio of the difference between the two outputs to the difference of the two trial inputs. This allows him to figure out the rate of change (in this case the constant of proportionality, or the slope) of the underlying (linear) function. Liam explicitly interprets the ratio he has formed as the unit worth of one guess: the amount the output will change corresponding to a change in one of the input. Liam then takes the output corresponding to one trial input he has selected as a reference and figures out how far that output is away from the target output. He then uses the unit worth of one guess to figure out how much he should change the input by in order to produce the change in output he just computed.

In episode two, Liam has refined his sense of how inputs and outputs co-vary. He has now found a way to quantify and explicitly leverage his intuitions about the underlying linear relationship that all the input-output pairs satisfy. Notice that the idea that a given input is "worth" a fixed amount in terms of its effect on the output is a refinement of the earlier qualitative versions of proportionality Liam used in episode one.

## Discussion of the Two Contrasting Focal Episodes

An important point of contrast between the two focal episodes is that in episode one, Liam's solution method is highly dependent on his inferences about a *particular* guess. However, in episode two, Liam realizes that his solution method is *general*, and depends only on determining the rate of change between *any* two input-output pairs. Further, he purposefully uses two trial values not for the purpose of converging to the solution to the problem, but for the purpose of determining an invariant (the rate of change) of the underlying functional relation which all input/output pairs must satisfy. Once he has determined this invariant, he uses it to deduce the unknown value that solves the problem.

## Analytical Framework

The key analytical move and insight made in this paper involves reframing the "strategy change" observed between episode one and two in terms of "conceptual change." To understand strategy change as conceptual change, we need to go deeper than a top-level description of the contrasting features of strategy one and strategy two. The task before us now is to find a way to describe the relevant shifts in conceptual understanding that allowed the strategy change to take place. Of course, we recognize that Liam has many other forms of resources that could potentially contribute to the construction of a new problem solving strategy (epistemological, meta-representational, etc.) in addition to the conceptual resources we will discuss in this paper. We focus on shifts in the activation and coordination of knowledge resources in this paper because even in this limited arena, there is significant analytical work to be done.

The first strand of analysis in this study involves recognizing that the approaches in the two focal episodes are qualitatively different and giving a characterization of some of the important dimensions of this difference. Some aspects of difference were discussed in the previous sections where it was noted that the move from "qualitative" to "quantitative" formulations of proportionality over the course of the sessions was particularly noteworthy as an underlying conceptual shift.

The focus of a second analytic strand is to give a characterization of and provide an argument for a set of relatively primitive and elemental knowledge resources, which allow will one to track processes of change in fine-grained detail. The underlying assumptions of the knowledge in pieces epistemological framework directly guide how the conceptual resources are identified in our analysis. In this analysis, we seek to identify the knowledge that was relevant to Liam and that he was drawing upon in solving the problems. To do this, one can consider the justifications he makes concerning his choices for trial values. This line of analysis has resulted in the identification of several conceptual resources that Liam activates and uses over the course of the sessions. Examples from the two focal episodes presented in this paper are given below.

| **Candidate knowledge resource** | **Description** | **Examples of resource activation in focal episodes** |
|---|---|---|
| Monotonicity | Larger inputs (in reference to previous inputs) result in larger outputs and smaller inputs result in smaller outputs. | All guesses in focal episode one fit this pattern. If a particular input resulted in an output that was too high, the next input was chosen to be a number lower than the previous input. Likewise, if a particular input resulted in an output that was too low, then next input was chosen to be higher than the previous input. *(Focal episode one)* |
| Sandwiching/In-betweeness | If an input yields an output that is too high and another input yields an output that is too low, then the true input must be in between these two inputs. | "Well it was actually definitely 9 if this [result for 8] was too low and this [result of a guess of 10] was too high. Unless it was a decimal number." *(Focal episode one)* |
| Qualitative formulation of proportionality | Small changes in input correspond to small changes in output. | This was "a little too high." [then he chooses a next guess that is two integer values lower]. (*Focal episode one)* |
| | Medium changes in input correspond to medium changes in output. | N/A in focal episodes one and two. |
| | Large changes in input correspond to large changes in output | "This is way too much" [and he follows up by choosing a guess that is a lot lower than the previous guess] *(Focal episode one)* |
| Half as a reference point | If an input yields an output that is about twice (or exactly twice) as much as the target output, then the next guess should be about half (or exactly half) as much. | This is "almost twice too much." [then he chooses a next guess that is nearly half as much] *(Focal episode one)* |
| Unit worth/Quantitative formulation of proportionality | A change of one in the input corresponds to a fixed change in output. | "So that means that for every one this one changes [notes the input column], this one [notes the corresponding output column] changes by three." *(Focal episode two)* |

*Figure 3.* A summary of "knowledge resources" identified in the analysis of Liam's justifications for choices of next trial values.

## Discussion and Findings

The main goal of this paper has been to illustrate how the emergence of a novel strategy in episodes of problem solving can be productively framed in terms of underlying conceptual reorganization. As we have seen, the landscape of the knowledge resources that students draw upon in employing informal problem solving methods is surprisingly rich. Through a preliminary analysis with the data from the case of Liam, we have seen that something as apparently simple as students solving word problems using informal strategies like guessing and checking actually can yield a striking complexity under analysis. The main contribution of this paper is an analytic framework that re-positions observed strategy changes for solving problems in terms of underlying conceptual reorganization. In the case elaborated in this paper, explicit candidate knowledge resources have been named that should allow one to track the dynamics of change between the two contrasting episodes discussed in this paper.

One of the challenges of tracking strategy change in terms of underlying conceptual reorganization is that the conceptual reorganizations are likely to be of a small scale and highly situated to the task at hand. In the data excerpts, we saw evidence that Liam had “invented linear interpolation” in a tabular context. Without being explicitly taught about functions, Liam implicitly recognized the “guess and check” chart he was generating as he solved problems as a tabular representation of a function. In inventing linear interpolation in this context, he discovered the tabular version of what might be stated in graphical terms as “two points determine a line” and the fact that once you have a point and the slope you can get to any other point on a line.

Since Liam was not familiar with “symbolic” or “graphical” representations of functions at the time of the sessions, one would not expect that he would spontaneously recognize and apply his linear interpolation approach in these other representational contexts. Hence, his understanding of “linear interpolation” is only a projection into the tabular representational context of a mature understanding of “linear interpolation.” Accordingly, the sub-conceptual grain-size posited by the “knowledge in pieces” framework is particularly well adapted to the goals of the analytic work in this line of research. A fine-grained and situated characterization of knowledge will be required to make sense of the emergence of Liam’s strategy in the tabular representational context.

## Future Research

Future analytic work grounded in this case study and other replication case studies will be needed to continue to identify other potentially relevant knowledge resources used by students. Preliminary analyses of a complementary classroom data corpus and data from the interviews with the five other pre-algebra students give evidence that the knowledge resources we have discussed in this paper are sometimes but not always activated in the solution strategies of other students. This certainly does not mean that students don’t “have” such resources. It may only indicate that they do not see them as relevant to the task at hand. One would hypothesize that classroom practices shape in fundamental ways what knowledge a student sees as relevant to activate as they engage with solving problems.

The analytic framework sketched in this paper naturally extends into at least two other additional strands. The work in the first two strands of analysis presented in this paper focused on (1) documenting that there was a change in the organization of Liam’s knowledge and (2) generating a vocabulary with which to describe that change. As suggested throughout this paper, the natural next analysis would focus on giving a genetic account of knowledge growth and change using the specific resources identified in this paper. Certainly, this would involve looking at the episodes *in between* focal episodes. Further, it is natural, both from the perspective of giving accounts of learning processes and from the perspective of designing instruction informed by such accounts, to ask what factors influence the process of conceptual change and by what mechanisms. Accordingly, a fourth analytic strand would focus on going beyond a description of the dynamics of change over the sessions in terms of the resources to proposing likely mechanisms that drove the process of conceptual reorganization forward. There are several potential candidates for significant mediators of the constructive process. Some examples of potential mediators include the role of the representational form in organizing the data obtained from individual trials, social interactions and questioning in the learning setting (both by the researcher and the student), the role of the activity of solving a problem in driving the inquiry, and the role of a students' prior knowledge and understanding. In addition, extending the analysis to other kinds of resources, such as epistemological resources and beliefs (Hammer, 2000), could lead to future insights about how and why strategies emerge in episodes of problem solving.

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# THE EFFECT OF CURRICULUM TYPE ON MIDDLE GRADES INSTRUCTION

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*In this article, we discuss differences between the mathematics instruction of CMP and non-CMP teachers in the LieCal project. There are three aspects of instruction that 200 6th grade urban classroom observations showed were strongly and differently related to the type of curriculum that teachers were using. These three aspects relate to the teachers' use of (1) group and individual work, (2) written narratives and worked-out examples, and (3) conceptually- and procedurally-focused instruction.*

## Introduction

Historically, curriculum has played a central role in educational reform. In all of the reform movements since the 1960s, curriculum has been used as a means to convey what and how teachers should teach (NCTM, 1989, 2000), and it has also been used to serve as an agent for instructional improvement (Ball & Cohen, 1996). While curriculum does not always dictate the content of instruction (Freeman & Porter, 1989), research has consistently shown that curriculum has a strong influence on the ways mathematics is taught (Remillard, 2005; Robitalle & Travers, 1989; Schmid et al., 2002). In fact, teachers often base their teaching approaches primarily on the ways the curricular materials are presented (Cai, 2005; Robitalle & Travers, 1989; Tarr et al., 2006). However, research has not documented specifically how curriculum materials actually influence classroom instruction.

The current discussion of how curriculum materials actually influence classroom instruction habitually has focused on differences between the use of *Standards*-based curricular materials developed through the support of the National Science Foundation (NSF) and the use of more traditional curricular materials developed through the support of commercial publishers (Hirsch, 2007; Reys, Robinson, Sconiers, & Mark, 1999; Tarr et al., 2006). *Standards*-based NSF-funded curricula claim to build students' understanding of important mathematics through explorations of real-world (or sometimes fanciful) situations and problems. These curricular materials are intended to align with the recommendations in the NCTM *Standards*, with a focus on the importance of thinking, understanding, communicating, representing, making connections, reasoning, and problem solving (e.g., NCTM, 1989, 1991, 2000). This view stands in contrast to a more conventional approach to curriculum that emphasizes the application of well-rehearsed procedures to solve problems, and stresses the memorization, recitation, and use of decontextualized facts, rules, and procedures. In this article, we report initial findings from our investigation of differences between the mathematics instruction of CMP and non-CMP teachers.

## LieCal Project

This research for this article was done as part of our LieCal project. LieCal (Longitudinal Investigation of the Effect of Curriculum on Algebra Learning) is a project that investigates differences between the effectiveness of the Connected Mathematics Program (CMP) and the effectiveness of more traditional middle school curricula (non-CMP) on students' learning of algebra. CMP is one of four *Standards*-based middle school curricula developed with funding from NSF (Lappan et al., 2002). In the LieCal project, we are studying the algebra-related teaching and learning of about 1400 students in 16 urban middle schools as they progress from sixth through ninth grades. Our overall goal for the LieCal project is to provide: (1) A profile of the intended treatment of algebra in the CMP curriculum with a contrasting profile of the intended treatment of algebra in the non-CMP curricula; (2) a profile of classroom experiences that CMP students and teachers have, with a contrasting profile of experiences in non-CMP classrooms; and (3) a profile of student performance resulting from the use of the CMP curriculum, with a contrasting profile of student performance resulting from the use of non-CMP curricula. In order to provide a profile of CMP classroom experiences and a contrasting profile of experiences in non-CMP classrooms, we are collecting data on how teachers use CMP and non-CMP curricula. In this article, we discuss differences in CMP and non-CMP classrooms in the LieCal Project.

## Method

The LieCal Project is being conducted in 16 middle schools in a district serving a diverse student population. When we began the project, 27 of the 51 middle schools in the school district had adopted the CMP curriculum while the remaining 24 middle schools used other more traditional curricula. Eight CMP schools were randomly selected from the 27 schools that had adopted the CMP curriculum. After the eight CMP schools were selected, eight non-CMP schools were chosen based on the comparable ethnicity, family incomes, accessibility of resources, and state and district test results.

An important part of the LieCal Project's examination of the fidelity of curricular implementation is classroom observations. The observations upon which this article is based were conducted in the 6th grade of the 16 middle schools. Subsequent observations in grades 7 and 8 yielded similar results. Using a pre-developed observation instrument[1], two trained research specialists observed each of the 50 participating classes (24 CMP classes and 26 non-CMP classes) 4 times a year, twice in the fall and twice in the spring. Due to space limitations, we have chosen to report only on the data from the 6th grade observations.

The LieCal observation instrument is designed to provide a comprehensive analysis of the instruction that transpires during each class. Two retired, highly experienced mathematics teachers were hired as research specialists to conduct the classroom observations. Over the course of the year, we checked the reliability of the specialists' coding three times. These three sessions revealed that the reliability of the coding done by the two specialists was quite high. The reliability achieved during the three sessions averaged 79% perfect agreement using the criterion that the observers' coded responses were considered equivalent only if they were identical (i.e., perfect match). The reliability averaged 95% using the following criteria: (a) If an item or sub-item was "scored" using an ordinal scale, then the specialists' coded responses were considered equivalent if they differed by at most one unit; (b) If an item or sub-item (e.g. representation) was "scored" by choosing from a list of alternatives all the words/phrases that characterize it, then the specialists' coded responses were considered equivalent if they had at least one choice in common (e.g. symbolic and pictorial vs. pictorial).

## Results

There are three aspects of instruction that our classroom observations showed were strongly and differently related to the type of curriculum that teachers were using. These three aspects relate to the teachers' use of (1) group and individual work, (2) written narratives and worked-out examples, and (3) conceptually- and procedurally-focused instruction. Our findings regarding these three instructional aspects are presented next.

*Teachers' Use of Group and Individual Learning*

The CMP curriculum is composed, to a large extent, of extended contextual tasks, called investigations. The teachers' edition of the CMP curriculum encourages teachers to organize their students in small groups to work on the investigations, which guide the students to explore important mathematical ideas and ways of thinking as they try to understand and make sense of real-world situations. Using the observation instrument of the LieCal Project, we documented ways that teachers organized students to engage in lesson activities. Two of the ways we documented were the use of small-group and individual learning. Nearly half of the CMP lessons (47 out of 100) involved students learning in small groups, but only 7.4% of the non-CMP lessons (7 out of 95) involved group learning (See Table 1). Accordingly, the CMP students also spent a larger percentage of lesson time engaged in group learning than non-CMP students. In particular, on average 15.4% of the total CMP lesson time was used for group learning, but only 2.3% of the total non-CMP lesson time. This means that, overall, CMP students spent about six times longer than non-CMP students on group learning.

What is surprising, however, is that when group learning was used in non-CMP classrooms, the time students spent in small groups was similar to the time spent in CMP classrooms. In fact, in the 47 CMP lessons that used group learning, the average number of minutes students spent in groups was 18.9 minutes, which is very close to that for the seven non-CMP lessons, 16.4 minutes. This result suggests that once teachers decide to use small-group learning, both CMP and non-CMP teachers engage their students in small group learning for about the same amount of time per lesson.

Table 1. *Group and Individual Learning in Both CMP and Non-CMP Lessons*

| | CMP Lessons (n=100) Minutes (n=5724) | Non-CMP Lessons (n=95) Minutes (n=4980) | Z | |
|---|---|---|---|---|
| Group work | | | | |
| % of Lessons | 47.0 | 7.4 | 6.02 | $p < .01$ |
| % of Minutes | 15.5 | 2.3 | 22.75 | $p < .01$ |
| Individual work (not on homework) | | | | |
| % of Lessons | 32.0 | 51.6 | 2.69 | $p < .01$ |
| % of Minutes | 10.7 | 12.6 | 3.15 | $p < .01$ |
| Individual work (on homework) | | | | |
| % of Lessons | 9.0 | 27.4 | 3.09 | $p < .01$ |
| % of Minutes | 1.4 | 6.5 | 16.15 | $p < .01$ |

From the opposite point of view, the relatively little use of group learning in non-CMP lessons implies that individual student work occurred in many more non-CMP lessons than in CMP lessons. There are two types of individual work. One is on homework, and the other is on non-homework activities. About half of the non-CMP lessons included individual learning that was on non-homework activities, but only about one third of the CMP lessons included individual learning that was not on homework.

CMP students worked individually on homework in 9 of the 100 observed lessons (9%), but non-CMP students worked individually on homework in 26 of the 95 observed lessons (27.4%). Both CMP teachers and non-CMP teachers assigned homework in about one third of their lessons. Therefore, in our study non-CMP students worked individually on homework in about 81% (26/32) of the classes in which homework was assigned. However, CMP students worked individually on homework in only about 27% (9/33) of the classes in which homework was assigned. So, non-CMP lessons were three times more likely than non-CMP lessons to have students working individually on homework.

*Teachers' Use of Written Narratives and Worked-Out Examples*

Even a cursory comparison of the CMP mathematics curriculum with non-CMP mathematics curricula reveals major differences between them. Customarily, the sections of non-CMP curricula are organized around worked-out examples of mathematics problems similar to the beginning exercises that appear at the end of the sections, rather than the application problems that appear later. The worked-out examples generally are connected by short narrative paragraphs that explain the worked-out examples, or that provide definitions, generalizations, and formulas that are based on the examples. On the other hand, the CMP curriculum contains very few worked out examples and almost no formulas. Instead, the curriculum is composed of a series of investigations that the students are expected to explore, often in groups. Each investigation comprises multiple paragraphs of written narrative interspersed with diagrams, tables, and unworked problems that the students are asked to analyze, solve, and discuss. In our study, we found that these two types of curricular presentations lead to very different patterns of textbook use by teachers and students.

Table 2. *The Purpose of Textbook Use**

| | % of CMP Lessons (n=100) | % of Non-CMP Lessons (n=95) | Z | |
|---|---|---|---|---|
| Students looked for problems in the text | 63 | 64 | 0 | NS |
| Students reviewed diagrams, charts or pictures | 28 | 1 | 5.10 | p<.01 |
| Students reviewed examples or find formulae | 5 | 16 | 2.30 | p<.05 |
| Students read from text | 49 | 14 | 5.09 | p<.01 |
| Students do not use text | 3 | 16 | 2.89 | p<.01 |
| Teachers drew examples from text | 17 | 47 | 4.34 | p<.01 |

*Student percents total more than 100 because textbooks can be used for multiple purposes in a lesson.

Table 2 shows ways that students used both CMP and non-CMP textbooks. The students' most frequent use of both the CMP and non-CMP texts was to look for problems in the text. These problems generally formed the basis for their instruction or their practice. This type of usage occurred in almost equal frequencies in the CMP lessons (63%) and the non-CMP-lessons (64%). We have shown elsewhere (Cai et al., 2009), however, that the overall cognitive level of the tasks posed by the CMP teachers was significantly higher than that of the tasks posed by non-CMP teachers. Therefore, opportunities to think conceptually and make connections were much different in the CMP classes, even though the frequency of use of problems from the text was the same in both types of mathematics classes.

The investigation of the other three purposes for which students used their textbooks revealed large and significant differences between the CMP and non-CMP classrooms. In particular, the CMP students in our study used their textbooks to review diagrams, charts or pictures in 28% of the CMP lessons we observed, while the non-CMP students used the textbook for this purpose in only 1% of the lessons. However, the percents were reversed when students used their textbooks to review examples or find formulae (5% of the CMP lessons and 16% of the non-CMP lessons). Also, the students read their textbooks in about half of the CMP lessons we observed, but students read textbooks in only about 14% of the non-CMP lessons.

These last three findings may be due, in part, to the fact that the CMP curriculum contains more written text and more diagrams and charts than the non-CMP curricula. It may also be due, in part, to the fact that teachers of non-CMP curricula presented and discussed the textbook's worked-out examples (or ones like it) much more often than teachers of the CMP curriculum (17% of CMP lessons and 47% of non-CMP lessons). This may have helped obviate the need for the non-CMP students to read the text. These two arguments are especially compelling when one considers that the brunt of the learning from non-CMP texts is often based on the worked-out examples and the subsequent practice problems. This is in stark contrast to CMP texts, in which the burden of learning usually resides in the students' own work on un-worked problems in the text, many of which are integrally dependent for their solutions on accompanying diagrams, charts, and pictures. Similar arguments can be made to explain why non-CMP students did not use their textbooks in 16% of the lessons we observed, but CMP students failed to use the their textbooks in only 3% of the lessons.

*Teachers' Use of Conceptually- and Procedurally-Focused Instruction*

CMP can be characterized as a problem-based curriculum. The focus is more on conceptual understanding than on procedural knowledge. It is expected that students will learn algorithms and master basic skills as they engage in explorations of worthwhile problems. On the other hand, the non-CMP curricula in our study include extensive sets of practice exercises, and the focus is more on procedural knowledge and basic skills than on conceptual understanding. Take the introduction to equation solving as an example, in the Non-CMP curriculum, equation solving was introduced symbolically by using additive property (add or subtract the same quantity on both side of the equation, the equality holds) and multiplicative property (multiple or divide a non-zero quantity on both sides of an equation, the equality holds). On the other hand, in the CMP curriculum, real-life contexts are used to help students understand the meaning of each step of the equation solving, as shown in Table 3 below (Nie, Cai, & Moyer, 2009).

Table 3. *Introduction of Equation Solving in CMP*

| Thinking | Manipulating the Symbol |
|---|---|
| "I want to buy a CD-ROM drive that costs $195. To pay for the drive on the installment plan, I must pay $30 down and $15 a month." | 195 = 30 + 15N |
| "After I pay the $30 down payment, I can subtract this from the cost. To keep the sides of the equation equal, I must subtract 30 from both sides | 195 – 30 = 30 – 30 + 15N |
| "I now owe $165 which I will pay in monthly installments of $15." | 165 = 15N |
| "I need to separate $165 into payments of $15. This means I need to divide it by 15. To keep the sides of the equation equal, I must divide both sides by 15." | $\frac{165}{15} = \frac{15N}{15}$ |
| "There are 11 groups of $15 in $165, so it will take 11 months." | 11 = N |

Table 4. *Sample Conceptual and Procedural Scales in the LieCal Observation Instrument*

1. *Example of Measuring Conceptual Understanding*

The teacher's questioning strategies were likely to enhance the development of student conceptual understanding/problem solving (e.g., emphasized higher order questions, appropriately used "wait time," identified prior conceptions and misconceptions).

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Does not ask questions | Asks low level questions or answers his/her own questions | Asks low level questions that lead to a dialogue between teacher and students | Asks high level questions but does not pursue the answers | Asks high level questions, and pursues their answers |

2. *Example of Measuring Procedural Knowledge*

The teacher worked out examples to demonstrate the steps of a mathematical procedure or solution process.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| Not at all | | Moderate Demonstration | | Extensive Demonstration |

The classroom observation instrument of the LieCal Project includes twenty-one 5-point Likert scale questions, which are designed to rate the nature and quality of instruction in a lesson. A factor analysis of the 21 questions revealed that five of the 21 questions rate the extent to which the lesson fosters students' conceptual understanding of mathematical knowledge, and another five questions rate the extent to which the lesson fosters students' ability to carry out mathematical procedures. The first question in Table 4 is an example of the type of Likert scale that was used to rate the likelihood that the teachers' instruction would help develop students' conceptual understanding of math knowledge. The second question in Table 4 is a detailed example of the type of Likert scale that was used to rate the likelihood that the teachers' instruction would help develop students' procedural knowledge.

The mean of the sum of the scores on the five questions that measure the likelihood that the teachers' instruction would help develop students' conceptual knowledge was 17.99 for CMP classes and 12.33 for non-CMP classes, which is statistically significantly higher for CMP classrooms than for non-CMP classrooms: $t(193)=10.05$, $p<.0001$.

The mean of the sum of the scores on the five questions that measure the likelihood that the teachers' instruction would help develop students' procedural knowledge was 14.70 for CMP classes and 17.16 for non-CMP classes, which is statistically significantly higher for CMP classrooms than for non-CMP classrooms: $t(193)=4.25$, $p<.0001$.

## Conclusion

In this article, we discuss differences between the mathematics instruction of CMP and non-CMP teachers in the LieCal project. There are three aspects of instruction that our classroom observations showed were strongly and differently related to the type of curriculum that teachers were using. These three aspects relate to the teachers' use of: (1) group and individual work, (2) written narratives and worked-out examples, and (3) conceptually- and procedurally-focused instruction. This article shows that the use of different types of curriculum materials in the LieCal project corresponds to major differences in teachers' classroom practice as it relates to these three aspects of instruction.

Based on findings from the LieCal Project, this article shows that the use of CMP and non-CMP curriculum materials can impact teachers' classroom practice in very different ways. Regarding group versus individual learning in classroom instruction, CMP teachers dedicate more time to group learning than non-CMP teachers, while non-CMP teachers use more individual learning. Because students in CMP classrooms more often work in small groups to perform cognitively demanding tasks, it is likely that CMP students are given more opportunities to interact and digest math concepts and ideas than non-CMP students. On the other hand, students in non-CMP classrooms have more opportunities to practice basic mathematical skills individually. Our classroom observations also revealed that teachers who use the CMP curriculum provide more opportunities for students to use diagrams, charts, and pictures than non-CMP teachers. For non-CMP lessons, both students and teachers are much more likely to use the curriculum materials to study worked-out examples. Finally, CMP classroom instruction is more likely to enhance students' conceptual understanding of mathematical knowledge while non-CMP classroom instruction focuses more frequently on mathematical procedures.

The data in our study do not establish a causal relationship between the different instructional practices we analyzed and the use of CMP and non-CMP curricula. Nonetheless, the differences that we highlighted between CMP and non-CMP teachers' classroom practices are closely related to the nature of CMP and non-CMP curriculum materials. Thus, there is good reason to believe that the choice of curriculum materials by a school or district is an important decision that should be made with the utmost care. When all is said and done, our study confirms that teachers tend to teach their lessons in ways that are compatible with the nature of the texts they use. Therefore, deliberate and close attention should be paid to the compatibility of potential curricula with the teachers' beliefs about the nature of mathematics instruction, and with the goals they have for their students.

## Endnotes

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[1]In developing the instrument, we adopted ideas from the QUASAR project, the Middle School Mathematics Study, the Evaluation Study of Mathematics in Context, and from Horizon Research, Inc.

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# EXPLORING PROCEDURAL FLEXIBILITY IN STRUGGLING ALGEBRA STUDENTS

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*Recent studies of procedural flexibility have identified promising instructional techniques can be effective in promoting flexibility with solving equations, but they have done so within controlled environments and using brief interventions. We describe results from a three-week course in algebra, in which these techniques were regularly employed as part of daily instruction. Written assessments and interviews of students who struggle with algebra suggest accuracy, rather than efficiency, is often a driving force when choosing solution methods. Prior instruction emphasizing one method appeared to inhibit flexibility when accuracy was not a concern, but students demonstrated flexibility for less familiar problems.*

## Introduction

For decades, researchers in the fields of mathematics education and cognitive psychology have been interested in the relationship between procedural and conceptual knowledge (e.g., Byrnes & Wasik, 1991; Hiebert & Lefevre, 1986; Rittle-Johnson, Siegler, & Alibali, 2001). A number of questions have been asked about how these two types of knowledge are linked, including which develops first and whether one is necessary for the other. However, recent arguments have emerged that challenge how the two types are defined and measured (Baroody, Feil, & Johnson, 2007; Newton, 2008; Star, 2005, 2007). In particular, Star (2005, 2007) argued that procedural knowledge is often conceived of and measured in ways that that are consistent with rote memorization. Yet, procedural knowledge can also be deep, as it must be in order to flexibly apply solution methods. The current study adds to the literature on procedural flexibility by exploring its development in a real classroom, with students who struggle with algebra.

### *Flexibility*

Verschaffel, Luwel, Torbeyns, and Van Dooren (2007) suggest that researchers have conceptualized flexibility in a variety of ways. Some use the term to refer to a person's ease in switching between solution methods, whereas others also include a person's tendency to select the most appropriate method in a given situation. In the current study we take the latter perspective, suggesting that flexibility develops slowly from knowledge of multiple procedures to the adaptive use of them (Blöte, Van der Burg, & Klein, 2001; Star & Seifert, 2006).

A number of recent studies support the notion of flexibility developing on such a continuum (e.g. Blöte et al., 2001; Star & Rittle-Johnson, 2008). Blöte and colleagues showed that second grade students can learn multiple ways of solving addition and subtraction problems, but the tendency to use alternative methods lags behind their knowledge and preference for them (Blöte et al., 2001). On the other end of the continuum, Star and Newton (under review) demonstrated that experts show knowledge of, frequent use of, and strong preferences for elegant and efficient solutions. The experts were in surprising agreement about which methods were better, even if they did not originally choose the method. When explaining why, they consistently pointed to structural characteristics of the problem (e.g., a certain number is divisible by another) and preferred methods that were more efficient, made use of important mathematical ideas, and reduced the chance for error.

Research by Rittle-Johnson and Star (2007) demonstrated that students who are first learning algebra can gain conceptual knowledge, procedural knowledge, and flexibility simultaneously. By comparing and contrasting two different solutions to the same problem, students in a treatment group (n = 36) gained more on measures of flexibility and procedural knowledge compared to a control group (n = 34) that examined the same two methods but with isomorphic problems. Given the brief intervention used by Rittle-Johnson and Star (4 instructional days), the significant gains in favor of the treatment group in flexibility are encouraging, but it is less clear whether this intervention can yield similar results over more extended periods of instructional time.

Klein, Beishuizen, and Treffers (1998) demonstrated positive effects of flexibility instruction over time in their study of 275 second graders. These researchers compared the impact of arithmetic instruction that promoted flexibility from the beginning, to instruction that emphasized procedural skill prior to promoting flexibility. Both groups of students demonstrated flexibility at posttest; however, promoting flexibility from the beginning of instruction seemed to be more effective. Students who learned flexibility and skill simultaneously outperformed those who focused first on skill alone. Furthermore, their findings suggest that early exposure to multiple solution methods does not hinder procedural competence; the groups performed similarly on a skills assessment.

Star and Rittle-Johnson (2008) also demonstrated that students can learn both skill and flexibility when they compared direct instruction to a more discovery-oriented approach. In their study of 132 rising seventh graders learning to solve linear equations, they found that the different types of instruction had similar effects on accuracy of equation solving but differential effects on flexibility. Namely, prompting students to solve problems in more than one way was most effective for increasing students' use of multiple methods, whereas direct demonstration of efficient methods was most effective at increasing students' use of efficient methods. Although the treatments impacted flexibility differently in terms of *use*, they were similarly effective in increasing flexibility in terms of *knowledge* (i.e., knowing that multiple methods exist and that some are more efficient than others.) This distinction is important and lends support to the notion that flexibility develops on a continuum.

*Current Study*

The current study builds on prior research on flexibility while addressing some of the limitations of that research. First of all, few studies at the secondary level have examined flexibility in real classroom settings, which often include pressures to cover large amounts of material in a limited amount of time, grading systems that emphasize accuracy over efficiency, and a wide range of student abilities. Second, few studies have examined how flexibility might develop for students who struggle with mathematics. Klein et al. (1998) reported that weaker second grade students were not confused by being introduced to multiple ways of solving problems, but it is unclear whether or not this finding would hold for weak algebra students, who may have struggled for years and built up a number of misconceptions that could impede learning. Finally, most studies have relied exclusively on written assessments to understand flexibility. Research suggests that knowledge of efficient procedures sometimes precedes the use of them (Blöte et al., 2001; Star & Rittle-Johnson, 2008), and interviewing students may help researchers understand why. The current study addresses each of these limitations.

Three research questions guide the current study. How do knowledge and use of multiple methods change during an algebra course focused on promoting flexibility? How does prior instruction in algebra impact students' flexible use of solution methods? How does prior knowledge of mathematics impact students' flexible use of solution methods in algebra? These questions are explored in the context of a three-week remedial/review algebra course offered in the summer.

## Method

### *Participants*

Two boys and four girls from a private school participated in the study. Xavier, Ricardo, and Nicole were all new to the school and were entering the ninth grade. A placement exam in algebra determined their need for this course. Annemarie and Naomi were entering tenth grade, and Yvonne was entering eleventh grade; these three students enrolled in the summer course at the recommendation of their prior math teacher. Performance during the course indicated that Nicole and Annemarie struggled most with the content, whereas Yvonne and Naomi struggled least. All students had previously taken a first course in algebra.

### *Measures*

*Algebra exam.* The final exam for the course served as both a pre-test and a post-test. It included 55 items, including linear and quadratic equations, systems of equations, graphing, and pre-requisite skills for quadratics such as factoring and simplifying with exponents and roots. Some of the items were designed such that the traditional approach might not be the easiest or most efficient.

*Intermediate assessments.* Intermediate assessments included homework, quizzes, and tests. Unlike the final exam, intermediate assessments sometimes prompted students to solve problems in more than one way, in order to determine whether or not they had knowledge of solution methods that were different than the ones they chose to use on their own, without being prompted. When prompted, the students were asked to indicate which method they thought was better by placing a star by that method.

*Interviews.* Students were interviewed three times during the course to assess their flexibility and attitudes about flexibility. The pre-interview and post-interview were identical, and they prompted students to solve problems in more than one way or to evaluate two methods of solving or graphing. The students were also asked about their views on learning more than one way of solving problems. An intermediate interview asked about the students' particular solutions from the first algebra test in the course. It also asked about their views on learning to solve problems in more than one way.

### *Procedure*

The three-week summer course included 14 instructional days plus one day devoted to the final exam. The class met Monday through Friday each week for two hours in the afternoon. The first week focused primarily on solving and graphing linear equations and systems of equations. The second week focused primarily on simplifying with exponents and radicals, as well as factoring. The final week continued with factoring and then focused on solving and graphing quadratic equations. A quiz and a test were administered for each of these units, and homework was assigned nearly every evening. Students were interviewed before the course began, immediately following the first test, and after the final exam. The pretest was administered on the first day of class, and the posttest was administered after the last day of class.

A typical lesson involved comparing and contrasting pre-worked examples of problems relevant to the day's topic. Having students contrast cases before discussing them may serve to deepen conceptual understanding by illuminating important features (Schwartz & Bransford, 1998). The features of the compared problems varied in such a way that one method might be preferred in some cases, but another method might be preferred in other cases. For example, when solving equations with parentheses, students might prefer to distribute the number in front of the parentheses when it is a whole number but they might prefer to multiply by the reciprocal first when the number is a fraction. These types of variations were meant to draw students' attentions to the structure of the equations rather than the superficial characteristics (e.g., the equation includes parentheses). However, students were not pressed to use a particular method. Instead, they discussed which methods they preferred, why they preferred them, and under what circumstances.

## Results

*Knowledge and Use of Multiple Methods*

When students were familiar with a particular solution method for a problem type, knowledge of alternate methods preceded use, confirming the findings of prior studies (Blöte et al., 2001; Star & Rittle-Johnson, 2008). However, students were more likely to use alternatives to the general approach with problems that posed difficulty, such as those containing fractions. As an illustration, on Test I all six students solved 7(n + 2) = 49 by distributing the 7 as a first step. On the same test, students were asked to solve -5(7x – 16) = 45 in two different ways, and all six showed knowledge of how to divide by -5 as a first step. Alternatively, three of the students solved $\frac{1}{2}(x - 15) = -4$ by distributing first, and the other three multiplied both sides of the equation by 2. As Xavier noted for this problem, "I think since there is a fraction, it would be easier just to use the reciprocal, in this case, instead of just working out the whole distributive property" (Xavier, intermediate interview).

When students were equally familiar with two different methods, two common reasons were cited for choosing a particular method in a given situation: efficiency and problem structure. With respect to the former, when graphing linear equations on Test I, Naomi and Yvonne both successfully used the slope-intercept method rather than plotting points. Although Naomi's intermediate interview revealed she had knowledge of both methods, she said she used the slope-intercept method "because it's faster." Likewise, Yvonne said her reason for not choosing to plot points was that "there is more involved, and it takes longer." On the other hand, it was quite typical that students noted the structure of the equation when providing a rationale for their graphing method. Although Ricardo chose to plot points when "the problem is really easy", such as for x + y = 10, he switched methods when the structure of the equation was more conducive to a different method. For $y = \frac{1}{2}x + 3$, he said, "Since it already had the slope and the y-intercept, I just found the y-intercept and went up from there, using the slope" (Ricardo, intermediate interview). Similarly, Xavier's rationale for the slope-intercept method was that "you know where to start, and from there you just to use the rise over run….it is ready to go" (Xavier, intermediate interview). And Nicole, who admittedly struggled with all graphing methods, successfully used the slope-intercept method for the same problem "because it was basically in that form" (Nicole, intermediate interview).

When solving systems of equations, students also attended to the structure of the problem in selecting strategies. At pretest, none of the six students attempted to solve the systems of equations, indicating weak or no knowledge of them. On Test I the students demonstrated both knowledge and use of two methods for solving linear systems, and their reason for switching between methods was often focused on the structure of the equations. For example, when one of the equations was already solved for *y*, students generally chose to substitute its equivalent expression into the remaining equation. As Xavier stated in his intermediate interview, "it's right there, what *y* equals, so you just substitute." Yvonne, Ricardo, and Naomi gave similar reasons for using substitution for this same problem. Focusing on characteristics of the equations was consistent with the experts for these types of problems (Star & Newton, under review).

For solving quadratic equations, students concentrated on a mix of efficiency and problem structure at posttest but not at pretest. At pretest, the students showed weak or moderate knowledge of quadratics, and only Naomi alluded to efficiency as a reason for choosing a particular method. During the course, students demonstrated knowledge of all three instructed methods for solving quadratics, but only Yvonne, Nicole, and Naomi attempted to *use* all three on the posttest. Most students cited efficiency as a rationale for a particular method, but some students mentioned structure. For example, Xavier stated in his post interview that one equation had a "perfect square" and another was "pretty easy to factor, it's already in a factorable form." Presumably, Xavier was referring to the fact that the equation was already set equal to zero. Nicole, who often preferred the quadratic formula because she admittedly struggled with factoring, said in her post-interview that her reason for preferring factoring for a particular problem was "because there's no coefficient."

*Impact of Prior Instruction*

At times, prior instruction seemed to limit the development of flexibility. It seemed students were less likely to use alternate methods when they were familiar with one that worked for a particular problem type, especially if they were comfortable with it. The most prominent example concerned linear equations of the type $a(x+b) = c$ where $a$ was a whole number. The course attempted to help students develop flexibility for solving problems of this type, presenting them with an alternate to "distribute first" when $c$ was divisible by $a$ (e.g., use the "divide first" method). Several students began the study having already developed automaticity with the "distribute first" method, and it appeared the exposure was so extensive that it negatively influenced flexibility for these problems.

Students' interview responses provide insight as to why knowledge of the "divide first" method did not lead consistently to its use. In her intermediate interview, Yvonne indicated that she still used the familiar "distribute first" method because "it was like instinct." Annemarie's and Ricardo's interview responses show a similar pattern. At the intermediate interview, Annemarie explained that while she knew how to use the alternative "divide first" method, she had still distributed first because "It's just the way I was taught." Similarly, Ricardo indicated that he still preferred to use "distribute first" because he was "used to it."

Unlike the other students, Naomi had prior instruction that promoted alternate methods for solving linear equations, at least for those involving fractions. In particular, she stated that one of her teachers taught her to "clear the denominator" when solving equations with fractions. In some cases, this strategy led to more efficient methods than the general approach, but not always. For example, at pretest she solved $\frac{2}{3}w + 3 = 10$ by first multiplying all terms by 3, obtaining $2w + 9 = 30$ as the next step. For the same problem at posttest, Naomi used the more traditional first step of subtracting 3 from both sides but then again cleared the denominator by multiplying by 3. In both cases, her methods were less efficient than the typical method of subtracting 3 and multiplying by the reciprocal. Although she did demonstrate knowledge and use of efficient algorithms during the course, her methods were clearly influenced by prior instruction that did not seem to emphasize efficiency. Instead, it seemed to be focused on minimizing error.

*Impact of Prior Knowledge*

Weak knowledge of algebra in some students interfered with their ability to implement and attend to efficient methods for certain problem types. For example, students in the class learned two ways to simplify multiplied exponential expressions with the same base (e.g., $a^3 \times a^4 = a^7$). One method was to expand all the terms and then find a single term that was equivalent to the expansion (i.e., $a \times a \times a \times a \times a \times a \times a = a^7$). The second method was to add exponents and keep the base the same. In this case the general algorithm, adding exponents, is also the more efficient one. However, some students were not comfortable with that method and chose to expand the terms in order to simplify the expression. When simplifying more complicated expressions, some students still used the expansion method. For example, Annemarie seemed to use the method to verify that exponents could be multiplied when raising a power to a power. Although Annemarie's method was inefficient, it did allow her to solve the problem correctly.

Although weak knowledge sometimes led students to use inefficient methods, other times it led to efficient ones. For example, when choosing to not to distribute on equations with fractions, Annemarie said, "Distributing fractions kind of scare me. Distributing them, I'm not good with fractions so I was afraid that if I distributed it, I would distribute the fractions wrong" (Annemarie, intermediate interview).

In fact, many of the students changed their methods when fractions were involved. For example, Ricardo expressed a preference for distributing in most cases but felt differently when 6/5 was in front of the parentheses. In this case, he preferred to multiply by the reciprocal of 6/5 "because this one was totally confusing….it was getting confusing multiplying 6/5 by this. It was easier just to get rid of it" (Ricardo, intermediate interview). Experts who took the same test were also concerned with avoiding fractions or any other arithmetic that may lead to errors. However, accuracy was not the dominant reason for choosing solution methods. For the experts, efficiency seemed to be the driving force (Star & Newton, under review).

Post-interviews support the notion that students who struggle with algebra are often more concerned with accuracy rather than efficiency. This focus may partially explain why knowledge of efficient methods sometimes preceded the use of them; if the method first learned by students was not problematic, they seemed less likely to try a different method. But despite not always choosing to use alternate methods, students in this study still viewed their knowledge of them as advantageous. Nicole suggested that knowing an alternate method was good "because in case you get stuck doing one way, you can always have a backup" (Nicole, post-interview). Ricardo, Xavier, and Yvonne had similar responses. Xavier and Yvonne added that being able to check your work was another advantage of having an alternate way to solve a problem. Only Naomi alluded to efficiency as an advantage. She said in her post interview, "Well knowing more than one way makes it easier to solve an equation just because one way might make solving it a lot harder, so if you know a different way, you could do it faster."

When asked about disadvantages, the students were again concerned with accuracy. In their post-interviews, the same possible disadvantage was offered by all six students. As Xavier stated, "Sometimes you could get the two methods mixed up." However, it was somewhat surprising that five of the six students insisted this was simply a possibility for someone else. As illustrated by Naomi, "People could do that, but I don't tend to get methods confused. Either I get it completely right, or I forget the method all together." Only Annemarie admitted that she might personally "mix up different methods." At the same time, however, she also expressed that the first method she learned did not always make sense to her. In those cases, learning a new method was particularly helpful. In the case of radicals, she stated that the one method her teacher taught was one she "really didn't understand and when you showed me the other method I was like, wow this is easier….I could understand it more and I always hated radicals, but now I am starting to get them" (Annemarie, post-interview).

## Discussion

Being able to flexibly solve problems is one of the hallmarks of procedural fluency (Kilpatrick, Swafford, & Findell, 2001), and recent attention to flexibility in the research literature has revealed some interesting findings about its development. The current study confirms and adds to those findings in several ways. The purpose of the study was to explore the development of flexibility with algebra within a real classroom setting and with students who struggle with algebra.

Findings confirm that students tend to acquire knowledge of alternate methods before they tend to use them (Blöte et al., 2001; Star & Rittle-Johnson, 2008), particularly if they have gained automaticity with one particular method. These results are consistent with Klein et al. (1998), who found that second graders who focused on skill before flexibility were less flexible that ones who focused on both at the same time.

When students encountered problems that were especially difficult or confusing for them (such as those containing fractions), they were more likely to use a new method for that problem. This finding is somewhat counter-intuitive, since one might expect students who are confused to prefer using one method only in order to minimize confusion. Yet, students in this study seemed grateful to have a "backup" method. In general, accuracy was often a driving force for deciding how to solve problems. This finding differs from experts who were asked to solve the same problems (Star & Newton, under review). For experts, efficiency (e.g., being fast, less complicated, easier to compute mentally) was a driving force, and salient features of the problem were cited as reasons for choosing one method over another.

When two methods were equally familiar and comfortable to students, they were more likely to cite efficiency and/or structure as reasons for choosing a method. Most students demonstrated little or no knowledge of graphing, solving systems of equations, and solving quadratic equations at pretest but, during interviews, consistently suggested they used a particular method because of the way the problem was posed to them or because a particular method was faster/more efficient. When students were not concerned with accuracy, their rationales for choosing methods were reminiscent of experts.

Taken together, findings of this study have implications for algebra classrooms. Comparing and contrasting methods for solving algebra problems seems to enable struggling students to solve problems they may not otherwise be able to. Furthermore, prolonged focus on one particular method for a problem type may inhibit flexibility with that type (Klein et al., 1998). The number of cases is limited in this study, but assessments and interviews suggest positive outcomes of instruction focused on flexibility. Although three weeks of instruction extends the time of prior studies, research is needed on how flexibility can be developed in a semester or year-long algebra course.

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# THE ROLE OF TASK IN EXPLORING ALGEBRAIC LINEARITY EPISTEMOLOGICAL OBSTACLE IN TECHNOLOGICAL ENVIRONMENT

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*This is a report of a study on high school Mexican students aged 16 and 17. The study seeks to unravel the role tasks when solved by CAS. Algebraic linearity epistemological obstacle features prominently in the tasks that were given to the students. The results showed that students were able to overcome the obstacle relating through the well designed task using CAS.*

## Introduction

For almost 3 decades, Researchers in mathematics education have reported that many secondary school students make algebraic syntax errors when solving problems in a pencil and paper environment. The reasons when students make such mistakes has been well documented such as (Matz, 1980; Booth, 1984).

Recent studies, (e.g., Kieran and Drijvers, 2006), on the use of technology, especially CAS, on the teaching and learning of algebra has indicated that students not only learn the syntax of algebra, but also reflect upon the underlying concepts involved in the symbolic manipulation. The use of technology in teaching change the kind of students-teacher interaction in the class so interesting tasks that students would enjoy solving has to be given (Kieran, 2007, p. 728). Technology on its own does not impact knowledge on the students, rather tasks whose solutions are interesting to students and rich in mathematics concepts should be designed and given to students. What then are the types of tasks that should be given to students when they are allowed to use technology to solve them? High school students achieve to overcome the algebraic linearity epistemological obstacle using technology in teaching? (Adapted from Monaghan & Pierce, 2004, p. 179.)

The role of solving tasks with the help of technology by the students of high school in Mexico is hereby documented. The aim of the study is to answer the question: “What is the influence of carefully drawn tasks such that students overcome the algebraic linearity epistemological obstacles when the tasks are solved with CAS?”

## Conceptual Framework

*Epistemological Obstacle*

The notion of epistemological obstacle arose in Bachelard (1948/2007) when he states: “When a scientific progress is being researched upon the scientific knowledge must be formulated in terms of obstacles, which are neither external nor due to the weakness of the senses, but rather in the very act of knowing “(p. 15). In education there have been many theoretical contributions to shed lights on the various types of obstacles. For example, Cornu (1991) affirmed that an epistemological obstacle is of the same nature as knowledge itself, while Brousseau (2002) notes that the origin of epistemological obstacles are intrinsic difficulties inherent in knowledge (p. 87).

According to Brousseau, an epistemological obstacle is an element of knowledge, which enables students to solve correctly mathematical problems of certain context. In this way the obstacle is established in the person but stops to be effective in solving problems of different context.

In mathematics education, overcoming an epistemological obstacle necessarily implies the occurrence of various interactions between a student and a medium (le milieu, which in the opinion of Brousseau), aims at giving students a task that eliminates any knowledge which acts as an obstacle (Brousseau, 2002). In this sense, Kieran, Guzmán, Boileau, Tanguay and Drijvers (2008) argue that technology can act as an agent that destabilizes some kind of knowledge in students, which acts as an obstacle.

*The Role of Task and Learning in Technology Environments*

Tasks are primarily problems. In the process of solving them, it is important that the people (students and the teacher) focus attention on the underlying mathematical concepts in the tasks. Franke, Kazemi, and Battey, (2007, p. 234) highlights the design of relevant mathematical problems which allow an exchange of ideas that lead to conjecture which are easily verified by students using technological tools in solve the problems.

The instrumentalist approach is one conceptual framework rich in theoretical elements to analyze processes of teaching and learning in technological environments (Drijvers and Trouche, 2008). Under this approach, the integration of technological tools in the classroom is seen as complex because it affects different aspects of education (Drijvers, Doorman, and Boon Gisbergen, 2008b) such as what didactic settings should be used in the classroom, How it should be used and the kind of tasks that should be posed to students to arise their interest in solving them.

To Drijvers and Trouche (2008, p. 368) an artifact is useless when a potential user does not know what kind of tasks that can be solved with it. The artifact suffers from a cognitive processing in the user when he is aware that the use of the devise can be extended (mental scheme) to solve other problems. When a user has developed various ways of using an artifact, an artifact becomes part of a useful and valuable instrument that mediate his mathematical activity. According to Drijvers and Trouche, before an artifact can becomes an instrument (instrumental genesis), a user needs to develop mental schemes which involve the ability to use the artifact properly and the knowledge of the circumstances under which the device is useful. More precisely, in the instrumentalist approach an artifact is an object (physical or symbolic), while an instrument is both the object and the mental scheme.

The instrumentalist approach in education led researchers to define three (closely related) concepts to explain how students learn meaningfully through solving tasks by technological tools. The concepts are: task, technique and theory, known in research as TTT (Artigue, 2002; Lagrange, 2002 & 2003). Technical means the manner in which tasks are solved. This manner does not have to necessarily be quasi-type algorithm or algorithm (Chevallard, 1999) since the use of a technique involves the conceptual knowledge of it; that is know why it is effective or not in solving tasks. Technique is a complex set of reasoning and routine work (Artigue, 2002). Lagrange (2002, p.163) states that tasks are after all, problems. Techniques play a pragmatic role in that the allow results to be generated. They also play an epistemic role, helping users of technology to understand the concepts involved in the task; they help in the generation of theory and serve as a source of further or emergent questions. Thus, resolution of tasks by students involves the development of theory related to the tasks.

## The Study

*Methodology*

The research focuses on nine students (aged 16 to 17) of a Mexican high school. Given their academic background having studied three courses of algebra in the second, third grade of secondary and in the first degree of high school in line with the existing Mexican syllabus, although in the traditional environments using paper and pencil, we can guess that they have imbibed algebraic epistemological obstacle of linearity. The students worked in groups each of which was made up of three students. Each of the students was given a printed copy of the tasks and a TI-voyage 200 calculator.

The applied tasks were designed taking into account persistent and recurring algebraic errors common to students of this educational level. The classical error, $\sqrt{a^2+b^2}=\sqrt{a^2}+\sqrt{b^2}$, due to the linear extrapolation from the rule $\sqrt{a^2\times b^2}=\sqrt{a^2}\times\sqrt{b^2}$. In general, each of the tasks given to the students has two possible solutions. Two errors or mistakes are committed in solving them: one of them is obvious (e.g. substitution error), while the other results from linear extrapolation from an algebraic rule.

The activities were designed to allow students reflect on algebraic errors related with linear extrapolation before and after using CAS. Thus, the purpose of the activities was to inquire what arguments students give in defense of their responses, before and after the use of CAS (calculator TI-voyage 200), to solve the tasks.

*Structure of the Activities*

a) Formulation of the problem;
b) Possible solutions to the problem;
c) Question about what the correct solution is;
d) Justifying their answer;
e) Use of the calculator and implementation of some its specific commands;
f) Reflection on the results given by the calculator;
g) Explanation on the results given by the calculator;
h) Reflections on the processes of the proposed solutions and student's initial response; and
i) Questions about what the correct solution to the problem proposed is.

The activity used in the study is herby reported. Due to space, only part of the data is discussed.

**Actividad 1**

En el zoológico de Morelia se construyó una nueva zona de alimentos. Los techos de los nuevos comedores tienen forma de cono, como se muestra en la Figura 4.

Figura 4

Para terminar con la construcción de la nueva zona de alimentos, sólo falta pintar por fuera cada uno de los techos. Los pintores encargados de realizar el trabajo, por experiencia, saben cuanta pintura se necesita por metro cuadrado de superficie. Ellos han encontrado en un libro de matemáticas una formula que les permite calcular el área superficial de un cono, lo que les ayuda para saber cuanta pintura necesitaran. La formula que encontraron es:

$s = \pi r^2 + \pi r\sqrt{r^2 + l^2}$, donde *s*, *r* y *l* son, el área superficial, el radio de la base circular del cono y la altura del mismo, respectivamente.

Dos de los pintores realizaron los siguientes cálculos, sabiendo que el radio es de 2 metros, la altura mide 3 metros, y $\pi$ lo aproximan a 3.1416.

| Cálculo A: | Cálculo B |
|---|---|
| $s = 3.1416(2)^2 + 3.1416(2)\sqrt{2^2+3^2}$ | $s = 3.1416(2)^2 + 3.1416\sqrt{2^2+3^2}$ |
| $= 3.1416(4) + 6.2832(\sqrt{2^2} + \sqrt{3^2})$ | $= 3.1416(4) + 3.1416\sqrt{4+9}$ |
| $= 12.5664 + 6.2832(2+3)$ | $= 12.5664 + 3.1416\sqrt{13}$ |
| $= 12.5664 + 6.2832(5)$ | $= 12.5664 + 3.1416(3.6055)$ |
| $= 12.5664 + 31.416$ | $= 12.5664 + 11.3270$ |
| $= 43.9824$ | $= 23.8934$ |

*Figure 1*. Exploration of the linearity epistemological obstacle ($\sqrt{a^2+b^2} = \sqrt{a^2} + \sqrt{b^2}$).

## Data Analysis and Discussion of Results

*Before Using CAS Technology*

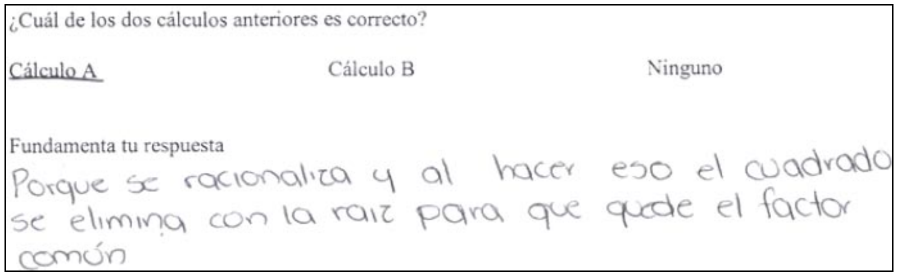
According to the students' justifications before using the calculator to answer the first question of the activity (Figure 2), it is noted that they considered $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ a valid equality. While all students identified the error or omission in calculating B, this did not lead them to automatically choose A as correct. Importantly, it was the separation of the square root $\sqrt{2^2+3^2} = \sqrt{2^2} + \sqrt{3^2}$ of A which led to the discussion as to whether such separation was correct or not. Below is part of the discussion among team 2, with regard to this separation of the square root.

S1: But here *[referring to the separation of the square root* $\sqrt{2^2+3^2} = \sqrt{2^2} + \sqrt{3^2}$]. Is it separable? I can remember it should not be separated. It goes with everything.

S2: It can be in two forms.

S1: Well yes. It's the same result.

At the end of first part of the activity (paper and pencil environment), all the three groups agreed that the correct solution was A. They argued that separating the sum of the square root simplifies the calculations. In particular, group 1 said that this allows them to rationalize (Figure 2).


¿Cuál de los dos cálculos anteriores es correcto?

Cálculo A Cálculo B Ninguno

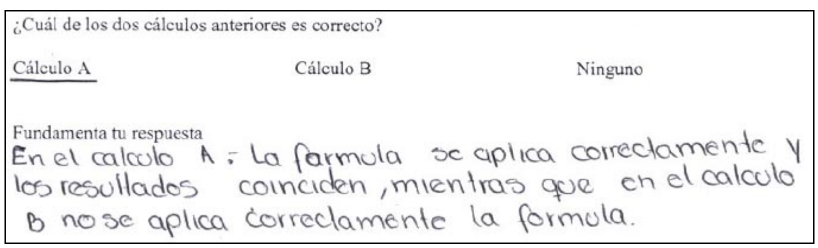
Fundamenta tu respuesta

Porque se racionaliza y al hacer eso el cuadrado se elimina con la raiz para que quede el factor común

*Figure 2*. The response of the group 1 to the first question of the activity before using the calculator.

Figure 3 shows the written record of the response of group 3 to the first question. It notes that the team identified the error or omission in calculating B, not the error in calculating A. In their discussion, it appears that they justified the separation of the square root, arguing that this will simplify the calculations. Here's part of their discussion:

S1: I would say no to this [*referring to the separation of the square root*]. This is not done when a formula is applied [*showing the error*] [...] It is not separable. Not in this kind.
S2: This can be [*referring to A*]. As you said [*pointing to student S1*] here the square root is separated [...] It is not altered.
S1: Just a little [...]
S3: One more step.
S1: Less
S2: Avoid a step.


¿Cuál de los dos cálculos anteriores es correcto?

Cálculo A Cálculo B Ninguno

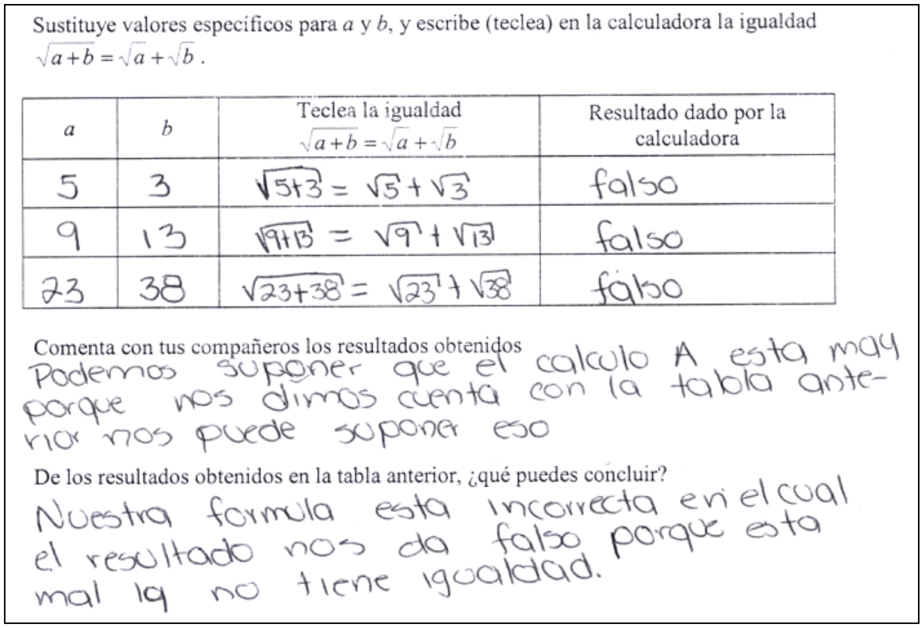
Fundamenta tu respuesta

En el calculo A : La formula se aplica correctamente y los resultados coinciden, mientras que en el calculo B no se aplica correctamente la formula.

*Figure 3*. The answer of group 3 before using the calculator.

*After Using CAS Technology*

After exploring equality $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$ (Figure 4) using the calculator substituting with some specific numerical values for *a* and *b* as chosen by the students. The results obtained by the students, allowed them to reflect on their initial response and thus change their mind.


Sustituye valores específicos para *a* y *b*, y escribe (teclea) en la calculadora la igualdad $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$ .

| *a* | *b* | Teclea la igualdad $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$ | Resultado dado por la calculadora |
|---|---|---|---|
| 5 | 3 | $\sqrt{5+3}=\sqrt{5}+\sqrt{3}$ | falso |
| 9 | 13 | $\sqrt{9+13}=\sqrt{9}+\sqrt{13}$ | falso |
| 23 | 38 | $\sqrt{23+38}=\sqrt{23}+\sqrt{38}$ | falso |

Comenta con tus compañeros los resultados obtenidos

Podemos suponer que el calculo A esta muy porque nos dimos cuenta con la tabla ante-rior nos puede suponer eso

De los resultados obtenidos en la tabla anterior, ¿qué puedes concluir?

Nuestra formula esta incorrecta en el cual el resultado nos da falso porque esta mal ya no tiene igualdad.

*Figure 4*. Part of the activity where calculator was asked to be used.

The exploration of $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$ by students, enabled most of them develop techniques that led to choose the correct answer to the problem (Group 3). This group developed strategies of solving the tasks, which are unlikely to have been developed in the paper and pencil environment. Although these students continued to use only natural numbers, they explored the equality using whole numbers (Figure 5).

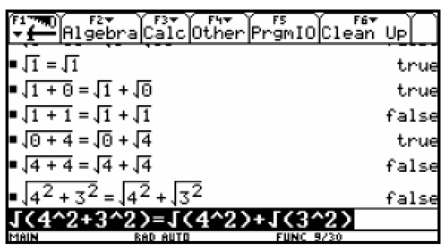



*Figure 5*. Exploration of equality by group 3.

After the equality $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$ has been explored using CAS, most students were able to solve the problem correctly. Some of them were even able to affirmed that the equality is true only when *a* or *b* is zero. Affirmations as the above are unlikely to be made by students of this level if they solve tasks in the paper and pencil environments.

## Conclusions and Final Reflections

It can be noted, in the study, that tasks designed allowed students to reflect on their solutions. Their reflection can be located on three crucial moments in the process of their solution: a) in paper and pencil environment, b) using CAS technology to verify their conjectures and c) in their answer to the questions under each task. Interestingly, the results given by the calculator motivated the students to discuss the veracity of their first sets answers which were based on prior their knowledge in the paper and pencil learning environments. The exploration of equality

$\sqrt{a+b}=\sqrt{a}+\sqrt{b}$ substituting *a* and *b* with certain numbers helped students to raise conjectures which are later validated using CAS.

The model proposed here is useful for activities in two settings (paper and pencil, technology) to solve tasks. The results show that the use of technology, and through the model used in the study, helped students to overcome the *algebraic linearity* errors. The results of the study however raise another question: Is the epistemological obstacle overcome by the students when using CAS technology to solve tasks permanent or temporal? This question will be addressed in a subsequent research under normal classroom environments using technology.

## Acknowledgments

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# THE EFFECT OF DGS ON STUDENTS' CONCEPTION OF SLOPE

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*This report is the first instalment of a broader study which investigates university students' conceptualisations of static and dynamic geometric entities. In this part we offer a refined look at the conceptualisations of two groups of students – one group which was taught using Dynamic Geometric Software and the other in a 'traditional' fashion. We use both APOS Theory (Dubinsky & McDonald, 2001) and Sfard (2008) to interpret learners' understanding of the slope of lines. Our data reveal that students using DGS developed a strong proceptual understanding of slope, which enabled them to solve problems in which slope could be seen a conceptual object. This report sets the stage for a look forward to how DGS may influence learners' process-object conceptualisation of other geometric representations of algebraic equations.*

## Background

Dynamic geometry software (DGS) offers the possibility for modeling mathematical concepts and processes related to topics across the curriculum (Scher, 2000; Sinclair & Jackiw, 2007). As many researchers have pointed out DGS also enables students to perform multiple actions and generate a large number of examples effortlessly (Hollebrands, 2007; Laborde, 1992; Mariotti, 2000). Although more widely used, and studied, in the context of the geometry curriculum, the dynamic and interactive features of DGS enable the designing of models appropriate for representing algebraic concepts. Such models are built using the geometric language of DGS such as *The Geometer's Sketchpad* (Jackiw, 1989), and thus offer geometric representations of algebraic concepts.

Additionally, these representations are fundamentally dynamic, representing relationships and behaviours over time; students are invited to explore these relationships through the dragging capacity of the software. For example, a model of an affine function might be represented both geometrically (as a line on a Cartesian coordinate system) and algebraically. If the line itself is draggable, students can investigate the relationship between the position of the line on the graph and the resulting algebraic equation, thus emphasizing the invariance of the slope as well as the changing value of the intercept.

While DGS research has mostly been conducted in the contexts of teaching and learning geometry, recent studies have focused attention on the possible benefits that the dynamic manipulation paradigm might have for other school topics (as well as for tertiary-level mathematics). For example, Falcade, Laborde, and Mariotti (2007) have shown that the use of *Cabri-géomètre* (Baulac et al., 1988) can support a covariational approach to functions. Similarly, Sinclair, Healy, and Sales (to appear) report on the use of dynamic functions within both *Sketchpad* and *Cabri*, and the way in which, compared to static functions, dynamic ones elicit more attention to mathematical aspects such as domain and range, the relationship between the dependent and independent variables, and the effect of asymptotes.

In this study, which draws on a wider investigation of the use of DGS in non-geometric areas of the curriculum, we focus on a much more specific element of the school algebra curriculum—that of slope. Given the importance of the notion of slope in early algebra, as well as in higher-level studies, as well as its convenient and widely-used geometric representation, we chose to study the effect of dynamic representations of slope on student learning. Prior research on slope has focused on the notion of slope as a ratio, and on the difficulty students have in constructing the concept of slope using the typical representation of the slope formula (Lobato & Seibert, 2002). Given the dynamic representation available in *Sketchpad*, and, in particular, the ability to drag a line (either by rotation or translation) and observe the dynamically-linked slope measurement, we decided to investigate whether such a representation might help students gain a better understanding of the slope concept as an object (and not just the end point of a calculation involving the rise and the run).

## Theoretical Perspectives

Prior research suggests that learners' engagement with computer software can facilitate certain mental constructions, particularly with respect to the process-object tension (e.g. Weller et al., 2003). Process and object understandings of mathematical entities lie at the centre of the inter-related theoretical perspectives which informed our study: the APOS Theory (Dubinsky & McDonald, 2001), Sfard's *reification* (1991), and the *percept-procept* distinction (Tall et al., 2000).

The APOS (Action, Process, Object, Schema) Theory postulates that learners' understanding of mathematical entities can develop from a *process* conception to an *object* conception through the mechanism of *encapsulation*. In the view of Dubinsky and McDonald (2001), an individual who imagines a mathematical entity, such as multiplication, as an action to be performed internally is said to conceive of that entity as a process. Accordingly, a process conception is recognised by qualitative descriptions which may describe actions though not execute them. For example, with respect to multiplication, imagining an action of collecting three groups of two objects to yield six objects would correspond to a process conception. If that process can be realised as a completed totality – three groups of two objects *as* six, rather than *to yield* six – then *encapsulation* of that process to an *object* is said to have occurred. Encapsulation of a process is a sophisticated step in an individual's conceptualisation. It requires appreciating the mathematical entity as a completed totality upon which transformations or arithmetic operations may be applied.

In a similar vein, Sfard (1991) describes the "quantum leap" that must occur in order that "a process solidifies into object, into a static structure" (p.20) in a learner's conceptualisation. Sfard refers to this leap as *reification*: "an ontological shift – a sudden ability to see something familiar in a totally new light" (1991, p.19). Further, Sfard (2008) identifies reification as a discursive process and describes it as "the act of replacing sentences about processes and actions with propositions about states and objects" (p.44). Relating the process-object distinction made both by Sfard and the APOS Theory to the idea of slope, we suggest that a process conception of slope might be recognised by an individual's description of a line that "goes up" by so much. In contrast, an object conception of slope could correspond to the description of an angle that may be scaled by any number.

Tall et al. (2000) offer a close look at the nature of mathematical objects and the intricacies of encapsulation or reification, and suggest that there is no "universal answer" to the question "how are [objects] constructed" (p.223). In particular, Tall et al. echo Sfard's (1991) observation that a structural (object) conception need not follow directly from an operational (process) conception, particularly in the case of geometric presentations. Tall et al. identify a distinction in how geometric 'objects' may be conceived: as "perceived objects" – *percepts* – or as "conceived objects" – *procepts*. In Tall et al.'s perspective, a perceptual understanding is recognised by an individual's attention to the "specific physical manifestations" (2000, p.228) of the geometric image. For example, a learner might conceive of a circle first as a percept – as a round, symmetric object – rather than as the process of finding all the points equidistant to the centre point.

Eventually, through *reflective abstraction*, a geometric *percept* may be constructed by an individual as a *procept*. Tall et al. (2000) suggest that procepts "are constructed as imaginary manifestations of the perfection of the definition, for instance, lines with no thickness that can be extended arbitrarily in either direction" (p.236). Further, they observe that "the construction of *perceived* geometric objects leads later to *conceived* geometric objects, which, though imagined in the mind's eye, and discussed verbally between individuals, are perfect entities that have no real-world equivalent" (ibid). With respect to the circle, a proceptual conception might correspond to the idea of the set of points which are equidistant from a centre point, and which necessarily form a round shape with infinitely many lines of symmetry.

We contend, in accord with Denis (1996) and Sfard (2008), that an object conception corresponds to 'descriptive' rather than 'narrative' discourse and suggest further that a pedagogical approach which connects algebraic representations with the percept of geometric entities may trigger a shift from narrative to descriptive discourse, and as such, promote encapsulation of that entity to a conceived object. Extending on prior research, we examine how DGS, which allows learners to act directly on objects such as circles and lines, may facilitate the leap toward the descriptive, from process to object conceptualisation of entities with both a geometric and algebraic representation, in particular with respect to *slope*.

## Setting and Methodology

### *Participants*

Participants were undergraduate liberal arts and social science students enrolled in a foundations course in quantitative reasoning. The course was designed as an up-grade to secondary school mathematics courses and focused on topics such as percentages, single variable equations, graphing lines, and problem solving. Two different sections of this course, Class A and Class B, respectively, which were taught by the first and second authors, were engaged in our study.

### *Comparing Instructional Representations*

Class A students were introduced to the notion of slope using dynamic representations of lines. These students used dynamic interactive sketches to visualize the change of slope measurement and the resulting geometric representation of the line. They were introduced to the notion of slope as a measurement to find out the steepness of a straight line, and discussed the relationship of numerical value of slope with its steepness. *Sketchpad* was employed to support the introduction and discussion with providing dynamic sketches of lines. Students interacted with the sketch shown in Figure 1. Specifically, they were asked to drag points A and/or B to find lines with slopes of -10, -4/5, -1/4 and 0. Dragging these points has the effect of tilting the line and generating several different examples of lines and their slopes.

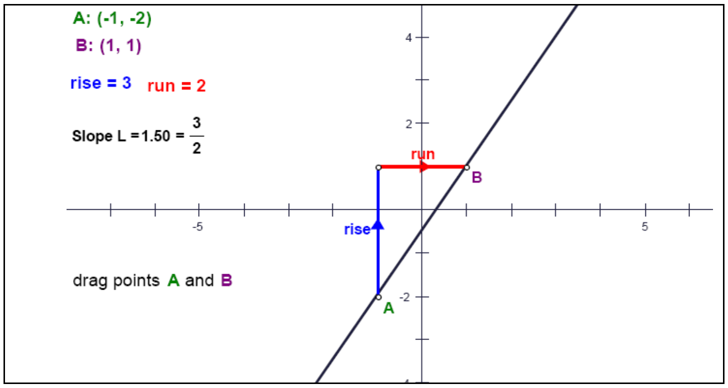


*Figure 1.* A snapshot of the sketch used by Class A.

Class B students were introduced to the notion of slope using a more traditional approach. These students saw static images of the typical triangle construction exemplifying ‘rise over run’ and the associated calculations. Students were exposed to a collection of different examples, including lines with different steepness as well as lines with negative slopes. Analogies and problems that reinforced the concepts of slope and steepness were also discussed. Students were also asked to reflect on and compare lines of different slopes, as in the task demonstrated in Figure 2 below.

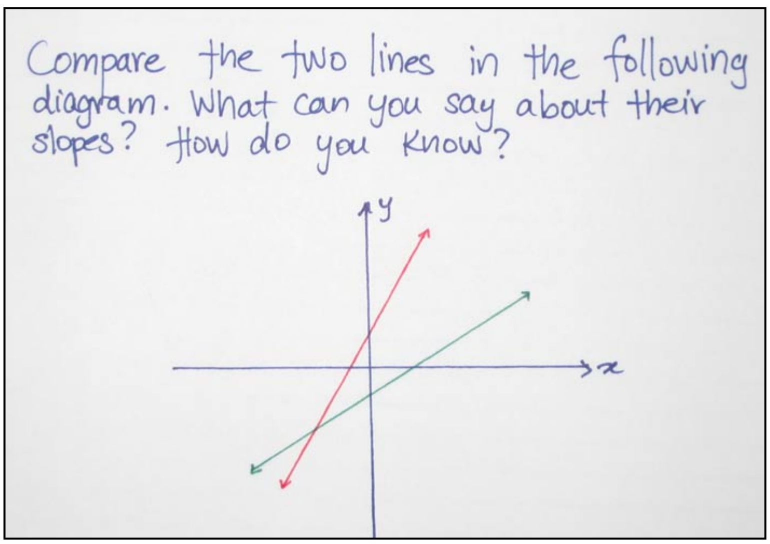


*Figure 2.* A snapshot of the sketch used by Class B.

Each class spent approximately 30 minutes on these slope tasks, and then moved on to investigations of linear equations and intercepts.

*Data Collection*

Data were collected from in-class pre- and post-tests. Pre-tests were given two weeks prior to the lessons on slope, and post-tests were given the week following instruction. Participants’ results on the pre-test informed the design of the post-test items. Pre-test items included questions relating to positive and negative slopes, as shown in Figure 3. It also included a question on identifying parallel lines given a list of equations, interpreting a story problem, and comparing the graphs of $y = x$, $y = 2x$ and $y = 2x+3$.

2. Circle the graph(s) that represent(s) a line with negative slope.

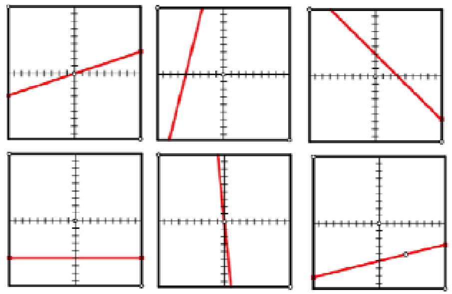

*Figure 3.* Shows positive negative slopes question of the pre-test.

Our intent in designing the post-test was to probe participants' conceptions about the notion of slope. In addition to repeating the question shown in Figure 3 above (with modifications of the actual order and shape of the graphs), we also included three additional questions shown in Figure 4 below: (1) A non-graphical question about the steepness of the slopes of different linear questions; (2) an example generating task; and, (3) a story problem where participants were asked to identify with justification whether the graph matched the story. This line of questioning was intended to present participants with non-routine problems relating to slope and to identify key features in participants' discourse that might indicate how they were conceptualising slope.

1. Consider the equations (a) $y = 3x+2$ (b) $y = 3x+10$ (c) $y = 2x+1000$ (d) $y = 2x-50$.
i. Which is/are steepest? Why?
**ii.** Which is/are less steep? Why?
2. The equation of a line is given as $y = 5x -3$. Provide three different examples of linear equations that pass through (0, -3). [An empty Cartesian graph was also provided].
4. Read the story below and discuss whether the story fits the graph.
*John was walking with his wife along the seaside. After about 5 minutes, he stopped to chat with his friend for a few minutes. He then began to run, to catch up with his wife.*

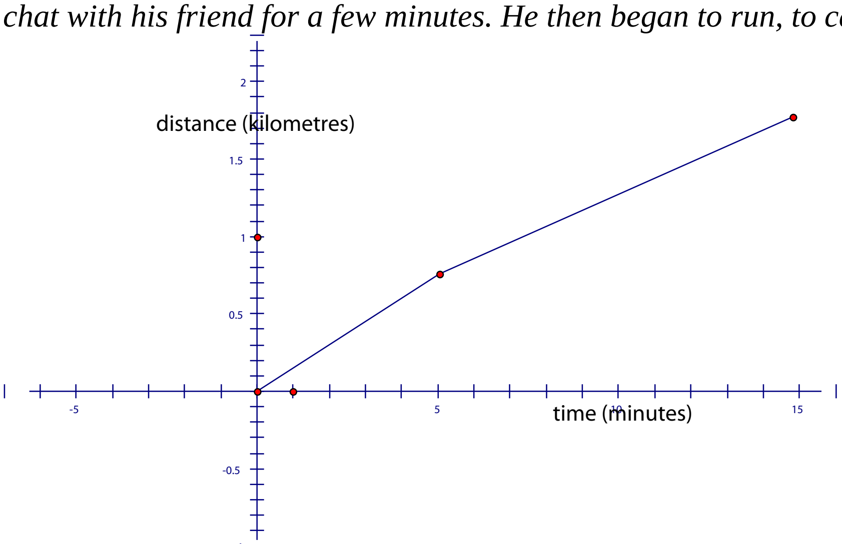


*Figure 4.* Three additional post-test problems.

## Results and Analysis

We present our results in two sections, first in terms of descriptive statistics on the pre- and post-tests, and second, on more qualitative interpretations of the patterns of student answers.

*Analysis of Pre- and Post-test*

Given that only the negative/positive slope question was repeated from pre-test to post-test, we limit our descriptive statistics to the comparison between the two classes on that item. The results are shown in the table below. Class A scores significantly lower on the pre-test compared with Class B, but then outscores Class B on the post-test.

Table 1. *Comparing Class A and Class B: Identifying Lines with Negative Slope*

| | Pre-test % | Post-test % |
|---|---|---|
| Class A | 36.36 | 87.5 |
| Class B | 77.59 | 83.87 |

These results indicate that Class B was already quite strong on the pre-test, and improved only slightly on the post-test. On the other hand, Class A was very weak at identifying lines of negative slope, and improved significantly after their instructional session in which they interacted with the dynamic sketch shown in Figure 1. On this sketch, the values of the rise and the run are shown, as was the value of the slope. Students were asked to drag the line continuously on the screen, so that they created a wide number of examples of lines along with their corresponding slopes. Given that the task asked them to create lines of given slope, they probably focussed more on the value of the slope, than on the values of the rise and the run. Therefore, these students did not need to coordinate the two numbers rise and run, and would have conceptualized the slope in terms of either bending forward or backward.

The lower pre-test scores on this item for Class A are consistent with their scores on other items in the pre-test; in other words, they were weaker than Class B on all items of the pre-test, and were also judged to be relatively weak by their classroom instructor. Given the significant difference in scores on this item from pre- to post-test, we decided to further investigate the students' understanding of slopes in contexts other than identifying positive or negative slope.

While comparative statistics on the remaining three items of the post-test are less useful, we include them here in Table 2 for completeness. We note that question 2, in which students were asked to generate equations having the same *y*-intercept, showed the only significant difference in favour of Class B. We found this result surprising, and will discuss it further in the following section.

Table 2. *Comparing Class A and Class B: Identifying Lines with Negative Slope*

| | Post-test question 1 | Post-test question 2 | Post-test question 4 |
|---|---|---|---|
| Class A | 64.29 | 48.81 | 69.64 |
| Class B | 75.81 | 79.57 | 59.68 |

*Participants' Discourse on Slope*

Tall et al. (2000) identify two distinct descriptive narratives relating to individuals' conceptualisation of geometric entities. On one hand, an individual may attend to visual cues elicited by the physical presentation of the entity – indicating a perceptual understanding. On the other hand, an individual may abstract from the visual cues to describe "the perfection of the definition" (Tall, et al., 2000, p.236) of the entity – indicating a proceptual understanding. Based on our analysis of participants' discourse relating to slope, students in Class A seemed to have developed a strong perceptual understanding of slope. Participants in Class A were able to reason visually, distinguishing negative from positive slopes visually, without relying on the rise over run formula.

Contrary to Class A's results, no change in participants' reference to perceptual aspects of slope or lines were observed in responses to post-test items from Class B. Rather, in support of our hypothesis, Class B participants seemed to rely more on narrative discourse, indicating a process conception of slope, in terms of the APOS Theory. For instance, in response to the "steepness" question on the post-test, the majority of students in Class B who answered correctly justified their response by saying that rise/run = 3/1 was greater than rise/run = 2/1. In contrast, the Class A participants who correctly answered did not make use of 'rise over run' narrative descriptions, and simply stated that 3 was greater than 2. In both classes, incorrect answers involved making comparisons between the values of the intercepts. This suggests that students were not so much confused about slope itself, but about the expression of slope within a linear equation.

Similarly, for the example-generating task illustrated in Figure 4, many participants in Class B superfluously introduced rise over run calculations, relying on manipulating the equation of a line to generate examples of lines with different slopes. Very few participants in Class B realised they could just change the value of the slope to generate a new example. Class B's reliance on manipulating equations is indicative of an operational, or process, understanding of slope. Rather than appreciating slope as an object that could be scaled or acted upon, many seemed to view slope as something which *needed* to be calculated – indicating a process understanding, in APOS terminology, or operational understanding in Sfard's (1991) perspective. Students in Class A who correctly answered this question did seem to generate equations without using the rise over run narrative (we infer this from the absence of the words 'rise' and 'run' in their responses). However, a larger proportion of Class A students simply left this question blank. Once again, we hypothesise that their inability to respond relates to a lack of coordination between the values of slope and intercept in the linear equation. In this case, having a procedural way of solving the problem seems to have benefited Class B.

Similar features were identified in participants' responses to the story problem illustrated in Figure 4. Again, participants in Class B were observed introducing slope calculations to justify whether or not the graph matched with the story. In contrast, participants in Class A were more likely to attend to the visual aspects of the graph, such as identifying stoppage time with a horizontal line segment (slope equal to 0). Participants in Class B were also more likely than participants in Class A to describe lines which "start at" a particular point, suggesting they viewed lines, as well as slopes, as entities which needed to be constructed – processes which needed to be carried out. The relatively strong performance of Class A supports the hypothesis that these students had built a strong perceptual understanding of slope, which they were able to use to interpret the visual graph accompanying the story.

## Conclusion

Based on our analysis of the students' responses to post-test questions, we suggest that the use of *Sketchpad* can help students develop a strong proceptual understanding of slope that emerges somewhat independently from the process-oriented conception inherent in the rise over run formulation of slope. This proceptual understanding corresponds to a descriptive narrative about the conceptual object of slope. The students in Class A, who had interacted with a dynamic graph, were able to use a descriptive narrative to distinguish positive and negative slope, to evaluate the steepness of lines, and to solve story-based problems. We argue that they were able to solve these problems because of their ability to see slope as an object, an ability that they had developed in their interactions with *Sketchpad* in which they directly acted on the graph to create different slopes (rather than acting on, say, the rise or run values of the line). The students in Class A solved these problems very differently than students in Class B, who depended far more strongly—and sometimes superfluously—on process-based conceptions.

Additional questions on the pre- and post-test that involved only algebraic manipulation (and not evaluation of visual graphs) indicated that the students in Class A were less successful in using their descriptive discourse in solving problems in which they had to coordinate between the slope and intercept values of a linear equation. Such problems may in fact require a more narrative discourse. However, we are pursuing further research to determine whether students can also develop an effective descriptive conception of slope within a linear equation, and coordinated with the role of the intercept, in order to be able to solve problems involving both slope and intercept concepts.

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# FROM CONCRETE MODELING TO ALGEBRAIC SYNTAX: LEARNING TO SOLVE LINEAR EQUATIONS WITH A VIRTUAL BALANCE

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*We are reporting on the results of a study undertaken with secondary school students in which a virtual version of a balance model is used to teach the students how to solve linear equations. The model is dynamic and enables the solution of term subtraction equations. We are studying the passage from the model to algebraic syntax and have adopted the perspective of mathematical sign systems (Filloy, Rojano & Puig, 2008), which incorporates student signic productions into the analysis as part of the interaction among the sign systems of algebra, arithmetic and the model.*

## Introduction

A good number of studies have focused on analysing the difficulties faced by students when learning algebraic syntax in order to solve linear equations. Some of the studies highlight the importance of reconceptualizing equalization as a sign of equivalence (Kieran, 1981; Kieran & Sfard, 1999). While others, underscore the need to learn how to operate with unknowns (Filloy & Rojano, 1989; Filloy, Rojano, & Puig, 2008; Herscovics & Linchevsky, 1991; Stacey & MacGregor, 1997; Vlassis, 2001). Some of the latter authors have specifically analyzed the role played by concrete modeling with a balance in the processes of building said syntax. Filloy & Rojano (1989) identified extreme cognitive tendencies in secondary school students, who at the beginning of their work with a balance showed, in some cases, a deep attachment to the model or, in other cases, a very quick detachment from the model going on to carry out the actions solely at the level of symbolic algebra. Vlassis observed in her study that use of the balance model helped students to learn the formal method of applying the same operation to both sides of the equation and that the main difficulties arose when it became a matter of having the subjects generalize the method to equations containing negative integers (Vlassis, 2002). Whereas Radford & Grenier (1996) found that a balance helps students to understand the rule for elimination of like terms.

One common element in several of the previously mentioned studies is that extending the numerical domain of equations to the set of integers represents a factor that obstructs generalization of the equation solution method. Gallardo (2002), Bruno and Martinón (1997) and Glaeser (1981) have analyzed the nature of the difficulty of incorporating negative numbers and their operativity into algebraic syntax. And in more particular terms, Filloy and Rojano have done so with respect to negative numbers and their relation to concrete modelling; they report the case of a student who spontaneously adapted the concrete model so as to solve equations with negative coefficients, which in principle was unexpected (that is to say that the ability to use the model to solve equations with negative coefficients was unexpected). These authors indicate that the foregoing is a manifestation of a cognitive tendency that consists of a deep-seated attachment to the concrete model, even in cases in which the modelling may be even more complex than the operativity itself at the symbolic level (Filloy & Rojano, 1989, pages 22 and 23; Filloy, 1991) that this may represent an obstacle to abstraction and generalization of the algebraic method. By the same token, Vlassis found in her research that the effect of having given students a concrete

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

meaning for manipulation of the equation terms lasted for several months after they had worked with the balance –considered a positive effect. This author admits however that the students will later on have to overcome obstacles related to the processes of abstraction, for instance such as in the case of solving equations that involve negative numbers and for which the balance model was not designed (Vlassis, 2002, pages 356-357).

The study reported here incorporates usage of an applets-based virtual balance model that differs from the traditional model (concrete or diagrammatic) in that it is dynamic and interactive and in that its expanded version (pulley balance) includes representation and solution of equations with term subtraction (SEP-ILCE, 2007) [1]. Another trait of this interactive unit is that it includes a section in which the balance is fixed and the user must choose the operation to be carried out with the terms of the equation (application of inverse operation so as to eliminate terms), which has the effect of helping the student to "abstract" the model's actions to the algebraic syntactic level. The results obtained suggest that after working on tasks with the interactive unit and worksheets that indicate a didactic route for mastering the syntax, the students are able to abstract the model's actions and recover them at the level of symbolic manipulation. The foregoing is achieved for broad families of linear equations, including those that contain terms with negative coefficients.

## Aim of the Study

The main purpose of the study is to research the extent to which work with the dynamic version of the balance referred to above helps subjects to abstract the actions undertaken with the balance at the level of the algebraic syntax associated with the solution of linear equations. In other words we are interested in investigating whether subjects are able to generalize the method of "doing the same thing on both sides of the equation" to increasingly more complex equation modes, including equations that contain term subtractions with positive coefficients. We have included arithmetic ($Ax = B$; $Ax \pm B = C$; A if a specific positive integer, B, C, and D are particular non-negative integers) and non arithmetic equation modes ($Ax \pm B = Cx \pm D$), in keeping with the classification of Filloy & Rojano (1989).

## The Virtual Balance and the Didactic Circuit

As previously mentioned the concrete model used consists of a virtual version of the traditional diagrammatic model. In the first part of the interactive unit, students work with the **basic balance** (see Figure 1).

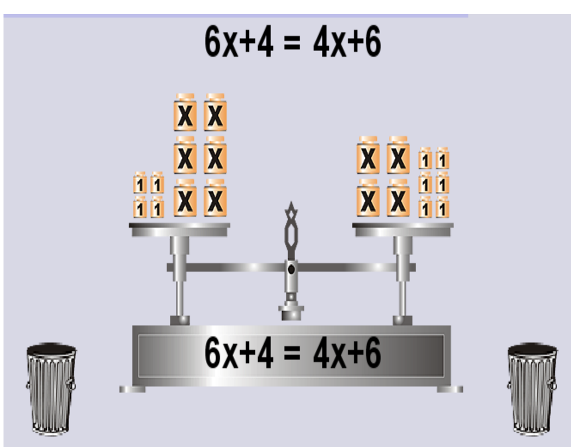


*Figure 1.* Basic balance.

Users can drag objects to add them onto either side of the balance (taking them from the piles in the middle) or to drop them. The objects available have either known weights (unit weight) or unknown weights (x). The idea is for students to understand the principles guiding the actions that maintain or re-establish equilibrium at each step. The sequence of scenes includes the following sections: a) *weigh objects*; b) *represent a given equation*; c) *find the value of the unknown weight*. The bottom section displays the equation that is to be solved, as well as the changes to the equation that result from the actions carried out on the balance, thus acting as feedback. Equations to be solved appear randomly, however the *level* button makes it possible to gain access to exercises containing increasingly more complex equations.

In the second part of the interactive unit, students work with **pulley balance** (see Figure 2). With the extended balance model one can represent and solve equations that include subtraction of positive coefficient terms, that is to say it is possible to remove and subtract weights. The terms that are subtracted are represented by weights that are placed in the upper pans. Objects can be dragged from the upper pans to the lower pans –be that the right or left pan– and vice versa. One can see that when going from an upper pan to a lower pan (or the other way around) or from the left to the right pan (or the other way around), the operation sign preceding the corresponding term in the equation changes (this can be observed in the equation that is displayed in the bottom section of the balance).

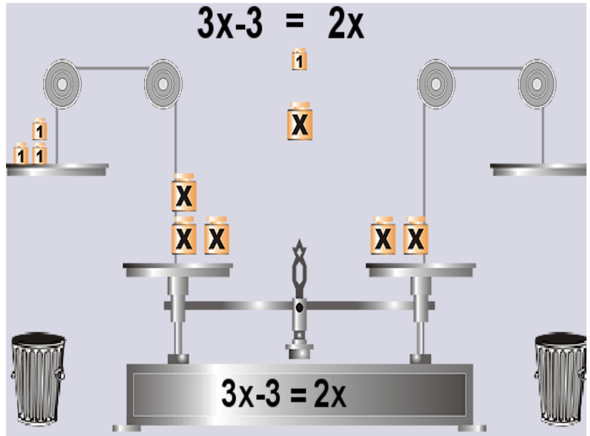


*Figure 2.* Pulley balance.

In the basic balance model, the didactic circuit consists of the following: 1) Familiarization with the balance (weighing objects); 2) representation of equations on the balance (correspondence between the elements of the equation and those of the model); 3) solution of equations with the dynamic balance, finding the unknown weight by eliminating objects from the pans (principles of manipulation that maintain the equilibrium); 4) solution of equations with the fixed balance, by choosing the inverse operation that is applied to the terms of the equation (recovery of the principles that maintain the equilibrium at the syntactic level: notion of algebraic equivalence) see Figure 3; and 5) solution of equations without the balance, transforming and re-establishing equalization at the symbolic level of algebra (automatization of actions at the syntactic level).

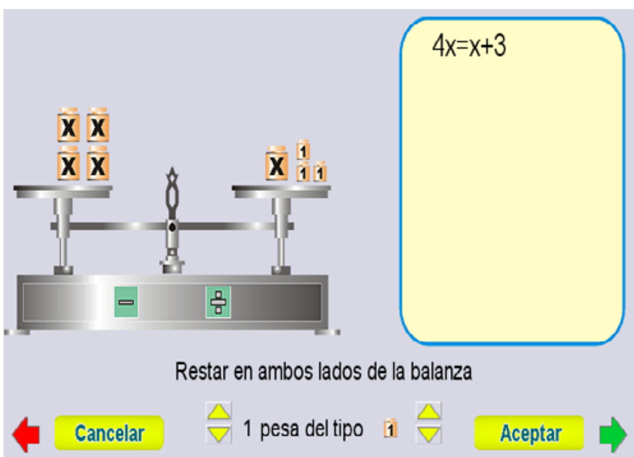


*Figure 3.* Fixed balance.

Steps 1-5 are repeated when using the pulley balance, thus extending the method to equations with term subtraction.

## Mathematical Sign Systems and Abstraction to Syntax

In the theoretical approach proposed by E. Filloy (Filloy, Rojano, & Puig, 2008; Kieran & Filloy, 1989; Puig, 2004) the notion of text is introduced to be used in the analysis of any sense production practice, for example when the learner interacts with a teaching model. From this standpoint essentially based on the semiotics of Pierce (1931-58), a distinction is made between *text* and *textual space,* a distinction that corresponds to that between *meaning* and *sense*. A text is the result of a reading/transformation labor made with a textual space, the aim of which is not to extract a meaning inherent in the textual space, but to produce sense (Filloy, Rojano, & Puig, 2008, page 125). The textual space is a system that imposes a semantic restriction on the person who reads it; the text is a new articulation of that space, individual and unrepeatable, made by a person as a result of an act of reading. Thus a teaching model is a succession of texts that are taken as a textual space to be read/transformed into another textual space as the learners create sense in their readings.

The interactive virtual balance unit is a teaching model and a textual space in terms of the afore-described theoretical perspective. When students interact with that textual space reading / transformation processes are unleashed, and sense is produced in those processes: the sense of the actions in the model and the corresponding actions at the symbolic level of algebraic syntax with the elements of the equation.

At the same time in this theoretical model symbolic algebra is considered a mathematical sign system (MSS), understood as a sign system (with its corresponding code) in which there is a socially conventionalized possibility of generating signic functions (Filloy, Rojano, & Puig, 2008, page 7). Attached to the MSS are the sign systems or strata of sign systems that learners produce in order to give sense to what is presented to them in a teaching model. According to Filloy the texts produced by readings that use different strata can be described in an MSS, but when that does not take place only the creation of a new MSS will make it possible. The process of creating new MSSs for that purpose is a process of abstraction and the new sign system is more abstract than those preceding it. In the case at hand, the new sign system is that of algebra and it is more abstract than the sign system of arithmetic. It is also more abstract than that of the intermediate strata constituted by the productions of the students themselves based on their interaction with the virtual balance model. Examples of that type of production will be showed in a subsequent section.

## Methodology and Data Collection: Sessions With and Without the Balance

The study was carried out with a group of eight secondary school students, 12 to 14 year olds, who had not received any instruction on the algebraic method for solving linear equations. Worksheets were prepared in keeping with the didactic circuit of the interactive balance unit, as were a pre and post questionnaire aimed at finding out what strategies the students used to solve the equations prior to and after their work with the balance. The items of both questionnaires correspond to arithmetic and non arithmetic equations.

The six class sessions in which the balance was used were undertaken in a computer lab during one and the same school term. The large screen display of the interactive unit was used for teacher explanations, pupils' individual participation, and collective discussions. Additionally, the students worked in pairs on the computers to solve the tasks proposed on the worksheets. The following section is a discussion of the outcomes obtained during the class sessions.

## Analysis of Outcomes

### *Sessions Using the Basic Balance*

In the first session, students worked with the scenes *"finding the weight of "x" and representation of linear equations on the balance"*. They manipulated the virtual dynamic model and were able to relate the meaning of balance –pertaining to the sign system of the model– with the meaning of the equality sign in the equation –pertaining to the sign system of algebra. However not all of the students were able to attain said level of understanding; when doing the tasks with paper and pencil, two of the students continued to write the equation transformations as chains of equalization (their own productions that are located within an intermediate stratus between the sign system of arithmetic and that of algebra).

During the *equation solution* session (second scene) the students demonstrated self confidence, navigating through the five complexity levels: from solution of equations with a single occurrence of an unknown (arithmetic equations), through to solution of equations in which the unknown appears on both sides of the equalization (algebraic equations). The worksheets, which students solved with paper and pencil, show how the subjects adapted the model and used it in solving equations containing coefficients greater than those supported by the virtual model. In this stage, we observed diagrammatic reproductions (drawings) of the model and the symbology used is a mixture of those diagrams with arithmetic operation signs (personal productions generated due to the interaction with the sign system of the model and incorporation of elements of the sign system of arithmetic).

In the third session, where students must *solve equations choosing the inverse operation*, we observed significant progress in algebraic syntax. The students were able to solve the equations, eliminating terms by applying the corresponding inverse operation and in the majority of cases there were no indications that the students had resorted to the balance model, not even diagrammatically. As of that point, we observed recovery processes of the actions undertaken in the model, at a more abstract level, producing sense related to the actions carried out with the elements of the equation. This more abstract level is the sign system of algebra, which enabled the students to carry out the reading/transformation of the texts constituted of the more complex linear equation modes (for instance, non arithmetic equations with negative coefficients that did not have meaningful referents in the balance). The foregoing was not possible for them when they remained at intermediate (more concrete) strata related to the model and/or to arithmetic. At this stage the solution characteristics were the following: use of the inverse operation; progressive usage of algebraic signs (students no longer use the model drawing, they rather make use of algebraic symbols), although in some cases we still observed certain reluctance to operating unknowns and a return to arithmetic strategies. The progress achieved in algebraic syntax as of the third session can be seen in Figure 4, which depicts part of two students' worksheets produced throughout several sessions.

| Item / Student | 6X+1=3X+7<br>X= 2<br>Session 2 | 10X+40=5X+50<br>X=2<br>Session 2 | 4x+1=3x+4<br>X=3<br>Session 3 | 6x+8=4x+26<br>X=9<br>Session 3 | 2X – 6 = - 4<br>X=1<br>Session 5 | 3X – 4 = -X +12<br>X=4<br>Session 5 | 6X+4=4X+10<br>X=3<br>Session 6 | 4+6X=-2+9X<br>X=2<br>Session 6 |
|---|---|---|---|---|---|---|---|---|
| FRANCISCO | X = 2 2 ©<br>6 6<br>7 0<br>-1 2<br>6 6 3 6<br>-3 0<br>3 | X 2 ©<br>50 10<br>- 40 5 5 10<br>10 5 0 | 1x=3 ©<br>4x=3x+3<br>1x=3 | 2x=18 (I)<br>6x=4x+18<br>2x=18 | X = 2 (I)<br>2X = 4<br>X = 2 | X = 4 (C)<br>3X+X = 12+4<br>4X = 16<br>X =4 | X=3 (C)<br>6X-4X=10-4<br>2X=6<br>6/2 = X<br>X=3<br>6(3)+4=4(3)+10<br>22 = 22 | X=2 (C)<br>4+2=9X-6X<br>6=3X<br>6/3 = X<br>X = 2<br>4+6(2)=-2+9(2)<br>16 + 16 |
| ALEXIS | X = 2 ©<br>X x<br>X x x x x<br>X x x x x<br>X x x x<br>x<br>xxxxx1 xxx<br>xx = 111111x<br>xxx = 111111 | X = 2 ©<br>~~Xxxxx~~ xxxxx<br>111111111 x111111<br>1111111111 x111111<br>1111111111=x111111<br>111111111 x111111<br>x111111<br>1111111<br>1111111<br>111111<br>2<br>5 10<br>0 | X = 3 ©<br>4x=3x+3<br>1x=3 | 2x=18 (I)<br>6x=4x+18<br>2x= 18 | X = 1 (C)<br>2X = -4+6<br>2X = 2<br>X = 1 | X = 4 (C )<br>3X+X = 12+4<br>4X = 16<br>X = 4 | X=3 (C)<br>6X-4X=10-4<br>2X=6<br>X=6/2<br>X=3 | X=2 (C)<br>4+2=9X-6X<br>6=3X<br>X=3X<br>X=6÷3<br>X=2 |

*Figure 4.* Examples of worksheets of Francisco and Alexis during six sessions.

*Sessions with the Pulley Balance*

In addition to progressing toward solution of a more extensive family of linear equations, working with the pulley balance enabled students to identify and learn the term transposition method. As such at the end of the study, participants could indistinctly apply that method, as well as that of “doing the same thing on both sides” or “using the inverse operation” method. Toward the second to last session, students solved all of the exercises and any reluctance for operating unknowns in non arithmetic equations had disappeared. At the same time, students had left behind usage of the model, and equations were solved in the sign system of algebra, although in some cases use of arithmetic operation signs persisted (for instance, ÷). At the syntactic level, the difficulty of applying the methods learned to a more extensive family of equations with negative numbers reappeared, especially in cases of signed numbers that are not suitable to modeling with the balance or in the case of negative solutions (such as, for example: $5x + 7 = -3x – 9$). The latter confirms the outcomes of Vlassis (2002), in the sense that the presence of negative numbers represents an obstacle for generalization of the method.

During the last session, which dealt with *solution of equations containing terms with negative coefficients, choosing the correct operation*, the virtual model made it possible to reinforce the way of working that the students had previously acquired; that is to say, transposition of terms. The foregoing can be attributed to the fact that in this section the model displays, on the left hand side of the scene, a representation of the actions that are carried out.

As mentioned before, the worksheet for this scene contains both arithmetic and non arithmetic equations, with additive structure and with term subtraction. We observed that the final work of the students contained both solutions attained using the “doing the same thing on both sides” method (for single-step equations), as well as the “term transposition” method (for equations with occurrences of “x” on both sides) (See Figure 4, session 6).

*Pre and Post Questionnaires*

Figure 5 is a synthesis of the outcomes of the pre and post questionnaires, while Figure 6 depicts some examples of the answers given by the students to items contained in both questionnaires. These outcomes confirm our classroom session observations, through the worksheets, dealing with the evolution of subjects on their road to mastering algebraic syntax in order to solve linear, arithmetic and non arithmetic, additive and term subtraction equations. In some cases, children could extend the method to equations of the form $Ax + B = - Cx + D$. However, there is no evidence to suggest a complete generalization of the method, for example to linear equations with a negative solution.

| PRE-QUESTIONNAIRE | POST-QUESTIONNAIRE |
|---|---|
| • Students apply arithmetic strategies to solve both arithmetic and non-arithmetic equations.<br>• They avoid operating on the unknown when solving non-arithmetic equations.<br>• Trial and error strategies predominated in the solution of non-arithmetic equations. | • Use of algebraic syntax predominates.<br>• Verification of solutions by numeric substitution.<br>• Special writings appear. For example, when solving the equation $5x+2 = -3x-6$, 3 out of 8 students operate on constant terms and terms with ‘x’, separately, as follows:<br>$5x+3x = 8x$<br>$-6-2 = -8$<br>$x = -8/8$<br>• Many still use the sign ÷: $10x=2$, $x=2÷10 = 5$ |

*Figure 5.* Pre-Post general outcomes.

| Item \ Student | *Gabriel* | *Alexis* | *Estephany* | *Francisco* | *David* | *Brenda* | *Isaac* | *Jonathan* |
|---|---|---|---|---|---|---|---|---|
| VII. 2x – 1 = x + 1<br>X = | VII. 2X-1=X+1<br>X= | II. 6 + x = 8<br>x= 2 | VII. 2X-1=X+1<br>X= 1 | VII. 2X-1=X+1<br>X=---------- | VII. 2X-1=X+1<br>X=--------- | VII. 2X-1=X+1<br>X=1 | VII. 2X-1=X+1<br>X=--------- | VII. 2X-1=X+1<br>X= +0 |
| | VII. 2X-1=X+1<br>X= -2<br>2X – 1 = -1 – 1<br>1X = - 2 | VII. 2X-1=X+1<br>X= 2 2x-x = 2<br>X=2 | VII. 2X-1=X+1<br>X=+-2<br>2x –x = x<br>+1+1 = + 2<br>x= +2 | VII. 2X-1=X+1<br>X= 2<br>2x-x = 1+1<br>X=2 | VII. 2X-1=X+1<br>X= 2<br>2x-x=+1+1<br>1x=2<br>x=2/1 = 2 | VII. 2X-1=X+1<br>X=2<br>2x-x=+1+1<br>X=2 | VII. 2X-1=X+1<br>X= 2 | VII. 2X-1=X+1<br>X= 2x – x = x = 1 + 1 = 2 |
| VIII. 5x + 2 = -3x – 6<br>X = | VIII. 5X+2= -3X-6<br>X= | II. 6 + x = 8<br>x= 2 | VIII. 5X+2= -3X-6<br>X=1 | VIII. 5X+2= -3X-6<br>X=---------- | VIII. 5X+2= -3X-6<br>X=--------- | VIII. 5X+2= -3X-6<br>X=--------- | VIII. 5X+2= -3X-6<br>X=--------- | VIII. 5X+2= -3X-6<br>X= ______. |
| | VIII. 5X+2= -3X-6<br>X=1<br>5X +3X = 2 + 6<br>8X = + 8<br>8 ÷ X8<br>5x1+2=3x1-6<br>7 = | VIII. 5X+2= -3X-6 X= -1<br>5x+3x= 8x<br>-6-2 = -8<br>x= -8/ | VIII. 5X+2= -3X-6<br>X= -1<br>5x + 3x2<br>8x = 8<br>x= 8/8<br>x= -1 | VIII. 5X+2= -3X-6<br>X= -1<br>5x + 3x = 8x<br>-2-6 = -8<br>8x = -8<br>x = -8 ÷ 8<br>x= -1 | VIII. 5X+2= -3X-6<br>X= 2 2x = 4<br>X= 2 | VIII. 5X+2= -3X-6<br>X= -1<br>5x+3x=-6-2<br>8x=-8<br>x=-8/8 =-1 | VIII. 5X+2= -3X-6<br>X= -1<br>5 – 2 = 3 3 -<br>3 – 6 = -3 x = -1 | VIII. 5X+2= -3X-6<br>X= 5x + 3x = - 6 – 2<br>8x = - 8<br>x = - 1 |

*Figure 6.* Students' answers to items VII and VIII. The highlighted correspond to Post Questionnaire.

## Final Discussion

According to Filloy when teaching is begun with a concrete model it is important to understand the actions executed, as well as to discover the syntax elements implicit in those actions. This process leads to abstraction of operations; that is to say recovery processes at the syntactic level. In the study reported on here, manipulation of the dynamic virtual balance model in the initial scenes made it possible to produce sense related to those actions in the model. Work with the scenes of the fixed balance and choosing the inverse operation favored production of sense at the level of algebraic syntax. In other words the fact that it was not possible to manipulate the balance fostered abstraction toward the sign system of algebra by way of discovering the implicit principles of preserving equality (balance). The foregoing became crystal clear in the post-questionnaire when the students solved the equations solely with paper and pencil and with no access to the interactive model. These outcomes serve to ratify those of other authors (Vlassis, 2002; Filloy & Rojano, 1989; Radford & Grenier (1996) who used a diagrammatic version of the balance in their research. The outcomes from the pulley balance section suggests that working simultaneously with addition and subtraction of weights also leads students to discover and abstract the rules of term transposition. That discovery broadened the students' level of algebraic competence with respect to solving linear equations, from the "doing the same thing on both sides" method to the vietic "term transposition" method, as well as to the rule for grouping like terms. The expansion also applied to the type of linear equations, given that the students ended up solving equations with negative coefficients, although several cases demonstrated the classic difficulties that students of that age group have in understanding and operating negative numbers.

## Endnote

[1] The pulley balance allows for modeling of equations that have an additive structure and of term subtraction equations. This means that only terms with positive coefficients that are added or subtracted can be modelled, while terms that include signed numbers, such as in 7 – (-2x) or 5x + (-3) cannot, unless the students are familiar with the syntax of negative integers that enables them to reduce the previous cases to 7 + 2x & 5x – 3, respectively. The foregoing is due to the fact that objects on the balance always have a positive weight. This applet-based interactive unit

was developed by the programming group “Descartes” at the Latin American Institute of Educational Communication (ILCE) in Mexico and is part of the interactive materials that the Ministry of Education (SEP) has distributed in the public secondary schools across the country. The experimental work was funded by CONACYT, Mexico (Project Ref. No. 80359).

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# NUMERICAL STUDY OF THE GRAPH THROUGH THE CONSTRUCTION OF ITS ALGEBRAIC EXPRESSION (CASE OF THE POLYNOMIALS)

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*Interpreting Global allows to explore the contents of the algebraic expression in order to identify visuals variables in the graphic, giving a categorical variable in the algebraic expression. However, developing the global interpretation in polynomials of degree higher than two seems to be a complex task, because the trace, to identify and interpret the correspondents visual variables. To overcome this problem, an alternative to explore qualitative and quantitative traces is reviewed here based upon "visual characteristics". This is enriched through the use of numeric representation analyzing the graph relations of deep-figure from the numerical point of view.*

## General Background

Duval (1999), mentions that mathematic visualization is not simultaneous in the field of perception; it is an intentional cognitive activity that produces a representation in two-dimensions (screen, paper, etc.). It shows the relationship between representational units that make up the figure, meaning that mathematical visualization presents only a mathematical object. These are "seen" through the organization of relationships between figures. In the case of the graph, it has two figures: deep-figure (meaning the Cartesian Plane) and the figure-form representing the trace.

In this regard, global interpretation (Duval, 1999) identifies the visual values to establish the relationships with categorical values of the algebraic expression. The treatment is essentially qualitative, leaving the deep-figure as a stable frame. This is an important issue in order to explore the contents of the representation, since the modification of deep-figure, dividing locally the unit of graduation, produces a change in figure-form. This activity modifies the unit of graduation, a change identification of the values.

Duval (1994), emphasizes the common principle of the visual organization for the numerical representation, in learning, which has two apprehension levels:

- "an apprehension resulting of a simple exposition cognitive of control.
- and an apprehension that result of the interpretation global". (pp. 6)

The first level is the most helpful in teaching, because it focuses exclusively on one approach to control, i.e. …, while the second level is ignored, it is required to expose the contents of the representation to identify the representational units, establishing connections with others, in particular the algebraic.

The objective of this paper is to establish a numerical analysis of the graph through the numerical table to enhance the algebraic construction. To pursue such a goal it is necessary to explore the contents of the numerical table (numerical representation) from a global interpretation requiring the use of treatments, thus benefiting the identification of numeric variables to establish connections with the categorical variables (from the algebraic representation).

## Enhancement of Global Interpretation

The global interpretation (Duval, 1988) stresses the changes in the algebraic expression corresponding to visual variables pertinent to interpret the graph, allowing the association of a visual variable with a categorical variable within the algebraic expression. It contributes to the identification and establishment of relations between the two representations.

Duval analyzes the behaviour of the straight line through the global interpretation, mentioning the possibility of a similar analysis for the case of the parabola. This paper has identified and reviewed the visual and categorical variables (algebraic expression) for second order polynomials. The study reveals the difficulty in discriminating the visual variables that characterize the polynomial, since the behaviour of the trace presents more variation than the straight line, increasing the number of numerical variables to be identified. With respect to the study of the cubic polynomial, the interpretation of its content through the analysis in the algebraic expression is performed. This task tends to be exhaustive due to modifications of the trace, obstructing the interpretation of the visual variables suffering many variations. However, these are identified for the cube term$(y = \pm ax^3)$. The analysis is complicated when involving linear and quadratic terms, except from the independent term that does not modify the behaviour of trace. An alternative is designed based upon qualitative information, for instance, the cubic polynomial around the origin in a vicinity small enough, whereby the term $a_3x^3$ is neglected in magnitude to the rest $(a_2x^2 + a_1x^1)$. This induces behaviour similar to $a_2x^2 + a_1x^1$ facilitating the visual identification of the visual variables with respect to the modifications of algebraic expressions, whereas the term independent is null.

Since the conditions used to discriminate the visual variables in the cubic polynomial, specifically for the terms $a_2x^2 + a_1x^1$ are different from those developed by Duval (1994), these are named "visual characteristics". The analysis to identify the "visual characteristics" of linear and quadratic terms is carried out through the selection of a small vicinity around the origin, where the term $ax^3$ is neglected in magnitude compared to the rest ($bx^2 + cx$), including the behaviour of a cubic polynomial, it is very "similar" to $bx^2 + cx$, i.e. the reader may reviews the cubic polynomial $Y(x) = ax^3 + bx^2 + cx$ , considering the functions $y(x) = ax^3$ and $h(x) = bx^2 + cx$ in order to identify the visual variables of the cubic term, as the "visual characteristics". (see Figure 1)

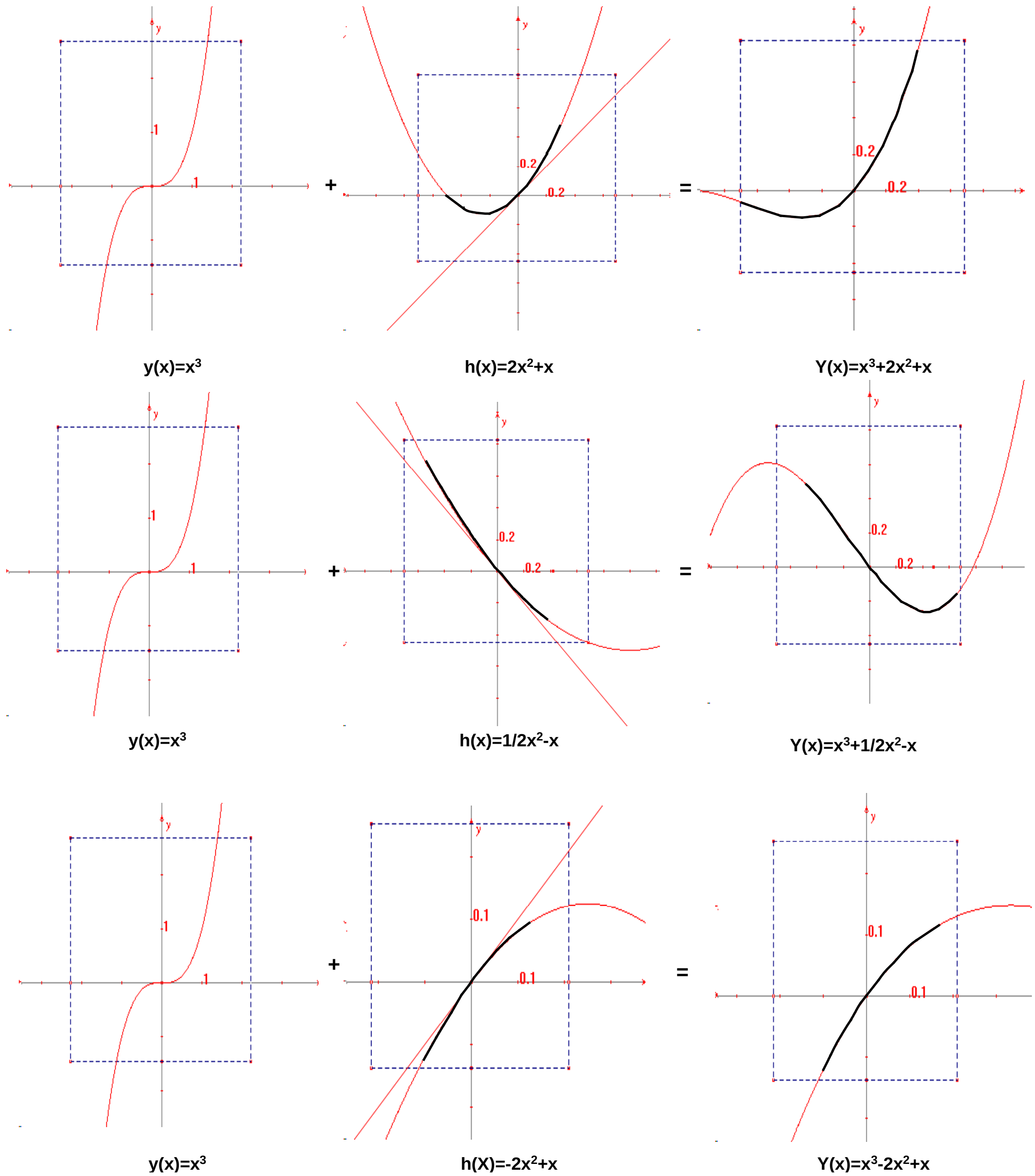


*Figure 1*. Examples of the behaviour of linear and quadratic terms for the polynomial ($Y(x) = ax^3 + bx^2 + cx$).

The "visual characteristics" identified in the vicinity of the origin are:

| Features Visual | Categorical Variables ($bx^2$) |
|---|---|
| **The line opened up** | **b>0** |
| **The line opens down** | **b<0** |
| **The stroke is close to its axis of symmetry** | **b≥1** |
| **The line moves away from its axis of symmetry** | **b<1** |

Moreover, in the behaviour of the linear term the trace is the tangent line to the parabola at the origin. This line is displaced to the right or left of the vertical axis with respect to the parabolic behaviour. For that analyzed behaviour, the line around the origin benefits the global interpretation through qualitative treatment to identify the "visual characteristics". It interprets the information from the global perspective, allowing the relations based on the changes of the categorical variables from the algebraic.

This approach strengthens the exploration of the trace within a region of the plane, identifying the information for the recognition of "visual characteristics", as well as the relation with the algebraic representation (categorical variables). Based upon this information it is suggested a global study in one region of the plane, whose information helps to identify relationships in the graph and the algebraic expression according to the modifications set out in this by using other representations from the identified information.

### Exposition of the Reason to Explore the Content Numeric in the Graphic, as well as the Need for Numeric Representation (table value), to Enrich the Information Identified in the Graphical Representation

Until now, the information has been analyzed using qualitative treatments in the graphical representation, however, the task to construct the algebraic expression of a graphic requires information of the numerical type, raising the possibility of analyzing the graphic through quantitative treatment, i.e., analyzing the relationship between deep figure and figure form from a numerical point of view. This led to exploration of the information contained in the deep-figure to establish relationships with the behavior of trace, requiring the use of treatments that contributes to the identification of numerical values, and thus establishing relations with the characteristics and visual variables. In this sense, the discrimination of the numerical values are closely related to the scale, since the characteristic and visual variables depend directly of the scale. Therefore is necessary to consider the same graduation in both axes. This is a global condition to identify the characteristics of visual variables. Based upon this premise, the variable and visual characteristics of the polynomial of degree two and three explore the deep-figure, allowing the identification of specific numerical values.

The quantitative treatment developed in the deep-figure generates numerical sequences, which are identified globally. Discrimination is performed through the "covariance". That consists of a displacement of $y_r$ to $y_{m+1}$ in coordinated movements $x_r$ to $x_{m+1}$, while maintaining the same graduation in both axes. This favours a particular behaviour of the numerical values, which are obtained through the movements in the x-axis (Figure 2).

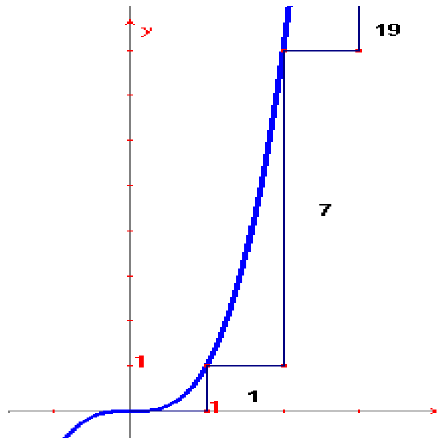


*Figure 2*. Treatment of the quantitative polynomial y = x $^3$.

The displacement takes place on the y-axis, generating the following sequence: 1, 7, 19, . . . , 3n $^2$ +3 n +1 (called a basic sequence). This sequence has an important role in the study of the cubic polynomic cubic, which is identified in global representation. To discriminate the visual value of the cubic term, depends on the number of repetitions that present the numerical sequence, so its algebraic representation is y = 1x $^3$.

To identify the sequence number in the numerical representation (numeric values) explores the coordinated behavior of the "x" values and the "y" values, i.e. for the composite column with "x" values is carried a subtraction operation of consecutive terms. Typically the increase in x is the unit, while the column's "y" values are implemented in terms of the consecutive subtraction of values (finite difference).

The exploration is performed with the information generated by finite difference. Table 1 shows the treatment to identify the quantitative information contained in the polynomial of degree three.

Table 1. *Treatment for the Global Quantitative Numeric Representation*

| x | $y = a x^3 + bx^2 + cx + d$ | $\Delta y$ 1° Diferencia | $\Delta(\Delta y)$ 2° Diferencia | $\Delta(\Delta(\Delta y))$ 3° Diferencia |
|---|---|---|---|---|
| 0 | d | | | |
| | | a + b + c | | |
| 1 | a + b + c + d | | 6a + 2b | |
| | | 7a + 3b + c | | 6a |
| 2 | 8a + 4b + 2c + d | | 12a + 2b | |
| | | 19a + 5b + c | | 6a |
| 3 | 27a + 9b + 3c + d | | 18a + 2b | |
| | | 37a + 7b + c | | |
| 4 | 64a + 16b + 4c + d | | | |

Diferencia Común

In the case of the cubic polynomial ($y = ax^3 + bx^2 + cx + d$), the quantitative analysis of the numerical representation generates three columns corresponding to the first, second and third difference (Table 1), These provide information to discriminate the numerical values of the visual variables, therefore the numerical values of the categorical values for the algebraic expression. For example in the third difference ($\Delta(\Delta(\Delta y))$) (Table 1, column 5), a common variance allows the identification of the numerical value for the coefficient of cubic term (a) using the general rule for the common difference "6a"

## Third Difference

| Constant | Categoric Value |
|---|---|
| 6 | a |

The second difference ($\Delta(\Delta y)$) (Table 1, column 4) allows discrimination between the numerical value for the categorical value of the coefficient of the quadratic term (b). The second difference consists of two digital sequences, the first of which is multiplied by the value of the categorical coefficient cubic term, while the second is multiplied by the value of the categorical coefficient quadratic term.

## Second Difference

| Categoric Value | Numerical Value |
|---|---|
| a | (6,12, 18,..., 6n) |
| b | (2,2,2,..., 2) |

Referring to column 3, which corresponds to the first difference ($\Delta y$) provides information to identify the categorical value of the coefficient of the linear term (c), consisting of three numerical sequences: the first one is multiplied by the value categorical coefficient of cubic term (a), the second sequence is multiplied by the value of the categorical coefficient of quadratic term (b), and finally the last sequence is multiplied by the value of the categorical coefficient of linear term (c), i.e. ;

**First Difference**

| Categoric Value | Numeric Squence |
|---|---|
| a | (1,7,19,37,..., $3n^2$+3n+1) |
| b | (1, 3, 5, 7,...,2n+1) |
| c | (1, 1, 1,..., 1) |

The use of finite differences allows identifying the numerical values of the visual variables and characteristics. These values are also present in the algebraic and graphic representations, which allow connections between the three representations from the interpretation global under the focus numeric.

## Conclusions

The global interpretation is a way to identify relevant information in the graphical representation in the numerical representation, to explore its implications in the algebraic representation, since the modification of a variable visual (graphic) or numeric sequence (numerical representation) involves a change in the categorical variables of the algebraic representation. In this sense, the numerical representation provides relevant information that benefits the identification of the numerical values of the coefficients to the algebraic expression.

The scale is a determining factor for constructing the expression of a curve and the visual identification of variables and the numerical sequence, i.e. if the scale is the same for both axes, visual variables are kept while the numerical sequences are altered, however, if the change of scale is only one axis then both variables are altered visual and numeric sequences. Therefore the study of the background figure-fund is an important aspect to be considered seriously when exploring the graphical representation and numerical representation.

The identification of the visual variables, as mentioned by Duval (1988), is an important activity to explore and interpret the contents of the representation, however, when scanning polynomials of degree greater than two, the task becomes difficult due to the complexity of the curves. Therefore, we discussed the need to explore qualitatively the trace at vicinity to evaluate its content. This vicinity is around the origin where the polynomial $f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$ behaves as a polynomial of the degree n-1 such as $a_1x^{n-1} + ... + a_n$. In other words the polynomial $f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$ tends to lose the term $a_0x^r$.

The analysis of the behavior of some trace of the origin allows the qualitative approach of identifying the "visual characteristics", and interpreting information from a global perspective, which allows connections to the categorical variables in the algebraic representation.

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# DEVELOPING ALGEBRAIC REASONING AMONG PRE-SERVICE TEACHERS

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*A three-week teaching experiment employed qualitative analysis to determine the growth of algebraic reasoning. Learning trajectories were determined across individuals and tasks. Results indicated that subjects increased their abilities to symbolically generalize and justify rules.*

## The Study

The focus of this study is to consider the effects of a three-week teaching experiment designed to introduce pre-service teachers [PSTs] in an elementary mathematics methods course to algebraic reasoning. The question of study relates to 12 PSTs' growth of algebraic reasoning over time, through a study of their learning trajectories when engaged in pattern finding activities. The conceptual framework is derived from the work of Powell, Francisco, and Maher (2003) and a previous analysis of the critical events that occurred during this teaching experiment (Richardson, Berenson, & Staley, 2008). Five critical events were noted beginning with the generalization of a recursive rule (level 1) and concluding with the generalization of an implicit rule with justification of the y-intercept and coefficient (level 5). The 12 Caucasian, female subjects were first semester, senior undergraduates in elementary education at a large state university in the Southeastern US. The related tasks asked PSTs to determine rules and justifications related to the perimeter of n-block trains made from pattern block shapes. Audio data were collected and transcribed for each diad and whole class discussions; artifacts and field notes provided additional data. Cross case analysis was used to analyze the data. The analysis of pre-service teachers' learning trajectories tends to indicate individual growth of algebraic reasoning over time while reminding the researchers that learning is not linear and that time is needed with related tasks for quality learning to occur. Using each week's result as a case informs the researchers of the varied knowledge bases and learning time lines of pre-service teachers including tasks that proved more or less difficult. Analyzing the second case of individual PSTs we capture the extent to which the tasks contributed to growth of reasoning over time. We conclude that it is important to find a series of related tasks that are geometric in nature and engage the PSTs in a community where ideas flow freely in written and verbal form. These are features of the teaching experiment that we attribute to its success in promoting algebraic reasoning among a methods class of elementary pre-service teachers.

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# YOUNG CHILDREN'S INFERENCE AND GENERALIZATION OF FUNCTIONAL RULES USING CUBE TOWERS: INITIAL FINDINGS

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## Introduction

Although there has been increasing emphasis on algebra in early grades in the past several years, we are still only beginning to appreciate young children's abilities to reason algebraically. In this study I investigated young children's understanding of functions. The central task was a variation on activities previously used to assess children's function abilities—numeric function machine game and visual growing patterns (see Carraher & Schliemann (2007) for a review)—in which cube towers serve as inputs and outputs in a function machine game. In this way students should be able to "see" relationships clearly without needing to calculate total amounts and thus be able to explore relatively complex rules.

## Method

Seventy-two second grade children were shown a series of input/output tables in the context of the activity described above. Students were also assessed with more traditional and/or non-visually based function tasks, for example finding the appropriate number of paddles for a given number of canoes, or inferring a $f(x) = 2x+1$ rule from a table of numeric data. Finally, students were administered standardized mathematics and language ability measures.

## Findings

Preliminary analysis indicates that the majority of the children could infer multiplicative (e.g., $f(x) = 3x$) and composite multiplicative and additive or subtractive (e.g., $f(x) =3x + 2$; $f(x) = 2x-1$) rules. Further, many children could also understand more complicated rules such as $f(x) = x(x+1)$ or $f(x) = x(x-1)$. Even children achieving relatively low mathematics ability scores and who performed poorly on other function tasks could identify mathematical functions using the activity. Many children could coherently express generalizations for rules; however this piece was challenging for others, particularly those with lower mathematics and language abilities.

## Conclusion & Implications

Data collected thus far suggest that in the context of this activity, young children are able to infer functional rules before a) they learn about those functions' corresponding operations formally, and b) they can succeed on other types of function tasks. More analysis is needed to understand correlations among performance on this task, performance on other function tasks, and mathematics and language ability scores. The results have implications both for curriculum development for young children and for approaches to algebra instruction in general.

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# EFFECTS OF THE CALCULATOR ON STUDENTS SOLUTIONS TO A PROPORTIONAL REASONING ASSESSMENT: IMPLICATIONS ON DIAGNOSING STUDENTS THINKING

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*This paper describes research conducted on a group of 38 college students entering a College Trigonometry course. A quantitative literacy test aimed at tapping into these students' proportional reasoning skills suggests that when permitted to use the calculator, students tend to more readily use a multiplicative mode of reasoning, whereas additive reasoning is favored when operations are performed by hand. These results have potential applications on making technology-embedded assessment decisions.*

## Introduction – Perspective and Framework

"*Proportional Reasoning* refers to detecting, expressing, analyzing, explaining, and providing evidence in support of assertions about, proportional relationships" (Lamon, 2005, p. 4). Students' thinking when faced with proportionality situations has been extensively researched over the past two decades (Behr, Harel, Post & Lesh, 1992; Clark & Kamii, 1996; Lamon, 1993, 2005; Longest, 2002; Steffe, 1994). Researchers have designed assessment instruments to evaluate students' proportional reasoning skills and raise teachers' awareness of children's thinking about ratios and fractions (Allain, 2000; Misailidou & Williams, 2003). The middle school curriculum relies heavily on proportional reasoning. Students' error patterns, misconceptions and intuitive reasoning about proportionality are particularly interesting for educators in middle grades and beyond (Cramer & Post, 1993; Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998). Indeed, research shows that students' ability to reason proportionally is a strong predictor of how well they thrive in algebra and geometry (Person, Longest, & Berenson, 2003). Of particular interest is the differentiation between two types of mathematical thinking when dealing with numbers: additive, and multiplicative thinking, which translate into absolute versus relative terms when dealing with a comparison setting (Lamon, 2005, p.29). Having to choose between these two reasoning is the ground for many students' mistakes in answering problems that involve ratios and proportions.

With the widespread use of calculators in the mathematics classroom, it is crucial to study the effect of such technology on students' thinking. Research focusing on students' learning mathematics with technology is already extensive (Penglase & Arnold, 1996; Forster, 2006; Heid & Blume, 2007). The 2008 Educational Technology Standards for Teachers clearly address the issue of assessing students' thinking while using technology by encouraging teachers to "provide students with multiple and varied formative and summative assessments aligned with content and technology standards and use resulting data to inform learning and teaching" (ISTE's Educational Technology Standards for Teachers, 2008). To align assessment designs with the technology standards, one needs to gain insight on how using technology affects students' responses on various forms of assessments.

*Research Focus/Questions:* This article contributes to raising awareness on how the calculator might affect students' answers within the scope of proportionality situations. This in turn will hopefully help teachers design thoughtful technology-embedded assessment strategies

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

when trying to diagnose students' reasoning. In particular we consider the following questions: Does using the calculator affect students' choice of reasoning patterns when solving proportionality problems? And if so, what does the calculator allow students to do differently?

## Methodology

The initial setup for the project was to assess a group of college students' quantitative literacy skills upon entering an introductory college math course. "Quantitative literate citizens […] are capable of interpreting and using information presented quantitatively" (Hastings, 2006, p. 56). These issues are currently highly in focus for higher education curriculum changes (Madison & Steen, 2003). A group of 38 college students entering a college trigonometry course were assigned a Proportional Reasoning test on the first day of class. This decision was based on the researcher's belief that acquiring proportional reasoning fluency is a major requirement to becoming quantitatively literate (Lamon, 2005, p.10). Quantitative literacy being especially at stake for students in courses below calculus (Hastings, 2006), it was important for this project to rely on comprehensive evidence of these students' ability to reason proportionally. Even though the test was initially designed to assess rising $8^{th}$ graders' proportional reasoning skills (Allain, 2000), it was selected over other proportional reasoning assessment instruments for its comprehensive approach to proportional reasoning, including comparison problems as well as missing value items. Indeed, reducing proportional reasoning to missing value problems would have provided only limited insight on quantitative literacy (Lamon, 2005).

The majority of college students will not encounter mathematics beyond pre-calculus, and the courses below calculus hold a population of students that calls for increased focus by the Mathematics Education Community due to its variety of backgrounds and ability levels (Hastings, 2006). However, the results described in this article are striking enough that they also might raise questions at the middle school and secondary school level.

The test, developed by Allain (2000), is comprised of ten questions from various sources that assess a variety of skills related to proportional reasoning. Half of the students were allowed to use the graphing calculator when taking the test, and half were not. The test was graded using the following rubric and students' answers were analyzed individually in random order, and then grouped according to the calculator split after individual grades had been assigned. The list below shows the categories of Allain's rubric, with Longest's (2002) additional category of a zero score for no response or a response that shows no mathematical effort, designed in combination with the test:

- 4 - Correct answer with evidence of appropriate strategy;
- 3 - Incorrect answer due to computational error with evidence of appropriate strategy;
- 2 - Correct answer without evidence of strategy or evidence of inappropriate strategy;
- 1 - Incorrect answer with evidence of inappropriate strategy;
- 0 - No mathematical response.

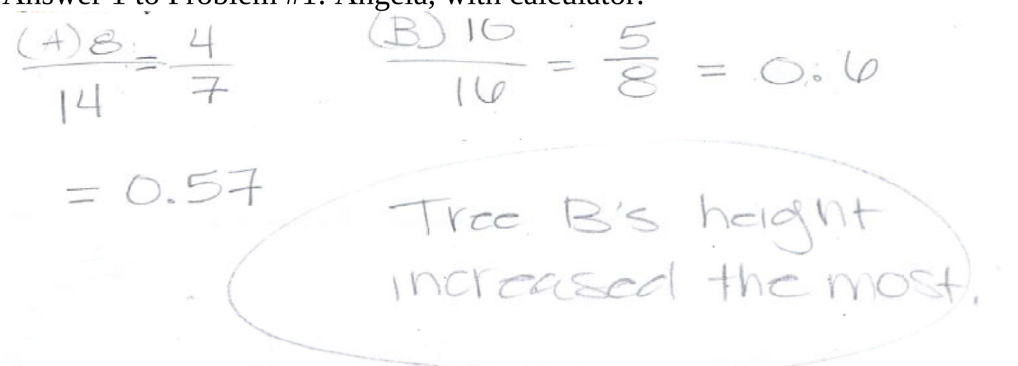
## Analysis of the Results

Examination of students' solutions shows that students tend to translate a problem into percentages or decimals when allowed to use the calculator, therefore adopting a multiplicative behavior, whereas students without the calculator more often resort to additive thinking patterns. For example, students were asked to solve the problem below:

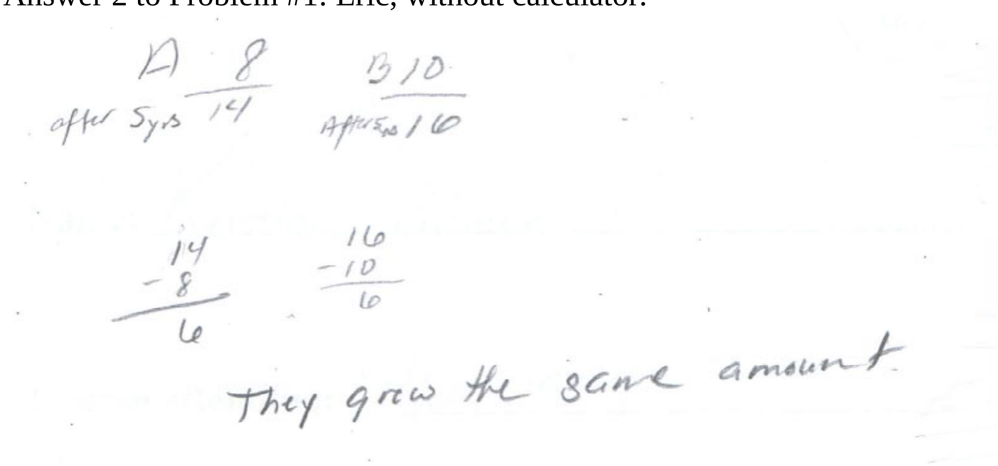
> <u>Problem #1</u>: *Two trees were measured five years ago. Tree A was 8 feet high and tree B was 10 feet high. Today, tree A is 14 feet high and tree B is 16 feet high. Over the last five years, which tree's height increased the most relative to its initial height?*

A typical answer using additive thinking is that both trees increased the same, by 6 feet. When using multiplicative thinking, a student is likely to compare the relative growth ratios of both trees and may conclude that either tree A or tree B grew the most relative to its initial height depending on how the ratios were compared. An example of both reasonings by students is given below:

Answer 1 to Problem #1: Angela, with calculator.




Answer 2 to Problem #1: Eric, without calculator.

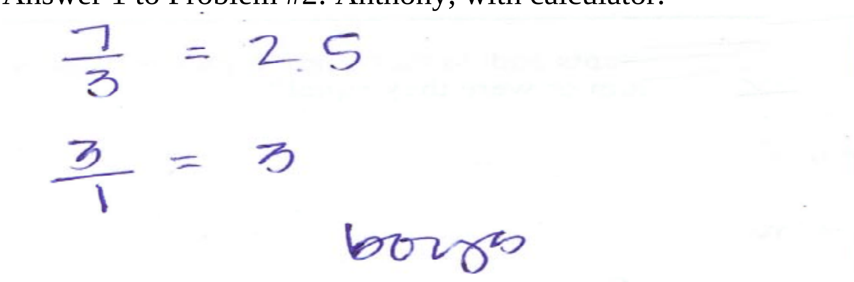



Eric's and Angela's answers were chosen here for their commonality in the group's responses to Problem #1. Among the students who were not allowed to use the calculator, 48% resorted to absolute thinking similar to Eric's in trying to answer the problem, and 48% chose relative reasoning. Within that group, 75% of the students who arrived at the right answer using relative thinking opted for an additive type of reasoning first (both trees increase the same amount of 6ft), and then a relative comparison to the initial heights of the trees (comparing 6:10 versus 6:8). Of the group with the calculator, only 31.5% chose additive thinking while 68.5% opted for multiplicative comparisons, none of which involved an initial additive step; others did not provide appropriate explanations for their answers to decide.

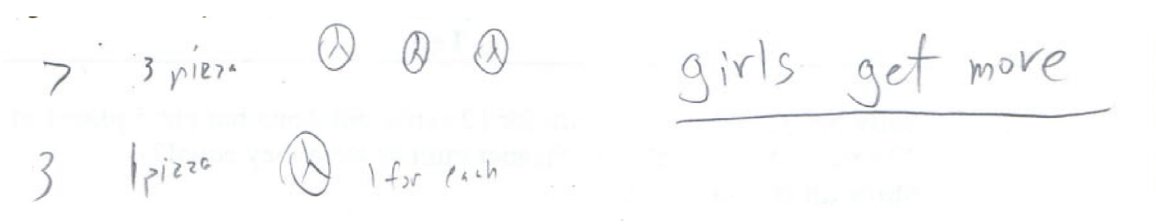
Choosing multiplicative reasoning over additive reasoning did not necessarily lead to the right answer. Indeed, students who used a ratio comparison often misinterpreted the final results when providing an answer, the same way Angela did. A further analysis shows that among the students who opted for multiplicative reasoning, 50% from the calculator group arrived at the correct solution, whereas 60% of those not using the calculator got the correct answer. This type of error due to numerical misinterpretation reflects another clear indicator of deficient quantitative literacy when the connections and transfers between numerical and contextual worlds are overlooked. This inability to interpret a result is better illustrated by students' most common answer to the following problem, probably one of the most difficult in the set assigned:

> *Problem #2: There are 7 girls with 3 pizzas and 3 boys with 1 pizza. Who gets more pizza, the girls or the boys?*

Answer 1 to Problem #2: Anthony, with calculator.




Answer 2 to Problem #2: Erika, without calculator.




For problem #2, 39% of the group with the calculator followed Anthony's path and answered that the boys were getting more pizza, whereas only 28% of the group without the calculator made that mistake. In this case, not having access to a calculating device forced the students to carefully think about the relationships among quantities, often in terms of sharing slices as Erika did. On the other hand, students with the calculator often simply plugged in the numbers in order of appearance in the text and compared decimals without thinking further about their contextual meaning. This type of error has been previously documented (Clark, Berenson, & Cavey, 2003; Person, 2004) and shows a disconnection between the contextual world of ratio and the numerical world of fractions. An important step to adequate proportional reasoning, thus improved quantitative literacy, involves the ability to go back and forth between these two worlds. This seemed more instinctive for the students when the calculator was not available. Like Erika, students without the calculator had a tendency to use drawings in their solutions, whereas no student using the calculator felt compelled to do so throughout the whole test.

## Discussion

The results above lead us to think that allowing the calculator on the proportional reasoning test had an effect on students' reasoning patterns. In particular the multiplicative reasoning patterns necessary to solve most proportional reasoning problems were encouraged by the use of the calculator. One should not conclude however that allowing the use of technology on such assessment will necessarily improve students' results. On average, the students from the group with calculators did not perform better than the group without, but the types of inappropriate strategies used were different. The absence of a calculator forced the students to think more carefully about the relationships between quantities. They often designed strategies that showed evidence of an elaborate thinking process and involved heuristics other than numerical manipulations. On the other hand, the students with the calculator more often misinterpreted the results they were finding even though their initial computations were correct. Students who did not use the calculator also provided more written explanations which we could use as a more transparent window into their thinking processes.

The tendency to choose multiplicative reasoning when using a calculating device has major consequences on designing assessments where the calculator will be available. Indeed, if the purpose of the assessment is to diagnose students’ ability to reason through a given problem, and choose an appropriate sequence of operations that will lead to a correct answer, one must be aware that the calculator is likely to condition students into adopting a multiplicative behavior, which may flaw the interpretation of the results in terms of students cognitive abilities: in such cases, opting for a multiplicative type of thinking would not necessarily show evidence of understanding why this particular thinking is appropriate in the given situation. Nevertheless, knowing the advantages that technology brings to the classroom, especially with the greater availability of visual representations and greater access to self-monitoring, one should not suppress it from the assessment process the way it has often been advocated in college courses. Rather, these results call for the development of technology embedded instruction where assessments are carefully designed to reflect its daily use in the classroom culture. It is important for teachers to learn how to use technology efficiently and know when to introduce it in the classroom for higher conceptual understanding: more studies like this one will help accomplish that goal so that technology may also be selectively part of assessing students’ skills, rather than banning it altogether from college courses for what it might “help” students accomplish. A diversity of modes of assessment is highly necessary to get a glimpse of students’ reasoning complexity. Due to the pervasive nature of technology in our society, quantitative literacy assessments should be highly informed by studies on cognitive technology.

## Conclusion

It is important to call for caution when trying to generalize the results described here. The pool of participants being small, the author wishes to encourage further research designs that may confirm or refute the validity of such findings. Of particular concern would be the tendency to generalize them to lower grades at which students’ developmental stages might affect their attitude towards the calculator, as well as their quantitative literacy skills and choices of reasoning. In other words, this paper is intended as a call for a larger scale study, be it to inform the higher education community on effects the calculator might have on students’ answers to assessments, or to raise questions on how one might be able to incorporate technology in assessment designs while still tapping into students’ cognitive skills.

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# TEACHERS' ESTIMATION OF ITEM DIFFICULTY: WHAT CONTRIBUTES TO THEIR ACCURACY?

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*This study used assessment items as a tool to assess teachers' understanding of student learning progression. We asked ten teachers to rank the difficulties of eleven items about the concept of linear measurement for elementary and middle school students. We compared teacher's rankings to the empirical order found in a field administration of the eleven items. Results indicated that only three teachers' ranking reflected the empirical order. Neither the amount of training teachers had on mathematics nor did the mere length of their teaching experience predict their accuracy. We discussed the relation among pedagogical knowledge, content knowledge, and teaching experience.*

## Background

This study is situated in a large, multi-year, multi-site project that focuses on teaching and assessing student knowledge of data modeling and statistical reasoning at elementary and middle school levels (e.g., Lehrer, 2007, Lehrer, Kim, & Schauble, 2007, Seeratan, et al, 2008). We conceptualize data modeling and statistics as encompassing knowledge of seven dimensions, and have developed construct maps (Wilson, 2005) to describe the hypothetical trajectory of learning progression along each of those dimensions. We used those construct maps not only as models for developing assessment items, but also as professional development tools to stimulate discussions about student performance. This paper focuses on one of the seven construct maps, namely, the theory of measurement construct that focuses on children's understanding of principles underlying (linear) measurement. Specifically, we suggest that evaluating item difficulty is a useful activity to learn about teachers' understanding of student learning progression, in this case, progression with regard to the concept of linear measurement.

## Perspective

The importance of understanding children's learning progressions is well-established and is clearly articulated, for example, by Davydov (2008) in his theory of developmental instruction. We consider this understanding a part of teachers' pedagogical content knowledge (Shulman, 1986). Understanding the learning progressions is a complex task. Teacher knowledge of the structure of a domain and of child development has been postulated as contributors to a sophisticated understanding of the learning progressions. Before we test the relation between teacher understanding of a learning progression and the other knowledge preparations of teachers, however, we need to have tools to assess their understanding of a learning progression. This study uses an assessment-based tool to probe into teachers' understanding of learning progression on the topic of linear measurement.

## Method

### *Participants*

Ten teachers were recruited from participants of our monthly professional development workshops during the year 2007-2008, held in a major city in the southwestern US. We sent out requests to the entire group of 26 teachers, and ten of them responded. Among these ten teachers, one was male. This male teachers and one of the female teachers were the only teachers having a degree in mathematics. One female teacher had prior training and experience in accounting. These three teachers were the only teachers among the ten who had teaching experience in mathematics only ($7^{th}$-$9^{th}$ grade mathematics). The remaining seven teachers all had training in educations and had taught all subjects. At the time of interviews, one of the seven teachers was a $7^{th}$ grade mathematics teacher; one was a coach for mathematics and science in an elementary school. All teachers had teaching experience of at least four years.

*Instrument*

A series of items assessing student understanding of linear measurement had been developed in the large project about assessing data modeling and statistical reasoning. We used eleven items in the research described here. The measurement topics covered by the items include the following: direct reading of a ruler, reading a ruler from starting positions other than zero, unit iteration, constant units, unit conversions, and unit partition in different fractions.

In addition to the eleven items, we prepared student responses for two of the items, one item on constant units and another on reading a ruler from starting position of one. We prepared four types of student responses for each of the items.

The theory of measurement construct map was also used during the interview.

*Procedure*

In December 2007 and February 2008, we interviewed the teachers individually on days not having the workshops. Prior to the interview, teachers filled out a questionnaire asking for information about their academic training and teaching experience. In the beginning of the interview, we asked teachers to reflect on their theories of how children developed concepts of linear measurement. We then presented them the eleven items, in random order. We asked teachers what understanding each of those items assessed. Following this question, we asked teachers to order the items according to their relative difficulty. We used prompts such as "Which item do you think students are able to complete successfully earlier than the others? Which item do you think would be the hardest for the students?" We asked teachers to try to put the items to distinct levels, but ties were also permitted. Teachers were also asked to justify their ordering.

After teachers order the items, we asked them to locate the items on the theory of measurement construct map. We asked that, if a student were completely correct on an item, what level on the construct map his/her performance would correspond to. Following this activity, we asked teachers to examine the selected student responses for two of the items, one at a time. Teachers were asked to explain student strategy indicated in each response, to locate the responses on the construct map, and to order them according to their sophistication levels.

*Administration of the Items*

We asked teachers of our existing professional development partnership in another site to help collect responses from their students to the same eleven items. Those teachers located in school districts in south central US. They administered the items in their classrooms during December 2007 to January 2008. Individual teachers determined exam time and duration for their classes. A field coordinator collected completed tests from participating teachers and mailed them to us. A total of 360 students responded to our tests. Student responses were coded and analyzed with a partial credit model to determine the relative difficulties of the items. The analysis was run using the ConQuest software (Wu, Adams, Wilson, & Haldane, 2007). The resulting order was the empirical order of item difficulty.

## Preliminary Results

We calculated Spearman rank-correlations between each teacher's ordering of the items and the items' empirical order. Three teachers' orderings were significantly correlated with the empirical order at the level of .05 (Spearman's rho > 0.648, n=11), indicating high accuracy. Among these three teachers, two taught all subjects and one taught middle school mathematics. The one who taught middle school mathematics was also the teacher with the least teaching experience (four years teaching 7th and 8th grade mathematics). Another one of these three teachers initially erred on a problem involving improper fraction, indicating that mathematics competence might not be directly related to knowledge of learning progression.

Moreover, mathematics training did not seem to be a significant contributor to teachers' accuracy. Between the two teachers with degree in mathematics, one tended to underestimate the more difficult items, but was relatively more accurate with the easier ones; the other tended to overestimate the easier items, but was more accurate with the more difficult ones. The mathematics coach had the lowest accuracy among the ten teachers, followed by another teacher who had teaching experience only at grade five and above.

Furthermore, teachers who were more accurate (i.e., having a larger Spearman's rho) were often more accurate with the easier items (e.g., between the two teachers with mathematics degree, the one who was more accurate with the easier items was also more accurate in general). Therefore, it appeared that teachers' difficulty with these set of items focused on ordering the easier items. Teachers appeared to have better sense of the difficulty for items that were more difficulty. The mathematics coach, however, was the only anomaly among the ten, because she overestimated the easier items and underestimated the more difficult ones, resulting in a negative rank correlation between her ordering and the empirical order.

Examining teachers' justification of their ordering indicated that many related their answers to their personal experience, i.e., what they personally found more challenging. This might explain why the teacher who initially erred on the problem involving a simple improper fraction could be accurate in her estimations – although she might not be good at mathematics, she might be a reflective learner/practitioner.

## Discussion

This study shows how assessment items of student knowledge can also be used to assess teachers' understanding of learning progression, which we consider to be a component of teachers' pedagogical content knowledge. We note that the pattern emerged out of our finding is consistent with the notion that pedagogical content knowledge and domain content knowledge are different. Our study also indicates teaching experience may not be directly related to teachers' pedagogical content knowledge, which in this case is in the form of understanding student learning progression.

As we are completing the analyses of the remaining interview data, which include a detail look at teachers' justifications of their orderings and their interpretations of the construct map, we hypothesize that teachers who are more accurate in their ordering would also tend to have better understanding of our construct map, which describe the learning progression of students about the concept of linear measurement.

We realize that, however, our study is exploratory, and future research will need to establish the validity of our tasks as an assessment tool for teachers' understanding of learning progression.

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# ASSESSMENT AS AN OPPORTUNITY TO LEARN: USING COLLABORATIVE TESTS TO IMPROVE STUDENTS' MATHEMATICAL UNDERSTANDING

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*Using the framework that assessment can provide an opportunity to learn, this study investigated the impact of collaborative testing on college students' understanding of statistics. In addition, because collaborative tests have the potential to mitigate test anxiety, students' attitude toward the treatment was studied. Results of the tests demonstrated that those students who tested collaboratively scored higher than students who tested individually. Moreover, students who tested collaboratively and were rated as low ability (based on a standardized entrance exam) outscored high ability students who tested individually. Additionally, results of the attitude survey showed an overwhelmingly positive view of using collaborative tests.*

## Theoretical Framework

Assessment, referred to broadly by Shepard (2001) as an emergent paradigm, can also be referred to as a framework to study students' understanding of mathematics. Assessment as an emergent paradigm refers to a conception of assessment that is more aligned with a constructivist perspective on teaching and learning. A focus on reform teaching in the 1980s and 1990s led to a gap between contemporary mathematics teaching and traditional assessment practices. Whereas the teaching and learning of mathematics had entered a reform paradigm that was determined by constructivist ideals, the assessment of mathematics remained in a traditional paradigm more closely aligned with behaviorist ideals (Greeno, Collins, & Resnick, 1996; Shepard, 2001).

One way to shift assessment into the constructivist paradigm is to provide students the opportunity learn by communicating with each other while completing an assessment task. Communication is a critical component of learning mathematics; in particular, communication is an essential component for learning mathematics with understanding (Fennema, Sowder, & Carpenter, 1999). In fact, NCTM highlighted communication as a necessary component when defining communication as one of the five Process Standards (2000). According to the communication standard, students must be able to—

- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
- analyze and evaluate the mathematical thinking and strategies of others;
- use the language of mathematics to express mathematical ideas precisely (p. 60).

Discussing others' ideas while defending one's own, can expose misconceptions, misinterpretations, and holes in one's thinking, leading to expanded understanding by all. "Through communication, ideas become objects of reflection, refinement, discussion, and amendment" (NCTM, p. 60). Communication in learning mathematics and the importance of assessment in determining the amount of learning that occurred are central features in learning with understanding. As such, they provide a solid foundation by which to frame a study of students' understanding of mathematics through the use of collaborative tests.

## Literature Review

Although assessment has many purposes, its use as a summative tool to evaluate, rank, and quantify student understanding is its most common purpose. Currently, researchers in K–12 (de Lange, 1999; Romberg, Zarinnia, & Collis, 1990) and higher education alike (Gold, Keith, & Marion, 1999; Shepard, 2001) agree that how students are assessed needs improvement. According to Shepard, "the purpose of assessment in classrooms must…be changed fundamentally so it is used to help students learn and to improve instruction, not just to rank students or to certify the end products of learning" (p. 1080).

A focus on assessment is grounded in the prominent role assessment takes in the learning process—prominent because one can only determine the degree to which learning has occurred if one assesses that learning. This role is also appropriate due to the fundamental nature of assessment on student performance not only in K–12 education (Chappuis & Stiggins, 2002; Shepard, 2000; Stiggins, 2002), but in higher education as well (Adams & Hsu, 1998; Ritter, 2000). Especially when assessment is used to *support* learning in addition to being used as a measure *of* learning, it is fundamental to the whole teaching/learning process. When assessment is used to support learning, teachers adapt the process and flow of information about student achievement in order to advance, not merely check on, student learning (Stiggins, 2002). This act of assessing can provide students with an opportunity to learn, and specifically, learn important mathematics (Steen, 1999; van den Heuvel-Panhuizen & Fosnot, 2001). When envisioned as such, assessment is an integral part of instruction (Black & Wiliam, 1998; Schoenfeld, 1997; Shepard, 2001; D. C. Webb, 2001).

While the main goal for using collaborative tests in this study was to provide students with an opportunity to learn, research has confirmed many other reasons for implementing collaborative testing. For example, taking tests in groups can reduce anxiety in testing situations. It is possible that anxiety surrounding test taking can serve to motivate students to rise to the challenge. However, it is much more likely that this kind of anxiety will interfere with thinking and eventually compel students to give up. According to Helmericks (1993), "[C]ollaborative testing operates to alleviate, or at least mitigate, examination anxiety, a major source of the debilitating math anxiety that …students endure" (p. 287). Thus the use of collaborative tests can allow students who normally might "freeze" to not only complete the test, but actually learn in the process.

Research has also shown that collaborative test taking promotes continued learning in that once students have the opportunity to learn, they can continue learning (Ittigson, 2002; Lehman, 1995; N. M. Webb, 1995). According to Webb, "[e]ven a small amount of collaboration may influence a student's understanding and performance" (p. 247). In addition, developing new understanding by building on other students' ideas is a form of learning as is giving explanations which encourage the explainer to justify, reorganize, and clarify his/her thoughts (Webb, 1995). Learning is not accomplished when students have given up. In a group testing situation, it is not acceptable to give up; "students put a lot of pressure on their group members and get very irritated if another member is not holding up his/her end" (Lehman, p. 4). So while peers do not allow each other to slack, it is because they hold their group members responsible for ensuring all group members understand the material.

In addition, collaborative tests require communication, which many agree is of the utmost importance not only in teaching and learning, but also in assessment (McConnell, 2002; NCTM, 2000). In fact, in NCTM's Assessment Standards for School Mathematics (1995), the authors state that assessment should enhance learning, and learning involves being able to reason and communicate mathematically. Communication such as this encourages support for each other's learning as opposed to competition. Traditional testing methods involve students working alone and are characterized by individual competition, primarily for grades. In contrast, collaborative testing engenders an “all for one, one for all” atmosphere conducive to learning, especially with the peer pressure for everyone to hold their own.

Based on the framework presented above, the author chose to investigate the effectiveness of using collaborative tests to improve students’ understanding of statistics and their perceptions of the treatment in general. In particular, the present study is guided by two overarching research questions: 1) In what ways does taking tests collaboratively increase students’ understanding of the material, and 2) What aspects of the treatment did students find especially helpful in terms of helping them understand?

## Methods

This study was designed to provide data sources that would document the influence of using collaborative tests as compared to individual tests. The primary source of data that was used was a series of three initial tests and three similar, but different retake tests measuring students’ understanding of basic statistics concepts. A secondary data source, an attitude survey, required students to rate ten statements on a five-point Likert scale and answer five open-ended questions regarding their perceptions of the treatment. Using a very simple two-by-two experimental design, this study compared initial test scores (taken either collaborative or individual) to retake scores (all taken individually) in both groups (reading horizontally) and compared scores on both tests between groups (reading vertically) (see Figure 1).

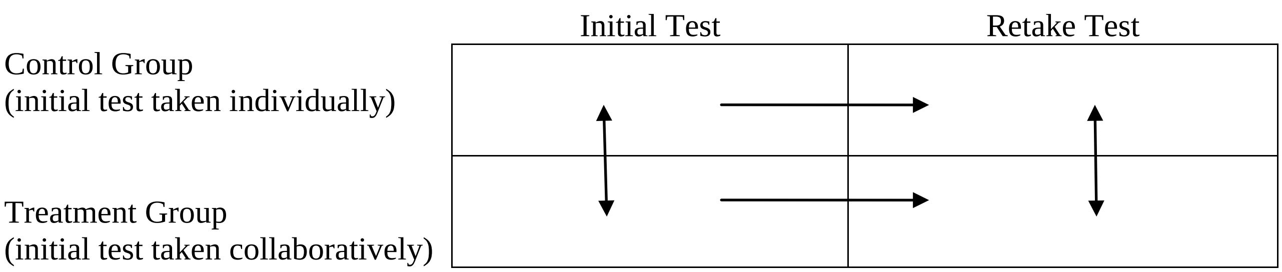


*Figure 1.* Two-by-two design to compare test scores between groups and between tests.

Forty-eight students enrolled in two sections of Quantitative Literacy at a two-year Arts College in the Midwest participated in the study. The median age of the students was 19, almost 70% were male, and the majority of the students were Caucasian. Both classes met twice a week for one hour and forty minutes, one class in the morning (the control group) and the other in the afternoon (the treatment group). The classes covered the same material each day and were taught by the same instructor.

The groups were deliberately assigned so that each group had at least one high ability student and to the extent possible, similar numbers of high and low ability students as measured by a median split of the scores received on an Asset test (ACT Incorporated, 2003) taken upon admission to the college. The Asset test is a national standardized test designed to “[obtain] academic and background data about advisees,…[place]-students in courses that match their interests and abilities,…[and help] students explore their educational and career options” (ACT Incorporated, 2009). The students remained in the same groups throughout the study except for occasions when more than one student was absent from a group.

Through the discussion of a reading based on Johnson, Johnson, and Holubec (1993) which highlighted the purposes, goals and methods of cooperative learning and collaborative test taking, the treatment group learned about the key features of cooperative learning and collaborative testing. In addition, the treatment group practiced, in their prescribed groups, the cooperative techniques by solving an in-class problem which involved multiple solution strategies and multiple answers. The students in the control section had no cooperative training and completed the same in-class problem individually. The students in the control section had no cooperative training and completed the same in-class problem individually.

All students took an initial test measuring their understanding of basic statistics concepts and then two days later, at the start of the next class period, they took a similar, but different, retake test on the same material. Each test contained two or three open-ended problem-solving questions that required students to explain, describe, or demonstrate how the problems were solved, and one or two procedural questions. The tests were all graded by giving partial credit if either (a) the answer was wrong but the process was correct or (b) the answer was wrong but the student was on the right track. Tests from both sections were graded together by the instructor with care being taken neither to look at the names nor to note in which section the student was enrolled. The students in the treatment group took all of the initial tests in their prescribed groups of three, while the students in the control group took the initial tests individually. Both groups took the retake test on the same material individually.

## Results

Table 1 lists the relative scores all for students on both the initial and retake tests. Reading vertically, the table shows the difference in students' understanding of basic statistics concepts between original scores and retake scores. Surprisingly, in only one case (control group, test 2) did students score better on the retake than on the original. In fact, students who tested collaboratively scored as much as almost 37% lower on all retake tests, whereas students who tested individually scored higher on two out of three retake tests, although at most only 5% higher.

Reading horizontally, in all but one comparison between the control group and treatment group, the treatment group scored higher, not only on all the initial tests, but also on two out of three retake tests. In particular, students testing collaboratively scored as much as 31% higher on an initial test and 12.5% higher on a retake test.

Table 1

*Mean Percents of Tests and Percent Differences between Control and Treatment Scores*

| Tests | Control group (Individual initial, individual retake) Mean | Treatment group (Collaborative initial, individual retake Mean | Percent of increase ↑ or (decrease ↓) between control and treatment scores |
|---|---|---|---|
| Test 1 | 78.0% | 86.8% | 8.8% ↑ |
| Test 1 Retake | 71.5 | 77.3 | 5.8↑ |
| Percent of ↑ or (↓) initial to retake | (6.5) ↓ | (9.5) ↓ | |
| | | | |
| Test 2 | 58.4 | 89.8 | 31.4↑ |
| Test 2 Retake | 62.8 | 75.3 | 12.5↑ |

| | | | |
|---|---|---|---|
| Percent of ↑ or (↓) initial to retake | 4.4 ↑ | (14.5) ↓ | |
| | | | |
| Test 3 | 56.0 | 84.9 | 28.9↑ |
| Test 3 Retake | 58.3 | 54.2 | (4.1) ↓ |
| Percent of ↑ or (↓) initial to retake | 2.3↑ | (30.7) ↓ | |

The results of the breakdown between high and low ability groups are shown in Table 2. The mean scores of and the mean difference (high minus low) between the high and low ability groups are represented. As might be expected, the high ability students scored higher on all tests taken individually. However, the low ability students who tested collaboratively actually outperformed the high-ability students in two out of three initial collaborative tests. And although the low-ability students scored lower than the high-ability students on the individual retake tests, they only scored on average, 3.6% lower. This is contrasted with low-ability students who tested individually and scored an average of 17.1% lower than the high-ability students.

Table 2
*Mean Test Percentages for High and Low Ability Students and Mean Differences between Abilities*

| | Individual | | | Collaborative | | |
|---|---|---|---|---|---|---|
| | Test 1 | Test 1 Retake | T1R-T1 | Test 1 | Test 1 Retake | T1R-T1 |
| High ability | 81.0% | 82.7% | 1.7% ↑ | 82.9% | 79.3% | (3.6↓) |
| Low ability | 75.0 | 62.1 | 12.9 ↑ | 90.7 | 75.6 | (15.1↓) |
| Mean difference | 6.0 | 20.6 | | (7.8) | 3.7 | |
| | | | | | | |
| | Test 2 | Test 2 Retake | T2R-T2 | Test2 | Test 2 Retake | T2R-T2 |
| High ability | 67.0% | 70.0% | 3.0 ↑ | 75.5% | 76.1% | 0.6% ↑ |
| Low ability | 51.0 | 50.0 | (1.0) ↓ | 88.8 | 74.3 | (14.5) ↓ |
| Mean difference | 16.0 | 20.0 | | (13.3) ↓ | 1.8 | |
| | | | | | | |
| | Test 3 | Test 3 Retake | T3R-T3 | Test 3 | Test 3 Retake | T3R-T3 |
| High ability | 68.2% | 63.7% | (4.5) ↓ | 93.2% | 56.4% | (36.8) ↓ |
| Low ability | 43.8 | 53.0 | 9.2↑ | 75.9 | 51.0 | (24.9) ↓ |
| Mean difference | 24.4 | 10.7 | | 17.3 | 5.4 | |

The secondary data source, the results of the attitude survey, showed that with only a few exceptions, students expressed very positive feelings about the treatment. The answers to both the Likert scale section and the open-ended question section strongly supported testing in collaborative groups. In particular, 93% of the students agreed or strongly agreed that they felt much less stressed when taking the tests in groups; 92 % of the students stated that working in groups made it easier to learn; and 70% agreed or strongly agreed that they have a better understanding of the material because of working in groups. The only suggestion students had to improve the collaborative testing process that was mentioned by numerous students referred to group composition. Forty-one percent of the students commented on the size, make-up, or consistency of the groups, but with no *one* suggestion standing out as agreed-upon by a majority of students.

## Discussion

In five out of six tests, students who had the chance to test collaboratively performed better than those students taking the tests individually. It is no surprise that students testing in collaborative groups during the initial test would score higher; the fact that they scored higher on two of the three individual retake tests strongly suggests that collaborative testing has merit. However, if collaborative testing really improves understanding it would be expected that the scores were higher on all three individual retake tests. Since all retake tests were taken individually, one could speculate that (a) students were not using the collaborative testing situation to help them learn and instead simply solved the problems by relying on the expertise of the high ability student in the group, or (b) the lower scores were due to the retake tests themselves, a possible limitation of the study.

However, the results of the high and low ability analysis were consistent with other research (N. Webb, Nemer, Chizhik, & Sugrue, 1998) in that the low ability students who tested collaboratively benefited the most from the testing treatment. They scored, on average, 11.9% higher on the individual retake tests than the low ability students who tested individually. An interesting result is that the high ability students who tested collaboratively scored an average of 14.3% lower on their individual retest. One could speculate that this was due to their perception that working in a group was not going to be beneficial to them; often high ability students prefer to work alone or only with other high ability students, and feel burdened when working with lower ability students. This is in contrast to low ability students who might perceive immediate value to testing collaboratively; due to their lower ability they often are much less confident about their answers and benefit greatly by the feedback they receive while testing collaboratively.

## Limitations

Some limitations exist within this study that could minimize the amount of strong evidence of the effectiveness of collaborative testing. One such limitation is the small sample size. Classes consist generally of at least thirty students, but student withdrawals resulted in class sizes of approximately 23 students. Another limitation is the limited time invested on training students how to learn cooperatively and test collaboratively. Training for these skills occurred at the same time as the study itself, thus the effectiveness of students testing collaboratively was compromised by the fact they were at the same time learning *how* to work together.

However, as can be seen by the scores, the students did improve over time; the scores of the second and third collaborative tests were approximately 30% higher than the scores on the individual tests, and 6% higher on the initial test taken collaboratively. This is consistent with the research that states that students need specific training over time in the skills required to work effectively in groups (Johnson et al., 1993).

The tests themselves also may have decreased the effectiveness of the study. For a test to accurately measure a group's understanding, it needs to include open-ended questions that require all group members to participate and contribute to the solutions. Otherwise, if there is only one right answer, the brightest student will often simply take the test and allow the others to copy. These types of tests, although effective when testing collaboratively, are extremely difficult to take individually. This would explain the lower scores on the retakes (which were taken individually), as all tests were written with collaborative testing in mind to maintain consistency.

However, two other limitations—if they could be accounted for in future studies —might actually strengthen the case for collaborative tests. A major limitation in this study was the transient student population. The multiple absences of numerous students made the consistency of groups nearly impossible in some cases. According to the research, for cooperative learning (or collaborative testing, in this situation) to work, students must be invested in the group; the group sinks or swims together (Johnson and Johnson, 1993). With so many students repeatedly absent, building this type of commitment was nearly impossible, as groups were constantly changing. Also, the absences meant that those students were behind when they did show up since almost no students returned to class with the current homework done. Thus, simply by improved student attendance, both group commitment could be developed and students could remain caught up.

## Conclusion

The results of this study has shown that students—especially low ability students—do increase their understanding of basic statistics concepts when testing collaboratively. Although the individual retake scores of students in the collaborative testing group increased only two-thirds of the time, the students' positive attitudes towards the treatment suggest that using collaborative tests is a method worthy of more study. Also, the large increase in scores of low ability students makes collaborative testing especially useful in teaching students whose strengths may not include math. Further research could be done with larger sample sizes, more collaborative training, tests specifically geared toward individual testing as well as collaborative testing, and a more consistent participant groups.

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# ASSESSING STUDENT UNDERSTANDING OF AREA MEASUREMENT


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Practitioners often notice that students have difficulty with area measurement. For instance, students often confuse the area of a rectangle with its perimeter. A review of the literature (e.g., Battista, Clements, Arnoff, Battista, & Van Auken Borrow, 1998; Doig, Cheeseman, & Lindsey,1995; Kamii & Kysh, 2006; Lehrer & Chazan 1998; Outhred & Mitchelmore, 2000; Reynolds & Wheatley, 1996) has indicated that student difficulty with the area tasks is often due to their lack of conceptual understanding of area as a quantitative attribute.

In order to assist teachers to better diagnose their students' conceptual understanding of area measurement, this presentation reports our effort in developing an assessment for student understanding of area measurement. This poster consists of three major parts:

1. *Description of the instrument development process*. Our process of developing the assessment follows a construct mapping approach (Wilson, 2005), which requires us to specify a hypothetical learning progression at the outset of the assessment development. This learning progression is used as a model to guide item design.
2. *Description of a calibration study of the assessment items*: We use the learning progression specified in the beginning to guide interpretation of student responses. Student responses are coded and analyzed using a partial-credit Rasch model. The quantitative results are supplemented with qualitative analysis of selected examples.
3. *Description of a teaching study of area measurement with embedded formative assessment*: We describe a teaching study that embeds the formative assessment items to support the teaching and learning of the concept of area measurement.

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# FIRST-YEAR SECONDARY MATHEMATICS TEACHERS' ASSESSMENT CONCEPTIONS AND PRACTICES

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This study investigated the factors that contribute to first-year secondary mathematics teachers' assessment practices as well as their conceptions about assessment. Given the fact that teachers spend about 40% of their time in assessment-related activities (Stiggins, 1988) coupled with the current educational reform calls for a much broader role of assessment in mathematics and the continued dominance of paper-and-pencil assessments (Shulman, 1986; National Council of Teachers of Mathematics, 2000), it is important to investigate which influences are important to first-year teachers' assessment practices and conceptions. Three case studies were conducted with first-year secondary mathematics teachers. Literature from the areas of teacher beliefs, beginning teachers, and mathematics assessment provided a lens to analyze the observations, interviews and classroom artifacts.

Results suggest that several factors influenced first-year teachers' assessment practices. Teachers' beliefs about mathematics and its teaching and learning, students and assessment were all intertwined and together impacted how they assessed student learning. The locale of their source of authority (external or internal) along with their perception of external constraints influenced their assessment practices. The teachers were also influenced by the practices of their colleagues and the school culture. Lastly, the presence of the researcher in the classroom played a small role in the assessment decisions and practices of the first-year teachers.

Despite their knowledge of reform efforts in assessment, their assessments were very traditional but looked different in each of the first-year teachers' classrooms. Jack heavily used traditional formal assessments from the school-created teacher test bank. Karen created her own assessments and experimented with nontraditional assessments like a project and a pair test. Angel modified her colleagues' traditional assessments and frequently used informal assessments like peer teaching and student questioning. Teachers thought assessment should provide all students a fair opportunity to show what they know and can do, align with instruction and include multiple sources. Furthermore, informal assessments like facial expressions and students' verbal explanations provided important information about student mathematical knowledge but did not contribute towards the student's final grade.

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# A FRAMEWORK FOR ANALYZING THE COLLABORATIVE CONSTRUCTION OF ARGUMENTS AND ITS INTERPLAY WITH AGENCY

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*In this report, we offer a framework for analyzing the ways in which collaboration influences learners' building of mathematical arguments and thus promotes mathematical understanding. Building on a previous model used to analyze discursive practices of students engaged in mathematical problem solving, we introduce three types of collaboration and discuss their influence on the building of mathematical arguments and student agency. The framework is exemplified using data from a study of the development of mathematical reasoning in an urban sixth-grade informal after school program.*

## Introduction

In recent years, policy makers and researchers have focused on the role of discourse in the mathematics classroom. The National Council of Teachers of Mathematics (NCTM, 2000) emphasizes the importance of communication in students' developing mathematical understanding and suggests that students be afforded the opportunity to share their ideas in a mathematical community and analyze and evaluate the ideas of their peers. Participating in mathematical discussions and reasoning about mathematics requires that students have opportunities to share and to discuss their ideas with others (Lampert & Cobb, 2003).

Collaboration is often viewed as learners supporting each other by offering missing pieces of information needed to solve the problem. Alternatively, a powerful type of collaborative work involves group members relying on each other to generate, challenge, refine, and pursue new ideas (Francisco & Maher, 2005). With this type of collaboration, rather than piecing together their individual knowledge, the students build new ideas and ways of thinking as a group. Martin, Towers, and Pirie (2006) refer to collective mathematical understanding as the kind of learning and understanding that transpires when a group of students work together on a mathematical task. They identify co-acting as a process through which an individual's mathematical ideas and actions are adopted, built upon, and internalized by others, thus becoming shared understandings rather than being limited to the individual. In co-acting, ideas are initially put forth by an individual student and are then picked up by others and built upon, thus they consequently become shared by the group members (Martin, Towers, and Pirie, 2006). In this report, we offer a framework for analyzing the ways in which collaboration influences learners building of mathematical arguments and thus promotes mathematical understanding.

## Theoretical Framework

Davis (1996, 1997) introduced a framework for analyzing the ways that teachers listen to their students. His framework involved three modes of listening: evaluative, interpretive, and hermeneutic. Powell and Maher (2002) extended Davis' categories to analyze "the discursive practices of learners in conversational exchanges" (p. 319) and referred to this phenomenon as interlocution. They identified the four properties of interlocution as evaluative (judging without participating), informative (seeking or providing information without judging), interpretive (teasing out the intention or meaning behind a partner's statement), and hermeneutic (engaging

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

and negotiating the partner in the interaction, participating in a shared project). Powell (2006) modified this framework for the purpose of analyzing students' mathematical discourse. Powell noted that negotiary discourse (hermeneutic) is particularly important in the development of *socially emergent cognition*. This phenomenon, as defined by Powell, is the "process through which ideas and ways of reasoning materialize from the discursive interactions of interlocutors that go beyond those already internalized by any individual interlocutor" (p. 33). Powell found that students' discursive interactions influence their mathematical ideas and reasoning.

When considering aspects of student collaboration and the reasoning that results from this collaboration, we must also consider the issue of agency and the nature of the interplay of mathematical ideas in the mathematics learning environment. Agency involves taking the initiative and making things happen in the classroom (Wagner, 2004). Human agency is exhibited when one creates one's own mathematical idea or extends an established idea (Pickering, 1995). Agency, then, is important in tracing the origin of mathematical ideas as well as the way that discourse, and the participants in the discourse, influence the ultimate mathematics that is constructed.

In this paper, we draw on these ideas to inform our conceptualization and analysis of student collaboration and the mathematics that results from various forms of student collaboration. Our framework highlights three modes of student collaboration, the discursive nature of each, and the significance and interplay of agency and neighbor interactions that take place during collaborative instances.

The first form of collaboration occurs when students engage in the *co-construction* of ideas. This is a form of collaboration in which the dialogue occurs in a back and forth nature (similar to the action that occurs during a ping-pong game) until the argument is built. In other words, the argument is simultaneously built from the ground up.

The second form of collaboration is that of *integration*. This form of collaboration is identified when a student's argument is strengthened using ideas from their peers. In other words the ideas, explanations, or representations of others are assimilated into their original argument.

The primary distinction between co-construction and integration is that in a co-constructed argument, the two interlocutors are creators of the argument. Without one of the participants, the argument would not exist. An argument that results form the process of integration, on the other hand, is established by its originator and is only enhanced by the other participants' contributions.

The third form of collaboration is that of *modification*. This occurs as students attempt to correct a peer or assist him/her in making sense of a model or argument that was originally expressed in an unclear or incorrect way. As the student attempts to make sense of the faulty argument and assist one's partner in seeing the error or sense-making, a sound argument is created that differs significantly in presentation and/or interpretation.

Figure 1 below illustrates the nature of the discourse and of the agency that is typical of each form of collaboration. Co-construction is typified by negotiary discourse, and all participants share agency in the discussion. Integration makes use of both informative and interpretive discourse. The original argument is interpreted by the second participant, who then enhances the argument in a way that informs the originator and allows the first participant to assimilate the information in a meaningful way. During integration, the originator of the argument is the principal agent, but the secondary participant influences the mathematical outcome and thus is a secondary agent in the discussion. Modification is characterized by interpretive discourse, as one student attempts to makes sense of another's faulty or flawed argument. This sense-making student also has the primary agency in the discourse, as he/she has the ultimate control in the mathematical outcome of the discussion.

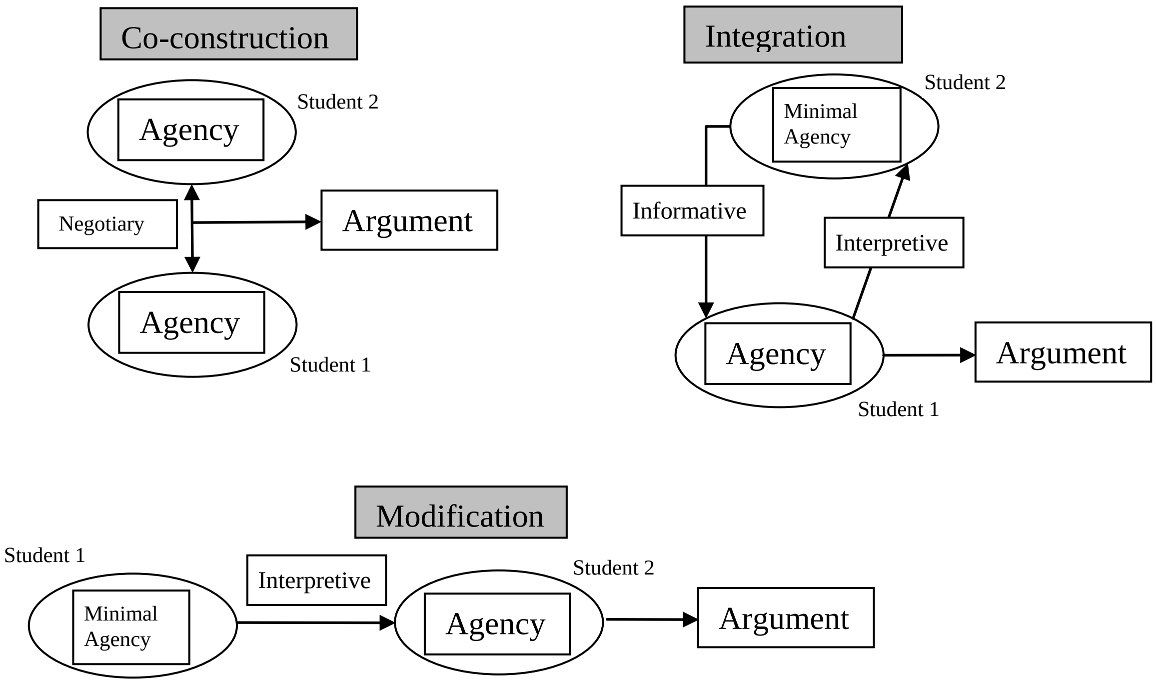


*Figure 1*. Three modes of collaboration.

## Method of Inquiry and Data Source

*Setting*

This research is a component of a larger, ongoing longitudinal study, Informal Mathematics Learning Project (IML), conducted through an after-school partnership between a University and an economically depressed, urban school district whose school population consists of 98 percent African American and Latino students. A goal of the program was to explore how mathematical reasoning develops in middle-school age students over time and under certain conditions. Students worked on challenging tasks, interacted with peers, and were allowed the time and opportunity to explore, explain, and discuss. The IML project spanned more than two and a half years (including summers) and included 30 sessions (each approximately 60-75 minutes in length). During each session a cohort of 24 students was asked to engage in open-ended problem solving working on stands of mathematical tasks, involving topics such as fractions, combinatorics, and probability.

In this paper, we report on the first five sessions of the IML program. Students, seated in heterogeneous groups of four, were given a strand of tasks dealing with fraction ideas in which they were asked to justify their solutions. For many of the students, the opportunity to work collaboratively on open-ended tasks was a new experience. With this in mind, tasks developed from earlier research had been found to promote collaborative reasoning and problem solving. Students were invited to build models of their solutions to the tasks using a set of Cuisenaire rods. The set contains ten colored wooden or plastic rods that increase in length by increments of one centimeter. After each task was posed, students worked in their small groups and were encouraged to build models and share ideas and conjectures. Students were then invited to the overhead projector to share their models and arguments with the whole class. Our analysis in this paper focuses on two groups of students working on two tasks during the second and fifth sessions of the program.

*Analysis*

Four video cameras were used to capture the activity of students working in small groups and presenting at the overhead projector. Video recordings and transcripts were analyzed using the analytical model outlined by Powell, Francisco & Maher (2003). The video data were first viewed repeatedly so that researchers could get a sense of the big picture. Next, the videos were transcribed and critical events were identified. Critical events consisted of students constructing justifications and defending these justifications. The critical events were organized into episodes which were organized in tables that traced the development of the justifications. Codes were developed to flag solutions offered by students, the justifications provided to support these solutions, and the occurrence of students collaboratively building arguments.

A modified grounded theory approach was used to analyze the data for collaboratively constructed arguments. Initial codes for the collaborative actions in building ideas emerging were organized into three categories: building on other's ideas, questioning others, and correcting others. Sub-codes of building on others' ideas included: expanding, redefining, and reiterating. Analysis of the patterns that these codes highlighted enabled the identification of patterns of interaction that were present. The result of our analysis was a coded, sequential narrative of both the justifications that students built collaboratively and the challenges that they made during the five sessions.

## Results

Examining the data across sessions, we found that the majority of arguments were built collaboratively. Upon further analysis, we discovered that the students collaborated in three different ways, and these modes of collaboration were ultimately crystallized and defined as outlined in the theoretical framework. A description of the three ways that students collaborated is organized by presenting an analysis of two episodes in which students collaboratively built an argument.

*Episode 1: Integration and Co-construction*

During the second session of the after-school program students were given the following task: *If I call the blue rod one, I want each of you to find me a rod that would have the number name one-half.* This is an impossible task since the blue rod is 9 cm. long. As such, the task generated many different arguments from the 24 students that included four different types of reasoning (contradiction, upper and lower bounds, cases, and inductive). The students worked for a few minutes and determined that such a rod did not exist. Chris lined up nine white rods under a blue rod and explained that you could find a rod whose length is one-third of the blue rod but not one-half. Danielle and Brittany began to build models of combinations of rods whose lengths were equivalent to the length of the blue rod (for example, a train of a yellow rod and a purple rod). After about five minutes Chris presented his partners with his model and explained:

Chris: If you take out four that's an even number but if you put the four back, that's not a half because it's nine, and nine is an odd number.

Danielle then presented the combinations of trains of rods whose lengths are equivalent to the blue rod (yellow and purple, black and red, light green and dark green, brown and white). Again, Chris explained using his model of nine white rods.

| | |
|---|---|
| Chris: | You can’t find a half of the blue one because if you put all white you only have nine so for nine you can’t really do it. And also if you put the white ones, you – it’s an odd number which is nine, and you can’t do it. |

Referring to the model that she and Danielle built, Brittany explained that none of the combinations of rods were a half because they were not the same length. In response, Chris explained that the blue rod is equivalent in length to nine white rods and nine is an odd number. The following dialogue then occurred:

| | |
|---|---|
| Danielle: | You can’t really divide an odd number; if the two yellow rods were the same length as the blue rod then they would be a half. |
| Chris: | Overall you can’t do it because if you use a white one, it is an odd number so you can’t divide by two. |
| Jeffrey: | Unless you get a decimal or a remainder. |
| Chris: | And you wouldn’t be able to do it anyway because none of these are even. |
| Danielle: | If only we had the two yellows and the yellows were shorter— |
| Jeffrey: | If purple was bigger— |
| Danielle: | Yeah, if the purple was bigger, then the green was kind of shorter so that the same color green could fit on it. |
| Jeffrey: | If the purple would have been a little, like half of the white, it would have been good. |
| Danielle: | Or if the light green was, um, it was, yeah, like a little bigger, then you could only have one up there or two. |
| Jeffrey: | The light green rod would have to be bigger than the purple rod. |
| Danielle: | But the purple has to be bigger; the purple has to be bigger equally because if you take another purple, you’re going to have to add a white. Yeah, you’re going to have to add a white; see, it don’t work. |
| Chris: | The thing we should say is that since we put the white cubes and we got an odd number, then if you have an odd number you can’t divide by two so you get one-half. So you get a decimal or a remainder so you can’t really divide it, right? |

In the above dialogue we see that two different forms of collaboration occurred simultaneously, integration and co-construction. Chris built a model using nine white rods and offered an argument based on a contradiction. Although his partners used a different line of reasoning, Chris continued to use his argument; however he integrated ideas from Danielle (*You can’t really divide an odd number*) and Jeffrey (*Unless you get a decimal or a remainder*) into his original argument. The integration of these ideas strengthened Chris’ original argument. Meanwhile, Jeffrey and Danielle co-constructed an argument based on upper and lower bounds.

Danielle offered the upper bound and Jeffrey the lower bound and together they determined that not rod existed between these two rods.

*Episode 2: Modification*

During the fifth after-school session students were presented with the task: *which is bigger one-half or one-third, and by how much?* Many students initially tried to name the orange rod one but realized that they could not find a rod whose length was one-third of the orange rod. They struggled with finding a model that they could use to compare one-half and one-third. Finally, Michael explained that the dark green rod would be named one, the light green rod one-half, and the red rod one-third. He used a model of the rods lined up next to each other to convince Ian. The students continued working on the task using Michael's model. Eventually Michael shared his "cake metaphor" described below.

Michael: And like, say, say you got five kids and each kid want a different slice [of cake] so one kid gets one-half [he pushes aside a light green] another kid gets this half [the other light green rod]—you only have three pieces left and the other three want small, well not small pieces, medium pieces [the red rods], okay, alright.

The researcher called Michael's attention to the discrepancies in his model (taking away two halves and then taking away three thirds) and walked away. Michael then asked Ian for help. Ian asked Michael a series of clarifying questions about his model. Through this questioning, he made sense of Michael's model and ideas and was able to explain it in his own words.

Ian: Oh, I get what your saying, so you're trying to divide them into different slices, like, if this is a cookie and your only here is a whole [the dark green rod], then when your friend comes over it's a half, you split it's a half [light green rods] and then when another friend comes over you split it into three slices [red rods].

Michael: This [he removes the dark green rod] it ain't a whole no more
So you just cut it right? So those are the three halves [red rods], this is the third.

Ian: Oh, this is the one-third [pointing to the red rods] and this is the half [pointing to the light green rods].

Michael: yeah

Ian: Oh I get what your saying now- if this is the whole cake, this is what you're trying to say, if this is a whole cake, alright so this is one whole and then you can divide it into three so this isn't a whole anymore, you have three pieces but then let's say this is the whole again—you divide this into one-half and you got two pieces instead of three—like one half is bigger than one-third, is that what you're saying?… yeah I get what your saying

Ian and Michael then used Michael's cake model to determine that one-half was bigger than one-third by one white rod or one-sixth. When the researcher rejoined them, both boys simultaneously explained the relationships and presented evidence to show that one-half is bigger than one third by one-sixth.

In this mode of collaboration, Michael initially built a direct argument to compare one-half and one-third using a cake metaphor. However, his explanation was faulty. In the process of making sense of Michael's argument and clearing up the inconsistencies (taking away two halves and having three thirds left), Ian developed a solid agreement for comparing the two fractions and used the model to find the difference.

## Discussion and Implications

In this paper, we extended Powell's (2006) framework for analyzing students' mathematical discourse by examining the effects of the four types of interlocution on the ways in which students collaborate when building mathematical ideas and justifications. In doing so, we offer a framework for analyzing student collaboration and the influence of this collaboration on agency. We show two episodes of students collaboratively building arguments as examples of the three modes of collaboration: integration, co-construction, and modification. In these episodes, it is evident that the nature of student's agency and the type of discourse in which the students engaged were interdependent and thus influenced the arguments that were constructed. The students' discursive interactions clearly had an influence on their mathematical ideas and reasoning. We concur with Powell (2006) that negotiary discourse promotes the development of *socially emergent cognition*, but also contend that informative and interpretive discourse are important in the discursive interactions of students and play a role in the ways in which ideas and reasoning influence students' justifications and mathematical understating.

Research points to the benefits of students working collaboratively on mathematical tasks and suggests that this collaboration influences individual mathematical understanding. We add to the literature by examining how specific types of discourse influence this collaboration and identifying three modes of collaboration that result. In addition, we show that each type of collaboration played a role in developing the strength or validity of the final argument. Analysis of the collaborative moves of students, then, is essential for the promotion of effective mathematical reasoning and argumentation in the learning environment. Ultimately, a more complete understanding of the mechanisms by which collaboration is encouraged can enable educators to facilitate the various forms of collaboration and promote agency and effective mathematical reasoning and argumentation in all students.

## Endnotes

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# COMPOUND EFFECTS OF MATHEMATICS DISCUSSION ON FIFTH GRADE MATH ACHIEVEMENT

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*Mathematics discussion has been suggested to improve mathematics achievement. The current study investigated the compound effects of students discussing mathematics with one another in both the third and fifth grades using hierarchical linear modeling. Results indicated that having discussion in fifth grade but not in third grade had a positive impact on mathematics achievement. The impact of discussion in both third and fifth grade was not found to be statistically significant from the impact of having no discussion in either grade. However, significant variability found in one variable may explain why a compound effect of discussion was not found.*

## Background and Objectives

According to Silver, Kilpatrick, and Schlesinger (1990), "mathematics deepens and develops through communication" (p. 15). Students gain a deeper understanding of the meaning of mathematics when they communicate with others about it (Goos, 1995; Lee, 2006; Pimm, 1987). Additionally, discussion has been shown to have a positive impact on mathematical achievement (D'Ambrosio, Johnson, & Hobbs, 1995; Grouws, 2004; Hiebert & Wearne, 1993; Koichu, Berman, and Moore, 2007; Mercer & Sams, 2006). Yet, there is evidence that discussion does not always have a positive impact on mathematics achievement (Kosko & Miyazaki, 2009; Shouse, 2001), which may imply that either discussion is not consistently effective in deepening mathematical understanding or that it is not consistently implemented to maximize its effectiveness.

A previous study conducted by the authors (Kosko & Miyazaki, 2009) investigated the impact of discussion on mathematics achievement using data from the Early Childhood Longitudinal Study (ECLS). Results showed that when accounting for prior achievement the difference between the two discussion groups in the study (*weekly* and *less than weekly*) was found not to be statistically significant. However, there was a statistically significant amount of variability in the impact of weekly discussion across schools. This variability was unable to be explained by the authors, even after the addition of covariates.

The previous study conducted by the authors was done with reference only to the impact of discussion in the fifth grade (Kosko & Miyazaki, 2009). The current study seeks to investigate the accumulated impact of discussion on fifth grade mathematics achievement. The large amount of unexplained variability in the previous study led the authors to question if mathematics discussion may take longer than one school year to positively impact math achievement. Yet to determine if this is actually the case, a new study had to be conducted. To date, the authors have yet to find a longitudinal study to investigate the compound effects of discussion on mathematics achievement. Therefore, the purpose and research question for the current study is as follows: Does the frequency of peer mathematics discussion in third and fifth grade have a compound impact on the mathematics achievement scores of fifth grade students.

## Theoretical Perspectives

Student discussion of mathematics has been stated as a means to deepen understandings of the mathematics discussed (Goos, 1995; Lee, 2006; Pimm, 1987; Silver et al., 1990). Students who understand mathematics more deeply should predictably perform better on mathematics achievement tests. According to several studies (i.e. Hiebert & Wearne 1993; Mercer & Sams, 2006; Stigler & Hiebert, 1997) students who were asked to explain and justify their mathematics in discussion had higher gains in mathematics achievement than students who were not asked to do so. Yet in some cases, student discussion of mathematics has been found to have a negative impact on achievement (cf. Shouse, 2001). As mentioned above, a previous study by the authors (Kosko & Miyazaki, 2009) found that the impact of student discussion on mathematics achievement varies significantly between schools. This means that in some schools the impact of achievement can be largely positive while in other schools it is largely negative. While some qualitative research (i.e. McGraw, 2002; Nelson 1997) suggests that it takes time for effective mathematics discussion to be successfully implemented, there is little to no quantitative data to support the impact of time or exposure on the effectiveness of discussion.

## Research Question

Does the frequency of peer mathematics discussion in third and fifth grade have a compound impact on the mathematics achievement scores of fifth grade students?

## Methodology

The current study analyzed data from the Early Childhood Longitudinal Study (ECLS) using a two-level hierarchical linear model (HLM). Hierarchical linear models are typically used when evaluating data nested in many groups (Raudenbush & Bryk, 2002). Since much of what is studied in education exists in hierarchical structures (i.e. students within classes, teachers within schools), HLM is particularly useful in studying differences within and between nested units. A two level hierarchical linear model looks at both the micro units at level-1 and the macro units at level-2. The micro units at level-1 (i.e. students) are nested within the macro units at level-2 (i.e. classrooms or schools). HLM-2 allows factors examined at the first level to be compared at the second level unit of analysis which they are nested in (see Raudenbush & Bryk, 2002 for further information). The purpose of using an HLM-2 model in the current analysis was to take into account the nested nature of the data (students nested within schools).

Third and fifth grade data from 3583 students in 1006 schools was used in the current analysis. The main variables included in the analysis were achievement scores and a teacher assessed item which asked how often the students involved in the study engaged in discussion about mathematics with other students. Teachers were required to complete this item for each student they taught who was a part of the ECLS study and was therefore used as an individual level variable rather than a classroom level variable. The variable addressing how frequently students discussed math with their peers was assessed in both third and fifth grade. For the purposes of simplifying the model for analysis, the variable was dichotomized at both levels to students who did have weekly discussions about mathematics and students who had discussions about mathematics less than weekly.

To investigate the compound effects of discussion on fifth grade achievement scores, four categories of students were compared in the analysis: students who did not have mathematical discussions with other students on a weekly basis in third grade or in fifth grade (*control*); students who did not have weekly math discussion in third grade but did in fifth grade (*disc_0_1*); students who had weekly math discussion in third grade but not in fifth grade (*disc_1_0*); and students who had weekly math discussion in both third and fifth grade (*disc_1_1*). These categories were included as dummy-coded variables in addition to other variables which served as covariates. These covariates included third-grade math achievement scores, race/ethnicity, gender, and socio-economic status.

**Results**

Results showed that students in the *disc_0_1* group had statistically significant higher scores than students who did not have discussion in either third or fifth grade. Students in both the *disc_1_0* and *disc_1_1* groups did not score significantly higher than students without discussion (*control*). At first glance this seems to indicate that there is no compound effect of discussion. However, the variable *disc_1_0* was found to have significant variability across schools, more than four times the impact of *disc_1_0*. Such variability indicates that weekly discussion in third grade had a largely positive impact on math achievement in some schools but also had a largely negative impact on math achievement in other schools. In turn, this variability may have affected the impact of *disc_1_1* on math achievement. Since weekly discussion in third grade was shown to have a negative impact on math achievement for many students, such discussion may, in part, counteract the generally positive benefits of having weekly math discussion in fifth grade. This would therefore explain why a compound effect of discussion was not found in the analysis.

**Discussion**

Mathematics discussion has been shown to have a positive impact on math achievement in some instances (D'Ambrosio, Johnson, & Hobbs, 1995; Hiebert & Wearne, 1993; Koichu, Berman, and Moore, 2007; Mercer & Sams, 2006; Stigler & Hiebert, 1997) and a negative impact in others (Shouse, 2001). The previous study conducted by the authors (Kosko & Miyazaki, 2009) showed a large amount of statistical variability in the impact of discussion that could not be explained. The current analysis sought to explain the variability in the previous study (Kosko & Miyazaki, 2009) and conflicting results of other studies (i.e. Shouse, 2001; Stigler & Hiebert, 1997). Although the results of the current study seem, at first glance, not to support a compound impact of discussion, the statistically significant amount of variability found in the impact of discussion in third grade is reminiscent of the results for analysis of the impact of fifth grade discussion (Kosko & Miyazaki, 2009). Therefore, the current study provides evidence that in a given year the general impact of student discussion on mathematics achievement can vary significantly between school setting if prior exposure to discussion is not taken into account.

One interesting result of the current study was that the impact of weekly discussion in the fifth grade without weekly discussion in the third grade (*disc_0_1*). This result could support claims made by Mercer and Sams (2006) who suggested that younger students may not inherently posses the skills necessary to maintain mathematics discussion without specific guidelines from the teacher. However, it is unknown how much guidance or structure these students received from the teacher in discussing mathematics with their peers.

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# THE EMERGENCE OF NORMS FOR MATHEMATICAL ARGUMENTATION: CONTRIBUTING TO A FRAMEWORK FOR ACTION

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*In this paper, we provide an account of the evolution of mathematical norms for argumentation that emerged during an ongoing teacher collaboration. The collaboration involves the mathematics and science teachers at Green Valley High School in the southwestern United States. As part of the collaboration, teachers were offered the opportunity to participate in college level courses offered at the school. In the fall of 2008, 21 of the 32 mathematics and science teachers chose to participate in a course that focused on functions and the covariation of the measures of two quantities. The activities that comprised the course were intended to serve as didactic objects around which productive conversations could emerge. Therefore, mathematical arguments developed around key significant mathematical issues. As a result, the norms for mathematical argumentation evolved during the semester.*

## Introduction

Balancing the tensions inherent in simultaneously attending to students' contributions and the mathematical agenda is a hallmark of deliberately facilitated discussions (cf. McClain, 2003). These discussions involve a plethora of decisions that must be made both prior to and while interacting with students. The image that results is that of the teacher constantly judging the nature and quality of the students' contributions against the mathematical agenda in order to ensure that the issues under discussion offer means of supporting the students' mathematical development. This view of mathematical discussions stands in stark contrast to open-ended sessions where all students are allowed to share their solutions without concern for potential mathematical contributions. In order to engage in the process of elevating discussions to the level of sophisticated mathematical argumentation, the teacher must have a deep understanding of the mathematics under discussion (cf. Ball, 1989; Ball, 1993, Ball, 1997; Bransford et al., 2000; Grossman, 1990; Grossman, Wilson, & Schulman, 1989; Ma, 1999; McClain, 2004; Morse, 2000; National Research Council, 2001; Shulman, 1986; Schifter, 1995; Sowder, et al., 1998; Stein, Baxter, & Leinhardt, 1990). This is critical in both being able to advance the mathematical agenda and in judging the quality and worth of student contributions. It requires decision-making in action concerning the pace, sequence and trajectory of discussions in order to ensure that the discussions are mathematically productive.

When focusing on students' offered explanations and justifications, the teacher is seen to actively guide the mathematical development of both the classroom community and individual students (Ball, 1993; Cobb, Wood, & Yackel, 1993). This guiding necessarily requires a sense of *knowing in action* on the part of the teacher as he or she attempts to capitalize on opportunities that emerge from students' activity and explanations. With this comes the responsibility of monitoring classroom discussions, engaging in productive mathematical discourse, and providing direction and guidance as judged appropriate. Similar pedagogical issues are addressed in Simon's (1995) account of the Mathematics Teaching Cycle that highlights the relationship between teachers' knowledge, their goals for students, and their interaction with students.

A focus on the importance of students' contributions also highlights the importance of norms that constitute the classroom participation structure. The importance attributed to classroom norms stems from the contention that students reorganize their specifically mathematical beliefs and values as they participate in and contribute to the establishment of these norms.

In the analysis in this paper, we focus on the discourse between and among the teachers and instructors in a setting in which the authors were the instructors. The analysis will make explicit the evolution of the discussions over the course of the interaction. In doing so we clarify the normative ways of speaking that evolved in the process. This work is significant in that it offers one framework for thinking about how to guide the evolution of productive mathematical argumentation.

## Setting

The authors are involved in ongoing teacher collaboration with a group of high school mathematics and science teachers from Green Valley High School[1]. Green Valley High School serves a student population of approximately 2,800. It contains grades ten through twelve. As part of the collaboration, in the fall semester of 2008, 21 of the 32 mathematics and science teachers at Green Valley chose to participate in a course that was taught on Monday afternoons at the school. The authors served as instructors for the course that focused on covariational reasoning. The teachers were able to earn three hours of college credit for their participation. In addition, all of the mathematics and science teachers in the school attended weekly curriculum planning meetings. The teachers were assigned to groups according to the primary subject they taught. For instance, all of the Biology teachers met together as did the pre-calculus teachers. This meeting time was supported by the Principal as evidenced by his arranging for common planning periods for the teachers. However, for the purposes of the analysis in this paper, data is taken only from the discussions that occurred during the Monday night class.

## Description of the Course

The course was designed to focus on a significant mathematical concept, that of covariational reasoning. Oehrtman and colleagues (Oehrtman, Carlson, & Thompson, 2008) have argued that covariational reasoning is foundational to high school mathematics and should serve as the organizing concept for all courses. We agree with this stance and therefore took covariational reasoning as our mathematical endpoint. In order to achieve the envisioned endpoint of covariational reasoning playing a significant role in instruction, we initially engaged the teachers in activities that were used in an Algebra I class where covariational reasoning guided the development of the mathematics. (These activities were part of a project conducted by Pat Thompson and his research team.) The teachers in the class worked through the series of activities *as students* and then reflected on their prior activity from their position *as teachers*. Following their mathematical investigations, they explored the classroom in which the instructional unit was implemented. This was made possible by video-based case development efforts from Thompson's ongoing grant. Two of the authors, McClain and Coe, had participated in the case development and were, therefore, well equipped to provide instruction based on the case.

Following the case investigation, class sessions turned to materials developed from Project Pathways that was funded by the National Science Foundation under Carlson's direction. She therefore took the lead on the instruction for this portion of the course. The materials begin with an investigation of proportional reasoning as it relates to functions, and they form the basis for an exploration of linear and exponential growth. Throughout the investigations the teachers were encouraged to consider how the measures of the two quantities co-varied in each situation. The grounding of the problems in contextual situations allowed the teachers to work in small intervals to investigate the phenomena (e.g. small increments of time). It also helped support a shift away from what Thompson (personal communication, 2008) calls "shape thinking." Shape thinking involves imagining the "path" of the phenomena and then seeing the graph as a "static" representation of the completed trace of the path. In other words, the graph is static and has already occurred. An example can be seen when students trace the path of a car given the time and distance it has traveled instead of trying to coordinate the measures of the quantities of time and distance.

It is important to note that throughout the semester, the teacher participants were constantly encouraged to *speak with meaning.* Elsewhere we have described speaking with meaning as ensuring that explanations carry meaning for all participants. This requires conceptually based conversation about quantity. The negotiation of the norms for argumentation that resulted lead to improved understanding between the teachers and deeper knowledge of the content. It is these discussions that provide the basis of our analysis.

## Methodology

The general methodology falls under the heading of design research (Brown, 1992; Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003). Following from Brown's characterization of design research, the teacher collaboration involved *engineering* the process of supporting teacher change. Like Brown, we attempted to "engineer innovative educational environments and simultaneously conduct experimental studies of those innovations" (p. 141). This involved iterative cycles of design and research where conjectures about the learning route of the teachers and the means of supporting it were continually tested and revised in the course of ongoing interactions. This is a highly interventionist activity in which decisions about how to proceed were constantly being analyzed against the current activity of the teachers.

The particular lens that guided our analysis of the data was a focus on the normative ways of arguing about solutions, or what Cobb and Yackel (1996) define as the classroom mathematical norms. Classroom mathematical norms focus on the collective mathematical learning of the teacher cohort[2] (cf. Cobb, Stephan, McClain, & Gravemeijer, 2001). This theoretical lens therefore enabled us to document the collective mathematical development of the teacher cohort over a period of time. In order to conduct an analysis of the communal learning, it is important to focus on the diverse ways in which the teachers participate in communal practices. For this reason, the participation of the teachers in discussions where their mathematical activity is the focus then becomes the data for analysis. The diversity in reasoning also serves as a primary means of support of the collective mathematical learning of the teacher cohort. An analysis focused on the emergence of classroom mathematical norms is therefore a conceptual tool that reflects particular interests and concerns (Cobb, et al., 2001).

## Analysis

Research on effective teaching often characterizes the teacher's classroom decision-making process as informed by the mathematical agenda, but constantly being revised and modified in action based on students' contributions. These characterizations take account of the students' contributions while attending to the mathematics. Attempting to balance the tension inherent in simultaneously attending to students' offered solutions and the mathematical agenda is the hallmark of deliberately facilitated discussions. A critical resource for the teacher in this process is therefore the *means of support* available to help him achieve his mathematical agenda. This support manifests itself in the form of the instructional tasks and the tools available for solving the tasks. For this reason, tools, notation systems, and student generated inscriptions all serve an important role in the mathematics classroom. However, it is not the tool (or the notation or the inscription) in isolation that offers support for the teacher. It is instead the students' use of the tools and the meanings that they come to have as a result of this activity (Kaput, 1994; Meira, 1998; van Oers, 1996). In this way, the tool is not seen as standing apart from the activity of the student. When designed, these objects must be thought of as didactic objects that will form the basis for reflection and discussion (cf. Thompson, 2002). For this reason, the teacher generated artifacts from the course served an important role in supporting the evolution of mathematical discourse.

As an example, one of the first tasks posed to the teachers in the Monday afternoon class is called the *Sprinter Task*[3]. In this task, the teachers first watched a video of Florence Griffith Joyner's Gold Medal 100 meter race. After viewing the race a couple of times, the teachers were asked to qualitatively track the distance from the start against time since the start. In this introductory task, the teachers had to begin to coordinate two quantities[4], distance from start and time from start. As part of the coordination, they were asked to create a graph and then explain the graph in terms of the two co-varying quantities.

As the teachers worked, one group of mathematics teachers was very focused on the accuracy of their graph. They were unable to think about the measures qualitatively and struggled to get exact measures for exact times by continually starting and stopping the video. They were reluctant to share their solution until they were sure that their graph was correct. Our goal was more global. We wanted the graphs to show a qualitative relationship between the measures of the two quantities that could be described generally with wording such as, "As time passes, her distance from start increases." We were also interested to know if the teachers viewed Griffith-Joyner traveling at a constant rate, an increasing rate or other, and we were interested in their understanding of the meaning of these. Our ability to support the teachers' ability to focus on the coordination of the measures of the two quantities depended upon their ability to reason about the situation, not read a graph in the canonical sense.

Unfortunately, we did not achieve our goal on this first task. This is not surprising either now nor was it at the time. The mathematics teachers in particular argued that scaling the axes was an important part of creating a graph. We eventually had to tell them to leave them unmarked. Even so, they were hesitant to present their results.

Although all of the groups of teachers were able to generate a graph that gave a qualitative sense of the how the measures of the two quantities co-varied, conversations at this point were characterized by telling. In this early phase of the class, the teachers were focused on the correctness of the answer and their contributions were a recitation of that correct answer and the procedure used to arrive at it. In addition, they did not question each other, but sat quietly as each group shared. This portion of the lesson took on characteristics of a "show and tell" instead of an intellectual conversation about ideas. We were, in fact, unable to prompt significant conversations at this point. We therefore describe the first mathematical norm that emerged as that of *argumentation as telling.*

A week later, we posed a question that involved the teachers watching another video and then creating a graph of the situation. In this video, a skateboarder skates back and forth on a half pipe. The *Skateboarder task* required the teachers to coordinate the boarder's horizontal distance from start with the time from start. In this task, the teachers had to attend to the quantities being tracked rather than the position of the boarder. This is in sharp contrast to the *Sprinter task* where Griffith-Joyner's position gave information on the distance from start. This task, therefore, focuses on the issue of shape thinking in that if teachers tried to rely on the *position* of the skateboarder, the graph would be incorrect as shown below in Figure 1.

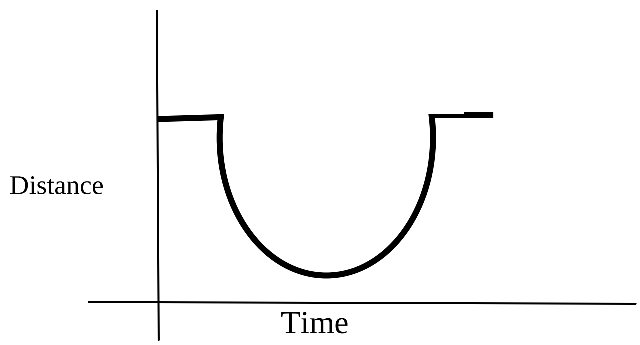


*Figure 1.* Path of the skateboarder on a half pipe.

The other significant issue that emerged was that of explicitly labeling the axes. For instance, using the descriptor "distance" to label the vertical axis does not clarify what quantity is varying. It could be the total distance traveled. When graphed correctly, the vertical axis should be labeled *horizontal distance from start* and the horizontal axis labeled *time from start.* For this reason, the graph is not the trace of the half pipe as shown above, but the coordination of the measures of the two quantities. Therefore, in creating their graphs, the different groups came up with differences in their interpretations as shown below (Figure 2).

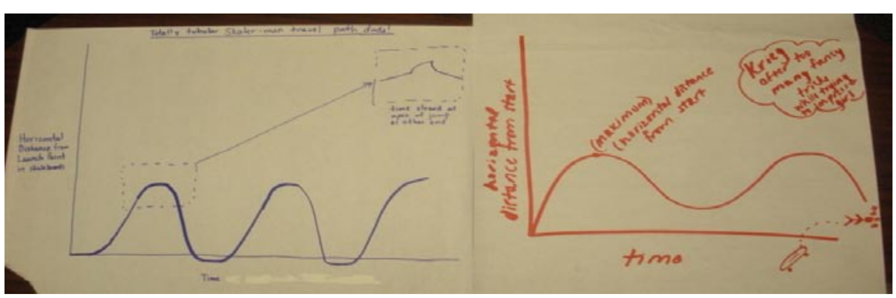


*Figure 2.* Graphs of the skateboarder's distance from start as a function of time.

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

When these were juxtaposed on the board, a lively discussion ensued. The teachers initially fell into their original mode of discussion by engaging in a show and tell. However, the differences in their graphs initiated a shift in the conversations such that the teachers began to take the position of defending their own graph. There was no effort to make a comparison across the different graphs—only to justify why their graph was correct. As a result, the teachers did not attend to each other's argument, but focused only on their artifact. There was no attempt to revise and modify the current graphs to move toward a more accurate representation. That only occurred at our initiation. The teachers just kept pointing to their own solution. As a result, the second mathematical norm that emerged was that of *argumentation as disagreement*.

As the course continued and the teachers continued to investigate situations where they had to coordinate the measures of two quantities, the instructors began to push the teachers to speak meaningfully about their graphs. In particular, as the teachers explained their graphs to the class, they were pressed to talk about what was happening to one variable as the other changed or varied. As an example, in the skateboarder task, it was insufficient to say, "First he went down and then across and then back up." A more meaningful explanation was, "as he dropped down the left side of the pipe, his horizontal distance from start did not change. However, as he moved along the base of the pipe, his horizontal distance from start began to increase. As he went up the right side of the pipe, his horizontal distance from start was the same." In other words, the explanation had to be in terms of the two quantities and how they are covarying. In subsequent problems the teachers and instructors began to renegotiate what constituted an adequate explanation.

In this third phase there was, therefore, a significant shift in that the teachers began to engage in conceptual explanations. However, their goal was not to engage other members of the class in a discussion. The teachers spoke *to* the other members of the class, not *with* them (elsewhere Lima and colleagues have made a similar distinction, (personal communication, June, 2008)). Although the teachers were able to reconceptualize their own thinking about a particular solution, they were still unable to understand what it might mean to speak so that others could comprehend their thinking. As a result, the third mathematical norm for argumentation that emerged was that of *speaking conceptually* ***to*** *another*.

The last mathematical norm for argumentation that emerged was that of *speaking conceptually* ***with*** *another*. In this fourth and final phase, the teachers continued to speak meaningfully or engage in conceptual explanations with their colleagues. However, the teachers were not only able to reconceptualize their own thinking about a particular solution but also do this while thinking about what it might mean to speak so that others might comprehend their thinking. This shift in argumentation was apparent in discussions of the *Bungee Jumper* task. In this task, the teachers were shown a video of a man bungee jumping off of a bridge. The task was to create a graph that coordinated the time since the jumper leapt from the bridge with his distance from the ground. By making the distance quantity the *distance from the ground*, teachers were unable to create a graph that was merely the trace of the jumper's path. Instead, they had to coordinate the variation of the time since he leapt with his distance from the ground. As the teachers shared their solutions, they used their explanations and questions to clarify for both themselves and their colleagues how the distance varied as time changed. In this process there was a concerted effort on the part of the teachers and the instructors to communicate clearly. It was during these conversations that the teachers began to speak conceptually **with** one another.

## Conclusion

In the analysis presented in this paper, we have provided an account of the evolution of the mathematical norms for argumentation that occurred in the course taught to the mathematics and science teachers at Green Valley High School. This work is significant in that it provides a framework for thinking about mathematical argumentation in the context of teacher development. This evolution is similar to data we have analyzed from other teacher development collaborations (see McClain, 2002). For this reason, we argue that we are providing the starting points for what diSessa and Cobb (2004) call a *framework for action*. diSessa and Cobb characterize a framework for action as a first step toward the development of a guiding theory. Therefore, we are not claiming that we have discovered a new theory. Our claims are much more modest. What we are offering is a first step in that direction. As a result, this pattern of evolution will be useful in our future work. The potential of its power as a theory is only determined by its use by others in similar and different situations. The messiness and complexity of teacher development in the context of design research "highlights the pressing need for theory while simultaneously making the development of useful theories more difficult" (p. 79). It is for that reason that we offer this process as a "way of looking" at this significant aspect of teacher collaborations.

## Endnotes

1. Green Valley is a pseudonym.
2. In this analysis, we purposely refer to the group of teachers as a *cohort* instead of a *community*. Documentation of the evolution of the cohort into a community is beyond the scope of this paper. We therefore take the "easier road" by not making assumptions about the nature of the relationships within the cohort at the time of this analysis. We choose to do so because of the importance we place on both establishing *communities* of teachers and verifying their existence with established criteria (cf. Wenger, 1998).
3. The initial tasks were designed by Scott Adamson and Ted Coe as part of their grant work with Pat Thompson.
4. By quantity we mean an attribute that can be measured or that one can imagine measuring.

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# MULTIPLE MEANINGS IN MATHEMATICS: BENEATH THE SURFACE OF AREA

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*This study applied thematic discourse analysis (Lemke, 1990) to a section of a middle school lesson focused on the relationship between the area of parallelograms and rectangles. This analysis provides a way to show the structure of the semantic relations between mathematical terms, shedding light on points of convergence and divergence between parallelograms and rectangles that can be, and often are, used for instructional purposes. Points of teacher dialogue where the semantic relations between mathematical terms might have been unclear to students are identified and particular attention is given to subtle shifts in the meanings of the terms* base *and* height*.*

## Background

Mathematical language presents several significant challenges to students (Schleppegrell, 2007). For example, mathematical terms are used with a different level of precision than terms in everyday language, in part because a mathematical definition provides both necessary and sufficient information about the term whereas the definition of an everyday term merely describes its meaning. This distinction was taken further by Poincaré who pointed out that a mathematical definition, rather than encapsulating an existing meaning, actually creates the mathematical entity in question (Folina, 1992). Linguistic challenges also arise when words are used differently inside mathematics classrooms than they are outside (Pimm, 1989; Thompson & Rubenstein, 2000), as is the case with *average, power, similar, right*, and even the word *or*. Another challenge of mathematical language, one that is especially relevant to the current study, is that a single term is often used in ways that have subtly different meanings. A mathematician using the word *inverse* may, depending on the context, be referring to an inverse function, an inverse operation, or the multiplicative inverse of a group element. We are not arguing that such uses of mathematical terms are inappropriate or undesirable because we recognize the value in a compact, versatile language that mirrors the myriad connections between mathematical entities. Rather, we are emphasizing that in the process of learning mathematics, which in part means becoming fluent in its language and meaning systems (Chapman, 2003), overcoming such obstacles is non-trivial for students. We contend that more detailed attention to such non-trivial aspects of mathematics learning can be helpful to teachers and researchers, a point we return to in the final section of this paper.

The fundamental geometric concepts of *base* and *height* provide another example of mathematical terms that are used with subtly different meanings at different times. The purpose of this study is to examine a classroom interaction that includes the terms base and height to see whether and how this subtle shift in meaning manifests in the dialogue.

## Theoretical Perspective

Michael Halliday's (Halliday, 1978; Halliday & Matthiessen, 2003) theory of systemic functional linguistics underlies the analytic methods we employ in this paper. A foundational assumption of this theory is that context and language use are intimately related: context influences language choice and language choice helps to construe context. Halliday described three metafunctions of language—ideational, interpersonal, and textual. Language is used to make sense of experience and in so doing serves the *ideational* metafunction; that is, it is used to give cues and clues regarding the meaning of *what* is being talked about. Language is also a means for acting out the social relationships of those *who* are using the language, thus serving the *interpersonal* metafunction. The *textual* metafunction refers to aspects of the organization of the language itself. Unfortunately, we are unable in the present paper to provide a full description of the theory, but we can make note of work in mathematics education that has taken up the general ideas of systemic functional linguistics (e.g., Atweh, Bleicher, & Cooper, 1998; Chapman, 2003; Morgan, 1998). For this study we focus specifically on the ideational metafunction—that is, the content of mathematics—though we recognize the value of research focused on other aspects of language and also recognize the artificial nature of isolating one metafunction from the others. That said, we, along with others (e.g., Steinbring, Bussi, & Sierpinska, 1998), feel that it is important to engage in forms of discourse analysis that focus on mathematical content and meaning.

Lemke (1990), who applied systemic functional linguistics to transcripts of science lessons, viewed language as "a system of resources for making meaning" (p. ix). From this perspective, language does not consist of mere grammar and vocabulary but is also seen as a semantic system or a system of meaning that allows us to create "webs of relationships" among and between ideas. As Lemke highlights, there are many ways to talk about ideas, but the underlying meaning or "pattern of relationships of meanings, always stay the same" (1990, p. x). It is through the patterns of semantic relations that meaning is construed. For example, a group of people may talk about the leg of a table using a wide variety of particular words and sentence structures, but the pattern across the particular instances will be the semantic relation that the leg is a part of the table (a MERONYM/HOLONYM relation). Lemke (1990) articulated a method for thematic analysis, described below, with the purpose of uncovering and examining such patterns. Examples of the semantic relations we drew on in this paper are displayed in Table 1.

*Table 1*
Semantic Relations

| Linguistic Terms | Description |
|---|---|
| MERONYM / HOLONYM | part of a whole |
| HYPONYM / HYPERNYM | subset of a set |
| EXTENT / ENTITY | space associated with an object |
| LOCATION / LOCATED | spatial relationship |
| SYNONYM / SYNONYM | equivalence relationship |

## Methodology

We employed Lemke's thematic analysis method as a lens that would bring into focus the mathematical content of a middle school lesson on area. The lesson comes from a sixth-grade classroom in an urban Midwestern middle school. The teacher, Robert, is elementary certified and had been teaching for seven years. Prior to the collection of the data presented here, Robert had not been a member of any professional organizations and the textbooks he used (there were several) can be described as conventional.

The analysis consisted of several stages. First, we selected the transcript excerpt from a larger corpus of classroom observations.[1] This selection was based on the pervasiveness of content terms used in the excerpt and the fact that a classroom observation of another teacher existed in our corpus that also dealt with the topic of area. (An article involving a detailed analysis of both of these transcripts is forthcoming). Second, we reviewed and filled in details (e.g., what "this" refers to) of the transcript. Third, we identified the content words that were central to the lesson and generated a *clean map* of the semantic relations between these terms. This clean map, which Lemke referred to as an ideal map, was based on our own mathematical understandings of the terms as well as the definitions of the terms presented in various mathematical textbooks. Fourth, we went through the transcript excerpt line by line and, for each occurrence of the identified content words, attempted to identify the semantic relation at play. Finally, we looked across the semantic relations of the transcript for thematic patterns and developed a *transcript map*. The processes of the analysis and the products of the analysis both contributed insights regarding the content of the interaction.

## Results

The mathematical terms that we identified for analysis were *rectangle*, *parallelogram*, *area*, *base*, *height*, *length*, *width*, as well as the non-technical term *bottom*. There is a HYPONYM/HYPERNYM relationship between rectangles and parallelograms since the former are particular instances of the latter. Area is an EXTENT of a polygon because it is a measure of two-dimensional space. The base of a rectangle or parallelogram is defined as one of its sides, thus forming a MERONYM/HOLONYM relationship involving base. In particular, a base is by definition a geometric ENTITY. The height of a rectangle or parallelogram, on the other hand, is defined as the distance between the given base and the line containing the opposite side and so is a quantity. This semantic difference is not always made explicit to students, as we shall see below, and may be confounded when the phrase "base and height" is used in a way that implies their interchangeability. Furthermore, the term *base* can also refer to the EXTENT of the base segment (as in the area formula "base times height" which calls for the quantities), and the term *height* can also refer to a line segment (often dotted) drawn between the base and the opposite side. Thus *base* is defined as a MERONYM of a rectangle or parallelogram but can also refer to an EXTENT (i.e., the length of the base segment). The term *height*, on the other hand, is defined as an EXTENT but is also used to refer to a geometric ENTITY (e.g., the dotted line from the base to the "top" of the parallelogram). This is captured in Figure 1, the clean map, by the fact that these terms appear twice under rectangles and parallelograms (with the term-as-defined above the other usage).

The semantic relations we have just described hold for both rectangles and parallelograms, which we see in Figure 1 because the overall semantic structures of rectangles and parallelograms with respect to area are quite similar. This structural similarity is one of the reasons that Robert, in the excerpt below, chose to teach parallelogram area by appealing to prior knowledge of rectangular area. There are, of course, important differences. For instance, *length* and *width* are terms that are associated with rectangles but not generally with parallelograms. Also, a side adjacent to the given base of a rectangle can be interpreted as a *height* (in the ENTITY-sense) of the rectangle, forming a MERONYM/HYPONYM relationship between the height and the rectangle. However, this relationship does not exist with non-rectangular parallelograms.

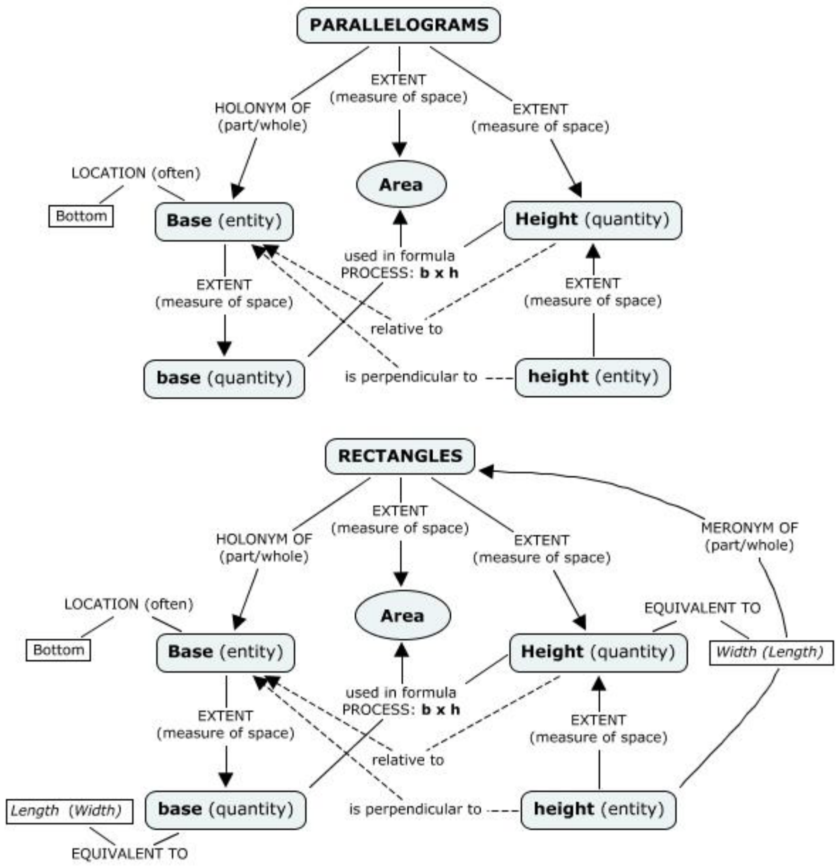


*Figure 1.* A clean map wherein rectangle is a HYPONYM of parallelogram.

Turning now to the classroom transcript, we join Robert after he has reviewed the area formula for rectangles and is about to transition to the development of the area formula for parallelograms. We use boldfaced text to draw attention to the content words in the transcript since these were a primary focus of our analysis.

Robert: OK, now it was important that you brought up parallel because the next one is **area** of a **parallelogram**. You just described to me a **parallelogram**, which was this: **rectangles** are **parallelograms**. [Student: Yeah, just turned.] **Parallelograms** have two sets of parallel sides. That's what a **parallelogram** is by definition. So what would you guess the formula would be for finding the **area** of a **parallelogram**? Adam?

Adam: *Length* times *width*.

R: *Length* times *width*. OK, aren't **rectangles--**, didn't we just decide that **rectangles** are **parallelograms**?

Ss: Yes. Yes we did.

R: We decided because I told you that (laughs).

S: Actually you decided.

R: OK, so using that, **area** would be equal to *length* times *width*. They call it a little different. Instead of saying *length* times *width*, they say **base** times **height**. They say **base** times **height**. [Ms: Face?] **Base**. We're going to make a **parallelogram** from our **rectangle**. On your picture of your **rectangle** I want you to make this…a diagonal line like this. [Draws on overhead a segment from the upper right corner of the rectangle to the interior of the bottom side.] We’re going to make a **parallelogram** from a **rectangle**. OK? Now, when I put this diagonal in what shape did I make?

Ss: Triangle.

R: Triangle. So what we’re going to do, we’re going to take this triangle off and we’re going to put it on the other side [draws a congruent triangle on the left side]. We’re going to put it over here. So if we have our **rectangle**, which is what we had before, and we had our three centimeters by five centimeters. We cut this section off and we just add it on to the other side. OK? So here is our diagonal.

S: That looks like a 3D figure.

S: That’s cool.

R: That’s how they’re getting your **parallelogram**.

From the first teacher turn (lines 1–5) we can identify several semantic relations. The first and last sentences of this turn imply that parallelograms have an area which, as was established in previous classroom interactions, is an EXTENT. Also, rectangles and parallelograms are related by the phrase “rectangles are parallelograms” in lines 3 and 7–8. By this the teacher means that all rectangles are parallelograms (i.e., the set of rectangles is a HYPONYM of the set of parallelograms), but the phrase is unclear because “are” can also be used to mean equivalence (e.g., rectangles *are* quadrilaterals with four right angles). Because of this ambiguity we will denote the relationship using the original term “are” in the transcript map (displayed at the conclusion of the third portion of the transcript). We also see in lines 3–4 a definition of parallelograms, with the implication being that a parallelogram is equivalent to or SYNONYMOUS with a figure satisfying the definition (i.e., containing two parallel sides).

Using reasoning based on the relationship between rectangles and parallelograms, Robert leads Adam to state the area formula of parallelogram, which the student describes as “length times width” (line 6). If we take “rectangles are parallelograms” to mean (correctly) that the set of rectangles is a subset of the set of parallelograms, then this reasoning about the area formula is flawed; the subset relationship would imply that rectangles have the same area formula as parallelograms but not the converse, as was assumed in the excerpt. If, on the other hand, we take “rectangles are parallelograms” to mean (incorrectly) that the two are equivalent, then their area formulas would necessarily be equivalent as well. In lines 12–13, the teacher modifies Adam’s statement to be base times height, noting that “they call it a little different” than length times width. This information from the teacher establishes a sort of equivalence between the terms *length* and *width* and the terms *base* and *height*. Lines 14–27 add another semantic relation between *parallelogram* and *rectangle* as we see that the former can be constructed from the latter. A rectangle, however, is already a parallelogram so what is meant is that a non-rectangular parallelogram can be constructed from a rectangle, but this is not explicit.

We continue in the transcript, picking up directly where we left off.

Robert: OK, now we said that the formulas were similar. We determined that because we said that the **rectangle** was a **parallelogram**, so we said that the formulas are similar. OK, this *bottom* section will be considered our **base** [shades in base on overhead]. OK, **base** is like *length*, so it is similar. That would be our **base** [points to the parallelogram on the projection screen]. Now, when we found perimeter of a **rectangle** what sides did we add? Or, what did we add together? If we were going to find the perimeter of this **rectangle** what would we add?

Students: The sides. The *length*.

R: Which would be what?

S: Fifteen.

R: Five, five, three, and three [points to the sides of the rectangle], correct? [S: Yeah.] So if we were going to find the perimeter we'd go around the outside. OK, this diagonal here is not the **height** of it [points to slanted side of parallelogram on screen]. It is on the outside but it's not the **height**. The **height** has to be perpendicular or at a right angle to the base.

This excerpt begins with the statement that the area formulas are "similar." We see in line 31 that base is "like" length and that they are also "similar." This is slightly weaker than the previous semantic relation, which implied that base and length were different references for the same thing. In line 30 the teacher states that the bottom of the parallelogram will be "considered" the base. The semantic relation in this situation is not clear, but there is some sort of identification or equivalence taking place between base and bottom. Furthermore, in lines 39–42 we see *height* for the first time and learn that it is not the slanted side of the parallelogram and so not a MERONYM. Its relationship to the base is articulated as one of perpendicularity. Since perpendicularity is a characteristic of actual entities and not quantities, this statement about the height implies that it is an ENTITY and not a quantity.

The relations involving height will be further developed in the next excerpt. Before then, however, we would like to point out the subtle semantic shifts occurring in lines 32–38. The teacher begins by asking what would be added in the calculation of a rectangle's perimeter, hinting in line 32 that sides are involved. Students respond that the sides are added. Semantically, however, it is not the sides themselves that are added but the EXTENTs or lengths of the sides. This subtle distinction is even more clearly confounded in line 38 when Robert points to the sides themselves but calls out their measurements. (Again, we do not wish to communicate that such action is negative, but merely to illuminate the fact that two different semantic relations are involved.)

We continue directly following line 42.

Robert: So if this was our **parallelogram**, the **height** would actually be this vertical distance here between these two lines [draws vertical segment in the interior of the parallelogram on the overhead]. OK, so this would be our **height**, from here down. Or it'd be from here down [draws vertical segment in the exterior of the parallelogram]. That would be our **height**. The **height** is actually just the distance between those two **bases**. So, what is the **area** of this figure here? Think about what we did with this [points to the cut-off triangle]. What does it have to be?

Student: Fifteen.

R: Fifteen. Didn't we just take this triangle and move it over here [points to rectangle]?

Ss: Yeah.

R: So doesn't the area have to be the same? [Ss: Yes.] Yeah, so we know the **base** is five and the **height** is three. These side *length*s may be different. They may actually be four or three point seven. They may be some other number. But the height has to be perpendicular to the base. It's got to be straight up and down for a parallelogram. OK?

We see in line 43 and again in line 47 that height is defined as a distance, which corresponds with the definition used in the clean map in Figure 1. This explicit discussion of height does not, however, correspond with the semantic relations in the previous excerpt in which height was characterized as a geometric ENTITY rather than a quantity. The notion of height and base as quantities reappears in line 54 when it is stated that "the base is five and the height is three." The notion of height and base as entities reappears in lines 55–56 when Robert reminds the class that "the height has to be perpendicular to the base." Thus this final turn is another example of the subtle and implicit shift in semantics involving the terms base and height. Figure 2 contains a transcript map based on all three excerpts discussed in this subsection.

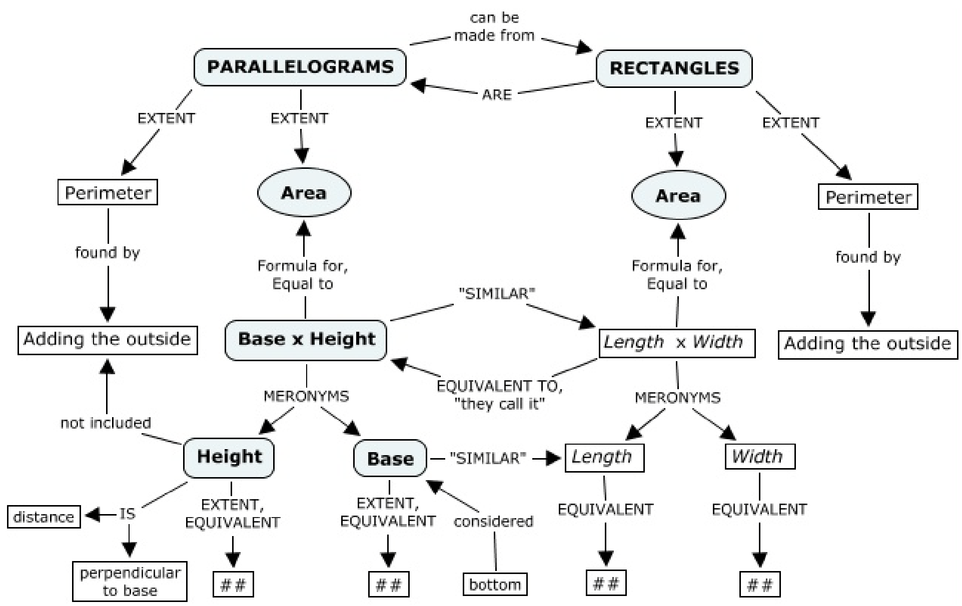


*Figure 2.* A transcript map based on Robert's excerpt.

**Discussion**

In this paper, we used thematic analysis to examine the semantic relations from a middle school lesson on the area of parallelograms. In generating the clean map of the relations between the pertinent mathematical terms (see Figure 1), we became acutely aware of the structural similarities between the semantic relations of rectangular area and those of parallelogram area, similarities that are often exploited in instruction as teachers and curriculum materials relate parallelogram area to rectangles or vice versa. The clean map also allowed us to see where the semantic differences lay between the two types of polygons, differences that could inform instructional decisions. Moreover, the clean map process exposed the difference between *base* (a geometric ENTITY) and *height* (an EXTENT) as mathematically defined, as well as the fact that the same terms *base* and *height* are used to refer to both physical segments and the lengths of those segments. Height has the additional distinction of having the semantic possibility of being a MERONYM of a rectangle but never of a non-rectangular parallelogram. All of these subtleties and possible points of student confusion appeared in the transcript from the sixth grade lesson. *Base* often referred to a side but was also used as a number and plugged into the formula (e.g., line 54). *Height* was explicitly defined as a distance (e.g., line 43) but then was referred to as the drawn-in segment and described as being perpendicular to the base (line 41); a property that only makes sense in reference to a geometric ENTITY. One point the teacher did try to be clear about was that the parallelogram's height was not its "diagonal" side. Perhaps this effort was a result of his experience with students thinking that height, because it is a characteristic of a polygon or because it seems interchangeable with base, is necessarily a MERONYM of that polygon. Indeed, all of these subtleties and implicit shifts in meaning involving height may be related to documented difficulties with the concept (e.g., Gutierrez & Jaime, 1999).

In summary, we have looked closely at the ideational metafunction of language from a lesson on area. We have seen that even the terrain of a topic such as parallelogram area, which appears smooth from a distance, can contain many potential potholes and pitfalls. If, however, researchers and teachers examine and come to better understand the structure and patterns of the semantic relations between mathematical terms, they can make mathematical language, and thus mathematics itself, more navigable for students.

## Endnote

1. This data was collected as part of an NSF grant (#0347906) focusing on mathematics classroom discourse (Second Author, PI). Any opinions, findings, and conclusions or recommendations expressed in this article are those of the authors and do not necessarily reflect the views of NSF. We would like to thank the teachers for allowing us to work in their classrooms.

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# LIMITATIONS OF APPEARING AS KNOWLEDGEABLE: AN EXAMPLE OF CONNECTIONS ACROSS MATHEMATIZING, IDENTIFYING, AND LEARNING

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*Identity has become a useful tool for making sense of students' mathematical learning. However, investigations of identity frequently consider students' stories told outside of the classroom. This paper proposes a lens on identity that examines how identity is enacted from moment to moment in the classroom, providing more opportunities to see how mathematical activity, identity, and learning are connected. Use of this lens revealed how one fourth-grade student's identification as someone who already knew limited her mathematical activity and had consequences for her mathematical learning.*

## Introduction

Many mathematics education researchers have turned to the notion of identity to make sense of connections across who students think they are, their mathematical activity, and their mathematical learning (e.g., Boaler & Greeno, 2000, Cobb, Gresalfi, & Hodge, 2009; Sfard & Prusak, 2005). This research has been productive in explaining variations in students' participation in mathematical tasks, in connecting affective factors to mathematical activity, and in establishing links between identity and mathematical activity. However, much of this research examines identity at a level that is removed from actual moments of mathematical learning. For example, researchers have asked students to reflect back upon their experiences in mathematics classes (e.g., Boaler & Greeno, 2000; Martin, 2000). While this delayed narrativization has yielded interesting connections between identity and mathematical activity, it does not examine identity in the moments of mathematical learning, which is when identity might operate on mathematical activity. Zooming in on mathematical activity as it unfolds from moment to moment during a lesson should provide significant opportunities to learn more about the relationship between identity, mathematical activity, and learning.

This paper proposes a lens for capturing identity in these moments of mathematical activity and then uses this lens to closely examine the identifying activity of one fourth-grade student as she worked with two peers to solve a mathematical task. In addition to illustrating how identity might be visible in activity, this case also explores how one student's persistent identification of herself as someone who already knew, in spite of her limited mathematical understanding, had consequences for her mathematical activity and her mathematical learning.

## A Framework for Zooming In: Connecting Mathematizing, Identifying, and Learning

### *Mathematical Activity*

Rendering identity visible at the scale of student activity requires a framework for studying student activity, both activity related to mathematics and activity related to identity, as it occurs during a lesson. Anna Sfard's (2008) commognitive framework provides tools for conducting this analysis. In Sfard's framework, the salient feature of mathematical activity is communication – communication among students and others and communication a student has with him/herself. Sfard focuses on communication because she defines thinking as communication with oneself. She notes that this communication does not need to be inner and it does not need to be verbal.

Defining thinking as communicating makes discourse the object of study, where discourse includes gestures and other nonverbal or nonlinguistic means of communicating along with what might be spoken or written. In the commognitive framework, participation in discourse pertaining to mathematical objects, whether mathematically appropriate or not, is called *mathematizing*.

The commognitive framework also provides a means for making claims about learning. Defining thinking as communicating suggests that learning, as an outcome, can be defined as a change in discourse. As students learn, they are working (even if tacitly) to change their communication with themselves and with others. The effectiveness of the process of learning can be evaluated by comparing the resulting change in discourse with the mathematically desirable discourse.

*Defining Identity*

A definition of identity must link to this framework for mathematizing and it must make identity visible as students engage in mathematizing. The definition proposed by Sfard & Prusak (2005) meets both of these conditions. They define identity as a collection of reifying, significant, and endorsable narratives about a person. They elaborate on this definition:

> The reifying quality comes with the use of verbs such as *be, have* or *can* rather than *do*, and with the adverbs *always*, *never*, *usually*, and so forth, that stress repetitiveness of actions. A story about a person counts as *endorsable* if the identity-builder, when asked, would say that it faithfully reflects the state of affairs in the world. A narrative is regarded as *significant* if any change in it is likely to affect the storyteller's feelings about the identified person. (p. 16-17, italics in original)

For example, statements like "I am a woman" or "I am a math person" are identities for me: I believe them to be true (or endorsable) and important (or significant) to how I feel about myself. In addition, these two statements are *reifications*: They no longer reflect actions I might have performed, such as wearing particular clothes or enjoying a mathematics class. Instead those actions are summarized and frozen into a label. Finally, these statements are also *narratives*.

My use of narrative draws upon the work of Ochs and Capps (2001). They studied narratives arising in everyday conversations, investigating how impromptu, co-constructed narratives help people understand themselves and others with whom they interact. They defined narrative as an account of life events. While some narratives can be quite extensive, Ochs and Capps specifically noted that narratives need not be lengthy: Even short sentences can also be narratives. Dino Felluga (2003) concurred, arguing that statements like "The road is clear" are narratives because they suggest some sequence of events. Similarly, statements like "I am a math person" are narratives because they conjure images of what a person has done or might do.

The connection between minimal narratives and potential actions captures an essential function of identities: They explain and predict an individual's activity. Their ability to do this is tied to the ways in which identities reify or freeze activity. According to Etienne Wenger (1998), reification is "the process of giving form to our experience by producing objects that congeal this experience into 'thingness'" (p. 58). As people tell stories, they tend to summarize the story's action into statements that describe or label people. These statements convert activity into human conditions. For example, a story about a high score on a math exam might become a reification through the statement, "Daren is smart." The story is no longer about one moment in Daren's life. Instead, the score on the test has been translated into an identity for Daren.

Thus far, I have discussed identifications as narratives that are directly uttered. However, individuals can also be identified through their actions and interactions. For example, the teacher could identify Daren as smart by asking him to share his work with the class. This request might specifically include a statement that Daren is smart or his smartness might be implied in the request and the teacher's subsequent reaction to his work as he displays it. This framing of identifying activity draws upon positioning theory. Positioning theory assumes that people use stories and position themselves and any others involved in the activity in the story (van Langenhove & Harré, 1999). Instead of relying only on reified narratives, identification through positioning is based upon the cluster of utterances and actions that are appropriate for that person in context of the story. For example, as the teacher invites Daren to share his work, she positions him through her words and accompanying gestures as smart.

Positioning theory also provides a means for individuals to negotiate their identities. Daren might agree with the identification of him as smart and as someone others can learn from, or he might think that someone else in the class has a better solution and that if he takes the risk of presenting, he will look stupid and not smart. He could dispute this identification of him as smart by refusing to present his work and/or pointing to someone else who he thinks has a better answer. Daren is not bound by this identification of him: His reactions can affirm or refute any identification of him.

Sfard's (2007) recent work on identity elaborates ways of identifying that are consistent with the indirect story-telling assumptions of positioning theory. Sfard has described three ways in which people identify: direct, indirect-verbal, and enacted. *Direct* identifying occurs as a person tells a reifying story about the identified person. I primarily referred to this way of identifying as I elaborated the definition of identity in the sections above. *Indirect-verbal* identifying is when a story is told about a person that does not include reifying statements. Finally, a person may identify him/herself or another through other activities that do not include story telling. Sfard calls this type of identifying *enacted*. As students interact in classrooms, they rarely directly identify each other. Instead, most identifications are enacted or in-direct verbal. Positioning theory's use of story to provides a framework for elaborating these last two ways of identifying.

Positioning theory also supports the specific vocabulary I will use to talk about identities. The notion of positioning emphasizes that identities are constructed by individuals in response to situations and are thus situated and dynamic. Rather than describe individuals as having identities, I draw upon Sfard's (2007) recent work and use the words *identifying, enacting identities, identification,* and *engaging in identifying activity* as a means of emphasizing the ways in which identities are constructed from moment to moment.

*Theoretical Links between Mathematizing and Identifying*

This framework of identifying has many parallels with the description of mathematizing. Because identifying arises from communication of stories, it, like mathematizing, arises in discourse. Both are also central to a definition of learning as a change in discourse: As a student engages in learning, the student both communicates about mathematics and identifies him/herself as a kind of learner of mathematics. His/her communication about mathematics simultaneously tells an identity story and provides the opportunity of changing the learner's mathematical discourse. These connections between identifying and mathematizing suggest that learning arises from the interplay of these two activities (Sfard, 2007). It is the goal of this paper to elaborate the interplay among identifying, mathematizing, and learning. Specifically, this paper explores the effects of the activities of identifying and mathematizing on one another and on the development of mathematics discourse.

## Background and Methods

The student discourse analyzed in this paper comes from one group of fourth grade students interacting during one mathematics lesson. This case was part of a larger study involving over 70 hours of videotape and accompanying student work arising from 28 mathematics lessons in one fourth-grade classroom. The videotapes were analyzed for evidence of student learning: For each lesson, student’s initial discourse was compared to their final discourse using four discursive features elaborated by Sfard (2008): word use, visual mediators, discursive routines, and endorsed narratives. When there were mathematically desirable changes between the student’s initial and final discourse along these four features, the video was further analyzed for mathematizing and identifying activity. The entire lesson was transcribed, capturing not only what was said but also any writing, gestures, and other nonverbal communication. Portions of the student discourse that involved mathematical topics were parsed into message units (Bloome, Carter, Christian, Otto, & Shuart-Faris, 2005).

For each message unit, the identifying analysis considered the range of possible meanings for the speaker and the listeners, seeking to answer questions about what story the speaker and listener(s) might construct from the unit and how each person might be identified or might identify him/herself within that story. Units that identified the focal student as a learner in a consistent way were grouped and labeled as a kind of learning.

The mathematizing activity of each message unit was also analyzed. This analysis focused on the ways in which the discourse of each member of the group resembled desirable mathematical discourse. This analysis used Sfard’s (2008) four discursive features as described earlier. As patterns of mathematizing emerged, message units were grouped and compared to the analysis of identifying activity. Message units that demonstrated patterns across mathematizing and identifying activity were grouped and labeled as a kind of learning. Finally, moments in which the student enacted these different kinds of learning were examined to determine their connection to mathematically desirable changes in the student’s discourse.

I have limited the findings in this paper to one group of students interacting during one lesson. This lesson and this group were selected because the interactions among the students and the mathematical learning of one student demonstrated important aspects of the interplay among mathematizing, identifying, and learning. While some features of the interactions in this group were typical of interactions observed in the classroom, many of the specific details of the activities in this group were unique to this group and this lesson. Thus, the moments of learning examined in this paper reflect the possibilities for activity and learning among students without making any claims about predicting activity or learning in other groups or for other students.

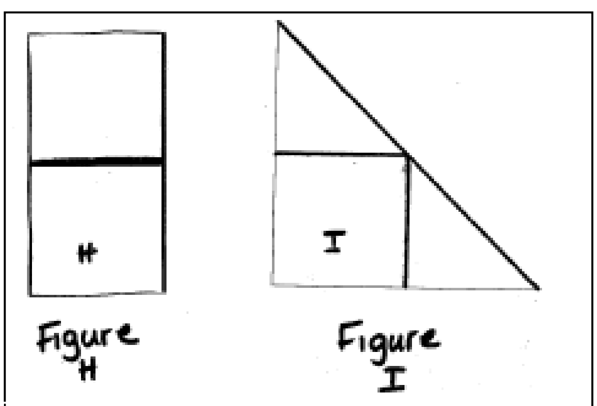


*Figure 1.* Scanned image of Figures H and I.

The group examined in this paper consisted of three students: Minerva, Bonita, and Jessica. The analysis in this paper focuses on Minerva’s learning and how the mathematizing and identifying in the group were connected to that learning. The lesson focused on how to determine the area of triangular figures. The students were given pairs of figures, one triangle and one rectangle (See Figure 1). The students were to determine which figure “covered more area.”

Each figure was partitioned by a square grid placed so that rectangle was partitioned into squares while the triangle (which had the same area as the rectangle) contained spaces that were squares and triangles.

## Findings

Throughout this lesson, Minerva was careful to identify as knowledgeable and in possession of right answers, although this identification stands in contrast to an analysis of her mathematical discourse, which suggests that her mathematical understanding was limited. What was remarkable about Minerva was not her efforts to be seen as knowledgeable and having the right answer – many students work to do this – but instead her skill at using this identification to mask her learning. This *covert learning* – presenting as knowledgeable and hiding learning – had consequences for the effectiveness of Minerva's learning. Her efforts to appear as knowledgeable meant that she worked on when and what to say, but not on exploring the mathematical discourse or probing the validity or consequences of the mathematical statements she uttered. As a consequence, Minerva's verbal discourse frequently seemed mathematically appropriate, but she was unable to construct and explain an appropriate solution to the task. This case of Minerva demonstrates how a focus on right answers and on identifying as knowledgeable can undermine opportunities to fully engage and thus learn mathematical discourse. The findings below provide one extended example of Minerva's covert learning followed by a summary of her other mathematizing, identifying, and learning activities.

Minerva's desire to appear as knowledgeable was most apparent when she resisted her peers' identification of her as someone who needed to learn. This moment occurred early in her group's work to compare the areas of Figures H and I (See Figure 1). The teacher asked the group how they would compare the areas. (Each excerpt from the transcript indicates the line number from the transcript, the speaker, any verbal discourse, and finally, in italics, any nonverbal communication or activity.)

200 Bonita Both the areas, they both have two.
201 Teacher How do you know?
202 Bonita Because two triangles make a square. *Bonita points at the 2 triangles in Figure I.*
203 Minerva No it doesn't
204 Bonita Uhhunh. *This is an affirmative utterance.*
205 Jessica Yes it does!
206 Minerva Nah hunh *This is a negative utterance, said in a sing-song way.*

Minerva's "No, it doesn't" (Line 203) was a direct contradiction of Bonita's statement that "two triangles make a square" (Line 202). Minerva may have disagreed that two triangles made a square or that the information was helpful in determining the area of the figures. In either case, it seemed that Minerva was not thinking about how triangles and squares would be useful in determining the area of the figures. Because successful completion of this task required students to make sense of the relationship between the triangles and the squares, Minerva's utterance in Line 203 demonstrated that she needed to learn in order to complete this task correctly.

As the interaction continued, Minerva's peers identified her as someone who needed to learn and as someone who could learn from them.

209 Jessica Let me see, let me show you
210 Bonita You cut this right off and put it there. *Bonita is holding Figure I. She points to one triangle in Figure I and motions next to the other triangle in the figure. Minerva watches her as she does this.*

211 Minerva Nah hunh
212 Jessica Let me see. See. Look it. Let me show her that it can make a triangle. *Jessica picks up two copies of Figure I. Jessica seems to misspeak here, uttering "triangle" instead of "square."*
213 Minerva Nah hunh. *Minerva watches Jessica.*
214 Jessica See. Look it. *Jessica places both figures together to make a large square.*
215 Minerva Nah hunh, nah hunh, nah hunh

In this excerpt, both Jessica and Bonita explicitly positioned Minerva as a learner and themselves as teachers. Bonita's demonstration of how to rearrange the figure (Line 210) and Jessica's insistence on showing Minerva how to make a square (Lines 209, 212, and 214) communicated their understanding that Minerva did not understand something they both understood.

Rather than respond with questions about the explanations, an acknowledgement of understanding, or a definite denial of the role of learner, Minerva reacted to Bonita and Jessica's positioning with a persistent, mocking "Nah hunh." Minerva countered each utterance of Bonita and Minerva with "Nah huhn," continuing for a full minute after the interaction above and uttering this phrase 32 times in total. Her "Nah huhn" had a sing-song quality and mocking tone that seemed to communicate that Bonita and Jessica's explanations were unnecessary and trivial. It seemed to be an attempt to portray her initial disagreement in Line 203 ("No it doesn't") as a jest or a joke and suggested mockery of the serious, explanatory tone used by Bonita and Jessica. Through her repeated utterance of "Nah huhn," Minerva seemed to communicate her rejection of the identity of learner.

In spite of Minerva's work to reject this identity, the timing of her utterances and her attention to Minerva and Bonita's activity provided her with an opportunity to learn from them. Minerva could have interrupted or ignored Bonita and Jessica. Instead, she carefully watched their demonstrations and timed her "Nah huhn" so they punctuated rather than disrupted Bonita and Jessica's explanations. Furthermore, in spite of Minerva's disagreement with Bonita's statement that "two triangles make a square," Minerva adopted this statement in her subsequent discourse. She incorporated these words into her first solution to the problem and she repeated these words on two separate occasions. In addition, all but one of Minerva's solutions to the task included the arrangement of the two triangles into a square. Minerva's use of Bonita's statement and arrangement of triangles suggests that Minerva learned from Bonita even while she resisted her identification as a learner.

This covert learning was not an isolated incident: There were four occasions in which close observation of Minerva revealed that she was unsure about the solution or how to proceed in the task or that she was not yet articulating the teacher-approved response to a question. In each of these instances, rather than admit that she didn't know, Minerva carefully and unobtrusively observed her peers and the interaction between her peers and the teacher. She then repeated the discourse of her peers that had the approval of the teacher. For example, Bonita stated and the teacher concurred that the areas of Figures H and I were the same. Eight turns later, Minerva articulated this same idea for the first time. Her use of this idea was in contrast to her previous solution which did not show the areas of the two figures as equivalent. Thus, Minerva seemed to be listening and learning from the interactions occurring at her group and using this learning to appear knowledgeable, but she was not overt about this learning.

While Minerva frequently repeated the discourse of others, her use of their discourse was limited. For example, she was able to use Bonita's statement that "two triangles make a square" to appropriately respond to the teacher's question about the area of Figure I and she cut out and connected the two triangles from Figure I into a square. However, one of her initial solutions to the task focused on demonstrating that the two triangles made a square and not on how that information contributed to a comparison of areas. Her final solution also placed the two triangles into a square, but, rather than compare areas, Minerva joined all of the pieces of both figures together and counted the total area, recording an answer of “4.” While four was the count of the area of both figures in terms of square units, this answer failed to compare the areas of the two figures. Minerva’s later work also demonstrated her understanding of area as an operation in which two figures were combined and total area was determined, rather than understanding area as a property of each figure that could be compared.

This limited understanding of mathematical discourse may have been tied to Minerva’s limited use of mathematical discourse. She was very skilled at identifying and repeating mathematical statements that were valued by the teacher, but she only used these narratives to reply to teacher questions. Her mathematical discourse was limited to brief responses, almost exclusively uttered only in the presence of the teacher. She did not talk mathematics with Bonita or Jessica. She did not explore the solutions of her peers, explain her own work (even when asked to do so by Jessica), or ask questions about the mathematics. Indeed, she asked only two questions throughout the lesson, both at the beginning of the lesson: one about the context of the problem and one about what to do. Her limited use of mathematical discourse provided her with few opportunities to explore the mathematical words, connect the words to representations, or examine the verity of the statements she used.

Perhaps if Minerva had been willing to identify as a learner, if she had been less insistent about appearing as knowledgeable and more curious, she might have done more exploration of the mathematical discourse. These changes in her mathematizing and identifying may have resulted in more effective learning.

## Conclusion

This case of Minerva’s covert learning demonstrates how mathematizing and identifying are intertwined and have consequences for learning. Minerva’s mathematizing communicated her identification as someone who knew answers: She could successfully respond to the teacher’s questions and she did not ask questions or request explanations. Minerva reinforced this identification through nonmathematizing moves when she resisted Jessica and Bonita’s identification of her as someone who needed to learn mathematics.

While these mathematizing and identifying moves were meant to demonstrate competence and knowledge, they limited Minerva’s mathematizing such that the changes in her mathematical discourse were primarily limited to adoption of the mathematical statements uttered by others. As a consequence, her mathematical learning was limited. She incorporated some mathematically desirable statements (i.e. “Two triangles make a square”), but her final discourse (which employed combining rather than comparing areas) was mathematically troublesome. Perhaps if Minerva had identified in a way that allowed her to ask more questions (of her peers or her teacher) or to explain her work to her peers, she would have engaged in more mathematizing and perhaps experienced more desirable learning.

Minerva’s identifying and mathematizing also had consequences for the learning of her peers. As she discouraged mathematical conversations between herself and her peers, she also limited the mathematizing and consequently the learning of her peers. Thus, Minerva’s activity has implications for mathematics pedagogy and curricula that emphasize group work and learning from conversations among peers. This case seems to imply that if students are to learn from each other, they must be willing to overtly identify to each other as learners.

This case of Minerva provides a perhaps extreme example of the consequences of focusing on right answers. It suggests that as teachers and schools seek to support students in learning mathematics, they may want to support students in valuing mathematizing and identifying activities that demonstrate questioning and exploring and a willingness to learn rather than quickly knowing. Eleanor Duckworth (2006) notes, “What you do about what you don’t know is, in the final analysis, what determines what you will know” (p. 67). How students mathematize and identify as they engage in mathematics will have consequences for the mathematics they learn.

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# CONSTRUCTING OPPORTUNITIES TO LEARN: AN ANALYSIS OF TEACHER MOVES THAT POSITION STUDENTS TO ENGAGE PROCEDURALLY AND CONCEPTUALLY WITH CONTENT

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## Introduction

This paper contributes to a body of literature that is concerned with supporting our understanding of how to facilitate whole-class mathematical discussions. In this paper, we share an analysis of four fifth grade mathematics teachers who are teaching with a worksheet that is designed to target students' understanding of fractions. Our analysis targets the relationship between particular teacher moves and the nature of mathematical ideas that get on the table in the classroom discussion. This paper makes a unique contribution because it looks across multiple classrooms in order to examine the nature of particular teacher moves that impact whole-class discussions.

Whole-class discussion is a key part of reform mathematics classroom practice, but one that is challenging and takes some time to master. As such, orchestrating whole-class discussions has been the topic of significant attention, although as a field we are still working to document practices that teachers can leverage which consistently support high-quality discussions. More specifically, while it is clear that the culture of the classroom significantly impacts the kinds of conversations that can take place amongst students (Cobb, 1999), teachers play a key role in the facilitation of whole-class mathematical discussions. Even within classrooms that create cultures of discussion, the difference between conversations that build to key mathematical understandings (as opposed to those that result in merely sharing ideas) appears to be related to the particular moves a teacher makes in the course of the conversation, and which create opportunities to engage with mathematics in particular ways.

Key challenges involve the tricky business of tracking the nature of the mathematical topic under discussion, the potentially fruitful and potentially unanticipated sidepaths, the sites to locate and address misconceptions, and the orchestration of students' own emotions and agency as contributors to these discussions. In her book *Teaching problems and the problems of teaching,* Maggie Lampert (2001) described the myriad conflicting goals that she often negotiated in the course of a single 10-minute discussion. Likewise, Deborah Ball (1993) has described the challenging task of unpacking the key mathematical ideas at play in the course of a discussion, and knowing how to most productively step in to either support the trajectory of the conversation, or redirect it.

This paper contributes to the growing body of literature that focuses on the relationship between particular teacher moves and mathematical meaning-making (Ball, 2001; Gravemeijer, 2004; Stein, Engle, Smith, & Hughes, 2009). In the work we present here, we focus specifically on the ways teachers' moves *position students* as mathematical meaning-makers, and *position content* as rules to be remembered versus ideas to be interrogated. Below, we unpack the idea of positioning and explain how it is used in this analysis. We then detail the methods and results of our study, and discuss implications for future work.

## Positioning in Practice

In this work, we leverage the notion of *positioning* as a lens for analysis. Positioning, as a mechanism, helps to bridge the space between the opportunities that are available for participation in particular ways and what individual participants actually do. In our work, we have examined two aspects of positioning: how students are positioned relative to content (disciplinary positioning), and how they are positioned relative to others (interpersonal positioning) (Greeno & Hull, 2002; Gresalfi, in press). For example, students are positioned relative to the discipline of mathematics through the ways in which content is organized to afford particular mathematical insights and understandings. This speaks not only to the design of different tasks, although design of course creates significant opportunities (Stein, Smith, Henningsen, & Silver, 2000), but also to the ways content itself is treated in conversation as a series of rules to be learned (*procedural engagement*) or a set of ideas to be interrogated (*conceptual understanding*) (Gresalfi, Barab, Siyahhan, & Christensen, in press), Likewise, practices around *doing* mathematics can create opportunities for students to become positioned relative to each other, such as being expected or obligated to convince or constructively challenge someone else (Carpenter & Lehrer, 1999; Lampert, 1990). In so doing, both what it means to do mathematics, and who is capable of engaging mathematics, is constructed.

Positioning takes place at two levels--as a moment-by-moment process through which particular students are given opportunities to participate in particular ways (Davies & Harre, 1999), and over time, as students become associated with specific ways of participating in classroom settings (e.g., (Holland, Skinner, Lachicotte, & Cain, 1998); (Wortham, 2004)). At the moment-by-moment level, positioning can be seen most easily in talk, as students and the teacher speak to each other about academic content, about themselves, and about their current work (van Langenhove & Harre, 1999)). When looking over slightly longer time periods, students can be positioned as being certain kinds of people through the emergent *participant framework* (Goffman, 1974; Goodwin, 1990; Herrenkohl & Guerra, 1998; O'Connor & Michaels, 1996)) of a classroom which shapes the ways that students are expected, obligated, and entitled to participate with content and with others in the classroom. For this paper, we consider two types of positioning:; the ways that students are positioned relative to others, or their *interpersonal positioning*; and, the ways they are positioned relative to content, or their *disciplinary positioning.*

*Interpersonal Positioning*

The idea of interpersonal positioning is closely aligned with van Langenhove and Harre's (1999) work on positioning, which concerns the fluid positions, roles, or characterizations that people make available for themselves and others through their talk. As Holloway (1984) states: "Discourses make available positions for subjects to take up. These positions are in relation to other people. Like the subject and object of a sentence…women and men are placed in relation to each other through the meanings which a particular discourse makes available" (p. 236, quoted in van Langenhove & Harre, p.16). Interpersonal positioning refers explicitly to the ways that students recognize themselves and others in relation to one another, for example, how are students expected or obligated to talk to each other? To challenge each other's ideas? Do some students become positioned as more or less competent than others?

One central way that students get positioned relative to one another involves their relative status—including aspects of status that are not necessarily relevant to academic work (Cohen & Lotan, 1995)). Interpersonal positioning also includes implicit and explicit comparisons between students, and between their ideas. Students' ideas might be positioned as equally valuable (even if they are not both accurate, c.f. (Lampert, 1990)), or one idea might be positioned as less important than another—for example, in a classroom that is more competitive. In some classrooms distinctions between who is “smart” and who isn’t may become a particularly dominant form of positioning, while in others, such distinctions are rarely made. In this paper, we attended to teacher moves that positioned students relative to each other in terms of their mathematical discussions; for example, teachers might position students as (productive) “critics” by asking them to “listen to this strategy and see what you agree with and what you disagree with.” Likewise, teachers might position students as “comparers” by setting them up to “see if your strategy is similar or different to the one you hear.”

*Disciplinary Positioning*

Students are also positioned relative to particular subject matter. Disciplinary positioning is related to both the affordances of the mathematical tasks, and the way those tasks are realized in the classroom. For example, in some classrooms, students are positioned primarily as *receivers* of knowledge (Boaler & Greeno, 2000). In such a classroom, students have opportunities to record and practice particular mathematical procedures, but possibly not to engage in acts of deeper meaning-making. In contrast, other classroom practices position students more actively and create opportunities for them to *construct* new information. In such spaces, students become both active consumers and producers of new knowledge and are positioned as having authority in justifying and determining the accuracy of their solutions (Gresalfi, Martin, Hand, & Greeno, submitted).

For example, Lampert (1990) describes an effort she undertook in her own classroom to position students to be both *courageous and modest* in their mathematical activity. By positioning students as courageous relative to the content—taking risks in sharing information that they weren’t necessarily confident about, and modest about their proposals (in other words, being ready and willing to accept suggestions and revisions to their ideas)—Lampert was positioning her students relative to the content of mathematics in ways that supported deep engagement with the subject matter. As a consequence, both the ways that students engaged with the content—sharing mistakes, listening to and offering suggestions about others’ work—and the actual content with which students were engaging—thinking about proof or rationales behind why particular decisions were meaningful—changed from the beginning to the end of the year. In this paper, we attend to teacher moves that positioned students relative to content by tracking utterances that positioned content as either “stable” or “shifting,” by noting the kinds of opportunities to engage content that were offered to students. For example, when teachers tell students information without asking for challenge or feedback, they are positioning content as unmalleable and students as receivers of information. In contrast, when teachers ask students to justify why a solution is sensible, they are positioning content as open to challenge, and students as active meaning-makers.

## Methods

The topic of this paper is four fifth-grade teachers and their students in a suburban school in southern Indiana. The data for this paper comes from one day of instruction in each of these four classrooms, during which time all students were working on the same worksheet about multiplying fractions. The worksheet was designed to promote discussion by posing open questions that could be answered in more than one way, and by asking students to explain to each other how they knew their answers were sensible. Specifically, the worksheet began with open-ended problems that targeted multiplication of fractions, and ended by asking students to explain if the fraction multiplication algorithm would work every time. These four teachers were participants in a larger project whose purpose was to support both small group and whole-class discussions about mathematical content. As a part of the project, all teachers had participated in four full-day professional development sessions whose purpose was to highlight key mathematical ideas with which students were grappling, and to devise strategies to support student discussion. The strategies that were developed in the professional development sessions primarily targeted key mathematical ideas that students might understand or have misconceptions about.

All four teachers used this worksheet in conjunction with the same lesson from their usual textbook, Everyday Mathematics. All teachers had their students begin by working collaboratively in groups of 3-5, and concluded with having students share and discuss their solutions collectively with the entire class. Beyond this common structure, teachers were free to discuss the worksheet however they pleased; as an example, some teachers had a member of each group share their solution while others selected particular students to share their work. Some teachers had common expectations for group work as a coherent part of their classroom practice, while others used collaborative group work less often in their everyday practice. Each teacher had between 25-30 students in their classroom. All classes were heterogonous, and the small groups were arranged heterogeneously.

Data was collected using three video cameras; two of which were focused on a small group, and the third of which followed the teacher. Data for this paper considers only whole-class discussion times, and draws from the teacher camera. All videoed segments of whole-class discussions were transcribed and entered into a qualitative database (NVivo8). Codes were developed using both *a priori* and emergent methods; drawing from previous work that targeted productive teacher moves, we began with a list of codes that we thought would be useful descriptors of teachers' moves. After watching the videoed segments of classrooms while using these *a priori* codes, additional codes were developed in order to capture emergent themes. A final coding scheme consisting of both forms of codes was then refined, and the codes were used on all four transcripts. The coding scheme documented both the kinds of teacher moves that were undertaken (for example, press, emphasizing expectations, telling), and the nature of the mathematical ideas those moves targeted (for example, making connections, procedural engagement, conceptual engagement) All coding was undertaken collaboratively between the first and second author, and disagreements between codes were discussed until agreement was reached.

## Findings

Our analyses revealed, unsurprisingly, that teacher moves that emphasized procedural aspects of mathematics were related to procedural mathematical ideas being introduced, while moves that emphasized conceptual aspects of engagement led to conceptual aspects of mathematics being shared. Specifically, 100% of teacher moves that emphasized procedural aspects of mathematics led to mathematical discussions that focused on procedures. An example of such an exchange (coded as "set expectations" "justify" and "procedure") is below. It begins with the teacher giving her students instruction on what a good drawing would look like in order to be convincing. The problem that the students were working on involved taking 2/3 of ½ of a pan of brownies. Some students divided the pan of brownies into halves, and then divided only one side of the pan into thirds, and arrived at the answer of 2/6. The teacher was concerned that students cut up the entire pan of brownies in order to "prove" that their answer is correct:

T: So, as you guys continue on, have you realized that once are working with this whole, if you divide it into more pieces equally, do you just do a part of it? Do you just divide a part of it? Will that work? It kind of worked, because you guys are smart enough that you can visualize and even though a lot of you did not divide both halves into sixths, you're smart. And you're like, "I know that's two-sixths, because you can see it in your head. But, as you get into more complex problems, you may not be able to see that so easily. So, it's very important that when you- that you keep that whole in equal-sized pieces; otherwise, our fractions aren't balanced."

This utterance was coded as procedural because although the teacher was focused on helping students to think that they needed to prove their answers, the source of the proof could be seen in students' accurate use of procedures; in this case, ensuring that they divided an entire figure equally, instead of visualizing the division of a figure into pieces. In this utterance there was little justification of why this was more convincing, except for the idea that it would help when problems became more complex. In this way, the utterance served to position students relative to content as mere executers of algorithms; they were not asked to critique solutions, but rather to ensure that they were able to perform solutions accurately. This positioning created opportunities for students to engage procedurally with content. As can be seen in the following exchange, procedural mathematical content was discussed following this utterance. In the exchange below, students were solving the next problem on the worksheet; one that involved taking 1/3 of ¼. A student had just come up to share her solution, and the teacher claimed that this solution was a wonderful proof:

T: Why does this prove it to me? Why does this prove it? Talk to me, guys. Talk to me. Why does this prove it? When I look at that entire pan of brownies, what do I see?

St: She divided it into fourths and then thirds?

T: OK, but what do I see? I'm the accountant. I don't know one thing about serving brownies. I just have to know how many- I have to know for sure how many brownies were sold, so I can keep my accurate records for your business. How do I know how many brownies that pan was cut into? Talk to me!

St: She divided it into twelve equal parts?

T: It's in twelve equal parts!

In this exchange, the teacher and the students were focused on the appearance of a problem, and a justification that focused on procedural aspects of understanding. Thus, the mathematical content that got established as a topic of conversation involved a procedure for proof: divide a figure into equal parts. These coupled exchanges detail the relationship between the ways that students are positioned to engage content, and their resulting opportunities to learn that content.

Likewise, students who were positioned to engage conceptually with content had opportunities to participate in discussions that focused on conceptual aspects of doing mathematics. Almost 65% of teacher moves that were classified as conceptual led to mathematical discussions that focused on conceptual understanding. An example of a teacher utterance that was coded as pushing on conceptual understanding can be seen below:

T: OK. Turn to that problem on the back page real quickly, 'cause that's sort of where the connection was. How can you multiply and end of with a smaller amount than you started with? How does that work? 'Cause normally if you multiply two times three, you're going to get six. How come we're multiplying, and we're getting something smaller?

In this utterance, the teacher focuses on the conceptual aspect of mathematics by asking students to consider why things work as they do. Thus, they are focused on justifying, rather than following rules. Specifically, students are positioned as investigators whose role is to make sense of mathematics. This positioning created opportunities for students to engage conceptually with content, as can be seen in the following exchange that led to conceptual mathematical content being discussed in the classroom. In the exchange below, students were working on answering the teacher's question of why multiplying by fractions makes things smaller:

St: Um, well, because fractions are less than one, so, um, like, um, multiplying is like, I mean, um, like, three times two. Three two times is six.
T: OK.
St: Um, so, one times point five, because point five is a decimal and decimals are (like) fractions-
T: Mmm-hmm.
St: That would be, um…one times a half and that's (a half)

In this exchange, a student was focused on trying to justify and explain how mathematics works. Although she did not produce a complete explanation, the topic of conversation shifted to justifying, rather than following rules. As could be seen in the procedural exchanges, these coupled exchanges detail the relationship between the ways that students are positioned to engage content, and their resulting opportunities to learn that content. Specifically, our analysis documents that teachers' moves which position students as meaning-makers lead to opportunities to engage in activities of meaning making, while moves which position students as following rules lead to opportunities to merely enact those rules.

Because our interest in this work is on better understanding how whole-class conversations lead students to have opportunities to engage mathematical meaning-making, our analysis has also detailed the specific moves that teacher make that are likely to support such meaning-making. Although space prevents us from detailing these practices in this paper, our presentation will detail the specific moves that were associated with conversational exchanges that focused on mathematical meaning-making.

## Conclusions

Understanding how students come to participate knowledgably with a domain and to see themselves as capable of doing so requires renewed attention not only to what students do, but to what they have opportunities to do. By attending closely to these opportunities, it is possible to see how students are positioned in moments of interaction relative to aspects of classroom practice. In this paper, we consider two types of positioning: how students are positioned relative to content (disciplinary positioning), and how they are positioned relative to others (interpersonal positioning). For example, students are positioned relative to the discipline of mathematics through the ways content is organized to afford particular mathematical insights and understandings. This speaks not only to the design of different tasks, but also to practices around doing mathematics. These aspects of mathematical practice can create opportunities for students to become positioned relative to each other, for example, as being expected or obligated convince or constructively challenge someone else. In so doing, both what it means to do mathematics, and who is capable of engaging mathematics, is constructed.

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# TEACHERS REFLECTING DIFFERENTLY: DECONSTRUCTING THE DISCURSIVE TEACHER/STUDENT BINARY

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*This session explores the ways that practicing teachers came to reflect differently regarding the discursive teacher/student binary during a graduate-level course entitled "Mathematics Education within the Postmodern." Using Dewey's concept of reflective thinking, as well as Foucault's discourse and Derrida's deconstruction, we show how the course provided new suggestions for the students as they continued their journey of becoming teachers. Through interweaving comments written by the students with concepts borrowed from postmodern philosophers and theorists, we illustrate how the teachers began to understand that teachers and students might indeed be described differently in the postmodern.*

## Introduction

Most, if not all, mathematics teachers, educators, and policymakers would agree that the documents produced by the National Council of Teachers of Mathematics (NCTM) over the past 30 years describe a different mathematics classroom than that which is experienced by most students in U.S. schools (see, e.g., NCTM, 2000). Although the impact of these documents in reforming mathematics teaching has been somewhat limited (see, e.g., Wilson & Goldenberg, 1998), research has shown that these documents have had an impact on how mathematics teachers define and practice "good" mathematics teaching (see, e.g., Wilson, Cooney, & Stinson, 2005).

Wilson, Cooney, and Stinson's (2005) research on the perspectives of seasoned mathematics teachers about good teaching suggests that efforts to reform mathematics teaching are seldom all or nothing affairs. Their research illustrated that even as seasoned teachers reformed (some of) their teaching practices that most often they continued to maintain a belief in the teacher-centered classroom and the infallibility of mathematics. It has been argued that the latter of these beliefs is counter to reform-oriented mathematics teaching, thus securing the continuation of traditional practices (see, e.g., Davis & Hersh, 1981; Ernest, 1998). To make it possible for teachers to create mathematics classrooms that are consistent with the constructivist, student-center objectives of reform-oriented mathematics teaching, we believe that teachers must be provided an opportunity to challenge and "trouble" both traditional mathematics teaching and the reform efforts themselves. In understanding the mathematics classroom as a pedagogical space for teachers and students to "reason together" (David and Hersh, 1981, p. 282) through the socially constructed discipline of mathematics (Ernest, 1998), we argue that postmodern (or poststructural) theory provides a *different* theoretical framework for teachers to trouble both traditional and reform-oriented mathematics teaching as they explore their own pedagogical philosophies and practices.

The value of postmodern theory is found in its awareness of and tolerance toward social differences, ambiguity, and conflict; it requires developing new languages, conventions, and skills to address the moral and political implications of knowledge (Seidman, 1994). In short, postmodern theory requires shifting the "focus from foundations and familiar struggles of establishing authority toward exploring tentativeness and developing scepticism of those

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principals and methods that put a positive gloss on fundamentals and certainties" (Walshaw, 2004b, pp. 3–4).

**Foucault's Discourse, Derrida's Deconstruction, and Dewey's Reflection**

As Foucault (1969/1972) reinscribed the concept *discourse*, he argued that discourses are not a mere intersection of words and things but are "practices that systematically form the objects of which they speak" (p. 49). That is to say, for Foucault, "discourses do not merely *reflect* or represent social entities and relationships; they actively construct or *constitute* them" (Walshaw, 2007, p. 19, emphasis in the original). Foucault (1976/1990), however, also conceived discourses "as a series of discontinuous segments whose tactical function is neither uniform nor stable" (p. 100), which provides for the occasion of developing different discourses—and, in turn, different knowledges. Thus, we are not forever doomed by discourses. In general, Foucault's (1969/1972) analysis of discourse replaces the concept of the "nature" of knowledge with the "discursive formation" (p. 38) of knowledge. His analysis rejects the "natural" or taken-for-granted concepts of knowledge found in humanism, such as Descartes' dualism of mind-body (which argues that the thinking subject is the authentic author of knowledge) or Comte's positivism (which argues for a "scientific" knowledge gained from methodologically observing the sensible universe) (St. Pierre, 2000). Foucault uncovered knowledge as a discursive formation through the means of performing an archeological analysis, which examines the history of a discourse. But rather than being concerned with uncovering the "truth" by an examination of facts and dates, it is concerned with the "historical conditions, assumptions, and power relations that allow certain statements, and by extension, certain discourses to appear" (St. Pierre, 2000, p. 496). In short, this methodology allows for the understanding of "how knowledge, truth, and subjects are produced in language and cultural practice as well as how they might be reconfigured" (St. Pierre, 2000, p. 486).

In other words, there is no origin, or understood in another way, no center to discourse. Derrida (1966/1978) argued that accepting discourse as having no center allows discourse to be open for the "movement of play" (p. 289). He defined play as the "disruption of presence" (p. 292). In this context, play rejects the totalization of humanism with its "dreams of deciphering a truth or an origin which escapes play" (p. 292). This movement of play provides more freedom. This reconstitution of freedom as play is implicated in Derrida's *deconstruction* of discursive binary oppositions (see, e.g., Derrida, 1974/1997). Although Derrida refused to limit the possibilities of deconstruction through definition (1983/1991, see also Derrida & Montefiore, 2001), others have described it as the methodology of exposing discursive binary oppositions defined interdependently by mutual exclusion, such as good/evil or true/false (Dillon, 1999). For Derrida, these binary oppositions shape the very structure of thought by constructing an "essential" center and authorizing presence—a center and presence that, it is assumed, will collapse if the binary opposition is undermined (Usher & Edwards, 1994). Within the context of mathematics education, some of these binary oppositions are: mathematical *T*ruths/mathematical *t*ruths, teacher/student, effective teacher/non-effective teacher, reform teaching/traditional teaching, mathematically able student/non-mathematically able student, high-level course/low-level course, and so forth.

The deconstruction of binary oppositions identifies the first term, the "privileged" term, as being dependent on its identity by the exclusion of the other term, demonstrating that primacy really belongs to the second term, the subordinate term, instead (Sarup, 1993). Deconstruction, therefore, involves unsettling and displacing (or troubling) binary hierarchies, uncovering their historically contingent origin and politically charged roles, not to provide a “better” foundation for knowledge and society but to dislodge their dominance, creating a social space that is tolerant of difference, ambiguity, and playful innovations that favors autonomy and democracy (Seidman, 1994). In short, deconstruction acknowledges that the world has been constructed through language and cultural practices; consequently, it can be deconstructed and reconstructed again and again (St. Pierre, 2000).

In the past 2 decades or so, the discourse of *reflection* has been identified as a crucial characteristic of exemplar teachers by numerous national, state, and local organizations, foundations, and boards (Rodgers, 2002). For example, the NCTM (2000) stated, “opportunities [for teachers] to reflect on and refine instructional practice—during class and outside of class, alone and with others—are crucial in the vision of school mathematics outlined in *Principles and Standards*” (p. 19). Mewborn (1999), in her study on reflective thinking among preservice elementary mathematics teachers, traced the emphasis of teacher reflection to Dewey, suggesting that he believed the primary purpose of teacher education should be to help teachers reflect on problems of practice. Although Mewborn rightly noted that there is little agreement as to the content and nature of Dewey’s reflective thinking in general, she did find some commonalities present within the literature, including that reflective thinking is qualitatively different from recollection or rationalization, and is both an individual and shared experience. Rodgers (2002) argued that reflection is not an end in itself but a tool used in the transformation of raw experience into meaning-filled theory—grounded in experience and informed by existing theory—to serve the larger purpose of the moral growth of the individual and society.

## The Course

Teacher reflection was a primary objective as I (the first author) planned the course “Mathematics Education within the Postmodern,” a graduate-level, mathematics education course. The course, a reading intensive seminar, began by engaging students in a brief overview of postmodern theory, reading book chapters by foundational French scholars such as Gilles Deleuze and Félix Guattari (1980/1987), Jacques Derrida (1966/1978), Michel Foucault (1976/1990), and Jean-François Lyotard (1979/1984). In addition, the students read book chapters and essays by education theorists who position their scholarship within postmodern theory, such as Patti Lather (2000), Robin Usher and Richard Edwards (1994), and Elizabeth St. Pierre (2000, 2004). This overview provided the foundation for students to begin an initial critical analysis of essays contained in Margaret Walshaw’s (2004a) edited book *Mathematics Education within the Postmodern,* essays that deconstruct and trouble the discourses of knowledge, learning, teaching, power, equity, and research, among others, within the context of mathematics education (for a review of this book see Powell, 2007).

The specific learning objectives of the course were for students to develop an introductory understanding of the philosophical underpinnings of postmodern theory and to explore and (re)position the philosophical and structural foundations of mathematics, mathematics teaching and learning, and research in mathematics education within a postmodern framework. The intended purpose was not to “change” their teaching practices per se, but rather to provide the opportunity for mathematics education professionals to reflect differently on mathematics, mathematic teaching and learning, and, in turn, their pedagogical practices in light of postmodern theory. In short, the purpose of the course was for students to take the familiar discursive binaries of mathematics education (noted earlier) and to undergo a deconstructive process, individually and collectively.

Twelve students (8 women and 4 men) took the course; all but one were part-time graduate students and full-time mathematics teachers, ranging from elementary to college, with 5 to 15 years of teaching experience. A daily written assignment for the course was to maintain a reading journal (i.e., annotated bibliography) that included written summaries of each assigned reading, student-selected significant quotations from each reading, and comments regarding the student's struggles with each reading and how it might (or might not) assist in her or his teaching (and research). The final for the course was a reflective, academic essay (eight text pages in length) in which each student was to discuss her or his understanding of mathematics education framed in the postmodern and her or his struggles with and remaining (or new) questions of such a framing.

### Teachers Reflecting *Differently*

No matter what the students' initial comfort level with the ideas of postmodern theory, in the following discussion we argue that their final reflective essays demonstrate that in most cases each student's thinking attempted to take a new "line of flight," in which they endeavored to "make a map and not a tracing" (Deleuze & Guattari, 1980/1987, pp. 11–12) of the meanings and *t*ruths of mathematics teaching and learning. Through using the first phase of Dewey's (1933/1989) five phases of reflecting thinking—suggestion—the discussion attempts to capture (some of) these new lines of flight, illustrating how these practicing teachers began to reflectively think differently. The discussion is not about tracking or documenting mathematics "teacher change." We understand mathematics teacher change to be a complex endeavor that most often occurs when teacher professional development opportunities are long-term, school-based efforts conducted within a community of learners that provide teachers opportunities to grapple with significant mathematics and to consider how students might engage with that mathematics (Mewborn, 2003). Like the NCTM *Principles and Standards* (2000), however, we believe that teaching is a continual journey. "Effective teachers" do not master teaching, but rather find themselves in a continuous state of growth and change (Mewborn, 2003).

Within Dewey's (1933/1989) reflective thinking phase of "suggestions, in which the mind leaps forward to a possible solution" (p. 200), we believe that postmodern theory offered these seasoned teachers the possibility of different suggestions as the familiar discursive binaries of mathematics education underwent deconstruction, and, in turn, motivated different suggestions. Given the space limitation of this paper, we focus the discussion on the discursive practices that classify and describe teachers and students through interweaving comments written by the students with concepts borrowed from postmodern philosophers and theorists, illustrating how the teachers began to understand that teachers and students might indeed be described differently in the postmodern (Hardy, 2004).

In a postmodern frame, a new suggestion emerges that attempts to pry loose the binary (Spivak, 1974/1997) teacher/student, deconstructing the binary both in identity and relations of power. Within postmodern theory, teachers and students are (re)defined as *subjects* rather than as individuals. The term *individual* is a humanist term that implies that there is an "independent and rational being who is predisposed to be motivated toward social agency and emancipation—what Descartes believed to be the existence of a unified self" (Leistyna, Woodrum, & Sherblom, 1996, p. 341). A postmodern perspective, on the other hand, defines the person as a multiplicitous, fragmented subject who is subjugated, but not determined, by the social structures and discourses that constitute the person.

This conception provides for a different suggestion of power and, in turn, agency. *Power* in a postmodern frame is reconstituted, not as an object that can be shared, deployed, or taken away, but as a dynamic and productive event that exists in *relations of power* (Foucault (1976/1990). Deanne (a pseudonym, as are all student names throughout) used this Foucauldian reconstitution of power when she argued that teachers can challenge discourses by the decisions they make in their classes, for their students, everyday. Deanne also wrote, “Teachers in a postmodern classroom (occupied by subjects who transfer power between the teacher and each other in order to gain knowledge) attempt to create a space where students [and teachers] can learn through communication with others in the class.” Similarly, Lauren wrote, “I must consciously acknowledge my students, not as objects, but as [subjects], using power, resisting power, and interacting with each other and with the mathematics.” While reconstituting power as “‘letting go’ of the control in their classroom” and allowing “for the possibility of being ‘found out’ as not being *the* authority,” Charles wrote: “Teachers need to embrace their lack of expertise. …By joining the learning process in the classroom, teachers can model the open-mindedness necessary for students so that they might begin questioning, discussing, and constructing their own mathematical knowledge.” This joining in the learning process allows for a different interaction between teacher and students—and mathematics—that supports the mathematics classroom in becoming a pedagogical space that is open for *“negotiation of intentionality*” (Valero, 2004, p. 49, emphasis in original).

Valero (2004) suggested that when students (and teachers) are defined as agents who negotiate the intentions of the mathematics classroom—using power, resisting power, interacting with each other and the mathematics—that real empowerment might take place. Here, empowerment is understood as *self*-empowerment: “a process one undertakes for oneself; it is not something done ‘to’ or ‘for’ someone” (Lather, 1991, p. 4). Within the context of a postmodern mathematics classroom, Valero claimed that empowerment is not passed from teacher to student through the transference of “powerful knowledge,” but rather might be defined in terms of the potentialities for students (and teachers) to participate in (i.e., to negotiate) the discursive practices of school mathematics. Sarah noted, “I hope to help my students empower themselves to overcome the discourses…to overcome the limitations society and our culture has put on them.”

Coupled with this different understanding of student and teacher empowerment was a different suggestion of understanding students and teachers as fragmented subjects. Lauren wrote: “I have been many in my life—there is no one *woman* who defines me. I am mother, wife, teacher, daughter, boss, and student—each time made anew by social context and relationships with others.” As these seasoned teachers began to understand themselves as fragmented subjects constructed through discourses and relations of power, they, in turn, began to view their students as fragmented subjects. For example, Nancy stated: “Educators should begin to look at their students as multiplicitous subjects rather than as individuals; it is important to remember students are not identical in math or English class, in sports or hobbies, at home or school.” She continued, “I need to accept my students as multiplicitous—each one coming to me with different levels of prior mathematical knowledge and different ways of learning.” Likewise, Susan wrote: “If nothing else, I have come out of this class knowing that students think differently, react differently, and position themselves differently; I need to recognize and respect these multiplicities.”

The multiplicitous is a key reconstitution of self, others, and knowledge found within Deleuze and Guattari’s (1980/1987) characterization of the rhizome. The rhizome, as described by Deleuze and Guattari, is not “reducible neither to the One nor the multiple. …has neither a beginning nor end, but always a middle (*milieu*) from which it grows and overspills” (p. 21). Fleener (2004), building upon the rhizome, argued for the importance of seeing teachers and students (and the mathematics curriculum) as multiplicitous, and that teachers should shift their “focus to the in-between, the relational, and the dynamic” (p. 213). Through “engaging the in-between, students build their own understanding, not as foundations, but as complex webs of the nexus of relationship in the abstract world of mathematics” (p. 214). Sarah began engaging in the in-between, writing: “Typically, in mathematics we think there is one right answer to a problem and focus on developing our students’ knowledge of how to get to that answer, [but] it is…the ‘in-between’ that matters the most.”

A new suggestion of the in-between brought about a different suggestion regarding the possibilities of classroom communication. Within the postmodern, Cabral (2004) claimed, language ceases to be regarded as a means of “communication,” but as the very process of constitution of the subject; that is, the discipline of mathematics, students, and teachers are constituted within a language community. Therefore, Cabral argued, “we need to stop talking and start listening to the student...it is through speaking that one learns and through listening that one teaches” (p. 147). Lauren wrote: “I will listen more, talk less. …Let the students guide the lesson, hear what they have to say, to me and to each other, about the mathematics, about their understandings, questions, and confusions.” Nancy noted, “Actively listening to students’ questions and concerns may lead to further areas of exploration outside of the daily…lesson.” Dorothy, a doctoral student, spoke about the importance of teachers listening to their students, and of students listening to each other: “There have been many times in my classroom when I could not understand the point a student was trying to make. It took another student, in different words, to relay the message so that I could understand.”

## Conclusion

The preceding discussion attempted to capture the different suggestions that engaging in the postmodern provided these seasoned teachers as they began to think differently about the discursive binary teacher/student. These suggestions motivated different classifications and descriptions for teachers’ and students’ identity, agency, and empowerment, and, in turn, a different suggestion of teacher and student participation in the mathematics classroom. There were several other instances in the teachers’ final essays in which other familiar discursive binaries were deconstructed or troubled. Some troubled the binary of mathematical-able student/non-mathematical-able student, while others troubled the effective teacher/non-effective teacher binary. And, in rare occasions, even the discursive binary mathematical *T*ruths/mathematical *t*ruths was troubled. For instance, Marcus, a doctoral student, wrote, “Are we confining ourselves and our students by the rules and laws of mathematics that do not allow for them to do the unexpected, to go beyond their own reality?” Likewise, Nicholas noted, “I was blown away by the thought that mathematics, something that I had found comfort in because of its absolute nature, was being viewed as a science of uncertainty that could not be defined by its absolutes any longer.” In general, the teachers limited their comments regarding the *t*ruths of mathematics, or, similar to Nicholas, somehow resisted reconstituting the “absolute nature” of mathematics. It appears that although mathematics has been argued to be the roots of postmodern thought (see, e.g., Tasić, 2001), to deconstruct the capital-T truths of mathematics might prove to be the most difficult deconstruction to undertake; it may be, nonetheless, the most important.

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# LEARNER-FOCUSED DISCOURSE IN LEARNING MATHEMATICS: A TEACHER'S PERSPECTIVE

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*This paper discusses learner-focused, whole-class discourse from an elementary teacher's perspective based on her thinking and practice in teaching mathematics. A learner-focused perspective based on agency, collaboration, and reflection frames this study of classroom discourse. Analysis of data obtained through interviews and classroom observations produced six central ideas involving reflection on "self" and the sharing and guiding of thinking that formed a key basis of learner-focused discourse. These ideas are discussed with examples of how they occurred and were supported by the teacher's thinking and actions. This learner-focused discourse is shown to be important to empower "self" in the learning of mathematics and to allow students to talk mathematically from and about their experiences and to make sense of mathematical ideas, mathematics in their lives, and their ways of thinking or learning.*

## Introduction

This paper is based on a two-year study of discourse that facilitates mathematical thinking as practiced in the classroom in the teaching of elementary school mathematics. The focus here is on the aspect of the study that investigated the teacher's perspective of learner-focused discourse in whole-class settings as a basis of facilitating the learning of mathematics.

## Related Literature

Many studies in mathematics education have dealt with various aspects of discourse that occurs, or ought to occur, in mathematics classrooms (e.g., Hiebert & Wearne, 1993; Hufferd-Ackles, Fuson, & Sherin, 2004; Knuth & Peressini, 2001; Lampert & Blunk, 1998; Sherin, 2002; Steinbring, Sierpinska, & Bussi, 1998; Wertsch & Toma, 1995; Wood, 1999). Sfard (e.g., 2000a & b, 2001) and Cobb, with colleagues (e.g., Cobb & Bauersfeld, 1995; Cobb et al., 1997; Cobb, Wood & Yackel, 1993; Cobb, Yackel, & McClain; 2000), in particular, have written extensively about discourse in learning mathematics. Some studies offered units of analysis in studying discourse, For example, Sfard (2000b) identified three foci that exist in any analysis of mathematical discourse – the pronounced focus (i.e. the words used by the interlocutor); the attended focus (i.e., what the interlocutor is looking at, listening to, etc.); and the intended focus (i.e., the inter-locutor's intention in contributing to the discourse). Ryve (2006) identified four types of math-ematical communication, i.e., discourse focused on: intrinsic properties – typically produced by students who ask "why"; identification of similarities – reasoning that relies on identified surface similarities; established experiences – discourse that does not fit into other categories; and non-mathematical – often about practical issues related to completing an assigned task. (Knuth & Peressini, 2001) identified "dialogic" and "univocal" as a basis to unpack the nature of discourse.

Some studies focused on the relationship between discourse and learning. For example, Cobb et al. (1997) examined the relationship between classroom discourse and mathematical development. They concluded that collective participation in classroom discourse, although useful to the learning of children, does not by itself determine that mathematical learning or

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development occurs. Other studies focused on the teachers' role, the tasks, and the learning community to support discourse. For example, Wood (1999) discussed argumentation as an interactive process of knowing how and when to participate in an exchange. She offered examples of strategies used by a teacher to establish the classroom norms necessary to lay the foundation for argumentation as a form of learning. In some studies, the teacher's role in relation to listening has been explored as an important aspect of discourse since the teacher must listen to students to sustain discourse. Wallach & Even (2005) identified five characteristics of how their participant heard or misheard her two students: over-hearing – when a teacher thinks she has head statements that were not made by the students; compatible-hearing – attributing meaning to what students are saying; under-hearing – not hearing or noticing cognitive progress made by students, e.g., believing that students have stumbled upon the correct answer, when in fact they have systematically arrived at a suitable solution; non-hearing – disregarding part of what students said; and biased-hearing – impacted by prior knowledge of students and views held by the teacher. Davis (1997) discussed three categories of listening – *evaluative*: listening for a particular, preconceived "right" answer or explanation, or listening to respond; *interpretive:* listening for sense-making, and for student understanding; and *hermeneutic*: listening to the speaker as a prelude to and as a component of a negotiation for meaning in a situation. He emphasized the importance of listening to discourse.

This increased interest in discourse in mathematics education in recent years can be linked to learning theories and reform recommendations such as those of the National Council of Teachers of Mathematics (NCTM, 1991) that emphasize the importance of a different form of classroom communication from that of traditional mathematics classrooms. NCTM (1991) describes this reform-oriented discourse as dealing with the ways of representing, thinking, talking, and agreeing and disagreeing as a way to learn about, and engage in, mathematics as a domain of human inquiry with characteristic ways of knowing. This discourse attends to reasoning and evidence for sense making and to the development of ideas and knowledge collaboratively. It allows students to create their own understandings. It provides opportunities for individual students to connect and integrate their mathematical learning. It makes thinking public and creates an opportunity for the negotiation of meaning and agreement (Cobb & Bauersfeld, 1995). It provides collective support for developing one's thinking, drawing it out through the interest, questions, and probing ideas of the teacher and others (Cobb, Wood, & Yackel, 1993). It enables students to articulate what they know as a way to clarify their own understandings (NCTM, 1991). It is important in helping students develop and sharpen their mathematical thinking (Watson & Mason, 1998). In general, this discourse involves a more dialogic-type of interaction.

Both the teacher and students have vital roles in this discourse process in order to initiate and sustain it (NCTM, 1991). But this form of discourse can be difficult for teachers to implement and manage. Kilpatrick, Swafford, and Findell (2001) explained: "Managing discourse is both one of the most complex tasks of teaching and the least thoroughly studied. Research needs to make visible teachers' considerations as they handle classroom discourse and the consequences of their moves for students' learning" (p. 346). Teachers who are able to implement meaningful, reform-oriented discourse offer a basis for such research in ways that could inform mathematics teaching and teacher education. This paper reports on a study based on such a teacher and provides insights of discourse that is learner focused from the perspective of the teacher.

## Theoretical Perspective

Theoretically, discourse can be considered from different perspectives (Steinbring, Sierpinska, & Bussi, 1998). Social constructivist or socio-cultural perspectives tend to be more common in conceptualizing it in studies in mathematics education. While these perspectives are applicable to this study, the focus, instead, is on highlighting the learner and the personal as central features of the discourse. Thus, a learner-focused perspective of discourse, as a basis to compare the teacher's perspective, is adopted based on Bruner's (1996) description of four "crucial ideas" of framing learning: agency, collaboration, reflection and culture.

Bruner (1996) explains: "The agentive view takes mind to be proactive, problem-oriented, … selective, constructional. …Decisions, strategies, heuristics – these are key notions of the agentive approach to mind" (p. 93). This means, "one can initiate and carry out activities on one's own" (p. 35). Thus, agency involves one being able to take control of one's own mental activity. In this context of agency, Bruner notes: "the child … [is] somebody able to reason, to make sense, both on her own and through discourse with others" (p. 57). This view of children as thinkers requires the teacher to give "effort to recognize the child's perspective in the process of learning" (p. 56). Agency, then, is about learner-focusedness and vice versa. This means that in learner-focused, classroom discourse, students are allowed to participate in ways that might include: initiation of a discussion; re-direction of discussions in a relatively teacher-unscripted direction; responses unanticipated by the teacher; responses with an element of creativity, students' intentions, and personalization; and expression of students' interests or agendas.

Bruner (1996) associates collaboration with "sharing the resources of the mix of human beings involved in teaching and learning" (p.93). He explains that agency and collaboration need to be treated together to account for the individual and the collective in learning. He notes:

> Mind is inside the head, but it is also with others. It is the give and take of talk that makes collaboration possible. For the agentive mind is not only active in nature, but it seeks out dialogue and discourse with other active minds. And it is through this dialogic, discursive process that we come to know. (p. 93)

Thus, agency and collaboration should be integrated in the design of a learner-focused classroom culture. For example, students should not only generate their own hypotheses, but also negotiate them with others—including their teachers. The authority for knowing mathematically must be shared between participants—either teacher and student or student and student—when constructing new meaning or developing students' understanding of mathematics.

Regarding reflection, Bruner (1996) describes it as: "not simply 'learning in the raw' but making what you learn make sense, understanding it … going "meta," turning around on what one has learned through bare exposure, even thinking about one's thinking." (p. 58). Thus, like agency, reflection is about learner-focusedness and vice versa. This means that learner-focused discourse, for example, should allow or prompt students to notice for themselves and to become aware of their own thought processes, that is, "to become more metacognitive—to be aware of how she goes about her learning and thinking as she is about the subject matter she is studying" (Bruner, 1996, p. 64). Finally, regarding classroom culture, Bruner (1996) suggests that it is crucial to support learning and is the way of life and thought that we construct, negotiate, and use for understanding and managing the classroom. For a learner-focused, classroom culture, this will include creating, for example, a supportive and non-judgmental attitude to allow students to feel comfortable to share their thinking and experiences.

## Research Process

A case study was conducted with an experienced elementary teacher and her Grade 3 class. The teacher's practice embodied social constructivist principles with discourse playing a prominent role. The teacher regularly engaged her students in whole-class and small-group discussions and inquiry-oriented mathematical activities. Whole-class discourse occurred consistently throughout the school year, sometimes for most of a lesson, for the first half of a lesson, or integrated with small-group activities. The whole-class discourse was usually centered on problem-solving tasks or introduction of a new mathematics concept from the curriculum. Data sources consisted of interviews with the teacher, weekly classroom observations throughout three consecutive school terms, and classroom artifacts. The open-ended interviews focused on the teacher's thinking about discourse and her discourse behaviors in the classroom. For example, she was prompted to talk about her understanding of, goals for, and role in the discourse; her goals for students in relation to discourse; her approaches to questioning, listening, and task selection; how/why she intervened during discourse; how she established the classroom context; and her understanding of mathematical thinking. The interviews and all whole-class discourses for the lessons observed were audio taped and transcribed. Field notes were made of learning tasks, board work, and non-verbal teacher-student interactions relevant to discourse. The artifacts obtained included relevant students' written work and teacher's notes.

Data analysis for the larger project focused on identifying characteristics of discourse and the relationship to facilitating students' learning and mathematical thinking. Initially, a process of open coding was carried out. Corbin and Strauss (1990) describe this as taking data and segmenting them into categories of information. Two research assistants conducted this open coding independently of the researcher, and independently of each other. Only after initial categories had been identified were the results discussed and compared and revisions made where needed based on disconfirming evidence. Coding included identifying: (a) types of questions/prompts that elicited mathematical thinking (guided by Watson and Mason, 1998) and reflection; (b) what the teacher attended to in students' responses; and (c) different teacher's actions and/or thinking that determined or influenced different features of discourse and the nature of students' participation and learning. Themes emerging from the initial coded information were used to further scrutinize the data and then to draw conclusions. Learner-focusedness emerged as an umbrella theme that characterized discourse in this classroom. The findings presented here focus on the nature of it based on the teacher's thinking and classroom behavior during whole-class discourse.

## The Teacher's Perspective of Learner-Focused Discourse

The teacher's perspective of learner-focused discourse during whole-class settings was evidenced in her classroom behaviors that promoted and supported students' reflection on "self" and her thinking that emphasized connections to the students' world in learning mathematics. The central ideas of this discourse (Table 1) are interrelated, but discussed separately here.

*Table 1*
Central Ideas of Learner-Focused Discourse

| |
|---|
| Learner-focused discourse involves students: |
| 1. Reflecting on (making connection to) personal "real-world" experience |
| 2. Reflecting on conceptions |
| 3. Reflecting on preconceptions |
| 4. Reflecting on thinking |

| |
|---|
| 5. Sharing thinking<br>6. Guiding thinking |

*Reflecting on/making connection to personal real-world experience* was a central aspect of discourse in this teacher's classroom. Students were required to reflect on their out-of-school, real-world experiences to decide on and identify what mathematics they embodied, as in these three cases. (1) Students reflected on their experiences to identify examples of mathematics. Their challenge was to decide on what was an example of mathematics in their real-world experiences. This was initiated by the teacher posing questions, usually at the beginning of lessons, such as: "Where is math in your world?" "Did anything happen in your life that involves math that you want to share?" "Who experienced a math situation since we met in class yesterday?" "What math is happening in your world since I've seen you last?" (2) In contrast to case (1) where a mathematics concept was not specified, for this case, students reflected on their experiences to identify a specific mathematics concept. This occurred during the introduction and discussion of a mathematics concept and involved students associating real-world applications or significance of the concept. For example, in introducing a discussion of the concept of one million, the teacher asked, "Where would you find the number one million used in your world?" During a discussion of a line graph, students drew on their experiences to respond to the teacher's question: "where have you seen it?" (3) Students reflected on their experiences to associate real-world meanings or interpretations as in the following situation: "We're going to look at numbers on the calendar and try to think of how many ways to make this number. …Is there anything that you can think of in your life that makes you think of 17?"

This focus on real-world experiences was also central in the teacher's thinking. The teacher emphasized the importance and connections of mathematics to students' world as a central goal for her students. She explained: "I want them to think it's important in their world, I want them to see it's around their world, and I want it to be positive." She elaborated that this involved:

> How they interpret it [math], how they use it,…where they were seeing it. …Viewing things that are math. … Thinking that math is not just numbers on a page or just work in school. … Seeing the real world connections and patterns. …I question them to bring it [math] back and make a connection to their lives, because a lot of times they don't do that. You have to link them with those questions. …I always tell them "see the math."

*Reflecting on conceptions* during discourse involved students thinking about what they knew about a mathematics concept based on their past experiences, in particular, what they learned in prior grades in school. The teacher prompted students to unpack a concept based on the conceptions they had constructed of it. For example, the teacher probed students' thinking about the shape making up the bar graph they were discussing. She asked, "How do you know it is a rectangle? Make me believe that it is a rectangle." Students were able to recognize the rectangle, but initially encountered a problem explaining why. To prompt their reflection, the teacher asked, "How did you know it wasn't a circle." This led them to talk about what the rectangle was not. The teacher then prompted, "Think about the art project we did," as a way for them to find the language to describe the rectangle, which they were able to do. This discourse allowed the students to reflect on what they knew, i.e., their conceptions, based on making comparisons and connections within and outside of their mathematical experiences.

*Reflecting on preconceptions* during discourse involved students thinking about their real or imagined preconceptions of a mathematics concept, that is, what they thought they knew about it before formally learning it, as in the following three cases: (1) Students reflected on a concept or process of which they likely held preconceptions. For example, at the beginning of a lesson on linear measurement, the teacher initiated the discourse with: “We are going to determine our height today. … How could we do that? … What tools do we need to use for math today to determine our height?” The students’ responses led to a discussion of both the tools and units of measurement based on their preconceptions. (2) Students reflected on a concept of which they unlikely held preconceptions, but to which they could relate. For example, during a lesson on representing a number numerically in different ways, the teacher asked: “But just talking about numbers, does anybody really know where numbers came from and why we have numbers?” (3) Students reflected on a mathematics concept they likely or actually had formed a preconception of, but had not explicitly thought of or articulated, as in the situation when the teacher asked: “What’s the biggest number you can tell me?”

*Reflecting on thinking* during discourse involved students thinking about their own thinking, that is, engaging on a metacognitive level, as in these four cases. (1) Students reflected on what they looked for, or thought of, in order to make sense of, or interpret, a mathematics concept. For example, the teacher asked, “We are going to take a look at how numbers are made up … what do you do when you read a large number?” (2) Students reflected on their problem-solving processes or strategies, as when asked, “What did you think of first when you read the riddle?” (3) Students reflected on their choices of tools to aid learning, as when asked, “Who used the place value mat? … Can you tell us why you chose to use that?” (4) Students reflected on the affective aspect of their problem-solving experience, as when asked, “How many people had a little bit of difficulty trying to solve it? … How many people found it a challenge?”

The preceding situations of reflecting on thinking and (pre)conceptions were also central in the teacher’s thinking. For example, she explained: “I always say, ‘Hands down, brains on.’ That’s my saying so that every brain can have a chance to at least think about what they know.”

*Sharing thinking* during discourse involved students expressing their ideas in their own ways. This was supported by the teacher’s questions, such as: How do you make sense of that? Can you say out loud what’s going on in your head? Do you have an “aha!”? What have you noticed? Can you explain that more? What do you know about this? Can you talk about it; describe it?

One unique aspect of the teacher’s goal for discourse that supported this focus on students’ sharing was her own learning. She explained: “I want to learn something. I want to be “aha’d!” and surprised. I want them to teach me something. … I’m not afraid to take a risk so I just put myself out there and see what I can learn too.” In addition to her learning, she explained: “Always, my kids know you have to explain the why, not just an answer. … I really want to know the process…how they got there. It tells me a lot more about the kids.” She helped “to bring them out, to get them to think they have something to contribute, … to make everyone feel important and have a voice.” She believed that the best way to listen was: “Get rid of an answer in your head as you focus on their answer.” She added, “They see me trying to figure out what’s in their heads as opposed to wanting them to figure out what’s in my head – the answer I want.”

*Guiding thinking* during discourse involved students taking the discussion in different directions based on their thinking and questions as in the following situations: (1) The teacher invited students’ guidance; for example, “We’re going to do patterns…what do you want to know about patterns?” (2) The teacher used students’ responses as a basis of follow-up questions for discussion. As she explained: “When I am planning a lesson, I think of some good questions as openers…to get them thinking about the concept and then from there the other conversational questions just come from what they are sharing or asking.” She would also “do an off-shoot on something one might say, take it somewhere else, and that will lead someone else to participate…in oral discussion.” (3) The teacher probed students’ ideas, for example, on occasions “When I know they have something that will help the others understand or something that I never thought of to bring into a lesson.” (4) The teacher provided a question box in response to students’ request. She explained, “They wanted their own claim to the lesson. …It was unexpected to hear clearly that they wanted more empowerment to ask the questions they wanted or to learn the things that they wanted.”

The teacher noted that “the math questions that the kids are asking” triggered conversation that developed in mathematically significant directions. Also, “Because they were given freedom to say, ‘tell me what you want to learn’… what triggers it is the interest from the kids.” She also supported students’ thinking when she repeated or paraphrased their responses. Her intent was often to check her understanding of what they said or meant; to get clarification of what they intended; to allow them to correct her interpretation; and “to spring off into another math topic.”

## Conclusion

This teacher’s perspective of learner-focused discourse embodies notions of agency, reflection and collaboration as discussed earlier in the “theoretical perspective” section. It provides one way of understanding this view of discourse that could enhance the teaching and learning of mathematics. At the center of this view are the students’ thinking and real-world experiences as a basis of their learning of mathematics. Students get to personalize mathematics, to see mathematics in their personal world, and to mathematize their personal world. In general, this view of discourse takes account of students' personal experiences, thoughts, and feelings. It capitalizes on and values students’ contributions to their learning. It provides opportunities for students to bring their own backgrounds, personalities, and beliefs into the construction of their mathematical knowledge. Thus, through it, students can come to realize that their ideas are valued and, as a result, have more authority over their learning and engage in more voluntary participation.

This learner-focused discourse empowers “self” and gives students “voice” in learning mathematics. Human agency, therefore, is of significant importance. This form of agency was also promoted by Boaler (2003) in terms of the “I”-voice. She promoted classroom discourse that prompts students to take initiative, to demonstrate human agency. However, the learner-focused perspective promoted through this study includes the unique aspect of reflection on “self” i.e., students become the subject in the process of their own reflection and learning. Thus, there is a connection between personal experience and conceptual sense making. In general, this study suggests that discourse framed in agency, reflection and collaboration can allow students to talk mathematically from and about their experiences and to make sense of mathematical ideas, of mathematics in their lives, and of their way of thinking or learning. This way of empowering self makes the learning of mathematics personal, real, relevant, important, and meaningful.

This study also suggests that in order to support this view of discourse, the teacher’s role has to be framed by the intent to listen to learn from the students and to listen to know them mathematically. For example, based on this Grade 3 teacher’s perspective, this required her: to be aware and accepting of alternative ways of thinking or approaching something; to be able to recognize the students’ logic or believe that there was a logic; to believe that she can learn something from the students; and to understand students’ knowledge of a concept, their strategies, their prior experiences with the concept and their preformed perspectives and understanding. This view to learn from and about the students is supported by an "intersubjective" (Bruner, 1996) perspective where the teacher applies the same theories to herself as she does to her students. For example, the teacher creates approaches that are as useful for students in organizing their learning as they are for her, i.e., approaches that support each other's learning. From this intersubjective stance, the teacher is also interested in what the student is thinking and thus is concerned with formulating a basis of discourse that she can use to satisfy this and facilitate the efforts of the student.

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# THE RELATIONSHIP BETWEEN THE WRITTEN AND ENACTED CURRICULA: THE MATHEMATICAL ROUTINE OF QUESTIONING

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*This study investigated the relationship between the written and enacted curricula through an analysis of their mathematical features, including (a) Mathematical Words, (b) Visual Mediators, (c) Endorsed Narratives, and (d) Mathematical Routines. The results from the examination of the mathematical routine of questioning are reported here. The study revealed similarities and differences between the questions included in the written and enacted curricula indicating the utility of this framework for documenting the characteristics of curricular implementation.*

## Background

The National Council of Teachers of Mathematics (NCTM) published a series of three standards documents beginning nearly twenty years ago that provided a new vision for school mathematics (NCTM, 1989; NCTM, 1991; NCTM, 1995). This vision advocated for a student-centered model focused on mathematical reasoning, problem solving, and communication. Subsequently, the National Science Foundation (NSF) funded the development of mathematics curricula based on the NCTM standards. These curricula, and the standards on which they were based, have been challenged to prove their effectiveness. The resulting evaluation and comparative studies have been criticized for inadequate documentation of the implementation of the curricula (e.g., Senk & Thompson, 2003; National Research Council, 2004) saying that in order to credit a curriculum for students' learning or blame a curriculum for lack thereof, some degree of fidelity of implementation must be established. Textbook-use diaries, table-of-contents implementation records, classroom observations, interviews, and surveys have been used in recent studies to document curricular implementation. What has been missing from these methods of accounting for implementation has been a focus on mathematics. That is, how is the mathematics presented in the written curricula, how is it enacted in the classroom, and what is the relationship between the two? In this study, I use the commognitive framework (Sfard, 2008) to compare the mathematical features of written and enacted curricula.

## Theoretical Perspectives

Commognition (formed from *communication* and *cognition*) treats communication (interpersonal exchange) and cognition (intrapersonal exchange) as two forms of the same phenomenon. It was developed to emphasize the relationship between these two processes. The commognitive framework proposes that *mathematics* is discourse about mathematical objects and *learning mathematics* is a change in participation in mathematical discourse. The framework suggests an examination of the discursive features of mathematics, including (a) Mathematical Words, (b) Visual Mediators, (c) Endorsed Narratives, and (d) Mathematical Routines.

Mathematical words are those that signify quantities and shapes (e.g., number) and those that highlight relationships between these quantities and shapes (e.g., equivalence). Visual mediators are artifacts created for the primary purpose of mathematical communication (e.g., symbols, graphs). Narratives include any text that is framed as a description of objects, of relations between objects or processes with or by objects, and which is subject to endorsement or rejection (i.e., being labeled true or false). Definitions, axioms, theorems, and proofs are commonly endorsed narratives in mathematics. Finally, mathematical routines are repetitive characteristics of mathematical discourse. A detailed comparison of these four mathematical features in the written and enacted curricula provides new ways to talk about curricular implementation that highlight the mathematics. The overarching question of this study is: What does an investigation of the key features of mathematical discourse, using the commognitive framework, in the written and enacted curricula reveal? Of particular interest in this paper: What does an investigation of the mathematical routine of questioning in the written and enacted curricula reveal?

## Methodology

I utilized the commognitive framework to investigate the relationship between written and enacted *standards*-based mathematics curricula. To this end, I made use of two primary data sources, (a) the written version of a *standards*-based mathematics curriculum and (b) an enactment of the same *standards*-based mathematics curriculum.

*The Connected Mathematics Project* (hereafter referred to as *CMP*) (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006) was used as the written curriculum. In particular, *Multiplying with Fractions*, an Investigation in *Bits and Pieces II: Using Fraction Operations* was used. The written curriculum is conceptualized here as the union of this Investigation in the Teacher's Guide and the Student's Guide.

I used videotapes of Investigation 3 in a sixth grade classroom as the enacted curriculum. This particular class is heterogeneous in mathematical ability (i.e., the students are not tracked). It is located in a middle school in a small rural town in the Midwest. The teacher of this particular class is a veteran *CMP* teacher. She has attended and conducted professional development for *CMP* and verbally endorses the curriculum. The discourse of the classroom (i.e., the union of the words and actions of the teacher and the students) is considered the enacted curriculum for this study.

My data analysis entailed, broadly speaking, comparing the mathematical features of the commognitive framework in the written and enacted curricula, and using the results of these analyses to investigate both the relationship between the two curricula as well as the usefulness of the framework to describe the results. Here, I report only on mathematical routines, and only on the routine of questioning.

## Findings

Routines are "well-defined repetitive patterns in interlocutors' actions, characteristic of a given discourse" (Sfard, 2007, p. 574). Questioning is a widely used mathematical routine in both the written and enacted curricula. The *CMP* Teacher's Guides contain a section entitled "Suggested Questions" for each Problem in the Investigation. Given the fact that learning mathematics is conceptualized here as changing discourse practices, the questions included in the written and enacted curricula are worthy of examination. That is, "What discursive practices are elicited through questions in the written and enacted curricula?"

The written and enacted curricula include 110 questions and 579 questions, respectively. A series of analyses was conducted on these questions: (a) Leading Words of Questions, (b) Elicited Answers to Questions, (c) Mathematical Processes addressed by Questions, (d) Questions that Address the Answer, and (e) Miscellaneous Questions.

*Leading Words of Questions*

For this analysis, the first word of each question in the written and enacted curricula was noted. Figure 1 summarizes the relative frequencies of the first words of questions that are present in at least 10% of either the written or enacted curriculum or both.

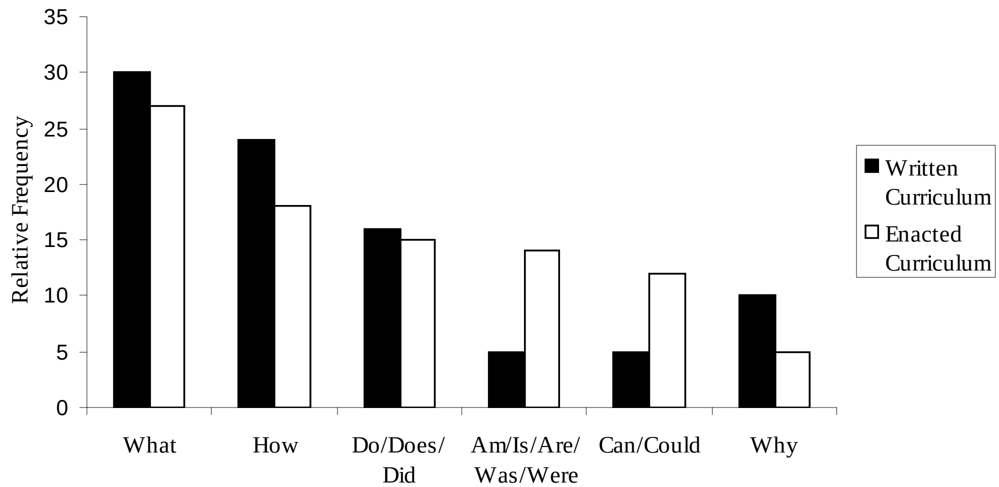


*Figure 1.* Relative frequencies of leading words of questions.

"What" and "How" are most common in both curricula. In both cases, however, the relative frequency in the written curriculum is greater than in the enacted curriculum. "Why" is not prevalent in either curriculum, but is twice as common in the written curriculum as in the enacted curriculum. Here, learning mathematics is defined as changes in participation in mathematical discourse. Given this, it seems that "open" questions (i.e., questions that require more than 1-2 word answers) would be preferable to "closed" questions, as they would provide both opportunities for students to engage in extended mathematical discourse as well as opportunities for the teacher to monitor mathematical learning. "How" and "Why" questions tend to be open, whereas the "Do" "Am," and "Can" families of questions are more likely closed. Contrasting these categories reveals that 34% and 23% of the questions in the written and enacted curricula, respectively, are open questions. "What" questions are not obviously open or closed and they appear in fairly equal frequencies in both curricula, therefore they were not included in this calculation.

*Elicited Answers to Questions*

The questions in the written and enacted curricula ask students either to say something or to do something. If learning mathematics is conceptualized as a change in ways of participating in mathematical discourse, then what students are expected to say and do is of critical importance. In particular, if students are expected primarily to "do," then it is questionable how much mathematics learning can take place. In both the written and enacted curricula, questions asking students to "say" something are much more common than questions asking students to "do" something (i.e., 84% and 86% "saying" questions, respectively).

The questions were classified into *what* the students are expected to say or do. When expected to "say" something, three types of responses are represented in at least 10% of either the questions in the written or enacted curriculum or both: (a) explanation, (b) yes or no, and (c) a number. Table 1 provides examples from the written and enacted curricula of questions eliciting each of these types of responses.

Table 1
*Sample Questions Eliciting Particular Response Types*

| Response | Written Curriculum | Enacted Curriculum |
|---|---|---|
| Explanation | *"Why does it make sense that the product [of reciprocals] is always 1?" (TG, p. 81)* | *S: "Why did you write twenty twos, though?" (Day 5)* |
| Yes/No | *"Do you think Takoda's strategy works?" (Student's Guide [SG], p. 38)* | *T: "So, is it fair to say that these two girls took each one of these thirds and split them into seven pieces?" (Day 1)* |
| Number | *"What fraction of a whole pan does Mr. Williams buy?" (SG, p. 33)* | *S: "So, how many thirds?" (Day 5)* |

Figure 2 summarizes the relative frequency of each category of response.

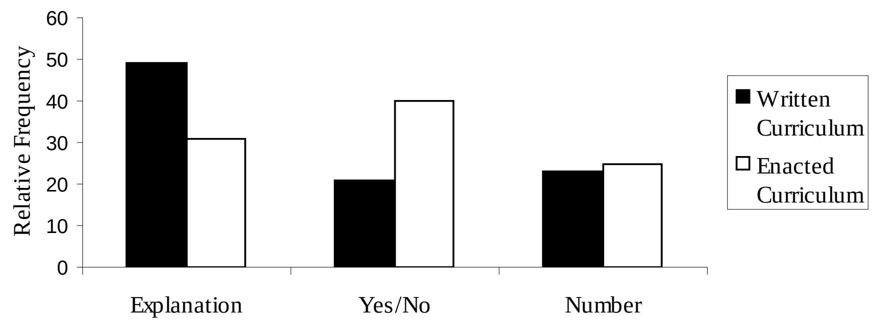


*Figure 2.* Relative frequencies of categories of elicited answers to questions.

Figure 2 indicates that explanations are the most commonly elicited type of response in the written curriculum, whereas Yes/No responses are slightly more common than explanations in the enacted curriculum. This analysis supports the conclusion from the previous analysis that "open" questions are more prevalent in the written curriculum than in the enacted curriculum.

When a question expects the students to "do" something, the actions include constructing a symbolic mediator (e.g., a number sentence); and constructing, manipulating, or indicating an iconic mediator (e.g., a diagram). Table 2 provides examples of questions from each category.

Table 2
*Sample Questions Eliciting Particular Actions*

| Action | Written Curriculum | Enacted Curriculum |
|---|---|---|
| Construct Symbolic Mediator | *"Ask groups to write the number sentences for the problems on their transparency." (TG, p. 72)* | *T: "Can you write that [number sentence] next to it?" (Day 1)* |
| Construct Iconic Mediator | *"How would you represent* $\frac{1}{4} x \frac{2}{3}$ *on a number line?" (SG, p. 34)* | *T: "Who can come up and show what that would look like on a long, skinny model?" (Day 3)* |
| Manipulate Iconic Mediator | *"What could you do in your drawing to make this clearer?" (TG, p. 60)* | *T: "Could you continue them as if this brownie were here, or this part of the goal was there?" (Day 4)* |
| Indicate Iconic Mediator | *"Where do you see this [the numerators] on the brownie pan drawing?" (TG, p. 61)* | *T: "So where is one whole ounce?" (Day 5)* |

Figure 3 summarizes the relative frequencies of the types of "doing" responses to questions in the written and enacted curricula.

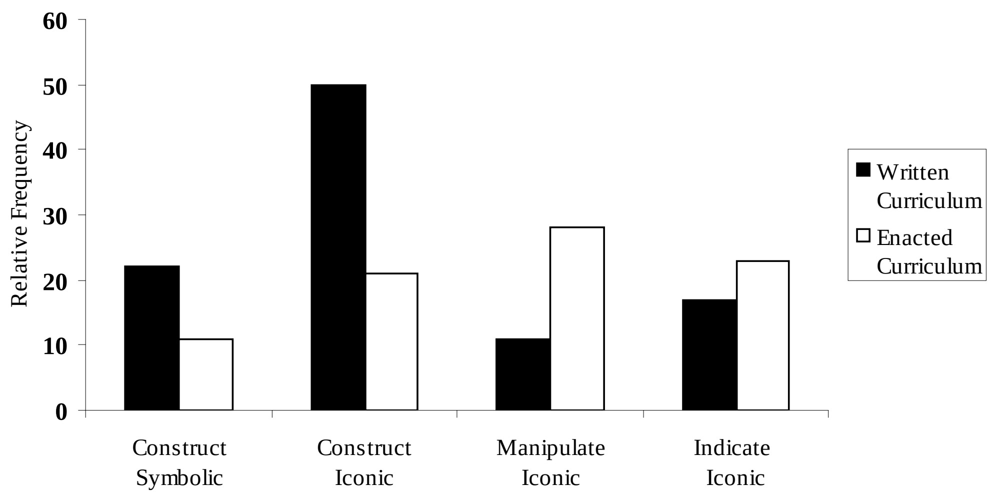


*Figure 3.* Relative frequencies of "doing" question responses.

Figure 3 illuminates several differences between the "doing" question responses in the written and enacted curricula. First, constructing iconic visual mediators is more than twice as common as any other expected "doing" question response in the written curriculum. In fact, half of all "doing" questions in the written curriculum expect this action. In contrast, several categories have relatively similar frequencies in the enacted curriculum. However, it should be noted that all three of the responses with the highest relative frequencies in the enacted curriculum involve iconic visual mediators.

*Mathematical Processes Addressed by Questions*

The mathematical processes addressed in at least 10% of the questions in either the written or enacted curriculum or both include: (a) estimating, (b) decomposing numbers, (c) using a model (including concrete, iconic, and symbolic mediators), and (d) using an algorithm. There are also questions that address mathematical processes in general. Table 3 includes sample questions from the written and enacted curricula which address each process.

Table 3
*Sample Questions Addressing Particular Mathematical Processes*

| Process | Written Curriculum | Enacted Curriculum |
|---|---|---|
| Estimating | *“Who can explain how they estimated $1\frac{1}{2}$ x $2\frac{9}{10}$?”(TG, p. 72)* | *T: “If I said I was going to get one half of two and nine tenths, Graham, how could we think about estimating that answer?” (Day 4)* |
| Decomposing Numbers | *“Would you want to use this strategy [distributive property] with the problem $3\frac{7}{8}$ x $2\frac{5}{6}$?” (TG, p. 77)* | *T: “Could I do ten groups of two and a third and then a half of a group of two and a third?”(Day 5)* |
| Using a Model | *“How does your drawing help someone see the part of the whole pan that is bought?” (TG, p. 60)* | *T: “So how far, what fraction of a whole mile is this one little piece right here [pointing to part of model]?” (Day 3)* |
| Using an Algorithm | *“What observations can you make from Questions A and B that help you write an algorithm for multiplying fractions?” (SG, p. 35)* | *T: “Do you think that these are the steps that we should tape - take?” (Day 3)* |
| General | *“What method did you use to solve the problem?” (TG, p. 81)* | *T: “Can you do this in a whole day, so how could you figure out what you do in a half of a day?” (Day 5)* |

Figure 4 summarizes the relative frequencies of the mathematical processes addressed in the questions.

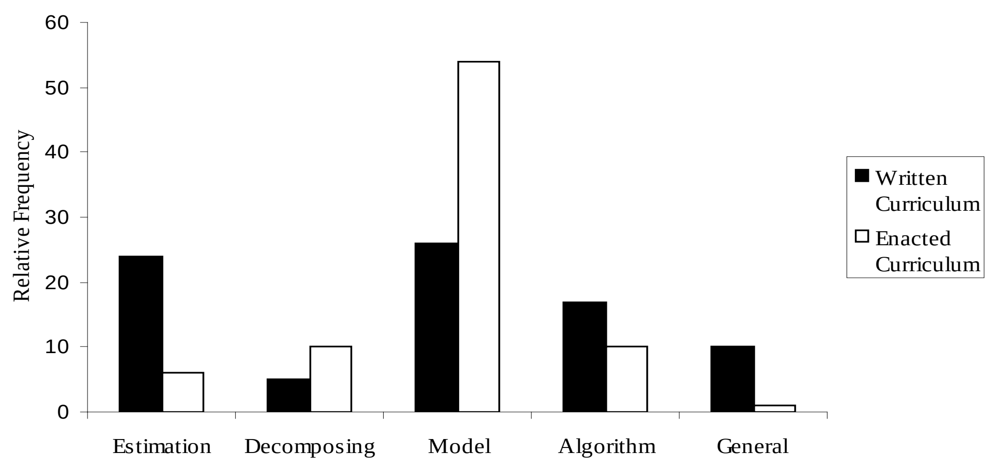


*Figure 4.* Relative frequency of mathematical processes addressed in questions.

Figure 4 indicates that more than 50% of the questions in the enacted curriculum address the use of a model. In contrast, 25% of the questions in the written curriculum address the use of a model and another 25% address the use of estimation. Questions addressing algorithms and general methods also represent at least 10% of the questions in the written curriculum. The same is true for algorithms and decomposing numbers in the enacted curriculum.

*Questions that Address the Answer*

Questions that address the "answer" are very common in both the written and enacted curricula. In fact, 45% of the questions in the written curriculum address the "answer" compared to 23% of the questions in the enacted curriculum. These questions take several forms including (a) asking for the answer, (b) asking for an explanation of the answer, (c) asking about the relative size of the answer, and (d) asking about the answer's relationship to another answer. Table 4 provides examples of questions from these categories.

Table 4
*Sample Questions Addressing the Answer*

| Question Type | Written Curriculum | Enacted Curriculum |
|---|---|---|
| Asking for the Answer | *"What fraction of a whole pan does Aunt Serena buy?" (SG, p. 33)* | *T: "What fraction of the candy bar did I eat?" (Day 1)* |
| Explaining the Answer | *"How did you come up with $2\frac{1}{2}$?" (TG, p. 71)* | *T: "How did you decide three eighths?" (Day 1)* |
| Asking about the Relative Size of the Answer | *"Does multiplication with fractions always lead to a product that is less than each factor? (TG, p. 67)* | *T: "Is your answer going to get bigger, or is your answer going to get smaller?" (Day 1)* |
| Asking about the Answer's Relationship to Another Answer | *"Who can use their model to prove that the answer $\frac{8}{12}$ is sensible?" (TG, p. 76)* | *T: "Do you think that it's one fourth or do you think it's showing one sixth of the whole bar?" (Day 3)* |

*Miscellaneous Questions*

Table 5 provides examples of several types of questions that each represent approximately 2-3% of the questions in the curricula.

Table 5
*Sample Miscellaneous Questions*

| Question Type | Written Curriculum | Enacted Curriculum |
|---|---|---|
| What is the Meaning of Fraction Multiplication? | *"What does it mean to find $\frac{1}{3}$ of $\frac{2}{3}$?" (TG, p. 60)* | *T: "What does that mean again, one third times one fourth?" (Day 2)* |
| Do You "Agree" with a Suggested Answer or Strategy | *"Do you agree with this answer and the reasoning?" (TG, p. 72)* | *T: "Do you agree with that, Jacob, or no?" (Day 2)* |

| | | |
|---|---|---|
| Does a Suggested Answer or Strategy "Make Sense"? | *"Do you think Paula's strategy of rewriting the mixed numbers as fractions is sensible?" (TG, p. 80)* | *T: "Could you guys live with that, does that make sense?" (Day 3)* |
| Do You "Understand" an Answer or Strategy? | ----- | *T: "Everybody understand what Katie did?" (Day 3)* |

## Discussion

The question addressed in this discussion is, "What does an investigation of the mathematical routines (in this case, the routine of asking questions) in the written and enacted curricula allow us to see?" That is, "What do we know now about the relationship between the written and enacted curricula that we did not know before?" Table 6 summarizes the findings from this investigation of questions.

Table 6
*Summary of "Mathematical Routines" (i.e., Questioning) Analysis*

| | Written Curriculum | Enacted Curriculum |
|---|---|---|
| Leading Words of Questions | "What" and "How" are the most common first word of a question in both curricula | |
| | 10% of questions begin with "Why" | 5% of questions begin with "Why" |
| | 34% of the questions are open | 23% of the questions are open |
| Elicited Answers to Questions | Approximately 85% of the questions ask students to "say" something rather than to "do" something | |
| | The most common elicited answer to "saying" questions is an "explanation" (49%) | The most common elicited answer to "saying" questions is "yes/no" (40%) |
| | Approximately 25% of the "saying" questions elicit a "number" in both curricula | |
| | 49% of the "saying" questions are open | 31% of the "saying" questions are open |
| | Nearly all elicited responses to "doing" questions involve visual mediators | |
| Mathematical Processes addressed by Questions | 26% of the questions address using a model and 24% address estimating | 55% of the questions address using a model and 6% address estimating |
| | Approximately 10% of the questions address the relationship between two or more mathematical processes | |
| Questions that Address the Answer | 45% of the questions address the answer to a Question | 23% of the questions address the answer to a Question |
| Miscellaneous Questions | Both curricula include questions that ask for the meaning of fraction multiplication, requests for agreement/disagreement, and whether particular mathematics makes sense | |

Table 6 and the more detailed analysis described earlier highlight many discursive similarities and differences between the questions included in the written and enacted curricula. Table 6 indicates (in several places) that a greater proportion of the questions in the written curriculum require an explanation (i.e., are open questions). If learning mathematics involves changing participation in mathematical discourse, then the opportunities that "open" questions provide for students to participate actively in mathematical discourse (i.e., beyond short answers) make a difference in students' learning.

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# OPEN, ONLINE, CALCULUS, HOMEWORK FORUMS: ARE STUDENTS POSITIONED TO LEARN?

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*Free, open, online, calculus forums are websites where students can post course-related queries that may be viewed and responded to by others. Is this a prescription for cheating or mastery-oriented help-seeking? I investigated one such site, FreeMathHelp.com, with a focus on the positioning of students as they sought help on their coursework. Two hundred exchanges on limit and related rates were collected and examined for evidence that students are contributing ideas and proposals for action, challenging and questioning others' proposals, and indicating that the issue was resolved. Of particular interest was the serendipitous finding that students sometimes engaged in self-reflection.*

## Background

Open, online help forums are websites where students can post course-related queries that are then visible to the public. These forums are "open" in the sense that they are not affiliated with any particular course or institution; members are attracted to them by necessity and interest. People learn of the forums' existence, access the web sites, and then choose whether or not to join in the conversation, either by posting a question, responding to an unanswered question, or contributing to an ongoing exchange. In this way, these forums alter the very nature of tutoring as it is traditionally conducted and transform it from a private activity between tutor and tutee to a public activity between people who share an interest in the subject domain. In this study, I explore how participating in this conversation engages students and tutors in mathematical and pedagogical discourse in ways not characteristic of traditional one-on-one tutoring.

## Theoretical Perspective

Traditional one-on-one, face-to-face tutoring is a popular form of instruction that offers benefits not generally provided by classroom instruction, mastery learning, computer-aided or programmed instruction, or computer tutors (Chi, 1996). One feature that distinguishes tutoring sessions from other means of instruction is the pattern of dialogue between participants. In contrast to the 3-step Initiation-Response-Evaluation dialogue frame that often marks classroom discourse (Cazden, 2001; Mehan, 1979), the dialogue in traditional tutoring sessions generally follows a 5-step dialogue frame (Graesser, Person, & Magliano, 1995): (1) Tutor asks question (or presents problem); (2) Learner answers question (or begins to solve problem); (3) Tutor gives short immediate feedback on the quality of the answer (or solution); (4) Tutor and learner collaboratively improve the quality of the answer; (5) Tutor assesses learner's understanding of the answer. This dialogue pattern occurs in a context in which the tutor explains a pre-determined set of topics to the tutee, often with the goal of providing remediation or augmenting an instructional explanation. Questions or problems are chosen by the tutor for the purpose of assessing the learner's understanding, and, in fact, the learners generally ask few questions during the tutoring session (Graesser & Person, 1994).

Alternatively, consider an instructional episode spawned by a particular problem that a student has encountered in coursework and has posed to a peer or more experienced other. In order to distinguish this type of instructional episode from traditional tutoring sessions, we have referred to them as “tutorettes” (van de Sande, 2007; van de Sande & Leinhardt, 2007a) and considered them as a form of student-initiated help-seeking (Nelson-Le Gall, 1985) that occurs in university help centers (face-to-face) and in online forums (computer-mediated). In these situations, the student selects the question(s) that begins the discussion, indicates the acceptability of the tutor’s responses, and generally decides whether the goals of the interaction have been met (i.e. whether the exchange was helpful). Thus, a typical dialogue frame (for a single student-tutor pair) for this type of encounter might look like the following: (1) Student asks question (or presents problem); (2) Tutor answers question (or begins to provide scaffolding); (3) Student gives feedback on the quality of the help; (4) Student and tutor collaboratively work on solving the problem; (5) Student assesses whether tutor’s responses were helpful.

Although tutorettes share many features of tutoring sessions (such as personalized instruction and support), they are also different in terms of initiation, goals, and instructional objectives. One key difference between online tutorettes and face-to-face tutoring encounters (either tutoring sessions or tutorettes) is the presence of an audience. Face-to-face sessions are generally conducted between a single student-tutor pair in relative privacy, whereas the exchanges in open, online forums are public and can be witnessed by others. In addition, in some forums, more than a single member can contribute to an ongoing exchange with alternative solutions, corrections, and commentary on mathematical issues as well as pedagogical approaches. The broader social dimension afforded by these open, online help forums reframes tutoring as a collective activity in which the exchanges become a public conversation between individuals who share a common interest in doing mathematics and helping others (van de Sande & Leinhardt, 2007b, 2008a).

## Methods

*Design*

In order not to disrupt this natural phenomenon in any way, this research was conducted by collecting a sample of 100 exchanges on the limit and 100 exchanges on related rates (both dating back from 4/29/08) from the archives of the calculus homework help forum on a representative site.

*Site choice and description.* The calculus forum at FreeMathHelp.com was selected because this site has an extensive history (archives dating back to 2005), includes a search mechanism for locating exchanges by a keyword or phrase, and is active in terms of daily postings and membership. In addition, the forum policies (such as achievement of member status) are explicit, and member “reputation” is an implicit mechanism of the social arena rather than being quantified (e.g. by others’ ratings of one’s postings or by some ranking determined by the forum site administrator) and made an explicit part of forum identity. In addition, all forum members can initiate threads in a discussion forum (e.g. as students posting mathematics questions) and can respond to others’ posts (e.g. as tutors providing help).

The prescribed etiquette for participation is located in a “sticky” that is the lead posting within each help forum. This covers administrative issues (e.g. posting to an appropriate category) and politeness (e.g. patience while waiting for response). In addition, there are three rules that specifically address the content and framing of posts: include problem context (“Post the complete text of the exercise”), show initial work (“Show all of your work [including intermediate steps that may contain errors]”), and attend to clarity (“Preview to edit your posts [to minimize errors]”).

*Topic choice.* Students enrolled in “introductory calculus” are exposed to a large number of topics from differential and integral calculus. Although the exact coverage of the syllabus will vary across programs and institutions, there is a large amount of overlap in the topics that are presented. Two such topics are the limit concept and related rates. Coming to grips with the limit concept is one of the first challenges that students in an introductory calculus course face. There is general consensus that many students struggle with constructing a coherent understanding of limit (Cornu, 1991; Cottrill et al., 1996; Tall, 1993; Williams, 1991) and that many fail to achieve this through instruction (Szydlik, 2000). Some part of student difficulty can be chalked up to the abstract nature of the limit concept; it is based on a never-ending process and therefore requires the contemplation of an infinite number of computational steps – a large conceptual leap from topics encountered in algebra and precalculus. In contrast, the topic of related rates encompasses a class of problems that involves the relationship(s) between two or more changing quantities, one of which is unknown and must be determined. These exercises generally appear in introductory calculus instruction as applications of implicit differentiation and the chain rule and are framed as word problems meant to reflect authentic situations with the solution to such problems scripted as a 5-7 step process. Yet, students struggle with this topic as well (Engelke-Infante, 2007; Martin, 2000). The concept of limit and related rates, then, were chosen as topics for this study to reflect the diversity of problem types characteristic of the subject domain and to capture tutoring interactions on topics that are challenging to students.

*Sample characteristics.* FreeMathHelp.com features participants’ profiles that include information on occupation, location, and interests. Whereas many student participants do not provide this information, the participating tutors in the calculus forum are self-reportedly students, educators, professionals, and retired mathematics professors. The most frequent tutor participants are from the United States, although there are representatives from a variety of other countries as well.

Although some tutors and students post more frequently, numerous tutors and students frequent MathHelpForum.com. The sample contained 100 related rates exchanges initiated by 65 different students, with responses from 18 different tutors and 100 limit exchanges initiated by 67 different students, with responses from 23 different tutors. There was some overlap in participants (both students and tutors) across the two mathematical topics: 17% of these students posted queries on both limits and related rates, and 63% of the tutors provided assistance for both topics.

*Coding and Analyses*

*Conversational complexity*. In order to characterize forum tutoring dialogue patterns, each exchange was assigned a participation code that tracks the number of participants, the total number of contributions in the exchange and the sequence of participation. For example, a code of 1231 would be assigned to a thread with four postings containing contributions from 3 different participants: a student (designated 1) posted a problem and then two different tutors (designated 2 and 3, respectively) responded, followed by a final contribution by the student. These codes permit one to catalogue exchanges that involve multiple conversational turns, multiple participants, and multiple contributions by a single participant. In addition, although the participation codes are agnostic with respect to the quality of the contribution (e.g. mathematical accuracy and depth, and pedagogical sensitivity), the codes do provide some indication of interaction within an exchange: for example, 1213121 is more likely to be an exchange in which two tutors are conversing with a student, whereas 1213232 is suggestive of dialogue between two tutors.

Based on these participation codes, each exchange received a conversational complexity index defined as the sum over code entries. For example, an exchange with participation code 1231 would have a complexity index of 1+2+3+1=7. While this index makes arbitrary use of the categorical indices – the numbers in the codes have *no* value beyond marking the sequence of participants – the index appears "well-behaved" (van de Sande & Leinhardt, 2008b). Lower sums correspond to exchanges that do not contain intense mathematical discussions and elements of pedagogical sophistication. Exchanges in which mathematical principles are invoked and perspicuous mathematical reasoning is present have higher indices (which is not to say that all discussions with higher indices are of high caliber, but simply that quality exchanges are marked by higher indices). Thus, the complexity index provides a rough guide as to which exchanges are "simple" versus which are "complex" that suffices for examining the effect of various positioning moves within an exchange.

*Positioning.* In order to explore aspects of student positioning in forum activity, each exchange was examined for the presence of three types of student activity that are consistent with active student participation and agency (Greeno, 2006): (i) assertions and proposals for mathematical actions (e.g. "THis is what i did. I don't know exactly how to solve for this but, my logic is that it'll go to zero eventually, because the bottom goes to infinity faster than the top so it'll to go zero. But i am not sure exactly if this is right can someone prove this or tell me if i'm intuatively correct. Thanks"), (ii) questions and challenges of others' proposals (e.g. "to find the derivative of this last part, i get that deriv of cos is -sin, but where does the -2 go. did you use the product rule. if so, what did you make 'f(x' and 'g(x)' for product rule?"), and (iii) indications of resolution coded according to strength (e.g. "i got it, forgot to factor, btw -14/sqrt(18) = -3.30, which is the correct answer" was counted as strong resolution, whereas "Thanks" was counted as weak resolution). Cohen's κ, a conservative statistic for establishing reliability, showed considerable inter-rater consistency on a sample of 20 exchanges: assertions and proposals for action (Cohen's $\kappa = 1$, standard error = 0); questions and challenges of others' proposals (Cohen's $\kappa = 0.77$, standard error = 0.22); degree of resolution (Cohen's $\kappa = 0.74$, standard error = 0.14); all differences in coding were resolved following discussion.

## Results

Although the forum participants acting as tutors are generally more experienced mathematically than forum members who bring their queries to the forum, students can position themselves with authority in an exchange in three ways: by contributing to the construction of a solution; by questioning or challenging the contributions of others; and by indicating that an issue has been resolved. In this section, we look at each of these three indicators of student authority in turn to reveal how students are positioning themselves as they participate in the forum. In the analyses, the threads are partitioned into sections with descriptive labels that are intended to convey and characterize the positioning of participants in that portion of the interaction.

### *Student Makes Assertion or Proposes Action*

Students participate in the forum for different reasons: because they have reached an impasse while attempting a problem, because they wish to confirm the accuracy of a solution that they have constructed, or because they have questions regarding an explanation that they have encountered in their studies. When a student posts a query on the forum, s/he can assume either a passive or an active position in the construction of the solution or explanation. One mark of active participation involves making assertions or proposing mathematical actions (even if these are hedged or couched in uncertainty), and the exchanges were examined for this aspect of positioning.

**Error! Reference source not found.** contains the percentage of exchanges in which the student made (or failed to make) an assertion or proposed a mathematical action within each topic by location in the thread (initial versus subsequent posting or both). The results indicate that students are generally positioning themselves as contributors to the discussions on solving the limit and related rates problems. The pattern of the location of these contributions in the thread is shared across topics, with students following the participation guideline to "show all of your work" in the initial posting 60% (limit) and 68% (related rates) of the time and contributing at some location in 69% (limit) and 79% (related rates) of the exchanges. Out of all 200 exchanges 74% involved students making some sort of assertion or action in the thread, and, of these, nearly three quarters involved exchanges with a complexity index greater than 6. That is to say, students proposed actions or made assertions in contexts that proved to have higher conversational complexity.

**Table 1.** ***Percentage of Exchanges Containing Student Proposals by Location in Thread***

| Topic | No assertion or proposal | Within initial posting only | Within initial and subsequent postings | Within subsequent posting only |
|---|---|---|---|---|
| Limit | 31 | 45 | 15 | 9 |
| Related rates | 21 | 54 | 13 | 12 |

*Student Questions or Challenges Assertion or Proposal Made by Others*

Contributions that question or challenge others' assertions or proposals are another indication of the way participants position themselves in the interaction. When a forum tutor offers advice, constructs part of a solution, or produces a hint, a student can either accept or question the information (just as contributions are either accepted or rejected in Clark's (1992) model of conversation). A student adopting a position as an active participant in a forum exchange may ask questions or challenge contributions as part of a self-regulatory learning strategy and in order to repair knowledge deficits.

Nineteen percent of the exchanges on limit and 20% of those on related rates contained a contribution in which the student questioned or challenged the contribution of a forum tutor. Only one of these exchanges in each topic had a conversational complexity index less than 7, and both of these had a participation code of 1212 (complexity index of 6). In other words, when a student introduced a question or challenge into a discussion, the outcome was an extension rather than a termination of the conversation. The patience and politeness that characterized this tutoring exchange are particularly noteworthy as an indication of how the forum functions as a tutoring environment in which students can safely challenge the contributions of more experienced others.

*Student Indicates that the Issue has been Resolved*

In the classroom, it is generally the teacher who is positioned to evaluate the understanding of the student(s) and makes the decision whether to continue or terminate a discussion (e.g. the IRE dialogue pattern in which the teacher asks a question and then evaluates a student's response (Mehan, 1979). Similarly, in tutoring sessions, the tutor is the participant who assesses the understanding of the student and decides whether to extend the discussion or move on to the next topic. In contrast, in an open, online forum, it is the student who initiates the exchange and who is ultimately responsible for deciding whether the goal of the interaction has been achieved to her/his satisfaction.

The perceived importance of indicating resolution within an exchange is underscored by the existence of automated “thank you” responses in some online help forums. For instance, MathHelpForum.com appends a “thanks” button to each post so that members can, with the click of a mouse, generate a response reading “The following users thank [name of contributor] for this useful post: [name of member].” This feature was introduced in the forum to support and encourage public recognition of the usefulness of member contributions.  However, because not all forums include this feature (including the forum chosen for the current study), and because students may indicate that an issue is settled in other ways, it is worthwhile to consider a broader range of resolution markers.

There are several ways that a participant can indicate that an issue has (or has not) been resolved. First of all, participants can be silent and opt not to further contribute to an exchange. Silence in computer-mediated exchanges may indicate acceptance or rejection of another’s contributions and does not offer evidence for (or against) the achievement of resolution. Thus, in the forum discussions, if a student does not return to the exchange beyond the initial posting or following tutor interventions, it is not clear whether the student feels that the issue has been settled or not. Exchanges of this type are referred to as “hangers” since other forum participants are, in some sense, left hanging regarding the helpfulness of their contributions. On the other hand, when a student does acknowledge tutors’ contributions, they can do so in either a weak or strong manner. For instance, an expression of appreciation, such as “Thank you,” indicates a weak level of resolution on the part of the participant since this may simply be a residual of polite manners, that is, a customary response to receiving assistance. In contrast, the contribution of mathematical actions (e.g. the presentation of a solution to the problem) and assessments (e.g. reflections on differences in understanding) are stronger indications that the issue has been resolved to the satisfaction of the student. Finally, an exchange can evince a lack of resolution, as when a student receives no response to a query or receives a refusal from forum tutors to provide further assistance. **Error! Reference source not found.** shows the number of exchanges for each topic in which resolution could not be determined (hangers), in which resolution was evident and the strength of the expression (weak versus strong), and in which there was no resolution.

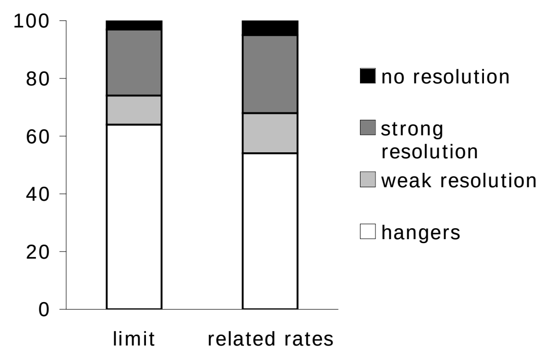


***Figure 2.*** **Student indications of resolution by topic.**

The majority of the exchanges that were not “hangers” (with unspecified resolution) exhibited resolution from the student’s perspective, in either a weak or strong manner. This means that roughly 40% of the exchanges showed some level of resolution, surely a level that is higher than most classroom exchanges in which the teacher has little indication whether students “got it.” In addition, the number of exchanges exhibiting characteristics of strong resolution outnumbered those in which only weak resolution was evident by a factor of two, a finding that is consistent with the amount of student activity in the forum. Over three quarters of the exchanges evincing weak resolution had low conversational complexity (index of 7 or less), so that exchanges in which a student received help and merely thanked the tutor(s) without demonstrating why the intervention was helpful were more likely to be brief transmissions of information rather than interactive discussions. Furthermore, there were very few exchanges (3 on limit and 5 on related rates) for which the issue was not resolved and in which the outcome of the exchange from the student’s perspective could be characterized as inconclusive or unhelpful.

## Conclusions

Free, open, online, help forums have transformed tutoring and instructional assistance outside of the classroom in a grass roots fashion. These help forums have grown up in response to a prevailing and universal need for accessible, efficient, and cost-effective homework assistance. The operational principle behind of these forums is that they connect students with a group of others who are willing to contribute their time, expertise, and support to help anonymous students arrive at solutions to course-related queries. Sites that are staffed by volunteers who spontaneously visit and come to the aid of students − the “Good Samaritans” of mathematics (van de Sande, C. & Leinhardt, 2008) − host collaborative dialogues so that tutoring in this venue is realized as a public and often many-to-one conversation between participants rather than as a private, one-on-one activity.

In some forums, students are positioned to learn calculus in nonstandard ways as they make mathematical assertions and proposals, question and challenge others’ proposals, and take initiative in demonstrating that the issue at hand has been resolved. As they work through course assignments and materials, students must both formulate and communicate their ideas, thoughts, and reasoning on the mathematics behind the solution to a problem to someone else through the computer. In addition, students in the forum may freely engage in self-reflection following tutor intervention. Student participation in such cases goes beyond the construction of a correct solution to the exercise to an analysis of the discrepancy between a novel and prior understanding that is then shared openly with other participants as a natural contribution to the exchange. This practice, consistent with the assumption of responsibility by students for their own learning efforts and advancement of understanding, has been observed in other online learning environments and appears to be facilitated by a computer-mediated mode of communication (Muukkonen, Lakkala, & Hakkarainen, 2005).

This study on authentic student-initiated online help-seeking positions us as researchers and educators to better understand how students are working on assignments outside of the classroom. In this emergent learning environment, many students are positioning themselves to learn and, through interaction with members of a larger mathematical community, taking away much more than just solutions to exercises. The impact such participation has on students’ broader mathematical endeavors and experiences remains to be addressed.

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# INTERWEAVING TASKS AND CONCEPTIONS TO PROMOTE MULTIPLICATIVE REASONING IN STUDENTS WITH LEARNING DISABILITIES IN MATHEMATICS

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*This case study examined the efficacy of tasks designed for promoting multiplicative reasoning in students with learning disabilities. Chad's (grade 4) construction of a mixed-unit coordination scheme was nurtured in the context of a teaching experiment with 14 students in two USA Midwest elementary schools. The analysis focuses on how a sequence of tasks, tailored to Chad's available conceptions, brought forth his transfer of a crucial mathematical idea to novel, realistic problem situations. We argue for the use of such conceptually tailored tasks as a means for promoting students' progress from what they know to a transfer-enabling stage (anticipatory).*

## Background

How can one design tasks that draw on available conceptions of students with learning disabilities (LD) in mathematics to effectively promote their conceptual understanding of a key idea, including 'transferring' this idea to a novel, realistic problem situation? The present study addressed this problem, which is particularly vital for students with LD who, too often, are left behind when facing the challenges of reasoning multiplicatively. A common denominator of LD is a severe discrepancy between the student's academic achievement and his/her normal or near normal potential (Mercer & Pullen, 2005). "Mathematics disabilities represent a learning disorder that has specific cognitive, behavioral, and potential neurological profiles" (Geary, Hoard, Nugent, & Craven, 2007, p. 83). Research indicates that the mathematical performance of a 9-year-old student with LD remains at about grade 1 level; the gap grows over time, as a 17-year-old student with LD performs at about grade 5 level (Cawley & Miller, 1989). In the USA, two Federal Acts—No Child Left Behind of 2001 and Individuals with Disabilities Education Amendment (IDEA) of 2004—have mandated that students with LD should achieve the same rigorous academic standards as their peers. However, while procedural, rote memory approaches are common practice in special education settings, conceptual knowledge of students with LD has not been studied methodically (Geary et al, 2007), particularly not their learning to reason multiplicatively.

This study focused on how carefully designed tasks/activities facilitate the understanding of *schema of correspondence* (Piaget, 1965) and *double counting* (e.g., Steffe, 1994), which are fundamental to multiplicative reasoning (Vergnaud, 1983, 1988, Nunes & Bryant, 1996). We were particularly intrigued by the difficult-to-grasp scheme of mixed-unit coordination. In this scheme, a learner has to figure out the number of singletons (units of One, 1's) or numerical composite units (larger numbers, CU) that, combined, constitute two sets of objects. For example, a child may be presented with 5 groups (e.g., 'sport teams') of 8 items (e.g., 'players') and 24 additional single items, and asked how many players are in all, or how many teams can be

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created if the 24 players were grouped, too. Mixed-unit coordination strengthens the understanding of what unit (singletons or CU) the student is operating on.

## Conceptual Framework

A constructivist perspective rooted in Piaget's (1985) and von Glasersfeld's (1995) core notions of assimilation, anticipation, and reflective abstraction, served as the overarching framework for this study. In particular, we used Tzur and Simon's (2004) recent distinction of two stages—*participatory* and *anticipatory*—in the construction of new mathematical conceptions via the mechanism of *reflection on activity-effect relationship* (Ref*AER, see Simon, Tzur, Heinz, & Kinzel, 2004). *Ref*AER* commences via assimilation of a task into the learner's extant schemes, which engender an anticipated global goal, and possibly sub-goals, toward which the learner carries out mental activity sequences with/on objects. The reflective process consists of two types of comparison the mind continually executes, through which learning—transformation in one's anticipation—occurs. Type-1 involves comparison between the learner's goal and the actual effect(s) of his/her activity. This allows the learner to notice effects s/he had not noticed before and relate them with the activity, hence re-structure their awareness (Mason, 1998, 2008). Type-2 involves comparison across his/her re-presented *AER* records. Common to both stages is the invariant anticipation of particular effects that follow an activity; the stages differ in the extent to which a learner has access to that anticipation. At the participatory stage a learner can only access an evolving anticipation if she is prompted for the activity that generates its effect(s). At the anticipatory stage, s/he can spontaneously access the anticipated relationship to consistently and properly employ it for solving similar tasks, that is, for the desired phenomenon that educators call 'transfer.'

The *Ref*AER* framework entails a particular stance toward design and implementation of tasks for promoting and assessing students' learning that elaborates on Simon's (1995) hypothetical learning trajectory notion. To promote learning, Simon and Tzur (2004) and Tzur (2008a, 2008b) suggested a cyclic process that necessarily begins with analysis of every student's available conceptions. To analyze such learning, Tzur (2007) proposed the fine-grained assessment method, in which tasks are sequenced from 'hard' ones (prompt-less) to 'easy' ones (prompt-inclusive). This method is consistent with several recent studies (Sullivan, Mousley, & Zevenbergen, 2004; Watson & Mason, 1998; Watson & Sullivan, 2008; Zaslavsky, 2007) as it focuses on identifying what in a learner's mathematics allows his/her to independently initiate activities that, via reflection, can bring forth intended changes in his/her conceptualizations.

The Explicitly Nested Number Sequence (ENS, see Steffe & Cobb, 1988) was the central content-specific notion that guided this study. ENS refers to a number scheme a child constructs in which abstract composite units (CU), integrated from units of one (1's), are embedded within and linked to one another. For example, 8 is nested within 9, is nested within 10, etc. Most importantly, such CU can be embedded within larger CU (e.g., 3 CU of 2 and 5 CU of 2 are embedded within 8 CU of 2). It is the ENS that underlies the foundational scheme, multiplicative mixed-unit coordination (mMUC), on which the present study focused. A critical aspect of the ENS is that the child's standard number sequence (1, 2, 3, 4, etc.) is used for counting two *distinct* types of units (CU, 1's) that must be explicitly differentiated prior to operating on/with them, particularly when it is possible to select and operate on the other.

## Methodology

This study was conducted within the larger context of the NSF-funded, *Nurturing Multiplicative Reasoning in Students with Learning Disabilities* project[1] (Xin, Tzur, Si, 2008). The second author (Ron) led a constructivist teaching experiment (Cobb & Steffe, 1983; Steffe, Thompson, & von Glasersfeld, 2000) to promote and study how seven pairs of 4th and 5th graders with LD construct multiplicative conceptions. The data presented in this paper focuses on five consecutive episodes with Chad and his partner Tara (pseudonyms), which took place at their elementary school's group education auditorium.

Data include the children's solutions to and the tasks posed in the context of a turn-taking, 'platform' game we call, "Please Go and Bring for Me …" (PGBM). Its basic version involves sending a student to a box with Unifix Cubes, to produce and bring back a tower made of a few cubes. After taking 2-9 'trips' for bringing same-size towers, students are asked how many towers (i.e., composite units, CU) they brought, how many cubes each tower has (i.e., unit rate, UR), and how many cubes (1's) there are in all (hereafter, $MT_N$ indicates $M$ towers of $N$ cubes; $7T_5$ means 7 towers of 5 cubes each). The PGBM game, and particularly its basic version, was designed to promote learners' anticipated creation of and differentiation among 1's and CU. These two anticipations are crucial if the learner is to ever construct the mental operation of multiplicative double counting (mDC, see Steffe & Cobb, 1998), that is, to operate with his/her single number sequence on two different unit types. A variation of PGBM, central to this study, was designed to promote construction of a multiplicative mixed-unit coordination (mMUC) scheme, which indicates the explicitly nested number sequence (ENS, see Steffe & Cobb, 1988). For example, a teacher may ask: "I covered $9T_6$ here and 18 cubes there. If you put all 18 cubes into $T_6$ and moved them under the other cover, how many *towers* will you have in all?"

Analysis began after each episode, with a team's discussion for noting major events and their significance. These notes served for planning tasks for the next episode(s) and for later, retrospective analysis. Next, team members read the transcripts of each episode and highlighted segments with critical events (teaching moves, changes in learners' anticipation), including conjectures as to why children solved or failed to solve a problem the way they did. This phase focused on two principal issues: the unit a child operates on and the operation s/he uses (Steffe & von Glasersfeld, 1985). Highlighted segments were then discussed retrospectively, to identify explanatory segments. All segments were organized in a story line (presented next) that interweaves theory-based tasks and prompts with our inferred models of the children's available and evolving conceptions.

## Analysis

We organize this section chronologically, along the 5 teaching episodes during which Chad progressed from having no idea of multiplicative mixed-unit coordination (mMUC) to the anticipatory stage of this scheme. This stage, as we predicted, enabled his independent solution of a novel, realistic task a month after the teaching took place (Episode 5). We briefly report on the first 2 episodes, as they served to lay the conceptual groundwork for Chad's learning during the 3rd episode. We culminate with data that indicate Chad's spontaneous use of the mMUC scheme to solve PGBM and realistic tasks (a week and a month later, respectively).

*Episode 1 – November 8, 2008*

The first teaching episode with Chad and Tara familiarized them with the PGBM game. All tasks were posed in the basic form—asking each child (in turns) to bring towers of cubes from the box at the top of the auditorium. After the towers were brought to the bottom of the auditorium and the *known* quantities were spelled out (# of towers=CUs, cubes in each=UR), the children were asked about the total number of cubes (1's). For $3T_5$ and for $3T_4$, Chad added the number of cubes three times; for $5T_5$ he used skip counting, apparently due to his familiarity with multiples of 5. Yet, for $4T_7$ he 'slowed down' and used counting-on from 7 into the second tower, from 14 into the third, and from 21 into the fourth tower. Although not yet fully intentional (anticipatory), Chad's solution indicated multiplicative double-counting (mDC) at least at a participatory stage. The episode culminated by the students sending Ron to bring $3T_9$, which he solved via skip counting (9-18-27) coupled with keeping track of CU on his fingers.

Using the PGBM basic form in *Episode 1* supported two key conceptions for Chad's later learning of mMUC. First, making one tower at a time oriented his awareness to the *anticipated* production of a *composite unit* via the activity of iterating the unit of One (1's). Second, it engendered his use of counting of CUs that he, or others (Tara and Ron), produced. Combined, these activities would underlie not only his mDC, but also differentiation and selection of the unit (1's, CU) he would need to operate on.

*Episode 2 – November 18, 2008*

The second episode focused on both promoting and assessing Chad's and Tara's *AERs* of *production*, *differentiation*, *selection*, and *operation* on 1's and CU. Starting with a task to assess a possible anticipatory stage of mDC for solving a measurement (quotitive) division task, we posed a 'What if?' form of PGBM: "Pretend we would put these 18 cubes in $T_3$, how many towers can you make before running out of cubes?" Chad spontaneously began counting, 1-2-3, 1-2-3, etc., and found he could make 6 towers. To see if Chad used mDC, Ron asked, "Could it be that [in your head] you did, 1-2-3 is '1', 6-7-8 is '2', and so on?" Chad confirmed, which led Ron to pose the follow-up task: "What if we added 9 more cubes and put them in $T_3$; how many towers would you have?" Here, instead of first operating on the 9 cubes and composing them into 3 towers, then adding those to the $6T_3$ (total = $9T_3$), Chad operated on 1's by adding the 9 and 18 cubes via counting-on, announcing the total was 27. Ron responded by first asking Chad if he meant '27 towers'; Chad confirmed. Ron asked him to make the towers, a task that was *particularly geared* toward orienting Chad's reflection on the actual effect of his composition of the 9 cubes into towers. As Chad completed his action and found there were only $9T_3$, he also noticed, and made explicit to himself and to Ron, his confusing of the units: "Oh, yeah, I meant 27 cubes." Indeed, Chad's inability to solve the mixed-unit task even within the context of producing and differentiating towers from cubes indicated that he has not yet constructed the mMUC scheme even at a participatory stage. It is this confusion among 1's and CU in a mixed-unit situation that became the focus of our ongoing analysis, and turned into a set of tasks designed for first promoting *explicit* unit differentiation.

*Episode 3 – December 2, 2008*

We began this episode to examine, and possibly claim, an anticipatory stage of a new conception. In particular, we asked both students to pretend they would bring $7T_3$ and figure out, in the absence of real cubes, how many cubes there would be in all. Already in answering the questions about the givens in the task ("How many cubes in each tower?") Chad's response ("7") indicated the aforementioned confusion of units. Unlike Tara who intentionally kept track of CU, Chad first just counted CU of 3 (1-2-3, 1-2-3, ...) to find the total. Thus, Ron replicated Tara's method—to possibly prompt Chad's re-activation of mDC. This intervention was useful, as Chad added keeping track to his counting of CU (1-2-3, 4-5-6, etc.). When Chad was done, Ron introduced the first among many statements of emphasis on the unit with which one operates as it relates (or not) to the answer requested in a task. That is, Ron initiated the socio-mathematical norm (Cobb, Yackel, & Wood, 1992) of *always* making explicit which unit a person is operating on and/or talking about.

The second task of this episode then moved to assessing the students' operation on CU. Ron asked Chad and Tara to pretend they brought 2 more $T_3$ and added them to the given $7T_3$, and to figure out how many such *towers* there would be in all. Consistent with our ongoing analysis conjecture, both students shifted to operating on 1's, figuring out there would be 27 cubes. Like in the previous episode, Ron asked to clarify if they found how many towers or how many cubes, and Chad realized ("Aha") that he responded in terms of cubes. Ron re-negotiated the aforementioned norm then moved on to a task designed for promoting unit differentiation.

The third task aimed to promote awareness to the difference between 1's and CU. Ron asked Tara and Chad to produce $9T_5$ and $7T_5$, respectively, and then name all differences and similarities they could detect between their sets. After announcing several non-mathematical ones (e.g., colors), Tara suggested and Chad agreed that common to both sets were 5 cubes in each tower. Next, Chad also noticed and used *counting-up of CU* to figure out a difference: "I got 7 [$T_5$]. I added 2 more and got 9 [$T_5$] so she got 2 more." Ron asked, "Two more what?" and Chad responded, "Two more towers." Then, Ron engaged them in solving how many cubes they had in both sets. Chad used mDC, skip counting by 5's and his 10 fingers (failed); later he used his 10 fingers and 6 of Ron's to figure out – 80. Thus, we turned to a task conjectured to bring forth Chad's first solution to a mixed-unit task, at least when prompted.

Ron showed Chad and Tara how he produced $6T_4$ (CU), then covered the towers. He then placed 12 additional cubes (1's) on the desk, and asked while pointing to those cubes: "If you put all of them in towers of 4, how many towers will we have?" Taking a few seconds to contemplate the task, Chad then spontaneously began manipulating the visible cubes into groups of 4, counted the groups, and (as he later explained his solution to Tara) used counting-on of $T_4$ to solve the task: "9." Ron repeated Chad's solution for Tara as well as to boost Chad's confidence: "If you put them in $T_4$, you have 3 of those [points to the groups of 4] and adding the $3T_4$ to the $6T_4$ will make 9."

Chad's (but not Tara's) successful operation in the mixed-unit situation included the following activity sequence: (a) selecting the singleton set (1's) to operate on, (b) composing them into groups of four 1's, selecting the proper quantity (CU) from the other set—6 *towers* (of 4 *cubes in each* with total of 24 *cubes*), and (c) operating 'additively' on the CU from both sets. We specify this sequence precisely because Chad was not yet able to anticipate it in its entirety in the absence of tangible objects.

For the last task, Ron produced $7T_5$ and covered them; placed 10 singletons nearby and covered them, then asked: "If we put those 10 cubes into $T_5$ and 'mush' them altogether with the $7T_5$, how many $T_5$ will we have?" This time, both Tara and Chad were at a conceptual loss. Quietly and persistently using his fingers, Chad seemed to mightily struggle with executing the activity sequence—but to no avail. After about 3 minutes, Ron decided to examine if Chad had constructed the mMUC at least at the participatory stage and, as a prompt, lifted *only the cover above the* $7T_5$. Soon after the towers became visible, Chad joyfully exclaimed, "9." In response to Ron's follow-up question Chad explained, while explicitly using his open hands to indicate a 'tower-of-five' on each hand: "Five right here (lays left hand on desk), and five right here (lays right hand), so that is 2 towers… so that is 1-2-3-4-5-6-7 [counting visible T5] and put 2 more like I did last time and [so we] get, 8-9."

There were three crucial pedagogical moves, explicitly informed by the *Ref*AER* account, in that last task of Episode 3. First was the increase in conceptual demand on the students' mental operations, via covering both sets of objects (1's, CU). This move proved powerful particularly when considering how the second move—prompting by lifting the cover—enabled Chad's completion of the activity sequence. Before the prompt, Chad could anticipate executing the first two components of the sequence (selecting 1's, composing them into CU), but not the third (selecting the $7T_5$ and adding the $2T_5$ resulting from the prior actions). Once prompted, Chad quickly reinstated his evolving, participatory stage of mMUC. The third move—asking Chad to explain his solution to Tara—opened the way for Chad's re-presentation of and reflection on records of his solutions to the last *two tasks*. It must be noted that Chad was not asked about the similarity between the situations; rather, he spontaneously and quite proudly announced they were alike. Thus, we argue that the reflection-orienting follow-up question opened the way for a *twofold* realization on Chad's part. He realized that his 3-step activity sequence, particularly the final step, produced an effect identical with his global goal in the task and that this activity-effect production was invariantly applicable across the last two tasks. Accordingly, we conjectured he might be able to independently solve such tasks in the next episodes. This conjecture, which amounts to saying we attributed to Chad an anticipatory stage of the mMUC scheme, was strongly confirmed (See Episodes 4-5).

*Episode 4 – December 9, 2008*

This episode consisted of 3 mixed-unit tasks, all posed within the context of PGBM in the absence of cubes. Chad solved all three with clear anticipation of the 3-step *AER* underlying the mMUC scheme. During the first task, ($5T_4$ & 8 cubes), he found out that 2 more towers would be composed, but lost track of the number of towers (not of the unit to operate on!) and added 4+2 to arrive at $6T_4$. He immediately corrected himself ($7T_4$) when Ron briefly lifted the cover above the $5T_4$. Then, Chad independently and straightforwardly solved the second ($7T_6$ & 18 cubes) and third ($9T_7$ & 21 cubes) tasks, though he experienced some difficulties explaining why the second would yield $10T_6$. Furthermore, unlike previous events in which a different answer by Tara would confuse Chad, her response (13) to the second task did not impact him whatsoever. We concluded that Chad had constructed the anticipatory stage and conjectured he would solve 'transfer' tasks in realistic situations in the same way.

*Episode 5 – January 13, 2009*

A month and a few holidays after Episode 4, Ron presented Tara and Chad with the following problem printed on a paper. "Grandma bakes chocolate chip cookies for birthday bags. She puts **6** cookies in each bag, and has already filled **9** bags. She now has **12** more cookies waiting to cool down. Once she'll put these cookies in bags, how many birthday bags ready for the party will she have?" Consistent with our theory-based prediction, Tara was unable to solve the problem (15, via adding 9+6), whereas Chad could again ignore her response, confidently announce, "11," and quite simply explain that grandma could fill 2 bags with 6 cookies in each, and hence 11 bags (9+2). Ever since, Chad solved any mixed-unit problem situation in a similar way and with much confidence.

## Discussion

This study contributed to two novel and important understandings. First, it showed how effective tasks designed on the basis of fine-grained assessment of evolving mathematical conceptions of a student with LD could be in promoting such students' learning. In particular, analyses of units (quantities—1's, CU, UR) a child operates on in situations that call for relating those units multiplicatively enabled designing a sequence of tasks, as well as creating on-the-spot follow-up prompts and questions that promoted transition to a participatory and then an anticipatory stage of mMUC. We claim that the latter, which is consistent with Steffe and Cobb's (1988) ENS construct, is the root for Chad's capacity to transfer his thought processes to the novel, realistic situation.

Second, this study demonstrated the possibility for effectively teaching students like Chad to properly reason (coordinate, or 'translate' quantities) in situations that are divisional in nature. That is, mixed-unit situations like those Chad successfully solved in different contexts (e.g., $7T_6$ + 18 cubes) require acting on the two sets by *segmenting* (Steffe & Cobb, 1998) the composite unit (18) with a given unit rate (6 cubes per tower, quotitive division) while maintaining the composite unit (set) of composite units (7 towers) of units (six 1's per tower).

## Endnotes

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# EXPLORING THE CHANGING PERCEPTION OF MATHEMATICS AMONG ELEMENTARY TEACHER CANDIDATES THROUGH DRAWINGS

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*Elementary teacher candidates enrolled in a mathematics methods course were asked to "draw math" at the beginning and end of the semester. Findings display the vision of mathematics that teacher candidates have before and after exploring teaching methods and implementing these methods with elementary students. In addition, it examines the specifics of the changes that occurred during the semester of methods and field placement experience.*

## Background

What is math? Often people reflect back to their vision of school mathematics, while others reflect upon its relevance to the real world when this question is asked. Vinter (1999) found that teachers often struggle to find the application of much of the math they teach. This can be due to the lack of meaningful experience with the content taught in the elementary grades (Ball & Bass, 2000). In addition to lack of experience with math, many elementary teacher candidates have high levels of mathematics anxiety (Swars, 2006). These factors can affect the impression of mathematics that teachers give to their students. Through examining their own perceptions of mathematics, teachers and teacher candidates can begin to explore how to deepen their own understanding, overcome anxiety, and connect the content to elementary students.

This article documents an elementary mathematics methods course, which begins by asking teacher candidates to draw math and write a few sentences describing the drawing. The drawings often involved students communicating and reflecting upon their emotions and past experiences associated with the content, including mathematics anxiety. In addition it became a theme of the course throughout the semester, inviting students to revisit their perceptions as various methods and content were introduced. This simple task provided insight into the perceptions teacher candidates bring to their teacher preparation programs and the impact that positive experiences with students and content can make upon these perceptions.

### *Drawing*

Student drawings have been used to examine students' perceptions about various content areas for years. In literacy, drawings of reading and writing have been used to understand their perceptions of the subject areas (McKay & Kendrick, 2001). Students' impressions and attitudes of scientists and science have been studied in many elementary and teacher candidate classrooms in order to understand student perceptions (Thompson et al., 2002). Drawing images before writing or verbalizing ideas can foster more creative responses and help generate ideas, because often language can slow down the creative process (Caldwell & Moore, 1991). Ideas can be explored through drawing without the cognitive demands often found when using language.

Art is often used in therapy, because thoughts and emotions can be expressed vividly through images (Lusebrink, 2004). These same techniques can be useful in supporting meta-cognition and addressing negative emotions often tied to mathematics by elementary teacher candidates. Drawings by teacher candidates of various subject areas can reveal dispositions, attitudes, and experiences related to a subject area. These drawings allow the artist to establish and reflect upon these attitudes and experiences in a non-threatening way (Rule & Harrell, 2006).

By acknowledging and giving voice to negative emotions and experiences, such as mathematics anxiety, through drawings, one is better able to deal with and move beyond those negative emotions and experiences (Rule & Harrell, 2006). This study relies on the theoretical understanding that the relationship between the conscious and unconscious mind can be expressed through images and thus given a voice where it otherwise might be ignored (Hillman, 1992). Often the negative emotions surrounding a concept, such as mathematics, develop in ways that distort original experiences. Instead reflection upon past experiences tends to reflect the current emotional attachment that has evolved over time and repeated experiences with that concept (Hillman, 1992). Sometimes images used to express the larger concept, such as mathematics, can be literature snapshots of a particular event and at other times they are more representational. However, both reveal a deep insight into the true relationship of the artist to the subject, such as mathematics (Rule & Harrell, 2006). By examining his/her own understanding and perception of a subject, the artists are better able to improve the negative emotions related to the concept and this in turn allows them to focus on the learning, without the obstacles associated with their negative experiential baggage. Watkins (1984) suggests that by investigating images and discussing feelings related to the images, the artists become empowered to engage actively in changing the negative perceptions related to the subject.

*Mathematics Anxiety*

Mathematics anxiety often begins in elementary school when students have negative interactions with the content and are taught by procedural, rather than conceptual teaching methods (Harper & Daane, 1998). Tying instruction to the exact procedures in the textbook, timed tests, hostile teacher behavior, embarrassing students in front of peers, only accepting one method of solving a problem, and lack of differentiation based on student needs are all factors that can contribute to mathematics anxiety (Swars, 2006). Hembree (1990) found the highest level of mathematics anxiety among college students came from elementary teacher candidates. To resolve mathematics anxiety, teachers need to have positive experiences with mathematics and see the purpose behind the mathematics they are teaching and have mathematical experiences with manipulatives and working in groups (Harper & Daane, 1998; Swars, 2006). Vinson (2001) found that exploring the conceptual content in meaningful ways with manipulatives before learning the procedural aspects of mathematics reduced the mathematics anxiety among teacher candidates. This impacts the way they teach their students.

When teachers are confident in their mathematics ability they spend 50% more time teaching mathematics than those who have mathematics anxiety (Schmidt & Buchmann, 1983). In addition teachers with math anxiety spend less time implementing standards based instruction and more time teaching to the whole class and assigning seat work (Bursal & Paznokas, 2006; Bush 1989). These activities perpetuate the notion that mathematics lacks real-world meaning.

*Perception of the Use of Mathematics*

Vinter (1999) found that many teacher candidates lack an applied understanding of mathematics and this in turn affects their ability to make the content meaningful for their students. Resnick (1987) suggests that many times teachers prepare students to do school math, but this is not the same as mathematics beyond the classroom. NCTM (2000) stresses the importance of problem solving, communication, and connecting math content, which requires teachers to have a deeper understanding in order to support these connections. Chappell and Thompson (1994) expressed the importance of the mathematical courses that teacher candidates experience in their preservice programs. The preservice program is crucial to the development of teacher candidate's beliefs, content knowledge and attitudes about the way math should be taught at various grades and their effectiveness as educators.

## Research Questions

1. What is the perception of mathematics held by elementary teacher candidates at the beginning of a mathematics methods course?
2. How do elementary teacher candidates perceptions of mathematics change during a mathematics methods course?

## Methodology

This study examined the symbolic representations of math drawn by teacher candidates at the beginning and end of a mathematics methods course. The drawings were analysed and categorized to explore the initial impressions of mathematics that teacher candidates bring to their teacher preparation courses and the changes experienced through opportunities to discuss anxieties, work with students, and explore various pedagogical theories for mathematics.

Sixty-two teacher candidates were enrolled in a mathematics methods course for elementary teachers at a midsized university in the southeast. The study took place over a period of two years with the same mathematics instructor, but four different sections all offered in the spring of their junior year. There were fifty-nine females and three males, fifty-five Caucasian, four African-American, and three Latino teacher candidates.

During the first class meeting, participants were asked to draw pictures of math. Teacher candidates were told to draw whatever came to mind and not to filter images. When they asked for further details about what to draw, the instructor simple advised them to draw what comes to mind about math. It was emphasised that the grades in the course would in no way be influenced by the drawings. Teacher candidates were asked to put the last four digits of their student identification number on the paper in order to compare pre-test and post-test results. Then they were asked to write a few sentences related to their drawing on the back of their paper.

During the semester teacher candidates spent one day a week in a practicum experience. In addition during the weekly three-hour mathematics methods course, each teacher candidate spent 30 minutes working with two fourth graders exploring various mathematics content. A portion of the methods course involved discussing the lessons learned from the fourth graders. In addition, teacher candidates explored various theories about teaching and learning mathematics, effective use of technology in mathematics, and the role of manipulatives in mathematics.

Throughout the class, there were informal conversations about the impressions of math that teachers bring to the classroom. Teacher candidates were able to evaluate the content and their experiences with their students with a conscious understanding of their lens that developed from their personal experiences as a learner. For example, those who understood algorithms were able to listen to students with misconceptions or invented algorithms. Teacher candidates who entered the methods course with hesitation due to their own struggles with math often found their struggles could help them relate and better explain the content to their students.

The same drawing activity was conducted the final day of class. Teacher candidates were asked to draw math and write a few sentences about it. In addition they were asked to write 2-3 sentences describing if and how the course changed their impressions. These drawings, and sentences were used to investigate the mathematics views held by the teacher candidates and the changes experienced over the course of the semester.

*Data Analysis*

The writing was used as needed for clarification of the meaning behind the drawings. Drawings were categorized in three ways: positive, neutral, and negative emotions; particular experiences and general meanings, and classroom, abstract or real world connection. Positive, negative and neutral were based on if the drawings or descriptions had specific emotional prompts, such as faces with smiles or tears. Next, the pictures were grouped by particular experiences or general meaning. Particular experiences were drawings in which one point in time was displayed, such as drawing on the board. When pictures simply had mathematical images such as a fraction, this was classified as general meaning, Finally, the drawings were categorized based on if the pictures showed images connected to the classroom, were abstract, or were connected to the real world. Pictures that were connected to the classroom displayed images such as books, teachers, or a white board. Abstract images were images such as fractions, multiplication problems and algorithms. Real world connections, had images such as shopping or cooking. This type of classification for analysis is based on the evaluation technique used by Rule and Harrell (2006). These categories were confirmed by a member of the mathematics education faculty. This faculty member evaluated and supported the initial findings, categories, groupings, and count. This confirmation provided validity of the interpretation. Then the analysis to determine the change was charted by grouping pre and post drawings and then analysing them for positive and negative changes. These findings were also confirmed by the aforementioned mathematics education faculty member.

## Results

*Initial Drawings*

Through the drawings and writings, teacher candidates expressed a variety of experiences and impressions of mathematics. A majority (32) of the experiences were negative. However, there were nine that were positive and twenty-one were neutral. All the positive drawings related mathematics to the content, real world examples, and puzzle. Most negative drawings related to the teacher candidates' emotions and experiences in school (see figure 1 and 2). For example, three people drew themselves at the board with question marks. Question marks seemed to be a common expression for teacher candidates to show their feelings of confusion. Also, many drew textbooks and jumbled ideas in their drawings.

*Figure 1*

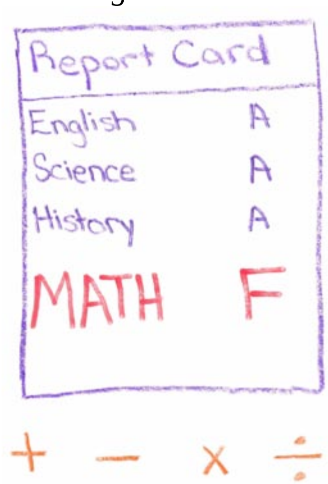


*Figure 2*


*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

Table 1 shows the positive, neutral, and negative emotions connected to the drawings. A common thread for many of the negative drawings was the struggle that teacher candidates felt in school mathematics. One teacher candidate explained, "I never understand it. I always feel stupid and like what the teacher says is a foreign language." Several wrote sentences expressing the desire to change these emotions in order to avoid negatively impacting future students.

Table 1
*Categories of Drawings*

| | Pre-test | Post-test |
|---|---|---|
| Positive | 9 | 38 |
| Neutral | 21 | 24 |
| Negative | 32 | 0 |
| Particular Experience | 33 | 19 |
| General | 29 | 43 |
| Classroom Setting | 28 | 22 |
| Read World Setting | 7 | 11 |
| Abstract | 27 | 29 |

After analysing the emotions in the drawings, were categorized based on if they referred to particular experiences or were more general (see table 1). Thirty-three drawings displayed a particular point in time. For example one teacher candidate drew tears and question marks around an illustration of herself. In her writing she explained that she remembers her fifth grade math teacher being angry with her. She went on to describe a time she was at the board in fifth grade and had no idea how to solve the problem. She described this situation as representative of her feelings of math. Twenty-nine drawings were math symbols or items related to math, rather than a specific memory related to math.

The final category evaluated if the images related math to the classroom setting, real world setting, or were more abstract (see table 1). Many teacher candidates drew content images such as shapes, numbers, and equations. The abstract images had a mix of written responses varying between positive, negative, and neutral emotions. The teacher candidates who drew themselves cooking, shopping, or building all expressed a passion and conceptual understanding of connections, real world meaning, and reasoning essential to mathematics.

*Drawings of Change*

The experiences of teaching mathematics to fourth grade students during the methods course as well as exploring content in the methods course from a Standards-based approach (NCTM 2000) were positive for the teacher candidates. Fifty-eight teacher candidates reflected in the final writing, that they now saw the connections between the real world and the content they were teaching. While five were still hesitant and concerned about their own understanding, they expressed growth and a more positive perception of math. Thirty-eight of the final drawings and writings showed a new perception that math is fun, meaningful, and makes sense. Teacher candidates expressed more confidence in teaching mathematics, but also more confidence in their own personal mathematical abilities.

> I always thought I was bad at math and dreaded this course, but now I see that math isn't just memorizing stuff the teacher says. It is talking about stuff, exploring different ways of doing things, and thinking about what makes sense.

I am actually good at it, now that I understand that.

When the emotions attached to the final images were compared to the initial drawings a positive growth in emotional affect and reduction of anxiety was seen (see Table 1). The drawings that showed positive emotions in mathematics contained images of collaboration, manipulatives, real world connections, and discussions. There were no changes towards negative emotional connections to mathematics in the drawings. Those that initially drew positive drawings kept these, but several included more images of collaboration and simplified the mathematics in the drawings to match the mathematics that they will use in the elementary classroom. In the final drawings the images of specific classroom experiences were positive, rather than the negative images initially drawn. In addition, drawings displayed more collaboration and meaningful learning (see figure 3).

*Figure 3*


## Implications

These findings contribute to the body of research on perceptions of mathematics held by elementary teacher candidates. Several key findings provide insight into teacher candidates' perceptions and have implications for teacher education programs.

This study provided further evidence of the negative experiences teacher candidates bring to the classroom (Swars, 2006). It goes deeper than simply recognizing these negative emotions and experiences. It examines how teacher candidates view the concept of math. The details in the images provide visual understanding of the anxiety experienced by many. One interesting finding was the discovery that the negative experiences are often related to the classroom rather than real world math. This aligns with the work of Nicol, who suggests this lack of real world connection negatively impacts their students (2002). When teacher candidates connected math with real world experiences, they viewed math in a positive light and displayed confidence in the content.

When teacher candidates changed their depiction of mathematic to a more positive image, this included images of discussions, manipulatives, understanding, and connections to the real world beyond the classroom. Even when classroom images were the focus of the drawing, the writings on the back referred to the importance of making connections. Those who expressed math in a positive light at either the beginning or the end did not make references to textbooks, isolation, or working problems on the board. Instead meaningful, connected, and engaging math was the focus. By exploring the methods for teaching elementary content within a context connected to working with students, teacher candidates expressed a change in beliefs about the concept of mathematics as displayed in their drawings.

The connection between teaching children and reduction of mathematics anxiety aligns with Harper and Daane's study (1998). Because conscious reflections on attitudes can alter negative complexes (Hillman, 1992), the attitudes may have been changed through the symbolic analysis, which was both a part of the unique methodology of the study and a conscious reflective process. This is one of the reasons that the images can be analyzed, but assumptions about why they changed cannot be made. By having teacher candidates draw their perception of math initially, they were more aware of this throughout the course. Their emotions attached to math were a natural part of the conversations about methods and their experiences with students. These teacher candidates were challenged to draw their own perceptions of math and many used this opportunity to express their own problems they encountered in math as a student. This in turn made them more aware of problems students might have throughout the semester.

Teacher educators need to allow time for teacher candidates to reflect upon the perceptions and beliefs they bring to the classroom. This reflection allows teacher candidates to acknowledge their biases and begin to explore how to create more meaningful experiences for students (Rule and Harrell 2006). In addition time for teacher candidates to make real world connections between math concepts should be an essential portion of elementary methods courses (Chappell & Thompson, 1994).

While this study offers insight into the perceptions that elementary teacher candidates hold, further studies are needed. Continuing this activity by having teacher candidates at the end of internship and inservice teachers draw math could show how these perceptions change and are refined as they progress in their development as teachers as well as the sustainability of the newfound positive perceptions. In addition, it would be interesting to compare the images of inservice teachers with the students they teach in order to see the correlation of perceptions. Asking teacher candidates to draw, write, and explain their conceptions of math can further understanding into the mathematics anxiety that many experience, the pedagogical stances they bring into teacher preparation courses, and their understanding of connections between concepts and real world uses of school mathematics. These understandings can provide insight into supporting teacher candidates as they develop positive affect, effective pedagogical strategies, and content knowledge for teaching mathematics. These are all critical areas of development to increase student achievement and end the cycle of mathematics anxiety.

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# TEACHING MATHEMATICS FOR UNDERSTANDING: THE SOCIAL CULTURE OF THE CLASSROOM

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*The purpose of this study is to examine the mathematics teaching cycle of two kindergarten teachers who took part in a professional development project that promoted culturally relevant pedagogy and teaching mathematics for understanding. The study examines if and how the teachers' lesson planning practices and their enacted lessons are consistent with the ideologies associated with culturally relevant pedagogy and teaching for understanding. This paper will focus on the social culture of the classroom, in particular how the teachers encouraged their students to develop their own strategies for sharing fairly and how those strategies were valued in the classroom.*

## Background

Recent reform efforts in mathematics education stress teaching mathematics for understanding in an environment that is accessible to all students (NCTM, 2000). Literature on the nature of teacher planning in light of reform efforts to change the teaching and learning of mathematics is sparse (Simon, 1995; Simon & Tzur, 1999). The majority of research on teacher planning is not specific to mathematics nor does it address how teachers attend to the cultural aspects of teaching and learning or what teachers need to do to promote learning mathematics for understanding (McCutcheon, 1980; Yinger, 1980; Zahorik, 1975). Additionally, research has not addressed how teaching mathematics for understanding and attending to students' cultural backgrounds can effectively be incorporated into teachers' lesson planning practices (Eisenhart, et al., 1993; Gutstein, Lipman, Hernandez & de los Reyes, 1997; Ladson-Billings, 1995; Putnam, Heaton, Prawat & Remillard, 1992).

In light of this gap in the research, this study aimed to examine the lesson planning practices and enacted lessons of two teachers who participated in a year long professional development project which promoted the development of culturally relevant pedagogy (Ladson-Billings, 1994) and teaching for understanding (Heibert, et al., 1997). The purpose was to examine if and how the teachers incorporated the ideologies associated with culturally relevant pedagogy and teaching for understanding into their lesson planning practices as well as their enacted lessons.

## Conceptual Framework

Due to the nature of the professional development project in which the teachers in this study participated in, this study is framed theoretically by tenets of culturally relevant pedagogy (Ladson-Billings, 1994) and Hiebert et al.'s (1997) dimensions of classrooms that promote teaching for understanding. Since the focus is on teacher planning in addition to classroom practices, Simon's (1995) mathematics teaching cycle will be used as a lens through which to view this phenomenon. This section provides details about each construct.

### *Culturally Relevant Pedagogy*

Research has emerged to examine the pedagogy of teachers that are successful at teaching students of color without disregarding their home culture. Ladson-Billings (1994) has coined the term culturally relevant pedagogy (CRP) and defines it as "a pedagogy that empowers students

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intellectually, socially, emotionally, and politically by using cultural referents to impart knowledge, skills, and attitudes" (p. 17-18). CRP does not adhere to an explicit list of behaviors that if performed will ensure the success of all students, but rather is an ideology that may look different in different classrooms. CRP is supported by three general tenets: (1) academic success is experienced by all students; (2) students nurture and sustain cultural competence; and (3) students develop critical sociopolitical conscientiousness (Ladson-Billings, 1995).

*Teaching for Understanding*

Learning mathematics with understanding can be described as the ability to make sense of new knowledge by relating it and connecting it to what we already know. By drawing on the research of four large projects that focused on students' conceptions of multidigit addition and subtraction, Hiebert, et al., (1997) were able to characterize classrooms that promote students' mathematical understanding. Hiebert, et al. (1997) calls these critical characteristics *dimensions* of classrooms that support mathematical understanding and they are: (1) the nature of classroom tasks, (2) the role of the teacher, (3) the social culture of the classroom, (4) mathematical tools as learning supports, and (5) equity and accessibility. This paper will focus on the third dimension, the social culture of the classroom, in particular how students are encouraged to develop and share strategies and if these strategies are valued in the classroom.

*The Mathematics Teaching Cycle*

The mathematics teaching cycle was chosen because it is grounded in the constructivists' view of learning, and as such, is consistent with CRP and teaching for understanding. The framework design for this study utilizes the three major components of the mathematics teaching cycle: planning (drawing on teacher knowledge and the creation of a hypothetical learning trajectory), teaching (the enactment of classroom activities) and assessment (formative). In each of these components, the researcher used tenets of CRP and dimensions of teaching for understanding as the lens through which to analyze the data (see Table 1 and Figure 1).

Teachers whose beliefs are consistent with CRP and teaching for understanding will draw on these constructs during each phase of the mathematics teaching cycle. For example, during the planning phase, one might expect that a teacher would draw on students' prior knowledge, students' out of school knowledge as well as the teachers' own mathematical knowledge. During the teaching phase, one might expect to see a classroom culture where students are encouraged to choose and share their own methods for solving problems and where those methods are valued. In addition, there would be evidence of high academic success by all students and students would be communicating about mathematics in ways that are consistent with their cultural ways of expression. During the assessment phase, one might expect to see teachers' attend to how students are communicating and reflecting on mathematics and how student learning informs the teachers' own knowledge. This framework was used to drive the design of the study as well as direct the analysis of the data to answer the research questions.

Table 1. *Features of CRP and Teaching for Understanding*

| **Features of CRP and Teaching for Understanding (TU)<br>(adapted from Ladson-Billings (1995) and Hiebert, et al. (1997) respectively)** | |
|---|---|
| **CRP:**<br>• High academic achievement is experienced by all students<br>• Cultural competence<br>• Sociopolitical consciousness | **TU:**<br>• Nature of classroom tasks<br>• Role of the teacher<br>• Social culture of the classroom<br>• Mathematical tools as learning supports<br>• Equity and accessibility |

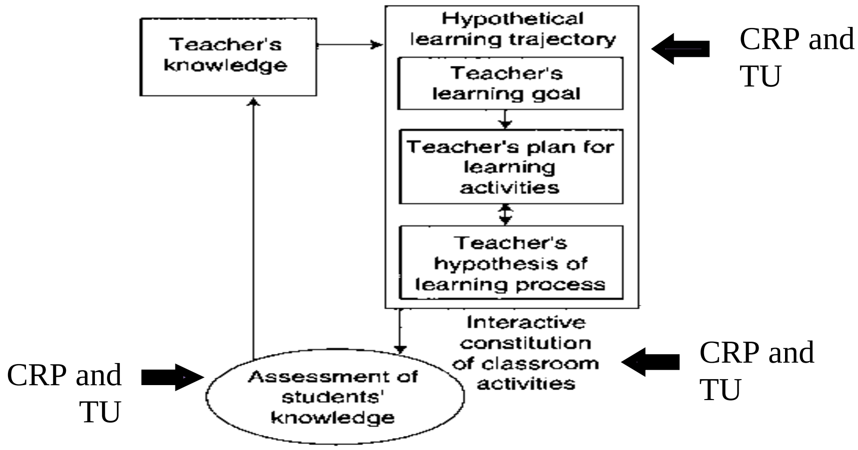


*Figure 1.* Mathematics Teaching Cycle (Simon, 1995).

**Research Questions**

1. What does a kindergarten teacher attend to in each phase of the mathematics teaching cycle?

2. Is there evidence of the ideologies associated with CRP and teaching for understanding with in each phase of the teaching cycle?

**Methodology**

*Context for the Study*

Participants in the current study were involved in Nurturing Mathematics Dreamkeepers (NMD), a professional development project for kindergarten through second grade elementary school teachers (for more information, see Marshall, 2008). Three cohorts of teachers from six local elementary schools were chosen to participate in the project. Each cohort of teachers attended professional development retreats for varying lengths of time (cohort I for three years, cohort II for two years, and cohort III for one year), where participants took part in various mathematical tasks aimed at improving not only their content knowledge and own mathematical understanding, but also their pedagogical content knowledge. In addition, teachers participated in a variety of activities where they were asked to engage in critical reflection on issues related to culture in the teaching-learning process.

*Participants.* This study involves two kindergarten teachers who were participants in the third cohort of teachers. Sarah is a white female in her twenty-ninth year of teaching kindergarten and pre-kindergarten. Pamela is a white female who has spent both of her two years of experience teaching kindergarten. Both teachers held undergraduate degrees in elementary education and had not received any specific training in mathematics instruction other than the curriculum training that was required by the county in which they teach.

*Sources of data.* Four sources of data were collected: a lesson planning interview, an observation of a lesson planning session, video-recordings of two consecutive math lessons, and a post-lesson reflective session. The lesson planning interviews as well as the lesson planning observation was audio recorded and transcribed. The lesson planning interview questions asked the teachers to describe and discuss what they attend to when planning their mathematics lessons. Questions were specific to content, types of activities, resources, assessment (both formative and summative), and student knowledge.

The math lessons and post-lesson reflective session were video recorded. The math lessons were described in five minute increments and portions were transcribed. The post-lesson reflective session was transcribed verbatim. The videotaped math lessons took place in the teachers' classrooms during their regularly scheduled math times for two consecutive days. On the first day, Sarah's math lesson was recorded first while Pamela observed, and then Pamela's math lesson was recorded while Sarah observed. On the second day, Pamela's math lesson was recorded while Sarah observed and then Sarah's math lesson was recorded while Pamela observed.

The reflective session took place four days after the second videotaped lesson. In the reflective session, the teachers discussed how their lessons supported or hindered their students' conceptual understanding of the mathematics and were asked questions regarding their students' out-of-school and in-school knowledge as well as if their enacted lessons followed the lessons they had planned.

*Analysis of Data*

The data was analyzed to describe what the teachers attended to in their lesson planning and what parts, if any, of their lesson planning and enacted lessons were consistent with culturally relevant pedagogy and teaching for understanding. In addition, the enacted lessons were compared to the lesson planning observation to determine if the lessons were consistent with what was planned. The four sources data were analyzed in three separate phases.

*Phase 1: Teaching planning.* First, the three transcribed sources of data (interviews, observation, and reflective session) were organized with respect to Simon's (1995) Mathematics Teaching Cycle and each piece of data was coded for instances of teacher knowledge, hypothetical learning trajectory (HLT), and assessment. Any parts of the data relating to teacher knowledge were then combined into one document. Similarly, new documents were created for parts of the data pertaining to HLT and assessment. The researcher then analyzed and coded each new document for what each teacher focused on in their lesson planning such as objectives, content, activities, materials, etc.

*Phase 2: Teaching for understanding and CRP.* During the second phase, each new document was analyzed for teaching for understanding and CRP. First, the documents for teacher knowledge, HLT and assessment were coded for evidence of teaching for understanding using Hiebert, et al.'s (1997) dimensions of teaching for understanding.

In order to analyze the data for evidence of CRP, the researcher initially coded with respect to the three tenets of CRP: high academic achievement, cultural competence, and sociopolitical consciousness. Once the data were coded, the initial codes did little to answer the research questions. More specific codes were created as sub-codes of the three tenets of CRP. Since CRP is an ideology, it may not be readily observable in a kindergarten classroom.

*Phase 3: Video-taped lessons.* In addition to the documents, the video-taped lessons were also coded for teaching for understanding and CRP. The lessons were viewed several times and then they were described in five minute increments. The lessons were viewed again and critical events were identified and transcribed. Then, the lessons were coded for evidence of teaching for understanding and CRP. The transcriptions included dialogue by the teacher and students as well as a description of any actions that were visible on the video recording. Lastly, the videotaped lessons were compared to the lesson planning observation to determine if the enacted lessons were consistent with what was planned.

## Findings

For the purposes of this paper, a specific dimension of teaching for understanding, the social culture of the classroom, will be described with respect to the teachers' mathematics teaching cycle due to the fact that this aspect of teaching for understanding was readily observed in the teachers' lesson planning and in their enacted lessons. Because of the nature of this dimension, one would expect to find evidence of classroom culture in teachers' lesson planning and in their enacted lessons. Therefore, this section will provide evidence of the teachers' social culture of the classroom first with respect to their lesson planning, and then with respect to their enacted lessons.

### *Lesson Planning*

*Planning for "fair shares"*. Sarah and Pamela did not plan collaboratively on a regular basis; however, because of the nature of data collection for the NMD project, they chose to plan their video-taped lessons together. The content they chose came directly from the North Carolina Standard Course of Study (NC DPI, 2003) for kindergarten: "The learner will share equally (divide) between two people; explain." This concept was also referred to by the teachers as *sharing fairly*. The teachers chose this content based on the fact that it was listed as an objective from the Standard Course of Study for the current quarter.

During the lesson planning session, the teachers began by stating the objective they wanted to cover. Sarah referred to a previous activity where her students struggled with the concept of equality and states, "So that's why I feel like we really need to do this." Sarah conveyed to Pamela the story *The Doorbell Rang*, by Pat Hutchins and stated that she had used the story in previous years and felt like it was relevant to the chosen objective.

The story is about a mother who offers her two children a plate of 12 cookies to share between themselves. After the children decide they will get 6 cookies each, the doorbell rings and in comes two more children. Now they must decide how to share the 12 cookies between four children. The story continues this way until each person has only one cookie and the doorbell rings one last time. Thankfully, it is Grandma with more cookies.

The teachers discussed using pretend cookies and plates to act out sharing the cookies fairly as in the story. Through their discussion, the teachers talked about how their students would learn fair shares and how to emphasize the idea of equality. They also discussed how to extend the activity for students who they believed could understand how to share an odd number of cookies. In thinking about strategies their students might use to share fairly, Sarah stated "…they'll be somebody…that will say 'Six? Oh that's three each. Eight? That's four each.' I have some that will do that." Other than this comment about students using known facts, they did not discuss specific strategies they would emphasize or other strategies they thought their students would use during the lesson.

### *Enacted Lessons*

*Day 1*. During the first day of the two day lesson, both teachers read the story *The Doorbell Rang* to their class and had the children act out the story by sharing 12 cookies between 2, 4, 6 and then 12 children.

Both Sarah and Pamela created a classroom culture where the focus was on students' methods for determining fair shares and not just the answer. Sarah and Pamela emphasized wanting to know students' strategies, frequently asking their students "What can we do to make it fair?" and "How did you know…?" However, neither teacher encouraged her students to reflect upon chosen strategies or how they were thinking about the mathematics involved.

The students in both Sarah's and Pamela's classes were encouraged to come up with their own strategies for sharing fairly and some students had the opportunity to share their strategies with the class during the first lesson. However, in Sarah's classroom, it was apparent that she did not value all of these strategies as she continually emphasized (and tried to get her students to emphasize) dealing out one cookie at a time until they were all gone, and then counting the cookies to determine how many each person had and if the amounts were equal. Sarah listened to the ideas offered and acted on some of them, but she was not content until a student suggested passing out one cookie at time.

For example, at the beginning of the story, Sarah's class is trying to determine how to share 12 cookies among two people. A student suggests giving each person three cookies. After Sarah passes out three cookies to each of two students, the class determines that there are still 6 more cookies left to share. Sarah asks her class "So how do we figure out how many more they can have? Who has an idea how we could figure out how to share it fairly?" Three different students suggest giving three of the remaining six cookies to each student to share the cookies fairly. Sarah verbalizes that this is correct, but continues to probe the class to find out "[What can I do] if I didn't know if there was enough for three and three? Is there another way we could do it?" Once the strategy of passing out one cookie at a time was offered, she was willing to continue on with the story.

In addition, Sarah did not ask her students to reflect on any of the ideas offered by their classmates. In one instance, a student suggested a strategy whereby she was able to count the cookies and determine how many each person should get. Sarah explained what she *believed* to be the student's thinking, however she did not attempt to have the student explain her strategy nor did she ask the class what they thought of this strategy. She continued to question the students and even put all the cookies back in the middle, until another student suggested giving one cookie to each person.

Pamela, too, encouraged her students to come up with and share their own strategies, yet she emphasized passing out one cookie at a time as well. Despite this, Pamela was more accepting of other strategies offered by her students than Sarah. Pamela was willing to act on students' strategies as long as they were correct. For example, when Pamela's class was trying to determine how to share 12 cookies among 4 students, a student suggested giving each person 3 cookies. Pamela modeled this strategy and even asked the student "How did you know to say give three to Adarian and three to Ally?" However, Pamela did not ask the class to reflect on the strategies suggested by students or the strategies she chose to model during the story. It was not clear if Pamela valued all of the ideas suggested by her students and in fact, only gave verbal praise to a student who suggested passing out one cookie at a time.

*Day 2.* On the second day, both teachers had their students work in pairs to share a set number of cookies between two people and then represent their answers in a table that they provided. In both classrooms, students were completing the activity, but they were not encouraged to discuss their strategies or explain how they determined fair shares.

## Discussion

In the context described by Hiebert, et al (1997), the social culture of a classroom that supports teaching and learning mathematics with understanding provides opportunities for students to choose methods that are meaningful to them. Additionally, Hiebert et al. (1997) state that "ideas, expressed by any participant, have the potential to contribute to everyone's learning and consequently warrant respect and response" (p. 9). By examining and reflecting on students' chosen strategies, teachers are valuing students' ideas and using these ideas as opportunities for learning. In this way, a mathematical community is created where students collaborate and communicate about important mathematics, thereby fostering mathematical understanding.

Both Sarah and Pamela created a classroom culture where the focus was on students' methods for determining fair shares and not just the answer. The teachers wanted their students to come up with strategies to solve problems and to communicate their ideas. During the planning phase, the teachers hypothesized about the importance of understanding equality and how their students might approach the chosen activities. Throughout the first lesson, both teachers encouraged their students to share their ideas and at times, both teachers acted on their students' suggestions about how to share the cookies.

However, the teachers did not appear to be open to the idea that all of their students strategies, both correct and incorrect, had value in the classroom. Both Sarah and Pamela focused on dealing out one cookie at a time when their students were in fact using more sophisticated strategies such as dealing out by two's or three's. Although it is unclear why the teachers did not choose to focus on the more sophisticated strategies their students came up with, during the post-lesson reflection they did state that they wanted to emphasize passing out one cookie at a time in order to explicitly offer a strategy for students who struggled with determining fair shares. Nonetheless, with respect to the strategies that were offered by their students, neither teacher encouraged their students to reflect upon and evaluate the strategies they used. In this way, the teachers did not appear to value the strategies offered by their students.

On the second day, the focus was less on the strategy that students were using and more on determining the correct answer when a certain number of cookies were shared fairly and filling out the table correctly. Although both Pamela and Sarah worked with individual groups of students to make sure they were sharing fairly, as students worked in their pairs, they used whatever strategy was meaningful to them and they were not asked to communicate about their strategies, only the answers.

*Implications*

With respect to the current study, it can be hypothesized that if the teachers had been more aware of the different types of sharing strategies appropriate for their students, they first may have been able to identify strategies other than dealing out by one at a time, and then secondly, may have been more accepting of those strategies. The findings from this study highlight the importance of teachers' knowledge on how students think about and develop mathematical concepts. Teachers can use literature on how students learn to inform their instruction. For example, in an article by Sally Roberts (2003) from the NCTM journal, *Teaching Children Mathematics,* teachers are offered ideas on how to teach fair shares and what strategies are typical of kindergarten students such as estimating, dealing out by ones, or grouping by twos or threes.

Furthermore, it should be emphasized that by merely having students share ideas, teachers are not necessarily valuing them. Teachers can allow students to not only share, but to also explain their ideas and use both correct and incorrect ideas as opportunities to develop students' understanding. By reflecting on and evaluating different strategies, students can build mathematical reasoning and extend their problem solving skills.

Teachers can choose appropriate classroom activities or tasks when they have information about how students may think about concepts and how those concepts can be developed over time. Through preservice methods courses and inservice professional development, teachers can gain insight into how students learn mathematics in order to accurately develop lessons that are meaningful and accessible to all students and that increase students' mathematical understanding. In addition, teachers can build on students' chosen strategies, encourage students to evaluate ideas, and use mistakes and misconceptions as learning sites for all students.

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# A CROSS-CULTURAL CURRICULUM STUDY ON U.S. ELEMENTARY MATHEMATICS TEXTBOOKS

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*The purpose of this paper is to report how U.S. elementary mathematics textbooks differ from other international curricula, specifically Korean national curricula. This paper further investigated the characteristics of the U.S. curricular within two opposite spectra, traditional and standards-based curriculum and extended the comparison with Korean textbooks. This study found that the Korean national curricula mostly emphasized on number and operation and both U.S. curricular was more advanced in geometry and measurement. The major difference between the U.S. curricula was its major focus on story and strategy based problems in standards-based textbook and more practice problems in traditional textbook.*

## Background

The findings from the TIMSS study brought considerable concern about the mathematics achievement of the U.S. students. One of the responses was the comparative curriculum studies with international textbooks in order to understand how the U.S. mathematics curricula differ from international averages. The textbook was a focus for two reasons. The first is teacher's high dependency of using textbooks in mathematics classrooms (Weiss, 1987 and Tarr et al 2006). The other reason was the importance of curriculum in terms of opportunity to learn, which means it is not taught unless it is introduced in the textbooks (Flanders, 1994; McKnight et al., 1987; NCTM Standard, 2000; Reys et al., 2004; Tarr, 2006).

## Theoretical Framework

Recognizing the need for the international curricular study, a number of research scholars examined the U.S. curricula in relation to students' mathematical achievement based on SIMS and TIMSS (Frase, 1997; NRC, 2001; Peak, 1996; Schmidt, W., McKnight, C., & Raizen, S.A., 1997; Reys, B.J., Reys, R.E., & Chavez, O., 2004). These researchers compared the mathematical content of textbooks on what content was introduced across participating countries. One major distinction found in these curricular analyses was that the U.S. curricular were not determined at the national level, as it was in most TIMSS countries (Frase, 1997). The above research scholars reported several problematic aspects of the U.S. mathematics curricula. Schmidt et al. (1997) described the U.S. textbook was 'a mile wide and an inch deep' (Schmidt et al. (1997)), which means the U.S. textbooks deal with too many topics in each grade with little emphasis on particular and strategic topics. He criticized covering too many topics as not facilitating students' higher achievement with deeper understanding. Flanders (1994) and Tarr et al. (2006) criticized the U.S. textbook focused less and late on algebra and geometry than international average. Another weakness was that the same topics were revisited repeatedly in the U.S. textbook (NRC, 2001 and Schmidt, 1997). They reported that in the U.S. curricula, multidigit computations were introduced over several years with one digit added each year, meanwhile, high performing countries introduced and develop such topic for students to master it within a specific grade level. These studies negatively valued this repetition because it defers students' mastery of concepts at a certain grade level. However, a Korean research paper which comparing the Korean national curriculum and Everyday Mathematics stated that repetition had a positive point. Seo et al (2003) argued that in Korean textbooks, mathematical concepts and skills were usually introduced once with little relation to the previous year, thus, the focus is placed on students' mastery of the concept at a specific grade level. They also criticized that this curriculum did not provide students with enough time and opportunities to learn important mathematical concepts and skills.

Other comparative studies criticized that in the U.S. textbooks, the topics persist over grade levels (Fuson et al (1998)). She reported that both the simplest and the most difficult multidigit addition and subtraction appeared and disappeared late (from 1 to 3 years) in the U.S. textbooks than other countries such as Japan, China, Taiwan etc. However, if extended this comparison to a Korean textbook analysis (Kang et al, 1998), even though Korea excelled in TIMSS, there was no difference between Korean textbooks and the U.S. textbooks, in terms of the first introduction time of multidigit number operation (Korean has 3.5 average grade level versus the U.S. has 3.4 grade level).

This comparative analysis implies that some factors other than the first instruction time possibly play a role in students' mathematical learning. Hence, there is a need for looking at how the concepts are developed within the chapter as Stein et al (2007) argued in his paper. He criticized that the previous curriculum studies primarily focused on what content was covered but rarely compared how the content was developed within the concept. He argued how the content was presented was also important because they set into motion different pedagogical approaches and different opportunities for students learning. For example, the organization of textbooks will be different depends on whether the developers believe that mathematics is best learned through student constructions or direct instruction and skill practice.

Along with above stated considerable critiques of the U.S. mathematics curricula, a number of mathematics educators developed elementary mathematics curricular to incorporate the ideas of the NCTM standards (1989, 1990, 1995). Even though the release of NCTM standard produced an entire new set of curricula there is not enough data analysis that compares international textbooks to the U.S. standards-based textbooks. The characteristics of the U.S. curricular described so far stood out more in the traditional U.S. textbooks. This brings an attention to the need for extending comparisons to both curricula, and to its organizations as well. Furthermore, it is necessary to broaden the comparisons cross culturally in order to evaluate the strengths and weaknesses of the U.S. curricula from an international perspective.

## Research Questions

In this paper, I extended the international curricula comparisons to Korea, which is one of the highest performing countries on TIMSS and made deeper comparisons with both the U.S. traditional and standards-based curricula. In addition to the topic placement in the textbooks, I also examined how three different textbooks - the Korean national, the U.S. standards-based and the U.S. traditional - were organized under three guiding questions. 1) What topics are placed in each textbook for both 1$^{st}$ and 4$^{th}$ grade? 2) How those topics are developed within the mathematical concept? 3) What is the main focus of the problems in the chapters (e.g. procedure based or strategy based)?

## Methodology

This paper investigated two mathematics textbooks currently being used in the U.S. elementary classrooms and the Korean national curriculum. One of the U.S. textbooks is research-based, developed through NSF funding and structured around the NCTM standards (standards-based). As a standards-based textbook, I selected the most recent edition of the Investigation series (2008) from Pearson: Scott Foresman, The other textbook is not research-based (traditional) and I chose 2007 Harcourt Math series which are widely used in the U.S. elementary schools. I considered it as traditional because although it references the NCTM standards, its structure is similarly to textbooks prior to the standards. As an extension of previous comparisons, I chose two fairly different US textbooks in terms of their characteristics and broadened the comparisons to the Korean national textbook which is dissimilar to the localized U.S. curricula. In order to analyze both lower grade levels and upper grade levels in elementary school I compared 1st grade and 4th grade of all three textbooks.

This comparison consists of three parts: 1) topic placement, 2) topic organization and 3) problem focuses. In topic placement analysis, the proportion of topic was main focus. In topic organization, I investigated the order of sequence of two major topics of elementary level, which were number and operation and geometry, and examined its highlighted mathematical ideas. In problem focus analysis, I classified all the problems as either 1) computation or procedure based problems when the problem asked for the answer or procedures without justification, or 2) story or strategy based problems when the problem was introduced with word problems and asked to the answer with reasoning or strategy. For the analysis of optic organization and problem focus I selected the multidigit addition and subtraction chapter for 1st grade and multidigit multiplication and division chapter for 4th grade because multidigit number operations are one of the most fundamental mathematical concepts for students.

## Result

Topic placement and organization of curricula differed from textbook to textbook across grade levels. There were unique features that stood out the most in each textbook and these features were somewhat different across grade levels. This paper reported the major differences among textbooks first and illustrated in depth comparisons later.

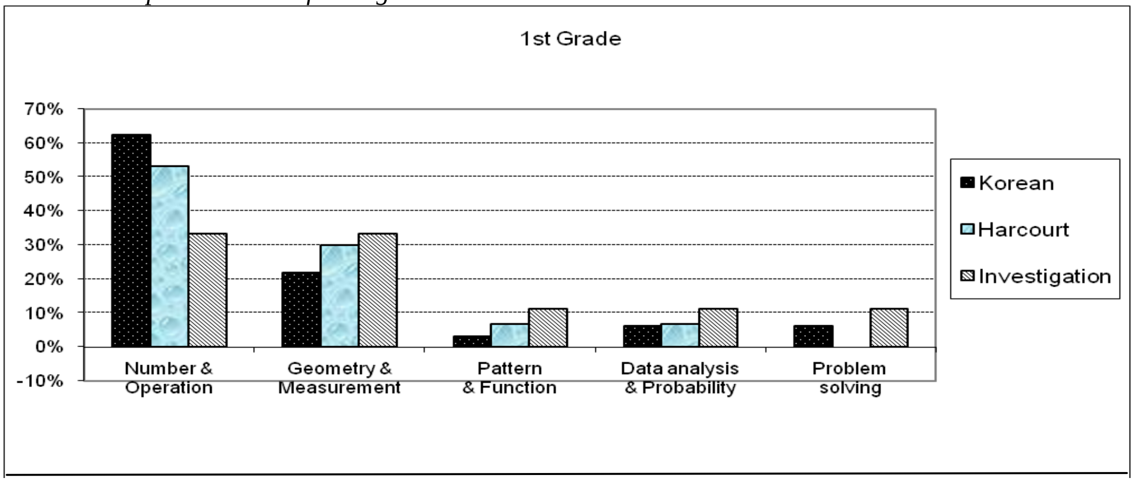
The most fundamental difference of the Korean textbook was its emphasis on the number and operation in both grade levels. It is not only because the Korean textbook dealt with the largest numbers across the grade levels, but also the Korean textbook highlighted the place value and number operations the most. For example, the Korean textbook included the activities such as counting objects, making groups of ten, and decomposing and composing numbers focusing on the combination of ten in 1st grade. Furthermore, in 4th grade, only the Korean textbook developed number sentences that contained all four operations (addition, subtraction, multiplication, and division) at the same time.

The highlighted features of Harcourt were first, the biggest number of topics observed from both grade levels. For instance, only Harcourt contained the fraction and probability concept for 1st grade and measurement of volume and circumference for 4th grade textbook.
The other was the emphasis of practice of number and operation. As shown in the table 2, Harcourt included the most number of practice problems for both grade levels.

In Investigations, activity based lessons and group discussion were the primary emphasis. In general, each chapter of Investigations started with student activities that included the mathematics concepts of that chapter and those activities were composed of group work. Another major focus of this curricular was the largest portion (about 35 %) of story based problems and the most flexible approach of using strategies.

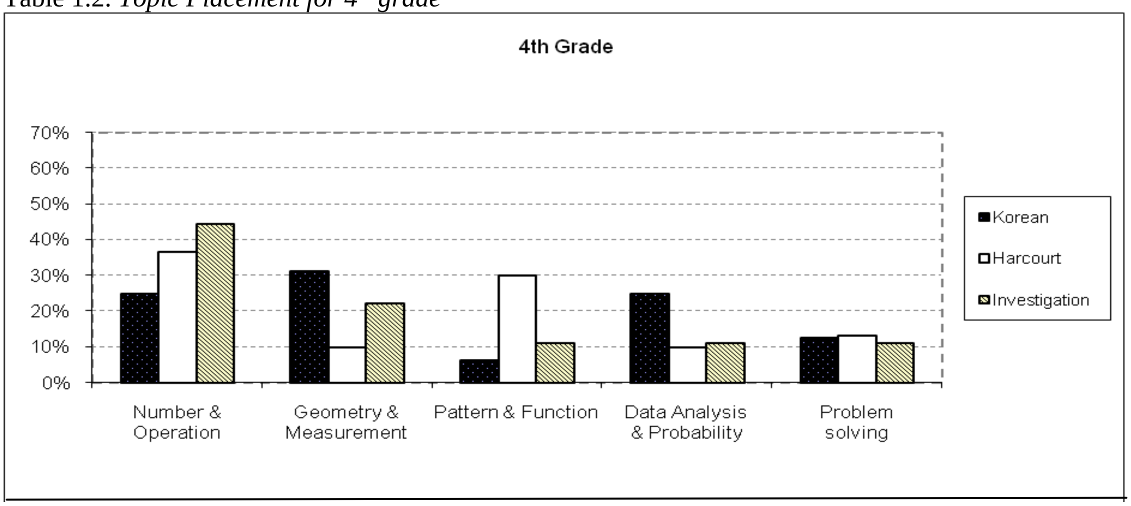
For in-depth analysis, the placement of topics was firstly examined. This section examined two aspects; what topics existed and the portion of such topic in each textbook.

Table 1.1. *Topic Placement for 1st grade*




This table illustrated that in the 1st grade curriculum. The Korean textbook focused the most on number and operations (62.5 %) than the U.S. textbooks (Harcourt - 53.3 % and Investigations - 33.3 %). This contrasted with the least emphasis of geometry of Korean textbook (21.8 %) than the U.S. curricular (Harcourt -29.9 % and Investigations - 33.3%). The critiques from the previous studies, less emphasis of geometry, stood out more in the U.S. traditional and Korean national curricula. However, the comparison was the opposite for the 4th grade curricular.

Table 1.2. *Topic Placement for 4th grade*




As shown in the table 1.2, Korean textbook included the least portion of number and operations (25.0 %) than Harcourt (36.7 %) and Investigations (44.4 %) at this grade level. The contrasting result by grade level continued to the geometry. The Korean textbook contained the most amounts of geometry (31.3 %) than both of the U.S. curricula (Investigations - 22.2% and Harcourt - 30.0 %) at 4th grade. The differences within other topics were not obvious at both 1st and 4th grade. The analysis of topic placement illustrated the proportion of each topic but it is necessary to examine in what order the topics were organized for deeper analysis.

Topic organization was examined focused on the sequence and the highlighted mathematics ideas under number and operation and geometry. In the analysis of the sequence, the central focus was how the same concepts were closely connected each other within the textbook especially when the same topics were revisited. First, the order of sequence was reported from 1st grade followed by 4th grade and key mathematical ideas were noted as the same order.

For the 1st grade, the Korean textbook included the least gap between number and operation concept and introduced such concept the earliest among three textbooks but it contained the least geometry concept. On the contrary, there was a largest gap in number and operation in Harcourt for this grade level but this textbook introduced the most advanced contents in geometry. Investigations was placed in between. For example, in Harcourt, chapter 20 introduced the addition and subtraction sums up to 20 but fraction came at chapter 21, then chapter 22-28 was all measurement. The similar leap was observed in Investigations. In this textbook, the number and operation concept was re-introduced five chapters later since it was introduced first time. This type of structure may be more difficult for the students to build number sense.

In the order of senesce of 4th grade, Investigations was placed in between Harcourt and the Korean textbook in terms of its distance. There were no significant gap different in number and operation but Investigations had the least gap among three textbooks. But when compared the content of number and operation, some major differences were observed. The first standing out feature was that only Harcourt included the concept of addition and subtraction of whole numbers (Chapter 1 – 4) at this grade level. Such concepts were not presented in other two textbooks.

Even though both of the Korean textbook and Investigations presented multiplication and division of multidigit number, there were differences between those two. Korean textbook dealt with multi-digit operations of multiplication and division as a mastery level, hence, such operations disappeared after chapter 1 and 2.On the other hand, both U.S. curricular introduced the concept of multiplication and division as a beginning level. This concept was introduced even later in Harcourt, which is in the middle of the book. Another difference within number and operation comparison was both Korean and Investigations began with multiplication first and related them to division later but Harcourt combined two concepts first and then introduced multiplication and division separately.

When compared the sequence of geometry in 4th grade, Investigations included the least gap than the other two. However, when compared the geometry concept of Harcourt and Korean textbook, the distance was bigger in Korean curricula. Chapter 17 through 19 of Harcourt covered lines, angles and plane figures and the chapters 28 to 30 covered measuring perimeter, area and volume of polygon. Harcourt provided explanation of general polygons first then moved to the measuring them at the end. On the contrary, Korean textbook explained learning angles and triangles first (chapter 4 & 5) and introduced quadrilaterals and other polygons at chapter 12. This structure provided less information to learn the relationships among polygons. Overall, Korean textbook introduced geometry concepts with least connection and relatively later compared to the U.S. curricula.

In addition to these differences, the highlighted mathematical concepts of each textbook were also dissimilar each other. This section was reported as the following order: 1) 1st grade number and operation, 2) 1st grade geometry 3) 4th grade number and operation 4) 4th grade geometry.

Firstly, the grouping number around 10 was highlighted the most in Korean textbook. When counting numbers, Korean textbook contained the activities to make a group of 10 and introduced only counting by ones and tens. Meanwhile, both the U.S. curricular introduced counting by twos, fives and tens equally without emphasis of grouping of tens. This pattern was repeated to the fact family. In Korean textbook, only the combination of 10 was introduced as a fact family but the U.S. curricular included fact family of 10 as one of other fact families.

How such concept was developed was dissimilar each other. Harcourt emphasized number operation the most than any other number concept hence it introduced the vocabulary and symbol first than the concept. However, both in the Korean textbooks and Investigations, number recognition, counting and composing numbers were explained first without symbols. In addition, Harcourt was composed of more operational problems than the conceptual problems, which will be discussed later in this section.

In Investigations, multiple strategies and conceptual problems stood out the most. This textbook introduced the highest numbers in counting (up to 200) and contained the smallest number of addition and subtraction (sums less than 20). It contrasted to the Korean textbook and Harcourt, which consisted of the same operation (sums up to 100) at this grade level. However, Investigations pushed more various strategies with the small number operations.

Yet, the different tendency was observed under the geometry. Compared to both of the U.S. curricular, Korean textbook included the least geometry contents and explained such concept limitedly. For example, Korean curricula only introduced three 3-D shapes (cylinder, cube and sphere) and three 2-D shapes (square, triangle and circle). However, both Harcourt and Investigations introduced more shapes such as cone, pyramid, hexagon etc. The central focus of geometry in Korean textbook was recognizing shape, and any other contents did not exist. On the other hand, both the U.S. curricula introduced geometric terms such as vertex, faces, sides and symmetry etc. and they also engaged with further geometric activities such as moving shapes. Within the U.S. curricular, the contents were pretty similar to each other.

Thirdly, in 4$^{th}$ grade comparisons, the difference among geometry concept was more significant than number and operation. Only Korean textbook contained separate chapters entitled 'order of operation (Ch 6)' which consisted of number sentences with mixture of all four operations (addition, subtraction, multiplication, and division), but none of U.S. curricular contained the same number sentences. Estimation related problems were placed separately under this chapter in Korean textbook, but in both U.S. curricular, the same concepts were embedded within the number and operation chapters. Even though Korean textbook contained the smallest portion of number and operation, those concepts seemed to be more advanced than the U.S. textbooks. However, the result was the opposite in geometry comparison.

The geometry comparison for 4$^{th}$ grade demonstrated the similar results as 1$^{st}$ grade but the difference became greater. For instance, Korean textbook did not contain the transformation of shapes and finding area of polygons but both the U.S. curricular were composed of measuring angles, perimeters, areas, volumes. Harcourt further added circumferences. No measurement was placed in Korean textbook and such concepts were introduced in 5$^{th}$ and 6$^{th}$ grades, which was 1 or 2 years later than the U.S.. Among the U.S. textbooks, Harcourt included the most number of topics such as congruent & similar figures, tessellation, circumferences, and diameter etc. which were not represented in Investigations.

A further comparable finding was that the U.S. curricular classified polygons based on the relationships across the shapes. Meanwhile, Korean textbooks tended to explain the properties of polygons limitedly connected the relationships among polygon. There was another dissimilarity that observed only at this grade level comparison, which was the different mathematical focus within geometry concept. The Korean textbook emphasized the angles of polygons hence it asked how to prove the sums of inside angles of triangle and quadrilateral but neither of the U.S. textbooks contained the similar problems.

The last analysis was to investigate the problem type of number and operation of each grade level. The result of each textbook was shown in table 2.

*Table 2. Analysis of Addition Problem Types for 1st and 4th Grade*

| | 1st Grade | | | 4th Grade | | |
|---|---|---|---|---|---|---|
| (N=total number of problems in the chapter) | Harcourt (N=220) | Korean (N=53) | Investigations (N=49) | Harcourt (N=145) | Korean (N=53) | Investigations (N=56) |
| Computation or procedure based problem | 200 | 36 | 31 | 129 | 49 | 36 |
| Story or strategy based problems | 20 | 17 | 18 | 16 | 12 | 20 |
| Percentage of story problems out of total number of problems | 9 % | 32 % | 36 % | 11 % | 22 % | 35 % |

Overall, for both grade levels, Investigations most often included story/strategy based problems and the U.S. traditional textbook had the largest number of practice problems but included the least number of story based problems. In the meantime, the number of problems was similar between the Korean and U.S. standard based textbook.

## Discussion and Conclusion

The results of this study brought some similar findings from the previous studies and dissimilar findings at the same time. Similar to previous studies (NRC, 2001; Schmidt, 1997), I found that the U.S. textbooks tended to revisit same topics in the previous year. Both in U.S. textbooks, multidigit addition and subtraction problems were represented in 4th grade textbooks, but, in the Korean textbook, the concept was introduced and disappeared only within the assigned chapter. Number sense and operations persisted across grade levels (NRC 2001) in the U.S. curricula but the U.S. reform curricula contained the most flexible way of number operation. Unlike the previous critique, both the U.S. textbooks were more advanced than the Korean textbook in geometry. Furthermore, the in-depth analysis of the textbook organization brought worthwhile educational implications in addition to the analysis of placement of topics.

The findings here highlight the demand of detailed comparisons of curricula from the opposite end of spectrum in U.S. and the need of broadening the study to cross cultural comparisons in order to improve written curricula. However, I believe it is also critical to develop the intended and implemented curricula for teachers who use traditional textbooks. This is urgent for teachers because it takes a long time and a lot of money to scale up reform curricula, even though researchers agree that reform curricula is more effective for students' mathematical understanding. How to use traditional textbooks is also effective if teachers can teach concepts with traditional textbooks. Therefore, research studies beyond the written curricula would help curriculum developers and researchers understand how curricula play out differently in student's mathematical learning and their achievement.

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# PROPORTIONAL THINKING: A CASE OF STUDY

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*a teaching proposal on ratio and proportion. A group of sixth-grade students (eleven years old,) of elementary education in México participated in the implementation of the proposal. The child of the case study was representative of those students in the group who had a lot of recourse to handling algorithms mechanically and whose elaborations made no sense at all, according to their answers to an initial questionnaire. A didactical program, developed in a problem-solving context for the research study, helped the child widen his qualitative thinking and strengthen his quantitative thinking about proportion.*

## Research Problem

The case study we present in this report was part of a research study carried out for a doctoral dissertation (Ruiz, E.F. 2002). Previously, other aspects and activities of that research have been presented and reported in various communications. The case study of our research is about a boy, Emilio, who solved ratio and proportion problems by having recourse to algorithms which made no sense and had no meaning at all.

By other hand, the topics of ratio and proportion beginning in the Primary School (Secretaría de Educación Pública, 2001 a y b, and NCTM, 2003), and they are the support for the other concepts, ( Clark, Berenson y Cavey, 2003).

We designed a teaching proposal embedded in this situation, with the aim of strengthening his establishing of solid connections between qualitative and quantitative thinking about proportion[1], so that he could improve his handling of algorithms by situating them into meaningful applications. The following question guided our research about Emilio´s case.

## Research Question

Does the extensive handling of qualitative aspects of ratio and proportion allow the student to widen quantitative relationships of these concepts as well as to improve the handling of her algorithms?

## Hypothesis

Enriching Emilio´s qualitative thinking -by using integrated verbal categories, recognizing the compensations posed between these categories, and involving the corresponding empirical and perceptual data- favors the significance processes he has developed by using algorithms for solving ratio and proportion problems.[2]

## Some Theoretical Antecedents

Piaget and Inhelder (1978a) pointed out, as a result of their experimental researches in education, that children acquire qualitative identity sooner than quantitative conservation. Thus, these authors made a distinction between qualitative comparisons and true quantification. According to Piaget and Inhelder (1978b), the acquisition of the notion of proportion always starts in a qualitative by using categories or classes of words. Our own interpretation of what is qualitative refers to what is based on linguistic recognitions by creating comparison categories such as big or small. Our interpretation is that what is qualitative consists of intuitive and empirical aspects as well, which are provided by our senses.

Piaget (1978) pointed out that the idea of order emerges during transition from the qualitative to the quantitative realm, although the idea of quantity is not yet present.

Piaget called these situations intensive quantifications. For us, this is what makes the transition from qualitative to quantitative thinking stand out.

On their part, Van den Brink and Streefland (1979) agreed with Piaget as to their research findings that qualitative aspects of thinking occur sooner than quantitative ones. However, Streefland usually had recourse to these findings in teaching contexts. In our approach for the designing of the didactical program as well as in the development of interviews for the case study of educational research we present in this report, we used that contribution by Streefland.

Research findings reported by Streefland (1984; 1985) emphasized that the early teaching of ratio and proportion topics must depart from qualitative levels of recognizing them. For that purpose, Streefland made use of didactical resources, which strengthen the development of perceptual patterns for supporting the corresponding processes of quantification. Streefland stated that qualitative reasoning evolves as the thinking of the child advances and he or she is capable of incorporating more elements for an analysis, which will allow him or her to consider different factors simultaneously.

Thus, since Piaget and Streefland took into account qualitative and quantitative thinking about proportion exhibited by their subjects under research, the rationale for our case study was strongly based on Piaget´s and Streefland´s findings. We based the didactical approach developed for our research on Streefland´s realistic mathematics approach.

Hart (1988) and her collaborators had reported results of their research studies on proportional thinking as well. They found out that most students who participated as subjects in their researches considered that it was difficult to solve mathematics problems that involved proportion. However, Hart and her team analyzed collected data and evidenced that younger students as well as pupils in secondary school with less success had a certain sense of “what is seen right” or of “what seems to be a distortion”. Hart designated the latter as a regulation from “common sense”, which we recognized as intimately involved in “qualitative thinking”. Moreover, Hart pointed out that the most advanced level of proportional thinking occurred in those subjects who had already constructed certain concepts.

We based the didactical context of our research on realistic mathematics education referred to by Streefland (1993). Realistic mathematics education has become a theory since reality is, in first instance, a source of information and the context for the application of teaching models, schemata, and notations-school productions that have an influence in social practice. This theory favors the development of research and practice of the teaching and learning of mathematics. Analogously, according to this realistic theory it is essential to link students’ learning periods by resorting to the “strategy of change in perspective”, which is characterized by the exchange of part of the information in the problem-situation being approached. Consequently, the possibilities for the reconstruction and production of problems become explicitly recognized by students, without losing their multifaceted conceptual richness, (Gueudet, 2007).

## Methodology

The research process of the case study of included integrating results from analyses of data collected from (a) his answers to an initial questionnaire, (b) a teaching program designed under a constructivist-didactical approach, (c) a final questionnaire, and (d) interviews of “didactical nature". The research instruments were tested in a pilot study of a one-year school cycle and definitively implemented during a ten-month period of fieldwork. In this case-study report we present relevant examples of the use of the research instruments.

*Participants*

Twenty-nine students of sixth-grade of elementary education in México, who were eleven years old, solved the initial questionnaire. We choose a child (Emilio), for a case study because he was representative of those students who, in the initial questionnaire, had a lot of recourse to handling algorithms that made no sense and who simultaneously exhibited few elaborations in the qualitative context. Throughout the development of the teaching experience, Emilio exhibited enrichment of his qualitative thinking and, in spite of making a lot of progress in the numerical context, he did not abandon the qualitative context of proportionality. He achieved a close harmony of both contexts.

*The Initial and Final Questionnaires*

The initial and final questionnaires were integrated by the same tasks, although their application had a different aim. The first questionnaire was applied for exploratory purposes, whereas the second one focused on evaluating the implementation of the teaching program. Eight months elapsed between the applications of both questionnaires: Thus, there was no influence of the first questionnaire on the students' answers to the second one.

The tasks included in the questionnaire did not involve the use of quantities for their solution: it comprised comparison activities that allowed the student recognize similarity relationships between figures.

*Didactical Program*

Figueras, Filloy, and Valdemoros (1987) defined *model* as a collection of teaching strategies which include meanings –of both technical and common languages–, didactical program, according to that definition, we designed several situations associated to "teaching models" so that Emilio could link his qualitative and quantitative thinking processes on proportion. We worked with those models at different stages of the research experiment, similarly to what Streefland (1993) pointed out in his realistic theory as to the "change strategy in perspective": We created a model and tried to get the best out of it in the light of an idea, so that we could retake it and use it for another idea.

## Analysis of Emilio's Progress by Comparing his Answers in the Initial and in the Final Questionnaires

In the initial questionnaire, Emilio exhibited a preference for using algorithms mechanically and very little work in the qualitative context. We observed that he almost did not use his common sense or visualization.

From the thirteen tasks posed in the initial questionnaire, he solved nine of them correctly.

The first two tasks in the questionnaire were designed so that Emilio could give justifications of his answers by strongly resorting to qualitative appreciations and not taking into account explicit quantities associated to the given relationships of proportionality. We employed squared paper in the next three tasks of the questionnaire to favor a transition toward quantification. The remaining tasks in the questionnaire involved quantified situations of ratio and proportion. In these last tasks, we provided Emilio with certain numerical values and asked his for new values. In some of these tasks we used a table of numerical values as a mode of representation for the recognition of external and internal ratios.

Now we present an analysis of two tasks Emilio answered incorrectly: task 1 and task 4. In task 1, the drawing of a car was presented and the student was required to select the correct reduced sketching of the original drawing (see figure 1). Emilio selected a sketching that did not correspond to the original drawing and he argued that his choice was the car C because it resembled best the original drawing. However, in the final questionnaire Emilio based his choice of the reduced drawing by having recourse to his intuition first and then by measuring each part of this drawing to obtain the ratios with corresponding magnitudes of the original drawing, although in his new explanation he mentioned again that "car B looks like the original car" and added that "it is similar, that is, proportional". Thus, we observed that, from the initial questionnaire to the final one, the expression "looks like" underwent a change of meaning for Emilio: he exhibited an understanding of the term "proportion" as the relationship of equivalence between two ratios (but he did not abandon his common sense, which was exploited throughout the teaching program).

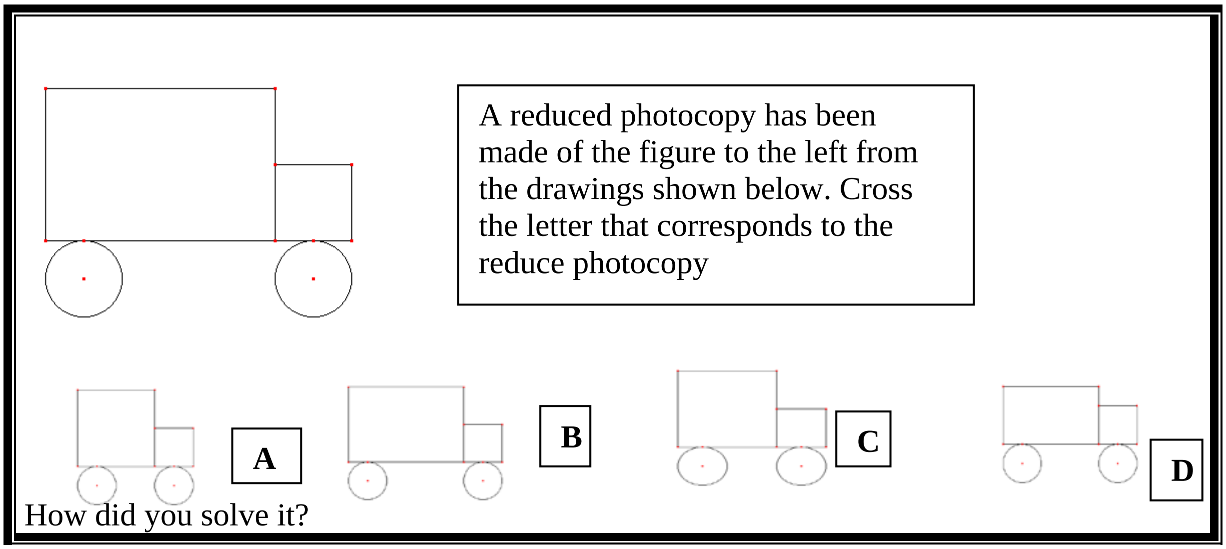
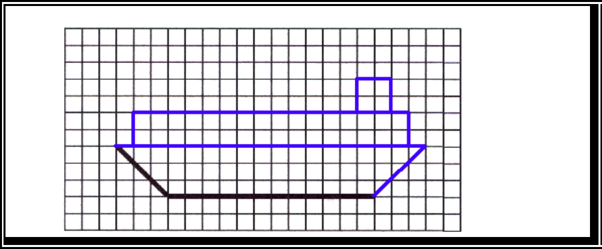



*Figure 1.* Task 1 of the initial and final questionnaires solved by Emilio.

The answer that Emilio gave in the initial questionnaire was: “*The car C is the one which best resembles the original one”*, and the one he gave in the final questionnaire was*: “Car B looks like the original car. It is similar, that is proportional.”*

We can ascertain this based on other collected evidence: for instance, Emilio did not solve correctly task 4 in the initial questionnaire, but he did in the final one. It is important to make the explanations he elaborated stand out in this research study; they are included in figures 2 and 3. In the task 4 we asked students to make amplification the original drawing, and we give them a portion of the amplified drawing.

As shown in figure 2, Emilio completed the drawing but he did not notice that he had amplified it twice and not thrice. As seen from figure 3, Emilio showed the establishing of equivalence between two ratios that were obtained from comparing two corresponding magnitudes from the middle portion of the ship.



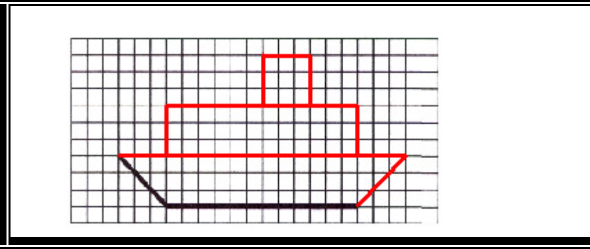
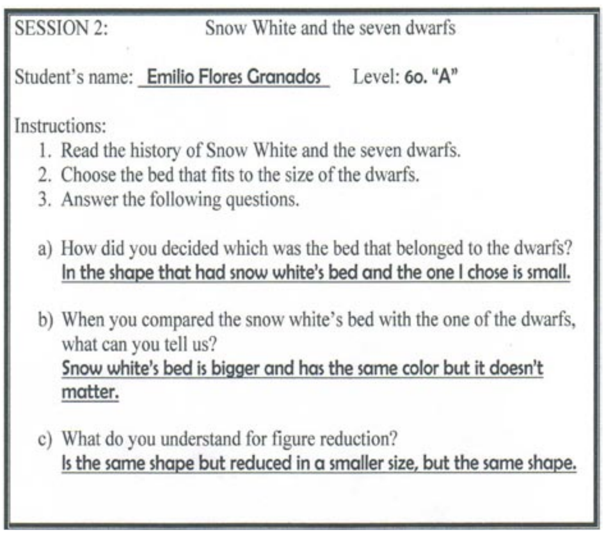


*Figure 2.* Task 4 solved by Emilio questionnaire.

*Figure 3.* Task 4 solved by Emilio in the initial the final questionnaire.

**Analysis of Emilio's Progress during the Development of the Teaching Program**

The solution of different tasks employed during the development of the teaching program, such as comparison activities, involved using quantities. These activities allowed Emilio recognize—by using very intuitive terms such as reduction and amplification—similarity relationships between figures and he could enrich his qualitative thinking. We worked with those notions by referring to concrete situations of the type of the experience of reproducing a drawing to scale and of the idea of using a photocopier. In the figures 4 and 5, we show two activities that are part of the Didactical Program and that Emilio solved them. In those activities is shown the notion that Emilio has about the proportionality, after some working sessions and the relation between magnitudes to establish ratios, as well as equivalences relations between ratios.



SESSION 2: Snow White and the seven dwarfs

Student's name: Emilio Flores Granados Level: 6o. "A"

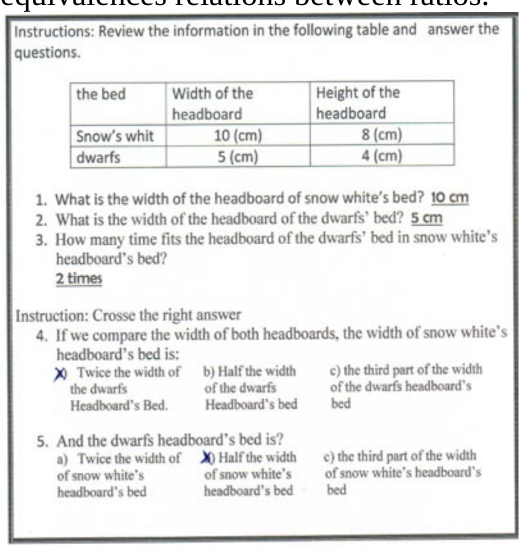
Instructions:

1. Read the history of Snow White and the seven dwarfs.
2. Choose the bed that fits to the size of the dwarfs.
3. Answer the following questions.

a) How did you decided which was the bed that belonged to the dwarfs?
In the shape that had snow white's bed and the one I chose is small.

b) When you compared the snow white's bed with the one of the dwarfs, what can you tell us?
Snow white's bed is bigger and has the same color but it doesn't matter.

c) What do you understand for figure reduction?
Is the same shape but reduced in a smaller size, but the same shape.



Instructions: Review the information in the following table and answer the questions.

| the bed | Width of the headboard | Height of the headboard |
|---|---|---|
| Snow's whit | 10 (cm) | 8 (cm) |
| dwarfs | 5 (cm) | 4 (cm) |

1. What is the width of the headboard of snow white's bed? 10 cm
2. What is the width of the headboard of the dwarfs' bed? 5 cm
3. How many time fits the headboard of the dwarfs' bed in snow white's headboard's bed?
2 times

Instruction: Crosse the right answer

4. If we compare the width of both headboards, the width of snow white's headboard's bed is:
X) Twice the width of the dwarfs Headboard's Bed. b) Half the width of the dwarfs Headboard's bed c) the third part of the width of the dwarfs headboard's bed
5. And the dwarfs headboard's bed is?
a) Twice the width of snow white's headboard's bed X) Half the width of snow white's headboard's bed c) the third part of the width of snow white's headboard's bed

*Figure 4.* Activity solved by Emil about proportionality using verbal categories.

*Figure 5.* Activity solved by Emil using quantities to establish ratios and equivalences relations between ratios

During the transition from qualitative to quantitative thinking, Emilio produced an ordering when comparing: he used the phrases "bigger than and smaller than". This finding agrees with what Piaget (1978) pointed out. Later on, Emilio took measures to make comparisons. First, he compared different objects by placing one figure over another and then by using a measure instrument. In terms stated by Freudenthal (1983), the resources exhibited by Emilio at this development stage of her thinking are called "comparers." After that, Emilio established relationships between magnitudes. He worked with natural numbers and employed fractions as well. Thus, at a very elementary level, he introduced his elf to the field of rational numbers. The boy of this case study could designate a ratio as a relation between two magnitudes and a proportion as an equivalence relation between two ratios. This designation agrees with definitions given by Hart (1988).

When the working sessions ended, Emilio showed he had achieved a close relationship between his qualitative and quantitative thinking. This relationship implied the sense she made of her work in the numerical context, which was not revealed at the beginning of his work. Eventually, when the teaching experience ended and the final questionnaire was applied. Emilio's meanings and quantification processes had been enriched. Now he could use a technical language in the designation context. He achieved a generalization stage in which new situations related to ratio and proportion were favored.

## Analysis of Emilio's Progress during the Interviews

Emilio was interviewed in three different occasions, once a week, after the teaching program ended and the final questionnaire had been applied. The main purpose of the interviews was to asses the teaching program. The interviews consisted of asking to solve new tasks which aims were similar to that of the didactical program and of the questionnaires. Additionally, the development of the interviews gave feedback to Emilio.

With the first tasks we posed Emilio during the interviews, through his solution processes we could observe how he kept qualitative aspects to the light of having worked quantitative aspects, and how important it was for his to use visual images as well as his perception ability. Through the next tasks in the interviews, we also investigated how he handled numerical tables to recognize ratios and express these as fractions. During the interviews he exhibited his use of internal and external ratios, his transition from one symbolic system to another, and his posing of a situation where the use of proportions would be necessary to solve it this first interview was closely related to the Snow White and the seven dwarfs teaching model. Next, we show the development and analysis of that interview. Emilio measured the length and the width of Snow White's wardrobe as well as the length and the width of each of the four drawings shown in the figure so that he could choose the required reduction. Once Emilio had chosen a wardrobe, he obtained the ratios between magnitudes of some of its parts and the corresponding parts of the original wardrobe. Now we show part of the interview with Emilio.

Interviewer: What did you base your choice of the dwarfs' wardrobe on?

*Emilio: I took measures and found out that wardrobe B is proportional to Snow White's because all their ratios are equivalent. (Emilio pointed to what she had written, "12/8 = 6/4=3/2.").*

*Interviewer: Will you please tell me how you obtained the ratios?*

*Emilio: By comparing measurements of Snow White's wardrobe with those of the dwarfs'.*

The numerator of each fraction measures certain part of Snow White's wardrobe: for instance, 12 is the length of the height, 8 is the length of the base, 3 is the length of one little window (she pointed to one of the drawings representing a decoration of the wardrobe), and 1.5 is the width of this little window. The denominators of the fractions are the measurements of the corresponding parts of the dwarfs' wardrobe. (The measurements Emilio mentioned are given in centimeters.)

Thus, Emilio established links to determine ratios based on taking measures. In another part of the same interview we could observe how he had recourse to his perception ability when he said, "*Wardrobe A is too long, C is very wide, and D is very little. Although I did take measures, I noticed that those three wardrobes did not seem proportional to Snow White's."*

Emilio exhibited that his handling of conceptual aspects was meaningful since he identified ratio as a relation and proportion as an equivalence relation between ratios. Moreover, we could notice that Emilio did not abandon the qualitative context, since he also used verbal categories and common sense to verify that his choice of the wardrobe was the right one. To this respect, he wrote that Snow White's wardrobe was equivalent to that of the dwarfs, and that as to their form they were equal although one was small and the other was big.

## Conclusions

Emilio exhibited a strong progress in relation to two important aspects:

1. The development of her qualitative thinking in relation to ratio and proportion.
2. The signification he gave to his using of algorithms.

During the processes of solving different tasks, Emilio exhibited how strong perceptual data became for his as well as how important it was for his to rely on his own experience. This is evidence about his achievements in the qualitative context of proportionality. The algorithmic work allowed us to explore the tacit recognition of the operators about which Emilio was thinking. These operators were natural numbers as well as fractions. The latter were used implicitly when multiplying certain value by a number and then dividing the result by another number, or vice versa, first dividing and then multiplying. In the context of what is now considered the construction of meanings, these—together with the processes of signification—were enriched. As to their designation, Emilio could eventually use the appropriate mathematical terms. Finally, he reached the point of constructing the concepts of ratio and proportion. This achievement was evidenced by the applications he made of those concepts in different contexts as well as by using their different modes of representation.

## Footnotes

1. The qualitative thinking is supported by linguistic recognition creating comparison categories such as big, small. In the qualitative is included the intuitive that is the supported in the experience, the empiric, in the senses. The quantitative refers to activities that allow the student to count, to measure, to use quantities in the procedures

2 According to Benveniste (1971), meaning is a “dictionary entry” and “a universal semantic category”; and sense is a semantic content, which is associated to particular constructions of language, it does not shape universal categories and usually keeps a close relation to specific modes of articulating them. Moreover, it is proper to emphasize that there is not a chronological sequence, or of precedence, in the development of sense and meaning. They are different semantic components, which complement each other.

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# TENSIONS OF ADOPTING ELEMENTARY MATHEMATICS CURRICULUM MATERIALS IN A DIVERSE DISTRICT

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How do school districts make decisions about the adoption of new elementary mathematics curriculum materials? Although these decisions are made annually at school districts across the country, little is known about the questions district leaders ask when adopting new materials, the kinds of evidence they consider, or the tensions and challenges involved in the decision-making process. In this paper, we present a case study of one district, River City (pseudonym), adopting new curriculum materials. We claim that the issues, questions, and tensions experienced in River City can inform the work of districts adopting new materials in the future.

River City is a district of 38 elementary schools serving approximately 14,000 students (~55% eligible for free and reduced lunch). In this district, textbooks are seen as just one among many tools teachers might draw on in designing instruction. The district mathematics leader articulated this vision of the relationship between teachers and curriculum materials and its relationship to the adoption process in the following quotation:

> I want teachers to have the freedom to make professional decisions based on their students and yet, I understand the need for that additional task to be taken off someone's plate who just doesn't have the time, the energy, or the knowledge….but I feel as if we were to purchase *a* material, then why would teachers ever read *Teaching [Children] Mathematics*? Why would they ever go to the NCTM web site? It almost appears as if we're turning our backs then on other things that could be valuable supplements, enhancements, or replacements.

The materials evaluation and adoption process initially proceeded quite smoothly. Without significant debate, the committee agreed on three programs for more in-depth consideration: *Investigations* (TERC, 2008), *Math Expressions* (Fuson, 2008), and *Math Trailblazers* (TIMS Project, 2008). It was in the process of considering these three programs that two key tensions arose. The first tension involved identifying a single set of curriculum materials that would meet the diverse needs of all of the *teachers* in the district. The second tension involved the levels of professional development and teacher learning required by each program and whether it would be best to adopt a high risk (in terms of demands on teacher learning), but also high reward (in terms of moving the district forward) program (*Investigations)* or to adopt a program (*Math Expressions*) that would represent more incremental and less risky change for most teachers, with the assumption that more teachers would be likely to implement a program that required less of them. It is important to note that members of the committee were not necessarily making judgments or assumptions about the willingness or the capacity of individual teachers for change, but instead about the capacity of the district and the curriculum programs to support the significant change that would be required. In the poster, we present more details about the process, as well as the eventual outcome.

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# TRANSITION FROM ARITHMETIC TO ALGEBRA: A TEXTBOOK ANALYSIS FROM AN ASIAN PERSPECTIVE

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Mathematics educators and researchers have increasingly recognized the gatekeeper role of algebra in preK-12 schooling (Carraher & Schliemann, 2007). Previous research highlights the importance of integrating arithmetic and algebra in early grades (Carpenter, Franke, & Levi, 2003). In Asian countries, students are exposed to algebraic thinking in textbooks much earlier than their U.S. peers (Cai et al., 2005). To investigate the transition from arithmetic to algebra, this study examined the features of introduction to algebraic thinking in selected Asian elementary mathematics textbooks.

The textbooks chosen for this study consist of two series of new *Standards*-based elementary mathematics textbooks in China and one series of widely adopted elementary mathematics textbooks in Singapore. Through the theoretical lens of Kieran (2004), this study adopted qualitative methods to examine and analyze the selected textbooks.

The main findings of the study include: First, introduction of algebra in Chinese and Singaporean textbooks is characterized by patterns and generalization. For example, in one Chinese textbook, students are asked to generalize from a popular Chinese nursery rhyme, *Count Frogs*, by filling in "n frogs, ____ mouths". Second, Chinese and Singaporean textbooks address difficulties and misconceptions of the equal sign and variables as a unidirectional operator and placeholders, respectively. Chinese textbooks present equations with numbers on both sides of the equal sign, followed by equations with one variable on one side of the equal sign. For example, 5=5, 5**+2**=5**+2**, and x=10, x**+5**=10**+5**. Singaporean and Chinese textbooks also present various forms of the multiplication sign. For instance, an expression of 3p can be written as 3×p or 3·p. Third, by emphasizing such topics as relating equations containing numbers alone with equations containing numbers and variables, and interpreting and simplifying algebraic expressions, Chinese and Singaporean textbooks help students make adjustments and transit from arithmetic to algebra.

This study extends our understanding of transition to algebra in early grades and has implications for curriculum development in algebra in the U.S.

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# PRE-SERVICE TEACHERS' KNOWLEDGE AND ATTITUDE PROFILES: RESULTS OF A MIXED METHOD DESIGN

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*There is a growing trend in mathematics teacher education on researching the nature of pedagogical content knowledge required for teaching. In what unique ways can rich mathematics problem-solving experiences contribute to pre-service teachers' understanding of subject matter and pedagogy? We research an elementary pre-service program that since 2004-2005 added a rich mathematics problem-solving component to its mathematics methods course. Specifically, we will report the results of a survey that studied elementary school pre-service teachers' cognition and affective responses at the end of the program. The conference poster will show case knowledge and attitude profiles of pre-service teachers. These profiles created from mixed methods survey data.*

There is a trend in mathematics teacher education on researching the nature of mathematics required for teaching (Ball, 2002). Many elementary pre-service teachers' classroom experiences have been negative, even when positive, have left the pre-service teachers with a narrow understanding of what doing mathematics involves (Cooney, 1999). Many researchers conceive that re-organization is one of the major roles of teacher education courses. They present varied tasks as forms of intervention and preparation. The program which we research uses rich mathematics problem solving. Teachers and students need to experience a *better, re-conceptualized* mathematics. "The underlying premise is … to allow them [teachers] to understand and reconstruct what they know with more depth and meaning" (p. 230, Ponte & Chapman, 2008). In our work we focus on ways to get teachers to engage in doing *warm* mathematics, specifically non-routine problem solving (Namukasa, Gadanidis & Cordy, in press). The study is guided by complexity research in education. In a manner consistent with most research on mathematical affect, complexity research asserts that the cognitive and the affective are closely interwoven. Complexity research also recognises the need for mixed methodologies (Day, Sammons, & Gu, 2008). The methodology for the study combines interpretive and survey study. Beginning 2004 we collected qualitative data from students' written work interview transcripts. In 2007, in an attempt to corroborate our methodology we designed a survey questionnaire on mathematical knowledge and beliefs. The survey is descriptive and explanatory. In 2008-2009, 150 candidates completed the pre-program and post program questionnaire for instruction and evaluation purposes. This poster illustrates pre-service teacher's knowledge and attitude profiles that were created from data from a small preliminary study with 20 participants carried out in 2007-2008. The profiles are an example of results from a mixed methods research and assessment design.

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# AFFECT AND RACIAL/ETHNIC ACHIEVEMENT DISPARITIES

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*This paper reports on an exploratory study on the role of affect in mathematics achievement disparities. Qualitative data was used to develop a survey (N = 513) to examine relationships between students' emotional experiences during class and their perceptions of group work, relationships with teachers and mathematics achievement. Analyses were conducted by racial/ethnic group. Results for African American students were significantly different than those for students from other groups, including: negative emotions were more associated with lower grades, and less frequent negative emotion was more associated both with positive views of group work and stronger relationships with teachers.*

## Introduction

Affect influences how students perceive assignments, relate to teachers, participate in class, and ultimately their achievement. This paper reports on exploratory research into how affective issues may contribute to achievement disparities in mathematics. More specifically, the study reported here considered negative emotions and analyzed how these emotions relate to students' perceptions of group work, relationships with teachers, and math grades.

The research was conducted at a medium-sized, urban, public high school with a diverse student body (no racial/ethnic group was in the majority). The school had a mathematics department that in many ways matched the ideals promulgated by leaders in the field of mathematics education. Teachers were predominantly mathematics majors from top schools and credentialed by leading programs. They worked collaboratively on developing norms across classrooms and improving teaching practices and materials. They were involved in professional organizations and maintained connections with the higher education community. They used high quality mathematical tasks for their courses and emphasized group work, multiple representations, and conceptual understanding. In addition, they made serving all their students a priority. Their classes were untracked. They had a block schedule. Extra help from teachers was available daily, before school, during lunchtime, and after school.

However, analyses of achievement trends found that African American students, males in particular, had not benefited to the same degree as students from other groups from their otherwise successful implementation of reforms. Differences in SES cannot account for these differences because the student population is from working class homes across races.

Department leaders reached out to the research community for help understanding and addressing the observed achievement inequity. The paper reported here is part of a larger project to help in these efforts. This school site and its mathematics department provide a particularly rich context to explore issues related to equity in education. It differs from the more commonly documented under-resourced schools, where there is often a lack of will or skills to address the challenges of promoting equity.

The initial focus of the research was to understand the school and the mathematics classrooms as social spaces. Toward that end 300 hours of observations were conducted over a three-year period, primarily in mathematics classes, but also in other courses, extracurricular and social contexts within the school. At the conclusion of each academic year interviews were conducted that explored students' views of themselves, their school, their attitudes toward mathematics, and their experiences in their mathematics classrooms (N = 60; African American N = 45). All data sources were open coded. This qualitative research had several important findings. First, literally, all the African American students interviewed reported caring about their grades and viewing mathematics as important for their lives (Davis, 2008). (Consensus on the importance of mathematics may be related to departmental norms regarding communicating mathematics' value to the students.) Additionally, students uniformly showed disappointment or dejection with low marks and protested when they thought they deserved higher grades. Despite their commitment to achievement the students frequently did not engage in mathematical activities as expected by their teachers. Many students reported and were observed experiencing and expressing intense negative emotion during mathematics class, including physical rage and crying.

Anger and frustration frequently emerged when students confronted assignments they could not understand. Also, many students spoke about fear of exposing incompetence. These reports fit with observational data. Students were often reluctant to share their approach to a problem unless they were confident it was correct and became agitated when pressed by their teachers and peers. Importantly, some factors seemed to be associated with less intense negative emotions. Students who understood and endorsed group-learning practices and had stronger relationships with teachers appeared to experience less negative emotion. It also seemed that these students engaged more frequently in group work activities as expected by their teachers. For example, one such African American student commented that he volunteered to present his work on problems when he was confused because he knew that he would get the support of his classmates in understanding the problem.

Our qualitative data (referred to above) focused heavily on African American students. It seemed that similar issues were playing out for students across racial/ethnic groups. However, it also appeared that African American students more frequently confronted the types of negative emotions described above. Moreover, we suspected that these emotional experiences might contribute to disparities in mathematics achievement and be alleviated by particular pedagogical approaches. Therefore, we investigated the role of negative emotions in student achievement across racial/ethnic groups and the degree to which students' perceptions of group work, personal connections to teachers, and mathematical identities were related to these negative emotions. We also examined how these relationships varied across racial/ethnic groups.

## Literature Review and Theoretical Perspectives

The focus of this study cuts across several different bodies of research. As is often the case with exploratory research, there is not an established body of directly relevant research. This review considers relevant studies on affect in mathematics, stereotype threat, and reform mathematics. It concludes with some theoretical considerations about emotion and cognition.

Research on affect in mathematics education has focused primarily on beliefs and attitudes and has not attended as much to more "hot" emotions, such as anger, fear, and shame (McLeod, 1994). Studies that do consider emotion have been mostly limited to the investigation of math anxiety. Although these studies have consistently found significant relationships between levels of anxiety and performance, they have either not considered race or found no differences between racial/ethnic groups (Ma, 1999). McLeod (1994) argues that researchers in this field have not attended sufficiently to the characteristics and experiences of their subjects. This limitation, in conjunction with investigating an insufficiently broad range of emotions may have contributed to finding no differences across racial/ethnic groups.

Other research suggests that there may be interactions between race, context, emotion, and cognitive performance. Stereotype threat research has shown that increasing the salience of an identity that is stigmatized in the domain tested can have detrimental effects (Aronson & Steele, 2005), including African Americans in mathematics (Steele & Aronson, 1995). Although research on causal mechanisms for these findings, has been inconsistent, anxiety-related emotions appear to be a contributor (Smith, 2004). However, stereotype threat findings are limited to specific populations and conditions, namely domain-identified individuals being tested in controlled settings on material that is at the edge of their skill level. In contrast, this study considers students' experiences in natural learning environments. Further, it explores how negative emotions operate in classrooms where race is salient on a day-to-day basis. Moreover, it looks at reform mathematics classroom that are highly social in nature where issues of identity, racial/ethnic identities included, are pushed to the forefront.

The public and social nature of mathematical activity in reform-based courses makes them interesting contexts to consider affect. Some researchers studying reform mathematics have found that these curricula foster achievement supporting affective changes (Boaler, 2002; Nichols et al., 1990; Stipek et al., 1998). However, these studies did not conduct analyses by race/ethnicity. Other researchers (Lubienski, 2007) have argued that issues of diversity have not been adequately addressed in the implementation of reform curricula. More specifically, some studies have found that teachers of African American students have difficulty establishing the desired discourse norms for these courses (Martin, 2000; Murrell, 1994). It is unclear what role negative emotions might have in the challenges observed.

There are several aspects of reform curricula that may make considerations of affect important for addressing equity concerns. First, these curricula employ problems that lower-achieving students have had more challenges with (Jordan, Kaplan, Nabors Olah, & Locuniak, 2006). Models of affect suggest that past experiences cumulatively precipitate relatively stable beliefs and attitudes that come to shape perception and behavior (Marshall, 1989; Schumann, 1994). Thus, students with histories of low achievement in mathematics (and by extension those students from under-served social groups) may have negative affective responses to the types of problems that are the focus in reform mathematics. Also, these curricula focus on types of mathematics most associated with math anxiety (Hembree, 1990). Finally, students are expected to share answers and justifications, which increases opportunities to expose not only miscalculations, but also misconceptions. Students who view these experiences as creating a risk of appearing incompetent may avoid participating in ways could help them learn (Covington, 1992). Negative emotions may also lead to prolonged deficits in the ability to complete assignments even when students re-engage (Carver & Scheier, 2005)

Studying the role of affect in learning, particularly emotion, presents methodological challenges. However, isolating affect from intellect is of questionable utility for developing an understanding of human behavior (Eder et al., 2007). Vygotsky (1986) wrote, "Their separation as subjects of study is a major weakness of traditional psychology" (p. 10). More recently other scholars have pointed out that education research in particular tends to treat affect as separate from cognition, which has limited a thorough understanding of affect (Malmivuori, 2006). We can make theoretical distinctions between cognition and emotion but the boundaries between them are not clear (Schoenfeld, 1994). Even while engaged in purely analytical processes our thoughts have an emotional valence (Schumann, 1994). These feelings influence our behavior and thinking and are connected to our identities and beliefs. We advocate a distinctly sociocultural view of emotion that seeks to understand how beliefs and identities are interrelated with affect and participation (see Evans, Morgan, & Tsataroni, 2006 for a related approach). Our model moves beyond looking at affect as an aspect of personality and the subject experiencing the emotion as the prime cause of that experience. Instead, it considers the ways contexts afford and support different beliefs and identities (Greeno, 1994) and how these are related to emotions.

## Research Questions

1. Are there differences across racial/ethnic groups in the frequency of negative emotion during mathematics class?
2. What is the relationship between negative emotion and mathematics achievement? Are there differences across racial/ethnic groups?
3. Are students' positive perceptions of group work, stronger connections with teachers, and identification with mathematics associated with less frequent negative emotion during mathematics class? Are there differences across racial/ethnic groups?

## Methodology

We investigated these questions through a survey that allowed us to make comparisons across broad segments of the population (N = 515; African American N = 115). The survey was administered to all students taking mathematics during a single semester. Participation in many classrooms was 100%. The survey contained four Likert questions about emotion that asked students to rate the frequency that they experienced the following during mathematics: worry about looking stupid, anger when confused, frustration when confused, and headache or stomach aches due to difficulty in mathematics. The content of these items was derived from interview transcripts and observation notes. Other scales included mathematical identity, which asked students about their commitment to the subject and sense of skill in it, and interpersonal connections with teachers, which asked about care and respect in student-teacher relationships (Marks, 2000). In addition, we asked students to provide five reasons why their teachers ask them to work in groups. Answers were coded into three categories: providing instrumental support for learning mathematics (IM), providing instrumental support for social learning (SL), and negative comments about group work (N). An additional pair of Likert questions asked students to report the degree to which they found working in groups helped them learn mathematics.

## Results

*Question 1*

Our results indicated that students from all racial/ethnic groups experience some negative emotions during mathematics class. There were no statistically significant differences in the levels of negative emotion reported between racial/ethnic groups

*Question 2*

Correlations between negative emotion and math grades for the entire sample were significant $r(470) = -.215^{**}$. African American females showed no correlation between negative emotion and math grades, $r(56) = .015$, while African American males showed a significant correlation, $r(50) = -.345^{*}$.

Breaking down the negative emotion variable into individual survey items we found two items contributed more strongly to the significant correlations for African American males, fear of looking stupid in mathematics class, *r*(51) = -.478** and frustration when unsure of what to do, *r*(51) = -.345*. White females (N = 20) were the only other group with significant correlations on both of these variables.

To investigate whether the relationship between math grades and 'fear of looking stupid' for African American students was significantly different from that found for the general population we tested the slopes of the regression lines for parallelism for math grades on 'fear of looking stupid' and found that the slope for African Americans was significantly steeper (*t* = -2.752, *p* = .003). To ensure that this significant finding was not confounded by gender differences, we also compared the slopes of the regression lines for African American males to the slope for all other males for these same variables. The difference between these slopes was also significant (*t* = -2.149, *p* = .033).

*Question 3*

Turning to the relationship between negative emotions during mathematics class and perceptions of group work learning exercises, we did not find significant correlations when examining the sample as a whole. However, analyzed by race/ethnicity, African Americans were the only group that displayed significant correlations between two of these variables, instrumental social (IS) *r*(103) = -.235* and negative comments (N) *r*(103) = .267**. These findings indicate that for African American students the awareness of teachers' intention to promote social learning through group learning activities was associated with less negative emotion and critical views of group work were associated with higher levels of negative emotion. A test for the parallelism of the slopes of the regression lines for African American students and non-African Americans students for negative emotion on IS found a nearly significant difference, with the regression line for African Americans having a steeper slope (*t* = -1.853, *p* = .064).

In the general population, higher levels of mathematics identity (MI) were associated with lower levels of negative emotion (MI *r*([513] = -.251**). Analyzed by race/ethnicity students from all groups displayed significant correlations between these variables, such that less negative emotion was experienced by more domain-identified students.

In the general population, there was a weak but significant negative correlation between interpersonal connection (IC) and negative emotion, *r*(513) = -.108*. Analyzed by racial/ethnic groups, only African American students showed a significant correlation between IC and negative emotion *r*(106) = -.269**. Analyzed by gender and race/ethnicity we find only African American and Latino male students showed significant correlations between these variables, but results for African American females were approaching significance (AAM: *r*[50] = .335*; LM: *r*[83] = -.244*; AAF r[56] = .252, *p* = .069). A test for parallelism of slopes for African-American students and non-African American students for the regression of negative emotion on interpersonal connection was approaching significance (*t* = -1.846, *p* = .065).

Group work support and interpersonal connection were both associated with lower levels of negative emotion for African American students. We also found that African American males showed the strongest correlations between these variables. Comparisons of the slopes for African American males and all other students for the regression of group work support on interpersonal connection found that African American males showed a steeper slope (*t* = 3.034, *p* = .002).

## Discussion

Not surprisingly, students from all racial/ethnic groups reported experiencing negative emotions in mathematics class. Although there were no significant differences in the amounts of negative emotion reported, there was divergence in how negative emotions were related to achievement across racial/ethnic groups. It is unclear if differences between groups are related to variation in how readily students acknowledge social anxiety, how they define negative emotions as expressed in the survey, and/or different responses to these emotions (or some combination). For example, fear of looking stupid can be experienced as a gripping anxiety or merely as something unpleasant to avoid. Also, some students may manage this anxiety (whatever its strength) by trying to take good notes and ask questions so as to insure that they master the material, whereas others may be motivated by this emotion to attempt to get recognized for some socially valued identity, such as class clown. In addition, we suspect that prior histories of difficulty with mathematics may be related to the strength of negative emotion experienced and how it is responded to. Longitudinal research that can disaggregate the role of factors related to race/ethnicity from achievement history is needed.

African American students showed a stronger relationship between negative emotions and perceptions of group work than students from other groups. Due to the prevalence of stereotypes about African Americans and intelligence (Steele, 1997) and students' awareness of the social consequences of school failure (van Laar, 2000) there may be more at stake and thus more negative emotions for African American students in school contexts where race is salient and emotional safety is lacking. Therefore, perceptions of learning exercises and their purpose may have a greater influence on African American students' emotions. That is, believing that teachers assign group work to promote learning or social development may help students view risk exposure as having an instrumental value, which in turn may lead to experiencing lower levels of negative emotion. Endorsing the learning practices may provide a much-needed proximal reason for engagement for those students who lack certainty about the distal value or likelihood of school achievement. Seeing group work as instrumentally valuable may help also students cognitively restructure interpretations of events in ways that down-regulate negative emotions when they emerge (Gross, 2002). Domain identification with mathematics and interpersonal connections with teachers may help students in similar ways. For example, believing that your teacher respects you and is committed to your learning may provide ways of framing struggle that decreases arousal and supports engagement and persistence.

Findings indicated that higher levels of mathematics identification were associated with lower levels of negative emotion. These results contrast with stereotype threat research that finds that only domain-identified individuals are subject to the performance-reducing effects of racial priming. This contrast suggests that perhaps the kinds of negative emotion measured in this study are not related to stereotype threat effects. Alternatively, these effects may play out differently *in situ*.

## Conclusion

This study suggests that negative emotions may play a role in achievement inequity in high school mathematics. It also offers promising results. More specifically, this study suggests that when students view the learning practices as having instrumental value and have interpersonal connections with teachers, they may experience less negative emotion. Further research is needed to develop validated measures for the kinds of emotions investigated in this study and to better understand the variability observed in the relationship between negative emotions and achievement across racial/ethnic groups. Additional qualitative research that adds needed complexity to the limited fixed categories of racial/ethnic identities used in these analyses is called for. It will be valuable to examine the different ways racial/ethnic identity is constructed and how these variations relate to the ways emotion influences participation in mathematics.

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# MATHEMATICS EDUCATION IN THE PUBLIC INTEREST: PRESERVICE TEACHERS' ENGAGEMENT WITH AND REFRAMING OF MATHEMATICS

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*Equity and social justice agendas in mathematics education are becoming increasingly central to researchers and educators. The Mathematics Education in the Public Interest*[i] *project implemented a mathematics course for preservice elementary and middle teachers where mathematical units are placed in contexts encouraging critical analysis and exploration of the world and connections between mathematics and students' lives. Preservice teachers' views about mathematics and mathematics teaching changed over time. Mixed-methods research indicated mechanisms supporting their engagement with and reframing of mathematics included: (1) Learning the relevance of mathematics; (2) Developing interest in mathematical applications; and (3) Changing prior assumptions and instructional goals.*

## Introduction

Projects such as the Dartmouth *Mathematics Across the Curriculum* project and the Indiana University *Mathematics Throughout the Curriculum* project have suggested the need for greater indisciplinarity and a strengthened mathematical infrastructure in the undergraduate curriculum. Quantitative literacy projects such as *Quantitative Reasoning in the Contemporary World* at the University of Arkansas have strong potential to help students make connections between quantitative information and their lives and interests outside the classroom. These projects help students to understand the relevance and interconnectedness of mathematics with other subjects and with the real world.

In recent years, an increasing number of mathematicians and mathematics educators have begun to ground mathematical investigations in meaningful personal and social contexts. Teachers and researchers have begun to document students' experiences and learning from this process, as well as their own experiences and learning. For example, teaching in a middle school classroom in a diverse Chicago school, Gutstein's class included mathematical studies of the distribution of the world's wealth, possible racism in housing data and mortgage loans, and random drug testing (Gutstein & Peterson, 2005). Based on his research, Gutstein (2007) suggested, "Students learned mathematics and began to develop sociopolitical awareness and see themselves as possible actors in society through using mathematics to understand social injustices" (p. 420). Turner and Strawhun (2005) described New York City middle school students' mathematical investigations of overcrowding at their school, concluding, "Not only did opportunities to engage in responsive action support students' sense of themselves as people who can and do make a difference, but using mathematics as a tool to support their actions challenged students' view of the discipline" (p. 86).

Consortiums such as the Center for the Mathematics Education of Latinos/as (CEMELA) have been extremely important to advancing equity and social justice agendas in mathematics education. CEMELA involves parents, school administrators, and teachers in a collaborative effort to improve the mathematical education of low-income Latino students by focusing on the interplay of the language, social and political issues affecting Latino communities. Thus far, little has been done in teacher education programs to prepare preservice teachers, in particular, for centering their future mathematics practice on equity and social justice agendas. In May 2008, the NSF-funded "Connecting Mathematical Funds of Knowledge Conference" held in Tucson, Arizona helped teacher educators consider what it means to support preservice teachers to connect children's mathematical thinking with children's and community funds of knowledge in the context of elementary mathematics methods courses. Such emphases, while still very uncommon, are beginning to take root in a small number of mathematics methods courses across the country. However, mathematics *content* courses engaging preservice teachers in learning mathematics in support of equity and social justice emphases, with an eye toward the relevance of mathematics in local and global communities, have been nearly non-existent.

The *Mathematics Education in the Public Interest* (MEPI) project initiated at Radford University has implemented a mathematics course designed for preservice elementary and middle school teachers where mathematical units are placed in contexts encouraging critical analysis and exploration of the world and connections between mathematics and students' lives. Based on mixed-methods research in the junior-level *Elementary and Middle Grades Mathematics for Social Analysis* (*Math for Social Analysis*) course, this article communicates changes in preservice teachers' views about mathematics and about mathematics teaching and mechanisms supporting preservice teachers' engagement with and reframing of mathematics.

## Theoretical Foundation for MEPI

> Democratic access to powerful mathematical ideas for social justice requires that students have comprehension of global conditions that are driving the global society and how mathematical and technical knowledge can be tools used to develop a more just world. (Malloy, 2008, p. 29)

For many years now, various forms of classroom or knowledge management, instruction, opportunities, and so forth, have been suggested as stratified across social classes (e.g., Anyon, 1980; Bowles & Gintis, 1976; Knapp & Woolverton, 1995, 2003; Moses & Cobb, 2001; Oakes, Joseph, & Muir, 2004; Secada, 1992; Tate, 1997). Among other things, content and pedagogies weak or lacking in cultural relevance for some students or stemming from Eurocentric perspectives (e.g., Atweh, Forgasz, & Nebres, 2001; Ladson-Billings, 1995; Lubienski, 2002; Rodriguez & Kitchen, 2005; Tate, 1995) have been offered as contributing to race and class divisions in access to knowledge. As a discipline, mathematics, "often regarded as the most abstract subject removed from responsibilities of cultural or social awareness" (Boaler & Staples, 2005, p. 32), has additionally been associated with such stratification. Historically, school mathematics is isolated from other subjects and from students' lives and interests outside of school. Mathematics is treated as independent from important social, political, and economic issues facing our communities and our world.

Regarding student learning, research has yielded largely positive support for reform practices of the kind supported in the U.S. by the NCTM *Standards* (1989, 2000). In an extensive, three-year comparative study of two schools in England, Boaler (1998) suggested that students who receive project-based instruction learn more, and different, mathematics than students receiving traditional skills-based instruction. In the U.S., relatively consistent evidence also exists that students using reform-based curricula perform equally well on tests of mathematical skills and procedures as comparison students using traditional curricula, and perform better on tests involving mathematical concepts and problem solving (Schoenfeld, 2002; Senk & Thompson, 2003). Schoenfeld further explained, "Reform appears to work when it is implemented as part of a coherent systemic effort in which curriculum, assessment, and professional development are aligned. Not only do many more students do well, but the racial performance gap diminishes substantially" (p. 17). Also, both male students and female students in reform-based school programs in the U.S. outperformed their counterparts in traditional programs; and for female students, all performance differences by program were statistically significant (Riordan & Noyce, 2001).

Equity and social justice agendas in mathematics education have become increasingly central to a growing number of researchers and educators in recent years. Recommendations for how to achieve equity goals almost always include requirements for setting high expectations and providing strong support for all students (e.g., Moses & Cobb, 2001; NCTM, 2000). But despite many great strengths, reform documents such as the NCTM *Standards* (1989, 2000) still do not go far enough.

Lubienski's (2002) criticism of *Standards*-based reforms (NCTM, 1989, 2000) focuses largely on multicultural considerations of discourse and the NCTM's general oversight of such considerations. Others have focused more on the absence in the *Standards* of a critique of societal inequities (e.g., Apple, 1992; Gutstein, 2003, 2006). Gutstein (2006) indicated the *Standards* embody a relatively narrow perspective on equity, discussing equity in terms of opportunity to learn, but not critiquing societal inequities behind the lack of those opportunities for many segments of the population both in the U.S. and abroad.

Social justice agendas help students to clarify issues, to understand the structure of society, and to justify or refute opinions, increasing learners' capacity to understand and also challenge oppressive social structures and power relations that perpetuate over time and across the globe (cf., Frankenstein, 1989). In a world where educational inequities and other inequities persist, the treatment of school mathematics as abstract, as independent of students' lived experiences, and as independent of moral and social obligations is short-sighted. We can do better.

Gutstein (2006) proposed an exploratory orientation toward building mathematics curriculum with integrated components of community knowledge, critical knowledge, and classical knowledge. The twelve characteristics of the Connected, Equitable Mathematics Classroom proposed by Goodell and Parker (2001) also support similar emphases in the rethinking of mathematics. The MEPI project foundation rests on an assertion that mathematics curriculum and instruction can be improved by maintaining overlapping objectives that, for example: (1) incorporate NCTM *Standards*-based reform practices, (2) are more culturally responsive, (3) make use of individuals' and groups' funds of knowledge, (4) engage learners' more fully, more meaningfully, and more responsibly with their communities, and (5) explicitly aim to achieve social justice locally and globally.

## *Math for Social Analysis* Course

In Radford University's Department of Mathematics and Statistics, we created a junior-level course for elementary and middle school preservice teachers, *Math for Social Analysis*. *Math for Social Analysis* is the third course in a three-course sequence of mathematics content courses for preservice teachers, each offered out of the Department of Mathematics and Statistics. All elementary and middle grades preservice teachers at Radford University are required to take *Math and Human Development I and II*, both of which are prerequisites for *Math for Social Analysis*. *Math for Social Analysis* is required for elementary education majors and recommended for middle school education majors—likely to become required for that group in the future as well. The course, and related research project "Mathematics Education in the Public Interest," maintain overlapping emphases on mathematics content, social critique, and community relations and actions. After completing this mathematics sequence, students later take a mathematics methods course in the education program.

In *Math for Social Analysis*, the connection between mathematics and the world emerges as students critically analyze social issues using mathematics. For example, one curriculum unit speaks to environmentalism, which includes topics on global warming, mountaintop removal in Appalachia, rainforest depletion, and water conservation. Students learn and raise questions about their own and global contributions to these problems. Another curriculum unit tackles the global economy, where students explore and mathematize such topics as poverty, the distribution of wealth, and sweatshop labor. Class activities, readings, guest speakers, and videos help connect students to the social issue, which promotes dialogue, and deepens their mathematical understanding as they grapple for solutions.

Additionally, our students complete a semester-long project, choosing between a research/teaching project option and a service learning project option. The research/teaching option is a small group project. The group raises an authentic question about the world and they research answers to their question. The group produces a research paper and several age appropriate mathematics lessons based on the research, and the project culminates with the group teaching one of the lessons to their classmates. The service learning option is an individual project in partnership with a local community-based organization's after school programs. Our students provide mentoring and tutoring assistance for 3 hours each week. This option requires the students create and teach five mathematics activities, two based on social issues, and they write reflections detailing their experience. The project culminates with a presentation to their classmates, reflecting on their experiences.

## Research Methods

This paper is based on mixed-methods research conducted with 77 preservice elementary and middle school teachers enrolled in three sections of *Math for Social Analysis* in 2007-2008. Three participants were male, and 74 female; 75 identified themselves as "Caucasian/White" on the survey. Further, 70 of the 77 participants were in elementary education, 3 in middle school education, 3 in special education, and 1 in early childhood education.

Data collection included:

- Pre- and post-surveys with all participants on their views related to the relevance of mathematics in understanding and solving social issues and to connections between mathematics and students' families and communities. Surveys included demographic items, Likert-type items, and open ended response items.
- 18 interviews (with 16 participants) conducted in the latter half of the semester to learn preservice teachers' background experiences with mathematics, views and attitudes about mathematics and its relevance, and learning and experiences in *Math for Social Analysis*
- Introductory journal reflections from all participants to learn their background experiences with mathematics, views and attitudes about mathematics and its relevance, and goals and expectations in *Math for Social Analysis*
- Final project assignments from all participants, including reflection papers addressing their attitudes toward mathematics, their experiences and learning in *Math for Social Analysis*, and the relevance of the course for their future teaching of mathematics

The author of this paper and colleague Jean Mistele each teach one or more sections of the *Math for Social Analysis* course each semester. For ethical reasons and to limit bias, we interviewed each other's students rather than our own. Data analysis involved coding and recoding the qualitative data, using categories such as “Math Anxiety/Confidence,” “Interest in Math,” “Views on Math and Social Issues,” and “Goals for Math Teaching.” We triangulated multiple quantitative and qualitative data sources to arrive at conclusions regarding the nature of preservice teachers’ experiences and learning in *Math for Social Analysis*. As we reviewed data and heard preservice teachers almost universally describing very different, and mostly very positive, experiences and learning in this course than in all previous mathematics courses, we became interested in how and why they characterized the course as they did. Our results section documents changes in preservice teachers’ views about mathematics and about mathematics teaching from the pre-survey to the post-survey and communicates mechanisms supporting preservice teachers’ engagement with and reframing of mathematics.

## Supporting Student Engagement with and Reframing of Mathematics

> My attitude about mathematics for a long time has been dread and confusion…I feel that by teaching math in this new method, people may better understand math because they will be able to learn by relating it to real life…This may also better people’s attitudes toward math by showing them its importance and relevance to their future. By using this method of teaching math, we have the opportunity to greatly change the way the world sees math for the better. (Mary, 12/11/08)

Students universally label this course as their first extended experience with learning mathematics in connection with multiple meaningful real-world applications and social issues. Many of our preservice teachers enter *Math for Social Analysis* describing high levels of mathematics anxiety. We have previously reported ways that preservice teachers’ mathematics anxiety levels subsided as a result of their engagement with social issues in the course (Mistele & Spielman, 2009). Survey results also provide evidence that preservice teachers’ views about mathematics and about mathematics teaching changed over the semester. They come to see mathematics as: (1) Increasingly useful for understanding and engaging with important issues and (2) Increasingly connected to home and community experiences.

*Using Mathematics to Understand and Engage with Important Issues*

One set of pre/post survey items measured preservice teachers’ views about the ways mathematics is, or can be, used to understand and engage with important issues. These items were measured on a 4-point scale ranging from strongly disagree to strongly agree. The 10-item scale (Scale reliability, Cronbach’s $\alpha = 0.80$) included items such as, “Having an understanding of mathematics makes people more powerful as citizens” and “It is my job as a math teacher to help students see connections between mathematics and social issues.” A t-test comparing pre- and post-survey means on this composite yielded a significant difference ($t = -5.229$, $p < 0.001$). At the end of the semester, preservice teachers reported significantly stronger agreement with scale items such as these, or respectively, disagreement with negatively worded items.

*Connecting Mathematics to Home and Community Experiences*

A second set of pre/post survey items measured preservice teachers’ views about the ways mathematics is, or can be, connected to students’ home and community experiences. These items were likewise measured on a 4-point scale ranging from strongly disagree to strongly agree. The 9-item scale (Scale reliability, Cronbach’s $\alpha = 0.81$) included items such as, “Getting to know students’ families and becoming familiar with their communities is useful for teaching mathematics” and “Home and community activities are good contexts for posing and solving mathematical problems.” A t-test comparing pre- and post-survey means on this composite yielded a significant difference ($t = -3.203$, $p = 0.002$). At the end of the semester, preservice teachers reported significantly stronger agreement with scale items such as these, or respectively, disagreement with negatively worded items.

*Mechanisms Supporting Engagement*

Overall, student responses thus far have been very positive and supportive of the *Math for Social Analysis* course, although to appropriately characterize in detail students' experiences and learning in the course would require greater depth than may be included in a conference paper. Based on qualitative data analysis, interwoven mechanisms supporting preservice teachers' engagement with and reframing of mathematics included: (1) Learning the relevance of mathematics to something they care about; (2) Developing interest in mathematical applications and in supporting their future students' interest and learning in mathematics; and (3) Shifting their perspectives on mathematics by changing prior assumptions and instructional goals. See Figure 1.

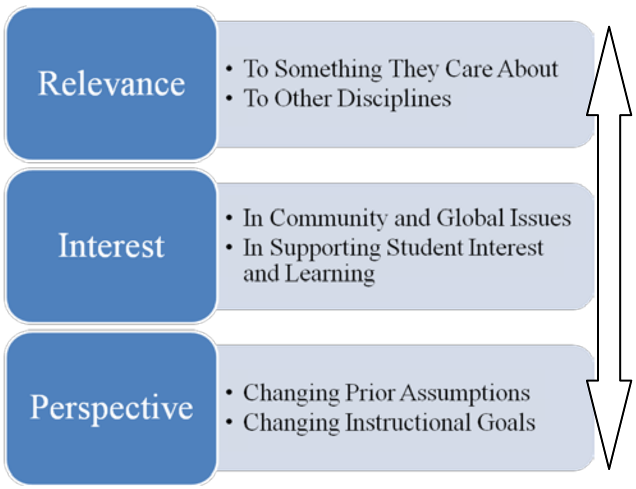


*Figure 1*. Mechanisms supporting engagement.

Classroom experiences helping preservice teachers connect mathematics to other disciplines and to social issues helped them develop a foundation for constructing new connections and applications in the future. Once preservice teachers started making connections between mathematics and other disciplines and social issues, they found it much easier to make additional new connections. Preservice teachers had often never considered relationships between mathematics and the things they cared about. By learning examples of how mathematics is related to social issues, this spurred many new possibilities for how to teach math using social issues. They saw a whole array of new possibilities for their future classroom practice. As teachers increasingly saw mathematics as relevant and important in social issues, they developed new teaching goals to help students integrate math with other subjects and the world outside of school.

Preservice teachers also re-examined their assumptions about the (ir-)relevance of mathematics as they learned mathematics in ways they had never before experienced. They used their new learning of interdisciplinary and social issue connections to mathematics to rethink their futures in mathematics teaching and reframe math as a discipline children can become excited to learn. Further, they developed a new sense of agency to create mathematical learning opportunities that students will find interesting and relevant. Preservice teachers' own interest and confidence in teaching mathematics in the future was tied to their knowledge of different ways to get students interested in the discipline. Their interest in mathematics was related to the ways they understood connections between mathematics, other disciplines, and their own lives and interests. The mathematics came alive to them in the context of meaningful applications.

## Implications and Future Research

Results from mixed-methods research in *Math for Social Analysis* are sufficiently positive to suggest the need for continued implementation and testing over time, on a larger scale, and at additional sites. Further research is also needed to learn more about the ways the *Math for Social Analysis* course, or similar courses, impact preservice teachers' mathematical understandings, as that has not yet been rigorously examined.

## Endnotes

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# MEETING THE NEEDS OF DIVERSE STUDENT POPULATIONS: FINDINGS FROM THE SCALING UP SIMCALC PROJECT

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*While research has shown the effectiveness of representational technologies in mathematics education, barriers to broad use remain. The Scaling Up SimCalc project has begun to address these barriers by considering the role of technology within a wider "curricular activity system." In this paper we discuss how we leveraged the representational and communicative infrastructure of SimCalc to meet the needs of a diverse student population, while we also met the needs of key stakeholders in the wider education system. This resulted in increased learning for a diverse group of students. We also discuss possible improvements to our intervention.*

## Introduction

Research has shown the effectiveness of using representational technologies in mathematics to scaffold and support student learning (Mayer, 2005; Marzano, 1998). However, there have been barriers to broad use, such as the perception that technology is too difficult to implement in diverse classrooms (Becker, 2001), and inconsistent findings on the benefits of educational technology in mathematics (Dynarski et al., 2007; National Mathematics Advisory Panel, 2008).

In this paper we report on a study that leveraged the effective aspects of representational technology while overcoming existing barriers to broad use. The study evaluated a particular instantiation of the SimCalc approach, which integrates interactive representations with paper curriculum and teacher professional development to increase students' opportunity to learn advanced mathematics. In designing the Scaling Up SimCalc study, we incorporated the perspectives of different stakeholders –students, teachers, and school districts– to minimize barriers to implementation and increase the chance of having the intervention used. We addressed teacher and district concerns regarding current policy demands (e.g. NCLB and accountability testing) and the need to meet local standards. We considered multiple teaching styles and designed materials so teachers with a wide variety of mathematical and technological backgrounds could use them. And, through representational technologies and scaffolded curriculum we met the cognitive, linguistic, and social needs of a diverse student population. At the heart of this approach is a refinement of our conceptualization of the use of innovative technology in the classroom. Whereas earlier work focused primarily on the representational and communicative infrastructure of SimCalc, the concept of a "curricular activity system" has emerged as being vital to successful scale up (Roschelle et. al., in review).

Scaling Up SimCalc makes an important contribution to the literature by providing very strong evidence that embracing these diverse perspectives increased student learning of advanced mathematics with a diversity of teachers in a wide variety of settings.

## Background

For over fifteen years the SimCalc project has had the goal of ensuring that all learners have access to complex and important mathematics, as expressed in the SimCalc mission statement "democratizing access to the mathematics of change and variation" (Kaput, 1994). The mathematics of change and variation emphasizes the concepts of rate and accumulation as thematic content that can be developed across many grade levels. A foundational belief of the SimCalc Project team is that reconceptualizing middle school and high school mathematics in light of the broader mathematics of change and variation developmental strand can yield a more coherent and fruitful mathematical experience for all learners, including those that have not traditionally been successful in mathematics (Kaput & Roschelle, 1997).

This view of how mathematics can be structured stands in contrast to the traditional mathematics curricula, which was laid out in the $17^{th}$ and $18^{th}$ centuries, and hasn't changed much since (Kaput & Roschelle, 1997). This is, in part, because the curriculum has "worked" to train a workforce where most jobs require little more than arithmetic, and few require deep understanding of advanced mathematical concepts. Today, the picture is different. Not only are there economic arguments for preparing more young people, of different races and backgrounds, to use complex mathematics on the job (National Advisory Mathematical Panel, 2008), but participation in society as an empowered citizen requires understanding the mathematics of change (Kaput & Roschelle, 1997), and increasing the number and diversity of those in the field of mathematics may even be vital in advancing the field itself (Gutierrez, 2007).

While the SimCalc research program has considered restructuring the mathematics curriculum as a way to achieve its goal of democratization, strict adherence to this goal in the short term may stand in the way of necessary reforms that can help many of the students who need this access the most—those in low-performing schools who are already likely to get worse instruction and less access to high-level content than their peers at high performing schools. In this study, guided by a curricular activity systems approach, we built upon the past successes of SimCalc, while taking an incremental approach to addressing what is taught.

In this paper, we will use *SimCalc* to refer to the Scaling Up SimCalc study (2005-2008) and the system of curriculum, software and professional development developed therein. The software, SimCalc MathWorlds® (hereon referred to as MathWorlds), is a simulation environment in which the user and the software co-construct mathematically meaningful objects and relationships. MathWorlds moves beyond simple interactivity and animations of math, and instead provides students with access to complex mathematics, and allows students to quickly conjecture, test, and iterate while preserving mathematical relationships and structures. This is very difficult to replicate in static media, where students may unintentionally violate mathematical principles in an investigation (Hegedus, 2005).

We next describe the results from the Scaling Up SimCalc study, and then report on those features of our intervention that most likely resulted in its success in helping a wide variety of students learn important mathematics.

## Results from the Scaling Up SimCalc Study

The Scaling Up SimCalc study found the SimCalc approach to be successful in meeting the needs of a diverse set of students and teachers. Ninety-five seventh grade teachers and their students across varying regions in Texas participated in a randomized controlled experiment in which they implemented a SimCalc-based three-week replacement unit. An analysis of the results showed a large and significant main effect with an effect size of 0.8 (Roschelle et al.,

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

2007; Roschelle et al., in review). This effect was robust across a diverse set of student demographics. Students who used the SimCalc materials outperformed students in the control condition regardless of gender, ethnicity,[1] teacher-rated prior achievement (we will discuss possible remedies for the trend of higher achievement students having slightly higher gain scores in the *Conclusion and Discussion*), and poverty level[2] (Figure 1). We provide a comparison of students in one particular region in Texas, Region 1, to other students in the study. Region 1 is in the Rio Grande Valley adjacent to the Mexican border, is predominantly Hispanic, and is one of the poorest areas in the United States. Consistent with our other data, we see that the students in Region 1 who used SimCalc had greater learning gains than students in the control condition.

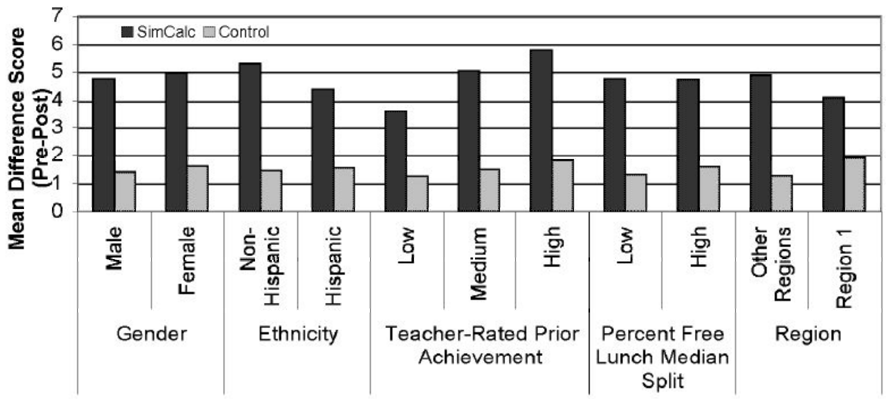


*Figure 1.* Mean student-learning gains by subpopulation group.

In the remainder of this paper we report on those aspects of the SimCalc curricular activity system that most likely led to these robust findings. In particular, we leveraged those features of the SimCalc environment that are consistent with the literature on under-achieving students (particularly those from non-mainstream backgrounds), while also meeting the needs of key stakeholders in the education system. We also discuss ways the integrated system could be improved to further close the gap for particular subpopulations.

## Research Foundations

In this section we describe some of the key features of SimCalc as a representational and communicative infrastructure. These features of SimCalc relate directly to what we know about effective instruction for all student populations, including students from non-dominant cultural and language backgrounds and other students who traditionally underperform in mathematics (e.g. Moschovitch, 2007b; Kaput and Roschelle, 1998).

SimCalc builds on students' existing competencies and experiences. The SimCalc approach differs from the traditional pre-algebra approach in several ways. Perhaps the most important is that SimCalc places motion phenomena at the center of learning (see Figure 2), enabling students to build on their existing cognitive and social competencies. Research with urban students (Monk & Nemirovsky, 1994) has shown that students tend to engage in "interval analysis" of motion simulations and interpret motion in a piecewise manner (e.g. "First the boy was going slowly, then he was running really fast, and then he stopped"). Further, all students, including traditionally low-achieving students, are capable of constructing rich stories about motion over time and can use narratives as a resource for interpreting graphical and tabular representations of motion as they build a qualitative understanding of calculus (Stroup, 2002). SimCalc allows students to play and replay a simulation of motion as many times as they wish, allowing more students to access these fundamental resources than is possible using traditional static media.

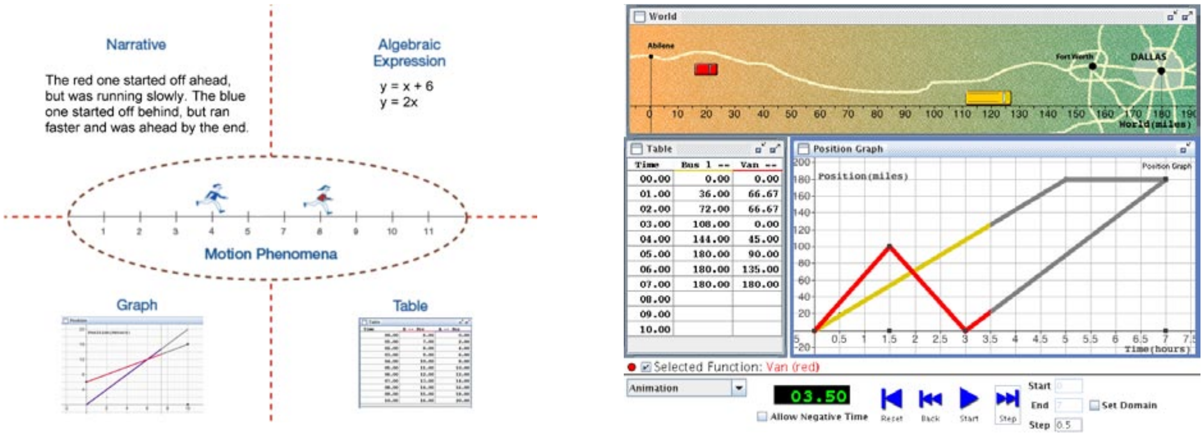


*Figure 2.* SimCalc linked representations (left); MathWorlds activity screenshot (right).

SimCalc supports multiple forms of representation and expression (see Figure 2). In SimCalc students study functions through linked motion, graphs, tables, and symbolic expressions. Research has found that complex mathematics is more learnable when students are not reliant on symbolic forms or dense textual descriptions, but can interact directly with a mathematical representation such as a graph, and immediately see the effects on other linked representations (Roschelle et. al., 2000). Moreover, providing access to multiple representations means that symbols can be introduced *after* students have experience with motion, narratives, tables, and graphs. In this way the symbols are *about* something, and can be understood as a compact and precise way of describing phenomena. By waiting to introduce the symbolic form, SimCalc is also not held hostage by what is symbolically or computationally simple. For instance, piecewise linear functions are quite complex to represent symbolically, and so are not introduced in most middle- and high-school curricula. However, interpretations of piecewise motions can help students understand the mathematics of change, and the narrative of an exciting race can provide exactly the context students can use to engage in deep mathematical thinking.

SimCalc supports communication and discourse. Making mathematical connections across different representations has social and communicative advantages. The four linked representations provide a shared set of referents for students and teachers to explore by replaying the motion or making changes in one representation to see the changes in the others. Students have opportunities to use a wider range of verbal and nonverbal communication acts, such as pointing: “See, right *here* the boy starts running faster.” Students also have opportunities to use the language of academic mathematics for a communicative goal (e.g., Does *going longer* refer to *time* or *distance*?). This goal- and meaning-oriented approach is consistent with best practices for learning language and with recommendations for supporting mathematical discourse (Moschkovich, 2007b; Swain, 2001) and is in contrast to traditional approaches to teaching academic language that rely on memorization of vocabulary lists.

## Scaling-Up: Meeting the Needs of the Educational System

Taking these research findings to the classroom on a large scale was a new challenge for the SimCalc project. Previously, the SimCalc approach was taught directly by either researchers or teachers who had been involved in long-term professional development or collegial arrangements with the researchers. In order to reach a larger audience of teachers and students, we needed a robust combination of curriculum, technology, and minimal professional development to leverage the benefits and while minimizing the chances of lethal mutations (Brown, 1991). This resulted in the emergence of a curricular activity system approach (Roschelle et al., in review), which helped us to address teacher, district and state constraints and realities while extending the SimCalc mission.

We designed a curriculum sequenced in a way that would be comfortable to most American teachers: breaking complex concepts into small pieces, starting with the smallest piece, and culminating with complexity. This approach differs from the “historical” SimCalc approach, where students are presented with a fairly complex problem, are asked to generate solutions for it, and through this process, learn concepts of rate and function—and other calculus related ideas. What we retained from the SimCalc approach built up over the years was a reliance on motion as a context for understanding function, and function as a way to think about rate. This, fortunately, aligned with Texas state-advocated approach. And of course, the curriculum is tied to the MathWorlds software, in which students are able to control simulations of motion and representations of graphs, equations, tables and actions are related.

We also focused on a small number of important activity structures, and provided supports for these in the written curriculum materials. For example, we incorporated the SimCalc tradition of having students predicting a motion by interpreting a graph, running the simulation to check their predictions, and explaining verbally differences or coincidences between prediction and simulation. This “predict-check-explain” model was not only discussed in trainings, but also written into each lesson in the student workbook, as one way to ensure students were exposed to the SimCalc approach, regardless of the teachers’ approach.

We used a fairly typical “week in the summer” model of professional development that met the time (and funding) constraints of a large number of districts and teachers. All teachers in the study received TEXTEAMS training, a two-day workshop on rate and proportionality developed by the Dana Center. SimCalc teachers received 3 additional days of professional development on the SimCalc curriculum. Over these three days, teachers became familiar with the SimCalc units and MathWorlds, and planned when they would teach the SimCalc units. The SimCalc pedagogy was modeled by the facilitator and included in the student workbook.

Deciding what mathematics to include in the units was a task of finding the intersections between the mathematics of change and existing national and state standards for 7$^{th}$ and 8$^{th}$ grades. The Texas Education Authority, through the Dana Center, was advocating an approach to teaching proportionality that was consistent with the SimCalc approach. Rather than presenting three numbers, and a procedure for finding the fourth, embedded in the equality among ratios ($a/b = c/d$), the advocated approach was to teach proportionality as a linear function of the form $y = kx$ (Stanley et. al., 2003). This provided the SimCalc project with the opportunity to connect the multiplicative constant *k* in the algebraic expression $y = kx$, the slope of a graphed line, the constant ratio of differences in a table comparing y and x values, and the experience of rate as “speed’ in a motion.

To ensure that students engaged with the mathematics in a variety of contexts, we grounded the unit in an overarching story framework–managing a soccer team. While use of real world contexts was consistent with prior SimCalc work (which has always been grounded in modeling the real world and students’ own experience of motion), having a single story framework was a departure from past research. This decision enabled us to start the units with typical linear and piecewise linear motions, and extend into non-motion contexts, such as mileage and money (oft used contexts in traditional math curricula and standardized tests). Though the units had overarching contexts, they were “context-light” in that the problems and software presented a highly simplified model of the real world, and these simplifications were made apparent to students. Knowledge of soccer, for example, was neither an advantage nor a barrier to understanding the problems. All the clues and grounding experiences necessary for solving the problem were contained in the simulation, so that all students regardless of cultural or socioeconomic background have the same opportunities to engage with the materials. Because SimCalc provides the phenomena to be studied, we leverage student knowledge of the “real world,” while avoiding inappropriate uses of their real world knowledge

## Conclusion and Discussion

In this paper we have shown how the Scaling Up SimCalc project integrated multiple perspectives to meet the needs of diverse student and teacher populations. Our focus on the representational and communicative infrastructure of SimCalc allowed us to create materials that were effective for students who are considered among the most at-risk for academic failure. By also incorporating a focus on the larger educational system, we were able to create materials that were used by a wide variety of teachers in a wide variety of settings. We believe that, as more innovations attempt to make a difference on a large scale, this focus on the overall curricular activity system will become crucial to successful scale-up.

We also note that, while our instantiation of SimCalc was successful in its goal of helping a wide variety of students learn important and complex mathematics, we believe that more can be done to further meet the needs of a diverse student population. We recognize that the data shown in Figure 1 indicates that there may be some disparities in learning among sub-populations of students who used the SimCalc intervention. For instance, students who were rated by their teacher as having low prior achievement had smaller gains than those who were rated as having high prior achievement, and there is a non-significant trend that Hispanic students had smaller gains than non-Hispanic students.

Detailed analysis of classroom interactions of a subset of the SimCalc teachers shows the importance of specific teacher moves that were used to scaffold discourse. Teachers who incorporated student ideas into their explanations (called “responsiveness”) and who engaged students in tasks that required cognitively complex intellectual work (similar to “cognitive demand”, Stein et al., 2000) had greater student gains than those who did not use such moves (Pierson, 2008). Providing additional professional development and support to allow all teachers to engage in these high-impact moves is likely to increase student achievement for underperforming sub-populations, as students with low prior achievement and students from non-dominant cultures and languages are those most likely to have impoverished classroom discourse. An additional component of discourse support is aiding students in acquiring an appropriate vocabulary (Moschkovich, 2007a; Olivares, 1996) including highlighting those words that have register-dependent meanings (Halliday, 1978; Pimm, 1987). Future work will consider creating a visual glossary of mathematical terms as well as general academic words (e.g. “predict,” “evidence”) to support students in using academic language appropriately.

To further aid in supporting productive discourse for our target students, we will investigate strategies that allow a reduction of the language load while maintaining the rigor of mathematical discourse. A productive strategy has been that of making expectations explicit, and providing scaffolding that aids students in meeting these expectations (Lee, 2005). This strategy is based on the finding that much of academic discourse is based on implicit norms (Gee, 2001; Lee, 2005), and students who are not aware of, or have cultural norms that are in conflict with, academic discourse norms are at a disadvantage (Ladson-Billings, 1995). By making norms and expectations explicit, *all* students will be able to more fully participate in the classroom discourse, while also engaging in rigorous academic thinking.


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## Endnotes

1. We focus on Hispanic students because they consisted of a majority of our student sample, there were negligible numbers of other minority groups in the study, and Hispanic students have traditionally underperformed in measures of mathematics achievement (Education Trust, 2003).
2. We take as our measure of poverty the percentage of the campus eligibility for the free and reduced price lunch program.

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# BECOMING A "LIBERAL" MATHEMATICIAN: EXPANDING SECONDARY SCHOOL MATHEMATICS TO CREATE SPACE FOR CULTURAL CONNECTIONS AND MULTIPLE MATHEMATICAL IDENTITIES

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*This case demonstrates how a multidimensional construction of secondary mathematics afforded Amelia, a young Latina immigrant, opportunities to participate in her mathematics classrooms in ways congruent with her cultural knowledge and salient identity as a "liberal." Additionally, this expanded version of school mathematics positioned Amelia's "liberal" ways of knowing as valuable tools for learning and provided her with opportunities for genuine self expression, motivating her continued engagement. This research contributes to the literature about culture and mathematics learning that is aimed at building from diverse forms of knowledge and cultured ways of being in an effort to support students who are often marginalized by traditional curricula and pedagogies.*

## Background

There exists a grave and immediate concern about how to encourage and support diverse students to engage more in school mathematics, to be academically successful with mathematics and to enroll in more upper-level mathematics courses. Some responses to this problem are the use of Culturally Relevant Pedagogy (Ladson-Billings, 1995), Culturally Responsive Teaching (Gay, 2000), pedagogy for social justice (Gutstein, 2006), Complex Instruction (Cohen, 1994; Cohen & Lotan, 1997), and Funds of Knowledge (Moll, Amanti, Neff & Gonzalez, 1992) as researchers and practitioners advocate for making connections been students' cultural knowledge and the knowledge needed to be mathematically successful and simultaneously create classrooms and use mathematical content that reframes cultural knowledge as necessary for the learning of school mathematics.

One mathematical success story has been explored in the research of Boaler and Staples (2007) and Horn (2002) and explained by the department's collaboration as a learning community and the use of Complex Instruction as a primary pedagogical tool in its mathematics classrooms. Many Railside students are achieving in mathematics and choosing to enroll in upper-level courses after they have met their graduation requirements. Previous research at Railside has used the Complex Instruction program, teachers' practices, and classrooms as the units of analysis. This work takes a different perspective by starting with students, specifically young immigrant women who achieved mathematical success at Railside. The goal of this work is to expand and build from what is already known about Railside High and Complex Instruction while changing the focus to students, their perspectives and experiences.

## Theoretical Framework

Research in mathematics education has recently begun using identity as a research construct in an effort to better understand the relationship between learning and culture within mathematical communities of practice (Boaler, 1997; Boaler & Greeno, 2000; Boaler, 2002; Cobb & Hodge, 2002; Cobb & Hodge, 2007; Martin, 2000; Nasir, 2002; Sfard & Prusak, 2005). This is important work, as attention to the identities students create within the local cultures of

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their mathematics classrooms has the potential to illuminate how students make sense of their school mathematical experiences and then make choices about how to act in relation to them. Additionally, understanding learning as a process that encompasses the construction of new ways of being provides for a unique balance between personal agency and influences from the broader communities in which students participate. This perspective prevents us from completely attributing students' failure or achievement to cultures located outside of school and simultaneously recognizes the role of the individual in academic pursuits.

The research that has been done about identity and mathematics learning has determined a reciprocal relationship between identity and practice. That is, within any learning community, inside or outside of school, students construct identities in practice. They shape self-understandings relative to the cultural activities afforded them, meaning that their participation affects the construction of their identities (Wenger, 1998; Nasir, 2002). Simultaneously, identities affect participation. Students come to a practice with ideas about who they are and their purposes for engagement which were situated in specific cultural contexts. Students understand practices and are motivated to practice in particular ways because of these identities (Nasir, 2002; Boaler & Greeno, 2000; Jilk, 2007; Sfard & Prusak, 2005).

However, there is little research about identity and mathematics learning that considers the multiple out-of-school communities in which students participate which shape ideas for how one thinks about herself and then acts. Martin's (2000) attention to the intersection of ethnic identity with students' mathematical identities is significant, because it acknowledges the culture inherently created in of all communities of practice (Cobb & Hodge, 2002). Ignoring culture means ignoring a critical component of students' lived experiences, which they use to shape their identities, and simultaneously contributes to the homogenization of young people by presenting an incomplete and inaccurate portrait of their lives.

This paper builds on and expands previous research about identity and mathematics education by foregrounding the life story of one young woman and her self-understandings relative to her lived experiences in Mexico, the United States, and her secondary mathematics classrooms. It considers the communities outside of school in which she participated as critical sites for identity construction and the ways in which identity is privileged and supported within the context of learning mathematics.

The research questions that framed this study were:

(1) What are the salient identities created by Latina immigrants who were academically successful in their secondary mathematics classes, and in which communities of practice were they shaped?

(2) How do Latina immigrants who were academically successful in secondary mathematics interpret their experiences with Complex Instruction through the lenses of their salient identities?

## Methods

*Context and Participants*

This case is part of a larger one-year ethnographic project focused on the cultural interpretations of secondary mathematics classrooms by Latina immigrants who attended a large urban high school as English Language Learners and successfully completed four years of college preparatory mathematics, including Advanced Placement Calculus. There was no specific site for this study. The stories the young women shared spanned time and contexts, both inside and outside of school, and both inside and outside of the United States.

Railside High School, however, was the one community that all participants had in common. They attended Railside during the 2000-2001 school year, the same year in which both Boaler (2002) and Horn (2002) conducted their research. During this school year Railside had approximately 1500 total students. Of these students, 35% were Latino, 25% African American, 21% White, and 18% Asian or Pacific Islander, and 1% American Indian. In 2000-2001, 12% of all students at Railside received English language services, and of these 68% were Spanish speaking (Education Data Partnership, 2007).

Additionally, these young women experienced four years of mathematics instructions steeped in the program of Complex Instruction. Based on the work of Elizabeth Cohen and Rachel Lotan at Stanford University, Complex Instruction (CI), is a framework that “enables teachers to teach at a high intellectual level” (Cohen et al., 1999) through the use of collaborative groups in heterogeneous classrooms. At the core of CI is an awareness of the structural inequities that are generated both in the larger society and within schools and classrooms, which often translate into an assumed hierarchy of competence. Complex Instruction aims to eradicate these hierarchies and to promote equal-status interactions amongst students, creating opportunities for all students to engage with and learn from rigorous mathematical tasks within a cooperative learning environment.

*Data Collection and Analysis*

Narrative methods (Lieblich, Tuval-Mashiach, & Zilber, 1998) were used to collect life stories told by these young women about experiences in their home countries, in the United States, and in their secondary mathematics classrooms. Other data included focus group meetings and parent interviews.

Narrative inquiry was used to shift the research focus away from teachers, programs and classrooms as the units of analysis and foreground individual students and their stories as primary data sources. This shift helps to dispel common views of Latina immigrants as a homogenous group and simultaneously affords access to the “self-understandings” (Holland et al., 1998, p. 8) and “representations of self” (p. 29) as described by the young women. Second, narrative inquiry is not often utilized in mathematics education research, therefore this frame expanded available information about students’ inner realities and the meanings assigned to them. Finally, in this work I am responding to a call by many to attend to out-of-school contexts as sites that influence students’ beliefs about self and mathematics in order to better understand the range of sociohistorical forces affecting students’ mathematical learning (Martin, 2000; Reyes & Stanic, 1988; Weis & Fine, 2000).

I analyzed the data by using a “content-oriented approach” (Lieblich, Tuval-Mashiach & Zilber, 1998), in which I dissected the original stories and analyzed smaller narrative sections aimed at uncovering the implicit content by asking about the meaning conveyed by the narrative, which traits or motives of the individual were portrayed, and the relevance of the images invoked by the author. I considered the distribution of themes across the story as a while and attended to emphasis placed on particular words or phrases and considered the emotion with which the young woman spoke.

Case studies provided me with the opportunity to examine the “local particulars” (Dyson & Genishi, 2005, p. 3) of each young woman’s identity and experiences as related to the more abstract phenomenon of learning mathematics. As a research design, case study emphasizes “the role of organizations, communities, crucial events, and significant others in shaping subject’s evolving definitions of self and their perspectives on life” (Bodgen & Biklen, 2003, p. 57). This focus coincides with my multi-level framework that includes multiple communities of practice in which each woman participated as important sites for identity formation.

## Results

Although her identities as a young woman and Mexican were important to Amelia, her identity as “liberal” was most salient to her. Amelia emphasized her “liberalness” in her decisions about the people with whom she associated and how she participated in the world. Amelia’s “liberal” identity did not have any political connotations. The meaning she assigned to “liberal” focused on voice and authority. Amelia felt that she acted “liberally” when she verbalized her ideas and opinions and made decisions that determined the trajectory of her life. Amelia constructed this salient “liberal” identity as she participated in both local communities in Mexico and as she moved across geographical and emotional borders throughout her life.

When Amelia described her secondary mathematics experiences at Railside High, she focused on the “multidimensional” (Boaler, 2004) nature of mathematics, especially the myriad ways to think, talk, and reason mathematically that were available to her. For Amelia, being a mathematics learner at Railside meant that she could discuss ideas, explore and justify alternative solutions, argue and reason. In these spaces where school mathematics was broadly constructed, Amelia’s “liberal” identity was useful and necessary. Her communication skills and desire to verbalize her opinions became strengths in these mathematical communities that required students to put forth ideas and justify reasons. Amelia’s “liberal” desires to determine her life decisions and paths were necessary skills when her mathematics teachers asked for multiple strategies for solving a problem. Indeed, Amelia’s “liberal” ways of being in out-of-school communities supported her mathematical participation, and her “liberal” identity became an intellectual resource for participating with and learning mathematics.

## Discussion

Amelia’s case demonstrates that the identities most salient to young people are not always constructed relative to major social structures such as ethnicity and gender. Identities are extremely nuanced and particular to an individual and her lived realities. It is therefore not useful and simultaneously potentially harmful to assume that all Latina immigrants fit into a homogenous group. While brown skin, long dark hair and a Spanish accent may place Latinas from the same continent, varied experiences intertwined with unique issues of immigration, bilingualism, religion, and generational status shape the identities created by Latinas.

Additionally, this case illuminates the cultural connections Amelia made between her salient identity as “liberal” and her participation in secondary school mathematics. Amelia’s focus on the multidimensionality of Railside’s mathematics program, framed by the use of Complex Instruction, argues for the expansion of secondary school mathematics such that students have opportunities to negotiate their use of cultural knowledge in service of learning mathematics. The knowledge and skills that young people cultivate while constructing salient identities become necessary resources for learning.

Finally, the opportunities available for Amelia to participate “liberally” in her mathematics classrooms helped her to engage authentically. As opposed to those who assert, “I am not a math person,” or students who claim that the norms for participation available in their mathematics classes do not coincide with the ways in which they think of themselves as young people (Boaler & Greeno, 2002), Amelia maintains that she could “be herself” in her mathematics classes. She could simultaneously be “liberal” and a mathematics learner. In fact, through her participation in four years of high school mathematics at Railside, Amelia created a new identity by bringing and using her "liberal" identity in the mathematics classroom and becoming a "liberal" mathematics learner in the process.

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# CURRICULAR REDESIGN, COFLECTION, AND DEMOCRATIC MATHEMATICS EDUCATION

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*In this paper, I examine teachers' tacit notions of equity as they engage in several efforts to reform their department's curriculum. Their efforts are examined for insights to implement equity-centered reform. Coflection, or joint inquiry, is a construct with four components: collective, deliberative, critical, and transformative. Coflection is offered as a means to promote equity-centered reform which encompasses teachers' professional development, curricular design, and construction of placement policies.*

## Introduction

Equity is a longstanding issue in mathematics education. For decades, equity has gained more attention with increasing prominence. In spite of gains in achievement on low cognitive demand items among students who traditionally underperform (American Institutes for Research, 2005; Tate, 1997), a significant achievement gap remains (Lubienski & Crockett, 2007). The achievement gap can be partially attributed to students' access to content or the curriculum. At the high school level, students' access to advanced mathematics content or courses is influenced by course offerings and policies such as tracking. When departments offer fewer and more rigorous courses, students experience higher performance (Lee, Smith, & Croninger, 1999).

In the face of low performance which is typically worse among underrepresented students, mathematics departments have engaged in various reform efforts to mitigate the problem. While some departments have been identified as *Organized For Advancement* (Gutierrez, 1999), others have implemented detracking (Oakes, Wells, Jones, & Datnow, 1996) to address the problems associated with the achievement gap. Still, others have sought reform through professional development programs (e.g., Silver & Stein, 1996). Each of these approaches has provided insights to the field. Yet, there remains a need to examine reform as a dynamic process in which assumptions about equity are explicitly addressed and challenged.

In this paper, I examine teachers' tacit notions of equity as garnered through a research study of department level curricular design. In particular, I consider the impact of their conceptions of equity on the curricular redesign that was implemented. The teachers' conception of equity comprise their expectations for underrepresented students as well as their expectations for how and for what students would use their mathematical understanding. Afterwards, I analyze how equitable the resulting mathematics program is as well as the barriers which contributed to inequitable outcomes. Finally, coflection is offered as an alternative approach to implementing equity-centered reform.

## Theoretical Framework

The teachers' conceptions of equity are informed by Secada's (1989) notion of equity as comparative, qualitative, and dynamic. In order to gauge equity, comparisons are made between two entities in order to judge what is fair or just. Equity is different from equality in that comparisons employed to evaluate equality consider if there is sameness or parity. The comparisons that are made are a matter of what is just; consequently, there is an

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acknowledgement that the comparisons are value-laden (Gutierrez, 2002). Equity is a dynamic construct and is context specific requiring ongoing and periodic evaluations. Additionally, what is characterized as equitable mathematics education comprises inputs (e.g., access to courses), processes (e.g., pedagogical approaches), and outputs (e.g., achievement scores). This approach to gauging equity was employed by Rousseau and Tate (2003) as they evaluated a mathematics program and identified barriers that impeded teachers' reflection on their practices.

The research reported here is also informed by Skovsmose and Valero's (2001) work which posits coflection as a means to promote democratic mathematics education and the work of Carol Malloy (2002). Coflection is a knowledge generating process and consists of four components – collective, deliberative, critical, and transformative. Coflection refers to the "thinking process by means of which people, together, bend back on each other's thoughts and actions in a conscious way, that is, people *together* [emphasis added] consider the thoughts, actions, and experiences they live as part of their collective endeavor, and also adopt a critical position toward their activity (Skovsmose & Valero, 2001, p. 48, 2001). Democratic education is characterized as education that is concerned with providing access and mathematical literacy (Malloy, 2002). Malloy outlines three benefits of democratic mathematics education - inclusiveness, mathematical understanding, and the ability to apply mathematics to problems.

With this framing in mind, I identify barriers that inhibited the design of a more equitable mathematics program. The resulting curriculum is evaluated to gauge how equitable or democratic it is. Then the barriers are investigated as I discuss how equity-centered reform focusing on coflection may have minimized their influence and yielded different, more-equitable outcomes.

## Research Design

The research was conducted in a secondary mathematics department in a small town in the Midwest. The department served 1500 students and consisted of 13 teachers with 12 participating in this study. The department implemented several efforts to reform its curriculum focusing specifically on low level classes which were disproportionately populated by underrepresented students. The data sources include field notes from a year of department meetings; field notes from classroom observations of five sections of low-level mathematics courses taught by three teachers; up to three interviews with each of 12 participating members of the department, administrators, and a guidance counselor; and school documents.

Constant comparative analysis (Strauss, 1987) was employed as I identified emergent themes. A list of initial codes was informed by the research literature and was used for the initial coding. The list of codes was revised as iterations of analysis were completed.

## Findings

The teachers held a one-dimensional conception of equity that focused primarily on providing students with access to content. They added and eliminated courses; however, they did not evaluate the pedagogical practices that were employed or how specific courses limited or increased students' access to advanced mathematics. The teachers' expectations were low and led them to eliminate content they classified as rigorous from courses. The efforts in which the department engaged did not produce a more equitable mathematics program. Rather, the department designed a new program of course offerings that limited students' access to advanced mathematics, perpetuated tracking, and neglected conceptual understanding.

The course offerings which resulted from the department's efforts yielded a mathematics program that was, perhaps, less equitable than the one they sought to reform. First, the new course offerings included two additional low-level courses. These courses were intended to provide students with opportunities to increase their fluency with skills and procedures or to prepare them for the first mathematics course for which a student could receive mathematics credit towards graduation. Second, the new course offerings included Modified Algebra, a course that by design, covered concepts in algebra with *less depth* than the traditional algebra course. Third, a Modified Geometry course was added; however, it excluded formal proof writing, thereby, limiting students' opportunities to engage in deductive reasoning.

While the new course offerings provided more opportunities for students to take more remedial courses, the course offerings also increased tracking, and stricter placement policies were implemented. Together, the Modified Algebra and Modified Geometry course constituted an additional track for students in the low level courses targeted by the department's efforts. Although this track permitted more students to take a geometry course, the track itself consisted of courses that were deemed, by the teachers in the department, as less rigorous and unlikely to prepare students for advanced mathematics course taking.

Moreover, the curriculum did not promote conceptual understanding. The Modified Algebra course addressed concepts with less depth, and the Modified Geometry course lacked formal proof writing and did not seek to improve students' ability to reason.

Two significant barriers to designing a more equitable mathematics program existed in the department's dynamic reform processes. The first barrier was the teachers' failure to examine their own beliefs and expectations for underrepresented students. Their conceptions of equity were narrow and reflected many commonly held and low expectations. Their low expectations were evidenced by their rationale to exclude formal proof writing; their rationale was that this was the *harder* content in the course. The goal became designing a course that students *could* pass. They did not examine their expectations for what their students could learn or how their own teaching might influence student performance.

The second barrier was the teachers' views about the purpose of mathematics. The teachers did not envision their students as users of mathematics far beyond arithmetic. They sought to prepare their students for everyday tasks that were restricted to purchasing and banking oriented tasks. They did not design courses that sought to prepare their students for college entrance examinations or mathematics- or science-based majors or careers. Nor were the students' preparation aimed at preparing them for decision-making and analysis needed to participate in a democratic society. The answer to the question that Secada (1989) asked about the intended ends of the students' mathematics education was that it would not to prepare them for advancement or economic or democratic participation.

## Discussion

In this section, I summarize how this research aligns with previous findings and outline how coflection employed as a knowledge generating process embedded in professional development might have minimized the negative, unintended outcomes. An exploration of how the four components of coflection promote democratic mathematics education is provided. The insights gained from this inquiry highlight the need for equity-centered professional development and reform efforts that explicitly confront teachers' conception of equity.

This study addresses areas that already have been studied. The curricular redesign efforts in the department revealed that the teachers held low expectations of their students. The students affected by the changes were primarily students of color and poor students, and the teachers holding low expectations aligned with previous research (e.g., Irvine & York, 1993). Analysis of student performance on national assessments reveal that despite some gains, the gains made by students of color are typically on low cognitive demand items. The focus of the department to design remedial courses and to improve procedural fluency yielded a program that did not promote conceptual understanding. The addition of more lower level courses may restrict the number of students who would take more higher level courses and stifle their performance since research higher performance is associated with a narrower more rigorous curriculum.

The department sought to improve students' performance in the low level courses and to have those students to take more mathematics. While a small percentage of the students would take geometry as a result of the new course offerings, the mathematics education that they received fell short of democratic education. Indeed, more students would take geometry. This access to new content speaks to inclusiveness – the first benefit of democratic mathematics education (Malloy, 2002). However, mathematics understanding and the ability to apply mathematics, the second and third benefits, were not foci or outcomes of the reform efforts.

Despite these disheartening outcomes, the challenge to create more equitable mathematics programs must not be abandoned. As mathematics teachers, mathematics educators, and mathematics teacher educators forge ahead, we must evaluate prior efforts and offer alternative approaches. Coflection is an alternative that I propose.

The four components of coflection speak to the barriers of equity-centered reform that I identified in the department's efforts. Coflection is collective; it takes place within a community with a shared endeavor. In this case, the community is the mathematics department, and the endeavor is reforming its mathematics program to address high failure rates amongst students of color and poor students. Coflection is deliberate with community members engaging in discourse aimed at solving the problems they have identified. Members of the department met frequently, at least once per month, and smaller groups of teachers met to discuss issues specific to particular courses. Thus, meeting to deliberate would be an unlikely problem for this department to employ coflection. The critical component of coflection, however, is essential and was missing from this department's efforts. While the members of the department met frequently and had a shared goal, they did not engage in critical analysis of the courses they designed or their rationale for such. Rather, their notions about the nature of mathematics, who can learn advanced mathematics, and the role of teaching in student learning were unchallenged. The absence of this critical component facilitated reform that was not more equitable. The goal of coflection and democratic education is also the fourth component of coflection – transformation. Transformation that advances equity, however, did not result. Yet, coflection appears to be an alternative that may have yielded more equitable outcomes had it been included as part of the reform process.

## Conclusion

Teachers who engage in reform must be willing to be critical of existing ideologies, practices, and policies. This study demonstrates that in order to create more equitable mathematics programs, having teachers who are willing to devote time, discuss problems, and make changes are insufficient conditions. Rather, reform must be equity-centered challenging long held notions about who can learn powerful mathematics and re-envisioning students' possibilities and how mathematics opens or closes the doors of opportunity. Coflection, with its goal of *transformation* that results from *deliberation* from a *critical* perspective to solve a shared endeavor by a *collective,* presents an alternative to equity-centered reform efforts that have been unsuccessful.

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# SEEING MATHEMATICAL LITERACIES IN AN AFRICAN-AMERICAN PARENT-CHILD INTERACTION

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*This paper provides an analysis of the mathematical ways of thinking present in a 15-minute interaction between an African American mother and her preschool son during a craft project as a way of opening up notions about the competence of young minority children and their parents.*

## Introduction

Much of the news reported about the achievement gap in mathematics is pretty bleak, even for young children. Studies have shown that while 66 percent of European American kindergarteners pass tests on reading numerals, counting past 10, sequencing, and comparing; only 42 percent of African American and 44 percent of Hispanic children pass similar tests (NRC, 2005). Similarly, other research has shown that poor children have a harder time solving problems mentally than well-off children do (Jordan, Huttenlocher & Levine, 1994).

Historically, researchers have attributed this gap at the start of school to a variety of factors, including inadequate preschool education (e.g., Graham, Nash & Paul, 1997) and inadequate support from parents (e.g., Starkey & Klein, 2000). The National Research Council (2005, p. 173) summed up this line of work in this way: "overall, the research shows that poor and minority children entering school do possess some informal mathematical abilities but that many of these abilities have developed at a slower rate than in middle-class children."

More recently, some mathematics education researchers have argued that the gap in performance between minority and low-income students and their majority peers may result from different mathematical values and practices in homes and schools as well as educators' inability or unwillingness to capitalize on the mathematical strengths children bring from home (Anderson & Gold, 2006; Baker, Street & Tomlin, 2006).

The purpose of this paper is to build on this later work by providing an in-depth analysis of the mathematical ways of thinking present in a 15-minute interaction between an African American mother and her preschool son as they worked together on a craft as part of a family involvement activity. To do this, we draw on the theoretical frame of multiliteracies (New London Group, 1996), which has been used by reading and language researchers to diversify notions of what it means to be literate. We believe this theoretical frame can be used within mathematics education to highlight the diverse practices involved in competent mathematical performances.

## Literature Review

A number of researchers concerned with the mathematical development of low-income and minority preschoolers have written about interventions that were designed to teach parents successful ways of developing the mathematical thinking of their children (Baker, Piotrkowski & Brooks-Gunn, 1998; Bryant, Burchinal, Lau & Sparling, 1994; Starkey & Klein, 2000). Although these studies demonstrated some success improving preschoolers' performance on mathematics assessments, the studies also began with the assumption that the research community and teachers had little to learn from low-income and minority parents. For example,

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

Bryant, Burchinal, Lau and Sparling (1994) compare children from "better" home environments to those in "poorer" home environments. The line between these two kinds of homes was determined by a survey, which asked about things like number of books in the home, organizational schedules, and family activities. Families that came out lacking in these measures were not seen as having alternative resources that had not been addressed by the survey. The authors of the study conclude that "determining how to improve the quality of Head Start child care and home environments are major challenges that still need to be addressed" (Ibid, p. 306). This notion that researchers must work to "improve" family lives makes it difficult to think about low-income and minority families as having strengths that researchers and teachers might tap.

Anderson and Gold (2006, p. 262) challenged this line of thinking with a study that examined the mathematical practices of low-income, minority children in informal settings, such as game-playing at home and school. They wrote: "Too often, teachers and schools fail to recognize or credit the knowledge, skills, and strategies that children bring with them from home – especially when a child comes from a family background that differs from that of the teacher's in social class, race, or ethnicity." Researchers are only just beginning to identify home-based knowledge, skills, and strategies that may be useful in early mathematical learning. Recent studies have identified some funds of knowledge (Gonzalez, Andrade, Civil & Moll, 2005) that families in some communities possess that may be drawn on in mathematics classrooms, such as knowledge of gardening (Civil, 2001). More work needs to be done that looks not only at the particular knowledge and skills that families in communities may have as a result of their work or home lives, but also how ways of speaking to and interacting with children in non-majority communities can be seen as sites for building mathematical competence, rather than as deficits.

A number of researchers have noted differences between the speaking patterns and norms of interaction of African American students and their majority teachers (e.g., Delpit, 1995; Heath, 1983). These differences have been explored in a number of ways. For example, Orr (1987), who studied the failure of African American students in mathematics and science at a private, progressive school, concluded that Black Vernacular English prevented students from thinking in mathematical ways. She suggested that this dialect, used by many African-American students in the school, did not have adequate vocabulary or grammar structures to support high-level quantitative thinking. This was harshly critiqued by linguists, in particular Baugh (1994); however, the notion that adult-issued commands and non-standard English inhibit mathematical thinking lingers. Other researchers have suggested that the rich, linguistic traditions in the African American community could be taken up and used in productive ways to further school learning (e.g., Delpit, 1995); however, much of this work has focused on language literacy learning. This paper seeks to bring this line of work into mathematics, examining the ways that an African American, low-income mother's comments, questions, and directions to her son can be seen as supporting his mathematical thinking and reasoning.

## Theoretical Framework

To contribute to the literature described in the previous section, this paper draws on the notion of multiliteracies to identify mathematical literacies present in a parent-child interaction. The goal is to name particular language and communicative practices that could be studied in other settings with attention to mathematical thinking and reasoning.

Multiliteracies, first articulated by the New London Group (1996), was intended to address both the multiplication of modes of literacy (visual, print, computer, etc.) and the increasing cultural and linguistic diversity present in many countries around the world. The theory was intended to expand notions of literacy beyond language and traditional representations of language to include the many ways that literacy is used by people in the world. In some ways, this work is similar to work in mathematics that has sought to identify and to value the ways that mathematics is used by people in informal situations (e.g., Lave, 1988). However, with the term “multiliteracies,” the New London Group made a conscious effort to include these alternative practices in the definition of what it meant to be literate in the world today. In particular, research drawing on multiliteracies has emphasized the visual aspects of language and has criticized definitions that focus only on the decoding and meaning-making of written words. Using this framework, literacy researchers have found that children identified as “struggling” in literacy classrooms often are capable of complex literacy practices when the definition of literacy is broadened (e.g., Cumming-Potvin, 2007).

## Modes of Inquiry

The data reported in this paper comes out of a larger study located within interpretive ethnographic traditions (Eisenhardt, 1988; Geertz, 1973). For the last two years, we have been studying the mathematical learning of children in a preschool classroom located in one of the most rural counties in Georgia. Oliver County Public School is a PK-12 school with fewer than 300 students. Most of the students are African American, and, because so many students qualify for free lunch, the school decided not to charge any of its students for meals. These characteristics make it an ideal setting to study the mathematical learning of underrepresented students in a *rural* (as opposed to the more-commonly studied urban) context.

Over these two years, we have visited the classroom weekly to observe both formal instruction and center time, where children engage in more open-ended play. During these visits, we wrote fieldnotes, audio-recorded conversations, took digital pictures, and collected student work. To supplement the written fieldnotes, all audio tapes have been transcribed. For this paper, we chose to focus on the fieldnotes and transcripts of a 15-minute interaction between a mother, Patrice, and her four-year-old son, Markus. In order to promote family involvement, one of the high school classes occasionally organizes activities for the preschool children and their parents. This activity, which involved making a paper plate scarecrow, was held just before Thanksgiving in the second year of the project. Although here we focus on just this interaction, our analysis was informed by our experiences in the classroom over two years as well as our analysis of the larger corpus of classroom fieldnotes for other purposes.

Little (2002) did similar work when she chose a short segment of conversation to analyze in her study of collaborative learning in teacher study groups. She notes that “there is crucial strategic value in looking closely at bounded segments of text” (p. 920) because the “mundane exchanges” of any moment reveal interaction patterns, ways of speaking, and shared values and expectations. The goal of this analysis is not to generalize to all of Patrice and Markus’s interactions and certainly not to all interactions between African American mothers and their children. As Geertz (1973, p. 23) has written, the point of ethnographic work is not to show “the world in a teacup.” Rather, the goal here is to provide an analytic model for examining informal conversations in order to identify mathematical literacies.

To meet this goal, we began our analysis by asking the following research question: What mathematical literacies did Markus engage in during this craft-making experience and how did his mother support this engagement? For this preliminary analysis, we did a content analysis of the mathematical knowledge and skills represented in the focal transcript as well as a discourse analysis of the conversational moves used by the mother and child.

## The Conversation

Due to the space limitations, we present the first seven minutes of transcript below in as much detail as possible (although we analyzed the entire interaction). To save space, markers of pausing have been eliminated, some parts of the conversation have been summarized instead of quoted, and some unrelated conversations with other children and parents have been removed. Although we could not present the whole interaction, we wanted to present a significant chunk in because there are few examples of parent-child interactions around mathematics in the literature. Many claims about parents' abilities to support their students in mathematics are based on interviews and surveys, rather than observations.

As stated above, this conversation occurred around an activity that asked parents and children to make a scarecrow out of a pre-packaged craft, which included paper plates, colored foam cutouts, and written directions. These directions included a black-and-white diagram of what the finished product was supposed to look like. Some of the foam cutouts were irregular, while others were triangles, circles, and rectangles.

*Patrice showed Markus the black-and-white paper with the directions on it. He looked at it in her hands as she talked.*

*Patrice:* "See this page?" *He nodded*. "See what's supposed to go at the top? See that triangle?"

*Patrice pointed to the drawing of the triangle on the paper. Markus looked at it and then at the foam pieces spread out in front of him. He reached for the large brown triangle that was the biggest part of the hat and was supposed to be glued on top.*

*Patrice:* "You see how that goes at the top*?" She pointed at the picture in the directions. Markus put the piece on the top of the scarecrow's head. Patrice picked up the piece that was supposed to serve as the brim.* "And this." *She held up the brim piece.* "This goes like that." *She pointed first to the picture on the diagram and then put the brim onto the scarecrow.*

*Patrice then picked up a rectangle of perforated yellow foam*. "See this? This is the hair. We got to take these apart. I guess." *She looked back at the diagram and then pulled one of the pieces off.* "Here, you can help me." *Patrice handed Markus some of the foam.*

*Markus:* "Where the hair go?"
*Patrice:* "Look on the picture and see." *She handed him the diagram.* "You see? Up under there?" *She pointed to the hair. Markus put down the foam and picked up the diagram and studied it.*
*Markus:* "Under the hat?"
*Patrice:* "Yeah. I'll take it apart for you and you can put it on." *She took back the foam and quickly ripped it apart, while Markus laid three pieces of foam down over the forehead of the scarecrow and tucked up under the hat, just like in the picture. He sat back.* "That's all the hair you want?"
*Markus*: "Uh-uh." *He started to put more pieces on.*

*Patrice put the strips she had separated into two piles in front of Markus.* "They got long hair and short hair." *She picked up the directions.* "You can use them so it can look like this, with the short hair in the front and the long hair can go on the back." *Patrice noticed that they had forgotten to put glue down on the paper plate. She took off the hair that Markus had put on, got a glue stick, and spread glue on the top. She then started to replace the hair.*

*Patrice*: "You see. You do the rest of them." *Markus started to lay down more strips of foam across the forehead of the scarecrow.* "Make sure it sticks now. Press it down." *Markus pressed on the hair.* "Can you get one more on there?" *Markus picked up another strand of hair and placed it in line* *next to the others.* "Now, you're going to put the long hair on the sides, but you need more glue."

*While Patrice got glue from another parent, Markus picked up the paper with the directions on it and traced the long hair on the sides with his finger. Then he looked back at his own scarecrow. Patrice took the glue stick and spread it on the sides and then put more glue on the top of the plate above the hair that had already been glued downed.*

| | |
|---|---|
| *Patrice:* | "Look at your picture. Remember." *Markus picked up the directions and studied them.* |
| *Markus*: | "The triangle goes on top?" |
| *Patrice:* | "Uh-huh. How?" *Markus pointed to the top of the scarecrow.* |
| *Markus:* | "Like that." |
| *Patrice:* | "Uh huh. You show me." *Markus picked up the triangle top the hat and placed it on the scarecrow to match the picture.* |
| *Patrice:* | "And then … this." *She picked up the brim of the hat. Without prompting, Markus picked up the directions and looked at them. After a moment he pointed to the bottom of the hat.* |
| *Markus*: | "It go right there." |
| *Patrice:* | "Okay." *She put glue on the bottom of the hat and then Markus put the brim on.* |
| *Patrice:* | "What else we got there?" *She picked up the directions*. "We need a nose, don't we? You got a nose?" *Markus looked at the directions* |
| *Markus:* | "Uh-huh." |
| *Patrice:* | "What kind of nose*?" Markus picked up the triangle piece that matched the black-and-white drawing in the directions.* |
| *Markus*: | "Orange." |
| *Patrice*: | "Okay. And what kind is it? … What it look like? … Shape." |
| *Markus*: | "That way." *He held the triangle so it was oriented the same way as in the picture.* |
| *Patrice*: | "Triangle, right?" *Markus nodded*. |

*Patrice put glue in the center of the face and Markus put the triangle on. The conversation continued as they finished the project.*

## Mathematical Multiliteracies

Throughout this conversation, Markus demonstrated literacies related to geometry, spatial reasoning, and representation. The Principles and Standards (NCTM, 2000, p. 41) asks that young students "analyze characteristics and properties of two-and three-dimensional shapes," "specify locations and describe spatial relationships," "apply transformations," and "use visualization, spatial reasoning, and geometric modeling to solve problems." In addition students are expected to "use representations to model and interpret physical, social, and mathematical phenomena" (Ibid, p. 67). Repeatedly, Markus studied the written diagram presented in the directions to place objects on the paper plate circle that represents the scarecrow's face. He began by placing the large triangle that represented the hat on the top of the paper plate, not only putting it in the correct place, but also orienting the triangle to match the diagram contained within the directions. He went on to place the brim in the correct place, as well as the hair, the eyes, the nose, the mouth, a flower, and a bow tie (some of this happened in the portion of the transcript not reported.)

Making these placements was not simple work for a four-year-old. In the written representation, the picture of the scarecrow was complete, in black-and-white, and much smaller than the craft Markus was making. Thus, to identify the correct piece among the foam cut-outs, Markus had to scale up the black-and-white image, disregard the color, and, often mentally transform the orientation of the cut-out to match the image in the diagram. This demonstrated not only literacy in reading a written diagram, but also in visualizing to solve geometric problems.

Some of the foam cut-outs were regular 2-d figures that preschool children are commonly asked to identify. (The eyes were circles; the nose and hat a triangle; and the long and short hair was differently-proportioned rectangles.) When working, Markus occasionally used the names of these shapes. When talking about the hat, he asked if "the triangle goes on top" and in the unreported part of the conversation, he described the center of the flower as a circle. However, when his mother asked him to identify the shaped of the nose, he did not use the word "triangle." He talked instead about its orientation. Because this question is prominent in the conversation (and most closely resembles the kinds of questions preschool teachers typically ask), one might assume that Markus has difficulty using shape names. However, his casual use of the terms in other contexts suggests that this is not the case, and rather, that it was the problem of orientation that he found more interesting at this time.

Another emerging literacy demonstrated by Markus in this conversation was his use of the written directions and the visual diagram to direct purposeful activity. In both prompted and unprompted moments, Markus returned to the written directions to make decisions about his craft. This demonstrated his expectation that directions are meaningful and that he is competent to interpret them. As in the reading and language multiliteracies work, this meaning-making around visual images can be seen as a literacy that is increasing important in an age of technical diagrams and machines. Despite its value, the interpretation of visual images and diagrams is a literacy rarely assessed in most standard assessments of preschool and kindergarten readiness.

Throughout the conversation, Patrice scaffolded her son's emerging mathematical literacies in a variety of ways. One frequent strategy she used was the making of "see" statements to direct Markus's attention and to model ways of thinking about and performing a task. She introduced the written directions by saying to Markus: "See this page? See what's supposed to go on top? See that triangle?" In the complete transcript, she told Markus to "see" a dozen times. These statements communicated to Markus what was important. Often, even when the statement was phrased as a question, Patrice did not expect Markus to answer, but instead looked at him to make sure that he understood what she was saying. In that opening question, after asking Markus "see what's supposed to go at the top?" she did not allow him to reply, but continued by directing his attention to triangle. She reinforced this move by pointing to the triangle on the written diagram. She then remained silent while he looked for the corresponding piece among the foam cutouts. When he located the correct piece, she reinforced the connection between the foam cutouts and the diagram by saying: "You see how that goes on top?" which was another question that Markus was not intended to answer (and, in fact, did not answer.) Some might critique Patrice's questioning here as not being sufficiently open-ended or as taking over the thinking for the child. However, her prompts can also be seen as promoting opportunities for Markus to make connections and to think. Later in the conversation, Markus picked up the diagram to decide where to place the hair on his own. It seems likely that he did this because his mother encouraged him to see the diagram as a source of information and provided him with the necessary support he needed to interpret it. This is another site where the concept of multiliteracies can open up ideas about competence. Markus does not necessarily need to be able to verbally articulate his thinking about mathematics in order to be seen as thinking mathematically. Competence can lie outside of spoken words.

Patrice also modeled her work interpreting the written directions as a way of figuring out what to do next. After she picked up the yellow foam that was to be taken apart to make the rectangles for the hair, Patrice looked back at the written directions to make sure her actions were correct. She did not explain this Markus, but simply performed it. This is a different kind of modeling then the sort often done by teachers of young children. Patrice did not over-dramatize her actions, narrate each step, or quiz Markus about what she was doing. Instead, she used the directions for information as an adult. Again, this could be seen as problematic because the modeling is not made explicit or because she is taking over the work of interpretation for Markus. However, Patrice's moves can also be seen as productive. Patrice's actions demonstrated to Markus that the reading of directions and the interpretation of a diagram is genuinely useful in the adult world, rather than *acting* as if this was the case. It is in many ways a much more genuine modeling of how adults solve problems in the world.

In addition to examining the mathematical literacies present, it is also worthwhile to think about the kinds of literacies that are not represented here, particularly those that mathematics educators and preschool teachers might expect to see in such an activity. For example, Patrice never asked Markus to count any of the foam cutouts in the project. The only reference to enumeration is when she asked Markus if he could add "one more" hair. (He did.) During this activity, both the teacher and the paraprofessional, who were helping children whose parents could not attend, repeatedly asked children to count eyes, noses, and hair. This emphasis is understandable given the focus on counting in many preschool standards and in many of the assessments that are used to make judgments about children. In some ways, Patrice missed an opportunity to help her son practice these often-assessed skills; however, by focusing on the kinds of mathematical literacies necessary for the task, she presented an image of mathematical literacy aimed at purposeful activity rather than as one imposed unnecessarily on the world. In similar ways, Patrice did not ever ask Markus to explain his thinking, as many mathematics educators might do. There are certainly drawbacks to this; however, it is important not to confuse the articulation of thinking with thinking itself in considering Patrice's ability to support her son's developing mathematical literacy.

In conclusion, it is unacceptable to dismiss what Markus does in this episode as demonstrating "some informal mathematical abilities," as the authors of *Adding it Up* appear to do. In discussions of the early achievement gap, the emphasis on counting, reading numerals, sequencing, comparing, and shape identification has created a narrow view of early mathematical literacy. To appropriately value and assess the mathematical literacies that all children bring to school, broader lenses must be used. The mathematical literacies demonstrated by Markus in this episode are in many ways more sophisticated than those required by many of the preschool assessments used to label low-income and minority children as behind. Systematically looking for diverse mathematical strengths may both broaden our conceptions of young children's mathematical literacies as well as challenge our notions about who is capable of supporting their children's learning.

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# A STUDENT'S CAUSAL EXPLANATIONS OF THE RACIAL ACHIEVEMENT GAP IN MATHEMATICS EDUCATION

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*This paper analyzes a clinical interview of one student's understandings of the causes of racial performance gaps in mathematics education. The topic was chosen because there is a literature on teachers' perceptions of racial gaps but little in terms of student perceptions. The particular student was chosen because her thought processes indicated an interesting tension between behavioral and structural explanations for the gap. Data suggest that this student privileged behavioral factors and justified her reasoning by drawing on prior in-school and out-of-school experiences in problematic ways. Implications of student perceptions of racial performance gaps for teaching and learning are discussed.*

## Background

The latest report from the Trends in International Mathematics and Science Study (TIMSS) shows that White and Asian students continue to excel relative to Black and Latino students in both elementary and secondary mathematics, thus confirming the persistence of disparate achievement patterns amongst racial/ethnic groups in the United States (Gonzales et al., 2008). Explanations for the achievement gap have spanned between what race scholar Cornell West (2001) calls "conservative behaviorism" and "liberal structuralism." Those falling in the first category explain the achievement gap in terms of personal responsibility, citing cultural deficits such as an impoverished work ethic passed on to children by their parents. Those in the latter camp find behavioral factors insufficient (Lee, 2002), instead explaining this phenomenon as an effect of racism at both an individual and institutional level (Oakes, 2005). Yet despite tendencies by some to advocate exclusively for either extreme (e.g., McWhorter, 2000), others contend that neither perspective in isolation fully explains the achievement gap (Noguera, 2008); that in fact, understanding the gap as a complex sociopolitical phenomenon implicates the relevance of both behavioral and structural factors.

One factor that has been shown to contribute to academic performance is teachers' perceptions of their students' academic capabilities. Secada (1992) has argued that teachers' perceptions tend to vary based on demographic variables, such as race, socioeconomic status, and gender. These disparate perceptions can result in unequal academic expectations for students, which in turn contribute to a widening of the achievement gap (Ferguson, 1998; Jussium, Eccles, & Madon, 1996).

However, while characterizing teachers' perceptions of the achievement gap continues to be a worthwhile enterprise, it is equally important that students' perspectives on this matter be explored as well. Making sense of the manner in which students' perceive and explain differences in mathematics achievement is significant not only because of students' status as critical actors in the classroom, but also because their beliefs have implications for teaching and learning. For instance, the reform movement in mathematics education encourages teachers to make collaborative learning a part of their pedagogy, an exhortation that is predicated on the idea that appropriately structured discourse facilitates student understanding (National Council of Teachers of Mathematics, 2000). But when students hold particular views on race and on the

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

achievement gap, their perceptions of their classmates may affect collaborative learning. The present study takes a first step in the direction of documenting student perceptions of disparate performance patterns by way of a fine-grained analysis of one student's causal explanations of such variations in mathematics performance.

## Theoretical Perspectives

Given the inherent complexity of the achievement gap, a study of a naïve explanation of this phenomenon must be geared to handle potentially conflicting explanations. In the context of his research on physics education, diSessa (1979) has claimed, "People are more fundamentally model builders than they are formal system builders….Their views are as much conflicting patchworks as they are coherent systems" (p. 251). When asked to explain the physics underlying everyday phenomena such as projectile motion, diSessa (1988) found that in lieu of a coherent, systematic explanation, people tended to provide multiple, occasionally inconsistent, explanations.

Although diSessa's ideas derive from a radically different domain, it has been shown that diSessa's theory can shed light on people's reasoning about social issues such as race and equity (Philip, 2007). Indeed, the notion of people's beliefs as "conflicting patchworks" has precedent in the literature regarding the cognitive underpinnings of stereotyping. For example, Katz (1981) has proposed that Americans embody an "attitude duality," whereby individuals can simultaneously endorse competing values like individualism and egalitarianism—even when doing so results in inconsistent reasoning. From this perspective, the messy way in which people might explain the achievement gap can be thought of as an entirely sensible cognitive process (Monteith, Zuwerink, & Devine, 1994).

Thus, it seems reasonable to expect ambivalence when asking people to explain complex phenomena, but that begs investigation into the factors that influence people's reasoning. Tversky and Kahneman (1974) have argued that biases in judgment are significantly affected by the "availability" of information. That is, people tend to draw conclusions about the frequency of an event based on the ease with which information can be retrieved from memory, and are consequently less likely to account for information that is not directly available to them in a particular context.

In the present study, analysis of data procured from an interview with a student seem to suggest that availability played a critical role in the subject's reasoning as she negotiated a wide range of explanations for the racial achievement gap.

## Research Questions

1. How do high school students explain the racial achievement gap in mathematics education?
2. What is the nature of their causal reasoning about the phenomenon, and what factors influence that reasoning?

## Methodology

Between 2002 and 2007 the author taught mathematics in a racially and ethnically diverse, urban public high school. In 2007 the racial demographics of the school were as follows: 74% Latino, 13% African American, 11% Asian, and 2% White. The racial achievement gap was explicit at the school, as African American students were grossly underrepresented in advanced math courses, while the opposite was true of Asian students. Latino students were proportionately represented, and the White student population was too small to warrant an evaluation. For this study the author recruited former students with whom he had developed strong bonds during his tenure and had since maintained consistent contact. The expectation was that the close relations would encourage the students to candidly express their opinions about a sensitive issue.

Semi-structured, one-on-one interviews were conducted because they facilitated the goal of understanding students' causal reasoning at a fine-grained level. A similar methodology has been used to study people's understanding of issues of race and social dynamics (e.g., Bonilla-Silva, 2003). Interviews lasted approximately forty-five minutes and began with the following prompt: "Research shows that in advanced math classes, Asian students are overrepresented and African American students are severely underrepresented. When I was a teacher I saw the same thing in my classes. How do you explain those trends?" As the interview progressed, subjects were presented a series of probes in order to investigate how being forced to grapple with a problematic context affected their reasoning.

Seven students were interviewed. Each of them had recently graduated high school and was enrolled in a highly selective university at the time of the study. Of the seven interviews, the interview with one student, Nikki (a pseudonym), was chosen for detailed analysis because it most clearly illustrated an individual attempting to juggle multiple causal explanations in complicated and intriguing ways. Nikki self-identified as Asian and was widely considered among her peers and the faculty as one of the top students at her high school. She had been the author's student for three years in the most advanced mathematics courses offered by the school.

## Results

Data suggest that as the interview progressed, Nikki did not settle on any one coherent explanation for the achievement gap. As Figure 1 shows, ten distinct "narratives" (A through J) were identified in the data that Nikki invoked in complicated and occasionally inconsistent ways throughout the interview. The narratives (A through E) that arose in response to the initial interview prompt fell on the "behavioral" side of West's (2001) behavioral-structural dichotomy. In fact, behavioral narratives were repeatedly called on throughout the interview. The most popular of these was Narrative A ("parental influence"), which Nikki called on five separate times during the interview. The lone definitively "structural" explanation, Narrative I ("teacher/counselor bias"), came up near the end of the interview.

A closer examination of the data also suggests a hierarchical organization of the narratives. The following excerpt, which occurred at the very beginning of the interview, illustrates this point:

Interviewer: So you were talking about…let me see if I get you right. You were thinking that because Asian parents – let's say that they emigrated from another country, so they're kind of new here – that you're saying that's why they push education to their kids?

Nikki: Yeah, they strive more for better education, and since…I think, like, for me, personally, parental influence has been a lot on me. If my parents hadn't, like, because even my parents are immigrants, and they tell me that they've come here to give us a better education, to make better…to make ourselves better. And, um, they tell me to, like, do good in class and take advanced classes and everything. And, um, that's what really made me go for it, honestly.

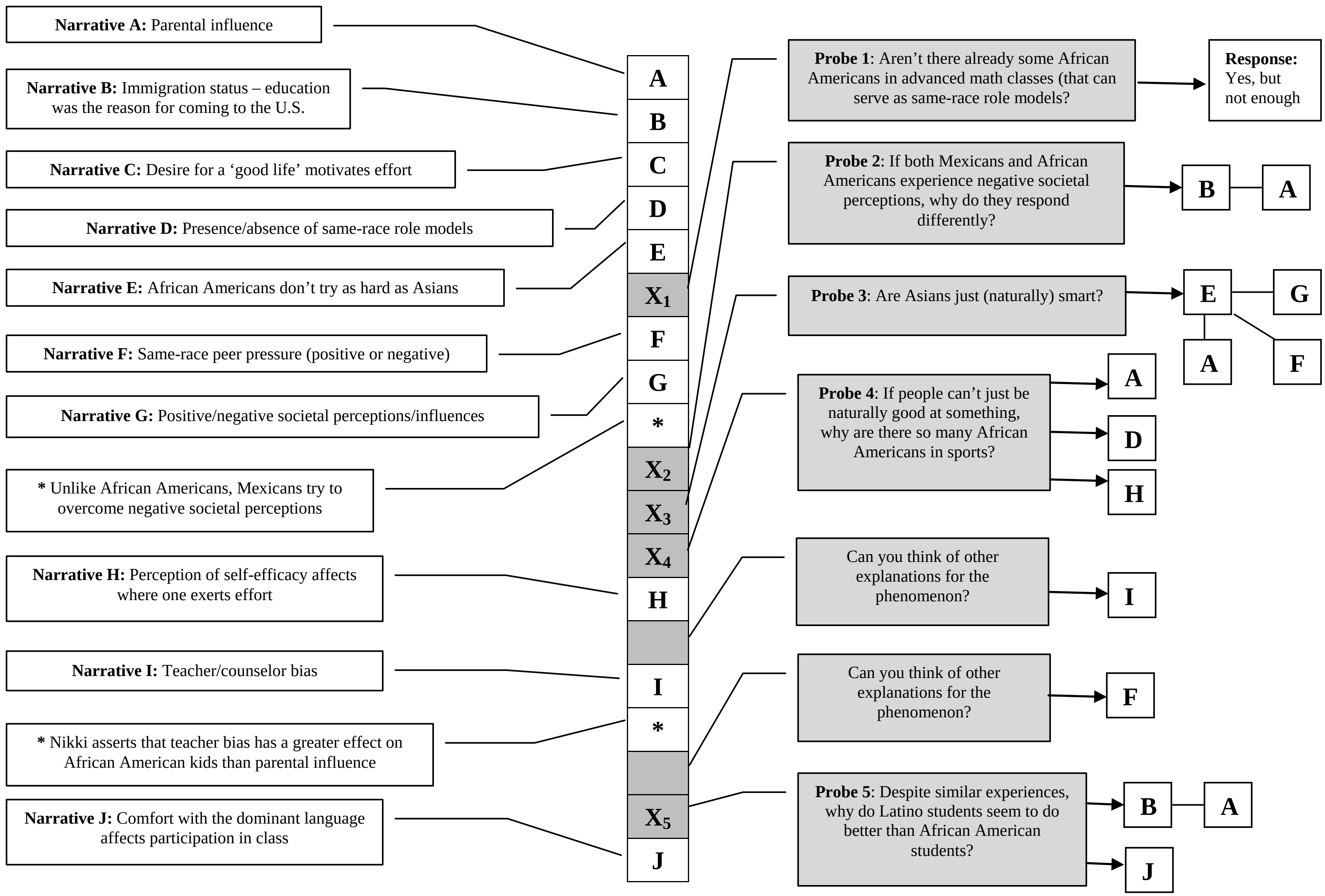


*Figure 1*. Overview of the interview with Nikki. The shaded boxes represent attempts to probe Nikki's reasoning.

A causal hierarchy emerged in Nikki's reasoning. She explained her academic success in terms of her motivation to "do good in class and take advanced classes" (Narrative E), but this explanation depended on other explanations. Nikki's motivation was a product of her parents' influence (Narrative A), and her parents pushed education because of their status as immigrants (Narrative B). Based on Nikki's reasoning, if her parents had not been immigrants, then they would not have been likely to push education, which in turn would have left her less motivated to succeed in school.

In addition to the hierarchical relationship between the narratives, the previous section of transcript highlights the way in which Nikki privileged Narrative E ("personal effort") as a prime determinant of success in mathematics. Her response to Probe 3 provided further evidence to support this claim:

Interviewer: One thing I've been trying to make sense of is, I guess, you know, why can't it just be that Asians are just better at math than African Americans? Why can't that be true?

Nikki: You can't really single out a race. It's just that Asians try a lot harder than African Americans, it's not that they're better at it, they just try harder. That's why they become better at [math].

But while Nikki touted behavioral narratives during the entire first half of the interview, when asked if there were other explanations for the achievement gap, Nikki provided multiple, first-hand accounts of biased teachers and counselors limiting the academic opportunities made available to African American classmates (Narrative I). Unclear as to how she reconciled her behavioral and structural explanations, the author asked Nikki whether parents or teachers/counselors had a bigger impact on African American students' achievement, to which she replied:

Nikki: I think both have similar impact, but probably teachers and counselors have a bigger impact, because they're the ones that set the standards for African Americans, for students in general. I think it's a really big impact from teachers, personally.

Interviewer: Why do you think that?

Nikki: Because, like, even if parents say, "Do good in school," if the teachers aren't paying attention, then *how* are they going to do good in school? If their teachers aren't paying attention to them, or if having talks with them about college or about other things, how are they going to know about those things?

To that point, Nikki had built a fairly consistent story around the behavior of students and parents, so it was interesting that she would demote it in favor of Narrative I. However, it was even more surprising that Nikki changed her mind once again when confronted with Probe 5. Given that Latino students at Nikki's high school were proportionately represented in advanced mathematics courses, the author asked her why Latinos and African Americans were achieving at different levels. This was her response:

> If they have the same group of friends…it's a lot probably coming from parents and a little bit coming from teachers. Because, well, Latinos if they just immigrated here, they're probably having a hard time getting to know English. At [her high school], I've seen that. So, um, they usually ask for help, I think, because their parents probably tell them, "If you don't get something, just ask for help." And African Americans already know English, so they think, "What am I going to ask for help for?" And then if the

> Latinos start talking to the teachers, they'll interact more and start speaking [with their teachers] about college and other stuff.

The primacy Nikki attributed to Narratives A and I changed once again. Just moments after arguing for a "really big impact from teachers," Nikki stated that "it's a lot probably coming from parents and a little bit coming from teachers." It is also noteworthy that Nikki shifts the burden for academic achievement from teachers to students when considering the issue in the context of Latino students. Recall that Nikki had previously criticized teachers for not "having talks with [African American students] about college and other stuff," but here she implied that teachers present academic opportunities to students when students reach out to teachers.

## Discussion

It is not altogether surprising that Nikki did not fixate on a single explanation for the racial achievement gap. The ambivalence Nikki showed in negotiating the ten narratives she generated reflects the complex nature of the phenomenon she was trying to explain (Katz, 1981). Moreover, the fact that nearly all of those narratives were of a behavioral nature is consistent with Bol and Berry's (2005) findings that secondary mathematics teachers tended to explain the achievement gap in terms of behavioral factors.

On the other hand, it was interesting that Nikki could vacillate so quickly between behavioral and structural narratives without appearing to consciously realize it. Why did Nikki make such a thorough case for Narrative I ("teacher/counselor bias") when the data indicate that she more likely favored behavioral explanations? And if in fact she actually attributed greater priority to Narrative I, why was it not among the first of her explanations?

One possibility is that Nikki simply did not immediately feel comfortable talking about prejudiced teachers in front of the author (who happened to be her former teacher), so she delayed mentioning Narrative I until halfway through the interview. Although the author and she had developed a family-like rapport over the years, the topic of race had never before been explicitly broached in one-on-one conversation. A related possibility is that Narrative I was actually a misrepresentation of her true beliefs, designed to provide balance to her point of view so that she did not come off as racist.

While more data would be needed to establish the validity of those conjectures, considering the role of Nikki's prior experiences in her reasoning may illuminate the issue. The data suggest that Nikki had a wealth of prior experiences regarding the academic consequences of both "parental influence" and "teacher/counselor bias," and yet for some reason she delayed mentioning the latter narrative. However, a potentially important difference between Narrative A and Narrative I is context. Because Narrative A depended on past interactions with her parents, it may be that the memories that gave rise to Narrative A were in some sense more "available" to Nikki than those that gave rise to Narrative I (Tversky & Kahneman, 1974). For although she cited specific examples in which she directly observed differential treatment of her peers with respect to race, those in-school experiences were of a markedly different nature than her prior experiences with her parents. Not only were the latter experiences more personal, but it is also likely that they were more frequent than the instances of bias she noticed in school. Calling on the experiences needed to vocalize Narrative I required Nikki to empathize with a largely unfamiliar perspective. And in the pressure of an interview setting, telling a story of her parents' influence may have been an easier story to tell because it did not require her to see the world through the eyes of another person or group. Whether or not experiences of an egocentric nature are more accessible is a question worthy of further investigation.

Another aspect of the data related to the availability of information was Nikki's tendency to over-generalize from her past experiences. In Narrative G Nikki described how people respond to negative societal perceptions of their racial/ethnic group: "Sometimes [African American students] adjust to what society thinks of them, and they don't try to overcome that" (marked with an asterisk in Figure 1). She also cited the existence of a negative perception about Mexicans being "lazy in academics and stuff," but went on to say, "But I know a lot of my Mexican friends: they're not lazy, they strive harder. They've overcome that perception."

Noteworthy here is that Nikki made this claim based not on her knowledge of her school's Mexican population as a whole, but instead on her knowledge of her Mexican *friends*. Because the overwhelming majority of her friends were also enrolled in advanced classes, generalizing from the slice of the population would result in an obviously skewed perspective. Nevertheless, Nikki reasoned based on the information that was available to her. Still, given the low enrollment in those courses overall, and the fact that the majority of Latino students at the school were not achieving at dramatically higher levels than African American students, it was interesting that Nikki did not explicitly acknowledge her sampling bias.

## Future Research

The findings of this study suggest myriad avenues for future research. An important issue that this study raises questions about is the possibility that some students may explain the racial achievement gap in behavioral terms. In spite of evidence that she had observed multiple instances of racial discrimination first-hand, Nikki ultimately placed the onus for getting a good education on students and parents. To what extent were her perceptions of the racial achievement gap a function of her background as someone who "made it" in a beleaguered educational environment? Given her background, one might argue that it was to be expected that Nikki would lean toward behavioral factors rather than structural factors. But might a high school sophomore repeating Algebra 1 explain the achievement gap differently? Interviews with students at varying levels of academic success may reveal different perspectives.

Also worth pursuing is how the notion of the "model minority" may influence student perceptions of the achievement gap. As a member of a "model minority," Nikki expanded the application of the label in a unique way. Historically, the "model minority" label has been reserved for Asian students (Wu, 2003), but Nikki stated that African American students respond to negative societal perceptions of their racial group by "conforming to the stereotypes," while Mexican students respond by "overcoming the stereotypes through hard work." Based on this datum, Nikki seemed to position Mexican students as a de-facto "model minority." Whether or not her framing of this population indicates an emerging trend in education, in addition to what it might mean for African American students, are both matters that deserve attention.

Finally, the tendency for students to explain the achievement gap in behavioral terms has serious implications for classroom dynamics. What impact might those perceptions have in classrooms where collaborative learning takes place? How should teachers account for students' perceptions in their pedagogy? In what ways can reform-oriented curricula and forms of classroom discourse militate against the effects of potentially negative perceptions among students? Exploring such questions would certainly benefit our understanding of teaching and learning, particularly in racially and ethnically diverse schools.

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# DOMINICAN PARENTS LEND THEIR PERSPECTIVE TO THE CONVERSATION ON MATHEMATICS TEACHING AND LEARNING

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*This study examines the perspectives of two Dominican mothers on the teaching and learning of mathematics. Through interviews and classroom observations by the parents, we learn about their mathematical learning experiences (both in formal school settings and in informal settings) in the Dominican Republic and their children's mathematical learning experiences (again, both in and out of school) in the United States. The mothers compare their own educational experiences to those of their children. The findings indicate that a more inclusive stance toward parents by the school would benefit the children's learning of mathematics.*

## Purpose of the Study

The purpose of this study was to give voice to the perspectives of two Dominican parents with regard to their own and their children's mathematics education with the intention of informing the school-home discourse around the teaching and learning of mathematics.

## Theoretical Perspective

This chapter builds on a small body of work that has been done about parental perspectives on the teaching and learning of mathematics. It is built on strands of research that examine issues of (a) the power relationship that exists between school and parent, and (b) the funds-of-knowledge that exist in homes and communities including the value that is (or is not) placed on knowledge the parent may possess.

The perspective of parents is one that is often missing from conversations of how to address the learning needs of children. Yet, parents are their children's first teachers, and their role as teachers continues to be an important one throughout elementary school and beyond. In spite of this, parents (particularly non-middle-class parents including immigrant parents whose facility with English may be emerging), and the homes and communities of many students are often positioned as a problem rather than a support for academic achievement (Foote, under review).

Research indicates that schooling in the United States maps best onto middle-class child rearing practices (Lareau, 2003). It is understandable that immigrant parents, schooled in an educational system distinct from that of the United States, may have different understandings and perspectives on schooling from parents schooled exclusively in the United States. Exploring and understanding the gap (or lack of it) between the mathematics learning experiences of parents and their children can only serve to support educators in understanding how to acknowledge and possibly build on parental experiences in the service of being more effective teachers for their children. Tapping into parents' actual perspectives may support educators in understanding what parents see as their role in supporting the mathematics learning of their children as well as how their own personal schooling histories inform that perspective.

Moll and Civil (Civil & Bernier, 2006; González, Andrade, Civil, & Moll, 2001; Moll & Gonzalez, 2004) and their colleagues developed the construct of "funds-of-knowledge" that countered the deficit perspectives often held by schools about poor or immigrant parents and families. In this work, power relationships between family/community and school are acknowledged and interrogated. The present study expands these ideas to include parental knowledge of educational systems (in this case that of the Dominican Republic) and how the lens of their personal educational experience influences their perspectives on mathematics teaching and learning as it plays out in their children's school in the United States.

The importance of including the parent voice in the discourse around mathematics learning is a position taken by Remillard and Jackson (2006). They state that it is important for preK-12 educators to understand the parent perspective so that they can work with parents toward common understandings. They further indicate that partnerships between parents and schools (including very importantly, teachers) must acknowledge (and I believe, fight against) the inherent power differentials that exist between the dominant culture (of which school is a force) and (in this case) the immigrant parent. Allexsaht-Snider (2006), suggests a question that is particularly appropriate to this study:

> If we do create more mathematics education contexts that bring together educators and diverse parents and link home, school, and community contexts, how can we help students, parents, and teachers find common ground and negotiate potential conflicts and areas of dissonance to create meaningful and successful mathematics learning for urban children? (p. 193)

This study explores one such avenue for finding common ground. Through this study I hoped to address the question of how understanding parental perspectives on the teaching and learning of mathematics might inform the discourse on classroom practice, so that eventually classroom practice can more effectively address the needs of some of the traditionally more vulnerable learners.

## Methods

*Researcher Position and Participant Selection*

I was a volunteer and researcher in two classrooms in an elementary school in New York City at the time of the study. I spent one half day a week in a Kindergarten classroom and one half day a week in the English component of a third grade. I was engaged in working on a separate research project with the classroom teachers and also assisted them in individual and small group instruction at their request. Although I did not know the parents at the onset of the study, I knew their children and I knew their children's teachers. Because of these relationships that I had formed with these teachers and students, I solicited parent participants from these two classrooms through letters sent home. A number of parents, all mothers, responded showing interest. For several of these mothers, it proved logistically impossible to set up interviews and classroom visits. Many of these mothers work long hours and do not have the flexibility at their jobs to come to school easily during the day. In the end four mothers participated in initial data gathering and three of those four were able to participate completely in all phases of data gathering; ultimately, one did not prove to meet the participant criteria of significant schooling in the Dominican Republic. Further discussion of participants is limited to the two who met participant criteria and completed all phases of data gathering.

*Setting and Participants*

Two mothers whom I call Silvia (child: Sarita, Kindergarten) and Vera (child: Victor, Grade Three), participated in this study. (All names of parents and children used in this chapter are pseudonyms). Both mothers were born in the Dominican Republic and currently live in the United States. Their children attend the same K-5 public elementary school in New York City.

Vera was raised and schooled in the Dominican Republic including attending university and receiving a degree in Accounting. She moved to the United States as a young adult and currently works as a school crossing guard. Vera is the sole adult living in her household. Vera identifies herself as able to understand, speak, read, and write Spanish, and to understand and speak a little English. In addition to her third grade son, Victor, Vera also has an older son who currently attends Middle School.

Silvia was raised and schooled in the Dominican Republic, moving to the United States when she was a young adult. Silvia currently cares for children in her home where she lives with Sarita, her kindergartener, another daughter, aged two, and her husband. Silvia identifies herself as able to understand, speak, read, and write Spanish, and to understand, speak, read, and write a little English.

Both mothers then have finished secondary school with Vera having finished a university course of study. They both have studied algebra, geometry and trigonometry in high school. Vera counts calculus, statistics, and accounting among her university courses. The participants in this sample of convenience prove to have extensive school-based experiences with mathematics and are therefore well positioned to note differences and similarities between the teaching and learning of mathematics in the Dominican Republic and the United States.

*Data Gathering*

Data gathering included (a) a preliminary interview with each mother, (b) a classroom observation of a mathematics lesson by each mother in the classroom of her child, and (c) a post-observation interview with each mother. The researcher took field notes during or immediately after each of the above sessions. The interviews were conducted in Spanish by parental choice. All interviews were audio-taped. The interviews were transcribed and translated into English for analysis.

In the preliminary interviews, participants were asked to provide personal information about their language proficiency in English and Spanish, their employment, their children living at home, (these data are reported above in the section on setting and participants), their own school history and that of their children. In addition to this basic information, participants were asked about their own personal history with mathematics, as well as their perspective on their children's experiences with mathematics, including interactions between parent and child around mathematics homework and mathematical activities more generally.

Each parent observed a mathematics lesson in the classroom of her child. Although the lesson observed in the third grade classroom was conducted in English, the teacher repeated much of what she said in Spanish for the benefit of Vera (whose language of preference is Spanish, although she understands English). The lesson observed in the Kindergarten classroom was conducted in Spanish.

Each mother participated in a post-observation interview either immediately after or within a few hours of the observation. In the interview parents were asked to give their perspectives on what they noticed in the classroom in general and in the mathematics lesson in particular with special attention to the teaching approach and goals, and the participation patterns. The participants were also asked to reflect on whether what they observed was consistent with what they had expected to see. They were asked as well what they had liked about the lesson and whether there were concerns that were raised for them. Finally, the parents were asked to reflect on how the teaching and learning that they had observed in their child's classroom compared with their own classroom experiences in the Dominican Republic.

*Data Analysis*

Upon completion of the data gathering, the entire data record (all interviews and field notes) was reviewed. Emergent themes were noted. The data record for each individual participant was then reviewed chronologically. Interim research texts (narratives) were constructed for (a) the initial interview, (b) the classroom setting on the day of the observation from the researcher's perspective, and (c) the post-observation interview (Clandinin & Connelly, 2000; Clandinin, et al., 2006). Using these interim research texts, chronological narratives were constructed with particular attention to the individual cases within this chronology. Finally themes were noted that cut across participants. This was done by re-examining the data from all participant responses to each question in each of the two interviews. In this way, the focus of analysis shifted from the individual case as the focus to the particular interview question as the focus. By providing another perspective on the data, a triangulated perspective was obtained. These themes and the interim research texts (initial narratives) were then used as the basis for constructing the final narratives found in the results section.

In an attempt to respect the perspectives of the parent participants, the results of the study were shared with them through informal conversation. The support of the teachers was likewise acknowledged by sharing with them, through informal conversation, the results of the study. This was done so that these important contributors to the research story had the opportunity to review the manner in which they were represented, modifying the picture if they thought it necessary. In the end, parents and teachers both were comfortable with their portrayal.

What follows is the story of two Dominican mothers and their perspectives on the teaching and learning of mathematics. These perspectives take into account their own mathematical learning experiences (both in formal school settings and in the informal setting of the home) in the Dominican Republic (and in the case of one mother in the United States as well) and their children's mathematical learning experiences (again, both in and out of school) in the United States.

## Results

In presenting the results, I will first present individual narratives of the perspectives of each of the two mothers regarding her own, and her child's experiences learning and using mathematics. I will then present individual narratives of the parent perspectives on the classroom and mathematics lesson.

*Parent Perspectives on Learning Mathematics*

*The case of Vera.* Vera describes herself as someone who loves mathematics and was an excellent student of mathematics. Algebra was an area that she particularly liked. Vera describes the school system in the Dominican Republic, where she graduated from university, as much more formal in many respects than that in the United States, from the more formal clothing worn by both teacher and student to the teacher directed manner in which the classes are conducted.

Vera reports that both her children (Victor, the third grader and an older child who is in eighth grade) like mathematics. Vera indicates that she works with Victor on word problems teaching him to look at the language in the problem. For example to know to add if you see "in all" in the problem, to know that if it says someone has more than another, you have to subtract. She also helps him to draw out or diagram the problem situation. In the case of a problem such as five children needed to share 100 balls, she would encourage him to draw five circles to represent the children. She would tell him that everyone needs to have the same amount so that they don't fight. When he was younger, she would encourage Victor to use beans or other small objects to model the problems and to work on calculations. Vera indicates that it is quite common in the Dominican Republic to work with beans or other small materials when doing math with young children.

Vera says that the notational system for standard algorithms in the Dominican Republic differs from that in the United States and has caused her some problems in working with her children. Once, for example, the older child came home with a system for multiplying that involved dividing the paper into small squares (something I recognize as lattice multiplication). Vera says she never understood that method. In addition to the notational issue, language use can also present difficulties for Vera. When she directly translated *"dos por dos"* into English as "two by two" instead of "two times two," Vera's older child said that wasn't the way he was learning in school and that his mother was confusing him.

Vera does not feel well informed about what Victor, her third grader, is doing in school. Vera reports she knows what's going on in the mathematics classroom mainly through homework. She feels that she should get some indication at least once a month as to the child's progress. No school work or tests come home. She sees only homework. This she says is in sharp contrast to the Dominican Republic where completed quizzes are regularly sent home. Since Vera works as a school crossing guard, she is not in a position to pick her son up at school and so does not (like so many other parents) have access to those quick, informal, after-school conversations with the teacher. Vera is concerned that Victor is beginning to show less interest in school this year and has tried to hook him up with an after school program to support his learning, but hasn't heard anything back from the school.

Vera is a single parent and as such is the sole earner as well as the person who manages all household finances. She has begun to talk to the older child about spending money and at times asks him to do shopping for her. Aside from this, she does not explicitly make a point of addressing mathematics in home and community situations. She takes a more organic approach. If something comes up that involves mathematics, they talk about it. They play cards and dominoes at home and Victor, the third grader, much like Marta's son, loves to play with and count pennies.

*The case of Silvia.* Silvia feels that she was a good student of mathematics, maybe not earning top grades, but making a solid showing. Silvia feels her kindergartener, Sarita, learns math easily and with enjoyment. Silvia describes herself as very involved in the mathematics homework that her daughter brings home from school. Sarita's father also helps her with her homework. Silvia supports Sarita's leaning by working with her using a variety of manipulative materials such as dried corn or beans. She recalls that this is how her own father helped her with her mathematics when she was a young child. In addition to using manipulatives available in the home, Sarita's parents also try to support her developing mathematics understandings by putting mathematics problems into contexts that she might find motivating, such as money. They try to find ways that are both interesting and understandable when explaining mathematics situations or contexts.

Silvia feels well informed about what is happening in school with Sarita. She picks up her child from school each day and has the opportunity to have a quick informal discussion with the teacher. In addition, Silvia cites parent-teacher conferences as time when she has an opportunity for more extended conversation with the teacher about her daughter's progress in school.

In addition to using mathematics while she helps her child with homework, Silvia identifies grocery shopping as an area in which she regularly uses mathematics, doing comparison shopping, for example. Sarita often accompanies her to the grocery store and she involves her in the shopping process, tying in mathematics by asking the child to select, for example four oranges or three pears. Silvia tries to buy games that are educational in order to support learning at home.

*Parent Perspectives upon Observing a Mathematics Lesson*

After both of the observations, I met with the mother to discuss her perspective on what she had observed in the classroom.

*The case of Vera.* Vera observed a lesson on volume of a rectangular prism. She liked the look of the classroom and how the children worked in groups. She was definitely surprised by the lesson. She had expected to see a lesson on computation and to see the students working individually out of a text book. She noted the difference between the structure for learning she observed in this classroom and what she had experienced as a student, which was primarily individual work done exclusively on paper, copying an example from the board and completing similar problems after hearing the teacher's explanation. Vera commented on the manner that the teacher had with the children, calling her accessible, caring, and calm. She noted that gathering the children in the meeting area supported them in being more focused and attentive.

Vera mentioned that there were wide attempts at participation, with many students volunteering during the mini-lesson, even if they were not all called upon. She felt that the children were very involved in the lesson both during the mini-lesson and afterward during the work period. This was an aspect of the lesson that Vera liked well. She did notice that after the mini-lesson, while the majority of children were working in table groups, that the teacher kept a small group with her in the meeting area and was able to focus more attention on their particular needs, supporting them in focusing on the activity.

Vera commented on how the learning environment was structured through what she described as play. She was impressed by the use of centimeter cubes to explore the concept of volume. She thought, however, that the teacher's explanation of the relationship of the base to the volume was confusing. Vera noticed that during the work period when the children were constructing a rectangular prism, for example, with a base of six, that they would draw the base as a 6 x 6 rectangle. She felt this confusion on the part of the students was due to what she saw as the teacher's confusing explanation. Vera thought that an exploration of more examples might have dispelled some of the confusion. Vera also thought that a review of the work at the end of the work period would have supported student learning. She noted that the need to go to lunch precluded this, but she felt that a discussion of the results the students had obtained as well as why they had obtained them would have contributed to the learning.

*The case of Silvia.* Silvia observed a lesson on comparison of numbers. Silvia was positive about her lesson observation. She too commented on the manner that the teacher had with the children which she found to be an encouraging one where the children were also supported in helping one another. She was impressed with how readily the children helped each other, how well they worked together, and how the teacher supported this atmosphere of cooperation that pervaded the classroom. She thought the classroom was inviting with much useful and interesting information about mathematics posted for the children to see. She was pleased with the level of participation as it seemed to her that every child in the class participated fully in the lesson. She liked the use of materials (pattern blocks in the routine portion, and number cards in the main activity) to support the learning.

Silvia found the class to be very different from her early childhood experiences in the Dominican Republic where the emphasis had been much more on simply play as opposed to play as a vehicle for learning particular content. She was impressed with the children's knowledge.

Silvia had expected that the work the children would be engaged in would be at a simpler level. She found the level of mathematical understanding demonstrated by the children including the range of numbers they were comfortable with to be well advanced over what she experienced in kindergarten. She thought that the dual focus in the use of games, both to entertain and to teach was a laudable one.

## Discussion

I will begin this section by discussing the initial perspectives that parents expressed in their first interviews. I will then discuss their reaction to and reflection on the lesson they observed in their child's classroom.

*Initial Perspectives*

I will begin this section by discussing two themes that emerged from the data: parental help at home at support from school.

*Helping at home.* Silvia and Vera both talked about how they used the same kind of manipulatives (dried corn and beans) in working with their children as were customarily used in the Dominican Republic. They also mentioned that she tried to reframe mathematics problems using interesting contexts or contexts that were motivating to her child. We see parents who see one aspect of their role as a parent to include fundamentally that of teacher.

*Support from school.* The perception of the support provided to the parents by the school (including the classroom teacher) varies from parent to parent. Silvia seems satisfied, even pleased, with the communication she has with the school. We can speculate that some of this may be due to the fact that her daughter is in Kindergarten and just easing into the school system. Vera is frustrated by the paucity of information she receives about her child's progress. Who knows if this lack of communication between home and school is contributing to what Vera identifies as a gradual lessening of interest in school on the part of her son, Victor. As Remillard and Jackson (2006) note, when parents are left out of discussions of mathematics learning, it can have a negative impact on their ability to be actively involved in their children's schooling.

*Reaction to the Lesson*

Both of the participants' experiences with mathematics in school were with traditionally taught mathematics. Based on their own schooling histories, both mothers expected to see more traditional mathematics lessons when they observed their children's classrooms. One might think that this would orient them toward resisting the teaching style they observed. This did not turn out to be the case. Both mothers were pleased by many aspects of what they saw in the classroom.

They both commented on being pleased that the teacher was using manipulative materials that supported the children in understanding the mathematics. Since they had both previously mentioned how it was typical in their experience in the Dominican Republic that small objects were used at home to support their developing mathematical understanding, the use of these materials may have been recognized as a support familiar to them.

In addition, they both commented on the teacher's manner with the children (saying similar things about the two different teachers); they liked how calm and supportive they were. This speaks to the importance of the connection between teacher and student for supporting learning. And more particularly, in the data being examined here, that parents recognize and appreciate this fact.

Vera alone pointed to a feature of the lesson that she thought was lacking. She recognized that simply engaging in the mathematical activity (building rectangular prisms with centimeter cubes) was not sufficient to develop a sufficiently deep understanding of the problem at hand. She recognized that a discussion following the activity would have supported the children in processing their work and arriving at a clearer understanding. Perhaps because she is the mother with the more advanced education (including a degree in accounting), Vera is positioned to understand the important role that discussion can play in learning. Whether this is the case or not, Vera's understanding of what is necessary for learning is impressive. This mother brings understandings to the learning of mathematics which could be exploited for the benefit of her child's learning.

These two mothers brought a keen eye to their observations of the mathematics lessons in their children's classrooms. Their comments are show a sophisticated understanding of children's learning needs. In addition to Vera's noting that a follow-up discussion to the mathematics activity would have supported student learning, Silvia understood and appreciated the level at which the Kindergarten children were functioning.

Here we see parents eager to help their children. The suggestions they have of more communication from the school and access to materials that can be used in the home to support learning are indicative of a desire to be involved in and support their children's learning. It seems that they recognize that having home and school learning environments better aligned would be supportive of their children's learning.

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# EXAMINING THE IMPACT OF DIVERGENT PEDAGOGICAL STYLES ON THE MATHEMATICS LEARNING OF AFRICAN AMERICAN MIDDLE SCHOOL STUDENTS

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*This study investigated the impact of two divergent pedagogical styles on the mathematics learning of African American middle school students. One of the teachers used her knowledge of students' background and culture, and aligned her instruction with their learning preferences to design meaningful and contextual learning experiences. In doing so, mathematics learning was positively impacted. Results suggest while knowledge of mathematics and pedagogy is crucial to teaching mathematics, an understanding of African American cultural style and learning preferences are also essential to effectively teaching African American learners.*

## Purpose of Study

The National Assessment of Educational Progress (NAEP) results show significant gains in mathematics achievement among African American middle school students (Tate, 2005). Even so, African American students are more likely to be enrolled in lower-level mathematics courses and experience a mathematics curriculum that stresses basic skills (Tate, 2005). As a consequence, African American students can expect to use computers for drill and practice, as opposed more meaningful activities such as simulations, demonstrations, or applications of mathematical concepts (Lubienski, McGraw, & Strutchens, 2004). Worksheets are also more commonly used on a daily basis in classrooms that serve African American students (Strutchens, Lubienski, McGraw, & Westbrook, 2004). The National Council of Teachers of Mathematics (NCTM) recommends students receive standards-based instructional practices (Lubienski, McGraw, & Strutchens, 2004). Standards-based instruction is characterized by an emphasize on conceptual understanding, mathematical reasoning, student engagement with mathematical ideas, multiple representations, collaborative investigations, and open discussion and writing (Goldsmith & Mark, 1999). Unfortunately, many African American students are less likely to experience these standards based approaches in mathematics classes (Lubienski, McGraw, & Strutchens, 2004).

This study investigates the impact two different pedagogical styles had on middle school students' mathematics learning. The study was conducted at Spartan Middle School, located in a rural county in a southeastern state. The two teachers involved in the study, Ms. Canady and Ms. Able, are both experienced mathematics teachers; Ms. Canady has taught for 14 years, and Ms. Able has taught for 12 years. Both are White women who graduated from the same teacher education program and they both grew up the community where Spartan Middle School is located. During the period in which the study took place, there were 514 total students at the school: (a) 258 Black, (b) 248 White, (c) 3 Hispanic, (d) 4 Asian, and (e) 1 unspecified. Two hundred eighty-nine students (56%) at SMS receive free or reduced lunch.

The pedagogical teaching styles of the two teachers are very different. Ms. Canady's teaching style can be described as an aggregate of high-demanding structure and a disciplined environment coupled with instruction that utilizes students' lived and cultural experiences. She feels it is her responsibility to know and understand her students' backgrounds and struggles, and

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she says she uses this knowledge to help her students succeed. Irvine and Fraser (1998) used the term “warm demander” to describe teachers like Ms. Canady. Ms. Able describes her teaching style as “professional,” and she appeared to maintain both emotional and physical distance from her students. She required all students to use her prescribed procedures and techniques to solve problem, with the main to goal being to get the right answer.

## Theoretical Perspective

The most prevalent mathematics classroom teaching pattern in American schools is the initiation-response-evaluation (IRE) pattern (Hiebert & Stigler, 2000). IRE is a teacher-centered pattern of teacher-initiated questions, student response, and teacher evaluation (Cazden, 2001). In classrooms where the IRE pattern is emphasized, teachers place little focus on having students explain their thinking, work through mathematical ideas openly, make conjectures, or develop consensus about mathematical ideas (Franke, Kazemi, & Battey, 2007). Mathematics educators have argued that an IRE teaching pattern limits students’ ability to fully understand and appreciate the complexities of mathematics because it emphasizes school-learned methods and rules (Boaler, 2000; Malloy & Malloy, 1998; Tate, 1995), without proper attention to meaningful understanding.

This instructional pattern is well documented in mathematics classrooms that serve African American students (Lubienski, 2002; Lubienski, McGraw, & Strutchens, 2004; Strutchens, Lubienski, McGraw, & Westbrook, 2004; Strutchens & Silver, 2000). Describing teaching parallel to the IRE pattern, Tate (1995) says this “foreign pedagogy” (p.166) has negatively impacted African American students. Tate suggests this “foreign pedagogy” has attributed to: (a) African American students being tracked into remedial mathematics; (b) low numbers of African American students in college preparation or advanced mathematics courses; and (c) fewer opportunities for African American students to use technology in school mathematics.

In contrast to the IRE teaching pattern, Franke, Kazemi, and Battey (2007) suggest mathematics teaching should be relational and multidimensional. In this sense, mathematics teaching takes the form of developing relationships between (a) students and teachers; (b) among students themselves and mathematics; and (c) engagement among students and teachers to develop mathematical understanding (Lampert, 2004). Mathematics teaching is multidimensional with respect to the interactions that occur between (a) teachers’ pedagogical content knowledge, (b) teachers’ beliefs about mathematics teaching and learning, (c) teacher understandings about students’ social and cultural contexts, and (d) creating an environment for mathematics learning (Lampert, 2004; Moschkovich, 2002). These interactions influence the ways teachers structure mathematics experiences for students.

We contend that for African American students, experiencing mathematics teaching and learning as relational and multidimensional requires teachers to know and understand these students’ cultural background, prior experiences, and contexts. In doing this, we give credence to culturally relevant pedagogy. Ladson-Billings (1995) defined culturally relevant pedagogy as a pedagogy that fosters meaningful classroom experiences that affirms students’ backgrounds and prior knowledge. Culturally relevant pedagogy provides a framework for making connections between using contextual situations for mathematics teaching and learning, connecting to students’ experiences, and linking mathematics to students’ social and cultural ways of knowing.

Culturally relevant pedagogy rests on three criteria: (a) students must experience academic success; (b) students must develop and/or maintain cultural competence; and (c) students must develop a critical consciousness. Culturally relevant teaching requires teachers to attend to students' academic, social and cultural understandings (Ladson-Billings, 1994, 1995, 2000). Consequently, mathematics teachers must demand, reinforce, and produce excellence in their students. Achieving excellence requires mathematics teachers to use students' social and cultural backgrounds as a bridge for developing mathematical understanding. Culturally relevant teaching allows students to use mathematics to critique the cultural norms, values, mores, and institutions that produce and maintain social inequities (Ladson-Billings, 1994, 1995, & 2000).

Culturally relevant teachers identify the resources that students bring to the mathematics classroom and these teachers construct experiences that utilize these resources to produce meaningful mathematical understanding (Gutiérrez, 2002). Therefore, effective culturally relevant mathematics teachers of African American students have strong mathematics content knowledge, pedagogical skills, and knowledge of African American cultural style and learning preferences. For this research, we used culturally relevant pedagogy as the foundation for examining the impact an IRE teaching pattern and a relational and multidimensional teaching pattern have on African American students.

*African American Students' Learning Preferences*

In order to engage in culturally relevant pedagogy for African American students, mathematics teachers must have an understanding of African American culture and accept that African American culture is a significant socializing force for African American students. African American students' mathematics identities are shaped by culture, learning preferences, and experiences with mathematics (Berry, 2003). Martin (2007) refers to mathematics identity as one's belief about "(a) their ability to do mathematics, (b) the significance of mathematical knowledge, (c) the opportunities and barriers to enter mathematics fields, and (d) the motivation and persistence needed to obtain mathematics knowledge" (p. 19). The development of a positive mathematics identity is essential if students are to sustain an interest in mathematics and develop persistence with mathematics.

Shade (1997) described the African American learning preference as a combination of holistic, relational, and field dependent learning styles. Holistic learners seek to synthesize divergent experiences in order to grasp the fundamentals of experiences. They are successful with content tied to a larger whole, and view cause and effect as separate entities. The kinesthetic mode is the principal mode of information induction; thus, concreteness is used to facilitate new learning (Shade 1997). Relational learning preference is characterized as freedom of movement, variation, creativity, divergent thinking, inductive reasoning, and focus on people. Field dependent learners require cues from the environment, prefer external structure, are people-oriented, are intuitive thinkers, and remember material in a social context (Shade 1997). Taken together, learning preferences of holistic, relational, and field dependent learners are directly related to African American culture.

Culture and ethnicity are frameworks for the development of learning preferences; however, other factors play a significant role in cultural and learning preferences (Irvine & York, 1995). The learning preferences of African American students suggest that these students should not only receive mathematics instruction that includes opportunities to learn mathematics in an abstract manner, but also instruction that embed relevant contexts, use concrete imagery, and provide experiences based on how mathematics concepts are related to each other. Teachers also need to understand the complexity of students' experiences, which may lead to doing things with students that are not mathematics, such as interviewing them, having them write autobiographies, and discussing their interests (Ladson-Billings 1997).

*NCTM Process Standards*

The NCTM Process Standards complement the learning preferences of African American students. Berry (2003) theorized the overlap between the NCTM Process Standards and African American learning preferences. The NCTM Process Standards are: Problem Solving, Reasoning and Proof, Communication, Connections, and Representation (NCTM 2000). Each of these standards focuses on how students should learn and use mathematics. The Problem Solving Standard is consistent with the learning preferences of African American students because it supports the notion that learners should have opportunities to experience mathematical problem solving in a social context and utilize various strategies to solve problems. The Reasoning and Proof Standard suggests that students make and investigate conjectures, develop and evaluate mathematical arguments, and select and use various types of reasoning and methods of proof (NCTM 2000). This standard aligns well with African American students' preference for expressive individualism, experimentation, and divergent thinking. The Communication Standard proposes that students organize and consolidate their mathematics thinking, coherently communicate their mathematics ideas to others, analyze and evaluate the mathematical thinking and strategies of others, and use the language of mathematics to express mathematics ideas (NCTM 2000). Complementary to the Communication Standard is African American learners' cultural and learning preferences towards oral expressions, social and affective emphasis (Boykin, 1986; Shade, 1997).

The Connections Standard recommends that students interconnect mathematics ideas within mathematics and outside of mathematics, and understand how mathematics ideas relate to and build on one another to produce a coherent whole (NCTM 2000). The Connections Standard supports the holistic view of learning suggested by Shade (1997), because African American learners need contextual experiences that connect mathematical ideas within and outside of mathematics. The Representation Standard suggests that students create and use multiple representations to organize, record, and communicate mathematical ideas; select, apply, and translate among mathematical representations to solve problems; and use presentation to model and interpret physical, social and mathematical phenomena (NCTM 2000). Since African American learners have a propensity for verve, mathematics learning and teaching should be stimulating and interesting as well as offer opportunities for hands-on experiences that promote interactivity.

## Methodology

There were a total of 100 students that participated in this study. Fifty one students (33 African American and 18 White) were taught by Ms. Canady, and 49 students (30 African American and 19 White) were taught by Ms. Able. To access the impact of the two pedagogical styles on students' mathematics learning, pre- and post-test were given to both groups of students. The tests were constructed from released items on the state mandated assessment. The pre- and post-tests were designed to be parallel to each other, with both having the same number and types of items. The tests were each composed of 10 items which consisted of a combination of multiple-choice open-ended problems. Both tests were reviewed by a mathematics educator, a mathematics education doctoral student, and a high school mathematics teacher for face validity.

The six multiple choice items were adapted from the state assessment released items. The state assessment gave students choices of solution to a given addition of integers "naked numbers problem." The pre- and post-test were adapted by giving students choice of whether the solution would be positive, negative, neither, or unsure. In addition, the students were asked to explain their thinking. Figure 1 is representative of the multiple-choice items with directions.

Using mental mathematics

- circle **positive** for the problem(s) that will result in a positive answer,
- circle **negative** for the problem(s) that will result in a negative answer,
- circle **neither** for the problem(s) that will result in neither negative nor positive answer,
- circle **unsure** for the problem(s) that you do not know what the result will be.
- Explain your answers.

1. $-24+78$ a. Positive b. Negative c. Neither d. Unsure

Explain your answer.

*Figure 1*. An example of a multiple choice item with directions.

On both the pre- and post-tests students analyzed the thinking of three persons who solved "naked numbers" problems. These problems were adapted from multiple choice items among the released items of the state assessment; the choices were reduced from five to three choices and were put into a context of solutions found by three students. Figure 2 represents an adapted problem that appeared on the pre-and post tests.

This class was given the problem: $7+(-12)$.

- Oren stated the answer is 5 because he subtracted.
- Robert stated the answer is -19 because it is an addition problem but he put the negative in front because of -12.
- Joe stated the answer is -5 because he said there are five more negatives than positives. Who is right? Why?

*Figure 2*. Post-test analytical problem.

Two open-ended items were on both tests. On the post-test, students were asked to solve, "Sam the snake slithers forward 4 feet and backwards 2 feet everyday. How many days will it take Sam to reach a rock 25 feet away from his starting point?"

The pre- and post-test measures were scored by two raters – one mathematics educator (one of the authors) and a high school mathematics teacher. The pre-and post-test measures were scored after the instructional units were completed. The raters used a sixteen-point rubric to rate the overall quality of the open-ended responses. For the items that required subjective ratings on the pre-test, the mean inter-rater correlation was .92. On the post-test, the mean inter-rater correlation was .96. These correlations were considered sufficiently high to provide reliable assessments of students' performance on these measures.

## Results

A two-way analysis of covariance (ANCOVA) was used to investigate the effectiveness of the two methods of instruction, controlling for group differences. Students were administered a pretest before the units were taught and a posttest after units were completed. The effect size was obtained to measure the amount of variance accounted for in achievement by the two types of teaching methods.

Prior to evaluating the different methods of instruction on student achievement, several assumptions underlying the two-way factorial ANCOVA were examined. Observations of skewness values and normal probability plots indicated that normality was satisfied. Levene's test of homogeneity of variance indicated that the variances of the two groups were not significantly different; $F(3, 96) = 0.16$, $p > 0.05$. Linearity between the covariate and dependent variable was shown to be satisfied by a linear regression of posttest scores on pretest scores. Also, interaction between the group and pretest variables indicated the interaction term was insignificant ($p > 0.05$), supporting the assumption of homogeneous regression slopes.

A two-way factorial ANCOVA indicated that the method of instruction students were exposed to had a statistically significant influence on their achievement, $F(1, 96) = 13.06$, $p < 0.05$. Within each respective group, African American and White students' scores were not statistically different. Parameter estimates indicate that, given two students with similar pretest scores, you can expect the student in the intervention group to have a score of 6.5 points higher than the student in the control group. Furthermore, the adjusted r-squared indicates 55 percent of the variance can be accounted for by group assignment.

## Discussion and Conclusion

While the results of this study may not be generalizable, we can not discount the fact that Ms. Canady's instruction appeared to have a strong impact on African American students learning of addition of integers. The reasons for this impact are hard to isolate but we speculate that the use of contexts for teaching addition of integers, students sharing and justifying their thinking, and Ms. Canady's disposition of a "warm demander" positively impacted students' motivation to do mathematics. Observations of both classrooms revealed that Ms. Canady's classroom was more engaging and respecting of students' perspectives. Hence, the difference might be found in the level of engagement of the students. The activities and engagement of Ms. Canady's pedagogical practices are strongly correlated to relational and multidimensional teaching, which appears to complement the learning preferences of African American learners.

The knowledge required for effective teaching is substantial. While having a solid foundation in mathematics content is essential, teachers must also be able to ascertain students' understanding of the mathematics content, and have a firm grasp of curricular goals (Schoenfeld, 2002). This means they must know and be able to teach problem-solving skills, represent mathematics concepts in multiple ways, connect mathematics concepts within mathematics and to other subjects, and be able to analyze students' thinking about mathematics. Schoenfeld (2002) contended that this is a gross underestimation of the knowledge and skills required to be effective mathematics teachers. This knowledge alone will not have the sustaining impact necessary for long-term effects on the mathematics teaching and learning of African American students (Martin, 2007). Effective mathematics teachers of African American students must also possess the tenets of culturally relevant pedagogy. These teachers must be "warm demanders," that is, they must demand academic excellence from their students while possessing culturally competence. Mathematics teachers of African American students must have knowledge of African American cultural style and learning preferences and how to use this knowledge to develop effective learning experiences in mathematic for African American students.

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# SUPPORTING ENGLISH LANGUAGE LEARNERS DEVELOP MATHEMATICS AND LANGUAGE PROFICIENCIES

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*Many teachers find themselves challenged to teach mathematics to students who are also learning English. Research in English as a second language recommends the use of objects, pictures, gestures, and language supports to help English language learners (ELLs) develop language and content. For example pictures and gestures can be used to augment a story similar to enacting a play. However, how can a mathematics teacher utilize these practices? The purpose of this study was to describe how a teacher used language structure to support two ELLs and how these students used the language structure to develop their mathematical thinking. Findings of the study suggest that modeling language structures for students enables ELLs to participate in mathematical discourse and deepen their understanding of mathematics through oral communication.*

## Introduction

Mathematics educators facing the challenge of raising the performance levels of students and are forced to examine how to accomplish this within growing culturally and linguistically diverse populations (Gebhard, 2002). Adler (2001) and Barwell (2005) described the challenges that teachers and students face when students are learning mathematics in a language different from their home language. Limited proficiency in the academic language of mathematics appears to be one reason why students that appear fluent in social English still have difficulty in mathematics (Irujo, 2007).

Words used in mathematics draw from two sources: everyday words with specific mathematical meanings and words specific to the domain (Irujo, 2007). A common strategy to decode the meaning of a word is to identify the root and build meaning from it. For example, *triangle* can be broken into the root *angle* and the prefix *tri*, which indicates three. Thus, triangle means three angles and describes a specific shape. However, individuals can not always discern the embodied concept when this strategy is applied to groups of words. A person who knows the meaning of least, common, and multiple may unpack the term *least common multiple* as the smallest familiar multiplication. This does not lead to a correct mathematical meaning and the individual may have difficulty understanding the concept embodied by the phrase. Interpreting academic mathematics language can be difficult for students with proficiency in English. However, these challenges compound when a student has limited English proficiency. The purpose of this research report is to describe how a teacher supported students who were gaining both proficiency in English and mathematics.

## Theoretical Framework

Gee (1991) theorized that discourse is socially constructed by groups of people to communicate ideas and this discourse is defined by the distinct language, thought, and actions shared by individuals in the community. Thus, social interactions, class structure, and curriculum combine to create the unique discourse of a mathematics classroom (Yackel, 2001). This discourse is created by the teacher and the ability of students to express their thoughts.

Classrooms with native English speakers are diverse and not all students have the same ability to articulate their ideas. However, the classroom discourse changes significantly when it includes students who are learning English. The teacher may ignore these students and rely on outside support to translate the classroom discussion or change his or her discourse to make the discussion more accessible to English language learners (ELL).

Monitoring vocabulary is an obvious way to modify discourse (Irujo, 2007). To further help ELLs develop a deeper understanding of words, a teacher can link those words to pictures, symbols, and actions. The choice of words and how those words are linked to sensory experiences determines the degree to which an English language learner (ELL) can construct meaning. Two considerations in word choice are helpful. First, concrete words that link to sensory experience are easier for children to learn. Second, certain words are key to understanding content in a lesson. Emphasizing concrete word choice and highlighting key content words are helpful ways that teachers can modify their discourse.

Echevarria, Vogt, and Short (2008) describe specific research-based techniques to help ELLs construct meaning in classrooms, including the use of objects, pictures, gestures, and language supports. The use of these techniques encourages ELLs to develop a vocabulary that goes beyond translating nouns from one language into another. For example, a rectangular prism is introduced by showing students several solids as examples with a discussion of their attributes. ELLs may learn to identify the shape with the term but the discussion of its attributes is often incomprehensible. However, if ELLs are given several solids to feel and provided language supports to describe its attributes (straight, flat, surface, edge, point, sharp, etc), they are more likely to appropriate descriptive language to characterize the rectangular prism's attributes. The word bank provided in this example enables an ELL with appropriate words that do not solely rely on memorized vocabulary.

Research and literature on instructing ELLs provide many examples of how the strategies described by Echevarria, Vogt, and Short (2008) can be infused in courses that focus on speaking, reading and writing. However, in mathematics there are fewer examples. The purpose of this study was to describe language supports that were used by a teacher to help ELLs gain proficiency in English and mathematics.

## Methods

This study took place in a classroom in the western United States. 62% of the students in the school spoke English as a second language. During the previous year, Olson and Salsbury worked with teachers in the school to incorporate language structures into their mathematics instruction through professional development. Braun was both a teacher leader who planned professional development with a team (Olson, Salsbury, Braun, Colasanti, 2008) and a researcher who conducted semi-structured interviews with ELLs over 18 months. Braun was curious about how she used language structures to support the learning of ELLs in her classroom and collaborated with Olson and Salsbury to characterize her practices.

Braun's fifth grade classroom had 27 students. She had 14 ELLs one of whom we will call Shi. Shi moved to the United States from Nepal during the summer of 2008. His native language was Chinese. Even though Shi's English was very limited, he found ways to communicate with other children, suggesting an outgoing personality. A second learner we will call Ricca. She moved to the United States from Mexico in 2006. Ricca participated in a larger study that examined how ELLs developed mathematical thinking while gaining English proficiency during the fourth grade. She spoke both conversational and academic English with hesitancy and difficulty. During the study, we developed a good relationship with Ricca through the monthly small group meetings and she gradually increased her spoken language during the small group meetings. At the beginning of the 2008 school year, Ricca was an intermediate speaker. Even though Ricca had a good relationship with Braun, she seldom spoke to Braun or her peers. Ricca was quite shy and would only communicate when she felt secure in her environment and confident in her content knowledge.

Qualitative methods with two case-studies (Merriam, 1998) were used to identify and describe the discursive practices that Braun used during her instruction. Data were collected from five sources during the fall semester. These sources included (a) reflections, (b) video recordings of classroom and small group instruction, (c) detailed lesson plans, (d) students' written interactions recorded on a smart board, and (e) informal interviews. Constant comparative methods (Merriam) were used to code, analyze, and collapse the data to display in a conceptual matrix with illustrative examples in each cell. These examples were used to characterize Braun's use of language structures.

## Results and Discussion

Braun made three instructional decisions to encourage students' vocalization of their mathematical thinking. First, she instituted a number talk at the beginning of the mathematics lessons a couple of times each week. During the number talk the students sat on a rug in front of a large sheet of paper. Braun wrote a problem on the paper that could be solved using mental arithmetic and after a short discussion, students thought about the problem silently, without paper and pencil. Then, Braun called on several students to share their solution strategy and recorded it on the paper using notation. Second, Braun wrote problems that would not be cognitively difficult for her fifth grade students. She wanted the students to be able to understand the process of a number talk which includes: mental math, sharing their thinking to the whole class and questioning each other. The third instructional decision was to provide students with a structure to share their strategy. This language structure was designed to provide students with limited English with the words necessary to enter a mathematical conversation. The language structure posted on the wall was, "I know __ x __ = __. This helped me solve __ x __ = __. The answer is ___." The language structure allowed students to focus on sharing their reasoning and establishing group norms. This technique provided a scaffold from which students could discuss complex mathematical ideas. Braun wanted students to be successful within the number talk and create a learning community in which students felt safe in sharing their ideas.

The following is an illustrative example of a typical number talk during the first month of school. Braun wrote 28 x 6 = on chart paper.

| | | |
|---|---|---|
| 1 | Braun | I have the problem 28 groups of 6 on the board. Who would like to |
| 2 | | make an estimate of the answer? Student A. |
| 3 | Student A | I know that the answer is going to be less than 180 because 28 is close |
| 4 | | to 30 and 30 times 6 equals 180. |
| 5 | Braun | Anyone else? |
| 6 | Student B | I know that the answer will be more than 120 because 20 times 6 equals 120? |
| 7 | Braun | Our answer should be between 120 minus 180 according the estimates |
| 8 | | from Student A and Student B try solving the problem. (Students sat |
| 9 | | quietly and solved the problem mentally. They took turns sharing their |

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| | | |
|---|---|---|
| 10 | | answers until several were posted on the chart paper.) Who would like |
| 11 | | to share how they solved this problem? |
| 12 | Student C | I split the 28 into a 20 and an 8. Then I multiplied 20 times 6. I know 20 |
| 13 | | times 6 is 120 from Student B. Then I multiplied 8 times 6. I know 8 |
| 14 | | times 6 is 48. 120 and 48 equals 168. My answer is 168. |
| 15 | Braun | Does anyone have questions for Student C? |
| 16 | Student D | How did you know that 8 times 6 is 48? |
| 17 | Student C | I know that 4 doubled is 8. I can multiply 4 times 6 which is 24 and |
| 18 | | then I need to double the answer. 24 and 24 is 48. |
| 19 | Student D | Ohhhh, I get it. |
| 20 | Braun | What strategy did Student C use and how do you know? |
| 21 | Student A | Splitting number because he split the 28 into a 20 and an 8. |

Braun initiated the number talk with a problem that students could easily solve (line 1) and engaged them in it mathematically by asking them to estimate the answer (line 2). Notice that students A and B use the language structure, *I know that,* followed by a reason (lines 3 and 4). Braun used the students' estimation to define the boundaries of acceptable answers, setting the norm that students need to estimate an answer and that the answer should be within the boundaries. Student C shares her solution, stepping out of the language structure (line 12) to describe the strategy that she used. The student then moves back into the language structure, *I know that,* to finish her description of her strategy (lines 12 and 13). Braun encourages students to question each other (line 15) and a student asks a question (line 16). Student C further explains (lines 17 and 18) which leads to an insight (line 19). Braun then reinforces the naming of a strategy (line 20), bringing closure to the number talk. Braun reflected that "the student named the strategy, splitting numbers. This type of naming became important because naming strategies helped students during the share time to express the concept that they were applying to the problem. It also allowed them to decide whether their strategy was different from ones that other students had shared." The teachers in her school decided to use consistent words for particular strategies. Here splitting is used to refer to the strategy of 'breaking apart'. This was a conscious decision to help ELLs and learning disabled students who became confused with multiple labels for a strategy.

During these early number talks, Shi watched and listened to other students during the beginning of the school year. He felt comfortable with one little girl and followed her lead through the daily routines. During number talks, Shi was able to figure out the answer to the problem, but he was unable to share his solutions or thinking with the rest of the class. After the first couple of weeks, he ventured to give the answer to the problem because he felt comfortable with numbers. However, with limited English he did not explain or share his solution strategy. Shi seemed to really enjoy math and felt comfortable with solving written problems. While there were times during the school day when Shi became uninterested in a lesson, this never occurred during math. Braun noted that Shi was engaged and appeared to use his background in math to build upon what the class was doing. He seemed to make connections to the ideas being taught. Like Shi, Ricca was also quiet. She kept her eyes on the speaker and watched Braun write on the chart paper. As both the structure of the number talk and the structure of the language became familiar, Ricca began sharing her thoughts using the language structure. Initially, when she tried to share her thinking during number talks, she became stuck and forgot her next step. After 30 seconds of wait time, Braun would ask her if she wanted to think about the problem for a little bit. Ricca always nodded shyly with relief. Occasionally Braun led the class back to the problem and asked Ricca if she was ready to share her thinking. She either finished expressing her steps or she asked for help from the class. When a peer helped her complete the problem by providing the needed language, Ricca listened intently trying to absorb everything that the student said. During the first quarter of the school year, Ricca shared her thinking only two or three times. During the second quarter, Braun introduced *cluster problems* as a strategy to help students solve more difficult problems mentally. Cluster problems were related to the original problem using the operations of multiplication or division. A combination of them could be used to solve the original problem often using the distributive property. For example, cluster problems related to 28 x 6 could include the following: 20 x 6; 8 x 6; 120 ÷ 6; 48 ÷ 6. Braun encouraged students to use one of the cluster problems to find the solution and if they finished quickly to use a different one. Students shared a solution strategy using each of the cluster problems. Initially, Braun provided the cluster problems for students to use and later students generated their own set, first as a class and then independently.

The following except illustrates how Ricca and Shi used language structures during number talk to share their mathematical thinking. Braun posted two problems, 184 ÷ 8 and ¾ of 24**,** on chart paper with four cluster problems (Figure 1).

**184 ÷ 8**
Cluster problems to help: 160 ÷ 8 24 ÷ 8 11x 8 1x 8

**¾ of 24**

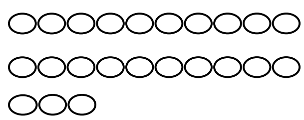

*Figure 1*. Two number talk problems posted on chart paper with four cluster problems.

Braun began the number talk by asking students for answers to the four cluster problems and students quickly provided them. Following the routine, students thought quietly for about two minutes and Braun asked for solutions which were recorded on the chart paper. Braun then asked for a volunteer to share a solution strategy.

| | | |
|---|---|---|
| 1 | Shi | 160 divided by 8 is 20. 24 divided by 8 is 3. 160 and 24 is 184. 20 and 3 |
| 2 | | is 23. Answer is 23. |
| 3 | Braun | Thank you, Shi, for sharing! |
| 4 | Student E | Shi, is learning English quickly. |

Shi quickly raised his hand and began using a slight modification of the language structure (line 1). Rather than beginning with, *I know that,* he simply stated the facts that he used, resembling the model provided by Student C (lines 12 to 14) in the previous example. As modeled by Student C, the classroom norm was to split the problem into two related cluster problems that were easy to solve mentally. Shi, following the classroom norm, articulated his answer to the cluster problem with the larger numbers (160 ÷ 8) before stating the answer to the cluster problem with smaller numbers (24 ÷ 8). Then he combined the pieces and stated his answer. Clearly, Shi followed the classroom norm that established an order for splitting numbers apart and used the language structure to articulate his mathematical thinking. His ability to communicate his thinking was recognized by a peer (line 4). From this peer student's perspective, it was unusual for an ELL to gain fluency so quickly.

After Shi's explanation, Braun moved the discussion to the second problem. She had recently introduced fractions to the class and wondered how they would apply their understanding of fractions in a new number talk. Previously, students found fractional parts of sets using unit fractions (e.g., find $^1/_8$ of 24).

| | | |
|---|---|---|
| 5 | Braun | Who would like to solve ¾ of 24? (Ricca raised her hand). Ricca. |
| 6 | Ricca | I know that 6 fours equals 24. And that 6 and 6 and 6 is 18. The answer |
| 7 | | is 18. |
| 8 | Braun | Three of the 4 groups of six is 18? |
| 9 | Ricca | Yes. |
| 10 | Braun | Questions? |
| 11 | Student G | Where did you get the six? (Ricca was silent.) |
| 12 | Student H | What was the question? |
| 13 | Braun | Where did Ricca get the 6? |
| 14 | Ricca | I divided (pause). I divided 24 by 4 is 6. |
| 15 | Braun | Student J, do you have a question? |
| 16 | Student J | No, I was going to help Ricca if she was stuck. |
| 17 | Braun | Oh, that was very nice of you. So, which number did Ricca use to |
| 18 | | divide the 24? |
| 19 | Student K | The 4 in the ¾'s. |

Ricca responded to Braun's invitation to discuss a mathematical idea that students in the class were beginning to understand. Ricca used the provided language structure, beginning with *I know that* (line 6). Then, stated the facts and her answer. It is interesting to note that Ricca was developing multiple ways to express an idea through her use of equals as seen in her statements (lines 6 and 7). Her thinking is less transparent for the students in the classroom, prompting Student G to ask a question (line 11). Ricca needs more time to think about her response. Another student, taking advantage of the silence, asks his teacher to repeat the question (line 12). This seemed to give Ricca the time that she needed to explain her use of division (line 14) to find the size of four equal pieces of 24. Anticipating that Ricca may need help articulating her thinking, Student J is ready to help (line 16). This exchange illustrates the collaborative nature of the classroom in which students worked together to solve problems and explain their thinking. Braun decided at this point to intervene and bring closure to the problem (line 17 and 18). While many of the students may not fully understand how Ricca found three-fourths of 24, Ricca was able to articulate her strategy in a way that some students understood.

## Summary

Shi and Ricca seldom spoke in the initial weeks of fifth grade. Shi was a beginning ELL student and did not know very many words. Ricca was classified as an intermediate ELL but was shy and lacked confidence in her mathematical thinking and ability to communicate in English. Like many ELLs they both listened intently to the other students during this silent receptive stage (Echevarria, Vogt, & Short, 2008). The number talks were ideal to support the language development and mathematical thinking of these ELLs. First, students were able to share their answer which was a single number. There was no expectation for an explanation and no judgment as to whether it was correct or incorrect. Thus, Shi and Ricca could provide a solution. If they made a mistake translating numbers into English, it was not an embarrassment for them. Second, the number talks were short (about ten minutes). The shortness of the number talks encouraged more intense concentration and focused listening to peers. Third, the language structure that Braun provided created a predictable pattern that they could follow and later use themselves. The early number talk problems were purposefully selected to pose little cognitive demand so that students could focus on articulating their reasoning using the language structure. Like many elementary school students, the students in Braun's class naturally split numbers into pieces and manipulated the larger place values before the ones and tens. This became a classroom norm that the ELLs adopted.

The use of a language structures helped the ELLs gain proficiency and confidence using English to express their thinking (Echevarria, Vogt, and Short, 2008). These structures also helped the two ELLs develop their mathematical thinking. Shi quickly used the distributive property to mentally solve a division problem and Ricca applied her understanding of fractions to find three-fourths of a set. Sfard (2008) illustrated the important role of discourse to the development of more sophisticated mathematical thinking. Providing language structures that are predictable and easily modified supported the ELLs' emergent mathematical discourse, which in turn allowed them to explore mathematical ideas in collaboration with their peers and teacher. This study suggests that language structures may not only help ELLs but also all students learning mathematics. The language structures help students explain their ideas in concise ways that their peers can easily understand. In doing so, these structures may also help solve a quandary that many teachers face: What do I do when students ramble in incoherent ways while trying to explain their thinking to their classmates? More research is needed to describe how language structures and other supports described by Echevarria, Vogt, and Short (2008) support the learning of mathematics.

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# THE POSITIONALITY OF AFRICAN AMERICAN GIRLS TOWARD MATHEMATICS: FROM THE VOICES OF THE GIRLS, THEIR PARENTS, TEACHERS, AND SCHOOL COUNSELORS

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*Despite recent process toward gender equity in mathematics and science education, the persistent underachievement among low-income African American girls remains a challenge. This presentation offers the preliminary outcomes of a longitudinal study examining the positionality of fifth grade students toward mathematics and science. From the data speak diverse voices, voices of African American girls, their parents, their teachers, and their school counselors, and they tell how the girls regard themselves as learners of mathematics.*

## Background

The centrality of mathematics for advanced degrees and economic advancement has been widely acknowledged. A comparison of the National Assessment of Educational Progress (NAEP) mathematics scores for students in fourth and eighth grade clearly demonstrates the difference in the mean achievement between Blacks, Hispanics, and Whites.

| NAEP Mathematics | 5th (1996) | 5th (2000) | 8th (1996) | 8th (2000) |
|---|---|---|---|---|
| Blacks | 243 | 247 | 280 | 274 |
| Hispanics | 251 | 253 | 287 | 283 |
| Whites | 282 | 286 | 310 | 308 |

Results of the Scholastic Achievement Test (SAT) mathematics exam reveal a gap of about 100 points between the lower-scoring Blacks and the higher scoring Whites. The gap that begins in fourth grade only gets larger as the years of schooling pass (Bennett, Bridglall, Cauce, Everson, Gordon, Lee, et al, 2004). Although research has been conducted on the African American student achievement (Foster & Peele, 1999; Murrell, 2002), girls and mathematics education (Kerr & Kurpius, 2004), and the impact of socioeconomic status on student learning, little is known about the relationship between teacher expectations and African American girls' self-perception as science and mathematics learners.

## Theoretical Perspective

Positionality, rooted in feminist scholarship, has been used to describe an individual's self-perceived social location that informs that individual's world-view. According to positionality theory, an individual's position in relationship networks defines that individual and also determines the amount of individual power (Cooks, 2003; Harley et al., 2002). According to feminist scholars, positionality is present in the classroom where power dynamics among teachers and students are affected by gender and racial differences (Johnson-Bailey, 2002). This

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position determines the level of power individuals possess and, if internalized, impacts their access to opportunities. Positional factors have been shown to affect knowledge construction, power, and relationships in and out of the classroom (Maher & Thompson Tetreault, 2001). Therefore, the researchers posited that African American girls' constructed cultural, gender, and class identities dictate their positionalities in relation to mathematics learning.

African American girls must negotiate both race and gender to succeed in school (Picken, 2002). "Living in the context of a larger African American community presents more choices, yet African American women still have to contend with devaluing messages about who they are, and who they will become, especially if they are poor or working class" (Tatum, 2003, p. 57).

Scholarship on positionality indicates that a teacher's positionality may affect teaching practices as well as students' experiences (Cooks, 2003; Rehm & Allison, 2006). The teacher's tendency toward certain behaviors with culturally diverse students, such as being open to cultural differences or tying to correct such differences, is a product of a teacher's views on diversity. Positionality is a salient element of all classroom dynamics. School personnel spend more time addressing the social skills of African American girls (speech and dress patterns), and less time promoting their academic skills. This is exacerbated for low-income African American girls (Morris, 2007).

Research is therefore needed to explore how the positionalities of teachers, counselors, and parents impact African American girls' positionality in relation to mathematics and science learning. The presenters are engaged in a three-year study funded by the National Science Foundation to investigate African American girls positionality toward science and mathematics.

## Research Questions

The primary research questions under study are as follows:

1. How do African American middle school girls position themselves as mathematics and science learners in relation to their gender and ethnic identities?
2. How do parents, teachers, counselors, and administrators position African American girls in relation to the girls' interest and achievement in mathematics and science education?

For this conference, the presenters will focus the component of the research questions that address mathematics.

## Method

The research participants were recruited from local elementary schools where the researchers had previously established working relationships. Thirty African American girls in the fifth grade were asked to join a cohort of female students who would be observed during their fifth grade and the two years following. Participants were drawn from the probable population at a neighborhood middle school where there was in place a magnet program for mathematics and science. The girls were asked to share their reflections on school experiences in mathematics and science. Of the three schools, one school was considered as "parallel", rather than "control," given that the study was not designed to test interventions. This school was already rich in community involvement, a situation atypical for the community, which is generally of low economic status.

Preliminary data sources included semi-structured interviews with teachers of mathematics and science content areas, parents, and counselors, and focus groups with the girls. All interviews were conducted at the schools. Interviewers were members of the research team, doctoral students and post-doctoral researchers in education areas.

Records of student grades and state tests scores were collected. During the summer, an institute was held for parents, teachers, and counselors from two of the schools. Data from the first cohort included observation, interview, and field notes, videotapes of classroom lessons, focus groups, and the summer institute. There is an advisory Board comprised of community members. During the second year, a new cohort of thirty fifth-grade girls was added to the study, comprising the second cohort.

Initial analysis was conducted using grounded theory, beginning with inductive analysis. The data was analyzed according to the data source, be it girls or teachers, parents or counselors. Hatch's (2000) inductive analysis method was used to analyze the data. Inductive analysis is a search for patterns of meaning in the data that guides the researcher to make general statements regarding the phenomenon being studied. Triangulation of the data was achieved by member checking, peer review, and confirmatory analysis using a review of the literature.

## Results

Triangulation of the data was achieved in part through the literature reviewed on African American girls and mathematics and science learning (Butler & Lakes, 2003; Foster & Peele, 1999; Kerr & Kurpius, 2004; Murrell, 2002; Tutweiler, 2005). Consistencies between this literature and the researchers' finding regarding African American girls' positionality in education provide some authenticity of representation, or a confirmation between what the researchers believed to have observed and what was actually observed.

### *Responses of the teachers*

During interviews teachers reported their observations of African American girls in classes in mathematics and science. They report that reading and mathematics are areas in which girls excel, but they are better in areas that involve language. They credit the boys in their classes as being more interested in science and mathematics than are the girls. Teachers hypothesized that the girls typically viewed math and science as a male domain. As one teacher explained,

> I think some girls kind of think that science is more, or a guy thing, cause like, you know. . . Of a lot of their experiences, like you know, who invented this or who invented that, it's a lot of guys, to the point where they don't feel like they fit in.

Teachers acknowledged that the mathematics instruction is based on rote memory, skills and computation, with less focus on higher order thinking, problem solving or hands-on activities, at the same time they are aware that their students would be more successful with hands-on experiences in the classroom. One teacher believed that among the resources that would influence mathematics skill development would be, "Definitely hands-on, definitely giving them something where they can feel it, touch it."

Observing that parental involvement was important to the girls' success, teachers note the significance of the home environment. Girls did better when the parents were supportive and the student was motivated. Early focus on school subject areas made a difference, especially when the work was taught in an earlier grade. Otherwise, teachers felt as if they had to start over at the beginning. Strategies that worked included group work and peer coaching, as did scaffolding of information. However, teachers believed that few of their female students would pursue mathematics in higher education or their careers.

### *Responses of the Parents*

Analysis of the data revealed that parents were indeed directly or indirectly involved in their daughter's mathematics and learning. To the extent that they viewed themselves as mathematics learners, parents were directly involved with their child's homework. Regardless, they were involved in setting up and monitoring homework time. Indirect parent involvement manifested itself in several ways. Parents were in communication with their daughter's teacher, were willing to seek resources to help their daughter in problem areas, and were sources of encouragement.

Another major finding focuses on parents' knowledge of their children and their learning process. They understand their children's learning process and the importance of teacher influence and teaching methods on learning. In terms of their daughters' engagement in mathematics instruction, they stated that they wanted teachers to make learning interesting, fun, understandable, applicable to real life, and hands-on.

*Responses of the Girls*

In their focus groups, the girls described their mathematics classes. In these rooms, the students worked on problems presented by their teacher or from their textbook. Their tools were paper, book, and pencils. The students listened, wrote, and answered questions posed by the teacher. Sometimes they worked on the problem solution with their team, and sometimes they worked alone or with the teacher's help.

To the girls, mathematics is a subject that is sometimes difficult and challenging, and sometimes it is easy. What's easy is, "the multiplication, the division, and . . .that's about it," to one girl. What's hard is " . . .to understand what you are doing." The girls note that a student who is good in math is one who studies and works hard. When asked to visualize somebody doing mathematics, the girls said " . . .knowing the steps and doing it right." "It looks like they are counting in their heads." The students believed that to be proficient in mathematics would require a great deal more schooling, perhaps even a college degree.

However, the girls did have moments of confidence when they knew the answer to the question or they got back a paper with a high score on it. When they knew the answer, they raised their hand quickly. Sometimes they would be so excited that they would shout out the information, but they knew that they might get in trouble for doing so. All of the girls preferred to be called on when they knew the answer. When they didn't know the answer, they were "embarrassed" or "nervous" or "scared." One fear was that their fellow students would think that they had not been "working hard or studying." As the girls struggled with their ignorance for a particular question, the teachers let another one help her or moved on to another student for the answer.

*Responses of the Counselors*

The outcomes of the study revealed that the counselors demonstrated low levels of awareness of their biases towards the students. Additionally it was found that the participants did not see themselves as advocates or agents for social change. Moreover, they did not understand their role in the advancement of mathematics and science learning for these students. For example, one counselor talked about the potential for her students to go to college, but she did not see herself as an agent of change for these students. Although another counselor stated, "There's a big world out there; they can do anything they want to," she also viewed the position of poverty as an obstacle to the girls' attendance at a four-year college, explaining that there are not many scholarships available to families of limited means.

All of the counselors evidenced an understanding of effective ways to engage students in mathematics and science, especially the importance of instruction with hands-on activities. Two of the counselors discussed utilizing prior knowledge and the girls' interests to increase student engagement and success in mathematics. They pointed out the motivation created by success. "When you're successful at something it makes you want to do it more . . .success breeds motivation." Counselors recognized the positive effect of parental involvement, and they noted a need for role models. "I think we see ourselves looking for someone like us."

Counselors are uncertain about their role in facilitating mathematics and science achievement in the girls. One counselor tried to make connections between the girls who were interested in the same things, who had similar career aspirations, but she saw her position as being little involved in the mathematics setting. Overall, it appeared that these counselors might unintentionally play a gate-keeping role in the way in which they position African American girls and place limitation on their potential. Their view of the obstacles faced by the students blocks a vision of a counselor taking a larger role in the students' mathematics experiences.

## Discussion

This study began by studying the girls, but the grounded theory revealed that the teachers' positionality toward mathematics is a critical component of any work regarding the achievement of girls in that subject. Among the strategies recommended for closing the Achievement Gap in mathematics is setting high standards (Bennett, 2004). What the teacher envisions for the girls' futures influences instruction. Teachers who see themselves as strong mathematicians are more confident about developing that capacity in their students. If they doubt that they can be a positive role model in mathematics, they can provide examples of others who are models of achievement, of female African American mathematicians and scientists.

One implication for the professional development of counselors and teachers working with African American girls is that the educators can address the girls' needs better when their own positionality is uncovered. This understanding could be advanced in the professional development context. Additionally, guidance counselors could be encouraged and taught to play a more active role advising and encouraging the girls' participation in mathematics.

All of the adult respondents noted that more hands-on instruction would benefit the mathematics learning of the students. Teachers need more training in providing this type of instruction. Moreover, they could be taught to look forward, taking a positive stance about teaching the girls what they will need for the future, rather than focusing on the deficits in their students' background knowledge.

We acknowledge that the girls are aware of how they are being positioned. They look to their teachers, parents, and counselors, for guidance and support to help them be successful. They want to be the kind of student that is regarded as capable and industrious. They know that boundaries for their horizons are deeply influenced by these important players in their educational lives.

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# THE IMPACT OF "MATH FOR SOCIAL ANALYSIS" ON MATHEMATICS ANXIETY IN ELEMENTARY PRESERVICE TEACHERS

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*Mathematics anxiety is prevalent in America and has been an important topic of researchers for educators in the past several decades. This paper describes ways mathematics anxiety levels subsided as a result of preservice teachers' engagement with social issues in a new course, Math for Social Analysis, designed to emphasize equity and social justice. Further, we make the case that contrasting models for studying mathematics anxiety generally fall into one of two predominant groups: deficit models and situated models. Our research raises new challenges to deficit models based on our observations that mathematics anxiety is situated within the classroom dynamics.*

## Introduction

Mathematics anxiety is prevalent in America and has been an important topic of research for educators in the past several decades. Ma (1999) found a significant relationship between anxiety in mathematics and achievement in mathematics was significant, as did other researchers (e.g., Satake and Amato, 1995). Mathematics anxiety is widespread in the U.S., including among preservice teachers (e.g., Buhlman & Young, 1982; Burns, 1998; Levine, 1996). This is problematic because, for example, mathematics anxiety among preservice teachers is associated with apprehension when faced with the prospect of teaching mathematics (Brady & Bowd, 2005; Gresham, 2008). Bursal and Paxnokas (2006) suggested nearly half of preservice teachers having higher mathematics anxiety believe they will not be able to effectively teach mathematics. Brush (1981) found mathematics anxious teachers tend to revert to traditional methods of teaching mathematics with a focus on basic skills, ignoring mathematical concepts.

We did not originally set out in our research to examine mathematics anxiety; however, the topic became salient in our research in a new mathematics course for elementary and middle grades preservice teachers. The course, *Elementary and Middle Grades Mathematics for Social Analysis* [*Math for Social Analysis*], incorporates National Council of Teachers of Mathematics [NCTM] *Standards*-based (2000) mathematics content and instructional approaches and uses relevant social issues to provide contexts for preservice teachers' learning of mathematics. Beginning with an overarching goal to understand preservice teachers' assessments of their learning and experiences in the course, numerous references to mathematics anxiety emerged. This paper describes ways mathematics anxiety levels subsided as a result of preservice teachers' engagement with the social issues. Further, we make the case that contrasting models for studying mathematics anxiety generally fall into one of two predominant groups: deficit models and situated models. Our research raises new challenges to deficit models based on our observations that mathematics anxiety is situated within the dynamics of the classroom.

## Math for Social Analysis

The *Math for Social Analysis* course is one component of the Mathematics Education in the Public Interest [MEPI] project.[i] The MEPI project has as its key objectives to support equity and social justice in mathematics education.

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

*Math for Social Analysis* is unique because we integrate mathematics, critical pedagogy and citizenship. In our course, preservice teachers and faculty identify social issues of personal or professional interest. Examples include rainforest depletion rates, poverty, and child labor. The preservice teachers identify and use multiple mathematical methods to better understand both the relevant mathematical content and the social issue. Each classroom unit of study or semester project includes actions which contribute to positive social change.

## Methodology

Results presented in this paper are based on research conducted in a medium-sized public institution located in southeastern U.S. We conducted open ended interviews with 14 White female preservice teachers enrolled in *Math for Social Analysis* in spring 2008. We asked preservice teachers to share their past experiences with mathematics and their thoughts concerning the course, mathematics as a discipline, and teaching mathematics in the future.

## Results

Our data analysis revealed an emergent theme which showed that *Math for Social Analysis* proved beneficial in reducing mathematics anxiety among our preservice teachers. For preservice teachers, a focus on social issues in the mathematics classroom: (1) Increased the utility of mathematics, (2) Redirected attention away from anxiety, and (3) Built confidence to teach.

*1. Increased Utility of Mathematics*

> You say “math” and people get psyched out...because math is one of those high anxiety subjects….Being able to answer why it is *really* important is really going to help some kids….All the word problems in math books I’ve ever seen are situations that are not gonna happen….By finding something that *matters* to the kids and something that they want to learn, where math is more of a tool to better understand it and not a means to get an answer, is going to help. (Megan)

Megan’s reflections are similar to those of several other students in *Math for Social Analysis*. According to some preservice teachers, the course led to decreased feelings of mathematics anxiety. Several expressed developing improved attitudes as a result of learning new relevance for mathematics. A focus on social issues gave new meaning to the mathematics.

*2. Redirected Attention*

> I hated [math]. I never really liked math….I just don’t think I was very good at it….*Math for Social Analysis*—I like the class a lot just because it gives a different spin. I kind of wish that math had been taught to me differently...like incorporating all the social stuff and hands-on learning ….When we did the whole rain forest ... I was like, oh my gosh, that many trees…but I’m learning math at the same time. (Amanda)

Some preservice teachers found their attention was redirected away from their mathematics anxiety and negative attitudes as they became immersed in social issues. In focusing on the social issues, they were at times surprised when they realized they were learning mathematics since they were feeling no anxiety.

*3. Built Confidence to Teach*

> I realized in my senior year of my Bachelor’s degree I would have to…[teach] math and that scared me. So a few years down the road I came back [to school for a graduate degree] and yes, [I] can teach math—I can do it now….[*Math for Social Analysis* has] shown me not only can I do it, but there are better ways….I want to take some of the social issues and make [math] relevant. (Janet)

Janet is not alone in finding her fears to teach mathematics were eliminated or greatly reduced. Some preservice teachers enjoyed learning mathematics in the context of social issues and no longer dreaded or feared mathematics or teaching mathematics.

## Discussion

In reviewing previous research on mathematics anxiety, we observed that much of it seems to fall within two predominant groupings—one based on a deficit model and the other based on a situated model.

Deficit models treat mathematics anxiety as a fixed, measurable quantity located inside the individual. Surveys are often used to evaluate the level of mathematics anxiety, and solutions to reducing anxiety often aim to improve the psychological or emotional state of the person. For example, Furner (2004) believes bibliotherapy is beneficial in overcoming mathematics anxiety. In bibliotherapy, books help educators guide the emotional development of their students as the students connect with the characters in a story. New books for children focus on mathematics anxiety so children do not feel isolated and alone with their fear of mathematics. Bibliotherapy is a form of psychological counseling that requires meaningful follow up discussion to be beneficial.

Another deficit model solution proposed by researchers is *systematic desensitization* (Furner, 1996; Hembree, 1990; Trent, 1985; Olson & Gillingham, 1980). This requires a gradual exposure of the mathematics anxious student to the mathematical concepts that are causing the student stress. Students are then taught appropriate coping skills. Sgoutas-Emch and Johnson (1998) recommend journal writing as an effective method to reduce anxiety, where the student can freely express feelings about mathematics. Hypnotherapy and cognitive skills training are additional techniques used to reduce test and mathematics anxiety. These methods require extensive training and counseling sessions to be effective, which may prove too costly and inconvenient for most students.

The problem with deficit models is that by identifying the problems within the individual, the models largely fail to consider, for example, the ways mathematics is socially constructed in the classroom or the ways mathematics as a discipline may be flawed. Situated models contextualize mathematics anxiety, contending that individuals' mathematics anxiety cannot be understood as separate from their experiences within the classroom or from the nature of the mathematics. People generally are not mathematics anxious prior to attending school (Williams, 1988). Research has suggested mathematics anxiety can be caused by teaching methodologies that do not encourage reasoning and understanding (Greenwood, 1984) and may also be a product of students' lack of mathematical understanding (Butterworth, 1999). To reduce mathematics anxiety, Cohen and Leung (2004) demonstrated the benefits of the reformed methods of teaching mathematics, as cited by the NCTM *Standards* (1989, 2000). However, Alsup (2004) indicated the particular instructional strategies used may not be as important in reducing preservice teachers' mathematics anxiety as the instructor's ability to communicate and clarify mathematical ideas, and the interconnectedness of mathematical concepts, while maintaining a calm and reassuring disposition.

Our research demonstrates a mathematics course with a focus on social issues can reduce mathematics anxiety among preservice teachers. We surmise from our research that situated models for examining mathematics anxiety may be more appropriate. By incorporating social issues into mathematics, this transformed the nature and relevance of the mathematics itself, thereby improving conditions for preservice teachers to engage with the discipline.

## Conclusion

We find it encouraging that mathematics anxiety can be reduced when preservice teachers learn mathematics in the context of social issues. Further research is needed that uses a situated model to examine mathematics anxiety. As previously indicated, we did not originally set out in our research to examine mathematics anxiety; therefore, interview questions were not explicitly intended to reveal preservice teachers' mathematics anxiety levels. To test the validity of the results presented in this paper, additional research is needed investigating mathematics anxiety in classrooms infusing social issues into the curriculum.

## Endnotes

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# ACCESS TO MATHEMATICS: A POSSESSIVE INVESTMENT IN WHITENESS

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A number of papers and articles call for more attention to race/racialized experience in the work of mathematics educators (see Martin, 2009; DiME, 2007). One way to interpret this call is to focus more on researching people of color. Another way to view it is to critique whiteness as invisible, neutral, or normal, and to make known privileges that come with this status. This shifts a researcher's lens to the perpetrators or oppressors of racism. However in doing this, it is important to expose systematic, structural, and institutional forms of racism. While individual racism has a face and can be quite explicit, institutional racism can be difficult to represent tangibly. Many governmental programs privilege whites even though they purport to be race-neutral. Lipsitz (1995) calls this the possessive investment in whiteness. One way of exposing this is by examining profits or advantages that racism affords whites over minorities while noting the processes that allow that advantage to take place. Perlo (1996) illustrates this by analyzing the exploitation of people of color through cumulative wage differentials for African American and Hispanic workers as compared to whites.

Employing a similar framework, this paper analyzes the wage-earning differential due to differences in mathematics coursework by ethnic/racial groups. These differentials are reported across 3 time points, 1982, 1992, and 2004. Of course, issues such as SES, gender, and geographic location are important, but they are not included for the purposes of this paper on racial differences.

## Whiteness

Lipsitz (1995) states that "a fictive identity of "whiteness" appeared in law as an abstraction, and it became actualized in everyday life." Much like Black is a cultural construction based on skin color, not biology, whiteness developed out of the reality of slavery and segregation giving groups unequal access to citizenship, immigration, and property. By giving whites a privileged position in relation to the "other", European Americans united into a fictitious community. Whiteness is a constantly shifting boundary separating those who are entitled to certain privileges from those whose exploitation is justified by not being white.

While many think of race in a Black/white binary, groups such as Jews, Native Americans, Asians, and Latinos have proved more difficult to classify in the racial hierarchy. In the 1840s and 1850s, California had debates about the status of Mexicans and Chinese. There were some Mexicans with considerable wealth and partners with whites, while the Chinese were exploited for work on the railroads and the field. It was decided that Mexicans would be considered white and the Chinese the same as Blacks and Indians. That decision determined who could become citizens, own land, marry whites, and other basic rights (Almaguer, 1994). To complicate things further, though Mexican Americans were considered white legally, they were denied rights and privileges that whiteness bestowed (Foley, 2002). Despite being ruled as white in courts, the government added a category of Mexican on the 1930 census, counting only 4% of Mexicans as white. This prompted the League of United Latin American Citizens (LULAC) to attempt to establish Mexicans as whites and considered it an insult to be counted Black or a "colored race". The organization turned its back on civil rights battles of the 1940s and 1950s with statements such as “tell these Negores that we are not going to permit our manhood and womanhood to mingle with them on an equal social basis” (Haney-Lopez, 2006, quoting Marquez, pp. 33).

In contrast to LULAC’s stance, the Chicano/a movement of the 1960s, rejected LULAC’s assimilation strategies. This movement found common cause with Blacks, Native Americans Chinese, and Vietnamese. They rejected whiteness and all it came to mean. The response from whites has been “Why do you insist on being different? Why do you have to be Mexican or Chicano? Why can’t you just be American” (Foley, 2002). The lure of whiteness and all that it entails has been a contested boundary for those in the Latino community, some seek it out others reject it. Now, Mexican Americans are not considered white and Chinese Americans are conditionally white at times, not at others, but clearly different from Blacks and Native Americans.

From 1878 to 1909, the courts in the U.S. heard twelve naturalization cases of persons seeking citizenship. Eleven of those cases were barred from citizenship including persons from China, Japan, Hawaii, as well as two mixed race applicants. The cases were argued based on reasoning of common knowledge, skin color, and the subdivision of the human race into five groups; Mongolian, Negro, Caucasian, Indian, and Malay (Haney-Lopez, 2006). White skin itself was not enough to not guarantee one’s property rights in whiteness (Harris, 1993). The courts later ruled that not even all Caucasians were white, cementing the cultural construction of whiteness (Foley, 2002).

Many ethnic groups have sought out equalization through citizenship, but when African American citizens still had to sit at the back of the bus and couldn’t vote, assimilation became the goal. And when the 1940 census stopped distinguishing foreign-born versus native-born whites, official assimilation as white became a possibility. As “not-yet-white” ethnic immigrants strive to assimilate as a way to attain whiteness, “immigrants of color always attempt to distance themselves from dark identities (blackness) when they enter the United States” (Bonilla-Silva, 2003, p. 271). For many immigrant groups the path to whiteness became not so much about losing one’s culture as becoming agreeing to the idea that Blacks were culturally and biologically inferior to whites, “Only when the lesson of racial estrangement is learned, is assimilation complete” (Morrison, 1997, p. 57). While there are certainly still markers for some (e.g. accents, dress, and cultural practices) that lead to prejudices, the fiction of whiteness brings real opportunities and access to particular people.

## White Privilege

White privilege is a result of the institutional and structural investments given to whites. Lipsitz (1995) coined the term possessive investment in whiteness over a decade ago. In this paper, he discusses federal policies in the United States that authorized attacks on Native Americans, restricting naturalized citizenship to “White” immigrants, slavery, and segregation. Many have discussed the legal challenges that weakened the Supreme Court’s decision in Brown vs. the Board of Education (Bell, 1979), but these past policies are still with us today through more covert yet racist systems. Many policies seem neutral, yet their effect is anything but that.

One example is the Federal Housing Administration’s (FHA) loan practices. From a confidential city survey to destroying housing in city centers affecting twice the percentage of African-Americans compared to whites in the 50s and 60s, these housing practices have shifting loan money and therefore future investment in real estate away from communities of color and towards whites since 1934. These practices served to drive up prices in white suburban communities, keeping people of color from benefiting. The development of highway systems as well as public policy cutting political precincts in half, served to reduce the political power of African Americans allowing garbage dumps and incinerators to be located in communities of color (Logan & Molotch, 1987). More recently, studies have shown that African Americans are 60% more likely than whites to be turned down for loans (controlling for credit scores), will be judged on dividend income more often, disqualified for loans at almost 3 times as much, and will receive conventional financing at ¼ the rate (Massey, 1994; Orfield & Ashkinaze, 1991). In fact, in some cases, high income Blacks were turned down at a higher rate than low income whites (Campen, 1991). Although these practices are not termed “Affirmative Action” they benefit whites at the detriment to Blacks.

In addition, tax policies of the 80s made taxation on goods and services higher than it was for profits from investments. Again, by connecting this to the above investment in whites owning their own homes and profiting from raised home values, whites will necessarily benefit from lowered taxes on investments. Similarly, Proposition 13 in California granted tax relief to property owners and reduced funds by $13 billion a year for public education and other social services (McClatchy, 1991). Businesses avoided between $3.3 billion and $8.6 billion in taxes per year (UC Focus, 1993).

Educationally, these same advantages have been invested in whites. Funding in schools is one way that whites maintain privileges. While the history of Brown versus the Board of Education is well know, the fact that we are now at similar levels of segregation in schools to the 1960s mean the problems of yesterday are still here (Orfield et al., 2004). Policies of school funding tied mostly to local property taxes have maintained differential funding for suburban schools at levels twice that for urban schools (Kozol, 1991). This well documented difference impacts teacher quality, curricula, building conditions as well as numerous other educational issues. Deficit ideologies of teachers are also well documented and affect students of color because they are framed as the other, similar to the prior discussion of those not included in whiteness (Perry, 1993). Additionally, outdated representations in history textbooks of Native Americans as a romanticized and almost dead people and slavery and civil rights issues for African Americans as a thing of the past, reproduce the appearance of neutral policies (Kivel, 2002). During the history of the U.S., the population of Native Americans went from 12 million to 237,000 and whites expropriated 97.5% of their land (Churchill, 1994). Yet, we don’t call this genocide in textbooks nor do we have representations of current day Native Americans in schools. The magnitude of those numbers should give one pause. These educational issues privilege the schooling of whites by providing more funding, not having to face advantages we live with today connected to a history fraught with killing and taking another people’s lands, and avoiding cultural deficit ideologies by maintaining the white norm.

Despite these investments in whites – generated through slavery and segregation and augmented by social reforms – a poll notes that 70% of whites believe that African Americans “have the same opportunities to live a middle-class life” (Orfield & Ashkinaze, 1991). There are numerous other policies such as social security, the GI bill, the college draft deferment, and legacy admissions at universities (Conley, 1999; Kivel, 2002). These forms of Affirmative Action for whites serve to guarantee that whites will continue to benefit from historic advantages. Meanwhile the attack on Affirmation Action counters policy trying to balance the playing field from such “neutral policies.”

## Exploitation of People of Color

Centuries of bestowing land, education, and stored wealth in the form of social security to whites means that whites born today begin their lives with more familial wealth stored in houses, educational attainment, and investments. Policies that place wealth in the hands of certain groups and take it away from other groups are a form of exploitation. The profits are essentially the benefits of racism to whites. Perlo (1996, p. 170) calculates the profits from racism as the "wage differentials against African Americans, Hispanics, etc., multiplied by the number of workers employed in private enterprises." He contends that although the median differential in 1991 against black workers overall was 32%, the difference for skilled craft workers was 25%. However, within each skill level, black workers tend to be consigned to the worst jobs – risking their health. This, along with the practice of categorizing the same occupations whites work at lower level for people of color, convince him to use the median differential for the calculation reasoning that there is a cost for subjecting Blacks to health risks and the devaluation of jobs for people of color.

Given this framing, Perlo (1996) calculates that the total profits of racism in 1947 were 56 billion dollars (in 1995 dollars). That number rose to 88 billion in 1972, 112 billion in 1980, and 197 billion in 1992. African Americans were exploited for 48, 60, 74, and 107 billion dollars in those years, while Latinos were exploited for 8, 28, 34, and 84 billion dollars. These numbers do not include the exploitation of white women or whites in poverty, even though they are certainly exploited by the white elite. Even so, these numbers speak to a widening divide between an investment in whites and people of color. They are a symbol of centuries of supposedly "neutral" policies, investing through education, home ownership, and maintenance of wealth in advantaging whites on the backs of African Americans and Latinos.

## Mathematics Education's Investment in Whiteness

We can calculate a similar statistic for mathematics education's investment in whites. Through the availability of AP classes in suburban schools, tracking students of color into lower mathematics coursework, and counselor's referring students to less advanced coursework different access and opportunities are available to students of color (Oakes, 2003). In addition, deficit ideologies of teachers create negative and sometimes hostile environments for students, when some teachers don't believe that students of color have the intellectual ability to think abstractly (Perry, 1993). This could produce what Smith and others have found to be racial microaggressions that lead to racial battle fatigue leading to psychological, emotional, and physiological consequences in the classroom (Smith et al., 2007). These processes serve to reduce mathematics coursework, college opportunities and earning potential for people of color.

In this paper, I calculate the exploitation of people of color, using a similar formula to Perlo, due to different mathematics coursework. This serves as a representation of the investment in whiteness that mathematics education continues to reproduce.

## Methods

The data used in this study were taken from national databases: High School and Beyond 1980 (and follow-ups), National Education Longitudinal Study (NELS) 1988 (and follow-ups), Education Longitudinal Study 2002 (and follow-ups), Current Population Survey (CPS) 1972–2005. This paper presents a secondary analysis of these data with respect to mathematics course completion. All dollar amounts are adjusted to 2008 dollars.

Though race is a constantly changing phenomenon, the data used in this study positions them as fixed groups (due to its use in the national datasets). Also, these national datasets mix ethnicity with race, but the data are presented using the terms from the survey, however problematic they may be.

The equation for the average yearly income for each ethnic/racial group is as follows:

% completed calculus x average earnings by mathematics coursework
% completed trigonometry/algebra III x average earnings by mathematics coursework
% completed algebra II x average earnings by mathematics coursework
% completed algebra I/geometry x average earnings by mathematics coursework
% completed low academic/no math x average earnings by mathematics coursework
+ % dropouts x average earnings for dropouts

Similarly taking this and multiplying it by the number of students each year gives the total investment by each ethnic/racial group.

## Results

The results begin with data across 1982, 1992, and 2004 on the highest level of mathematics completed by racial/ethnic group. In addition, I included dropouts since the dataset only included the highest level of mathematics completed for high school graduates. The analysis then shifts to the income levels ten years after graduation by mathematics achieved and by race/ethnicity. These data are then used to calculate the investment in whites over students of color, ten years after graduation, over a lifetime, and across generations. I want to caution readers in that these numbers should be seen as approximations since, as with all national datasets, survey questions can change over time, under or over estimate results, and have missing data. Still the results speak to major differences.

Table 1 shows the highest level of mathematics completed for high school graduates in 1982, 1992, and 2004 by ethnic/racial group. The right column also presents the status dropout rate. The dropout rates are a major underestimate, but use the only data available across all three years (for an understanding of the problems with dropout rates in national datasets see Orfield et al., 2004).

Hispanics, American Indians, and Blacks took advanced mathematics at lower rates across years and were more likely to take no mathematics than Asians and whites. While this data could be used to reaffirm students of color lack of interest in mathematics, cultural deficit theories, or a discussion of racial "gaps", I urge readers to remember the literature on whiteness and privilege cited earlier in understanding that these are institutional differences generated over the history of this country to advantage whites and that a number of other factors need to be considered as well. First, in urban schools researchers have documented the lack of AP mathematics and science courses available. Even if students of color wanted to take this coursework, it is not available to many, often because they do not have teachers certified to teach AP mathematics. Also, tracking has routed students of color to lower levels of mathematics for decades (Oakes, 2003). One more factor among many is that counselors steer students of color away from college preparatory courses. These three factors, though there are many more, speak to differences due to institutional racism, access to higher quality mathematics instruction to whites, rather than cultural myths of students of color.

Table 1. *Highest Mathematics Completed by Year, Ethnicity/Race, and Average Yearly Earnings 10 Years after High School Graduation*

| | Graduates | | | | | |
|---|---|---|---|---|---|---|
| | Advanced | Middle | | | Low Academic /No Math | Dropouts |
| Year and Ethnicity/Race | Calculus | Trig/algebra III | Alg 2 | Alg 1/Geo | | |
| 1982 | | | | | | |
| White | 6.8% | 22.9 | 18.9 | 30.9 | 20.5 | 8.8 |
| Asian/Pacific Islander | 15.4 | 37.6 | 19.1 | 17.2 | 10.6 | 2.2 |
| Black | 2 | 11.2 | 18.5 | 27.7 | 40.6 | 11.3 |
| Hispanic | 2.6 | 11.3 | 13.8 | 33.1 | 39.1 | 16.8 |
| American Indian/ Native Alaskan (AI/NA) | 2.3 | 8.6 | 13.4 | 27.8 | 48 | 25.1 |
| 1992 | | | | | | |
| White | 11.5 | 28.9 | 26.9 | 20.8 | 11.9 | 4.3 |
| Asian/Pacific Islander | 22.1 | 32.4 | 23.9 | 12.5 | 9.2 | 4.6 |
| Black | 6.9 | 18.7 | 23.5 | 32 | 18.9 | 7.6 |
| Hispanic | 5.0 | 23.9 | 26.3 | 30.9 | 14.2 | 10.9 |
| AI/NA | 1.0 | 13.7 | 28 | 28.5 | 28.9 | 18.2 |
| 2004 | | | | | | |
| White | 16.2 | 39.1 | 23.8 | 16.4 | 4.6 | 4.7 |
| Asian/Pacific Islander | 33.8 | 35.5 | 17.5 | 11.1 | 2.1 | 4.6 |
| Black | 4.9 | 36.3 | 31.5 | 18.7 | 6.6 | 10.5 |
| Hispanic | 7.0 | 28.1 | 31.9 | 26.5 | 6.4 | 13.4 |
| AI/NA | 5.4 | 16.8 | 40.8 | 23.7 | 13.3 | 15.2 |
| Avg. Earnings by Math Coursework (2008 dollars) | 42,625 | 39,581 | 35,014 | 30,447 | 28,163 | 23928 |

Using the mathematics coursework data and the average earnings 10 years after graduation (2008 dollars), I calculated the average earnings by ethnicity/race. Multiplying the percent of whites who completed calculus in 1982 (6.8%) by the average income of that group of students ten years later ($46,625) along with the percent who completed Trigonometry/Advanced Algebra (22.9%) by their income ($39,581) and so forth results in the average wage earned for whites across mathematics course work completed and dropouts. In doing this across years for each ethnic/racial group the calculations result in the average earnings ten years after graduation (or not). The numbers for these calculations can be seen in the first column of table 2. This gives a sense for the earning potential for different groups according to mathematics coursework.

In Table 2, I present the total investment for each ethnic/racial group based on the number of students in each high school class multiplied by their average earnings. To compare the investment with that of whites, I used the same number of students (as each ethnic/racial group) multiplied by the average earnings for whites. The last column compares total and adjusted investments, represents the financial advantages given to whites through mathematics preparation (rounded to the nearest million). This does not include statistics for people of color earning lower wages with the same education working in the same job, SES, or other factors that would make these differentials greater. Therefore, these calculations should be considered an underestimate.

Table 2. *Annual Earnings due to Mathematics Coursework and Total Investment, by Year and Ethnicity/Race*

| Year and Ethnicity/Race | Average Earnings from Mathematics Coursework (2008 dollars) | Percent of High School Population | Total Investment Based on Mathematics Coursework (millions) | Adjusted Investment for whites with same Percent Population (millions) | Net Investment in whites (millions) |
|---|---|---|---|---|---|
| 1982 | | | | | |
| White | 33,129 | 79.9% | 119,507 | -- | -- |
| Asian/Pacific Islander | 36,187 | 1.3 | 2,124 | 1,944 | -179 |
| Black | 29,830 | 11.6 | 15,623 | 17,350 | 1,728 |
| Hispanic | 28,753 | 6.3 | 8,179 | 9,423 | 1,244 |
| AI/NA | 29,458 | 1.0 | 1,330 | 1,496 | 166 |
| 1992 | | | | | |
| White | 34,887 | 72.7% | 91,804 | -- | -- |
| Asian/Pacific Islander | 36,533 | 4.5 | 5,951 | 5,683 | -268 |
| Black | 32,021 | 11.9 | 13,793 | 15,027 | 1,234 |
| Hispanic | 30,862 | 10.0 | 11,171 | 12,628 | 1,457 |
| AI/NA | 31,107 | 1.2 | 1,351 | 1,515 | 164 |
| 2004 | | | | | |
| White | 36,405 | 62.3% | 91,907 | -- | -- |
| Asian/Pacific Islander | 38,289 | 4.5% | 6,982 | 6,639 | -344 |
| Black | 33,481 | 13.3% | 18,044 | 19,621 | 1,576 |
| Hispanic | 32,215 | 15.0% | 19,581 | 22,128 | 2,547 |
| AI/NA | 32,771 | 0.9% | 1,195 | 1,328 | 133 |

There is a surplus in favor of Asians (see table 2). However, whites having more access and opportunity to take mathematics courses than African Americans results in yearly advantages of 1.23-1.73 billion dollars. For Hispanics the range is from 1.24-2.55 billion dollars, in part due to an increase in population. And finally, for Alaskan Indian/Native American the range is from 133-166 million dollars. This accumulation of wealth in favor of whites over Native Americans is for less than 50,000 high school students. These numbers are for only one year of work, ten years after high school, and do not include the years in between the dates included.

Over the course of the 23-year span included in this study, using the average yearly advantage (see column 1 in table 3), whites are advantaged over Blacks (38.4 billion dollars), Hispanics (41.1 billion dollars), and Native Americans (3.84 billion dollars). For a typical 40-year work-life the totals range from 6.67 to 71.4 billion dollars. This means for the life of one high school class, the effects of differential mathematics access results in over 144 billion dollars invested in white students. Notice as well that Asians still show advantages over whites totaling 5.69 (23-year span) and 9.90 (40-year work-life) billion dollars. Importantly, these numbers do not consider that differentials at one point are more likely to grow due to raises, accumulated income, and interest.

Table 3. *Average Investment in Whites over 23-year Span and 40-year Work-life*

| Investment in whites over | Average (over 1982, 1992, & 2004 in millions) | Cumulative over 23 years (1982-2004) | Work-life (40 years) | Work-life * 23 years |
|---|---|---|---|---|
| Asian/Pacific Islander | -247 | -5,693 | -9,901 | -227,724 |
| Black | 1,670 | 38,414 | 66,808 | 1,536,588 |
| Hispanic | 1,785 | 41,077 | 71,438 | 1,643,093 |
| AI/NA | 166 | 3,837 | 6,674 | 153,507 |
| Total | 844 | 83,329 | 144,921 | 3,333,188 |

The last column in table 3 reports the aggregate investment for the 23 years of high school students across an estimated 40-year work-life. The total advantage for whites is over 3 trillion dollars. This number more or less represents the investment in a generation of mathematics students over their careers. However, it does not include wage differentials afforded whites that would likely advantage whites even over Asians.

## Discussion

These differences should be appalling. This makes the work we do as mathematics educators so crucial in challenging so called neutral policies, in bringing more AP courses to urban and poor communities, and merely helping teachers care for, expect more from, and push students of color to take more mathematics.

Approaching this issue generationally, we can see how income, home investments, educational attainment, and mathematics knowledge serve as huge advantages passed on to whites. This historical perspective allows a framing of achievement of Latinos and African-Americans despite numerous obstacles. This different framing moves us out of thinking about deficits and failures to achievement, successes, and the struggle to be educated.

While many are looking at mathematics education from an economic perspective as far as competitiveness in the global market, we haven’t invested enough in African Americans, Latinos, and Native Americans. There are plenty of children to be educated that we are not reaching. While an economic outlook is one slant on this issue, the differential in investment poses a moral problem more than any other. Why do we as a society allow differential funding to schools? Why do we allow students to be put in lower tracks robbing them of access to higher-level mathematics? How can we begin to address such glaring inequities? The answer may be that these policies privilege whites and many are not ready to give up these advantages.

Another perspective might ask, is all of this mathematics, and everything it gives access to, necessary for college admission and citizenship more generally. Although the incomes are different, this could be due more to college admission than using mathematics in careers, which implicates a false gatekeeper. Bob Moses termed access to mathematics as the next civil rights issue and according to this analysis, he is certainly right. But maybe the mathematics required for higher education is merely keeping some from further educational opportunities. The policies of Harvard and Stanford in not counting high school work in calculus, though for a very different reason than educational access, might signify a first step towards slowly lowering the gate for students of color.

## Conclusion

These results indicate that we invest considerably more in whites than Blacks, Hispanics, and American Indians showing the systematic racism that educational institutions reproduce. The numbers indicate one way that historic and current mathematics education practices serve as an investment advantaging whites. In another sense, it raises the need of informing parents of color of the importance to seek out access to mathematics for children. As currently constructed, it serves as a gatekeeper to universities, mathematical careers as well as economic capital and independence.

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# COUNTER NARRATIVES OF AFRICAN AMERICAN EXPERIENCES IN MATHEMATICS: THE CASE OF ANDREA

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> How do I commit myself to do work that is predicated on a belief in the power of the mind, when African-American intellectual inferiority is so much a part of the taken-for-granted notions of the larger society that individuals who purport to be acting on my behalf, routinely register doubts about my intellectual competence (Perry, 2003, p. 5)?

## Introduction/Purpose of the Study

Perry's quote is a stark articulation of the dilemma that continues to surround African American children and adolescents in the United States. Lemons-Smith (2008) similarly poses, what *really* prohibits schools and teachers from providing cognitively demanding, high-quality instruction for all students, regardless of race, class, gender, language, culture, or other characteristics? It is a question that lies at the heart of educational underachievement broadly, and specifically within the discipline of mathematics education. Over the last decade or so much has been written regarding the mathematics education experiences of African American students (Berry, 2005; Hilliard, 1995; Ladson-Billings, 1997; Martin, 2000; Moody, 2004; Stinson, 2006; Tate, 1995; Walker, 2006; Walker & McCoy, 1997). These scholars have attempted to give voice to Black students and those who teach them – two groups who are often silenced or superficially addressed within the context of mathematics education research. These scholars' work does not subscribe to entrenched deficit theories nor provide artificial prescriptions or strategies for "fixing" the mathematics performance of African American learners. Rather, they provide a culturally embedded perspective of African American students and their engagement in the mathematics teaching and learning process.

This paper describes a case study that looked critically at the connection between African American students' K-12 mathematics experiences and how they view the discipline of mathematics and themselves as learners of mathematics. Specifically, this paper focuses on one African American student who participated in the study. I will begin by discussing the theoretical framework and study context. Then, I will present excerpts of the student's narratives to illustrate how she perceived her mathematics experiences, views about mathematics, and views about herself as a learner of mathematics. The student's narrative reflects her voice, and therefore is in first person. Brief school and teacher background information precede the narrative.

## Theoretical Framework

This study sought to provide a voice to African American students and their perspectives and experiences in mathematics. Given the historical silencing and negative representation of African Americans in education and society generally, Critical Race Theory was selected as the lens through which to view and affirm their life stories. Critical Race Theory has at least five defining themes: (a) the centrality and intersection of race and racism, (b) the challenge to dominant ideology, (c) the commitment to social justice, (d) the centrality of experiential knowledge, and (e) the interdisciplinary perspective (Solorzano & Yosso, 2001).

Within education Critical Race Theory challenges the dominant ideology of race and racism and its interplay within school structures, processes, and discourses (Solorzano & Yasso, 2001). Critical Race Theory provided the context for considering how the African American students in this study negotiated their K-12 mathematics experiences.

## Methods

How are K-12 mathematics experiences connected to one's mathematics-related views? Specifically, views about the discipline itself and oneself as a learner of mathematics. The study consisted of ten African American undergraduates, one of which is the focus of this paper (Andrea). Participants represent a convenience sample and were solicited from various minority organizations on a university campus. Participants' attitudes toward mathematics, performance, and grades in mathematics were not a criterion for selection.

Data sources consisted of semi-structured interviews, which were audio taped and lasted about three hours, and handwritten autobiographies. The interviews and autobiographies solicited information about (a) the participants' academic and personal histories, (b) experiences in the mathematics classroom as it relates to curriculum, instruction, and classroom culture, and (c) views about mathematics, school, and oneself as a learner of mathematics. Interpretational analysis (Gall, Borg, & Gall, 1996) was used to analyze the transcribed interviews and autobiographies. The data analysis involved open coding in which common and divergent participant responses were identified. From these responses themes that encompassed and summarized the data were established. These themes were central in describing the connection between African American students' mathematics experiences and mathematics-related views.

## The Case of Andrea

Andrea is nineteen year-old biochemistry major. She received her K-12 schooling in public, majority White schools. The demographic composition of Andrea's math teachers was: High school (all White males); middle school (Two White males, a White female); and elementary school (all White females).

*Andrea: In Her Own Words*

> For the most part I like math. I like it because I'm good at it. It's not just that I get good grades, but also because I understand it. I can explain it to other people and make them understand it too. You don't really realize how much you understand something until you try to teach it to someone. Then you realize you really know this stuff. I didn't really focus on math until the sixth grade when I tested into the Extended Learning Program (gifted). Since the sixth grade I've been the only African American in my math classes. At first being the only African American didn't bother me because I was young, but as I got older I started realizing that the students didn't think I was as smart as them. Nor did the teachers expect as much of me. I could tell that by how they interacted with me. My teachers seemed to make me work harder than my white classmates. Even though I was a good math student, my teachers didn't really call on me much. In many instances if I was called upon and gave the right answer, my teachers' reaction was exaggerated, as if they didn't expect me to respond correctly. I found this particularly odd given the level of math courses I was enrolled in. They acted as if I was in remedial math. My peers would seem surprised if I scored an A on an assignment. I guess because I was black they thought I was dumb or something. I enjoyed participating in the minority math and science summer program. It was cool being around other smart minorities like me. No

> one questioned your ability. Everyone was on equal footing. I felt as though most of my math teachers held lowered expectations for me. If I asked them a question they would go back to the very basics and explain it to me as if I were stupid. It was as if I didn't know that a negative times a negative is a positive. If a white student asked a question they would explain it to them assuming they understood the basics. They didn't insult them by reviewing pre-algebra concepts. My teachers lowered expectations made me want to try harder. Since they don't think I'm as smart as the next white person, I'm just going to show them. I vividly recall my ninth-grade teacher making race-related comments and jokes that were inappropriate and made me feel uncomfortable. I don't feel that I received the same quality and quantity of instruction as my white classmates. When I asked for help they'd half explain it or wouldn't explain it and pointed me to the book. Whereas, when white students asked for help they'd explain it with no questions asked. It's like they expected me to work harder even though they didn't expect much of me. If a white student made a C or D on a math assignment the teacher would tell them they needed to get their grade up. If I made the same grade they wouldn't say anything. It's as if they thought a C or D was the best I was capable of doing. Their expectations of me were set from day one. Sometimes when I got a problem right my math teacher's reaction would be exaggerated. It was so transparent. No one was really hostile toward me or outright rude, but I was just there. They didn't ignore me, but they didn't acknowledge me. They would listen to what I had to say, but it wasn't valued as much. The expression on their faces was different when they were listening to white students. It's like they were listening differently, more attentively.

## Results/Discussion

In considering how Andrea's mathematics experiences are connected to her mathematics-related views, the following three themes emerged: (a) teacher characteristics, (b) self-positioning, and (c) resiliency. In this section I will discuss each of these themes.

### *Teacher Characteristics*

Analysis of Andrea's interview and autobiography revealed generally negative perspectives about her mathematics teachers and instruction. Andrea's depiction of her mathematics teachers is interesting when considered within the context of Gloria Ladson-Billings' work on culturally relevant teaching. Ladson-Billings (1995) asserts that culturally relevant teachers exhibit the following broad qualities with respect to the underlying propositions: (a) Conceptions of self and others suggests that culturally relevant teachers hold high expectations for all students and believe all students are capable of achieving academic excellence; (b) social relations infers that culturally relevant teachers establish and maintain positive teacher-student relationships and classroom learning community as well as are passionate about teaching and view it as a service to the community; and (c) conceptions of knowledge suggests that culturally relevant teachers view knowledge as fluid and facilitate students' ability to construct their own understanding. Ladson-Billings' work illuminates an instructional ideology for facilitating the academic success and cultural competence of African American students. Hence, it is salient for contextualizing Andrea's mathematics teachers. As it relates to culturally relevant instruction her teachers receive a grade of "F." Their conceptions of knowledge, conceptions of self and others, and social relations do not demonstrate a commitment to providing equitable mathematics instruction to *all* students. It is interesting to note that during Andrea's K-12 mathematics career all of her mathematics teachers were White. Certainly race is not a determinant in whether or not one demonstrates the qualities of a culturally relevant teacher; however the observation is striking. Embarking on this line of thought raises questions about the broader impact of teacher and school demographics. To what extent are those demographics linked to the existence or absence of culturally relevant teaching? What other dynamics account for Andrea's experiences in mathematics?

In addressing these and other questions related to Andrea's experiences, one must also consider the very personal nature of perceptions. The personal nature of perceptions is evidenced in the two other themes revealed in the data: self-positioning and resiliency.

*Self-Positioning*

Despite the absence of mathematics teachers that can be defined as culturally relevant, Andrea expressed generally positive views about the discipline of mathematics and herself as a learner of mathematics. While this is true for Andrea, it is often not the case for students of color. Her interviews and narrative suggests that the way in which she positions herself within a macro and micro context is a contributing factor in how she views mathematics and herself. In several instances she noted that school was only a small piece of her life, who she was, and relegated it to micro status. In contrast, she drew heavily upon home, family, and external forces in shaping her belief system. Perhaps if school had been assigned a more prominent position in Andrea's being, the lack of culturally relevant teachers may have yielded a more significant impact. That self-positioning is also implicitly linked to the third theme – resiliency.

*Resiliency*

Despite encountering less than empowering teachers Andrea successfully negotiated schooling structures and excelled in the mathematics classroom. She preserved and developed a strong mathematics background. She explicitly contributed her ability to do so to strong family and extended family support. This support appears to be a key factor in her resiliency and ability to challenge the persistent myths and stereotypes surrounding African Americans in mathematics. Rejecting widely held notions of deficiency enabled Andrea to embrace positive views about mathematics and herself as a learner of mathematics.

Andrea did not reflect the more popular theories related to African American achievement. For example, she did not view her academic personas and success as "acting white" (Fordham & Ogbu, 1986). In addition, her perspectives did not reflect stereotype threat (Steele, 2003). That is, she did not rebuff characteristics not typically associated with African Americans. Andrea held a positive self-concept and did not view her success in mathematics as incongruent with her racial identity. Hence, that positive self-concept facilitated her resiliency and ability to combat messages of inadequacy.

## Concluding Thoughts

In considering how one's K-12 mathematics experiences are connected to their views about mathematics and themselves as learners of mathematics, teacher characteristics, self-positioning, and resiliency emerged as salient themes. I opened this paper with the quote: "How do I commit myself to do work that is predicated on a belief in the power of the mind, when African-American intellectual inferiority is so much a part of the taken-for-granted notions of the larger society that individuals who purport to be acting on my behalf, routinely register doubts about my intellectual competence (Perry, 2003, p. 5)?" I would argue that Andrea was successful in rejecting persistent notions of inferiority and excelling in mathematics. Her positive self-concept, affirmative internal dialogue, and strong support system provided the foundation for resisting institutional barriers and perceptions set forth. I would argue that Andrea is the rule, not the exception and success in mathematics is a norm associated with African Americans. Hence, additional research that provides counter narratives to disaffirming messages about African Americans in mathematics is warranted. Collectively, researchers, schools, teachers, and students can articulate a counter discourse that acknowledges the pursuit of excellence in mathematics is an endeavor that can be accomplished by ALL.

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# AN ACTIVITY SYSTEM ANALYSIS OF MATHEMATICS TEACHING PRACTICE IN AN URBAN HIGH SCHOOL

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*Through use of an activity theory (AT) analytic and explanatory frame, we examine and articulate the ways in which one teacher, April Lincoln, a mathematics teacher in a large urban school, consistently facilitates reading strategies in her Algebra I classroom. After mapping April's practice onto the AT framework, we will focus our discussion on the relationship between three key components of the activity system – subject, mediating tools, and object – with particular attention to April's personal history as a struggling reader and how this experience appears to influence her pedagogical choices.*

## Introduction

There is a movement underway in the mathematics education research community that seeks to better understand students' mathematics schooling experiences by examining these experiences through sociocultural and historical perspectives (Atweh, Forgasz, & Nebres, 2001; Boaler, 2000; Martin, 2000, 2007; Nasir & Cobb, 2007; Secada, Fennema, & Adajian, 1995; Yackel & Cobb, 1996). A guiding tenet of this movement is the acknowledgement that the learning and teaching of mathematics is not 'culture-free'; a complex mix of historical, political, and cultural forces determine that students at specific intersections of societal communities (racial, ethnic, economic, linguistic, geographical) *experience* mathematics differently than students positioned at other intersections, and these differential experiences contribute to differences in performance on measures of mathematics achievement and competence. A thread of this work focuses on examining teachers' mathematics instructional practices from sociocultural perspectives and seeks to conceptualize and study mathematics learning environments that are equitable and responsive to the needs and experiences of all learners. Of particular interest to many are the ways mathematics teachers in urban schools engineer effective mathematics learning environments in school contexts that are often characterized as challenging and difficult due to history of low academic achievement as measured by students' performance on standardized assessments. Through use of an activity theory (AT) analytic and explanatory frame, we examine and articulate the ways in which one teacher, April Lincoln, a mathematics teacher in a large urban school, consistently facilitates reading strategies in her Algebra I classroom. After mapping April's practice onto the AT framework, we will focus our discussion on the relationship between three key components of the AT map – subject, mediating tools, and object – with particular attention to April's personal history as a struggling reader and how this experience appears to influence her pedagogical choices.

## Theoretical Framework

### *Activity Theory*

Activity theory is an analytic and explanatory framework that is useful for research of human behavior (Engerstrom, 1987). This framework first emerged from the scholarship of Vygotsky, Leont' ev, and Luria in Russia in the cultural-historical school of psychology (Minick, 1997). Nardi (1996) contends that research using AT can shed light on the complex experiential unity of

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

individual cognition and social activity; furthermore, AT is perceived as a useful framework to view and understand the complexities of mathematics classrooms in varying contexts (Radford, Bardino, & Sebena, 2007; Walshaw & Anthony, 2008, Anthony & Clark, 2008). The particular and unique characteristics of U.S. urban schools, namely high enrollments of minoritized students, historical trends of low performance as measured by standardized assessments (and, consequently, a palpable assessment culture), high teacher attrition, and the consistent cycle of instructional and curricular 'reforms', demand that teacher practice be examined in ways that more fully acknowledge the complexities of teaching in these contexts. In that the AT framework is designed to explain activity through a broad range of intertwined influences, we found it particularly useful in our efforts to examine mathematics teachers practice in urban schools.

An activity system is the unit of analysis in activity theory. The minimum elements of an activity system, according to Engestrom (1987), is an object, subject, mediating tools (including psychological tools), rules, community and division of labor (Figure 1). The "subject" is the individual or group of individuals involved in the activity, the "object" is the motivating problem or reasons behind why the subject participates in the activity, and it is what connects individual action to collective activity. The "tool" includes people and artifacts that act as psychological tools, mediating activity between the subject and the object. The rules, community, and division of labor components add the historical aspects of mediation that Vygotsky omitted from his work (Engestrom, 1999). Subjects are members of social groups or "communities" that have explicit and implicit "rules" or norms that provide guidance to acceptable interactions among system participants. Rules and mediating artifacts that are accepted by the community mediate relations between the subject and the communities they are a part of. "Division of labor" refers to the tasks and responsibilities that are constantly negotiated among participants of the activity system (Cole & Engestrom, 1993). By considering the influences of rules, community, and division of labor on an activity, Engestrom's model includes both historical and situated aspects of human activity. The model also represents the motive behind situation-bound actions that individuals within the activity system are a part of (Engestrom, 1987). The "outcome" of the system refers to the outcomes or results of the activity.

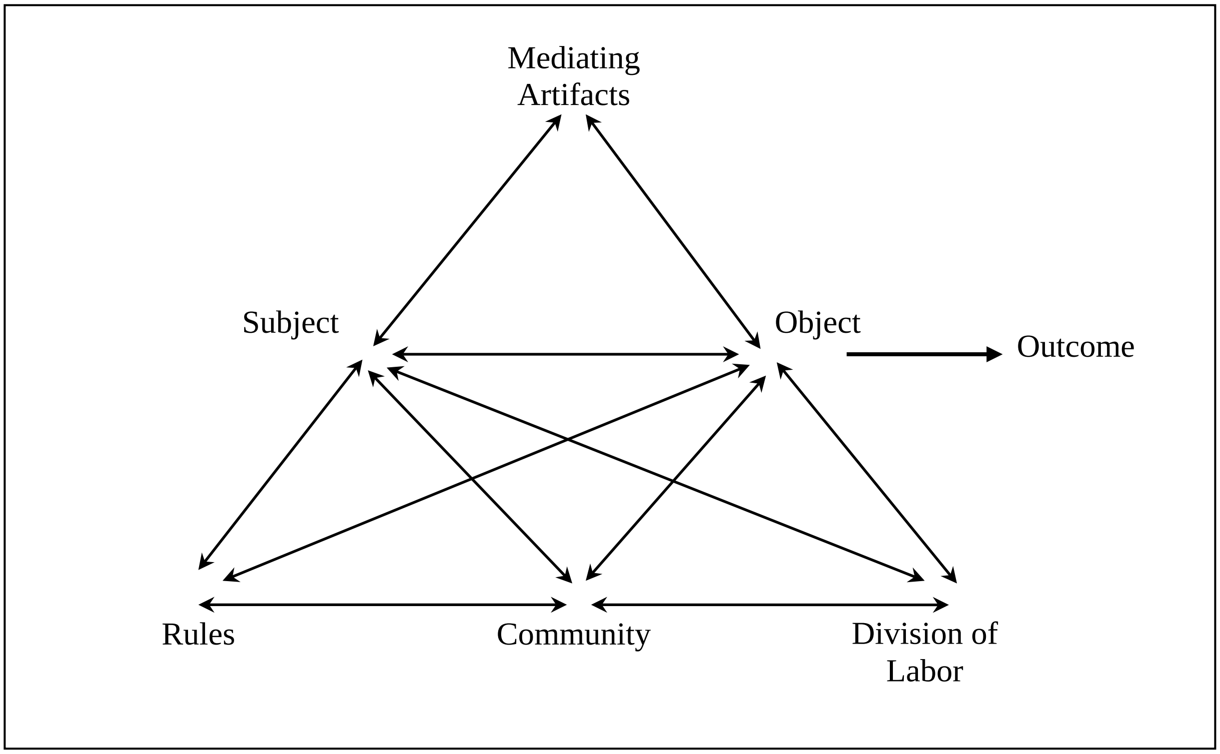


*Figure 1.* Representation of an activity system. Engestrom, Y. (1987).

*Teachers' Personal History as Mediating Tool*

A powerful aspect of examining mathematics teacher's practice through the AT framework is its capacity to broaden traditional notions of the resources and tools mathematics teachers draw on and use to do their work. When mathematics instructional practice is viewed through sociocultural and sociohistorical lenses, 'non-mathematical' resources and tools, namely knowledge of students' lived experiences and community histories, as well as teachers' personal experiences and community memberships, must be examined side-by-side with more traditional physical and intellectual resources (i.e., teachers' mathematical knowledge, teachers' general pedagogical knowledge, teachers' content-specific pedagogical knowledge, curriculum materials). Wenger (1998) states that teachers are far more than "representatives of the institution and upholders of curricular demands" (p. 276); they are doorways into the adult world. Wenger (1998) contends, "Teachers… constitute learning resources, not only through their pedagogical or institutional roles, but also (and perhaps primarily) through their own membership in relevant communities of practice… This type of lived authenticity brings into the subject matter the concerns, sense of purpose, identification, and emotion"(p. 276). In keeping with this perspective, teachers' lived experiences and community memberships (such as gender, as members of various ethnic and racial groups in racialized societies, as once young students of mathematics, as local community members) serve as mediating tools that they employ in their efforts to teach mathematics. Through implicit and explicit means, and through their personal story, teachers communicate to their students what it means to construct a healthy mathematical identity (Martin, 2000, Clark, Johnson, & Chazan, 2009). Minoritized students in the U.S., particularly African American and Latino students, are engaged in a unique, continuous process of negotiating multiple traditions, cultural frames, and identities, some of which arguably have been perceived and/or constructed as incompatible, in an effort to function in U.S. society (Boykin, 1986; Du Bois, 1903). It is reasonable to believe, therefore, that teachers, particularly mathematics teachers of marginalized, minoritized students, play an important role in assisting their students negotiate and reconcile (Wenger, 1998) real or perceived conflicts or dilemmas in their identity formation in general and their mathematics identity formation in particular.

## The Case Studies of Urban Algebra I Teachers Project

In 2004, the University of Maryland's Center for Mathematics Education with funding from the National Science Foundation embarked on the Case Studies of Urban Algebra I Teachers Project through the Mid-Atlantic Center for Mathematics Teaching and Learning. The main purpose of the project was to document the practices, instruction, and perspectives of 'well respected' teachers of algebra in two urban high schools that were mainly comprised of large populations of African American and Latino students. The focus of this paper, April Lincoln, a teacher at Erasmus High School, is one of six mathematics teachers participating in the project. All teachers in the study were African American except for one African American male. The range of teaching experience across teachers was vast – two years to over 20 years.

The research team structured observation and interview protocols around three main themes: 1) teacher's "sense of purpose" of teaching mathematics, 2) the teaching of algebraic concepts, and 3) the teaching of data analysis. Each teacher was observed roughly 30 times across the three themes (most of which were videotaped) and interviewed eight times over the course of a year.

*Erasmus High School*

At the time of data collection (2005), Erasmus High School enrolled approximately 2000 students in grades nine through twelve. Forty-five percent of Erasmus students were African American, 43% were Hispanic, 7% were White, and 5% were Asian. In 2005, 27% of Erasmus students scored at or above the 'percent proficient' cut score on the mathematics portion of the state mandated assessment. Thirty-five percent of students scored at or above the 'percent proficient' cut score in reading during the same year. Student mobility was considered 'high' at Erasmus during 2005, with 20% of students enrolling in school after the start of the school year and 17% leaving Erasmus before the end of the school year. In 2005, 31% of Erasmus classes were taught by teachers that did not have highly qualified teaching status. Attendance rates have been historically high at Erasmus and approximately 76% of 9$^{th}$ graders entering Erasmus in 2001 (4 years prior to projected graduation year) graduated. (It should be noted that by 2008, student performance in mathematics and reading increased dramatically to over 75% of students at or above 'percent proficient' in both areas.)

## Data Analysis Methods

Data analysis techniques utilized in the development of this case study included traditional case study methods that consisted of an iterative process that proceeded from more general to more specific observations (Creswell, 1998). The authors individually studied and coded interviews and observations and identified instances that map onto the AT framework. In some cases, relevant instances did map cleanly onto an element of the AT framework; in other cases the authors did not agree on the how a particular instance should be coded. In the latter case, the authors negotiated until an adequate agreement was reached.

We first present a short biography of April Lincoln, followed by discussion of elements of the activity system that structured April's efforts to facilitate reading strategies in her algebra classroom. We conclude with a discussion of our interpretation of how April's experiences as a struggling reader may influence her commitment and capacity to engage in this practice.

## April Lincoln

April Lincoln is an African American female in her late forties at the time of data collection. She described herself as being a very quiet and shy person, and, throughout her interviews, projected a sense of modesty and humility. Initially, it was one of her professors that suggested she should consider pursuing a career in teaching mathematics and April commented that she was 'shocked' by his recommendation. She did not follow his suggestion upon graduation from college and began working in private industry. During her career in private industry, one of April's children said to her one day "Mom, you act like a teacher". April stated that, "A light went off in my head" at that moment and, as a result, April started teaching kindergarten at the local YMCA. After teaching kindergarten for a few years, April began teaching middle school. After two years at the middle school, she began teaching mathematics at Erasmus High School. At the beginning of this study, April was entering her fifth year at Erasmus.

April described her performance as a school student as mixed – she excelled in mathematics but and struggled in courses that demanded considerable amounts of reading and writing. She repeatedly mentioned that mathematics was the only subject matter that she felt competent and powerful. Despite her challenges with reading and writing, April enrolled in graduate school to pursue a master's degree and, after experiencing continued difficulty in reading and writing, was diagnosed with a reading disability. April's reading difficulty lead to problems with her reading the consent form to participate in the research project which resulted in the interviewer reviewing the document for April. April also experienced difficulty reading and comprehending the prompts associated with this research project.

April projected a very empathetic character and appeared to have a deep sense of concern for her students. April viewed her students as an extension of her children and approached teaching them with her parental instinct and the same level of concern she has for children. She strongly believed that she must better prepare her students to read in order for them to pass the mathematics portion of the high school assessment because "it is all they do is read and write" (Interview, September 28, 2005).

## Activity Theory Activity System: Reading Strategies in the Mathematics Classroom

Research exploring the prevalence of reading strategies in secondary classrooms indicate that few mathematics teachers view incorporate reading strategies into their practice for a host of reasons (Davis & Gerber, 1994), however April was committed to incorporating reading strategies in her algebra class. Of the six teachers in the study, April's classes enrolled the highest percentage of students identified as having learning or behavioral difficulties, resulting in many students with Individualized Learning Plans (IEPs). In April's classes with large numbers of students with IEPs, an additional teacher from the Special Education department was assigned to support April in class. April stated, "I've taken two reading classes. And my whole task this year is to get them to do more reading in mathematics…the [high school graduation test] is nothing but reading and writing, so my whole thing this whole year, we gonna read, we going to write, we going to explain. And uhm… that's what I'm pushing them to do…" (Interview, November 16, 2005). We witnessed three types of reading strategies in April's classroom during the year of data collection: 1) explicitly attending to words that have multiple meanings inside and outside of mathematics classroom, 2) identifying root words of key mathematics vocabulary to find meaning, and 3) utilizing problem solving heuristics that explicitly contain a 'step' related to reading comprehension. Table 1 contains elements of the activity system related to April's facilitation of these strategies in her classroom.

Table 1. *Activity System Elements Related to April's Facilitation of Reading Strategies*

| Element | Elements in April's practice |
|---|---|
| Object | Facilitation of reading strategies in the mathematics classroom |
| Subject | April Lincoln, resource teacher, students |
| Mediating tools | High school graduation assessment, algebra curriculum guidelines, April's experiences as struggling reader, students' achievement history, April's reading courses, students' behavioral problems |
| Rules | Pacing schedule for algebra curriculum guides, students' IEPs |
| Community | Other Algebra teachers, school administrators, other teachers in school |
| Division of labor | April Lincoln, resource teacher, reading support classes |

## Discussion

Our analysis of the activity system around April's facilitation of reading strategies in her class suggest that a primary mediating tool was her concern that her students would perform poorly on the state assessment due to their reading struggles. Additional mediating tools surfaced in our analysis, including April's personal struggles as a reader. April was very candid and open with students about her reading struggles, and described to students supports and techniques she used in her daily life to function. It was evident that April felt it important for students to know of her struggles and her story, and how she has managed to be resilient, successful, and highly functional despite the challenges she faced. Her story complicates notions of the skills, knowledge, and resources effective teachers must possess. As representatives of academic excellence and upholders of curricular demands, it is not common in teacher education circles to consider that some teachers have overcome considerable learning challenges and use these experiences as instructional resources. Had April not been a struggling reader, would she have consistently incorporated reading strategies in her classroom? Our interpretation suggests that her experiences were a significant mediating tool between her and her facilitation of reading strategies in her algebra classroom. There was no evidence that other teachers in her school were engaging in similar strategies.

April's story, however, leaves many critical questions to explore, including:

- Is it important for students who may have a history of low performance on standardized measures of mathematics achievement to be exposed to resilient teachers who have overcome adversities?
- Can this exposure support struggling students' mathematics identity formation?

- Does April's story simply reify characterizations of mathematics teachers in urban schools as low quality and poorly prepared or does her experience actually position her as better qualified in these particular contexts than teachers that may not have had her experience?

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# FRAMEWORKS FOR STUDYING MATHEMATICS COMMUNITIES THAT SUPPORT AND RETAIN WOMEN DOCTORAL CANDIDATES

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By examining statistical data only, it seems that women are making great strides in overcoming their under-representation in mathematics (Monroe et al., 2008). However, statistics do not tell the entire story. Qualitative studies (see Monroe et al., 2008; MIT Report, 1999) suggest that women still face barriers in academia. Therefore, the research study described in this poster will focus on qualitatively exploring graduate mathematics departments that have a history of producing women mathematicians. By examining the recruiting and educational practices within these communities and linking those practices to students' and faculty member's perceptions and experiences within their departments, the study hopes to reveal successful practices for producing women mathematicians.

In an attempt to examine the practices and environment of a mathematics department, the researcher has employed both the "Communities of Practice" (Wenger, McDermott, & Snyder, 1998) and "Stewards of the Discipline" (Golde, 2006) frameworks. A community of practice is a "group of people who share a concern, a set of problems, or a passion about a topic, and who deepen their knowledge and expertise in this area by interacting on an ongoing basis" (Wenger et al., 2002, p. 4) while the stewards of the discipline framework focuses on what it means to pursue Ph.D.s. The two frameworks will be combined with gender issues from research.

These frameworks will provide a glimpse into mathematics graduate programs with high percentages of women. It is important to examine the departments because at this level math becomes a choice. Women and men either choose to study it or they choose not to. The frameworks will help us to understand why specific educational environments are appealing to large numbers of women choosing to study mathematics. If either of these frameworks is successful at helping us understand the highest level of education for women, it may be beneficial to examine mathematics classrooms at the K-12 level using this type of framework for improving women's participation in mathematics at all levels.

This poster will demonstrate how the communities of practice framework is related to the stewards of the discipline and how it will be used in the analysis of the data. The poster will also show how research-based gender issues will be superimposed with the frameworks. There will be examples from the data analysis to help exemplify the use of the two frameworks.

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# THE REGRESSION OF HIP-HOP

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*This poster presents the development and implementation of an activity in which community college students engaged with a standard algebra topic in the context of hip-hop music. Inspired by an article in The New Yorker, the authors developed an activity in which students used data to explore the question "Has Timbaland surpassed the music production record of Dr. Dre?". The poster presents the motivation for creating a context of interest to students, the collaborative effort to develop and implement the activity, student work, student interviews, and a student written article in the college newspaper. We also provide the curriculum and graphs created by students and instructors, and summaries of our reflections for other instructors who might to pursue similar work.*

## Background and Motivation

Teaching Algebra at a community college is about providing students with opportunities to develop conceptual understanding and procedural fluency, but it is also about helping them develop motivation for learning and identities as students who are capable in math. We teach in a context in which a student-centered, reform-oriented, Algebra curriculum has been developed, Lesson Study is an important part of professional development, and concerns about educational equity shape our work. In this context, we created the Hip Hop activity as a replacement activity in the Algebra curriculum, with the goal of providing students a relevant context in which to learn about lines of best fit (LOBF).

## Mathematical Context: Algebra and Linear Regression

Linear functions constitute a major focus of our curriculum, including the use of LOBF to model data. Our Algebra course also aims at general learning outcomes, including communication, problem solving, and multiple representations. In this activity, students organized "raw data" (number of songs each man produced) in tables and graphs, and they used lines of best fit to model the data. Then they predicted when Timbaland surpasses Dre, and when Timbaland's production reached twice that of Dr. Dre. In an extension activity, LOBF were used to motivate solving systems of linear equations.

## Implementation, Reflections, and Future Work

The activity was implemented in both authors' classrooms with similar results. Unlike some activities we have implemented, here students had significant knowledge of this context, and they eagerly suggested other quantities that could be used to compare the relative success of these producers. Some students struggled to organize the raw data, construct lines of best fit, and write the equations of the lines, likely because they lacked experience with such activities. However, the group work and whole class discussion components of the activity served as important sites for learning about linear equations, rates, graphs, estimation and prediction. Different groups necessarily graphed different lines, but were able to make similar predictions about future production and to interpret the slopes of lines as production rates. Future versions of the activity will include the examination of different quantities that measure relative success, as well as the investigation of other contexts and questions generated by students.

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# MATHEMATICS LITERACY WORKERS' IDENEITY WITHIN A COMMUNITY OF PRACTICE

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*This study examined the identities of mathematics literacy workers of the Young People's Project (YPP) Chicago within the context of their mathematics literacy work. YPP Chicago is a youth-led organization and mathematics initiative that is an outgrowth of and in partnership with Robert Moses' Algebra Project. Wenger's (1998) framework, communites of practice (COP), was utilized in this study. Through the data, I found that there were several identitites constructed by MLWs and CMLWs due to their engagement within this COP. CMLWs and MLWs themselves as change agents, doers of mathematics, and an authority and role model in flagway trainings.*

In the past two decades there has been a major emphasis to increase understanding of rural and urban youths in mathematics. Proficiency in mathematics has become a fundamental requirement for students taking advanced mathematics courses in schools and "more influential on income at age 24 than 30 years ago" (Carpenter & Bottoms, 2003). As urban schools work toward improving student achievement in a variety of disciplines, like mathematics, some scholars have suggested a possible avenue for improving student outcome – allowing students to take a more active role in the mathematics teaching process. Research has shown that student involvement in school change impacts student learning (Mitra, 2003), youth development and produces new identities (Mitra, 2004).

Wenger's communities of practice espouses that learning requires extensive participation within a community where its members are engaged in a set of relationships over time. Communities of practice exist all around us, they are an important part of our everyday life, and individuals are part of a number of them both implicitly and explicitly. There are three dimensions of communities of practice: (1) mutual engagement, (2) joint enterprise, and (3) shared repertoire. Moreover, there are three modes of belonging in identity formation: (1) engagement, (2) imagination, and (3) alignment. For this study, the workshop training was the context for the community of practice. Through these modes of belonging, I was interested understanding mathematics literacy workers' formed their identities. The perspectives that guided this work were as follows: (1) Learning occurs by doing or through practices in social activity (Lave & Wenger, 1991); (2) Identity formation occurs in a social context and is contingent on ones levels of engagement in that context. I engaged participants in mathematical tasks and prompts to solicit their interpretations of how they were influenced by their work in YPP overall and the workshop training in particular. This paper describes how CMLWs and MLWs came to understand their own construction of identity as a result of their community outreach work and participation in the workshop training.

Results from data revealed a wide variety of experiences that influenced how mathematics literacy workers came to their work at YPP. Each of the participants' identity were constructed differently through various aspects of the outreach work they engaged in and the way they felt it came through in their overall purpose with the Young People's Project Chicago. Each participant's identity was displayed through very specific goals they had for themselves and for others in doing the outreach work. One of the ways that identity was constructed was mathematics literacy workers defined as authoritative figures. Math literacy workers as an authority were defined by participants aspiring to be role models, capable of shaping how children saw mathematics, and demonstrating a high level of understanding among peers. Mathematics literacy workers also saw themselves as change agents, capable of changing how others perceived mathematics in their community.

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# SEX DIFFERENCES IN PERFORMANCE ON THE SAT I QUANTITATIVE SECTION

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*This theoretical paper demonstrates that gender gaps in performance on the SAT I quantitative section (about one third of a standard deviation) have little to do with college readiness, but rather are due to the misaligned content of the instrument as well as the environment in which the exam is administered. The findings of this research have far-reaching implications for the design and administration of standardized mathematics tests and in particular for the SAT, which is used for determining admission to many colleges as well as the awarding of scholarships.*

## Introduction

Performance differences between the sexes across mathematical content areas, problem types, and various instruments have been previously documented (Gallagher & Kaufman, 2005; Hyde, Fennema, & Lamon, 1990; McGraw, Lubienski, & Strutchens, 2006; Willingham & Cole, 1997) and generally attributed to biological and/or sociological influences. Also it has been demonstrated that stereotype threat can contribute to performance gaps by causing females to falter in the face of complex numerical reasoning tasks or to forego tedious calculations and opt to guess at an answer while their male counterparts are unaffected (Quinn & Spencer, 2001; Spencer, Steele, & Quinn; 1998). Since the SAT I is supposedly designed to predict the success of college freshmen, students' scores play a significant role in many institutions' admissions and scholarship decisions. However, the disparity in scores between college-bound males and females, particularly on the quantitative section, leads to inequities in terms of access. It is asserted here that the conditions under which the high-stakes SAT I is administered are conducive to stereotype threat. Further, this paper argues that the predictive validity of the SAT I quantitative section is questionable because it includes a significant amount of content that not only favors males, but also is not reflective of first-year college mathematical subject matter.

## Theoretical Perspectives

This literature review examines possible reasons for divergence in performance (biological and sociological), differences between the sexes' performance in particular mathematical content areas and problem types, differences in use of mathematical problem-solving strategies, and stereotype threat's effect on performance. The review then examines the validity of the SAT quantitative section in predicting college success.

*Possible Reasons for Divergence In Performance*

The literature shows three main lines of reasoning to explain the differences in male and female performance on mathematical tasks: biological and sociological (covered here) and psychological (covered later in the section on stereotype threat). While biological arguments have in recent literature largely been discredited, they still exert an influence in contemporary discourse and thus cannot be ignored.

*Biological*

In terms of sex differences in performance on cognitive tests, arguments have been made that males for biological reasons have a higher aptitude for mathematics than females, which would account for performance gaps on tests such as the SAT I quantitative section. Hormones may play a role in enhancing some abilities, as levels of testosterone in males have been shown to be causally linked to spatial-skills performance (Janowsky, Oviatt, & Orwoll, 1994), and female-to-male transsexuals have demonstrated marked increases in their spatial abilities after being given large amounts of testosterone (Van Goozen, Cohen-Kettenis, Gooren, Frijda, & Van De Pol, 1995).

Another line of reasoning is that males have greater variability in mathematical performance than females which leads to more males at the far right of the statistical distribution (e.g., at high levels of performance) than females. Studies conducted in the late 1970s and early 1980s with nearly 50,000 junior high students who had taken the SAT I quantitative section as part of talent searches, revealed that boys exhibited a “substantial” sex difference in mathematical reasoning, and further suggested, because the participants were adolescents, that genetic differences (e.g., deficiencies in female genetics) were the reason for this difference (Benbow & Stanley, 1980; 1983). Feingold (1992) found that 12th-grade males also exhibited more variable performance than females on the SAT I quantitative section, and suggested that this greater variability, combined with the medium effect size of central tendencies favoring males (moderate differences in average scores between the sexes), could lead to even greater effect sizes in the right tails of the ability distributions for the sexes (larger differences in average scores for high-ability males and females).

However, this domination by males at the far right of the ability distribution (e.g., at the high end of ability in mathematics performance) has diminished over the last two decades in the U.S. and internationally (Brody & Mills, 2005; Feingold, 1994; Monastersky, 2005; Willingham & Cole, 1997), which suggests that sociocultural factors may play a role in gender differences in achievement. In general, current evidence for a purely biological basis for differences in mathematical performance between the sexes has been described as “weak, at best” (Wilder, 1997, p. 14).

*Sociological*

A different argument is that sex-related differences in mathematics performance are due mainly to environmental influences. In this view, differential socialization of boys and girls via prejudicial treatment, social norms, and the expectations of parents, teachers, and fellow students, results in the divergent development of boys and girls and sex-role stereotyping (Baker & Jones, D. P., 1992; Eccles & Jacobs, 1986; Fennema & Peterson, 1985).

For example, in the area of spatial abilities (where males have a documented edge), it has been shown that taking part in related activities (especially those specific to completing spatial tasks) is essential to the development of the spatial ability, and that females tend to participate in these learning situations less often than males (Baenninger & Newcombe, 1989). Further, it has been demonstrated that when females are given the opportunity to train with visual-spatial tasks, their performance improves (Vasta, Knott, & Gaze, 1996). In addition, international studies have shown that sex differences in performance on mathematics tasks decrease as females are provided more access to advanced training and better jobs (Baker & Jones, D. P., 1992).

Therefore, the evidence suggests that sex differences in mathematics performance are not immutable. Instead, rather than a dichotomy of nature (biological) or nurture (sociological) as possible reasons for divergence in performance, it would seem that differences in performance are a result of a unique and complex blending of influences and opportunities.

*Differences between The Sexes’ Performance In Particular Mathematical Content Areas And Problem Types*

Sixth- and seventh-grade girls perform better in problems of number sense, problems of estimation, and those that involve patterns, while their male counterparts perform better in geometry and ratio/proportion, and on problems that employ figures (Lane, Wang, & Magone, 1996). These differences in performance continue through high school—for example, studies of students taking college entrance exams (e.g., the SAT and ACT) revealed that females perform better on algebra items that involve familiar algorithms or computation, while males perform better in geometry, mathematical reasoning, word problems, and items including figures, graphs, or tables (Doolittle & Cleary, 1987; Harris & Carlton, 1993).

In a review of studies up to 1985, the observed trend was that among high school students, males were somewhat better than females in solving word problems, and females were better or at least equal in computational skills (Stage, Kreinberg, Eccles, & Becker, 1985). A 1990 meta-analysis of 100 studies revealed "a slight female superiority in computation, no gender difference in understanding of concepts, and a slight male superiority in problem solving" (Hyde, Fennema, & Lamon, 1990, p. 147). Further, despite relatively similar male and female mathematical abilities at the elementary level, males' ability in geometry seems to accelerate past and beyond that of females by the end of high school (Leahy & Guo, 2001).

Differences in the ability of spatial visualization may be the key to differences in mathematical trajectories for the sexes. Males have been shown to outperform females on geometric tasks involving spatial visualization, but males and females exhibit roughly equal performance in logical reasoning ability and in the use of geometric problem-solving strategies (Battista, 1990). Males exceed females by some of the largest differences in performance on items that involve mental rotation of three-dimensional objects (Casey, Nuttal, Pezaris, & Benbow, 1995; Willingham & Cole, 1997), and also perform better on items based on measurement of two-dimensional and three-dimensional objects (perimeter, area, surface area, volume), which also involve spatial visualization (Garner & Engelhard, 1999; Li, Cohen, & Ibarra, 2004).

Higher ability in spatial visualization may also play a role in a higher ability to retrieve mathematics facts. College males (undergraduates from the 1996–97 academic year) have been shown to be faster at math-fact retrieval on an achievement test (the Computer-based Academic Assessment System, now known as the Cognitive Aptitude Assessment System) (Royer, Tronsky, Chan, Jackson, & Marchant, 1999). It has been suggested that this higher retrieval speed may be linked to males having greater flexibility in their choice of problem-solving strategies, because they have the option of employing a spatial approach that might be more appropriate for some items. On the same items, someone with limited spatial ability would be forced to rely upon a perhaps less effective approach (Casey et al., 1995).

Another possibility to explain the difference in male and female mathematics performance is that math-fact retrieval ability itself plays a pivotal role in solving more complex problems that require more cognitive load. That is, those who can retrieve information more quickly or more effectively would perform better on items requiring recall of a mixture of concepts and/or procedures (Paek, 2002; Royer et al., 1999). Further, it has been demonstrated that on SAT I quantitative items, males take less time to solve problems than females (Paek, 2002). But, regardless of whether math-fact retrieval ability is an effect that favors males (due to higher ability in spatial visualization) or is the cause for their better performance on certain mathematics tasks, it certainly would provide an advantage on a timed test.

*Differences in Use Of Mathematical Problem-Solving Strategies*

A study of high school students working problems from the quantitative section of the SAT I (Gallagher, 1992) found that females were more apt to rely on standard algorithms traditionally presented in classrooms, while males were more inclined to use insight. Females demonstrated less prior knowledge than males and used fewer mathematical strategies on SAT mathematics items, even when both groups had similar backgrounds in terms of mathematics courses taken and grades received (Byrnes & Takahira, 1993; Paek, 2002). It has also been shown that on SAT mathematics items, males use unconventional strategies (including logic, estimation, or insight) significantly more often than females, and females use conventional strategies (such as an algorithm, assigning values to variables, or "plugging in" numbers to a formula) significantly more often than males (Gallagher & De Lisi, 1994; Paek, 2002).

These findings would certainly tend to suggest that males would have the advantage in situations where problems cannot be solved by the more traditional strategies presented in school, and would help to explain why females generally have better grades in school but underperform on tests such as the SAT I quantitative section. One study that used course grades rather than standardized tests to measure differences in mathematics achievement between the sexes noted that "when differences are found, they almost always favor girls, and these differences are quite consistent across samples of varying selectivity for junior high through university mathematics courses" (Kimball, 1989, p.199). The study's findings, which are contrary to male and female achievement patterns on the SAT I quantitative section, suggest that situational variables (such as the testing environment or method of administration) might play a role in performance gaps between males and females.

*Stereotype Threat's Effect on Performance*

A landmark study that introduced the term *stereotype threat* (Steele & Aronson, 1995) suggests a plausible explanation for some of the differences in male and female mathematics achievement when measured by classroom test grades and by standardized tests. Stereotype threat is defined in the study as a feeling of "being at risk of confirming, as self-characteristic, a negative stereotype about one's group" and further, stereotype threat "may interfere with the intellectual functioning of these students [affected by stereotype threat], particularly during standardized tests" (Steele & Aronson, 1995, p. 797). The initial study focused on black students who were found to be burdened by the stereotype of having less ability than white students in terms of general intellectual aptitude.

Steele and Aronson's findings are relevant to this study of performance gaps between the sexes on standardized mathematics tests, because females may also deal with stereotype threat. In particular, females are susceptible to the stereotype threat that they are publicly perceived to be less able in mathematics than males. Steele and Aronson note the broad applicability of stereotype threat: "This threat can befall anyone with a group identity about which some negative stereotype exists, and for the person to be threatened in this way, he [or she] need not even believe this stereotype" (Steele & Aronson, 1995, p. 798).

In a later experiment, the effects of stereotype threat on female mathematics performance was confirmed among college females who excelled at math and identified strongly with the subject (Spencer et al., 1998). In that study, on an extremely difficult test (composed from the advanced GRE in mathematics), females underperformed compared to males when informed before they took the test of historic sex differences in test performance, but females achieved as well as males when told that the test was gender-insensitive. The results of the experiment contradicted the hypothesis of female genetic deficiency suggested previously by Benbow and Stanley in the early 1980s (Spencer et al., 1998). Additionally, the experiment's findings confirmed what Steele had concluded in an earlier study: that “stereotype threat may be a possible source of bias in standardized tests, a bias that arises not from item content but from group differences in the threat that societal stereotypes attach to test performance” (Steele, 1997, p. 622).

*The Effects of Priming Stereotype Threat*

Stereotype threat can be primed in various ways. One is the condition of *evaluative scrutiny.* In this situation test-takers know their results will be available to others, such as parents, teachers, administrators, or colleges. On a standardized test such as the SAT, some degree of evaluative scrutiny is always present. Another way to induce stereotype threat is the condition of *identity salience.* This condition can be thought of as “the likelihood that the identity will be invoked in diverse situations” (Hogg, Terry, & White, 1995, p. 257). Participating in a mixed testing environment (for example, males and females together) or having to identify one’s sex prior to a standardized test, as is the norm, can invoke identity salience.

The effects of these two causes of identity salience have been measured in studies conducted by the Educational Testing Service (commonly known as ETS) and the College Board. In one experiment, researchers administered the mathematics section of the Graduate Record Exam general test to males and females on an individual basis (GRE Board, 1999). In this study, the gap in scores between males and females was less than half of the gap from the regular administration of the GRE general test ($d$ =.40 versus $d$ =.97) that same year, in which students tested in a mixed environment.

Another study measured the effect on performance of identifying one’s sex before a test (College Board, 1998a). The experiment focused on students taking the Advanced Placement Calculus AB exam. For those who indicated their sex on a standard background information sheet *before* the test, the performance gap effect size ($d$ = .41) between males and females was more than triple that for the males and females who identified their sex *after* the test ($d$ = .12). This study, along with the ETS study, clearly demonstrates that the condition of identity salience, while subtle in nature, can play a major role in inducing stereotype threat.

*The SAT and Its Present Validity in Predicting College Success*

Approximately 92 % of four-year institutions require SAT/ACT scores from potential students and 75% routinely use these scores in making decisions on admissions (College Board, 2002). However, in Hyde, Fennema, and Lamon’s 1990 meta-analysis of gender differences in mathematics performance on various instruments (100 studies total), the effect size favoring males was by far the greatest on the SAT I, and this performance gap persists today.

The College Board has offered various reasons for differences in the performance of males and females on the SAT I quantitative section. Essentially, the College Board suggests that females are generally less prepared than males in terms of mathematics (College Board, 1998b). However, the validity of the exam is suspect, in terms of gender, as it has been demonstrated that the SAT I quantitative section consistently under-predicts the college success of females compared to males. Studies have shown that while males consistently score higher (a third of a standard deviation) than females with similar mathematical backgrounds on the SAT I quantitative section, females perform on par with males, later, in like college mathematics courses (Bridgeman and Wendler, 1991; Wainer and Steinberg, 1992).

One reason for the lack of predictive validity with the SAT I quantitative section may lie with the content of the exam, which is seemingly not well-aligned with college readiness standards. Approximately 40% of the test is devoted to geometry, measurement, and data analysis items (Achieve, 2007), content strands where 12th-grade males tend to outperform their female counterparts on the NAEP (McGraw et al., 2006). However, the typical core curriculum mathematics course required for students at universities is college algebra or its equivalent, a mathematical area where women perform on par with men (McGraw et al., 2006). Further, a recent national curriculum survey of post-secondary mathematics instructors (ACT, 2007) revealed that algebra skills were ranked as the most important, while measurement, geometry, and probability/statistics skills were considered least important for success in college.

Also, in another study on college readiness (AAU & Pew Charitable Trust, 2003), the content strands of geometry and measurement accounted for only seven out of 81 mathematics standards for success in college (the rest covered by algebra) and probability/statistics was not considered a necessary prerequisite for entry-level mathematics courses. Therefore, since the SAT I quantitative section contains a large percentage of items that are for the most part considered irrelevant to college freshman success and tend to favor males over females, the fact that the exam over-predicts the future success of males while under-predicting the same for females should not be unexpected.

## Conclusion

Evidence to date suggests that performance gaps between males and females on standardized mathematics tests, such as the SAT I quantitative section, are influenced by a complex mix of sociological, psychological, and (to a lesser extent) biological factors unique to each individual. However, the typical core curriculum mathematics course required by universities is college algebra or its equivalent, a content area where females perform on par with males (according to NAEP data). Yet, a large percentage of the SAT I quantitative section is devoted to other content areas which are not essential to freshmen success and favor males (also according to NAEP data), which in combination with a testing environment conducive to stereotype threat potentially lead to existing sex differences in performance on the exam.

Since high-stakes decisions involving admissions to colleges and awarding of scholarships are often based on the results of this single testing instrument, the College Board should reexamine the content of the exam to ensure that it is aligned with and accurately reflects the topics necessary for early college success in mathematics. Further, in order to lessen the effects of stereotype threat, ETS should consider shifting the administration of the SAT I to individualized (or same sex) testing and moving identification of one’s sex to after or well in advance of the exam.

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# PROBLEM-SOLVING STRATEGIES OF GIRLS AND BOYS IN SINGLE-SEX MATHEMATICS CLASSROOMS

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*This research report examines patterns in middle-grades boys' and girls' written problem solving strategies for a mathematical task involving proportional reasoning. The students participating in this study attend a coeducational charter middle school with single-sex classrooms. 119 6th grade students' responses are analyzed by gender according to the solution strategy they used to arrive at their final response to the task. 42.7% of participating girls' responses are classified as either using a purely multiplicative strategy, as evidencing emergent proportional reasoning, or as evidencing mature proportional reasoning. 60.5% of participating boys' responses are thus classified. 52.5% of girls are classified as relying on a purely additive strategy for their final response, as compared with 29.5% of boys.*

## Objectives/Purposes of this Study

This research report examines patterns in middle-grades boys' and girls' problem solving strategies for a mathematical task involving proportional reasoning. The students participating in this study attend a coeducational charter school with single-sex classrooms. All of the students in this study attended 6th, 7th, or 8th grade mathematics classes (6th grade mathematics, pre-algebra, or algebra) in single-sex classrooms. Instruction in single-sex classrooms is a mandate of the charter for this school, formed by parents seeking an alternative to local public middle schools. The research reported in this paper is part of a more broad investigation of the impacts of single-sex middle-grades education on mathematics learning, mathematics achievement, student academic self-concept, and parent, teacher, and student perspectives of the experience of being educated at this charter school. For this paper, we are looking specifically at the written and, in some cases, video-recorded solution strategies of girls and boys as they attempt a mathematical task centered on proportional reasoning.

## Perspectives/Theoretical Framework

Equity issues in mathematics education are currently receiving much attention, as evidenced in part by the NCTM Board of Directors' emphasis on equity as a strategic priority and by the upcoming special issue of JRME on equity in mathematics education. Of the many aspects of equity to consider, this research report proposal contributes to understandings in mathematics education about gender and mathematical thinking. Much of the work in the mathematics education research community on gender and mathematics thinking focuses on a narrow, restrictive view of mathematics achievement by using mathematical performance on standardized tests to compare girls and boys. While such studies can provide insight about certain questions regarding girls' and boys' mathematical performance, one of the problematic issues with basing findings on students' mathematical performance on a standardized test is that results from such studies are easily (mis-)extrapolated to include implications about not just performance, but achievement, ability, and talent. In addition, findings resting on outcomes from an assessment in which answers are coded either right or wrong disregards students' thinking about the mathematical tasks on the assessment.

## Research Questions

1. How do girls' strategies for solving a mathematical task compare to other girls' strategies?
2. How do boys' strategies for solving a mathematical task compare to other boys' strategies?
3. How do the girls' strategies for solving a mathematical task compare to boys' strategies?

## Modes of Inquiry

*Participants*

The initial phase of this research involved selecting a sample of students to participate in a videoed task-based interview. 23 students (12 girls, 11 boys) were selected for this task, in which students attempted to solve and explain their strategies for solving a mathematical task involving proportional reasoning. Following these task-based interviews, a total of 162 students, grades 6-8, participated in writing responses to the mathematical task. Table 1 shows the distribution of these 162 students across gender and grade level. The reason there are so many $6^{th}$ graders relative to the $7^{th}$ and $8^{th}$ grade students is that the school has decided on a growth process whereby new classes are added at the $6^{th}$ grade level.

Table 1. *School Student Population by Grade and Gender*

| | Girls | Boys | Total |
|---|---|---|---|
| $6^{th}$ grade | 61 | 58 | 119 |
| $7^{th}$/$8^{th}$ grade | 24 | 19 | 43 |
| Total | 85 | 77 | 162 |

For the school's first year, there were about 20 $6^{th}$ graders, 20 $7^{th}$ graders, and 20 $8^{th}$ graders. At the time of our data collection, there were still about 20 $7^{th}$ graders and 20 $8^{th}$ graders, but a larger class of $6^{th}$ graders had been admitted. Because there are so many more $6^{th}$ graders than $7^{th}$ and $8^{th}$ graders, the data in this paper concentrate on $6^{th}$ grade students.

*Contexts*

Students participating in this research attend a public coeducational charter school in the southeast. Classes at this charter school are conducted in single-sex classrooms (the one exception is $8^{th}$ grade geometry, which is coeducational and offered before school so as to not violate the school's charter). The school currently has approximately 220 students in grades 6 through 8, and class sizes are at most 20, with class sizes from 15-20 being habitual. Thus, this charter middle school is atypical from local public middle schools in both school and class size. Further, this charter middle school has a homogenous student population both in terms of race and SES; more than 90% of the students are white and no students qualify for free and reduced lunches. Though this school is not representative of local public schools, it does offer a rare glimpse into single-sex education in non-private settings in the U.S.

*Data Collection*

The data of primary focus for this paper (written responses to a mathematical task) were collected in May, 2008, near the end of the academic year. Students participating in a task-based interview (in October, 2007) were selected by their mathematics teachers; researchers asked teachers to select several students in each of their classes. Upon completion of the initial analysis of these interviews, researchers distributed the task in written form and asked all students to respond to the task. Participating students responded to this problem:

> The capacity of an elevator is either 20 children or 15 adults. If 12 children are currently in the elevator, how many adults can still get in? (Johnson & Herr, 2001)

This particular problem was chosen for several reasons: first, this problem addresses one of the main ‘big ideas’ of middle grades mathematics, proportional reasoning. Second, this problem allows for many distinct solution strategies. Third, students only using the traditional algorithm of setting up two equal ratios and cross-multiplying to solve for the unknown quantity will not arrive at the correct response if they use only the numbers given in the problem. This is because the information given in the problem relates to the full capacity of the elevator and the capacity that is occupied, but the problem asks for the capacity still available for use. The fact that, when students only use numbers given in the problem with a traditional algorithm, they do not arrive at the correct answer facilitates analysis, as discussed below. Fourth, the context of the problem is one to which the participating students can all relate and which has a high degree of gender neutrality (no names are used, no one in the problem is referred to by gender, and both women and men frequently use elevators).

## Analysis

The student responses were analyzed in two ways: responses were first analyzed separately by gender and responses were sorted according to the students’ final answer. There were some students who did not have a clear final answer, and there were some students who offered an answer with no written strategy. The remaining task attempts were put into categories coinciding with the students’ final answer. For instance, all the 6$^{th}$ grade girls whose final response was 6 adults were placed into the same category. Once the range of final answers from 6$^{th}$ grade girls was established, each of the strategies leading to these final answers was examined, with researchers tracing students’ thinking according to the sequencing of the work on the page and the flow of the mathematical computations. In several cases, there were also illustrations which frequently gave insight into students’ thought processes. During this phase of analysis, researchers articulated students’ strategies for each response and noted similarities and differences in students’ strategies.

The second level of analysis occurred across final answers, and focused on the spectrum of student solution strategy from purely additive reasoning to appropriate use of proportional reasoning. This spectrum is based on Lamon’s (2005) work on levels of sophistication of student solution strategies. While we do not explicitly place students’ work samples in different levels of sophistication, we do place them on a spectrum from additive to multiplicative strategies, using a four-part scale: purely additive strategies, emergent proportional reasoning, purely multiplicative strategies, and mature proportional reasoning. This scale is admittedly problematic in a sense because the scale uses both strategies and reasoning as sorting variables. The rationale for this is that a student’s use of a multiplicative strategy (such as setting up a proportion and cross multiplying) is not necessarily indicative of proportional reasoning. Such students could have simply followed a memorized procedure. Thus, only those papers with clearly articulated reasoning and clear strategies are placed in the categories of emergent proportional reasoning or mature proportional reasoning.

## Results

Tables 2 and 3 show the results of the analysis of final responses by gender. Of the 61 $6^{th}$ grade girls, there were 11 different final responses. Of the 58 $6^{th}$ grade boys, there were also 11 different final responses. Looking at the $6^{th}$ grade as a whole, of the 119 students, there were 15 different final responses to the question.

Table 2. *Distribution of $6^{th}$ Grade Girls' Responses (n = 61)*

| Final Response (# of adults) | Number of Each Response | | Strategies for each response |
|---|---|---|---|
| No work/rationale | 4 | (6.6%) | 1 attempt at a multiplicative strategy, 3 papers have no work |
| 8 or 3 | 1 | (1.6%) | 1, an additive strategy |
| 4.6 | 1 | (1.6%) | 1, attempts a multiplicative strategy |
| 23 | 6 | (9.8%) | Additive strategies; several multiplicative strategies attempted and rejected |
| 9 | 9 | (14.8%) | All multiplicative strategies |
| 8 | 7 | (11.5%) | Additive strategies, although several multiplicative strategies attempted |
| 7 | 10 | (16.4%) | All additive strategies |
| 6 | 5 | (8.2%) | All strategies used proportions |
| 5 | 2 | (3.3%) | 2 purely additive strategies |
| 4 | 7 | (11.5%) | 1 purely additive strategy; 6 strategies involving multiplicative reasoning; 2 strategies using a 2:1 ratio |
| 3 | 7 | (11.5%) | Additive strategies; several multiplicative strategies attempted and rejected |
| 2 | 2 | (3.3%) | 2 proportions attempted |

Table 3. *Distribution of 6th Grade Boys' Responses (n = 58)*

| Final Response | Number of Each Response | | Strategies for Each Response |
|---|---|---|---|
| Unclear | 4 | (6.9%) | Mainly strategies involving multiplication and division; some additive strategies |
| No work/rationale | 2 | (3.4%) | |
| 25 | 1 | (1.7%) | 1 multiplicative strategy |
| 13 | 1 | (1.7%) | 1 additive strategy |
| 9 | 4 | (6.9%) | 4 multiplicative strategies |
| 8 | 5 | (8.6%) | 3 additive-only strategies |
| 7 | 7 | (12.1%) | 5 additive-only strategies |
| 6 | 10 | (17.2%) | Multiplicative strategies; two strategies used LCM (15,20); one strategy assigned a weight for children |
| 5 | 4 | (6.9%) | 1 multiplicative strategy; 4 additive strategies |
| 4 | 8 | (13.8%) | 4 strategies using a 2:1 ratio |
| 3 | 9 | (15.5%) | 5 multiplicative strategies, 3 additive strategies |
| 2 | 2 | (3.4%) | 1 strategy using a 3:1 ratio |
| 1.666… | 1 | (1.7%) | Divided 20 by 12 |

In Tables 2 and 3, the category of no work/rationale means that the student states a response but there is no mathematical work shown and no written rationale for the response. This category *does not* include those students who show mathematical work but do not give a written rationale.

Table 4. *Strategy Classification of Girls and Boys*

| Response Classification | Girls (n = 61)* | | Boys (n = 58)* | |
|---|---|---|---|---|
| No work/rationale | 3 | (4.9) | 4 | (6.6) |
| Purely additive | 32 | (52.5) | 18 | (29.5) |
| Emergent Proportional Reasoning | 12 | (19.7) | 18 | (29.5) |
| Purely multiplicative | 9 | (14.8) | 13 | (22.4) |
| Mature Proportional Reasoning | 5 | (8.2) | 5 | (8.6) |

*Quantities in parentheses indicate percentages.

In Table 4, the category 'No work/rationale' corresponds to the same category for Tables 2 and 3. That is, these papers gave a response but showed no work and stated no rationale for their response. The category 'Purely additive' refers to the strategy the student relied on to get their final response. As will be discussed in the presentation, there are several students in this category that attempted strategies involving multiplication and/or division, but ultimately resorted to strategies using only addition/subtraction for their final response. 'Emergent Proportional Reasoning' papers were classified as such because of their images, their words, and their use of ratios. These students indicated that they were aware of the necessity for ratios but were not quite sure how to find and/or use meaningful ratios for this problem. Several of these students assigned arbitrary ratios (like 2:1) and solved the problem using this assigned ratio. Many of these students indicated uncertainty in their response, but none resorted to purely additive strategies. 'Purely multiplicative' strategies are those whose responses include setting up a proportion and cross-multiplying to solve, or calculating a ratio and using multiplication to get a response. Many of these students got a response of 9 adults. Those students whose work is classified as 'Mature Proportional Reasoning' indicated an understanding of the process they were doing and what their results meant; further, their process was appropriate to responding to the question—that is, it made sense mathematically. This category of 'mature' does not mean that this is the highest level of proportional reasoning possible; rather, it indicates that, from the student work samples in this study, these responses were making sense of the problem.

## Discussion/Conclusion

There are many ways to analyze student work samples. In addition to the analyses above, we could also group student work according to the specific strategy employed, sorting those students who assigned a ratio of 2:1, for instance, in one group and those students who subtracted 12 from 15 in another. We could also analyze according to the strength of the rationale or reasoning indicated; this could help with the issue of hidden non-reasoning in the purely multiplicative category in Table 4. The problem with this type of analysis is that very few papers, particularly boys' papers, used words or other indications of their reasoning. On the other hand, many girls who used purely additive strategies still explained their thought process in words. In our presentation, we will share several examples of both girls' and boys' work for each category in Table 4.

The selection of the task became important to the types of analyses we could do and the insights we could get from those analyses. The fact that students who set up a proportion and solve (usually getting a response of 9 adults) do not get the correct result helped us to distinguish between students who were reasoning about whether their result made sense and those who were not. If you figure that more than half of the capacity of the elevator is already taken up, then it becomes clear that fewer than half of 15 (7 1/2) adults can still get on. That is one reason why students using multiplicative strategies are not necessarily assigned a category indicating proportional reasoning. Indeed, several student papers classified as emergent proportional reasoning evidenced a higher level of reasoning than many papers classified as multiplicative. Thus, the results in Table 4 should not be viewed as a spectrum from lower sophistication levels to higher sophistication levels.

Returning to our original questions for this study, comparisons of girls' and boys' strategies, we can see that 57.4% of the girls either showed no work or used purely additive strategies to arrive at their final response. By contrast, about 36.1% of boys either showed no work or relied on purely additive strategies for their final response. Some of the students who ultimately used additive strategies also attempted a multiplicative strategy but abandoned it; even still, this discrepancy between the portion of girls relying on purely additive strategies for their final response and the portion of boys doing so is the most striking discrepancy in our results.

Though more comparable, there is still a discrepancy between the percent of girls classified as emergent or mature in proportional reasoning and that of boys (girls had 27.9% so classified as compared to 38.1% of boys). Of the ten papers classified as Mature Proportional Reasoning, only 2, both of whom were girls, set up a proportion to solve the problem.

Findings from this study, involving student work samples from just one problem, are clearly not generalizable. We can make some inferences for these students for this mathematical context, but applying results from this study more broadly is problematic. Our results point to the importance of mathematical communication and having these students articulate their reasoning and their rationales. We say this because students clearly indicating some degree of proportional reasoning were also thinking about the problem in reasoned ways. Further, it could be that those strategies classified as purely multiplicative involved some degree of reasoning beyond following an algorithm; just because this reasoning was not evidenced on paper does not necessarily mean that the reasoning was not present at all.

Many questions remain: why did so many more girls use additive strategies to get their final response? What level of reasoning was evidenced in students' discarded strategies? How was the use of imagery related to gender and to reasoning level? What are the strengths and limitations of this kind of methodology for understanding students' reasoning more broadly than student standardized performance? We will share our reflections on these questions and invite discussion on these issues during our presentation.

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# THE ROLE OF ADVISING IN THE ACADEMIC DECISIONS OF WOMEN MATHEMATICS MAJORS AT A LIBERAL ARTS COLLEGE

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*Research suggests that the support of "significant others" such as family, teachers, and professors plays a large role in women's participation in non-traditionally female careers. Using the method of case studies, we discuss the experiences of three undergraduate women mathematics majors at a small liberal arts college and describe the role "significant others" played in the development of their mathematics identities and their choices as undergraduates. Furthermore, we emphasize the role advising by college faculty played in these women's knowledge of career options and how this has influenced their future career plans.*

## Introduction

Research shows that the support of "significant others" plays a large role in women's participation in non-traditionally female careers. Women who are successful in mathematics and who pursue mathematical careers often cite having a family member who served as a role-model in mathematics or feel that they were directly encouraged in mathematics by family members or teachers beginning at a relatively young age (Fox & Soller, 2001; Zelden & Pajares, 2000). Furthermore, women who choose to leave the field of mathematics often describe a lack of mentoring or advising from their university professors (Herzig, 2004; Stage & Maple, 1996).

It is likely that the feedback women receive from significant people in their lives plays a rolein the development of their mathematics identity, i.e. their tendency to identify as a "mathematics person." Carlone and Johnson (2007) have developed a framework in which they use the concept of one's science identity as a lens for interpreting the science experiences and choices made by successful women of color in science. These scholars view one's science identity as consisting of three overlapping dimensions: *competence*, *performance*, and *recognition*. The dimension referred to as *recognition* is developed from both one recognizing herself, as well as one being recognized by others, as a science person. These scholars' work suggests that "recognition by others" played a critical role in these women's science identities and career paths.

It is clear from the literature that the encouragement and recognition of others plays an important role in the choices and career decisions of women in non-traditionally female careers. In this paper we describe a case study of three undergraduate women mathematics students at a small liberal arts college in the Midwest. We discuss how support from their professors and other significant people in their lives have influenced the development of their mathematics identities and their choices as undergraduates. We further note the role advising played in their knowledge of career options and how this has influenced their future career plans.

The study described in this paper is part of a larger study being conducted to look at what motivates undergraduate women mathematics majors to choose to earn a Bachelor's degree in mathematics and what factors influences their future career choices. Although participants from the larger study are from universities of differing types, we found that the emphasis on faculty advising at the small liberal arts school was worthy of closer investigation. As such, this paper focuses specifically on these women's experiences.

## Method

The participants in this study were undergraduate women mathematics majors at a small liberal arts college in the Midwest. All of the participants had either junior or senior class standing at the time of the interviews and were earning non-teaching, mathematics degrees. Given the small size of the college, all students at this school fitting these criteria were invited to participate in this study. We had approximately a 50% participation rate.

The main source of data for this study was a series of three in-depth, phenomologically-based interviews with each of the participants. "In this approach interviewers use, primarily, open-ended questions. Their major task is to build upon and explore their participants' responses to those questions" (Seidman, 1998, p. 9). We followed the Three-Interview Series protocol as suggested by Seidman (1998). During the first interview, we collected data on the participant's pre-college mathematics experiences. The second interview consisted of learning about the participant's mathematical experiences while in college. During the third interview, we asked the participants to reflect on their past experiences with mathematics and explored how these experiences may be influencing their future career goals. In addition to the data collected through the interviews, we also collected information about these students' college grades and ACT/SAT scores. The purpose of collecting this data was to compare their perceived academic achievement with their actual academic records. Previous research suggests that women frequently underestimate their accomplishments (Fox & Soller, 2001).

All of the interviews were audio and video-recorded. After the interviews were transcribed, each of the two researchers independently analyzed the transcripts and created profiles for the participants (Seidman, 1998). Once the profiles were created, we noted common themes emerging from the data and developed a coding scheme grounded in the data (Strauss & Corbin, 1998). This collaborative method allowed for researcher triangulation. Once the codes were agreed upon, the primary investigator coded and analyzed all of the data with respect to these themes. Because only one researcher conducted the final coding, consistency in coding between researchers is not a concern.

## Results

In this paper we will mostly focus on the role that advising played (or did not play) in these women's academic and career decisions. Furthermore, we will be utilizing the word "advising" rather loosely. The term "advising" may reference advice that these students received from family members, teachers, professors or other significant people in their lives. Moreover, advising may be done either formally or informally through casual interactions.

At this particular liberal arts college, formal advising is taken quite seriously. All incoming freshmen are assigned an academic advisor. If the student has chosen a major prior to beginning his or her college career, the student is assigned an advisor in that department. Otherwise, the student is encouraged to select one or two academic areas that they might have an interest in and the administration then assigns the student to an advisor in one of these disciplines. The role of this advisor is critical because each semester before the students are able to sign up for their classes for the following semester, they must meet with their advisor and have the advisor sign off on their choice of courses for the following semester. This method ensures that students are receiving proper advising with respect to their course selections.

We will now describe each student's case separately, focusing specifically on the role that significant others in their lives played with respect to the development of their mathematics identities, their choices with regards to mathematics, and their future career plans.

*The Case of Paige*

Paige developed a strong mathematics identity at a young age. As early as freshman year of high school, Paige knew that she would major in mathematics. She claims that she probably knew it subconsciously even before that. Mathematics was always considered "her subject" and she knew mathematics was what she wanted to do. She admits, however, "I didn't know what I was going to do with it, but I knew I was going to be a math major."

Many teachers encouraged Paige in mathematics through the years and in high school Paige remembers being the "math person" amongst her group of friends, but the person who clearly contributed the most to her mathematics identity was her mother. Paige recalls that her earliest memory of mathematics was her mother using candy to help her learn arithmetic. As the years went on, her mother continued to support her and encourage her in mathematics. Her mother felt that she, herself, was strong in mathematics and was happy to see her daughter choose such a field because she felt Paige would have more opportunities going into a non-traditional discipline for women. Paige claims that she never really received discouragement from doing mathematics from anyone, and jokes, "I'm pretty sure my mom would hurt them if they [did]."

Because Paige was so far advanced in mathematics, she had completed Calculus I by the end of her sophomore year in high school. For her junior and senior years of high school she commuted to the local community college for her mathematics courses. Her instructor at the community college was very supportive and even encouraged her one semester to take both Calculus III and Differential Equations at the same time.

When Paige began college as a freshman, she was already taking junior level mathematics courses. Both she and her mother were a little concerned about her taking such advanced courses as a freshman, but she had already completed all the freshman and sophomore level courses by then. During her first semester at college, she took a 300-level Linear Algebra course. She remembers feeling behind because she had not taken the "introduction to proofs" course yet, which was only offered during the winter semester, and the instructor did a number of proofs in the Linear Algebra class. Though this was a difficult transition for her, she remembers the professor never treating her like she was unable do it. Because this college is a small school, all of the mathematics professors know who the students are. Therefore, the professor knew that she was simply a freshman in a junior-level class and that she had just not taken the "proofs" course yet. He provided encouragement and positive feedback to her to ensure her that this was the source of her difficulties in the course.

Paige feels like her experience as a mathematics major was a positive one. She enjoyed her mathematics classes more than any of her other classes, for the most part, even though she claims she spent much longer doing homework for her mathematics classes than her other classes. Through her time in college, Paige decided that she preferred applied mathematics to pure mathematics, and she especially enjoyed statistics. As a result, Paige has decided to go to graduate school to earn a Master's degree in statistics. At the time of the interviews, she had been accepted to a number of programs but had not yet decided which program she would choose. After earning her Master's degree, Paige plans to find a job working for a company as a statistical consultant.

Initially, Paige had no idea of what to do with her degree in mathematics. She admits, "I know there's a bunch of … hidden jobs where it doesn't say Bachelor's in math or whatever on the requirements… but I don't know what I would actually be eligible for." Once she decided that she wanted a career in statistics, Paige started looking into possible jobs by searching the internet and claims that she realized that to get a good job in statistics, she would need to get a Master's degree.

At the beginning of her junior year, Paige says that she was encouraged by her professors to apply for REU programs for that summer. An REU (Research Experiences for Undergraduates) is an NSF-funded program in which undergraduate students are funded to spend their summer doing research. There are currently 57 REU programs in the mathematical sciences and these programs tend to be very competitive. Paige ended up participating in an REU program in statistics. Her particular project was done in collaboration with a company to help them solve a real-life problem, which allowed Paige to contribute to mathematics in a real way. Prior to this, Paige had already decided that she wanted to leave academia and be in the "real world" and she felt like this REU experience really gave her a taste of this. The experience helped her to decide that working for a statistical company would be a good career for her. She claims, "I didn't know that you could do statistical modeling before I did this."

In addition to being encouraged to participate in an REU by the mathematics faculty, Paige also has received substantial encouragement and support from her advisor in applying for graduate school. He advised her to apply to a wide range of programs, including some really prestigious ones. This made Paige feel really good because she said that even though he was encouraging her to have a "back up," by suggesting that she apply to some competitive programs too, it demonstrated to her that he saw her as being a strong candidate.

Although Paige spoke of both advantages and disadvantages of attending a small college, she feels like some of the advantages were that she got to know her professors really well. She says that she talks with her mathematics professors at least once a week, but not about mathematics. They talk more about what REU or graduate programs that she has gotten into. She feels like the professors have shown a sincere interest in her. The mathematics faculty also host a dinner once a year for all the mathematics majors, but Paige has always had a conflict and has not been able to attend. Paige says that she liked her experiences as a mathematics major on this campus and really appreciated that everyone in the mathematics department knew her. She further makes a point to emphasize that they did not just know *of* her, but they knew her personality, too.

*The Case of Mandy*

Unlike Paige, Mandy did not decide to be a mathematics major until sometime during college. In all actuality, Mandy never really consciously made the decision to major in mathematics; she sort of just "fell into a math major," as she puts it. When beginning college, Mandy did not know what to major in. Because she had expressed an interest in mathematics, the college gave her an advisor in the mathematics department. Each semester her advisor encouraged her to take more mathematics (and science) classes. By the end of her sophomore year, she was required to choose a major. She had never expected to major in mathematics, but at this point she had already completed so many courses for that degree that she decided that it just seemed natural for her to choose that as her major.

Though no one in her family has earned a degree in mathematics before, Mandy's family has definitely shown an interest in working with numbers, since many of her family members are accountants or engineers. Mandy claims that often at her parents' house, she, her sisters, and her parents all fight over the puzzles in the newspaper. She says they all enjoy Sudoku and jigsaw puzzles as well. Despite her family's interest in such things, Mandy credits her interest in mathematics to the teachers that she had in high school, especially her A.P. Calculus teacher. During her last two years of high school is when Mandy feels like her interest in mathematics really began to develop.

Mandy took Calculus II her first semester in college. This class went really well for her since she had learned most of the material in her A.P. Calculus class in high school. She recalls, "I think it was helpful and kind of kept my confidence up in my math ability." During her sophomore year of college, Mandy had a similar experience to Paige in that she took a proof-based course in cryptography prior to having taken the "introduction to proofs" course. She remembers studying with a friend a lot in that class and they both worked with the professor a lot since neither had learned how to write proofs before. Mandy recalls that the professor knew they were both in this predicament and feels like he went out of his way to help the two of them be successful in that class.

Mandy has had both good and bad experiences with some of her mathematics classes and admits to sometimes thinking about switching her major. At the same time, she figures that she is so close to finishing her degree in mathematics, she may as well just keep going. She says though that sometimes she is not even sure she likes mathematics anymore. She explains that one reason she feels this way is because she does not know what she can do with a major in mathematics. She believes that the biggest challenge of being a mathematics major is not knowing what to do with the degree afterwards. Mandy feels like with other disciplines it is more clear-cut; all of her friends know what they are going to do with their degrees after they graduate. She says it is also hard because when people ask her what she is going to do with her degree, she has to tell them that she still does not know.

Although this is frustrating to her, Mandy was completing her junior year at the time of the interviews, and is hopeful that she will figure out what she wants to do when the time comes. Her advisor suggested that she apply to REUs to get ideas of what she might like to do. She applied to four of them. At the time of the interviews, she had heard back from two of them and had been accepted to both. She has decided to participate in an REU that has a focus in using mathematics to model biology. At this point, Mandy does know that she does not want to focus on pure mathematics, but would rather apply mathematics to other disciplines, so she feels like this program will be a good fit for her. Mandy recalls that when she told her advisor in the mathematics department that she got into one of the REUs, he told all of the other mathematics professors the news. She was amazed at how excited he was for her and remembers thinking, "Oh wow, dude, they're awfully excited."

Being accepted into this REU has really made a difference in Mandy's mathematics identity. When asked, Mandy says that she would not say that she is "good" at mathematics, just "okay." She admits though that now that she has gotten into some REUs, it has made her feel more confident with her mathematics abilities. "I'm not sure if math is for me; if I should continue with it. But I guess like having something to do for the summer makes me feel like I can succeed, so I think that's helping a lot." She also says that she thinks the mathematics professors at her college would say that she is good at mathematics.

Mandy says that she really likes the professors in the mathematics department and claims that this is part of the reason she decided to major in mathematics. She says the professors are all very approachable and personable and that they all know her name. This makes her feel more comfortable to ask questions in class and to ask them for letters of recommendation. They give her advice on what classes to take and have encouraged her to apply for REUs for the summer. Mandy says that she drops by the mathematics faculty offices at least once a week and peeks her head in just to say "hi" to the professors. She also thinks it is really neat that her advisor sometimes tells her stories from when he was an undergraduate and that once she went to his house for a game-night as an event through her church.

*The Case of Nicole*

Nicole became interested in mathematics as early as elementary school. She claims that none of her K-12 teachers were ever really an influence on her. Rather, Nicole believes that her father, a high school mathematics teacher, played the largest role in her interest in mathematics. She remembers him bringing home puzzles and logic games for her to do as a child. Nicole is very close with her father and considers him to be her biggest role model. This gave her a close connection to mathematics at a young age.

By fifth grade, Nicole was put a year ahead in mathematics and by eighth grade she was being bussed to the high school to take a high school geometry course. During her senior year of high school, Nicole took Calculus I and II at the local college. She remembers this being the first time mathematics was ever difficult for her. She explains though that she did not think of this as a bad experience, but simply as an adjustment. It was actually during her senior year of high school that Nicole made the decision to major in mathematics. Prior to that, she had thought about majoring in one of the sciences, like biology. When she began taking mathematics classes at the college level, however, she really enjoyed them. She felt like she received a lot of encouragement from her father on this decision.

Nicole has a strong mathematics identity and considers herself to be good at mathematics. She claims that the material in mathematics classes can be challenging, but then follows up by saying, "I didn't really have a lot of trouble with getting grades and that kind of thing. But I think *other* people maybe had trouble with that." She says that she knows mathematics is a challenging subject for most people, but that it is not extremely challenging for her.

When thinking about specific classes, Nicole remembers certain mathematics classes that she did not enjoy very much, but thinking back on her entire college experience as a mathematics major, she claims, "It was *all* a good experience." The thing that Nicole enjoys the most about being a mathematics major is the feeling of being able to do well in mathematics classes, and being able to do something that most people are not good at. She feels like there really was not anything she did not enjoy about the experience. "Even when it was *hard* I, I wouldn't say that I was *not* enjoying it."

Unlike the other two participants, Nicole did not communicate much with her mathematics professors outside of class. She claims that she normally only went to office hours about once a semester. Nicole claims that she did not go very often because she really did not have the need to go; she generally did not have specific questions to ask. She also says that she would not consider any of her mathematics professors as mentors to her since she did not communicate with them much and did not ask them for help with decisions.

Similar to the other two participants, however, Nicole initially did not really know what she wanted to do after college. She claims there are so many things one can do with a mathematics degree that it is overwhelming. At the same time, she seemed very unclear as to what those careers might be. Her father had discouraged her from becoming a high school mathematics teacher in their home state because of job instability, and she really did not know where to go from there. She had thought about a lot of different career options, such as engineering, physics, or veterinarian school. She decided though that careers in engineering and physics would be boring and that she wanted a career with more public interaction. As for veterinarian school, Nicole knew that veterinarian school is really hard to get into and felt like she had not taken the right classes to get in. In the end, she decided that she wanted to stick with mathematics.

One day while on Facebook, Nicole saw an advertisement for a program called “Math for America.” The program consists of one year of schooling, which would conclude with her earning an MAT. After that she would be required to teach in New York City public schools for four years. The program also has professional development opportunities and hosts social activities. Since Nicole really did not know what else to do after graduation, she decided to apply for this program. She felt like this program would be good for her because she would be able to earn her Master’s degree for free, plus the program pays quite well. Also, she would not have to commit to a life-long career now, since she really did not know what she wanted to do with her life yet.

Originally, Nicole was determined not to go into teaching. She had thought if she ever went into teaching, she would teach at the college level. She says that she definitely does not see herself remaining as a public school teacher for a long period of time. Nicole sees this opportunity as just that, an opportunity, and does not see this program as leading to a future career. At one point Nicole says she almost sees it as being “a big, long camp.” It almost appears that she is using this program as a way of putting off trying to decide what to do with her degree.

Despite her uncertainty with what she wants to do with her life, Nicole really never has asked for advice or guidance from her professors. She has asked for letters of recommendation once she had decided where to apply, but did not solicit advice prior to that. Even with her father, she often told him what she was thinking about doing, but he never gave her much advice or further suggestions of what else she could do. She really feels like she made most of her decisions all on her own, and she appears to be happy with that.

It is interesting that though all three participants were students in the same department, Nicole did not utilize the advising that was clearly available to her the same way as the other two did. Nicole had the same advisor as Paige, and Mandy had a different advisor, so we know that the difference is not a result of an over-zealous advisor. Furthermore, of the three participants, Nicole had the highest mathematics grades, highest college GPA, and highest ACT/SAT scores, so we doubt that the professors did not show interest in advising her. From conversations with her, we believe that the lack of advising she received was completely her choice.

## Discussion

From our data, we can see that “significant others” played a large part in the development of these women’s mathematics identities. Each of these women began developing a strong mathematics identity prior to entering college, which seemed to formulate as a combination of being successful in mathematics and being perceived as good at mathematics by others. While in college, all three of these women felt like their mathematics professors viewed them as high achieving, which contributed to their views of themselves as a “mathematics person.” Some of the women even spoke of their achievements, such as their acceptance into an REU program or a

Master's program, as a form of being recognized by others as someone successful in mathematics.

The role that meaningful others played in these women's development of their mathematics identities is consistent with the findings of Carlone and Johnson (2007) who determined that recognition by others was a key factor in the formation of science identities of women of color. Furthermore, Adhikari and colleagues (1997) found that the women in their study were more likely than men to report valuing the support of others as important to their success in mathematics. Because women are often socialized to want to please people, it may not be surprising that women tend to place a large emphasis on the recognition and support of others in the development of their mathematics identities.

Although the mathematics faculty at this liberal arts college played a large role in the development of these women's mathematics identities, there is another critical role that they played for two of these students. The faculty provided useful suggestions that helped these women determine what career options were available to them. All of the participants clearly stated that originally they did not know what they could do with a degree in mathematics other than teach. Furthermore, they had each made a conscious decision that they did not want to pursue K-12 teaching. The two participants who utilized the advising available to them were able to expand their knowledge about future career options, leading them to more informed and purposeful future career decisions. The participant who did not utilize faculty advising chose to pursue teaching despite her initial decision against it. This choice appeared to be more of a "stalling tactic" until she figured out better what other career options were available to her.

It is not uncommon for college students graduating with degrees in mathematics to not know what career options in mathematics are available to them. Piatek-Jimenez (2008) found that senior-level mathematics majors often are not aware of many potential careers and that they had a very shallow understanding of the careers that they were aware of. Therefore, regardless whether or not a student has a strong mathematics identity, if he or she does not know what careers exist, it will be difficult to enter a career in mathematics. Consequently, it is important for students not only to receive encouragement in mathematics but also to obtain career advising. Furthermore, our work suggests that more research programs, such as REUs, and internships in industry would serve the discipline of mathematics well in retaining our majors in the field.

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# GENDER DIFFERENCES IN LUNAR-RELATED SPATIAL UNDERSTANDINGS

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*Gender differences were examined in spatial understandings of 70 females and 53 males who studied the Moon through observations, journaling, and geometric spatial modeling. Understanding was measured through analysis of pre/post-test results of a Lunar Phases Concept Inventory (LPCI) and a Geometric Spatial Assessment (GSA). Results showed both genders making gains on five of eight science LPCI domains and on three of four mathematics LPCI domains. Males scored significantly higher than females on the Geometric Spatial Visualization LPCI domain items. Females made gains on GSA domains, Periodic Patterns and Cardinal Directions, while males made only Periodic Patterns gains. The GSA filtered out understanding that might normally be missed.*

## Background

This research concerns an examination of gender differences in geometric spatial understanding of 123 middle school students after participation in an inquiry unit. The lessons within this unit were adapted from an integrated mathematics and science curriculum called Realistic Explorations in Astronomical Learning (REAL).

## Objective and Theory

Research has shown that students have difficulty understanding the cause of lunar phases (Abell, Martini, & George, 2001; Baxter, 1989; Lightman & Sadler, 1993; Trundle, Atwood, & Christopher, 2002; Zeilik & Bisard, 2000). "Instructors believe they have successfully taught lunar phases, only to find that the majority of students cannot answer questions related to the concept" (Lindell & Olsen, 2002, p. 1). The most common misconception of the cause of phases is the belief that the Earth casts a shadow upon the Moon. Fanetti (2001) stated that there was a correlation between students' ignorance of Moon-size, Earth-size, and Earth-Moon distance that feeds this misconception. Some researchers examined links between students' understanding of phases and spatial ability. For example, Reynolds' (1990) and Wellner's (1995) claimed that students were more likely to report a correct cause of phases when they had well-developed spatial skills. Others reported findings connecting spatial ability to success on assessments, such as the Force Concept Inventory (Hake, 2002) and molecular rotation exams in Chemistry (Pribyl & Bodner, 1987).

Mathematics education literature has well documented that males outperform females on assessment tasks that focus on spatial visualization (Ben-Chaim, Lappan, & Houang, 1988; Battista, 1990; Casey, Nuttall, Pezaris, & Benbow, 1995). Sex differences with regard to spatial ability were found in preadolescent children by Kerns and Berenbaum (1991). "Boys again performed better than girls on all tests. The sex difference was significant on Geometric Forms, Mirror Images, and 3D Mental Rotations" (p. 391). Linn and Petersen (1985) determined that males outperformed females at all age levels on mental rotation tasks and to a lesser extent on spatial perception tasks. Black (2005) "hypothesized that mental rotation is the most important in understanding Earth science concepts that are associated with common misconceptions" and stated that “humans are handicapped by their single vantage point from Earth of the moving bodies in outer space” (p. 403).

Therefore, based on this mathematics education literature, if males have better spatial skills than females, then males should show higher scores on science assessments that have spatial components. *Few studies speak directly to gender differences on lunar phases assessments.*

I claim that one cannot completely understand lunar phases without a developed understanding of four mathematical spatial concepts. The four mathematical concepts are defined within the context of its relation to lunar understanding. They are: (a) *geometric spatial visualization* (visualizing the geometric spatial features of the Moon/Earth/Sun system as it appears in space above/below/within the Moon/Earth/Sun plane), (b) *spatial projection* (projecting one’s self into a different Earthly location and visualizing from that global perspective), (c) *cardinal directions* (documenting an object’s vector direction from a given position, i.e. North, South, East, West, etc.), and (d) *periodic patterns* (occurring at regular intervals of time and/or space, i.e. periodicity of celestial orbits, phases, percent illuminated).

The first spatial term, geometric spatial visualization, has a mental rotation component since as one visualizes the Moon/Earth/Sun three-body system in space above/below/within, one must also manipulate and consider the motion of the system itself. The second spatial identity, spatial projection, not only includes the idea of visualizing the environment from one’s own reference point, but also visualizing the environment of another from their reference point. Spatial projection also has a mental rotation derivative since one must also mentally manipulate the movement of the sky throughout a day’s viewing due to the rotation of the Earth on its axis. The final two mathematical terms that are instrumental in understanding lunar related concepts are cardinal directions and periodic patterns. Learners must be able to distinguish cardinal directions in order to document an object’s vector direction in space as a function of time from a given position. The final mathematical term is that of periodic patterns. For this paper, periodicity comprises the idea of something occurring at regular intervals of time and/or space. Within the lunar context, some examples of periodic patterns materialize in lunar orbital cycles, illumination, and in altitude angles.

Giedd, Blumenthal, Jeffries, Rajapakse, Vaituzis, Liu, Berry, Tobin, Nelson, and Castellanos (1999) conducted a study with 145 normal children where their brains were scanned using magnetic resonance imaging at two-year intervals. The researchers found that although 95 percent of each child’s brain structure was formed by the age of 6, a second wave of brain development occurred in preteens just prior to puberty (age 11 in girls, and 12 in boys). Giedd et. al (1999) as cited in Bryce and Blown (2007) showed in their research that “different anatomical regions of the brain mature differently in childhood and typically by several years, e.g. favoring earlier development in girls for those areas which handle verbal fluency, handwriting, and face recognition; favoring earlier development in boys for those which handle spatial and mechanical reasoning, and visual targeting” (Bryce and Blown, p. 1657). These varied rates of brain development in girls and boys are of particular interest since the subjects in my study were of this preteen age where this newfound wave of brain development occurs. The following study was conducted with 123 US preteen (average age of 12), seventh graders.

## Methodology

This paper reports the effect gender has on measurable learning via an integrated mathematics and science, inquiry Moon unit adapted from a NASA/IDEAS funded curriculum called *Realistic Explorations in Astronomical Learning* (Wilhelm & Wilhelm, 2007). A veteran female teacher enacted the unit within her seventh-grade science classrooms with 70 females and 53 males during the Fall 2007 term. All lessons and activities were drawn from the REAL curriculum to accommodate the school district's mathematics objectives regarding using two-dimensional representations of three-dimensional objects to visualize and solve problems; and applying geometric ideas and relationships in areas outside mathematics, such as science. The district's science objectives concerning the relationship of the Earth's movement and the Moon's orbit to the observed cyclical phases of the Moon were also addressed.

In this seven week unit, which purposefully included spatial geometric activities, students explored the lunar phases through observation, journaling, sketching, two- and three-dimensional modeling, and classroom discussions. The lunar unit began with students conducting daily Moon observations. Students were expected to record in a journal at least two sentences per daily entry to communicate what they viewed regarding the Moon and sky. Further explorations focused on geometric configurations and scaling of the Earth/Moon/Sun system, where students modeled these geometries in both two and three dimensions. In-class lessons were conducted where students, working in groups, used modeling techniques and ratio concepts to discover the number of Earth diameters between the Earth and the Moon. The general method of teaching was inquiry-based, and most student groups were self-selected and gender-mixed.

After daily Moon observations had ceased, student journals were completed and a guest instructor guided the students through a "Moon finale". The Moon finale was a hands-on modeling activity in which students were asked to create the geometric configuration of the Earth and Moon (Styrofoam balls), given a fixed Sun (overhead projector light), for a number of chosen Moon's phases. This three-dimensional (3-D) modeling was followed with a two-dimensional (2-D) drawing activity where students represented their 3-D configurations within a 2-D (paper and pencil) space.

This study was quantitative where data collection involved the administration of the Lunar Phases Concept Inventory (Lindell & Olsen, 2002) and the Geometric Spatial Assessment (Wilhelm, Ganesh, Sherrod & Ji, 2007), pre- and post-implementation. The LPCI assesses eight science and four mathematics domains (see Table 1).

Table 1. *LPCI Domains*

| Scientific Domains | Mathematical Domains |
|---|---|
| **A-**Periodicity of Moon's orbit | Periodic Patterns |
| **B-**Periodicity of phases | Periodic Patterns |
| **C-**Moon's direction around Earth | Geometric Spatial Visualization, Spatial projection |
| **D-**Moon Motion | Cardinal directions |
| **E-**Phase and Earth/Moon/Sun positions | Geometric spatial visualization |
| **F-**Phase-sky location-time | Cardinal directions |
| **G-**Cause of phases | Geometric spatial visualization |
| **H-**Phase effect with location change | Spatial projection |

This research focused on the development in students' mathematical and scientific content knowledge from pre- to post-implementation. Questions pursued were:

1) What lunar-related mathematics and science content knowledge will be developed by students through inquiry experiences?

2) What gender differences will be observed in learned, lunar-related mathematics and science content knowledge?

The GSA (a 16-item, multiple choice test) was administered pre and post implementation for the purpose of filtering out mathematical spatial understandings of the same four math domains shown in table 1, *but not posed within a lunar context*. Example GSA test items are shown in figure 1.

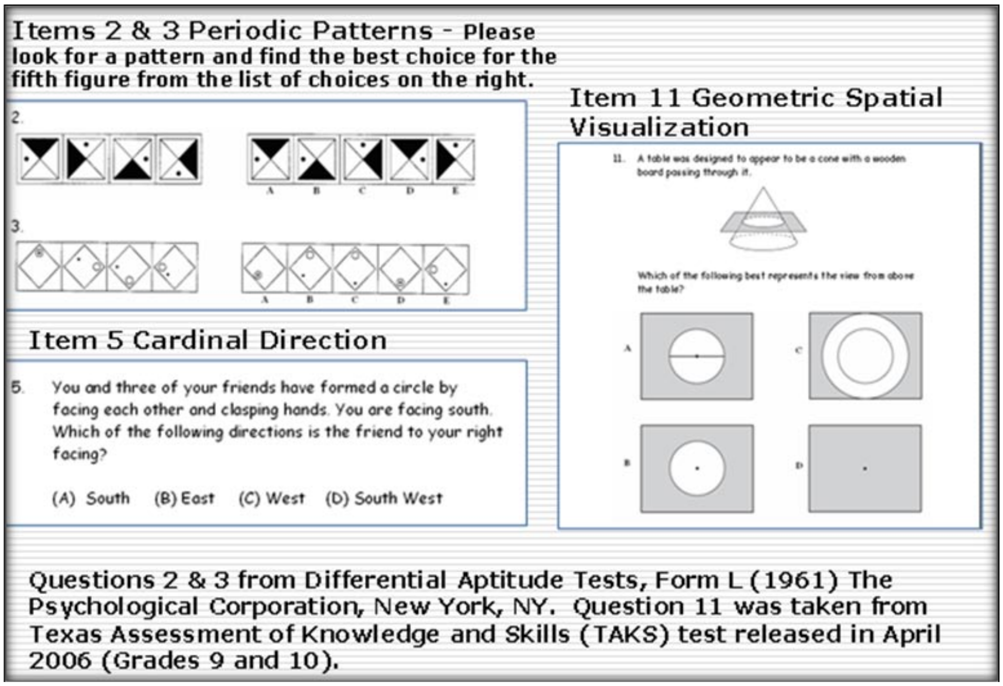


*Figure 1*. Example Geometric Spatial Assessment Test Items – Items 2 & 3, 5, and 11 that assess math domains periodic patterns, cardinal directions, and geometric spatial visualization, respectively.

## Data & Analysis

*LPCI Results*

The LPCI pretest was given to 53 males and 70 females prior to any observations and the posttest was administered seven weeks after the initial lunar viewing. The mean pretest score was 31.2% correct and the mean posttest score was 52.9% correct. A repeated measures ANOVA revealed a significant increase in the mean values from pre to post on overall test scores, $F(1,122) = 288.5$, $p < 0.001$, partial $\eta^2 = 0.703$. A one-way ANOVA showed no significant difference between groups on the pre-LPCI scores, $F(1, 122) = 3.816$, $p = 0.053$. Both groups made similar significant overall gains from pre to post. The significant percentage gain score for males and females were 23.6% and 20.2%, respectively. A one-way ANOVA showed a significant difference between groups on the post-LPCI scores, $F(1, 122) = 10.133$, $p = 0.002$, mainly due to males scoring 24.8% higher than females on domain E (science-*phase and*

*Sun/Earth/Moon positions*, mathematics-*geometric spatial visualization*) of the posttest (see Table 2).

To test for significant differences from pre to post on individual domains, a repeated measure ANOVA was conducted. Results showed males and females making similar significant gains on science domains: A (*period of orbit*), B (*period of phase cycle*), C (*direction of orbit*), and G (*cause of phases*); and on mathematics domains: *periodic patterns* and *spatial projection* (Tables 2&3).

The largest gender gap was found to exist within the LPCI-domain E (science-*Phase-Earth/Moon/Sun positions*, math-*geometric spatial visualization*). No significant increase in understanding by any group was observed on concept domains F (science-*Phase-sky location-time*, math-*cardinal direction*), D (science-*Moon motion*, math-*cardinal direction*), or H (science-*Phase effect with location change*, math-*spatial projection).*

Table 2. *Percentage Correct on Pre-Post LPCI by Science Domain*

| Science Domain | | All % | | | Male % | | | Female % | |
|---|---|---|---|---|---|---|---|---|---|
| | Correct Pre (SD) | Correct Post (SD) | Gain | Correct Pre (SD) | Correct Post (SD) | Gain | Correct Pre (SD) | Correct Post (SD) | Gain |
| A | 28.5 (34.5) | 70.3 (37.2) | 41.8** | 32.1 (36.8) | 80.2 (33.0) | 48.1** | 25.7 (32.7) | 62.9 (38.7) | 37.2** |
| B | 43.4 (31.3) | 59.4 (27.2) | 16.0** | 50.9 (28.2) | 62.3 (27.0) | 11.4* | 37.6 (32.6) | 57.1 (27.3) | 19.5** |
| C | 54.5 (40.0) | 92.7 (21.9) | 38.2** | 51.9 (37.9) | 93.4 (19.7) | 41.5** | 56.4 (41.6) | 92.1 (23.5) | 35.7** |
| D | 35.7 (33.0) | 41.1 (38.4) | 5.4 | 36.8 (32.7) | 46.2 (37.8) | 9.4 | 35.0 (33.3) | 37.1 (38.7) | 2.1 |
| E | 25.8 (27.9) | 60.7 (32.5) | 34.9** | 29.6 (31.1) | 74.8 (29.9) | 45.2** | 22.9 | 50.0 | 27.1** |
| F | 7.9 (14.2) | 7.3 (15.1) | -0.6 | 8.8 (14.8) | 5.0 (12.0) | -3.8 | 7.1 (13.8) | 9.1 (17.0) | 2.0 |
| G | 22.0 (28.7) | 50.0 (38.9) | 28.0** | 26.4 (30.3) | 58.8 (40.1) | 32.4** | 18.6 (27.1) | 43.6 (37.0) | 25.0** |
| H | 38.6 (32.5) | 43.9 (29.0) | 5.3 | 38.7 (30.4) | 37.7 (29.3) | -1.0 | 38.6 (34.2) | 48.6 (28.2) | 10.0 |

*$p < 0.05$; **$p < 0.001$

Table 3. *Percentage Correct on Pre-Post LPCI by Mathematics Domain*

| Mathematics Domain | All % | | | Male % | | | Female % | | |
|---|---|---|---|---|---|---|---|---|---|
| | Correct Pre (SD) | Correct Post (SD) | Gain | Correct Pre (SD) | Correct Post (SD) | Gain | Correct Pre (SD) | Correct Post (SD) | Gain |
| A, B – Periodic Patterns | 35.9 (25.2) | 64.8 (25.2) | 28.9** | 41.5 (25.2) | 71.2 (22.9) | 29.7** | 31.7 (24.7) | 60 (25.9) | 28.3** |
| C, E, G – Geometric Spatial Visualization | 34.1 (20.6) | 67.8 (21.5) | 33.7** | 36.0 (24.6) | 75.6 (20.9) | 39.6** | 32.6 (17.1) | 61.9 (20.2) | 29.3** |
| D, F – Cardinal Directions | 21.8 (16.8) | 24.2 (20.4) | 2.4 | 22.8 (16.9) | 25.6 (19.3) | 2.8 | 21.1 (16.8) | 23.1 (21.3) | 2.0 |
| C, H – Spatial Projection | 46.5 (26.3) | 68.3 (18.1) | 21.8** | 45.3 (24.0) | 65.6 (16.4) | 20.3** | 47.5 (28.0) | 70.4 (19.2) | 22.9** |

** $p < 0.001$

*GSA Results*

The purpose of the GSA was to filter out spatial understandings that were not posed within a lunar context. The GSA pretest was given to 54 males and 67 females prior to any observations and the posttest was administered eleven weeks after the initial lunar viewing. The mean pretest score was 49.4% correct and the mean posttest score was 56.2% correct. A repeated measures ANOVA revealed a significant increase in the mean values from pre to post on overall scores, $F(1,120) = 21.29$, $p < 0.001$, partial $\eta^2 = 0.151$.

A one-way ANOVA showed a significant difference between gender groups on the pre-GSA scores, $F(1, 120) = 6.322$, $p = 0.013$. The significant percentage gain scores from pre to post for males and females were 5.8% and 7.5%, respectively. A one-way ANOVA showed no significant difference between groups on the post-GSA scores, $F(1, 120) = 3.125$, $p = 0.08$.

To test for significant differences from pre to post on individual GSA domains, a repeated measure ANOVA was conducted (Table 4). Results showed males and females making similar gains on geometric spatial visualization and spatial projection, although neither was significant. However, with genders combined, there was a significant increase in means from pre to post for geometric spatial visualization. Males and females achieved significant gains on periodicity where males made a 7.9% gain while females scored a 10.5% gain. Females' gain score nearly doubled that of males' for cardinal direction (males-insignificant 7% gain; females-significant 12% gain). While the interaction effect between gender and time was not significant, one cannot help but notice a closing gap from pre to post between males and females. Although females scored lower than males on every pre-GSA domain, females made gains that brought each of their post-domain items up to or beyond those of the males' pre-scores.

There appears to be inconsistency when comparing the results of the LPCI- and GSA-mathematics domains. For example, all LPCI mathematics domains showed both groups making significant gains from pre to post *except* for the math domain, cardinal direction. However, females made significant gains from pre to post on the GSA-domain, cardinal direction. In order to understand how this might be possible, one must examine the LPCI items that have the embedded cardinal direction domain. For example, LPCI item-1 requests the learner to consider a waxing crescent Moon in the sky immediately after sunset. One would need to know the Sun had just set in the west and that since the crescent Moon was still visible, it would be close to setting in the western sky as well. In order to choose the correct response, the learner would have to take into account the moving setting Sun and Moon, and realize the rise and set times of a waxing crescent Moon. The GSA did its job of filtering out the mathematics of only cardinal direction (i.e., not posed within a lunar context) and assisted in showing measurable understanding of this concept whereas the LPCI did not.

Table 4. *Percentage Correct on Pre-Post by GSA Domain*

| Mathematics Domain | All % | | | Male % | | | Female % | | |
|---|---|---|---|---|---|---|---|---|---|
| | Correct Pre (SD) | Correct Post (SD) | Gain | Correct Pre (SD) | Correct Post (SD) | Gain | Correct Pre (SD) | Correct Post (SD) | Gain |
| Periodic Patterns | 52.9 (23.3) | 62.2 (25.2) | 9.3** | 58.8 (22.8) | 66.7 (24.3) | 7.9* | 48.1 (22.7) | 58.6 (25.6) | 10.5* |
| Geometric Spatial Visualization | 55.8 (29.9) | 61.8 (28.3) | 6.0* | 58.8 (28.0) | 64.8 (30.5) | 6.0 | 53.3 (31.3) | 59.3 (26.4) | 6.0 |
| Cardinal Directions | 46.5 (27.3) | 56.2 (26.3) | 9.7* | 49.5 (29.3) | 56.5 (26.2) | 7.0 | 44.0 (25.4) | 56.0 (26.5) | 12.0* |
| Spatial Projection | 42.6 (23.2) | 44.6 (23.7) | 2.0 | 45.8 (23.7) | 48.2 (24.2) | 2.4 | 39.9 (22.6) | 41.8 (23.2) | 1.9 |

*$p < 0.05$; **$p < 0.001$

The other apparent contradiction concerns spatial projection. Both genders made significant gains from pre to post on this LPCI-domain. However, the GSA-domain of spatial projection showed neither males nor females scoring significantly higher from pre to post. Upon examination of the GSA questions assessing this domain, it was found that the questions did not appropriately filter out the spatial projection concept. For example, GSA test item-13 requests the learner to consider two boats approaching a Texas flag (one approaching from the east and the other from the west). "*The captain on the boat approaching from the east saw the star on the Texas flag to be on the right side of the flag, what side of the flag would the captain of the other boat observe?*" This question not only had the spatial projection domain embedded, but also cardinal direction. The question was easily modified by replacing east/west with right/left.

## Conclusions and Importance

No significant increase in understanding by any groups was observed on concept domains F (science - *Phase – sky location-time;* math - *cardinal direction*), D (science - *Moon motion;* math - *cardinal direction*), and H (science - *effect of lunar phase with change in Earthly location;* math – *spatial direction*). This needs to be considered as the integrated unit within the REAL curriculum is modified.

This research indicates that mathematics and science learning of lunar related concepts can be significantly improved by both sexes in an inquiry environment. Even though the males scored significantly higher (from pre to post) than females on the LPCI domains concerning geometric spatial visualization, both groups made the highest gain scores in this LPCI mathematics domain. Males and females made similar significant gains on science domains: A (*period of Moon's orbit around Earth*), B (*period of Moon's cycle of phases*), C (*direction of Moon's orbit*), and G (*cause of lunar phases*), and on LPCI mathematics domains*: periodic patterns* and *spatial projection*. The LPCI pre-tests showed no significant difference between gender groups at the $\alpha = 0.05$ level; however, the LPCI post-tests did show a significant difference between groups. This significant difference between groups was mainly due to science domain E (*phase and Sun/Earth/Moon positions*) where males scored an 18.1% higher gain score than females, and due to the mathematics domain, *geometric spatial visualization*, where males scored a 10.3% higher gain score than females. More research is needed to better understand why males seem to perform significantly better than females on this particular domain, but this finding also seems to be consistent with the Bishop (1996) research results. Perhaps this finding can be explained by the faster maturation rate (during these particular years of development) of the male brain's anatomical regions which handle spatial visual reasoning as reported by Giedd et. al (1999). Both males and females achieved significant gain scores on the GSA periodic patterns domain, while only females scored significantly higher from pre to post on GSA cardinal direction domain. Females had scored significantly lower than the males on the pre-GSA, but the gap narrowed by the time of post-GSA where no significant difference between gender groups was observed.

This research is unique because it will further the literature concerning students' lunar related mathematical and scientific understandings, especially regarding gender differences. The development of the GSA helped to filter out understanding that might normally be missed within a science classroom.

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# BLENDING PERSPECTIVES: STUDENT MEDIATIONS OF GEOMETRIC AND NUMERIC REASONING TO MAKE SENSE OF SIMILARITY AND SCALE

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*Although studies have documented student difficulties with similarity, a gap exists between documented visual insights of younger children and the quantitative inadequacies of older ones. This study investigated the intermediate strategies used by middle school students to construct rectangular and non-rectangular similar figures. Seven types of construction strategies were identified which indicate that multiple types of reasoning were used. Results suggest that visual perception is not entirely primitive and the consideration of visual perception as a powerful indicator and supportive extender of conceptual understanding is warranted. Experiences with scaling complex figures may encourage students to bridge intuitions and numeric strategies.*

## Rationale and Purpose of the Study

Research studies have shown repeatedly that similarity is one of the most difficult contexts for proportional reasoning (e.g. Kaput & West, 1994). Even young students have useful visual intuitions about proportion (e.g. Lehrer et al., 2002). However, studies also document the quantitative inadequacies of the older ones (Hart, 1988). Karplus, Pulos, & Stage (1983) highlighted this disconnect by documenting that students struggle to remember and utilize procedures and symbolism for numeric strategies, which seem to replace rather than extend visual perceptions and intuition. Studies have been done to characterize the nature of this student difficulty (Chazan, 1987), but the gap remains.

In strictly numerical contexts for proportional reasoning, this gap between useful intuitions and mature proportional reasoning has been narrowed, and work has been done to identify intermediate qualitative strategies such as building up, norming, or unitizing (Lamon, 2007). The identification of these strategies gives us some power to hypothesize about intermediate conceptions of similarity; however, these strategies are not entirely translatable to a geometric context. Given the position of similarity at the crossroads of geometric and proportional reasoning, it is likely that the progression of students from visual to *proportional thinker* (Cramer & Post, 1993) can be better described if, in addition to attending to numeric proportional reasoning, we acknowledge that students also bring geometric and spatial understanding (van Hiele, 1986) to bear on similarity tasks. In a review of middle-grades texts, it was found that similarity tasks often include only simple polygons such as rectangles or triangles (Lo, Cox & Mingus, 2006). In further study (Cox, Lo & Mingus, 2007), it was hypothesized that scaling these figures did not help students understand the continuous all-directional nature of scaling but instead focused on isolated instances of scaling.

Although the research literature suggests that students struggle to develop abilities to reason proportionally and to make sense of similarity, the fact remains that some students actually do develop these abilities. How these students advance from using visual and additive reasoning strategies to using multiplicative proportional reasoning on similarity tasks is an open question. Before we can study the transition, however, we must admit that we have only incomplete theories about what intermediate student strategies would look like on these tasks. Lamon (1993) argues for

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research that identifies the ideas that students have that contribute to proportional reasoning, and investigates the contexts and models that "offer more explanatory power" (p. 42) to students in their work.

This study is focused on identifying and describing potential extenders for visual intuitions about scale by analyzing the strategies that students use during clinical interview to construct similar figures. Results are shared here in answer to the questions:

1. What strategies do students use to construct similar figures and what types of geometric and numeric reasoning are indicated by these strategies?
2. How does the complexity of the figure to be constructed influence student reasoning about construction?

## Theoretical Perspective

This study assumes a constructivist perspective on the inquiry into student conceptions and the modeling process. This has two implications for the study at hand. First, there is the direct implication that without observations of students themselves, no theory can stand apart from the limitations of the mathematical understanding and biases of the researcher (Cobb & Steffe, 1983). By observing students interacting with the ideas behind the theory, we open the theory up to the unexpected (Cobb & Steffe, 1983). Thus, the method of clinical interview (Cobb & Steffe, 1983) was chosen as the primary method of data collection.

The second implication is in registering the significance of the data that are collected. It is possible to have as a goal the empirical vetting of a theory, marking instances where the predictive power is great and where it is not. However, another goal, responds to Vergnaud's (1987) challenge to "understand better the processes by which students learn, construct or discover mathematics and to help teachers, curriculum and test devisers, and other actors in mathematics education to make better decisions" (as quoted in Confrey & Kazak, 2006, p. 311).

## Theoretical Framework

Proportional reasoning has been investigated through two major types of tasks: comparison tasks and missing value tasks (Lamon, 1993, 2007). Lamon (1993) outlined a conceptual progression for the development of proportional reasoning that stemmed from visual and intuitive solutions growing through successful preproportional strategies up into mature proportional reasoning. This progression was useful in describing and organizing the numeric strategies that were used by students during interviews. However, because students begin with a visual intuition about similarity, it is insufficient to focus only on instances of numerical proportional reasoning in student strategies. In order to capture other forms of reasoning that students may use on similarity tasks, it was imperative that a geometric lens also be used. van Hiele's descriptions of reasoning at levels 0 and 1 provided such a lens while analyzing student strategies on these tasks.

## Methodology

*Population*

A population of students in a Midwestern, urban school district was identified to target racial, economical, and academic diversity. The inclusion of diversity in the sample for study was not intended to highlight differences between groups of students, but rather to ensure that a broader extent of prior student experience and knowledge is included in the results. For example, students from urban areas may have significantly different experiences related to geometric proportionality (such as reading bus maps) that influence conceptions of scale or correspondence. Alternatively, lower socio-economic status may indicate a more limited access to technology, photographic or otherwise, and a different repertoire of imagery that others take for granted. Furthermore, because the majority of student data related to research on proportion and similarity are more than a decade old (Lamon, 2007), it is likely that technological advances, such as the availability of publishing software, photo enlargement machines, and multi-dimensional video gaming systems may alter potential student imagery and visual acuity, and provide a more fertile ground for developing quantitative strategies.

*Data Collection*

An assessment, the revised Similarity Perception Test (rSPT), was administered to a group of 91 seventh-grade students for sampling purposes. This instrument provided information about students' visual perception of shape, correspondence, and size transformation and helped to divide students into subgroups according to their responses. A stratified purposeful sample (n=21) that included the most common as well as unique response patterns was selected for task-based interviews. This method of sampling intentionally included individuals who exhibited varying abilities, perceptions, and strategies.

During the interview, students were given up to six construction tasks. Each task involved one of three different shapes shown in Figure 1: a rectangle with an embedded square, an L-shape, or a heart. Students were asked to draw an enlarged or shrunken version of the figure with a given scale factor (*k*) and in most cases were given a scale factor, generally 2. All students started with the rectangle with the embedded square task. If successful, students were given another version of the task. In this case, students were asked to draw another version of the figure "somewhere in the middle" of the original size and the size of the image. It was notable that many students applied different strategies to scaling the rectangle and scaling the square embedded in the rectangle. Thus, during analysis, tasks involving this figure were analyzed in two distinct parts: scaling the rectangle (Double Rectangle/Middle Rectangle) and scaling the embedded square (Embedded Square (*k*)/Embedded Square(M)).

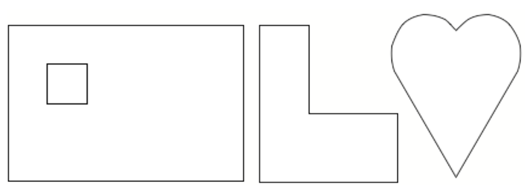

*Figure 1.* Figures depicted in construction tasks.

*Data Analysis*

Interviews were transcribed using Transana (Fassnacht & Woods, 2005), a software package used to transcribe and organize data. Student constructions, including drawings and measurements, were all digitally scanned. Analysis started with individual responses to individual tasks. It is possible that one student used two distinct strategies on one task. In this event, both strategies were analyzed separately. Descriptions of each strategy used by each student on each assigned task were written. All of the strategies used by an individual student were compared, noting similarities and differences in particular uses. Then, all of the strategies used on each individual task were compared in the same way. In order to validate these general descriptions and types, they were compared again to original student responses and revised when necessary. General strategy types were then devised to organize the strategies. The theoretical framework described above also helped to order the types with respect to the sophistication of implied reasoning.

## Results

In this section, I will provide the results relevant to the research questions stated earlier. I will first describe the seven identified types of construction strategies. Although defined here, there is not space here to elaborate on each of the strategies observed. More detailed illustrations will be provided during the presentation. Second, I will present an overview of the strategies used on each of the tasks to gain insight into how the figure may influence the strategies selected and reasoning used.

*Strategy Types*

Seven construction strategy types were observed in this study: Avoidance (AV), Additive, Visual, Betweening, Pattern Building (PB), Unitizing, and the Functional Scaling strategy (FS). Two of the seven strategy types, *Avoidance* and *Additive*, were similar to strategies documented for proportional reasoning. Reponses that indicated no meaningful engagement with the problem were considered indicative of an avoidance strategy. Responses that featured students scaling lengths using a constant additive approach were considered indicative of a classical additive strategy resembling those documented by Lamon (1993) and many others.

There were other strategy types described in the literature that diverged from standard in the case of similarity. For example, a *visual* strategy was identified by Lamon (1993) as a primitive approach to proportional reasoning and was aligned with the additive strategy as non-constructive. However, in the context of similarity, there was cause to differentiate this strategy more from an additive approach. The main distinction between additive and visual strategies is that visual strategies in this context can indicate sophisticated conceptions of proportional growth, and can be quite constructive. While it is true that for some students, a visual strategy is more akin to a guess, this is certainly not the case for all students. Visual strategies can incorporate a range of simple to sophisticated concept imagery regarding the constant of proportion, correspondence, and dilation. The additive strategy is more accurately depicted as primitive and non-constructive.

Elaine's construction of an enlarged heart in Figure 2 is an illustration of how intuitive conceptions of proportional growth can be constructive. In this image, the original heart was traced and a larger version was constructed around the outside like a frame. Elaine's image was reasonably similar to the smaller original heart and was based on only visual measurements. *"All I did was look at the heart; the design of what it was drawn. I looked at it while I was drawing, too. I was trying to make it exactly like it was."*

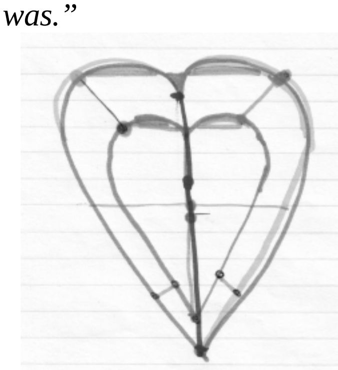

*Figure 2.* Elaine scales the heart.

Elaine's concept imagery related to dilation was robust and did not occur accidentally or by chance. This was not the only instance where Elaine demonstrated her intuitions about geometric proportion and dilation, which were remarkably reliable. Elaine's drawings, Figure 2 included, illustrated intuitions she has about correspondence and the implications of this intuition on the way she visualized scaling. The correspondence lines between the two hearts that Elaine drew in Figure 2 were not constant in length. While not made explicit during the interview, this might illustrate an informal understanding of proportion—the distance between corresponding points on the original and image depends multiplicatively on the distance the original is from the point of dilation. Elaine suggested in another situation that when this distance was held constant, the result was visually displeasing to her and that the original and scaled figures did not look like the same shape. This type of intuition was not evident when a student took an additive approach and accepted the result.

In some cases, it was apparent that students were making use of visual judgment to mediate numeric strategies. This combination of numeric and visual reasoning was indicative of a new type of strategy, *Betweening*, which is characterized by the remediation, both during and post-construction, of a numeric strategy so that it conforms to visual expectations. Betweening was used primarily, though not exclusively, to remediate a constant additive construction. Students, noticing distortion in their constructions, adjusted side lengths so that they varied by different additive amounts.

*Pattern-building* is an umbrella term for the use of oral or written patterns without indicating an understanding of the functional nature of the scale factor. In the context of construction, three pattern-building strategies were observed: Angle-matching, Median Length Finding, and Tiling. All of these strategies require the use of existing angles, lengths and shapes as tools in constructing a new shape. In the case of Median Finding, the lengths of a shape become numeric tools by which a student can interpolate intermediate lengths of similar figures. In all three cases there were limitations of the applicability and reliability of the strategy.

*Unitizing* and *Functional Scaling* were the most sophisticated strategies observed. Unitizing, as observed in this study, is closely related to tiling, but differs in that a student acts on lengths as units rather than entire figures. In addition, it is not limited to use with figures that tile the plane. Students using the Functional Scaling strategy multiplied select original lengths (dimensional, secondary, and space) by the identified or indicated scale factor to determine corresponding image lengths before construction. Students knew how long the lengths would be before they even began drawing them. This is not always the case when students used pattern-building strategies or even unitizing strategies.

*Overview of Strategy Use*

If a student was successful with a given strategy, it is possible to imagine that they would continue to use this strategy regardless of the figure they were constructing. This was not the case. Students who were successful using a Functional Scaling strategy on the first task did not necessarily do so on all of the tasks. In fact, most ($n=17$) students were able to apply the Functional Scaling strategy on the double rectangle task yet only four out of twenty-one students used Functional Scaling exclusively. As the figures became more complex and as the scale factors were changed from whole numbers to non-whole numbers to numbers less than 1, fewer students applied the strategy and began to use other types tempered by visual judgment. At the point where the strategy broke down, students utilized less sophisticated strategies, or made modifications to their constructions using visual judgment. Visual judgment, used in concert with other strategies, was used as a tool for mathematical reflection and evaluation.

For example, Andre applied the strategy on three out of five tasks, but resorted to a visual strategy when embedding the square in the medium rectangle and when scaling the heart figure. Andre's drawing of the embedded square is shown in Figure 3. His difficulty is more related to spatial reasoning rather than a lack of proportional reasoning. He applied a scale factor (1.5) functionally to all edges of whole-number lengths, but did not transfer that strategy to the space between the square and the rectangle. All attempts at placing the square within the rectangle were done visually, and the square changed in dimension to accommodate his visualization as he made three attempts at moving the location—each attempt shown in this figure.

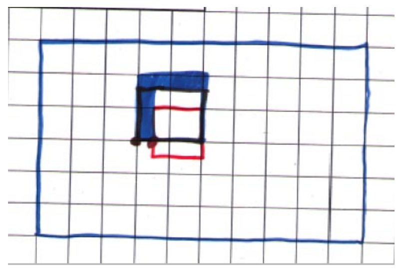

*Figure 3.* Andre attempts embedded square (M).

This leads us to the second research question regarding the influence of the complexity of the figure. In Figure 4, profiles of the strategies used on each task are compared. When organized by task rather than individuals, the strategy profiles are also varied in both unexpected and expected ways. For example, the two L-shape tasks should have skewed results. Only students who indicated using an additive strategy on some or all of their previous responses were asked to double the L-shape. Thus, it is expected that the profile for this task would be skewed toward less sophisticated strategies. Only students who had shown proficiency on other tasks or who finished tasks more quickly were asked to reduce the size of the L-shape. Thus, it is expected that the profile for this task would be skewed toward more sophisticated strategies. In fact, neither skew is observed. Reducing the L-shape inspired the greatest variety of strategies with no single strategy dominating student responses. Doubling the L-shape did have the highest frequency of additive strategy use, but this is to be expected. A few students used a visual approach, but students tended to favor the Functional Scaling strategy—even if, like Chris, they did not use Functional Scaling on previous tasks.

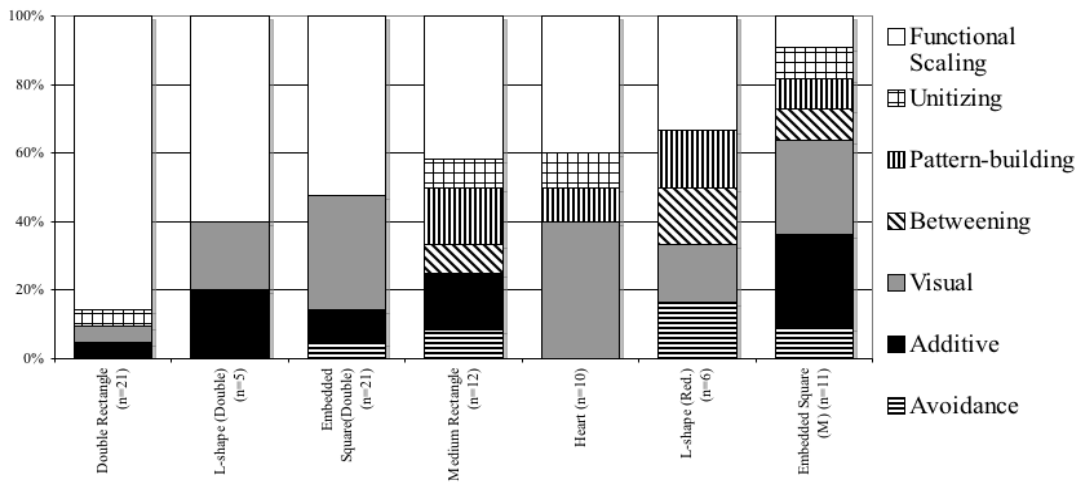


*Figure 4.* Summary of strategy by task.

The tasks incorporating whole scale factors (DR, L-shape [Double], Embedded Square [Double]) are clustered at the top end when the profiles are ranked according to the percentage of students utilizing the functional scaling strategy. As the scale factors change to non-whole numbers, fewer students utilized this strategy in lieu of a variety of other strategies. A smaller percentage of students (42%) utilized the functional scaling strategy on the Medium Rectangle (MR) task, a task that is different from the DR task only in scale factor. No students avoided solving the DR task, but this behavior emerged in the MR profile along with the pattern building strategy. Pattern building emerged as a strategy on the MR task even though it was not used on the DR task; it was a strategy used by students to reduce the L-shape but not to double it. In fact, on the L-shape Reduction task, a task that incorporated non-whole factors less than 1, there was much variety in student strategies. No particular strategy type seemed more prevalent than any other.

On the other end of the proposed spectrum were students who utilized a visual strategy. The number of students who utilized a visual strategy did not seem to be impacted by a non-whole scale factor. On tasks where students used whole and non-whole factors to scale the same figure, the frequencies of visual strategies were remarkably close. However, more variation is noticed when tasks of differing figure type are compared. Very few students used visual strategies to scale each of the rectangles, but 40% of students used a visual strategy on the heart task, 27% of students used a visual strategy to embed squares inside the double rectangle, and 36% to embed squares inside the medium rectangle.

## Discussion

To return to the original research questions regarding the strategies students choose to use when constructing similar figures and the influence of the complexity of the figures on reasoning used, two conclusions are possible. First, students in this study used a variety of construction strategies that have not been previously classified according to existing research frameworks including some that seemed to mediate numeric strategies with visual judgment or reasoning. The seven types of strategies are related to the literature on intermediate strategies for proportional reasoning, but as hypothesized, indicate as well the use of geometric and spatial reasoning.

Second, the complexity of the figures to be scaled did influence the strategies and reasoning types used by students. Depending on the nature of the figure being constructed, students were required to attend to a variety of characteristics including angles and different lengths of different varieties. The three figures (rectangle with embedded square, L-shape, and heart) used incorporate *primary*, *secondary,* and *gap* lengths. Primary and secondary lengths are both measurements of drawn lines within the figure, generally edges. A *primary* length is a length that defines the height or width of the entire figure. In the case of the rectangle, all four edges were defined as primary because they frame the figure and determine both horizontal width and vertical height. All other lengths including edges or drawn lines within the figure are *secondary*. A *gap* length measures the width of a gap in the figure not represented by a drawn line.

By increasing the complexity of the figures in the interview protocol, it was possible to manipulate the characteristics that students are required to attend to. One interpretation of the variance in the use of visual strategies is based on this complexity. As the lengths to be scaled became more oriented toward secondary and space lengths and away from primary lengths, the visual strategy was used more frequently. Furthermore, students who began with a numeric strategy—whether additive or multiplicative—utilized their visual judgment to mediate these strategies when the resulting image did not match their expectations.

These conclusions suggest that visual perception is not entirely guess-related or primitive in this context for proportional reasoning. Two implications follow. First, the consideration of visual perception as a powerful indicator and supportive extender of conceptual understanding in this area might be warranted. Second, there is strong evidence that providing students with complex figures to scale may encourage students to mathematise their visual perceptions and increase their ability to attend to the quantifiable features of shape and the numeric relationships between then.

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# ELEMENTARY CHILDREN'S 3-D VISUALIZATION DEVELOPMENT: REPRESENTING TOP-VIEWS

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*The Spatial Operational Capacity* (*SOC*) *framework* (*Yakimanskaya, 1991; van Niekerk, 1997*) *guides a long-range design research study to develop spatial skills in elementary-age childrenusing Geocadabra, a dynamic computer interface. Learners engage in activities that move among 3-D models, 2-D conventional and semiotic* (*abstract*) *representations, and verbal descriptions of figures. In this presentation, we focus on children's development of top-view coding as they engage in activities that move among the SOC representations, their extension of this knowledge to invent non-conventional numeric coding systems and their spontaneous connections to their daily mathematics programs. Social-constructivist instructional approaches undergird the classroom ecology.*

## Introduction

This paper, written in the voice of researcher, A, demonstrates how 3rd and 4th grade children developed mastery at top-view coding to create assembly puzzles to challenge each other and invented their own coding systems to account for holes and overhangs in figures they constructed using loose cubes and Soma pieces (Weisstein, 1999). We share how they continued to use the coding to create assembly diagrams for 3-D figures and recognized connections to other strands in their everyday mathematics curriculum. The ongoing study is conducted in a dual-language urban elementary school within one of the largest public school districts in the mid-southwestern United States.

## Theoretical Frameworks and Perspectives

*Spatial Visualization*

The National Research Council's report, *Learning to Think Spatially* (2006), identifies spatial thinking as a significant gap in the K-12 curriculum, which, they claim, is presumed throughout but is formally and systematically taught nowhere. They believe that spatial thinking *is* the start of successful thinking and problem solving, an integral part of mathematical and scientific literacy. The National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* (NCTM, 2000) supports this view. In their early years of schooling, students should develop visualization skills through hands-on experiences with a variety of geometric objects and use technology to dynamically transform simulations of two- and three-dimensional objects. Later, they should analyze and draw perspective views, count component parts, and describe attributes that cannot be seen but can be inferred. Students need to learn to physically and mentally transform objects in systematic ways as they develop spatial knowledge. From a purely academic perspective, the importance of visual processing has been documented by researchers who have examined students' performance in higher-level mathematics. For example, Tall et al (2001) found that to be successful in abstract axiomatic mathematics, students should be proficient in both symbolic and visual cognition; Dreyfus (1991) calls for integration across algebraic, visual and verbal abilities; and, Presmeg (1992) believes that imagistic processing is an essential component in one's development of abstraction and generalization.

*Spatial Operational Capacity Framework*

The spatial operation capacity (SOC) framework (Yakimanskaya, 1991; van Niekerk, 1997) that guides our study exposes children to multiple representations through activities that require them to act on a variety of physical and mental objects and transformations to develop the skills necessary for solving spatial problems.

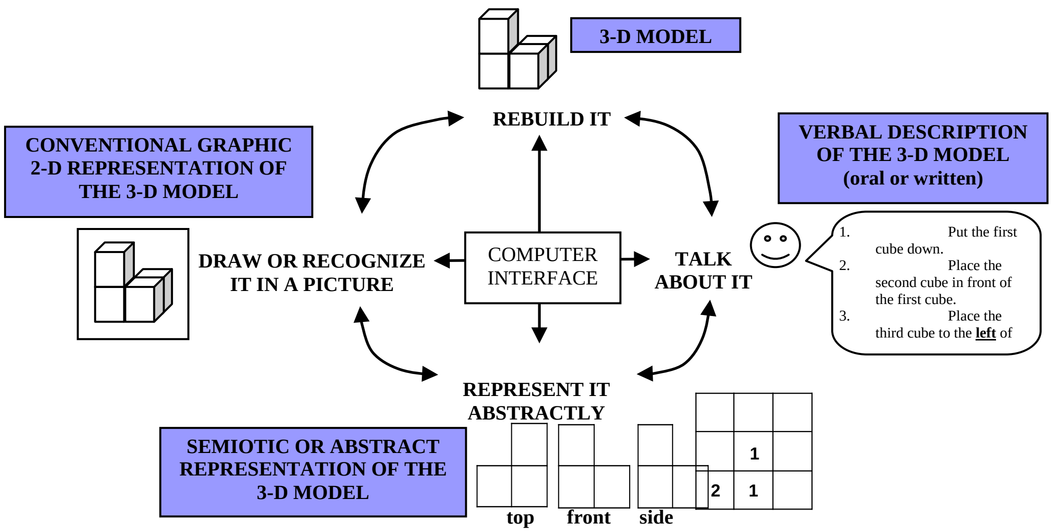



*Figure 1.* Multiple representations within 3-D visualization.

The SOC model (see Figure 1) uses:

- *full-scale* figures, that, in our study, are created from loose cubes or Soma figures, made from 27 unit cubes glued together in different 3-cube or 4-cube arrangements (see Figure 2);
- *conventional-graphic* 2-D pictures that resemble the 3-D figures;
- *semiotic* representations (Freudenthal, 1991) such as front, top and side views or numeric top-view codings that do not obviously resemble the 3-D figures; and
- *verbal* descriptions using appropriate mathematical language (Sack & Vazquez, 2008).

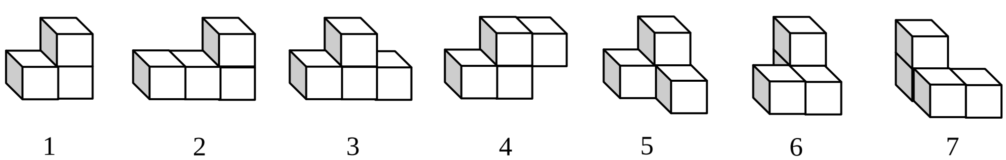



*Figure 2.* The Soma set can be made by gluing unit cubes together.

We also utilize a *dynamic computer interface*, Geocadabra (Lecluse, 2005), a tool that was not available when the SOC framework was originally developed. Through the Geocadabra Construction Box module, complex, multi-cube structures can be viewed as two-dimensional conventional representations or as top, side and front views or numeric top-view diagrams (see Figure 3). Whereas one can move around a three-dimensional model to see it from other vantage points, one may see various views of a dynamic computer-generated figure through its ability to be rotated in real time. The Geocadabra computer interface serves as a mediator of knowledge (Borba & Villarreal, 2005) rather than as a unique form of representation.

*Design Research Methodology and Data Sources*

Instructional decisions are guided by the design research methodology of Cobb, Confrey, diSessa, Lehrer, and Schauble (2003). Our intent is to support and give an account of young children's development of spatial reasoning and ultimately to create curricular resources that may be integrated into the elementary-level mathematics curriculum. Each lesson we enact is part of a design experiment in which the research team hypothesizes learning outcomes, designs instructional activities to support the outcomes, and enacts the lesson. Social constructivist approaches provide us access to student understanding in that we encourage verbal explanation and justification orally and where appropriate, in writing. During the retrospective analysis following each lesson, the research team determines the actual outcomes and then plans the next lesson, which may be an iteration of the last lesson to improve the outcomes, a rejection of the last lesson if it failed to produce adequate progress toward the desired outcomes, or a change in direction if unexpected, but interesting, outcomes arose that are deemed worthy of more attention. Data sources include formal and informal interviews, video-recordings and transcriptions, field notes, student products and lesson notes.

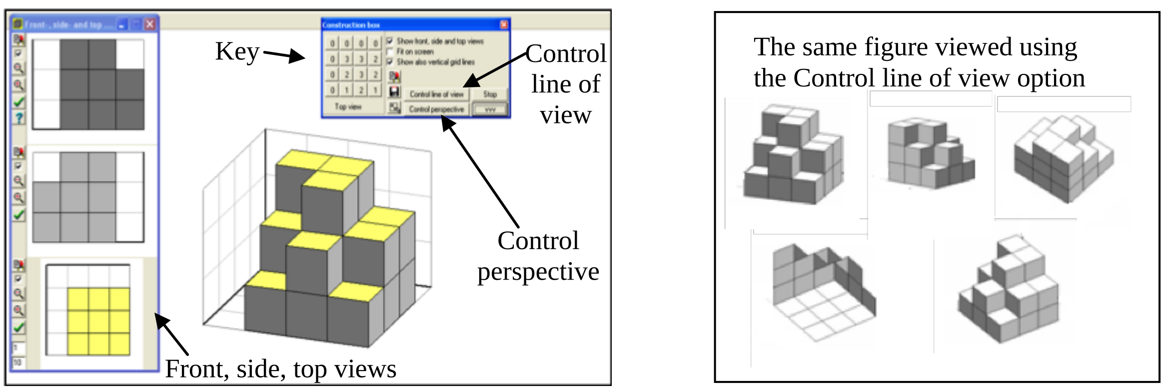



*Figure 3.* The Geocadabra Construction Box.

*Context and Classroom Ecology*

During the 2007-2008 academic year, teacher-researcher A, teacher B, and co-teacher C formed the research team that worked with a third-grade and then a fourth-grade group of children weekly (one hour per group) in teacher B's classroom during an after-school program. English and Spanish parent/guardian and student consent-to-participate forms were sent home to parents of all after-school third and fourth graders. All respondents were accepted into the program. Teacher B had taught mathematics and science to all fourth-grade participants during their entire third-grade year. Due to staffing changes for the third-grade class, she taught all core subjects to half of the school's third-grade students. Consequently, some of the third-grade participants in the after-school SOC program were not her students during the school day. However, all participants became attuned to her behavioral and communal expectations very quickly during the first month of the research program. She expected all students to develop independence by asking each other for help or support before asking the teacher, and to treat each other respectfully. Students expressed their understandings, justifications, confusions or frustrations safely in front of their peers. We rarely gave away answers or explanations. Students constructed meaning and representations for themselves. Furthermore, our design incorporated learning experiences that challenged each child according to particular readiness, interest and learning profile. Our strong attention to differentiated instruction ensured "processes and procedures that ensure effective learning for varied individuals" (Tomlinson & McTighe, 2006, p. 3). This environment supported problem-solving and fostered creativity in our participants while as researchers we were able to make sense of student understanding of 3-D structures.

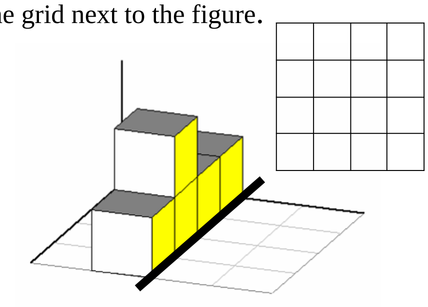
The stimuli presented to students began with simple reproductions of any of the given SOC representations into any or all of the others using single Soma figures. Later we concentrated on figures assembled with combinations of Soma figures together with rigid, congruent transformations. Activities included various game components and tasks using Geocadabra in which students had to identify the individual Soma figure(s) that made up the figure, or (de)code a top-view mapping to (re-)construct an assembly of Soma figures.

In the next section, we focus on how the children's interest in top-view coding led to invention and connections to other aspects of their daily mathematics work.

## Results

Over the course of six lessons spanning November-December, 2007, students worked with the Geocadabra Construction Box to master numeric top-view coding of 2-D conventional images. Students initially rebuilt the figures shown in a customized manual (van Niekerk, 2008) using loose cubes and then created the complete structure on the computer using Geocadabra. For an example, see Figure 4. They coordinated the top-view codings with the 2-D figures that emerged on the computer's screen. To check for understanding, we de-selected "Hide spatial model." The student then entered his or her predicted top-view coding on the screen and then clicked to un-hide the computer figure to compare with the printed 2-D figure.

Build the following figure and its mirror image on your screen. Write the correct numbers in the grid next to the figure.



Geocadabra screen for this task:

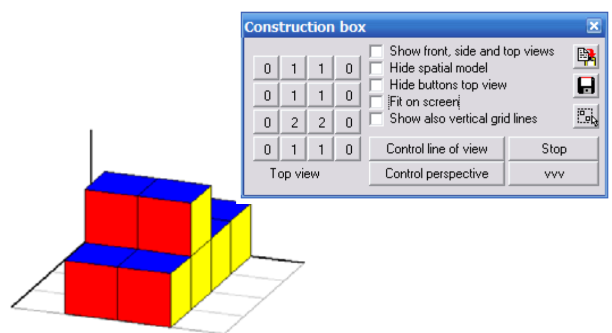



*Figure 4.* Mirror image task

A custom-created Geocadabra module, the Extended Construction Box, allowed students to construct figures with spaces and overhangs (see Figure 5). One places individual cubes or linked combinations of cubes along axial lines in a 3-dimensional octant. We challenged the children to create a numeric top-view coding system that included holes and overhangs since we were not aware of a conventional coding system of this nature. In this problem setting, students tested each other's invented codings to see if they could re-create the corresponding 3-D figures; challenged each other and offered suggestions for improvement in writing, to engage the verbal representation; and, compared and rated the invented codings. These activities provided the children opportunities to tackle open-ended problems without known solutions.

We had discovered that our students ascribed to two different but conceptually appropriate interpretations of the conventional top-view coding system (as in the original Construction Box). For example, in Figure 6, some children said that the 2 in the front left position represented a stack of 2 cubes. Others said that the 2 represented a cube on the second level, implicitly knowing that there is a cube (or a stack of cubes) supporting that cube from below. These interpretations came into conflict when students tried to decode someone else's invented coding for structures with empty spaces or overhangs especially if one's initial interpretation of a grid number was different from the coder's interpretation. For example, one student used (↑2), where the arrow meant one empty space and two cubes above the empty space, which aligns with the first interpretation described above. Another used (**3) to mean two empty spaces and one cube

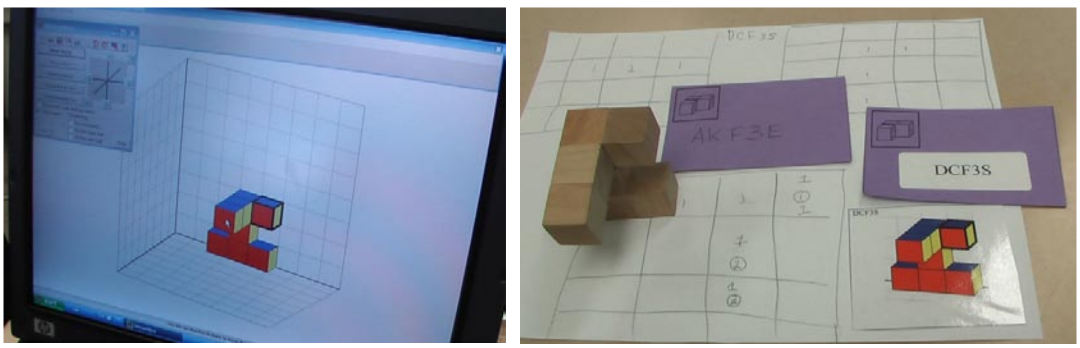
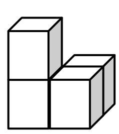



*Figure 5.* Extended Construction Box, student-created task card, coding and 3D figure.

on the third level. His coding sprang from the second interpretation of the conventional coding described above. The plethora of invented codings created class-wide confusion. Our video clips show how Teacher B and the class negotiated to select and refine a new class-wide convention that reflected the conventional coding and included holes and overhangs. Examples of various student-created codings are shown in Figure 8. Sarah's code (Fig. 8(c)) was selected with a modification to change the square to a circle to denote the number of empty spaces.



| | | |
|---|---|---|
| | | |
| | 1 | |
| 2 | 1 | |

*Figure 6.* Conventional coding system developed through the Construction Box.

The traditional and class-wide coding conventions continued to be used to create assembly plans for multi-Soma figures. By presenting this as a puzzle-creation and solving activity, the children challenged each other and checked each other's work against the task cards they created using the Extended Construction Box. They set their own levels of difficulty and carefully recorded their assembly diagrams carefully. See Figure 7 for examples of student work.

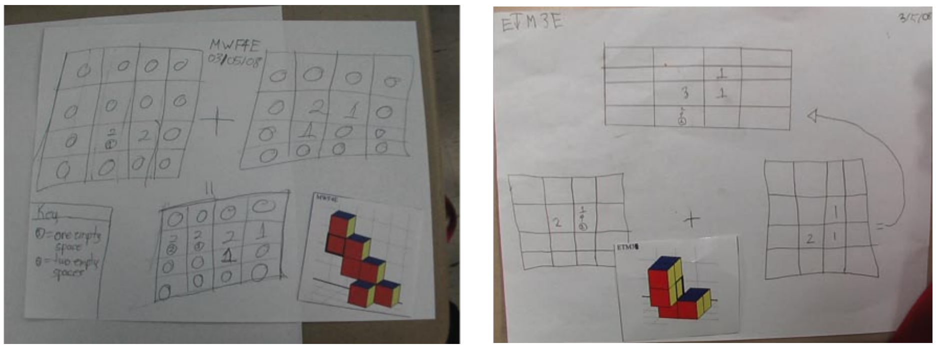


*Figure 7.* Student-created task cards and assembly puzzles.

For the end-of-year party, we decided to order a cake designed by the children, by combining all seven Soma figures into a rectangular base shape. Considering factor pairs, they determined possible dimensions for the base. Everyone was successful creating patterns on 3x8 or 4x6 bases. Vena, a third-grader, believed she would be able to construct a figure on a 2x12 base. We encouraged her to explain why this was not possible. She simply shrugged. When the program resumed the following year, Vena immediately returned to her self-created problem of building the cake on a 2x12 base. She generally finds mathematics difficult but this problem is hers and she will probably wrestle with it until she convinces herself that it cannot be solved. In separate incidents, Sarah (third grade) and Debra (fourth grade) noticed that Somas #1, #5, #6 and #7 all had the same 3-square footprint and that these pieces could be interchanged within their cakes to produce many more different patterns. This observation became an opportunity for the two groups to transfer to their academic class-work. They successfully completed tabular representations to show how many different permutations these interchanges would produce. In our presentation, we will share examples of students' assembly diagrams and a video-clip of a particularly interesting "cake" in which the ends of the student's rectangular base rotate up to lock into the 27-unit cube.

## Discussion/Conclusions

*Re-invention and Invention*

In this learning environment, through small-group and whole-class discussion, students formalized, or re-invented concepts that later became the foundation for further mathematization. Although these concepts may be new to these students, they generally constitute age-old mathematics that form the basis of school mathematics curricula. Freudenthal uses the term "re-invention" (1973, p. 120), sometimes known as discovery-based instruction for this approach to learning mathematics. Through debate and negotiation, the children adopted a particular system to be the class-wide convention for all to use. More than re-inventing mathematics, our students invented a new mathematical coding system. Our confidence in managing such an approach comes from cumulative years of providing our students opportunities to share problem-solving strategies and solutions with their peers in order to deepen their knowledge of the concepts at hand.

*Connections to Classroom Concepts*

The permutations connection that arose out of the cake-designing activity is one example of how our activities integrated with classroom-mathematics concepts. Others include figures created using loose cubes or Soma figures that exhibited repeated or step-wise patterns. These presented opportunities for children to calculate how many unit cubes were in the structure. Finding the total number cubes by in multiple ways, such as by horizontal and then vertical slicing, developed students' number and spatial senses symbiotically. A final check was established by looking at the totals of the numbers in the top-view coding grids. These activities helped establish a foundation for the concept of volume. When the children returned the following year, we extended the permutation connection from the cake activity to finding and representing as many 24-cube rectangular boxes as possible. We expected students' records-of-action to be tables showing length, width and height, but they immediately reverted to drawing their figures using top-view numeric coding. They all agreed that the numbers in the top-view grid represented the heights of the boxes. In reflection, the connection to volume was implicit during our first year of the study, especially when the children were developing mastery with conventional top-view coding. Now in our second year of the study, as we work with a much larger group of third-graders, we are making the volume and top-view grid connection explicit by asking "how many" in almost every activity. We concur with Tall et al (2001), Dreyfus (1991) and Presmeg (1992) in the importance of integrating spatial visualization with symbolic forms at the concept-development stage of mathematical learning.

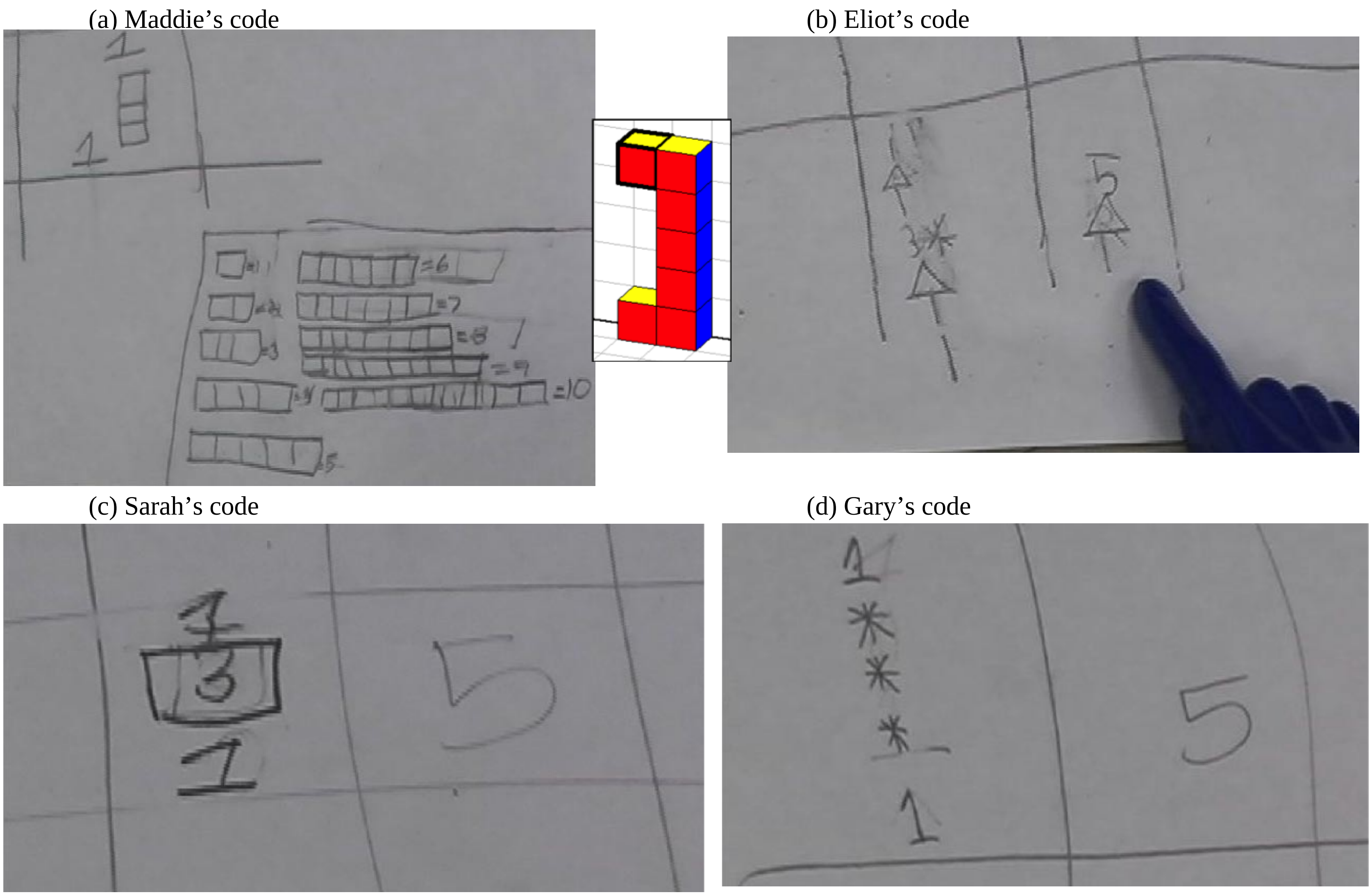



*Figure 8.* Top-view numeric coding systems to represent holes in cube figures.

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# ELEMENTARY PRESERVICE TEACHERS' AREA CONCEPTIONS INVOLVING THE NOTION OF PERIMETER

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*This study investigated elementary preservice teachers' conceptions about the area of parallelograms through the processes of shearing and squashing. When asked to compare areas of parallelograms, the preservice teachers revealed two opposite misconceptions along with lack of a dynamic understanding of the quantity of area: 1) increasing perimeter decreases area and 2) increasing perimeter increases area.*

## Introduction

When asked about the area of a two-dimensional shape, do you first focus on the boundary of the shape or the region that the shape covers, or both? Regardless of whether the question is about the area or the perimeter, both the boundary and the region covered may be in our primary consideration rather than either of them in an exclusive manner. Interestingly, when asked to compare areas of parallelograms, many of the elementary preservice teachers in our mathematics content course paid attention to the boundaries, more specifically, to the changes in the boundaries from a related rectangular shape. Similarly, in response to the question about the perimeter of a shape, many of the preservice teachers attempted to change the shape so that they could easily produce the area and use the changed shape to find the perimeter measure. This paper shows elementary preservice teachers' various attempts to relate area and perimeter.

## Lessons Taught and Theoretical Background

This study is based on the lessons for the second mathematics content course that is required for those who pursue or major in elementary education (K-8). The author taught two sections of the course where 43 students attended in total. Most of these students were learning the school geometry that was planned and instructed especially for elementary preservice teachers for the first time.

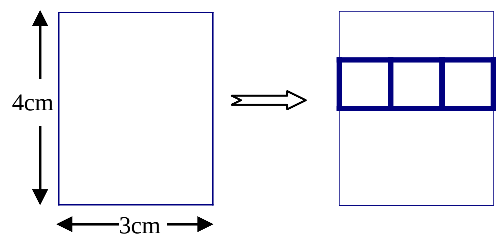
We began by describing how the topic of area measurement was introduced in our mathematics content course for elementary preservice teachers and what research results contributed to shaping our lesson. The preservice teachers in the course were first asked to find the area of a 3cm by 4cm rectangle. Every student answered 12[i] and many of them justified the answer using the area formula $length \times width$ or $base \times height$. The use of the area formula, especially starting with the area formula from the very beginning stages of the topic, has been criticized as leading to difficulties and poor understanding of area measurement (Zacharos, 2006). Three salient shortcomings were reported with respect to the use of the area formula (Baturo & Nason, 1996): reinforcing the perception of area based on the boundary of a shape; avoidance of generating the unit of area; and likely disregarding of the array notion of multiplication due to the dominant notion of multiplication as repeated addition. Along with

* I would like to thank Diane Dowd for her reading of this manuscript.

these concerns, three research ideas affected our lesson plans: (1) Schwartz's idea of area as an attribute of a quantity: "The confounding of these two attributes [perimeter and area] of shape is a serious obstacle to the learning of area measure" (Schwartz, 1996, p.9); (2) Simon and Blume's idea of the quantitative reasoning involved in the evaluation of the area of a rectangle: "Important aspects of this [quantitative] reasoning include the anticipation of a rectangular array of units as the structure of the area quantity" (Simon & Blume, 1994, p. 472); and (3) Steffe's idea of multiplicative reasoning as a way to coordinate two levels of units (Steffe, 1994).

*Area of a Rectangle*

To address the preservice teachers' dominant use of the area formula when justifying their answer of 12, the instructor started her instruction on the topic of area by focusing on the following questions: Why do you multiply 3 by 4 to determine the area? Where can you see the answer of 12? What does your answer refer to? These questions intended to encourage the preservice teachers to generate a unit of area while differentiating it from a unit of length and to produce the measurement of area based on a multiplicative way of thinking (Figure 1).




- There are four rows and each row has three squares.
- There are four rows of three 1 by 1 squares.
- 12 is the number of squares.
- When 4 is multiplied by 3 to find the area, 3 represents the number of squares in each row.

Figure 1

*Area of a Triangle*

Based on the conceptual understanding of the area of a rectangle, our preservice teachers were asked to produce the area measure of a triangle and explain why they need to know the length of the segment perpendicular to the base of the triangle in order to produce the area (Figure 2).

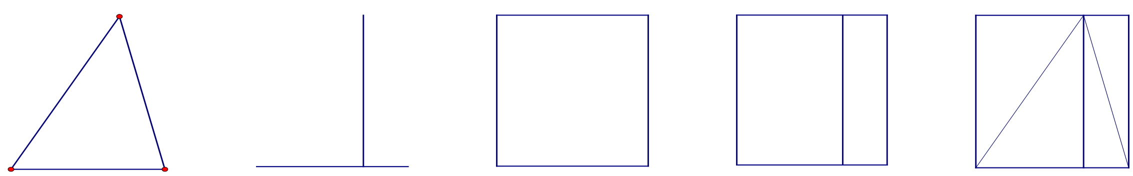


Figure 2

*Area of a Parallelogram*

The preservice teachers were then asked to find a way to determine the areas of parallelograms by relating the parallelograms to rectangular shapes. One student produced a rectangle that shares one side with a given parallelogram by applying a moving and combining principle (Figure 3).

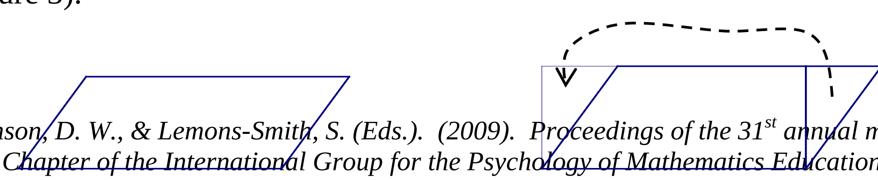




Figure 3

As another way to relate a rectangle to a parallelogram for finding the area measure, a shearing process was introduced using toothpicks. The shearing process is illustrated in Beckmann's textbook (2008) as follows: "Start with a polygon, pick one of its sides, and then imagine slicing the polygon into extremely thin (really, infinitesimally thin) strips that are parallel to the chosen side. Now imagine giving those thin strips a push from the side, so that the chosen side remains in place, but the thin strips slide over, remaining parallel to the chosen side and remaining the same distance from the chosen side throughout the sliding process" (p. 601). According to Cavalieri's principle for areas, the sheared shape has the same area as the original shape.

### Emergence of the Notion of Area Measurement with a Concern about Perimeter

Our preservice teachers seemed to have no problem in recognizing that changing a rectangular shape into a parallelogram that is not a rectangle through the shearing process results in preserving the area but changing the perimeter. However, many of the preservice teachers had difficulty implementing the inverse way of shearing, which is transforming a parallelogram that is not a rectangle into a rectangular shape. They desired to keep the perimeter of the parallelogram when shearing, which is referred to as squashing[ii].

Such a way of confounding shearing and squashing seemed to lead them to conjure up the *same A-same B* intuitive rule[iii] with respect to the process of squashing. That is, assuming that shearing preserves the area of shapes, they proceeded to the process of squashing and concluded that if a shape is changing while preserving its perimeter, the changed shape must have the same area as the original (Figure 4).

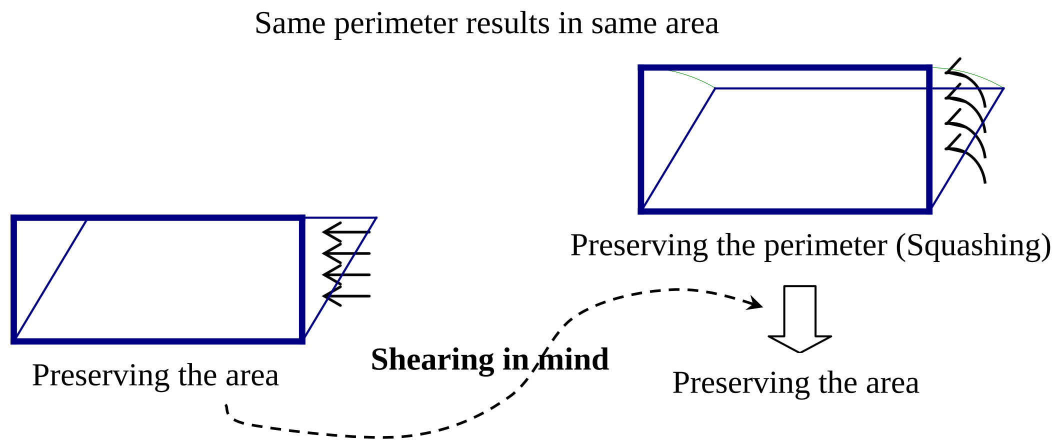



Figure 4

In addition, many of the preservice teachers showed a perception that area determines perimeter, which is the opposite of the above idea that perimeter determines area. In response to the question about determining the perimeter of a shape, they seemed comfortable using a *same A-same B* intuitive rule. That is, they changed a given shape into the one that has the same area as the original and determined the perimeter of the changed shape (e.g., Figure 5), or they calculated the perimeter by applying the procedure they would have followed to determine the area of the shape (e.g., Figure 6).

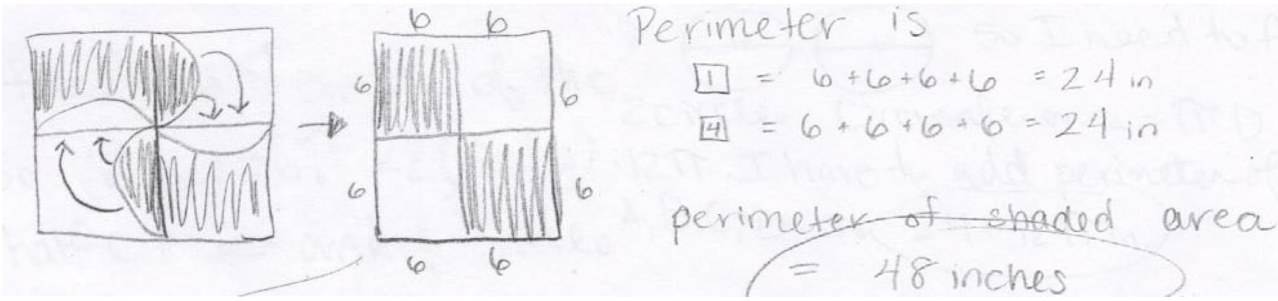



Figure 5

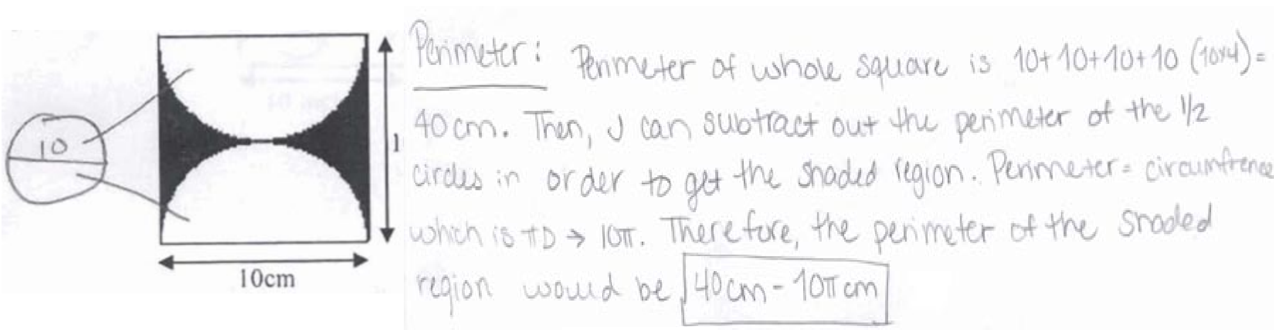
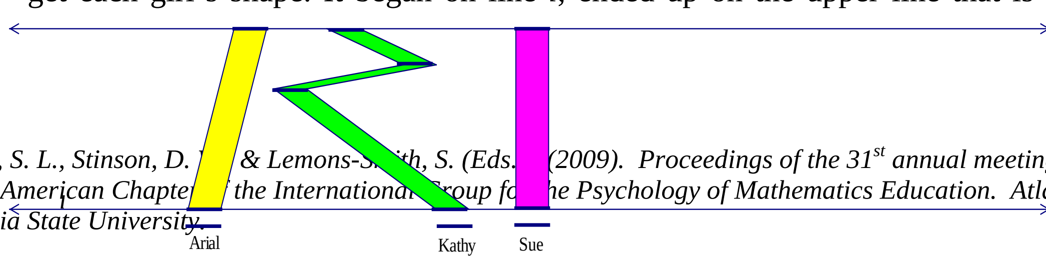



Figure 6

**Investigating Changes in Area by Looking at Changes in Perimeter**

In order to further investigate preservice teachers' conception of the relationship between perimeter and area, we proposed the following problem to the students in our mathematics content course for elementary preservice teachers.

> Arial, Kathy and Sue were planning to create a shape for planting on an empty plot. They were told to make a shape using a 3-foot long stick. Arial created her region by sliding the stick slightly toward the right, as shown at left below; Kathy made one by sliding her stick using a zigzag motion, as shown in the center; and Sue created a rectangle by sliding her stick straight forward, as shown at right below. When sliding their sticks, all of the girls kept their sticks parallel to the line *l*. The girls were wondering which region (if any) has a larger area than the other. Elaborate your reasoning as you comment on the girls' wondering about the regions. [Note: The stick is represented by the horizontal segment show below line *l*. This stick was slid directly upward, as shown in this picture, or upward and to the left or right, to get each girl's shape. It began on line *l*, ended up on the upper line that is




*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

parallel to line *l*, and remained parallel to line *l* throughout its journey.]

We found a group of preservice teachers had difficulty visualizing using the given stick to create two-dimensional shapes. Many of the preservice teachers in the group thought each shape itself represented the 3-foot long stick. This kind of response suggests a static perspective of area (Baturo & Nason, 1996), which is perceiving area as an amount of region that is enclosed within a boundary. In other words, the boundary of the shape is needed to determine the region for which the area is being asked, so the preservice teachers would need to have the boundary specified before investigating areas of each shape.

There was another big group of preservice teachers who attempted to compare the areas of the shapes based on a relationship between the boundaries of the shapes. Interestingly, two opposite ways of thinking emerged: "Increasing or preserving the perimeter decreases the area" vs. "Increasing the perimeter increases the area". Let's closely look at theses two ways of reasoning.

<u>Argument 1:</u> Increasing or preserving the perimeter decreases the area

- Anne: "Sue's stick [rectangular shape] would be a bigger area because it is fattest. The more you shorten the height the less the area is. Below (Figure 7), each shape has a base of 7 but the height changes every time because it is getting more slanted. This causes the area shrink."

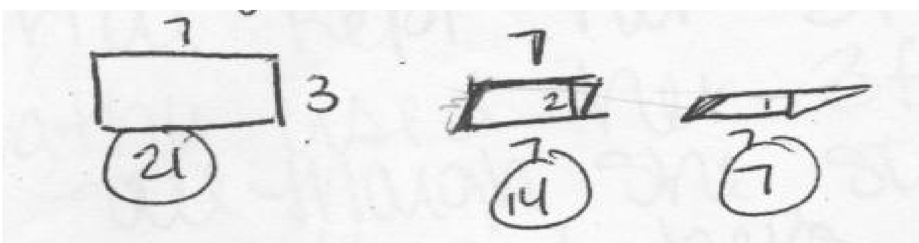
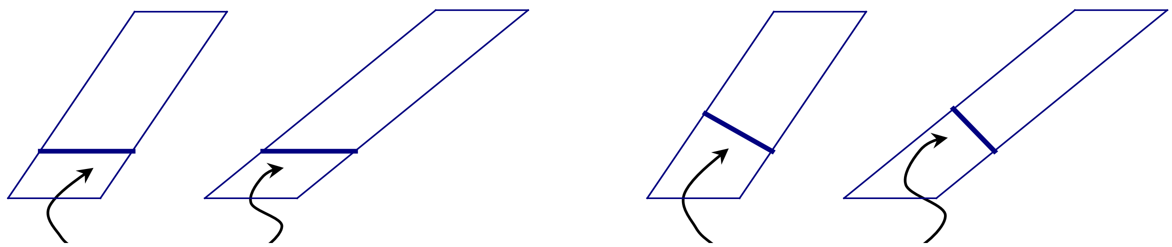



Figure 7

- Molly: "Kathy's [zigzag shape] is smaller because it is zigzagged. … Kathy's stick got much thinner and therefore having a smaller area in those smaller parts."

Both Anne and Molly's reasoning suggests that they are perceiving that (1) shape gets thinner as its sides become slanted and (2) a thinner shape has a smaller area than a thicker shape. However, they also took opposite positions in that Anne seems to have preserved the perimeter while Molly seems to view it as having changed. That is, Anne implemented squashing, instead of shearing, with respect to Sue's plot [or rectangular shape] in order to produce Arial's shape [a parallelogram that is not a rectangle]. On the other hand, Molly seemed to employ the shearing process properly, in that she did not express any view of the height decreasing. Anne and Molly's mentioning about being thinner as the rectangular shape slanted indicates they may have viewed the *width* of the shape as a segment that is perpendicular to each slanted side, and this view of a thinner *width* caused them to determine that Arial's area was smaller than Sue's. Notice that the term *width* can be interpreted two ways in a parallelogram (Figure 8).

Suppose a student perceives a parallelogram as getting "thinner" through the shearing process as it becomes more slanted. That may suggest the student is focusing on the segment that is perpendicular to the slanted sides of the parallelogram as the width of the parallelogram (e.g., Figure 8 (b)). In that case, the student is likely to think the thinner shape has the smaller area unless the student notices that the two measures—of the "thinner" perpendicular width and a slanted side—will be used to determine the area. On the other hand, if a student sees the width of a parallelogram in terms of the segment that maintains the distance between the two slanted sides, she or he would not have trouble noticing that the degree of the slant does not affect the area unless the student confounds shearing and squashing.



(a) The width is a segment that maintains the distance between the slanted sides

(b) The width is a segment that is perpendicular to a slanted side

Figure 8

<u>Argument 2:</u> Increasing the perimeter increases the area

- Crystal: "Kathy [zigzag shape] has a larger area than the others. All 3 sticks are covering the same distance. However, Kathy's if you were to lay it out straight is much longer than the other two. They all have the same width, but different length so Katy's has a larger area."
- Maggie: "Sue [rectangular shape] would have the smallest area because the shortest distance between two points is a straight line. We calculate the area of their plots by multiplying 3 by the length of their plot (Kathy we would have to use segments). So the 3 is constant. Arial's [parallelogram that is not a rectangle] length is a little longer because she goes at a diagonal. Lastly Kathy's length would be the longest because she zigzags back and forth. So it would go smallest to largest area respectively, Sue, Arial, and Kathy."
- Joel: "Arial's area is bigger than Sue's area because the line *a* is bigger than line *b* in my picture [shown at the right, Figure9]. The area of *a* is $1\times\sqrt{2}$ whereas the area of *b* is $1\times 1$ which is less. As for Kathy the same logic should hold for her as well and she should be bigger than the other 2, so long that the width of all the lines are the same."

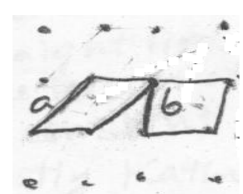



Figure 9

All of the students holding Argument 2 focused on the changes of the lengths of each shape. By length, they meant the side or the measurement of the side of each shape that goes from the line *l* to the other line that is parallel to *l*. It is interesting that they all considered the width as remaining the same in every shape; that is, unlike the previous group of the students who thought

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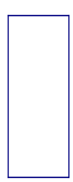
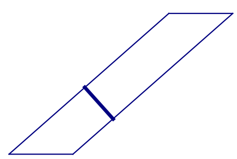
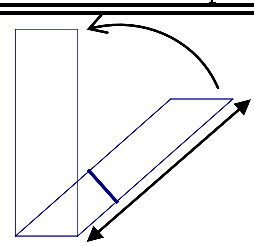
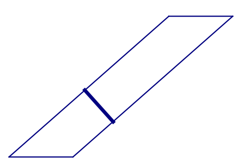
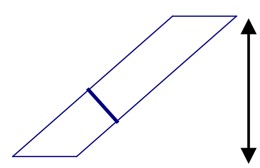
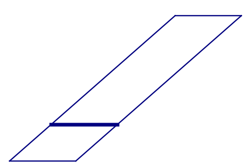
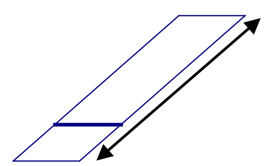
increasing or preserving the perimeter decreases the area, they were not concerned about the shapes getting “thinner” as they become more slanted. Therefore, the lengths became the only factor they considered to compare the areas between the shapes. They reached conclusions about the area comparison only by looking at the lengths of the sides that are not width, the given 3 feet, and by squashing the shapes into a rectangular shape. Joel’s drawing clearly shows his confounding of shearing and squashing.

## Results

This study researched elementary preservice teachers’ conceptions about area through investigating their ability to implement the shearing process and to differentiate it from the squashing process. Three different ways of confounding shearing and squashing were found. The following table summarizes how those ways of thinking led the preservice teachers to mistakenly both conceive the areas of parallelograms and relate the areas to the perimeters.

| Original shape | What is the width of the new shape? | What is considered in order to determine the area of the new shape? | Which process is being intended, shearing or squashing? | Misconception involved in the process employed. |
|---|---|---|---|---|
|  |  |  | - Mixing squashing and shearing<br>- The width of the sheared strip was changed but the same height was assumed for the area comparison. | Preserving the perimeter decreases the area. |
| |  |  | - Shearing<br>- The conception of the width of the sheared strip changed. | Increasing the perimeter decreases the area. |
| |  |  | - Squashing<br>- The width of the sheared strip was preserved. | Increasing the perimeter increases the area. |

## Discussion

This study shows that investigating the shearing and squashing process while differentiating them can provide a good ground for students to develop a dynamic and static perspective of area at the same time. Baturo and Nason (1996) argued that “Area needs to be considered from two perspectives, namely static (a description of something at a certain point in time) or dynamic (a mapping or function from one thing to anther). … Underlying this dynamic perspective of area is the notion of the definite integral of differential calculus. However, the dynamic perspective is often not included in curriculum documents, thus limiting students' understanding of area." (p. 238-239). Many of our elementary preservice teachers had difficulty implementing the shearing process flexibly. We suggest that it may be due to an inability to generate two-dimensional shapes (plots shown in the stick-plot problem, especially the zigzag plot) using a one-dimensional attribute (the given definite length of stick) because the study reveals that our preservice elementary teachers do not have a good picture in their minds of what is to be sheared. In addition, confounding of and difficulty in differentiating the shearing and the squashing indicate that our preservice teachers' perspectives of area are inflexible in that the squashing process requires them to perceive area through reasoning about the perimeter, whereas shearing permits them to conceive perimeter through reasoning about the area. The impact of balancing these two perspectives of area on preservice teachers' conceptual understanding of area measurement needs to be investigated further.

# THE ROLE OF PRIOR KNOWLEDGE IN THE DEVELOPMENT OF STRATEGY FLEXIBILITY: THE CASE OF COMPUTATIONAL ESTIMATION

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*The ability to estimate is a fundamental real-world skill where strategy flexibility is particularly critical. Here, we consider the role of students' prior knowledge of estimation strategies in the effectiveness of interventions designed to promote strategy flexibility across two recent studies with 5th and 6th grade students. Results indicated that students who exhibited high fluency at pretest were more likely to increase use of estimation strategies that led to more accurate estimates, while students with less fluency adopted strategies that were easiest to implement.*

## Introduction

Estimation is a critically useful skill in everyday life and in mathematics, as encapsulated in the "Adding It Up" report from the National Research Council: "The curriculum should provide opportunities for students to develop and use techniques for mental arithmetic and estimation as a means of promoting deeper number sense" (2001, p. 415). Unfortunately, current instructional methods have not been particularly effective at supporting estimation knowledge. It is well documented that a large majority of students have difficulty estimating the answers to problems in their heads (e.g., Reys, Bestgen, Rybolt, & Wyatt, 1980). Given the challenges of mentally computing estimates, it is especially important to have a broad repertoire of estimation strategies and to select the most appropriate (often, computationally easiest) strategy for a given problem and goal. Thus, students' difficulties with computational estimation partially results from a lack of strategy flexibility in this domain.

Strategy flexibility is defined as (1) knowledge of multiple strategies and (2) adaptive use of strategies, based on accuracy and efficiency. Knowledge of multiple strategies has clear benefits for learning and performance; for example, learners with knowledge of multiple strategies at pretest are more likely to learn from instructional interventions (e.g., Alibali, 1999). People who know multiple strategies also learn to choose among them based on their accuracy efficiency; adaptive choice is a fundamental feature of problem-solving expertise and is also a fundamental mechanism supporting learning and development (Siegler, 1996).

### *Flexibility and Strategies for Estimation*

There are numerous strategies that can be used to compute estimates. Of particular interest here are students' knowledge of and use of three strategies for estimating two-digit multiplication problems. One commonly taught strategy is *round both*, which involves rounding both numbers to the nearest multiple of ten. Another strategy that can be used is *round one*, which involves rounding only one number to the nearest ten. Finally, a third strategy that we explore here is *truncation, or trunc*, which involves covering up or ignoring the ones digits and multiplying the tens digits and subsequently adding two zeros to the resulting product. Note that *trunc* is a less familiar strategy than the other two, but it is relatively easy and fast and has been advocated for by researchers on computational estimation for these reasons (e.g., Sowder & Wheeler, 1989).

Flexibility in estimation includes choosing the most appropriate strategy for computing an estimate for a given problem. Choosing an appropriate strategy in estimation is complicated by the

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presence of multiple, at times competing goals. On the one hand, it may be desirable to generate an estimate that is close to the exact answer. However, on the other hand, one may seek to compute an estimate using the strategy that is computational easiest. Ease is particularly important for estimation because one often must estimate mentally. (Note that it could be argued that ease of computation is a subjective and individual judgment; however, in our prior work with middle-school students (Star & Rittle-Johnson, in press), we have shown that *trunc* and *round one* are both easier (e.g., faster) strategies to implement than *round both*.)

The studies described here explore the development of students' flexibility for computing estimates. We were interested in students' learning of *round both*, *round one*, and *trunc*, and when and how students began to use these strategies to optimize for ease and/or for proximity for given problems.

*Prior Knowledge and Flexibility*

In our prior work, we have identified interventions that reliably lead to gains in flexibility. Our interventions built upon cognitive-science research suggesting that comparing multiple examples is a fundamental pathway to flexible, transferable knowledge (e.g., Gentner, Loewenstein, & Thompson, 2003). However, in a recent study, it became clear that students' prior knowledge may impact the effectiveness of interventions designed to promote flexibility (Rittle-Johnson, Star, & Durkin, 2008). We found that students who were not initially familiar with one of the target problem-solving strategies learned less if they were exposed to multiple strategies simultaneously, rather than sequentially. This study raised an interesting question that is not well explored in the existing literature on flexibility. If an instructional goal is to promote flexibility, then is it more effective to teach novice students multiple strategies from the beginning, or should learners develop initial fluency with one strategy before increasing their repertoire to include multiple strategies?

Our prior work would suggest that learners need initial familiarity with one strategy before they can become flexible in the use of multiple strategies (Rittle-Johnson et al., 2008). Research on analogical reasoning also provides indirect support for this position: learning from comparing unfamiliar examples is often difficult for young children (e.g., Gentner, Loewenstein, & Hung, 2007) and for college students who do not receive additional instructional support (Schwartz & Bransford, 1998).

*Present Studies*

The goal of the present paper was to continue our exploration of the role of prior knowledge in the effectiveness of interventions designed to promote flexibility, this time in the case of computational estimation. In Study 1, 65 fifth graders began the study as fluent users of the *round both* strategy, while in the Study 2, 157 5th and 6th graders began the study with moderate to low prior knowledge of strategies for computing mental estimates. Note that elsewhere we report on the effects of the intervention on improving flexibility (Star & Rittle-Johnson, in press); here, our interest is in prior knowledge and flexibility.

## Method

*Participants*

In both Study 1 and Study 2, participants were 5th and 6th grade students. Study 1 was conducted in an urban, private school (School A), and Study 2 was conducted in the same school as well as in a small, rural school (School B).

In Study 1, students in four classes of 5th graders (n = 65; 33 girls) participated. Students' mean age was 10.88. The fifth grade was comprised of a majority of Caucasian students and 25% minority students, of whom 18% were African-American. Approximately 10% of students at School A receive financial aid. All students were taught by the same mathematics teacher and had received some prior instruction on estimating answers to multiplication problems.

In Study 2, participants were fifth- and sixth-grade students from two schools. In School A, 69 fifth-grade students participated (32 girls). There were four fifth-grade mathematics classes (all taught by the same teacher) at the school. Students' mean age was 10.6 years; a majority were Caucasian (23% minority, with 13% African-American). At School B, 45 fifth graders and 46 sixth graders participated. At School B, 5th grade students' mean age was 10.7 years (range: 10.0 years to 11.8 years) while sixth grade students' mean age was 11.8 years (range: 11.0 years to 13.1 years). There were two fifth grade classes (taught by the same teacher) and two sixth grade classes (taught by the same teacher). Most of participating students were Caucasian. Approximately 36% of students at School B received financial aid. Across schools in Study 2, teachers had not taught computational estimation in any of the classes, although some students had received limited instruction on computational estimation in previous grades. Three students were dropped from Study 2 because they were absent from school and missed more than one intervention session. Thus the analysis below for Study 2 includes data from a total of 157 students.

*Materials*

*Intervention*. The interventions in Study 1 and Study 2 were largely the same. Students were presented with a packet of worked examples, showing hypothetical students' estimates and estimation strategies for multiplying two-digit integers. The worked examples focused on the three estimation strategies discussed above. Each packet contained 32 worked examples, with questions at the bottom of each page prompting students to reflect on the estimation strategy or strategies demonstrated on that page. In addition, practice problems were integrated into each packet, where students were asked to compute estimates and answer questions about their choice of strategy. Students also received a brief whole-class lesson and a brief homework assignment each day.

*Assessment*. The assessments for Study 1 and Study 2 were very similar. Within a study, the same assessment was used as an individual pretest and posttest and was designed to assess procedural knowledge, flexibility, and conceptual knowledge. The *procedural knowledge* measure assessed knowledge of how to estimate, using both whole-number multiplication problems (six problems, such as 12 x 24 and 113 x 27) and transfer problems that involved decimal numbers or division (six problems, such as 1.19 x 2.39 and 102 ÷ 9). *Flexibility* was assessed in two ways. First, flexible *use* of strategies was assessed by examining students' strategy use on the six whole-number multiplication problems. Second, flexible *knowledge* of strategies was assessed by items designed to tap students' ability to recognize, implement, and evaluate multiple strategies for computing estimates. Flexibility *knowledge* items fell into three categories: (a) Knowledge of multiple strategies; two questions asked students to compute estimates in three different ways; (b) Recognize and evaluate ease of use; two questions assessed whether students knew which strategies were computationally easier to implement; and (c) Recognize and evaluate closeness of estimate; four questions assessed whether students knew which strategies resulted in an estimate that was most proximal to the exact value. Finally, *conceptual knowledge* items assessed students' knowledge of core concepts related to estimation. The items focused on definitions of estimation as well as acceptance of multiple strategies of estimation and multiple values of estimates and were modified from past research (Sowder, 1992; Sowder & Wheeler, 1989).

*Procedure*

In both Studies 1 and 2, the study occurred during one week of students' regular mathematics class. The procedures for the two studies were very similar. On the first day, students completed a 30-minute written pretest and then were provided with a 10-minute introduction lesson by a member of the research team. On Days 2 and 3, students were divided into pairs to work on the intervention packet. During the partner work, the pairs of students were asked to first explain their answers to the explanation prompts verbally to one another and then write down a summary of their answer on the packet. At the conclusion of each class, students were given the same brief homework assignment to practice estimating. In Study 1, students received a wrap-up lesson and completed the posttest on Day 4. In Study 2, students received the wrap-up lesson on Day 4 and completed the posttest on Day 5. Students in Study 2 were given additional time to work on the packets because they were expected to need more time given that they were less familiar with the target content.

## Results

We begin by describing students' prior knowledge of estimation strategies at pretest and then explore the extent that students' prior knowledge impacted the development of strategy flexibility.

*Knowledge and Strategies at Pretest*

*Measures of conceptual knowledge, procedural knowledge, and flexibility*. In Study 1, students' scores at pretest were quite high on all measures, indicating substantial knowledge of estimation strategies prior to our intervention. At pretest, students had quite advanced levels of both procedural knowledge of estimation and procedural flexibility, as well as some conceptual knowledge of estimation. In contrast, students in Study 2 began the study with substantially less knowledge of estimation strategies and concepts. For example, Study 2 students on average were able to generate accurate estimates for 4 or 5 of the 12 pretest procedural knowledge items, whereas students in Study 1 on average were able to generate accurate estimates for 12 of these items.

*Students' strategies on familiar procedural knowledge items*. Differences in prior knowledge could also be seen in students' estimation strategies at pretest on the whole-number multiplication problems. In Study 1, almost all students began with considerable fluency with the *round both* strategy; 92% of participants used *round both* on at least one problem. Over one-third of Study 1 students were also familiar with *round one*. In contrast, only 49% of students in Study 2 used *round both* on any problem at pretest, and only 17% used *round one*.

*Flexibility Knowledge at Posttest*

In both Studies 1 and 2, students made gains in their flexibility knowledge as well as in procedural and conceptual knowledge. The similar gains in flexibility knowledge came about despite the stark differences in prior knowledge of estimation. In Study 1, students' scores on the flexibility knowledge measure rose from 73% to 89%, while in Study 2 the gains were from 46% to 68%.

However, looking more closely at the three subscales in the flexibility knowledge measure, a more complex picture of the differences between Study 1 and Study 2 emerged. First, consider the multiple ways subscale, which assessed students' knowledge of multiple strategies for generating estimates. Study 1 students' score rose from 75% to 92%, while Study 2 students' scores improved much more dramatically, from 24% to 62%. For example, one question in this subscale asked students to generate an estimate for 12 x 36 in three different ways. At pretest, 77% of Study 1 students were able to generate an estimate for this problem in at least two ways; at posttest, 98% were able to do so. On the same problem, only 11% of Study 2 students were able to generate an estimate in at least two ways at pretest, while 45% were able to do at posttest. Study 1 students were almost at ceiling at pretest in their knowledge of multiple strategies, while Study 2, who began the study knowing fewer strategies, experienced much more substantial gains.

A somewhat different picture emerged when examining the ease subscale, which assessed students' knowledge of which strategies led to estimates that were the easiest to compute. Students' scores on the ease subscale were comparable at pretest across the two studies, but Study 1 students' scores grew substantially from 62% to 89% while Study 2 students' gains were more moderate, from 58% to 72%. For example, one question on this subscale asked students to evaluate whether *round both* or *round one* was easiest to use for computing an estimate for 27 x 39. (*Round both*, 30 x 40, is easier to mentally compute than *round one*, 27 x 40.) Study 1 students' scores on this item grew from 67% correct to 91% correct, while Study 2 students' scores grew only from 70% correct to 73% correct. The other question on the ease subscale asked students to evaluate whether *round both* or *trunc* was easiest to use for computing an estimate for 172 x 234. (*Trunc*, or 1[redacted] x 2[redacted], is easier than *round both*, 170 x 230.) Study 1 students' scores on this problem grew from 59% to 86% correct, and Study 2 students' scores went up comparably, from 45% to 71%. Students in both studies made comparable gains in their recognition of the relative ease of the *trunc* strategy, but Study 1 students made greater strides in their ability to identify the relative ease of *round one*.

Finally, consider students' evaluation of which strategies provided more proximal (i.e., closer) estimates. Students' gains from the two studies were quite similar, from 76% to 86% (Study 1) and from 56% to 69% (Study 2). Thus, students in both studies made similar strides in their ability to evaluate strategies based on which yields the closer estimate. For example, students were asked to evaluate (without computing the exact value) whether *round both* or *round one* gave the closer estimate for 34 x 42 and 9 x 48 (*round one* is closer for both problems; 92% and 83% of Study 1 students answered these two questions correctly at posttest, as compared to 74% and 65% of Study 2 students at posttest) and whether *round both* or *trunc* gives a closer estimate for 21 x 39 (*round both* is closer; 86% correct at posttest in Study 1 and 72% correct in Study 2) and 31 x 73 (*round both* and *trunc* give the same estimate; 96% correct at posttest in Study 1 and 66% correct in Study 2). While Study 1 students' performance was higher on all of these items in this subscale, gains from pre- to posttest were quite similar for Study 1 and Study 2 students, indicating similar growth in students' ability to think about estimation strategies and proximity.

Our comparison of students' scores on the independent measure of flexibility knowledge suggests the following with respect to the role of prior knowledge in the development of flexibility. First, students with low prior knowledge in Study 2 made the greatest gains in their knowledge of multiple strategies. Study 2 students began with relatively little knowledge of strategies other than *round both*, and as a result of the study, increased their knowledge of *round both* as well as *trunc* and *round one*. Second, in addition to learning new strategies, Study 2 students also gained an appreciation of the relative ease of *trunc* over *round both* for some problems. In contrast, Study 1 students made relatively small gains in their knowledge of new strategies (likely due to a ceiling effect), but showed superior performance on all subscales and greater gains on questions relating to which strategies were easiest for computing estimates for given problems.

*Flexibility Use at Posttest*

To further explore the role of prior knowledge in the development of flexibility, we examined students' *use* of estimation strategies on the whole-number multiplication problems at posttest. Below we consider students' use of multiple strategies, as well as their ability to select the most appropriate strategy for a given problem on the posttest.

*Use of multiple strategies*. Study 1 students chose to use *round both* quite frequently on the posttest. Recall that 92% of Study 1 students used *round both* on at least one pretest problem; 100% of students used this strategy on at least one problem on the posttest. Use of *round one* on the posttest also increased; 51% of students used this strategy on at least one posttest problem. Interestingly, use of *trunc* fell among Study 1 students; while 14% of students used *trunc* on at least one problem at pretest, only 5% did so at posttest. Among Study 2 students, use of *round both* jumped to 77% of students at posttest, from 49% of students at pretest. Similarly, use of *round one* increased to 17% of students, and use of *trunc* increased to 23% of students. In addition, Study 1 students were more likely to use multiple strategies on the posttest. 53% of participants used at least two of the three target strategies (*trunc*, *round one*, *round two*) on at least one problem on the posttest, as compared to only 29% of Study 2 students.

*Choice of appropriate strategies*. In addition to use of multiple strategies, we also considered whether students switched to a more appropriate strategy on a given problem. Of interest were two potential switches that students could have made.

First, students could have switched from *round both* to *round one* on problems where *round one* was more appropriate. On two problems, *round one* was easier to implement than *round both*. In addition, for problems 1 and 2 in Study 2, *round one* yields a closer estimate than *round both*. To what extent did students in Studies 1 and 2 who used *round both* on problems 1 and 2 at pretest switch to *round one* at posttest? For this analysis, we only considered those students who showed some fluency with *round both* at pretest -- those who used this strategy on at least one pretest item (92% of Study 1 students and 49% of Study 2 students). Within this subset of participants, 25% of Study 1 students switched from *round both* to *round one* on problem 1 and/or 2, as compared to only 5% of Study 1 students.

Second, we also investigated whether students switched from *round both* to *trunc* on problems where *trunc* was appropriate. In our prior work, we have shown that *trunc* is easier to implement than *round both* (Star & Rittle-Johnson, in press). We coded whether students switched from *round both* (at pretest) to *trunc* (at posttest) on problems where *trunc* was easiest. As above, we only considered students who showed some fluency with *round both* at pretest. Results indicated that only 3% of Study 1 students switched from *round both* to *trunc* on one or more problems, while 19% of Study 2 students made this switch. Note that our interpretation of students' decision to switch or not to switch to *trunc* is complicated by the fact that, while *trunc* is easier to implement in problems 3-6, *round both* produces the most proximal estimate on problems 4, 5, and 6 (and the same estimate as *trunc* on problem 3). Study 1 students' reluctance to switch to *trunc* can be seen either as a reflection of these students' prioritization of proximity goals or their strong preference for *round both* in spite of the greater ease of *trunc*.

## Discussion

The goal of the present paper was to explore the role of students' prior knowledge of estimation strategies in the development of strategy flexibility. We report the results of two very similar studies, conducted with students with quite different prior knowledge profiles. Study 1 students began with significant fluency with the *round both* strategy, while Study 2 students had substantially less fluency with *round both* or with any estimation strategy. Our results indicated that prior knowledge did impact the development of flexibility, but in rather complex ways that these studies did not fully explicate, as we elaborate below.

First, there is some evidence that prior knowledge can be a boon to the development of strategy flexibility. Students from both studies made comparable gains on the independent measure of flexibility, but Study 1 students made greater improvements in their ability to identify the relative ease of *round one*. Although Study 1 students relied heavily on *round both* at pretest, they also showed greater familiarity with *round one* than did Study 2 students, which likely supported their ability to learn the relative merits of *round both* and *round one* in terms of ease. These results are consistent with our prior work suggesting that learners may need initial familiarity with one strategy before they can become flexible with multiple strategies (Rittle-Johnson et al., 2008). In addition to these gains in terms of flexibility *knowledge*, Study 1 students also were superior in flexibility *use*. Study 1 students used a greater diversity of strategies on posttest estimation problems, and they were more likely to switch from *round both* to *round one* on posttest problems 1 and 2—problems where *round one* is the most appropriate strategy.

However, in other ways, the impact of significant prior knowledge was not as widespread as we might have hypothesized. Students with lower prior knowledge made greater gains in their knowledge of multiple strategies and made comparable gains in learning the relative merits of the *trunc* strategy in terms of ease and closeness. In addition, Study 2 students were more likely to switch strategies from *round both* to *trunc*, which we interpret as a choice to optimize strategies based on ease of computation.

These findings have important implications for the assessment of flexibility and for interventions designed to promote flexibility. First, our results underscore the value of including measures of both knowledge and use in assessing flexibility. Our prior work indicates that knowledge develops prior to use (Star & Rittle-Johnson, 2008), suggesting the importance of knowledge measures to tap emerging flexibility. Similarly, in the present study, our investigation of students' strategy use showed that Study 1 students used a limited repertoire of strategies for solving posttest problems, yet our independent flexibility measures indicated that these students did develop sophisticated knowledge about the relative ease and closeness of various estimation strategies that was not reflected in their strategy choices.

Second, when considering the role of prior knowledge in the development of strategy flexibility, there are intuitive explanations for how prior knowledge can help or can hinder learning. On the one hand, students with high prior knowledge may be reluctant to adopt new strategies, given their fluency with (and likely preference for) a small set of known strategies. On the other hand, students with minimal prior knowledge may be overloaded by attempts to teach multiple strategies (and the pros and cons of each) at the onset of learning. Our results do not fall into one or the other side of this issue. Rather, an important take-away is that students' prior knowledge plays an important role in the development of strategy flexibility but in ways that are subtle and not completely understood. In particular, prior knowledge did not make students more or less willing to learn about or to adopt new strategies, but rather prior knowledge served as a filter through which students attend to or failed to attend to strategic information about problem solving methods. Study 1 students, who already possessed an easy-to-implement strategy for computing estimates, seemed driven to switch because of the proximity appeal of *round one*, while Study 2 students, who did not have an easily executable strategy at pretest, were attracted to the ease of execution offered by *trunc*.

In conclusion, our results indicate that prior knowledge plays an important but complex and nuanced role in the development of strategy flexibility. Flexibility can and should be an instructional goal for all students, but efforts to promote this outcome must carefully consider students' prior knowledge and the ways that such knowledge might promote or hinder students' knowledge of multiple strategies and their ability to select the most appropriate strategy for a given problem.

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# MATHEMATICS LEARNING AS COMMUNITY SERVICE

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*In this paper we present a case study where middle school children, working in an after-school setting, develop knowledge (a) about their community and (b) about mathematics and use digital media to create and share literacy-based and mathematics-based digital stories as community service—that is, as a way of sharing knowledge with, and for the benefit of, others outside of educational settings.*

## Introduction

The role of community in education, and the sharing of knowledge as community service, is not a new idea. Educational institutions typically develop programs that encourage school-community and school-home connections. For example, the Ontario Ministry of Education has published booklets such as *Helping your Child Learn Math – A Parent's Guide* and *Helping your Child with Reading and Writing – A Guide for Parents* that help make home-school connections. Also, all Ontario high school students are required to complete a minimum of 40 hours of community involvement activities as part of the requirements for an Ontario Secondary School Diploma. In addition, high school students in Ontario can apply two co-operative education credits towards their core graduation requirements. However, despite such programs, it is probably fair to say that for the most part students see and experience their learning as a school-based activity, done within and for school, rather than for the purpose of sharing their knowledge with or for the benefit of others.

There are important implications for education if we view knowledge generated in educational settings as something to be shared with others in our community: (a) it enhances a sense of audience, motivating and giving purpose to student learning; (b) it increases the importance of skills needed to communicate with wider audiences; (c) it provides an opportunity for students to give voice to the things that concern them; (d) it creates school-community links by opening public windows into school learning; (e) it creates self-community links; and (f) it creates a setting for a meaningful application of the multimodal broadcasting capacities of digital media.

## Theoretical Perspective: Narrative and Agency/Identity

The case study positions middle school students as community storytellers of personal learning and growth, and offers opportunities for them to experience "narrative reconstruction" as they reflect on their lives, their learning, their choices, their past experiences and their goals for the future (Hull, 2003, p. 232). As Hull points out, "The ability to render one's world as changeable and oneself as an agent able to direct that change is integrally linked to acts of self-representation through writing" (p. 232). When adolescents are given opportunities to share their "identity texts" with peers, family, teachers and the general public through media, they are likely to make gains in self-confidence, self-esteem and a sense of community belonging through positive feedback (Cummins, Brown & Sayers, 2007). Hull (2003) urges collaboration among

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educators, researchers, and community organizations to "find space and time to think expansively about the interface of literacy, youth culture, multi-media, and identity" (p. 233).

There is ample research on the role of narrative in the construction of personal agency and identity (cf. Ochs & Capps, 2001). Bruner's (1994) studies of narrative indicate that changes in conceptions and representations of self are typically associated with "turning points" in personal narratives. Bruner identifies turning points as "thickly agentive … whose construction results in increasing the realism and drama of the Self" (p. 50). There is a dialogical relationship between narrative and self: to shape our narrative is to shape ourselves, and vice versa. There is also a dialogical relationship between narrative/identity and community. Narratives are social artifacts and "the narrated self is constructed with and responsive to other people" (Miller & Goodnow, 1991, p. 172). Stories change depending on the audience, and a personal knowledge story aimed for a school-based audience can change when the audience is the wider community. When the audience is the community, the narrative becomes more of a public performance. Hull & Katz (2006) note "the power of public performance in generating especially intense moments of self-enactment" (p.47). Digital (unlike oral or solely print-based) stories potentially enhance the power of narrative to transform as they can be easily broadcast, creating a stronger sense of audience and performance.

In this research we also consider parallels between the arts and mathematics: between what makes for "a favourite book or movie" and what makes for "a favourite math idea or activity". This leads us to look to the performing arts to understand students' repertoires for organising and expressing the mathematical ideas they seek to communicate to one another and to their worlds outside of the classroom (Gadanidis & Borba, 2008; Gadanidis, Hughes & Borba, 2008). Bauman & Briggs (1990) suggest that digital stories, because they are forms of social interaction, are best analyzed from a framework that recognizes the dialectical relationship between performance and its wider social context. Hull & Katz (2006) add that "digital stories, because they of necessity layer multiple media and modes, complicate our understandings of textual performance as it is linked to the development of identity and agency" (p.47).

It should be noted that the knowledge shared by students is framed as "identity texts" (Cummins et al, 2005). That is, they are seen less as impersonal documentaries of knowledge and more as personal narratives of experience. The goal is to give students voice and agency in the context of community, and thus provide opportunities for students not only to learn subject matter but also to explore its and their place in the world around them. Thus, even stories that seem to be subject-based, like a new way of understanding a mathematical concept or problem, can also be seen as identity texts. Students who engage in developing a conceptual understanding of mathematics are also engaging in developing their mathematical identities (who they are and what they do when doing mathematics) as well as their view of mathematics (what mathematics is and how/why one engages with it). And when students author stories about experience and share them with the wider world, they are developing their identities within community.

## Methodology

*Setting for the Study*

The case study involved twelve middle school students at the Alderville First Nation Learning Centre, which had a lab of 10 desktop computers. Two additional laptops were provided by the research team for student use. Photo Story 3, a free download from www.microsoft.com, was installed on all computers.

The students typically attended a daily afterschool program organized by the Alderville First Nation Student Services, from 3:30 to 5:30, where they worked on homework and other educational activities. The research team was made up of two researchers and one research assistant. At least one of the researchers and the research assistant were present at all sessions. Also present at all sessions was one of the teachers from the Alderville First Nation Student Services and an educational assistant.

The twelve students participated in an afterschool program in which activities were designed by the research team, consisting of seven two-hour sessions and a culminating public performance of their work. During the first four sessions, students learned about and created digital stories of (a) various Alderville First Nation themes (like the Black Oak Savannah ecological sanctuary or the life of Ojibway marathon runner Fred Simpson) and (b) their experiences with rich mathematical tasks. Then students worked for three sessions to write poems and song lyrics based on the themes of their digital stories. They also worked with Aboriginal recording artists Tracy Bone, J.C. Campbell and Dave Mowat to add melodies to their poems and turn them into songs. In the last evening of the program, the students and recording artists performed the songs and digital stories for the students' community, at the Alderville Community Centre.

In this research we purposely involved students in the creation of both digital stories about their community and digital stories about their mathematics experiences. We assumed that students would have a better sense of story in the context of their community compared to the context of mathematics, and we wanted to draw parallels between the two contexts, to help students transfer their narrative skills about community to their mathematical storytelling. To reinforce this paralleling, we used George Ella Lyon's "Where I'm from" poem as a model for telling personal stories about community and about mathematics.

*Data Collection and Analysis*

Given that the case study involves only seven two-hour sessions with students, this research is exploratory in nature. Our general research interest in this case study is to investigate the elements that come into play when an educational program orients learning as community service and to develop a nascent conceptualization of "learning as community service". More specifically, we are interested in: (a) how community-oriented learning shapes what students learn and how they communicate their knowledge, and (b) how the public performances of students' knowledge shape the relationship between the educational institutions and the community.

We used a case study method, which is suitable for collecting in-depth stories of teaching and learning. The case study method is also appropriate for studying a 'bounded system' (that is, the thoughts and actions of participating students or the learning/community connection of a particular education setting) so as to understand it as it functions under natural conditions (Stake, 2000). The 12 students (along with the researchers, teachers, and recording artists) constituted a case. As well, individual students, and their digital stories, were considered as individual cases. The analysis was qualitative, in keeping with the established practice of in-depth studies of classroom-based learning and case studies in general (Stake, 2000). Case study data consisted of (a) field notes, (b) students' writing, (c) interviews with students, (d) interviews with one of the teachers, (e) interviews with one of the recording artists, (f) the digital stories created by students, and (g) the lyrics written by students. Because of the complex blending of multimodal data elements, we used the digital storytelling analysis method of Hull and Katz (2006) of developing a "pictorial and textual representation of those elements" (p.41)—that is, columns of the spoken words from recordings juxtaposed with original written text, the images from digital stories, and data from interviews, field notes and lyrics. This facilitated the "qualitative analysis of patterns" (p.41). The analytic methods included thematic coding (Miles, 1994) and critical discourse analysis (Fairclough, 1995). The data was read and coded for major themes and sub-themes across data sources, and the codes were revised and expanded as more themes emerged. In the authoring of the digital stories, we were particularly interested in moments that might be interpreted as "turning points" (Bruner, 1994) in the representation of identity and/or the conceptual understanding of subject-based knowledge. Part of the analytic process was to use "turning point" moments to construct narrative lines, based on diverse sources of data (Hull & Katz, 2006). Like Hull and Katz (2006), who also researched cases of digital storytelling, we relied on the work of linguistic anthropologists Bauman and Briggs (1990) and their "agent-centred" view of verbal performance (pp. 67-71), and adapted their framework to characterize the ways in which speakers can establish textual authority. A cross case analysis was conducted to compare/contrast the cases of individual students and to compare/contrast digital stories about Alderville and about mathematics.

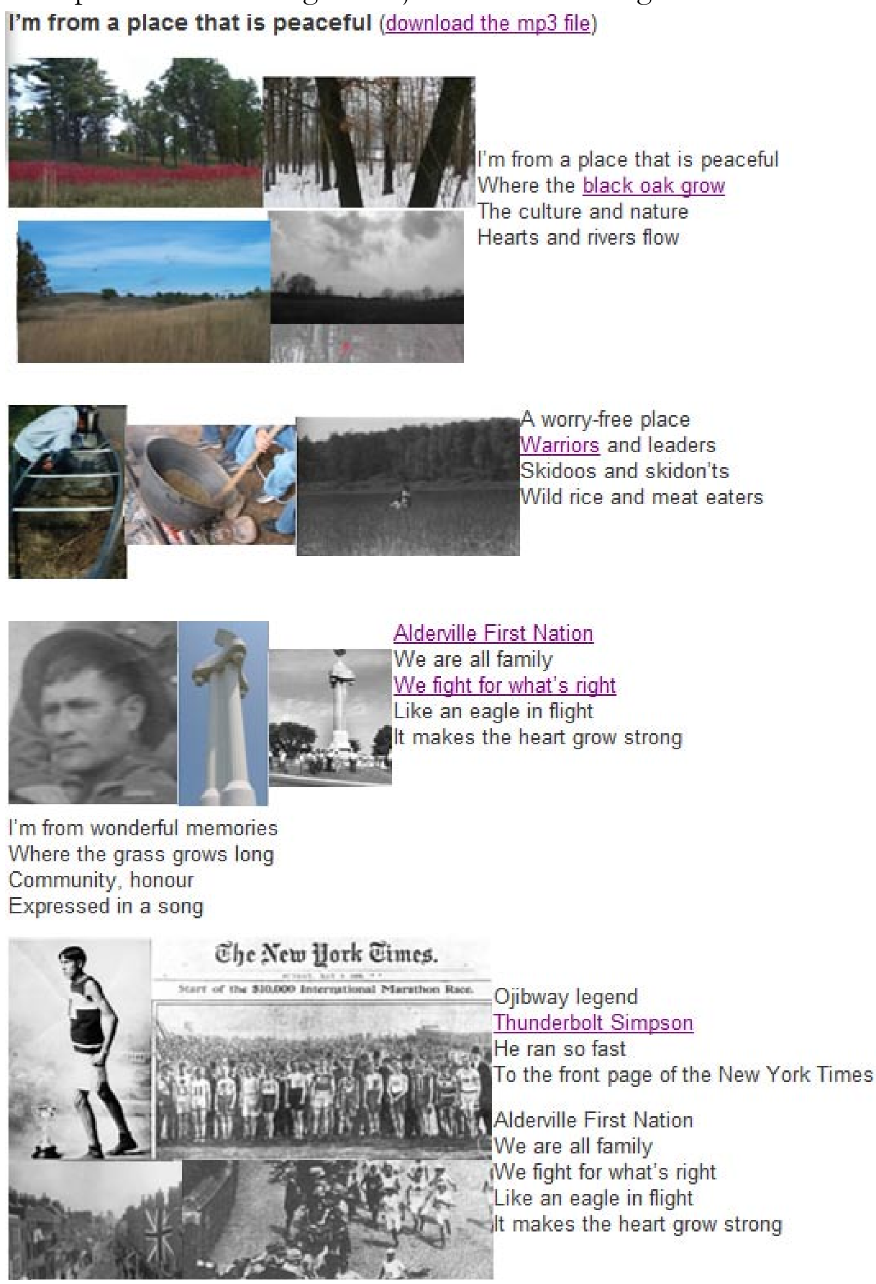


*Figure 1.* Alderville song.

## Findings and Discussion

A more complete discussion will be provided at the PMENA conference. Given the space restriction, we limit our discussion to a subset of the results. We will discuss the case study in terms of three themes that are evident in the data collected and are also key ideas in our theoretical approach: identity, community and turning points.

*Identity*

*Alderville.* Aboriginal identity was a theme identified by the Alderville Student Services staff. There were significant problems here twenty years ago. The language was lost, there was really no tradition and there were a lot of social problems that went along with that. So this community has really worked at it. In the last 15 years there has been a real momentum. There is a real effort to bring some of that back. And it translates into I think they have a better sense of who they are and because of that they want to achieve, not necessarily be a straight A student but be proud of who they are. One of the recording artists commented on the students' sense of identity as aboriginal children.

> It educated me. I was never brought up in the culture. I'm just learning myself. So they know a lot more than I do. I learned a lot through their lyrics, about things they do in their community and how proud they are, and it really brings me a lot of joy and hope for our future because they are so proud of who they are because when I was young, the years that I was young and in school I didn't feel as proud.. I was very moved by the kids and how proud they are of their community, very proud of it.

In the song *I'm from a place that is peaceful* (see Figure 1), the students describe some of the Alderville themes we discussed in Session 1: the Black Oak Savanna, the wild rice harvest, the cenotaph erected to honour the Alderville soldiers who died in First World War, and the marathon runner Fred Simpson. Some of the pictures that students used to create their digital stories were part of the historical archive of one of the Alderville councilors. Many of these pictures were used to illustrate the themes in the song. Not all of the students were familiar with all of these themes, and not all to the same degree.

> I think most of them know about the Black Oak Savanna, that it's a protected area, and it's important to the community. They wouldn't be able to tell you what grows there, how big it is, all the details but they know something, they have an idea. They all know about the Pow Wow, some traditions. Some families get really involved with it. Five of them have outfits and they dance the Pow Wow. They have been brought up to know that. About half of them are strongly involved with different traditions that are going on in the community. These kids would feel quite comfortable talking to you about what the traditions are. Maybe a little less about Fred Simpson, they would know a little less about him. On Remembrance Day I lay down a wreath on behalf of Student Services, every year since I started working here and a lot of the kids would be there, doing the drumming or they are coming with their parents. Now the kids from Roseneath would know more, they have a native liaison there, but not all the others. So they were learning. They were learning and sharing.

*Mathematics.* The authoring of "Where I'm from" digital stories and poems about Alderville set the stage for students writing similar poems about mathematics experiences (see Figure 2). Students responded well to a mathematics that involved hands-on activities and dramatic interpretations of concepts and ideas. In the sum of odd numbers activity, students used linking cubes to represent odd numbers as a growing L pattern. The fitting of consecutive Ls to form a growing square offered students a visual and tangible representation of the proof that the sum of the first N odd numbers in NxN. They used a similar pattern to explore the sums of consecutive even numbers. In the spherical geometry activity, they also used balloons to model a sphere and used pens to draw lines on the balloons to explore straight and parallel lines on a sphere. Students also used drama to communicate some of their ideas. One of the Alderville Student Services staff commented:

> It's no surprise to me that kids will respond more to hands-on, dramatic interpretations of math, as opposed to math out of the textbook. We did none of that but we did a lot of drama, using the computers, trying to access math in a different way. So that's quite interesting.

Parents reported that students enjoyed the math sessions. "The parents thought that the kids seemed like they were really enjoying it. They were looking forward to it. 'Oh, my child really likes math'.

The mathematics activities also involved problem solving and opportunities to explore and experience complex mathematics. The sum of odd numbers activity, which can also be found in upper high school mathematics (sequences and series), was made accessible to middle school students through the use of concrete materials. The spherical geometry activity is one that is typically not in the public school mathematics curriculum. Although we live on a sphere, the geometry of school mathematics is typically limited to flat surfaces. The combination of complex and imaginative mathematics offered students new ways of looking at what mathematics is and what it means to do mathematics. One of the student services staff noted:

> I think they learned to be open to new ideas. I think that's always a good thing. They're receptive to let's give it a try. I like to see that in our kids, they're willing, to give it a shot. And they were at least willing to look at the idea of math in this way. They didn't really question it. They showed up, and 'alright, what are we doing tonight?'

One of the recording artists commented:

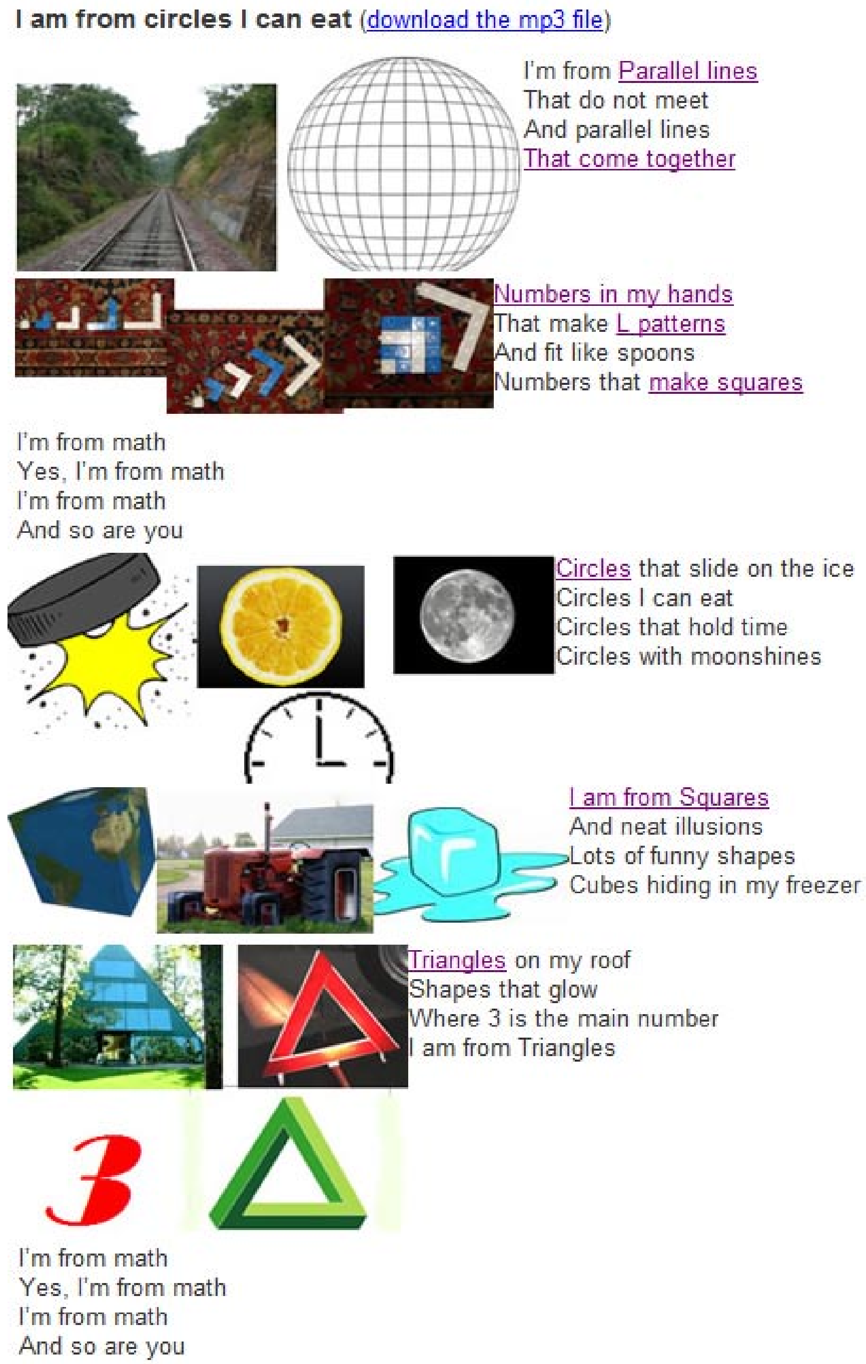


*Figure 2.* Math song.

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

> But I look at it different now. I look at math totally differently. I was looking at one of the towels upstairs and they have lines and I was thinking about parallel lines (laughter). It makes me look at the world and things differently.

*Community*

When asked what role community service played in the program for students, one of the student services staff commented that "I honestly believe that that's the biggest part of what we did."

One of the recording artists expressed the opinion that the experience would have a lasting effect on the community. "They'll be talking about this with their friends and their families. And they talk about it to other people."

In the evening following the 7th and last session of the program, students, along with the performance artists, performed their songs for their community at the Alderville Community Centre. Videos of their performance are available at http://www.edu.uwo.ca/mpc/alderville.html. It should be noted that our initial *community service* plan was that students would create digital stories and share these through the project website. However, our intent to *share* students' work beyond the classroom setting also meant that we were open to opportunities that emerged to share students' work in other ways. The opportunities that emerged included the following: Dave Mowat, who was our local resource for the history of the Alderville First Nation was also a recording artist; the research assistant on the project had both a mathematics and a music background; we had recently worked with aboriginal recording artist Tracy Bone on another project; the Alderville Learning Centre was adjacent to the Alderville Community Centre, which had a stage and sound system; and one of the researchers (second author) has a drama background.

*Turning Points*

We identified a number of turning points during the course of the project.

- *The performative pull of community oriented learning.* As discussed above, our initial plan was for students to create digital stories and share these through a website. Our community focus led us to unexpected community collaborations that expanded the project's performative orientation.
- *Seeing math differently***.** The mathematics activities and their integration with the arts help shift teachers', students' and recording artists' views of math. This shift is evident in the poems and digital stories authored by the students, where mathematics incorporates playfulness and imagination. As one of the recording artists commented, "I look at it different now. I look at math totally differently."
- *Strengthening identity***.** Although the (Alderville or mathematical) themes expressed in the poems and songs came from various individuals, the final product was appropriated by all students. As one of the recording artists commented, "It educated me. I was very moved by the kids and how proud they are of their community, very proud of it."
- *New relationship for the community***.** The project created a new relationship between two universities and a First Nation community. Past research relationships with the community focused on research based on surveys. As one of the teachers commented, "This is something different that we haven't done before. We've partnered before but we've never had a university come in and try that with the kids and it was really a lot of fun. People are sick and tired of doing surveys."

- *Raised expectations of what students can do***.** The teachers expressed surprise at both the artistic and mathematical ability of students. One teacher commented, “I was surprised with the lyrics they came up with … the way they looked at math and music and how they incorporated them.”

## Looking Ahead

The learning as community service focus of the Alderville First Nation project, although small in scope, offers some potential for new ideas in mathematics education. The community focus, and the community as audience for school learning, has the potential of drawing classroom teaching and learning towards performative directions. One question in our minds is how the methods of this project might be used in the more structured environments or regular classrooms. Towards this end, we are presently using the methods of this project in mathematics classrooms in a K-8 elementary school, over a more extended period of time and with seven different classrooms.

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# LOW FLOOR, HIGH CEILING: PERFORMING MATHEMATICS ACROSS GRADES 2–8

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*In this research we take the view (as an elementary school teacher noted) that mathematics "can be discussed with your family and friends just like you would a favourite book or new movie". Such a view—rather uncommon in our culture, where mathematical ideas are rarely shared or discussed beyond the confines of mathematics classrooms or communities of mathematicians—leads us to consider parallels between the arts (whose aesthetic qualities support social and imaginative interactions) and mathematics: between what makes for "a favourite book or movie" and what makes for "a favourite math idea or activity." It also leads us to look to the performing arts to understand students' repertoires for organising and expressing the mathematical ideas they seek to communicate to one another and to their world outside of the classroom.*

## Introduction

When parents ask children "What did you do in math today?" it is not uncommon for children to reply: "Nothing" or "I don't know", or to mention a topic like integers or geometry, without elaborating. To change this situation, we imagine that we would need to work on at least two fronts, which we discuss below: (a) mathematics experiences worth talking about, and (b) skills for communicating mathematics beyond the classroom.

Mathematics Worth Talking About

In our experience in mathematics classrooms, as teachers and as researchers, we have found that mathematics activities that have a low floor and a high ceiling often tend to create mathematics experiences worth talking about. By a low floor we mean that the mathematics knowledge prerequisites for engaging with the activity are kept to a minimum. By a high ceiling we mean that the mathematics activities lead to or can be extended to include much more complex mathematical ideas and relationships. For example, the L pattern shown in Figure 1 can be used as a starting point for exploring the following questions, which (a) range from patterning in grade 2 to the study of sequences and series in grade 11, and (b) for the most part can be made accessible to younger students using concrete representations (Gadanidis, Hughes & Borba, 2008):

- How does the pattern grow, and what would the $10^{th}$ stage look like?
- Notice that the first 5 stages fit together to form a 5x5 square. What is interesting about this?

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

- If we were to construct the first 10 stages, how many blocks would we need?
- What is the sum of the first 5 odd numbers? The first 10? The first N?
- What if we added 1 block to each stage? What changes?
- If we were to construct the first 10 stages of this new pattern, how many blocks would we need?
- What is the sum of the first 5 even numbers? The first 10? The first N?

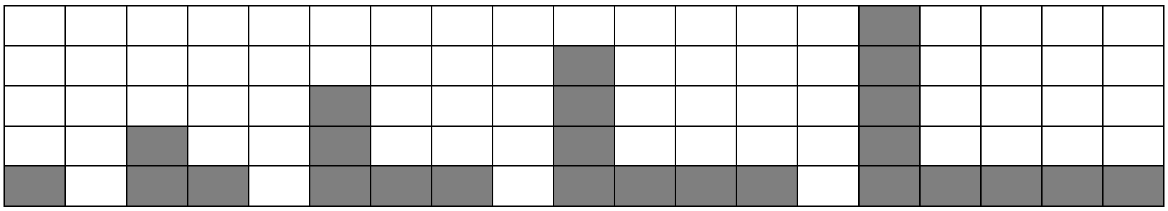


*Figure 1.* L pattern.

Gadanidis and Borba (2008) have looked at mathematics through a performing arts lens, using the performance model by Boorstin (1990) for analyzing movies. Boorstin suggests that we get three distinct pleasures from watching a movie, which we will paraphrase to suit our mathematics context: (a) the pleasure of experiencing the new, the wonderful and the surprising in mathematics; (b) the pleasure of experiencing emotional mathematical moments (either our own, or vicariously, those of others); and (c) the visceral pleasure of sensing mathematical beauty.

*The pleasure of experiencing the new, the wonderful and the surprising in mathematics.* The L pattern activity offers a number of surprises. Isn't it neat that the pattern also represents the odd numbers? Isn't it neat that the sums of these odd numbers are square numbers, and that these can be represented physically as squares? Isn't it neat that we can also add the odd numbers by pairing the first with the last, the second with the second last, and so forth? And this world can grow, by imagining variations on the original pattern.

*The pleasure of experiencing emotional mathematical moments (either our own, or vicariously, those of others).* Students who engage with this activity become excited about the patterns they are noticing and share their ideas and their excitement with others. Students may also experience moments of frustration or anxiety, or other feelings, and share these feelings as well. One student commented, "I felt like I was a lot younger because I haven't played with these blocks for a very long time." Another said, "I liked this more because I was more challenged than regular math." We can imagine a classroom where emotional moments become visible and valued, making mathematics more of a human (thinking + feeling) endeavour.

*The visceral pleasure of sensing mathematical beauty***.** We recently did the L patterns activity with 180 elementary preservice teachers in an auditorium setting (using blocks in small groups). When we showed them a digital simulation of the Ls fitting together, there was an audible visceral reaction from the teachers ("ooh", "ah", and laughter). We witnessed a similar reaction in a fourth-grade classroom. Zwicky (2003), commenting on this square pattern formed by the Ls—this visual proof that the sum of the first N odd numbers is $N^2$—says that such patterns draw our attention, and invite us to "Look at things like this" (p. 38). Sinclair (2001) notes that an aesthetic math experience often involves a sense of pattern or a sense of fit.

*Communicating Mathematics beyond the Classroom*

Our research objective is to explore the concept of elementary school students as "performance mathematicians" (Gadanidis & Borba, 2008; Gadanidis, Hughes & Borba, 2008).

Our research questions are: (1) how might classroom mathematical ideas and experiences be structured to increase their performative potential? (2) how might (a) performance arts methods and (b) digital communication affordances (like the multimodal nature of new media and the read/write capabilities of wikis) be used by students for organizing and expressing the mathematical ideas they seek to communicate to one another and to the wider world? We intend to create a parallel between the classroom focus on performance and the methods and methodology of our research, by relying on performance ethnography methods (Denzin, 2003, 2006; Dicks, Mason, Coffey & Atkinson, 2005; Madison, 2006; McCall, 2000).

When we see a movie we like or read a book we enjoy, we typically share our experience with others. When we do this, we don't retell the whole story. Rather, we might share what was new or fresh about the story, how the plot's turns might have surprised us, and how the movie or book made us feel. In other words, we concisely relate some of the *pleasures* of the experience identified by Boorstin. In our work with students, we aim to focus students' retelling (or performance) of math experiences on the pleasures of (a) the mathematically new, wonderful and surprising, (b) their emotional moments, and (c) the sensing of mathematical beauty.

## Examples of Math Performances

The construct of "students as performance mathematicians" is new in mathematics education and offers a fresh perspective on what school mathematics might be and how students might experience the subject. Some mathematical films do exist (like the 1960's short films produced by filmmaker René Jodoin at the National Film Board of Canada: *Spheres* (Jodoin, 1969), *Dance Squared* (Jodoin, 1961) and *Notes on a Triangle* (Jodoin, 1966)) and some mathematical songs are used in schools (like *Zero, My Hero* by Schoolhouse Rock (Schoolhouse Rock, 1973)). There do exist popular movies about mathematicians, such as *Good Will Hunting* and *A Beautiful Mind*; however, these movies are not performances of mathematics but rather narratives of the social adventures of mathematicians. Some digital examples of student "artistic" mathematical performances can be found on the Web, particularly images of geometry art (typically using tessellations), and student-produced mathematical videos that are just starting to appear on the Web (on YouTube, for example). For the most part, however, it is fair to say that the idea of students creating mathematical performances—digital or otherwise—as a way of communicating their ideas within the classroom and to the world beyond is new. In today's mathematics classrooms "mathematical performance" is associated with testing and standards and "digital mathematics" is associated with using technology to model mathematical concepts or with eLearning, not with students' aesthetic experiences and artistic expressions of mathematics.

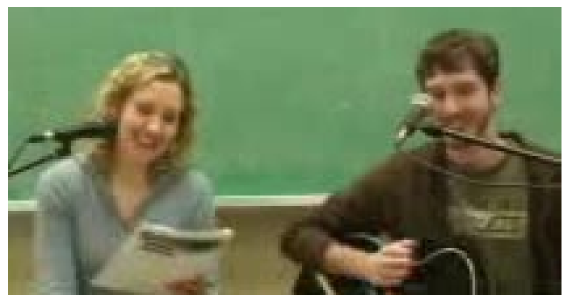
A recent venue for math performances is the Canada-wide *Math Performance Festival* (available at http://mathfest.ca). Below we briefly describe and discuss three math performances submitted to the *Festival.*

*Little Quad's Quest*

*Little Quad's Quest* (available at http://www.edu.uwo.ca/mathscene/lq/lq1.html) is a five-part shadow theatre performance created by a class of fifth-grade students. Little Quad, the protagonist, is in search of his identity. All of Little Quad's quadrilateral friends (Square, Rectangle, Rhombus and Trapezoid), unlike Little Quad, have their own special names.

How do we find instructional time to help students create such performances? Creating a performance like *Little Quad's Quest* takes much longer than 'covering' the concept of quadrilaterals in a more typical fashion. However, the process of creating such a performance integrates mathematics concepts with language arts, visual arts, and drama. The extra time spent is valuable and meaningful instructional time. Perhaps more importantly, such settings offer students opportunities to think creatively and imaginatively about mathematical concepts. They also afford personal and emotional connections to mathematical ideas.

*Measuring the Millimetres to YouMeasuring the Millimetres to You* (available at http://www.edu.uwo.ca/mathscene/pst/pst5.html) is a song written and performed by preservice teachers at The University of Western Ontario. In this romantic ballad, two friends are saddened because of the great distance (100,000 mm) that separates them. Then, they realize that 100,000 mm is the same as 10,000 cm. And, if they divide by 10 and by 10 again, they are really not that far apart: only 100 m.



This song provides insights into the Metric system in ways that are humourous, emotional and difficult to forget.

*Now I'm a Trapezoid*

*Now I'm a Trapezoid* (available at http://www.edu.uwo.ca/mathscene/geometry/geo1.html) is a song by a triangle that lost her head. Saddened by this loss, the triangle laments that it's now a trapezoid. The song is one of three songs in a Geometry Idol contest setting. The second song is *Triangles Rock*, and the third song is *Hexagon*.

A triangle loses its 'head' and becomes a trapezoid. Does this view of shapes and their relationships make a difference, for students, for teachers, for you? How might the student who sings the song see triangles and trapezoids differently? How might she 'feel' differently about these shapes, and about mathematics in general? What if the triangle lost all three of its vertices? What might it become? What if the triangle was created by cutting vertices off of another shape: what might this shape have been?

## Methodology

*Setting*

The study is, at the time of writing, in progress and involves seven teachers and students in grades 2, 5 and 8. The study uses mathematics activities that engage their imagination and motivate them to share their learning with others. Starting with the same core mathematical ideas (one of which is the L pattern exploration discussed above), teachers work with the project team to develop activities suitable for students at the different grade levels (2, 5 and 8). Students are provided with models and opportunities to 'perform' (communicate) complex mathematical ideas in compelling ways, using the Arts (story, drama, poetry, song, and artwork). Some models we have used so far include: (a) retelling experiences that students find surprising, (b) embedding the core activity in a fairy tale (where the Big Bad Wolf tempts Little Red Riding Hood with chocolate bar pieces in the shape of Ls), (c) rewriting George Ella Lyon's poem *Where I'm From*, using their math experiences as a base to write *I'm From Math*, and putting some of the poems to music to create songs.

*Timeline*

*Table 1*

Research Plan for January to May of 2009

| **2009 DATES** | **DESCRIPTION** | **NOTES** |
|---|---|---|
| January | Full-day math/planning session<br>• Teachers engage in (1) doing mathematics and (2) in artistically performing what they have learned/experienced<br>• Teachers collaboratively plan the first set of activities | |
| January-March | Teachers engage students in the first set of activities<br>• Each activity consist of 3-5 math classes (or approximately 3-5 hours of mathematics instruction)<br>• Students are provided with models and opportunities for communicating their ideas and knowledge in artistic ways | The research team will observe classroom activities and may also assist with instruction as needed by the teachers. |
| End of March | Half-day math/planning/reflection session<br>• Teachers reflect on classroom experiences<br>• Teachers engage in doing more mathematics<br>• Teachers collaboratively plan the second set of activities | |
| March-April | Teachers engage students in the second set of activities<br>• The activities consist of five math classes (or approximately 5 hours of mathematics instruction)<br>• Students are provided with models and opportunities for communicating their | The research team will observe classroom activities and may also assist with instruction as needed by the teachers. |

| | | |
|---|---|---|
| | ideas and knowledge in artistic ways | |
| May | Public performance<br>• Students perform their math performances during a afternoon where parents and the public are invited<br>Half-day math/planning/reflection session<br>• Teachers reflect on classroom experiences<br>• Research team presents preliminary data analysis for discussion and feedback<br>• Teachers and research team create a performance to communicate some of the ideas and/or results of the project | |

The research team, in collaboration with the teachers in the study, will design mathematical experiences that afford students opportunities to engage imaginatively with mathematics. Such mathematics "activities" are treated as drafts, to be adapted for use in each classroom and then revised based on teacher/researcher reflection for future use. The culminating student task in each activity will be the creation of a mathematical performance that depicts some of the mathematical and aesthetic qualities of students' real or imagined experiences. We are thinking about a performance mindset that spends proportionally longer on exploration (called rehearsal in theatre) that is not about "getting it right" but about exploring the possibilities of the text, stage, setting and characters. This approach to learning borrows from Frye (1988, 1990) whose understanding of the aesthetic was located in such considerations as "what if" "let's pretend" – those aspects of the psyche not usually valued at anything but the earliest levels of education. The performances will in some cases be created by individual students, in some cases by small groups, and in other cases by a whole class. The performances may take the form of a poem, reader's theatre, role play, improvisation, song, drawing or painting, or a combination of some of these forms. These live performances will occur in the classes and also at a culminating afternoon, where parents and the public will be invited to attend. All performances will be videotaped, and some (with student and parental permission) may also be shared publicly through the Math Performance Festival website.

Ethnographic data will be collected through classroom observation and field notes, audio or video recording of selected classroom activities, and individual and focus group interviews of students and teachers.

We intend to create a parallel between the classroom focus on performance and the methods and methodology of our research, by relying on performance ethnography methods (Denzin, 2003, 2006; Dicks, Mason, Coffey & Atkinson, 2005; Madison, 2006; McCall, 2000). The research team will use content analyses (Berg, 2004) of research data to build up piece by piece coherent, data-centred "stories" (Emerson, Fretz & Shaw, 1995). McCall (2000) (building on the work of Emerson et al) suggests that in writing an ethnographic performance script, the researcher "must read and reread their field notes or transcripts" to "create and elaborate analytic themes" and "organize some of these into a coherent story" (p. 427). Our ethnographic performances will take either digital form, for example, as video recordings of the performances we script, cast and perform, or as multimedia stories using digital storytelling tools like Photo

Story or iPhoto, which use zooming and panning on still pictures accompanied by narration and/or music. These ethnographic performances will then be shared with teachers and students, thus returning these "stories" to the classrooms from which they emerged (McCall, 2000; Denzin, 2003). We also intend to post our performance ethnography scripts in the project wiki and encourage teachers and students to edit these scripts to add their own interpretations of events—to add their own voices, perspectives and stories, leading to the creation of new digital performances. We will also invite teachers and students to become cast members in these performances.

An ethnographic approach is appropriate for our research as we seek to enter into "close and relatively prolonged interaction with people … in their everyday (classroom) lives" (Tedlock, 2000, p. 456) in order to investigate a community of mathematical performance in each of the schools as well as their relationships of participants across schools, and over a three-year period. The reliance on performance for (a) students to communicate mathematical ideas and (b) researchers to communicate research ideas about mathematics teaching and learning in a performance-rich setting provides a way of communicating about mathematics and mathematics research more publicly, bringing mathematics to the world beyond the classroom and mathematics education research beyond the scholarly community. This paralleling of methods of doing and researching mathematics will also serve to build a performance community of students, teachers and researchers. In this sense, the researchers are immersing themselves and their methods in the performance tools and methods of the subjects of their study.

## Findings and Discussion

The study is currently in progress and some of the classroom activities are underway. However, we do not have sufficient data on which to report at this stage. The project will be completed by June 2009 and we will be able to provide details of the findings at the 2009 PMENA Conference.

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# COMMON PLANNING IN THE MIDDLE SCHOOL: WHAT DO 7TH GRADE MATHEMATICS TEACHERS DO?

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*This paper describes preliminary analysis from data obtained during the common planning time for a team of 7th grade mathematics teachers. The focus of the planning time was primarily about what the teachers were going to teach that day or in the near future. Later in the semester we began to see changes in this focus towards discussions about tasks, students, and activities that would best suit the needs of various groups of students.*

Curriculum standards are sweeping across the education landscape (Sandholtz, Ogawa, & Scribner, 2004). In the last two decades, state level interventions have encouraged the adoption of standards (Marzano & Kendall, 1997; Ravitch, 1996; Tucker & Codding, 2001), due to federal mandates such as *No Child Left Behind* (2001). Most states have launched some form of standards-based instruction with the expectation that the reform would improve student achievement and equality of educational opportunity (Berger, 2000; Buttram & Waters, 1997; Sirotnik & Kimball, 1999; Sutton & Krueger, 1997). According to Pappano (2007) teachers need collaboration to help them implement standards-based instruction.

Teacher collaboration is a primary factor in teachers' ability to implement change in their instruction towards more effective pedagogical strategies (Briscoe & Peters, 1997; Gajda & Koliba, 2008). Collaboration serves as a catalyst to teachers' abilities to reflect on their practice, is vital to continuous teacher learning (Riley, 2001), helps teachers find the courage to take risks in their practice through developing and providing new learning experiences for their students, and fosters growth in teachers' pedagogical and content knowledge (Briscoe & Peters, 1997). These differences caused by increased collaboration aid teachers in working towards more inquiry-based, or standards-based, instruction.

It is important for researchers to understand that collaboration is necessary but it alone is not sufficient for teacher change and learning (Briscoe & Peters, 1997). Collaboration can certainly help facilitate teacher change, but to create an environment that is conducive to change, teachers' individual commitment to change is equally important (Briscoe & Peters, 1997). Teacher collaboration should include active learning in which teachers engage in activities such as observing other classes, collaborative planning, and reviewing student work together (Garet, Porter, Desimone, Birman, & Yoon, 2001). These characteristics have a positive relationship "to changes in teachers' knowledge and skills and changes in practice" (Graham, 2007, p. 6).

Principals and other school administrators have tried to promote collaboration by giving teachers additional time to plan together during school hours (DuFour, 2004). However, most of the research on common planning time has focused on elementary, special education, language arts, and science teachers. However, mathematics teachers are also increasing their use of common planning. This paper describes preliminary analysis of a research study that answers the question: What do 7th grade mathematics teachers focus on when they plan collaboratively?

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

## Relevant Literature

Several researchers have examined the effectiveness of collaboration, including common planning (Briscoe & Peters, 1997; Graham, 2007; Shachar & Shmuelevitz, 1997; Tonso, Jung, & Colombo, 2006). This planning should have a focus on students and student learning in order to be successful (Tonso, Jung, & Colombo, 2006). One way to make sure that planning is focused on student learning is to use specific techniques to engage teachers in reflection of their practice. Smith (2001) uses a reflective teaching cycle in her professional development with teachers. This cycle includes building teacher knowledge, planning a lesson, observing implementation, and reflecting on the lesson. Briscoe and Peters (1997) describe collaboration that shapes teacher change using three assertions that resemble Smith's approach. The first assertion, similar to building knowledge and planning a lesson, is brainstorming. This is an important process that assists teachers to learn content and pedagogical knowledge from one another. Assertion two, like observing, provides teachers with the knowledge that a colleague would be there to try similar activities and discuss successes and failures provided teachers with courage to take risks they would not otherwise have taken. Finally, the third assertion, like reflection, is that teachers should participate in meetings that provide a valuable opportunity to consider what worked and what did not. These experiences rejuvenated teachers and encouraged them to continue to use problem-centered activities (Briscoe & Peters, 1997).

Shachar and Shmuelevitz (1997) also use a method of collaboration similar to the reflective teaching cycle. In their research, teachers engaged "in cooperative planning of lessons, implementation of plans by one teacher in the group while the others made systematic observations, followed by a feedback session based on the observations made during the lesson" (Shachar & Shmuelevitz, 1997, p. 58). The researchers found that the teachers who reported effective collaboration had a higher level of self-efficacy and efficacy in their ability to promote students' social relations in the classroom.

## Conceptual Framework

Brown, Arbaugh, Allen, and Koe (2000) identified three ways teachers address issues related to mathematics content during common planning time: (1) Scope and Sequence; (2) Talk about Tasks; and (3) Working through Tasks. *Scope and Sequence* is the time teachers spend discussing mathematics topics taught and the order in which they are taught. *Talking about Tasks* refers to the time teachers spend discussing specific tasks or activities that were or would be used in instruction. Finally, *Working through Tasks* is the time teachers spend actually working through a task as a student would. The study reported that teachers spent the majority of their time taking about tasks, describing and reflecting upon ones they had already used including a focus on student difficulties with the tasks. The teachers spent a moderate amount of time discussing past and future issues of scope and sequence and a minimal amount of time working through tasks. When the teachers did focus on working through tasks, a mathematics educator initiated and led the conversation.

## Methodology

The research is being conducted in the context of Project ISMAC (Improving Students' Mathematical Achievement through a Professional Learning Community), a school-based professional development project designed to increase teachers' mathematical content and pedagogical knowledge, while building a mathematics education community among the entire mathematics department at College Middle School, one mathematics educator, and two graduate students. The project includes professional development workshops, the facilitation of weekly grade-level planning meetings, demonstration lessons, and classroom observations. This paper focuses on work with a team of 7th grade mathematics teachers during their weekly grade-level planning meetings.

College Middle School is a small urban school enrolling approximately 640 students. In 7$^{th}$ grade, there are two math teachers, Ms. Bell and Mr. Williams, and one special education teacher, Mr. Sanders. The special education teacher supports a classroom teacher during two periods and independently teaches one mathematics class with six students. The classroom teachers instruct five mathematics classes of 20 – 25 students and are provided with one period for planning. Common planning of mathematics occurs on Wednesday.

During the 2008-2009 school year, we facilitated 14 planning and collaborative meetings. During the meetings we wrote field notes and audio-recorded the conversations. A common set of field notes constituted the data for this report. We used the categories established by Brown et. al. (2000) to initially organize the data.

## Findings

There are many topics this team of 7$^{th}$ grade mathematics teachers talked about during collaborative planning. They talked about non-planning as well as planning topics. Non-planning topics include professional growth or concerns, expectations for students, and classroom management and students issues. What follows are the planning topics they discussed using Brown et. al. (2000) three categories: (1) scope and sequence, (2) talking about tasks, and (3) working through tasks.

*Scope and Sequence*

The scope and sequence discussions were typically teacher initiated and dominated the focus of most meetings. Teachers focused attention on two main topics: pacing and presentation of the lesson. The teachers would concentrate their efforts on unit pacing and what they would be teaching that day. These discussions were based on the state unit previews and curriculum maps, what topics would be on the next district assessment, and what was in their instructional materials, i.e. book, unit, and page number. On at least two occasions, one of the teachers expressed anxiety over thinking too far into the future because she may forget what we discussed.

The teachers were concerned about how to preview lessons, use mathematical reflections, modify tasks, and talk about vocabulary. They concentrated on gaps in student knowledge, questioning, and what mathematical procedures and methods would be the most effective. However, these comments were rarely supported by rigorous observations of student thinking. Instead, the teachers defended their ideas by expressing opinions about how the students were feeling or why they were not succeeding on the tasks and assessments.

*Talking about Tasks*

Mathematical tasks are defined as mathematics problems or activities that students will engage in during a lesson. When the teachers focused on mathematical tasks or activities, it was generally to talk about what they would be using for their lessons that day or week. They would only discuss how to use tasks after we prompted them. In only one case did a teacher talk about a task he completed with his class.

In the beginning of the fall semester, the teachers would show what they had found from instructional resources or had been given from other teachers. They also relied on the researchers as additional sources for classroom activities. Their discussions did not include questions about the mathematical goals of the tasks, how the students would approach the task, what the students would have problems with, or what questions the teachers could use to support student learning. In the beginning of the semester, the only time the teachers spoke about these things was when one of the researchers introduced them into the dialogue. As time progressed, the teachers began to take on more responsibility for these types of discussions.

The two classroom teachers, Ms. Bell and Mr. Williams each started approaching the mathematical tasks in different ways. In September, Ms. Bell talked about her warm up problems by saying what they were and what she would tell the students. By November, she spoke about what her warm up problems by not only showing what they were, but how she expected her students to use different models to solve the problems. In the beginning, Mr. Williams was using tasks from other teachers or getting suggestions and lesson plans from the researchers. By mid-October, Mr. Williams was bringing his own ideas to the meetings. He had begun to read his instructional materials and think about what activities would be most appropriate for his students. However, the researchers still worked to press Mr. Williams on the mathematical goals of the tasks to help clarify the content, and to think about appropriate questioning and student thinking.

*Working through Tasks*

There was only one instance of the teachers working through tasks. This was prompted by the district secondary math coach as she previewed an upcoming unit on Geometric constructions. The mathematics coach wanted the teachers to explore the constructions as a student would. Unfortunately, Mr. Williams and Mr. Sanders were more concerned with finding the answer using procedures and having the necessary tools to teach the lesson. In contrast, Ms. Bell tried to work through the investigation, but we ran out of time. It was unclear if this session was really about working through the task as a student would, but it was the closest we could find in this category.

## Discussion

The research on teacher collaboration suggests that when teachers are provided with opportunities to plan and review student work together they are more likely to deepen their content knowledge and change pedagogical practices. In our study, the teachers used their common planning time to foster collaboration and learning. Although we used the categories developed by Brown and her colleagues (2000), our findings differed in several ways.

We found that teachers spent most of their planning time talking about scope and sequence in general, and what they were going to teach in particular. The teachers were very concerned about the pacing, the sequence of contents, and less interested in choosing inquiry-based activities. This concern may be due to pressures from the state and district for the school to make Adequate Yearly Progress (NCLB, 2001) – something the school has not met in the last six years. As the district continues to push the use of a mandatory pacing guide, teachers are less inclined to consider student thinking and using inquiry-based activities because these decisions may not align with mandated materials.

Whereas Brown et al (2000) reported teachers focusing on the past, present, and future, our data shows the common planning focus very much in the present. The teachers did not spend time reflecting on past lessons or tasks to assess their students' understanding of the material or to think about possible changes that would make the activities more successful in the future. Research says that effective collaboration should focus on student thinking and learning

(Pappano, 2007; Tonso, Jung, & Colombo, 2006). Ms. Bell and Mr. Williams chose mathematical topics and tasks based on state and district resources. They did not work through the tasks to determine the level of difficulty, the appropriateness, or the potential problems the activities could cause for their students. Similar to Brown et al's (2000) observations, the only time the teachers talked about how the tasks would be used and how their students would react was when a researcher asked them. Otherwise, the teachers were trying to figure out what they considered to be the best way to present the material instead of starting from student thinking to figure out how to build on their students' knowledge and experience to facilitate learning.

We are starting to see changes in the teachers' focus during common planning time. This was accomplished in part through our facilitation of deeper discussions as we asked teachers to predict what their students would struggle with and why. They are realizing that by focusing more on student thinking, they can stay true to the district guides and textbook. They are also working to modify mathematical tasks to best suit their classes' needs, but these observations remain largely about groups of students instead of individuals. Thus, teachers can make more progress towards improving instruction by considering particular student's thinking when planning.

Teachers may need support in learning how to effectively use common planning time, and one way to do this is to have a facilitator present who can ask questions, probe the teachers thinking, and lead them into talking about tasks and doing tasks. We will conduct further research to understand our role in this process and how we may best support the teachers in focusing their common planning time on matters that will more directly impact their students' learning. Further research also needs to be done to understand the depth of teacher change and their movement towards discussions about tasks and working through tasks.

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# TYPES OF REPRESENTATIONS VALUED IN A HONG KONG EIGHTH-GRADE MATHEMATICS CLASSROOM

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## Background

The significance of linking and translating among multiple representations (e.g symbolic, graphical, numerical, and matrix) has been paid a great deal of attention over the past few decades because this mathematical practice enables students to look at different facets of intricate mathematical concepts in depth (Arcavi, 1995; Keller & Hirsch, 1998; Knuth, 2000).

In this study, data for one Hong Kong classroom, which is a subset of data collected as part of the Learner's Perspective Study (LPS) (Clarke, Keitel, & Shimizu, 2006), will be analyzed to examine how different types of representations can be practiced in the classroom. Specifically, I will analyze the data to address the following questions: 1) how does the teacher employ different types of representations related to systems of linear equations when problem solving within different contexts; and 2) what affect does teacher-preference for one type of representation have on students' choices to make use of different representations in problem solving?

## Methodology

A comparative case study will be employed during the Spring of 2009 to explore how a teacher utilizes different types of representations in varying problem contexts and how this practice influences students' choice of representations in solving systems of linear equations. Eleven consecutive lesson transcripts will be examined and a coding scheme will be developed to specifically identify instances of the following: (a) different type of representations being used, (b) evidence of students and/or the teacher connecting and translating among multiple representations, and (c) different problem contexts involved in the classroom. In addition, five different student post-lesson interviews and three interviews with the teacher will be analyzed in terms of which types of representations are viewed as important for understanding concepts.

## Analysis and Findings

Analysis will provide information about the impact of a teacher's portrait of representation on students' understanding of mathematical concepts. This study will provide useful implications for teachers and researchers because, although Hong Kong classroom culture may differ from that in the U.S., the practices can be employed as catalysts for discussion and reflection on the practices of the U.S. classrooms and the values that underlie them (c.f., Clarke et al., 2006).

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# STUDENTS' INTERPRETATIONS OF PROBABILITY DISTRIBUTIONS IN A SIMULATION ENVIRONMENT

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*In this paper we present findings across multiple data sources of how young learners (ages 8-12) often interpret probability distributions within the context of a probability microworld environment. We provide a brief description of the software, followed by details about several research observations made in multiple investigations of student explorations with this probability simulation package. The paper concludes with a discussion of a next generation innovation for representing a theoretical probability distribution in the software.*

Introductory probability lessons typically attempt to build from at least one of two primary student intuitions: 1) the concept of equiprobable outcomes, 2) and the law of large numbers (Batanero, Henry, & Parzysz, 2005). Breaking sample spaces into equiprobable outcomes and viewing the probability of an event as a part-to-whole proportion is the primary paradigm known as the classical approach. This approach does not apply well to inherently non-uniform or continuous situations, and the combinatorial techniques can sometimes be difficult for young learners (see Jones, Langrall, & Mooney, 2007 for synthesis of research on student's understanding of sample spaces via combinatorial reasoning). In contrast, the frequentist approach appeals to the law of large numbers by describing the probability of an event as being a limiting proportion of a large number of quasi-identical trials. With access to more advanced technologies, teachers are encouraged to use an empirical or frequentist introduction to probability through computer simulations (e.g., Batanero, Henry, & Parzysz, 2005; Jones et al, 2007). This approach gives little guidance for neither interpreting probability for a single or small numbers of trials, nor few guarantees for relative frequencies in the long run.

The pseudorandom number generators in technology tools use a function dependent on a defined distribution as the basis for its input to generate subsequent "random" outputs. Thus, in a technology environment, students can model probabilistic situations based on assumptions about a theoretical distribution, simulate an experiment to generate a large amount of data, and manipulate and represent the data in various ways that would be nearly impossible to do within the time constraints of school curriculum and instruction. Thus, technology offers a rich medium for designing tools and studying students' reasoning about theoretical probability distributions as well as empirical distributions from simulated data, and many researchers have designed various software environments and studied students' learning of probability. Several of these researchers have documented how students are able to make connections between distributions of data from a simulation and the theoretical distribution described in the model with particular attention to the effect of the number of trials (e.g., Abrahamson & Wilensky, 2007; Konold, Harradine, & Kazak, 2007; Pratt 2000; Stohl & Tarr, 2002).

In this paper, we are focusing on the specific designs used in *Probability Explorer* and, in accord with suggestions from Clements (2007) concerning iterative curricula and tool design research, we are taking a retrospective examination across several studies. We are specifically interested in how students' have interpreted theoretical probability distributions within simulation environments. Others have used computer tools to examine students’ thinking about probability distributions in two-stage experiments or compound events that relied on combinatorial reasoning (e.g., Kazak, 2006; Abrahamson & Wilensky, 2007; Konold et al, 2007). Our focus is thus on students’ thinking about a probability distribution with one-stage experiments or simple events. In such contexts, how do students interpret probability distributions when using *Probability Explorer*?

### Modeling a Theoretical Probability Distribution in *Probability Explorer*

The current version of *Probability Explorer* (*PE*, Stohl, 2002, v.2.01) was a result of prior iterative research and design studies (Drier, 2000a, 2000b, 2001; Stohl & Tarr, 2002). *PE* is designed to allow students to explore the numerical representations of an underlying probability distribution as well as the numerical and graphical representations of the distribution of results from repeated trials (Drier, 2000a, 2001). A probability distribution is currently represented in *PE* with a finite set of outcomes, each of which has an integer “weight”. Various numerical representations of this distribution can be accessed through a “Weight Tool” that allows students to examine and create distributions that build from part-to-part and part-to-whole reasoning (see Figure 1). Traditional scenarios of fair coins, dice, and bag of marbles are easily represented; bias coins and dice and several real world scenarios (such as the weather) can also be represented. To conduct a simulation, students decide the possible outcomes for a simple experiment, how many of these to combine into a compound experiment (1, 2, or 3), the number of trials to conduct, how to arrange data, and which graphical or numerical representations to view for analysis (Figure 1). In addition, all data representations update dynamically *after each trial* to facilitate students analyzing data during a simulation, rather than only viewing representations of data in an aggregate static form (Drier, 2000a).

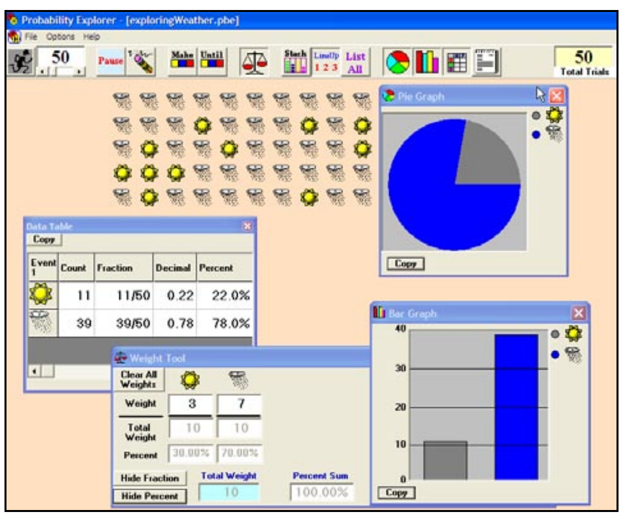


*Figure 1*. Screenshot of probability distribution in Weight Tool and empirical distribution from sample of 50 trials.

As an experiment is being defined, the probability distribution is stored, and can be altered, through a Weight Tool (Figure 1). The metaphor of “weight” was used to help students understand the process of assigning probabilities to an outcome. “Heavier” outcomes are more likely to occur, while “lighter” outcomes are less likely to occur. Weight is measured in units of whole numbers. To facilitate the instantiation of the “weighting” process, students can click on an object in the Weight Tool to increase its weight. Each click corresponds to an increase of one in the weight. By default, students view the distribution of weights as a count. This view of the distribution will allow them to think about the part-to-part relationship between the outcomes. This level of thinking is also aligned with children’s early fractional thinking when they only consider the “parts” of a fraction (numerator) rather than the “part” in relationship to its “whole” (denominator). A part-to-part display is also similar to the concept of odds and can be useful for distinguishing between the odds and probability of an event. Because theoretical probabilities rely on both “part” and “whole,” the weights can be displayed with fractions and percents.

## Exploring Probability Distributions

In several prior and on-going studies with students ages 8-14 (e.g., Drier, 2000a, 2000b; Stohl & Tarr 2002; Tarr, Lee, & Rider, 2006; Weber et al, 2008) it has been observed that many students use similar approaches to interpret the probability distribution. In each of these studies, students were working in small groups (2-3 students per computer) with *Probability Explorer*. Students’ interactions with the software and each other in the primary studies were videotaped and analyzed for critical events (Powell, Francisco, & Maher, 2003). A constant comparative method (Strauss & Corbin, 1990) was then utilized to look for patterns in critical events within the individual studies, followed by an interpretation cycle across the different studies (Lesh & Lehrer, 2000). Each example presented represents similar activities observed across studies.

*What is the Role of a Probability Distribution?*

In the current design of *PE*, most tasks in which students initially design an experiment involve an equiprobable distribution (e.g., fair coin, fair die, choosing up to 8 possible outcomes), in which the Weight Tool defaults to assigning Weights of 1 to each outcome. The most common ways that students interpret an equiprobable distribution is to state that the chances are “50-50”, even when there are more than two equiprobable outcomes. When pressed on this interpretation, many students use language such as “no difference”, or “equally likely” or “all fair” or “anything can happen”. These vague expressions seem to indicate their awareness of equal probabilities but give no real indication of how they perceive the role of a distribution in a simulation. Thus, it is important to consider ways in which students interpret distributions in the context of a simulation of equiprobable as well as nonequiprobable distributions. Two common interpretations have been found to be prevalent:

1) Students often imagine a hypothetical experiment where the sample size is equal to the total weight and they explain that the empirical distribution should be equal, or almost equal, to the assigned weights. For example, Carmella (age 9) designed an experiment with two equiprobable outcomes, the sun and the rain.

| | |
|---|---|
| Carmella: | It means that if you were to press this [points to the “run” button] twice, then one of them would be the sun and one of them would be the rain, most likely. |
| Teacher: | Most likely. Okay and why is that most likely? |
| Carmella: | Because the weight is one and one. And then the total weight would be two. And one is divided, and two is divided into one. And that's most likely because there is no guarantee. |

2) Students often describe the weights in terms of an imaginary box or bucket filled with the number of each items equal to each assigned weight. For example, when Jasmine (age 9) designed a weather situation where it would be twice as likely to be sunny than to rain, after a lot of struggles, she assigned 24 to "lightening" and 48 to "sun."

Teacher: 48 and 24. So what do these numbers mean here? 48 over 72?

Jasmine: Forty-eight over 72 …Oh, there are 72 suns and lightening bolts put in the box. Forty-eight of them are suns. Twenty-four of them are lightening bolts. And children put in that many because they think out of 72 days … there are going to be 48 sunny days and 24 thundering days.

Of course Jasmine also used the first strategy to apply the distribution to an imagined empirical situation. Neither of these strategies is surprising, as it is common for students to describe the probability of an event in both ways, the first representing a typical empirical probability interpretation, and the second a classical counting approach to computing probability.

In many studies, when conducting a simulation where a total weight is known from the Weight Tool or suggested from the task context (e.g. knowing there are n marbles in a bag but not the exact distribution of colors of marbles), students gravitate towards using the total weight (or $n$) for an initial sample size. Then, when these same students are conducting simulations with a hidden Weight Tool and the experiment suggests no integer weights or a total weight (e.g., many fish in a pond), students often are faced with a dilemma and do not know what sample size to choose for their experimentation. At this point a teacher or peer typically has to tell them they need to choose a sample size, or one is suggested.

Students readily accept a strong tie between the Weight Tool and trial results. This is a strong start in forming intuitions about the connections between the theoretical distribution and empirical data. Students are not surprised by some variation between samples and their expectations based on the Weight Tool. However, a big question is what tolerance do they have for this variation? How do they form intuitions about this variability? In fact, in many studies, observing surprising variations provoked students' playful exploration (see Lee, 2005).

A key lesson learned is that the integer values in the Weight Tool are highly suggestive to students, both in interpretation and sample size choice for running a simulation. And it may be these interpretations that drive student expectations for a rather close match between the empirical data and the distribution of weights.

*Modeling Situations with the Weight Tool*

When modeling a situation, it is not uncommon to observe students using an additive approach to create an equivalence relation of different sets of weights for a situation. For example, if given a real bag of marbles with 4 red and 6 green, students would initially believe that weights of 1 and 3 could be used in *PE* to model this situation since 6-4 = 3-1. As students gain more experience or are old enough to readily apply multiplicative reasoning, they will correctly use weights proportional to the context or to a different set of weights. In addition, when asked to create an equivalent experiment including setting a new Weight Tool distribution using weights different from a previous experiment, students will often alter the order in which possible outcomes are entered in the Weight Tool but correctly maintain the proportionality.

Of real interest is that regardless of a students' maturity in weight equivalence reasoning, they appear to expect that equivalent distributions should give similar empirical results. Students often realize after collecting empirical data that their "equivalent" weights designed with additive reasoning do not correctly model the context. For example, Brandon and Manuel (age 11) used the Weight Tool to create a model (Pink: 4, Yellow: 2, Blue: 4) for a spinner with pink and blue sectors each 40% and the yellow sector of 20%. They subsequently ran several trials to collect data from their spinner experiment, used the pie graph, decimals and percentsto analyze data and test the “goodness” of their model. They most often ran multiple sets of 100 trials and occasionally a larger number of trials. After Brandon and Manuel were convinced that their 4:2:4 model was accurate, a teacher challenged them to design a model of the spinner using a total weight of 50. Manuel typed in 20:10:20 in the Weight Tool. Brandon claimed, “that’s not right” and Manuel said, “I bet you a billion dollars it is.” The teacher–researcher asked Manuel to convince Brandon that 20:10:20 could be used to model the spinner. Manuel struggled to explain how the weight model was in proportion to the original weights of 4:2:4 or the spinner regions. Brandon decided to run simulations in *PE* to “see if it still comes close, as long as we have the same percentages.” He first ran 100 trials with the pie graph and data table open and after 60 trials said, “That looks pretty right.” When the 100 trials were complete [showing a 34:23:43 distribution] he said, “Okay, that’s right.” Brandon continued to run sets of 50 and 100 trials and compare percentages of the theoretical distribution in the Weight Tool with the empirical data shown in the pie graph and data table. Brandon ran several sets of 50 and 100 trials before he was convinced that the empirical data supported the weights of 20:10:20 to model the spinner.

A key lesson learned is that students seek relatively stable and similar repeated empirical results, in graphical and numerical form, as a means of comparing the goodness of a model of a probability distribution and for comparing the equivalence of models.

*What Does a Weight of Zero Do?*

Since the Weight Tool uses whole numbers to model a distribution, students are often faced with situations when zero is used. Some students seem to be able to connect this with empirical results where that outcome does not, or could not occur. For example, Jasmine had designed an experiment with four different icons (tails, circle, hexagon, volleyball) with weights of 1. When asked if they each had the same chance, she used the Weight Tool to illustrate her thinking.

Jasmine: They each have one. But they wouldn’t have the same chance if someone did [she changed the 1 under the tails to a 0] that. Then there would never be any of those [tails]. Or how about this? [she changed the zero under tails to be a two] now it’s more likely to get the tails because there are two out of five. But there’s only one circle, one hexagon, one volleyball out of five.

However, many students do *not* initially interpret a weight of zero as meaning the associated outcome is absolutely impossible, just highly unlikely. These students willingly run relatively large samples in search of this outcome occurring and are surprised when it does not happen. For example, Dean and Lydia (8 years old) were modeling a situation where they were choosing whether to play soccer or baseball. When asked to design the chances so they were *certain* to play soccer no matter how many times they ran the experiment, they gave the soccer ball a weight of 12 and the baseball a weight of 0. Looking over at their screen, Jon turned to his computer and used weights of 19 and 0. When asked if the different computers would give different results since they used different weights, Jon thought his weight of 19 made him “more certain” to get a soccer ball than the other computer. Dean promptly said “it doesn’t matter since we both gave baseball 0.” After 100 trials and all soccer balls, Lydia was surprised and then, after a pause, noted “it doesn’t matter what number you use as long as you give it all to the soccer ball.” Similarly, Amanda thought if she used one and zero as the weights for heads and tails in a coin toss, there would be more heads, but that a “few tails” could occur. She ran a simulation with these weights and after about 400 trials decided tails would not occur because "it's like there are none in the bucket."

Some of the students' initial interpretations of a weight of 0 demonstrate that the number zero itself may be too abstract to interpret in a meaningful way. One possible interpretation is that students think that the mere listing of the outcome in the Weight Tool asserts its possibility. But how is this possibility coordinated with a weight of zero? What if students are envisioning the weights as the "usual" distribution of outcomes in a sample of size equal to the total weight, but being random means, to them, that there is some variation in these sets. It is in these variations where any outcome listed in the Weight Tool can occur. So a zero weight may be interpreted as the outcome not occurring in the "usual" sample, but the mere listing of the outcome makes it a candidate for appearance in the random "errors". Note that this would mean a subtle but real separation of probability and variation.

It is also worth noting that when building the distribution in the Weight Tool via placing marbles in a bag, students have never been observed in believing, say, a yellow marble will possibly occur if they do not place a yellow marble in the bag, even though a yellow marble is listed with a zero probability in the Weight Tool after construction. So whatever students are thinking, it seems to change if they construct weights via a more physical representation.

A key lesson learned is that a weight of zero is not directly interpreted as an impossible outcome, although a physical representation (bag of marbles) seems to eliminate this difficulty. In addition, the expectation that an outcome with zero as a weight can occur with a small chance may be related to students' imagination, and expectation, of a hypothetical experiment with results similar, but not exactly, like a probability distribution.

## Reflections on Design Improvements for Modeling a Probability Distribution

The results from the various studies using *PE* can inform the next iterative cycle of design. Specifically, what Weight Tool redesigns may better help students coordinate representations of the probability distribution with those of the empirical results? The first response may be to keep the Weight Tool essentially the same, but to add on the ability to view the distribution in the same bar or pie chart as is available for the empirical results. Being able to watch a static pie chart of the theoretical distribution beside the wiggly, slowly stabilizing pie chart of results from a running simulation would probably prove to be quite useful to a student in solidifying connections between the two via some governing law of large numbers. However, students' interpretations may be artifacts of the design of the Weight Tool using whole number weights.

Since the bag of marbles has shown promise in being a powerful metaphor, we would like to draw from the strengths of that, but we want students to move beyond (or avoid altogether) a total weight approach to their experimentation. Thus, we would need so many marbles that counting the total would be impossible, even as we maintain the appropriate ratios of subset cardinalities. Not only would this approach hopefully discourage the thought that a particular sample size is optimal, but it may even suggest the desirability of a large sample size.

The current Weight Tool was designed based on research of students' tendency to relate probability in terms of part-to-part relationships rather than part-to-whole. Thus, an initial conjecture for the design was to have weights displayed and entered in part-to-part format. While that early design was built to help students where they may be starting, we also need to consider a design for the Weight Tool that is robust enough to lead the students where they are going—inferential statistics. An issue of particular importance to statistics is developing a successful transition from different representations (relative frequency tables, etc.) of discrete sample spaces to those (density functions, etc.) of continuous spaces (Lee, 1999).

## Granular Approach to Probability Distributions

In upgrading *Probability Explorer*, we propose a *granular density* paradigm to replace the integer Weight Tool. A unit amount of "sand" will contain grains, each grain being equally likely to be selected in a trial. Thus, the primary concepts of equiprobable and large numbers are integrated in all scenarios. As a student decides on the *n* possible outcomes for the sample space, the Distribution Tool will default to *n* bins of equal width. The content of each bin represents the probability for that outcome. A unit volume of sand will be in a large container with the goal of redistributing it to the bins below to assign a probability to each outcome. The user has the option of auto-distributing the sand equally to all bins or manually "pouring" an amount of sand into each bin (Figure 2), either through typing in numerical values or manually operating the spigot. When the Run button is pressed to execute a number of trials, for each trial, a grain of sand will be randomly picked and illuminated.

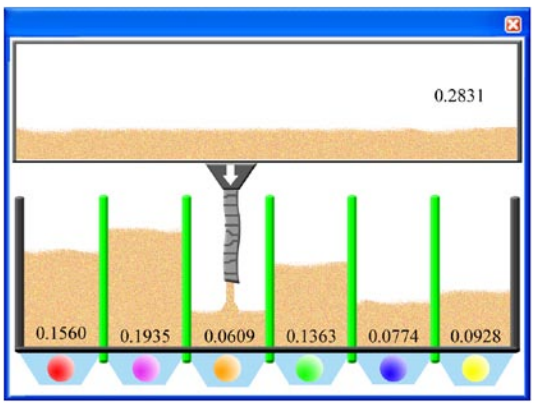


*Figure 2.* Redistributing sand into bins.

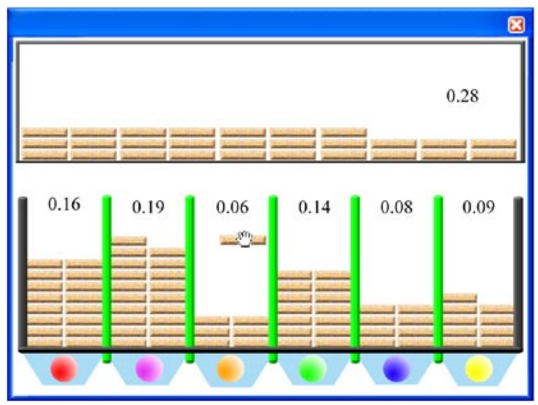


*Figure 3.* Redistributing sand as bricks.

To build off the powerful metaphor of "marbles in a bag", the unit of sand can also be discretized into "bricks" that can be distributed to the bins by either dragging a brick, clicking on the outcome icon to auto-move a brick, or typing in a numerical value to indicate the quantity of bricks to place in an outcome bin (Figure 3). In order to stay true to our desires to give students access to the ability to model a probability distribution in a part-to-part manner, the bricks allow a user to conceive of the total sand as divisible into a small number of equal parts. Distributing bricks of sand then becomes a partitioning task of sharing *r* bricks among *n* outcomes. Of course, we recognize the potential for *n* bricks to also promote a total weight approach to experiments. However, the bricks can be "broken" back into sand grains. Thus, whether the sand is contained in "bricks" or not, the entire probability distribution is made up of a very large number of tiny grains of sand. We believe this large number will reduce the total weight approach in students' data collection and may be suggestive of collecting large sample sizes.

As students become more sophisticated in their use of the sand bins, they will be able to drag the bin dividers to adjust the width of a bin. This can be done prior to redistributing sand from the unit container or after sand has been poured. In the later case, students should notice that the amount of sand in each bin is invariant. We conjecture that having the bins being adjustable in width can strengthen the notion of probability being stored in area with height being a byproduct of density. This step from the discrete to the continuous, with this move from storing distributions as a finite set of numbers to storing them in density functions, is quite weak in most curricula. The granular density paradigm may provide a nice tool for this transition. Further design and research will help us know.

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# WHAT DOES IT MEAN TO UNDERSTAND STATISTICS MEANINGFULLY?

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*This paper reports results of a study in which models and modeling perspectives were used to investigate what it means for university students—graduate students in education—to develop meaningful understandings of foundation-level statistics concepts and skills. Our research findings indicate that while graduate students in education can be very capable of earning excellent grades in introductory statistics courses, their actual understandings of the course materials tend to be quite shallow. Results indicate that most of the students participating in the study were essentially unable to use statistics to evaluate, describe, or otherwise make sense of data or the relationships between them.*

## Why Focus on Data Modeling?

Most future-oriented statements of mathematics curriculum goals identify data modeling as a topic that should be receiving priority attention. At the same time, for a wide range of students from elementary school through graduate school modern data modeling concepts and tools are becoming both accessible and empowering. One reason that data modeling is so accessible is that many of these concepts, skills, and abilities involve straightforward extensions of basic ideas in elementary mathematics. The second is that new technologies for model conceptualization, design, representation, and communication have strongly impacted both the capabilities and accessibility of data-driven models as "reflection tools" (Hamilton, Lesh, Lester, & Yoon, 2007) for supporting the development of mathematical knowledge.

As we enter the 21st century, these same types of next-generation technologies are also changing the nature of work in many industries at ever-increasing rates. These two developments (in tools for schools and tools for work) have caused us to adapt the designs of our statistics courses in the following ways:

- We mean to introduce the kind of problem solving situations in which some beyond-school type of "mathematical thinking" is needed for success (Lesh, Caylor, & Gupta, in press).
- We encourage deep and reflective thinking about the levels and types of "mathematical thinking" that are needed in those kinds of problems (Lesh, Hamilton, & Kaput, 2007).
- We want to extend the ways that these concepts and abilities can be thought about, learned, documented, and assessed (Lesh & Lamon, 1994).

Tools on the drawing boards now may be expected to deliver unforeseeable potentials and innovations with unique and new requirements for modeling expertise. At a time when we should be striving to help our students to prepare for a dynamic and technologically demanding future, many of our best graduate students in education (as well as students in the other sciences) emerge with only the most superficial "cook book" conceptions of relevant statistics concepts and procedures.

## What Theoretical Framework Supports Our Research?

As many readers will recognize, the theoretical framework for our work is known as a models and modeling perspective on mathematical problem solving, learning, and teaching (MMP). Details about MMP have been described in a variety of recent publications (e.g. Lesh & Doerr, 2003). In the context of the research reported here, one of the most important characteristics of MMP is its emphasis on the fact that, in virtually every area of learning or problem solving where researchers have investigated differences between effective and ineffective learners or problem solvers, the results typically indicate that the more effective people not only do things differently, but they also see (or interpret) things differently. For example, expert teachers not only do things right, but they also *do the right things—at the right time, for the right reasons, and with the right people.*

Consequently, because judgments about when, where, why, and how to do things depend on how experiences are interpreted, and because mathematical and scientific interpretations tend to be referred to as models, MMP treats the development of powerful interpretation systems as one of the most important characteristics of what it means to develop expertise in many fields– including teaching and researching mathematics and science education for all citizens. The primary goal of our MMP research is to develop models of teachers' and students' modeling abilities because these models are expected to be embodied in useful tools. As those tools are subsequently shared, tested, revised or rejected, then so will the underlying models, resulting in fundamental changes in our understandings of learning and teaching in the mathematical sciences.

## What Students were Involved in the Project?

The project involved 108 graduate students at two large research universities in the midwestern United States. Students participating in the study had just completed their first course in "quantitative methods" for their doctoral degrees in education and the information we report includes only results from students who had earned A's or B's in the course. In order to focus on students who were users of (not specialists in) quantitative research methodologies, we also eliminated from the study any students whose primary studies were mathematics education, science education, or research design.

The course that the students had just completed was a traditionally taught course that covered statistics concepts corresponding to the left hand column of Table 1. The course covered topics ranging from means and standard deviations up through correlation, regression, hypothesis testing, and analysis of variance. The right hand side of Table 1 corresponds to topics and coverage in an introductory statistics course that we are developing. In order to gather base-line data relative to traditionally taught students' understandings, skills, and fluency in using statistics to understand data we introduced these students to the types of modeling and analysis tasks that we use in our experimental classrooms.

Table 1. *A Comparison of Topics Emphasized in Traditional Statistics Textbooks and the "Big Ideas" Currently Emphasized in Data Modeling Approaches to Statistics*

| Traditional Approaches | Topics Emphasized in Data Modeling Approaches |
|---|---|
| • Variables and graphs<br>• Frequency distributions<br>• Measures of central tendency: mean, median, mode<br>• Measures of dispersion: standard deviations and variance<br>• Probability theory<br>• Binomial, normal, Poisson distributions<br>• Sampling theory<br>• Estimation theory<br>• Hypothesis testing<br>• Chi-squared test<br>• Curve fitting<br>• Correlation theory<br>• Analysis of variance | • Measuring the expected occurrence of uncertain events (probability)<br>• Quantifying qualitative information (or in other ways organizing, systematizing, or mathematizing information)<br>• Transforming data (making comparisons or combinations)<br>• Operationally defining quantities or characteristics that cannot be measured directly<br>• Aggregating qualitatively different types of information<br>• Operationally defining "centers" and central tendencies for collections of data<br>• Operationally defining "spread" for data collections<br>• Developing complex rules from simple rules<br>• Developing rules to describe patterns or trends (e.g., to make predictions)<br>• Measuring unaccounted variation (error)<br>• Measuring relationships between two data sets (correlation)<br>• Developing rules to describe how well data fit known distributions<br>• Sorting out sources of variation (analysis of variance)<br>• Comparing collections of data (testing hypotheses) |

## Which Research Methodologies Were Used?

The type of whole-class teaching interviews that we used were developed and refined in a series of NSF-supported projects that have come to be known collectively as The Rational Number Project (e.g., Post, Lesh, Cramer, Behr, & Harel, 1993). Whole-class teaching interviews are specialized versions of the kind of teaching experiment methodologies that mathematics educators pioneered during the 1970s (Kelly & Lesh, 2000). They were intended to focus in on many of the kinds of understandings that had emerged in past research when investigators were administering Piaget-style clinical interview—but this interview method was also intended to be scalable for use with larger numbers of students. To accomplish these goals, students were asked to record their answers to our questions as they do on most paper-and-pencil tests; however, the questions were presented to the class-as-a-whole by an experienced interviewer who used brief oral descriptions, concrete materials, computer-generated graphics, and other media designed to support the questions.

Many of the questions in the whole-class teaching interviews were based in contexts that involved model-eliciting activities (MEAs). As the name suggests, MEAs are activities in which learners create products which involve the development of models (or conceptual tools that embody models), rather than producing short mathematical solutions to previously mathematized textbook word problems. In general, MEAs are designed to be simulations of "real life" and authentic (to the students) problem solving situations which usually require sixty to ninety minutes for students to complete. Solution processes generally involve a series of modeling cycles in which current ways of thinking are iteratively expressed, tested, and revised or refined—so that auditable trails of documentation are automatically generated as the process progresses and which reveal important information about the nature of students' evolving ways of thinking.

When MEAs are extended across ninety-minute periods, students have a sufficient amount of time to go through multiple cycles in which they express → test → revise their current ways of thinking. However, when MEAs were used as the contexts for whole-class clinical interviews, students had only a few minutes to respond to each problem or problem situation. So while the students being tested had the typical MEA benefits of being presented with authentic problems rich with data, they did not enjoy the usual concomitant benefits of multiple modeling cycles and peer collaboration.

## Results and Discussion

The whole-class interviews revealed that more than 80% of the students expressed the strong opinion that "The main thing I need to know from my statistics courses is how to calculate correct answers for the kind of problems that are in [my textbook]." Paradoxically, most of these same students also said that they themselves didn't plan to do any such calculations for their own dissertation studies, "I'll get someone else (who is good at statistics, but doesn't need to know details about my study) to tell me which SPSS program to use, how to plug in my numbers, and what the results mean." In other words, most of these students really were not expecting to be anything more than intelligent consumers of statistics routines and results that would be provided and interpreted by others (i.e., quantitative analysis experts) who were not expected to be familiar with details about the relevant studies.

Less than 10% of the students exhibited more than superficial understandings of the fact that all computational formulas presuppose models—models that are based upon assumptions that may or may not be appropriate for a given situation. Nor were those students aware of the fact that small changes in underlying models and assumptions applied to data can often lead to large differences in results and analyses. Due to space limitations, we next turn to one only of the many assessment iterations included in our study and use it to illustrate the following general findings of this research project:

- Few of the students were able to estimate the magnitude of biases introduced by inappropriate coding or aggregating information.
- Few students were able to estimate the consequences of inappropriate thinking about (and measuring) basic constructs such as distances, centers, and spread.
- Few students were able to visualize or describe how different computational formulas are related to one another.
- Few students were able to visualize or describe how different computational formulas may be based in completely different ways of thinking about (and measuring) basic constructs.

- Few students were able to modularize or unpack formulas in order to show how different constructs (such as correlations and regression lines) are related to one another—or how components of these formulas corresponded to basic.
- Most students chose to use procedures that they judged to be (politically?) “correct” rather than choosing procedures whose relationships to the given situations were reasonable.

For the most part the students in this study simply were not aware of implicit assumptions that their choices of procedures presupposed. Virtually all of our results point to the fact that most of these students believed choosing appropriate data acquisition and analysis methods was basically a multiple-choice problem where the possible choices would be listed in their statistics text—rather than collecting data and choosing analytical tools based upon considered and reasonable assumptions (models) of the situation at hand.

We administered an established MEA, the “Test Scores Problem” (c.f., Lesh, 1987, p. 326) to the students in our studies in order to gain a useful window into participants’ understandings of the statistics and statistical tools that they had apparently mastered the semester before. In The Test Scores Problem, students are presented with a table of data that provides a number describing the quantity of work each student-artist has produced as well as a value that describes the quality of the work. In the MEA, the story goes like this

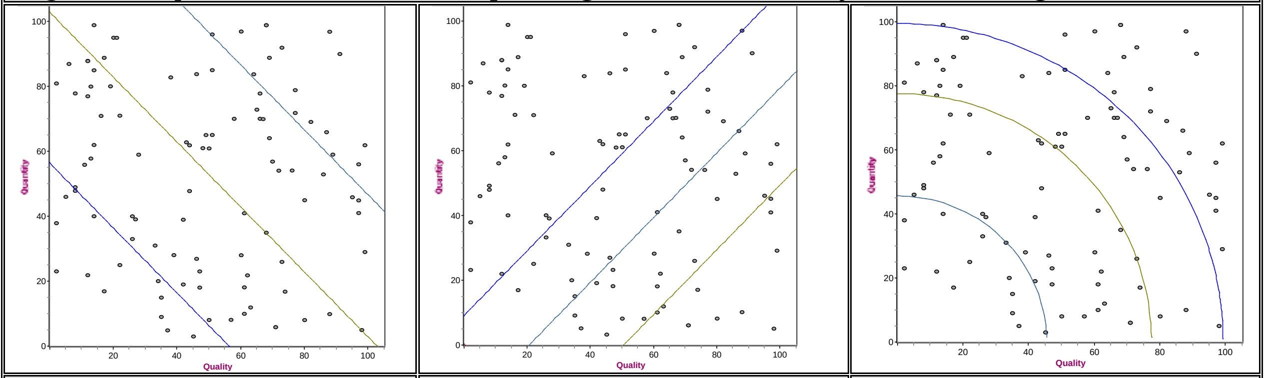
> … the chairman of the art department explains the system ... “The quantity score focuses on "product objectives." We count how many projects each student completes satisfactorily. The quality score focuses on "process objectives." We count how many different tools and techniques each student uses.” (Lesh, 1987, p. 326)

In our tests, the interviewer briefly displayed and described the given information about both quality scores and quantity scores that were earned by students in an art class. Next the interviewer briefly described five suggestions that “students in another class” had made about ways to combine these quality and quantity scores to obtain an overall achievement score for students in the art class. The five suggestions were:

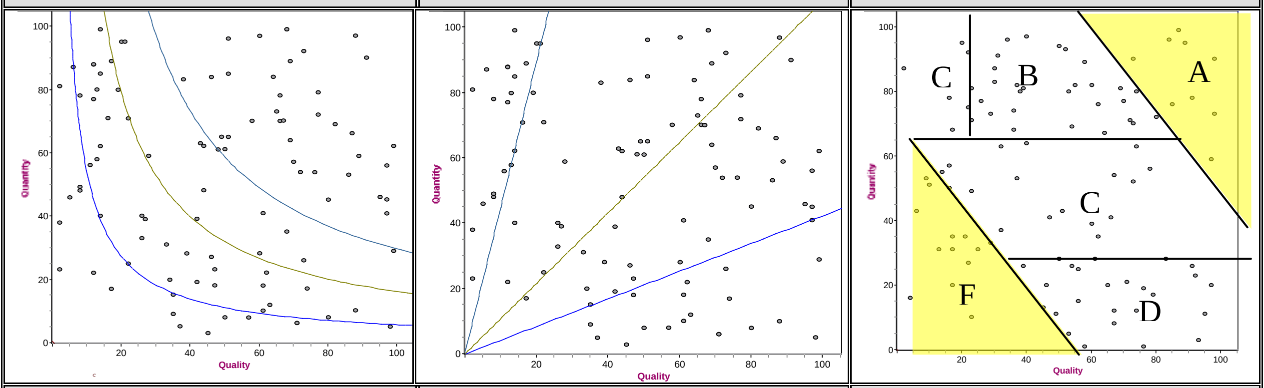
- Combine the scores using Sums: Quality + Quantity
- Combine the scores using Differences: Quality – Quantity
- Combine the scores using Products: Quality x Quantity (i.e., Total Quality)
- Combine the scores using Quotients: Quality ÷ Quantity (i.e., Quality-per-Unit-of-Quantity)
- Combine the scores using Vector Sums: the square root of (Quality$^2$ + Quantity$^2$).

The interviewer then quickly discussed the graphs shown in Figure 1—and pointed out the equivalence classes that are formed when each of the preceding combinations is used to calculate overall achievement scores. Finally, the interviewer showed Figure 1F and asked the students whether they thought that it would be okay to use such a diagram to assign overall scores to the art students—instead of assigning scores using an algebraic or arithmetic formula.

*Figure 1.* Equivalence classes corresponding to six different ways of combining scores




A. Equivalence based on addition | B. Equivalence based on subtraction | C. Equivalence based on vector sums




D. Equivalence based on multiplication | E. Equivalence based on division | F. Equivalence based on locations

More than half of the students thought that Figure 10F was not an appropriate way to solve the problem, and almost all of the students thought that "the more quantitative methods" were going to be "more correct." Even more interestingly, over half of the students expressed the view that, if methods like the one shown in Figure 1 are used, then you are no longer doing quantitative research. Yet, Figures 1A-E clearly showed how qualitative assumptions underlie even the simplest quantitative methods. The students interviewed appeared to be almost completely unaware of the fact that "choosing a computational procedure" involves "describing a situation mathematically"—and that these descriptions are based on relational assumptions that are the most important factors determining the appropriateness of employing the procedures associated with them.

## Conclusions

Even though the 108 students who participated in our studies had earned A's or B's in an introductory statistics course that they had just completed, their performances on the MEA-type questions that we posed demonstrated that their understandings were severely restricted. Their understandings were apparently excellent as long as questions they responded to involved standard word problems in which the relevant data were presented in forms that fit the computational procedures that the students were expected to use—and as long as the choice as to computational procedures only depended on “knowing rules for [course curriculum-based] socially acceptable behavior.”

These A-grade students were not at all adept at developing their own mathematical descriptions of problem situations. They had little awareness of alternative ways to “operationally define” constructs such as centrality, variation, distance, or other relevant metrics that underlie the computational formulas that they appeared to have learned. They were remarkably challenged in describing and visualizing how different computational formulas are related to one another. They were not able to deconstruct formulaic representations or discuss how different constructs (such as correlations and regression lines, or two-sample hypothesis tests and analyses of variation) are related to one another—nor how components of these formulas correspond to basic statistical concerns.

As another telling example, consider that throughout the statistics courses that these students had completed, they had been allowed to develop their own one-page “formula sheets” where they could write notes to themselves about any of the facts or formulas that they thought that they might need to use on tests. We also allowed the students to use their “formula sheets” during our interviews to serve as scaffolds for their reasoning and explanations. After the interviews had been completed, we collected and analyzed these formula sheets and scored them according to how much they emphasized “conceptual simplicity” relative to “procedural simplicity.” The results of this analysis showed that most of the formulas that the majority of students used were focused primarily on *procedural* rather than *conceptual* simplicity. Furthermore, the extent to which students emphasized conceptual simplicity was strongly correlated (.48) with high performance on the MEA interview questions.

In some ways, results of the preceding study seem somewhat paradoxical especially when juxtaposed with results that we have reported in past publications (Lesh & Harel, 2003; Lesh, Yoon & Zawojewski, 2007; Lesh, Caylor & Gupta, in press; Lesh, Middleton, Gupta & Caylor, in press). On the one hand our past research has shown that when classroom learning curricula emphasize model eliciting activities then average-ability students routinely *invent* more powerful ideas, procedures, and explanations than their past performances on tests suggest that they could be *taught*. In a related manner, the results reported here suggest that when model eliciting activities are used as *contexts for assessing* students’ understandings of important data modeling ideas, the meanings that many high-achieving students have developed are often seemed to be remarkably narrow and shallow.

Two primary facts may help explain this apparent discrepancy. First, when students engage in model-eliciting activities, their solutions typically evolve through several modeling cycles. Therefore, the final responses that MEA students produce can be similar to the “$n^{th}$” draft of a paper that an academician might write: while the first draft may be seriously lacking, the processes of product evolution and adaptation can result in an excellent and insightful $n^{th}$ revision. In contrast the evidence generated in typical classroom assessments tends to be students’ 1$^{st}$ and only iteration of problem solution—because traditional word problems seldom if ever provide students with the types of data or the expressed challenge that would cause and assist them in assessing the plausibility and sensibility of alternative ways of thinking about the problem.

Second, the word problems contained in most textbooks and tests mainly require students to know rules about when and how to use a list of routine procedures. As such, students are seldom required to develop their own mathematical descriptions of situations; they are seldom required to access the assumptions that underlie alternative operational assumptions; and they are most often asked to know about which rules are to be used when to demonstrate the algorithmic, rules-based, “correct” solutions. In this way, dogmatic and formula-driven solutions are imposed on situations where they do not fit, and alternative means for describing, analyzing, and assessing problem situations remain uninvestigated.

This important distinction between “choosing a correct computational procedure” and “developing a sensible mathematical description” is probably the most essential difference between the types of data modeling that we emphasize in this paper and introductory statistics as it is taught in traditional courses. In courses that employ MEA-like, open-ended, and “generative” (Stroup, Ares, & Hurford, 2005) learning activities, students tend to develop much deeper understandings of the important mathematical relationships inherent in the system under investigation. Adding to our concerns, this distinction tends to become even more important when technology-based tools are central to statistics instruction. In this case, when modern technological tools become primary instructional foci, then problem-solving goals are even more likely to seem to be about choosing and using correct algorithmic routines, about punching in the right numbers, and about stating results in ways that “just seem to fit” with the status quo.

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# EXPLICATING THE MULTIVALENCE OF A PROBABILITY TASK

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*This article demonstrates that a famous task found in probability research is multivalent (i.e., has many interpretations). More specifically, the multivalence of the sequence, likelihood, experiment, and question elements will be the main focus of this investigation. Further, this article demonstrates that certain individuals answer the famous task according to an interpretation not in accord with researchers' intended interpretation of the task. Utilization of a novel theoretical framework, the Task Interpretation Framework (TIF), will aid in investigating unintended interpretations of the task.*

## Introduction

Certain prior research on the comparative likelihood task (Kahneman & Tversky; 1972; Konold, 1989, 1991, 1995; Konold, Pollatsek, Well, Hendrickson & Lipson, 1991; Konold, Pollatsek, Well, Lohmeier & Lipson, 1993; Lecoutre, 1992; Tversky & Kahneman, 1974), hereafter denoted the CLT, consists of interpretations and hypotheses based on multivalent elements of CLT responses. In a general sense, elements of the CLT *response* are interpreted by *researchers*, whereas elements of the *task* are interpreted by *respondents*. However, it is possible for researchers to attempt to interpret responses for evidence of how respondents interpret the task. Motivated by the notion that respondents may have interpreted that task differently than intended and interested in the interpretation of the task by respondents, the author proposes a focal shift in mathematics education research concerning the CLT. Instead of continuing to base research on multivalent elements of CLT responses, the author creates a new theoretical framework, the Task Interpretation Framework (or TIF), with which to analyse CLT responses based on multivalent elements of the CLT. In doing so, it will be demonstrated that certain individuals do not respond to the task they are presented.

## Task Interpretation Framework (TIF)

*Multivalent Elements of the CLT*

The CLT is multivalent. However, the degrees of freedom of the CLT dictate the multivalence be examined in terms of different elements of the task. Specific to this research, four elements of the task—sequence, likelihood, experiment, and question—will be investigated, and will be described below in accordance with an example of the CLT seen in Figure 1.

> *Which of the following sequences is the least likely to occur from flipping a fair coin five times. Justify your response.*
> *a) THTTT b) THHTH c) HHHTT d) HTHTH e) equally likely*

*Figure 1.* CLT example for elemental examination.

*Sequence element.* Consider the four sequences presented in the CLT: While each of the sequences can, and is intended to, be seen as a sequence of heads and tails for five flips of a coin

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

(i.e., THTTT interpreted as: tail on the first flip, then head on the second flip, and then tail on the third, fourth and fifth flips), the four sequences may be interpreted in other ways. For example, THTTT, THHTH, HHHTT, and HTHTH are, concurrently with the first interpretation, sequences with a ratio of four tails to one head, three heads to two tails, three heads to two tails, and three heads to two tails, respectively. From this interpretation, the sequence THTTT has a different ratio of heads to tails (4:1) than the sequences THHTH, HHHTT, and HTHTH (3:2). As such, THTTT is different than the others. The ratio of heads to tails is not the only plausible interpretation of the sequences presented in the CLT. There are other attributes associated with sequences of heads and tails derived from flipping a fair coin that may also be concurrently employed when interpreting sequences.

The number of switches (or alterations), the length of the longest run, and the combination of the two attributes (i.e., switches & longest run) are examined. As such, the sequences of THTTT, THHTH, HHHTT, and HTHTH, which coexist with the ratio of heads to tails interpretation, also coexist as: (1) sequences that have two switches, three switches, one switch, and four switches, respectively; (2) sequences that have a longest run of three, two, three, and one, respectively; and (3) sequences that have two switches and a longest run of three, three switches and longest run of two, one switch and a longest run of three, and four switches with a longest run of one, respectively. As evidenced, there exist a variety of meanings for the sequences of H's and T's that are seen in the sequence element of the CLT. The sequence element of the CLT is multivalent.

*Likelihood element.* Similar to the sequence element of the task, the likelihood element of the task is multivalent. First, there is the colloquial use of the word likelihood. For example, 'in all likelihood Jim has passed his exam.' In this particular use of the word 'in all likelihood' implies that it is probable, or perhaps very probable, that Jim will have passed his exam. That being said, 'very probable' is yet another colloquial usage of the notion of probability, which can be closely connected to 'certainty.' This procedure, of changing from one colloquial word for probability or likelihood to another adds to the multivalence of the task. Further, there is tremendous usage of colloquial terms for probability in the English lexicon, and usage of these words begins at a very young age also adding to the multivalence of the task. As such, the multivalence of the colloquial representations of likelihood may bring high levels of coexistence of interpretations to the likelihood component of the CLT.

The second non-colloquial domain of usage for the word likelihood is in the field of probability, or in a more general sense the field of mathematics. The precise nature of mathematics may, at first, appear to clarify the coexistence of multiple interpretations; however, the usage of likelihood is not as precise as one may expect. Likelihood, within the domain of mathematics is seen more as a synonym for words such as: probability, degree of certainty, or frequency. However, there is a particular distinction when likelihood is used in the formal sense in mathematics. "The concept [likelihood] differs from that of probability in that a probability refers to the occurrence of future events, while a likelihood refers to past events with known outcomes" (Wesstein, 2008). Thus, usage of likelihood and probability as synonyms, formal or informal, will not recognize an inherent temporal distinction, i.e., the coexistence of multiple interpretations. Further compounding the temporal multivalence of the likelihood element, the task employs an informal representation of likelihood, yet expects formal use of likelihood in completing the task.

*Experiment element.* The experiment element for the CLT is the flipping of a fair coin five times, and as will be shown, contains a number of coexisting interpretations. While the experiment may not appear as a source of much confusion, there exist a number of plausible interpretations. More specifically, the derivation of the four particular sequences is one manner in which it can be shown that the experiment element of the CLT is multivalent, because interpretations of how the sequences were derived for the task are not entirely clear. The sequences could have been derived from: flipping one coin twenty times, or twenty coins all at once. There are other plausible combinations of coin flips and coins: five coins flipped at once on four separate occasions, four sets of five different coins flipped all at once, four different coins flipped one at a time for five times simultaneously, four different coins flipped subsequently one at a time for five times, or four sets of five coins flipped simultaneously or subsequently one at a time. Given coin flips are independent events it would not matter how the outcomes were derived; however, it must be recognized that independence is one of the key elements being examined in the implementation of the CLT. As such, it is plausible that an individual answering the task may be influenced by the derivation of sequences.

The temporal notion of the likelihood element also influences the experimental element of the CLT. Individuals' who interpret likelihood in reference to past events will be aligned with an *a-posteriori* (or frequentist) perspective to the experiment, whereas individuals who interpret likelihood as probability will be aligned with an *a priori* (or classical) perspective to the experiment. Thus, the coexistence of the experiment 'having been' conducted and 'to be' conducted also exists in the experimental element of the task.

As shown, the sequence element of the task, the likelihood element of the task, and the experiment element of the task are multivalent. However, the argumentation presented above is all based on the assumption that the question posed in the task is being appropriately interpreted. As such, the question element of the CLT is now investigated.

*Question element.* There are viable, coexisting interpretations of the question posed in the CLT. For example, the question does ask the subject to determine which of the sequences presented is least likely to occur, but sequences presented may be pitted against the other three sequences in the task for likelihood comparison, or may be pitted against all thirty-two possible outcomes for five flips of a fair coin. That said, the coexisting interpretations do not have to be based solely on the question element of the task. The multivalence of the question element of the task can also be derived from the answer, likelihood, and experiment elements of the task, whether treated in some sort of unison or singularly. Alternatively stated, and as an example, an unintended interpretation of the sequence element of the task implies an unintended interpretation of the question element of the task: If a subject interprets the sequences of heads and tails as a ratio interpretation, that individual is answering an unintended interpretation of the question element of the task. As another example, if an individual interprets likelihood in an unintended manner, then they are also not interpreting the question element of the task in an intended manner. The task of achieving the intended interpretation for the question element is based on achieving intended interpretations for the answer, likelihood, experiment, and question elements, treated singularly or in some type of inter-elemental arrangement. Given the argumentation for the multivalence of the elements of the CLT, the author contends that the question element of the CLT is perilously multivalent. Probabilistically stated, the chances of answering the CLT as the researchers intended is unlikely. Alternatively stated, it is likely that a subject answering the CLT is answering an unintended interpretation of the task.

## Research Method

To determine which interpretation has occurred for an individual completing the CLT, inferences can be made by examining responses made by the individuals who have completed the task. Consider an individual who after completing the task comments, 'One of the four sequences had a different number of heads to tails.' The author contends that it is more likely that the individual has interpreted the sequence element of the task in terms of a ratio interpretation. Further, and as another example, the reading of an individual who comments 'the longer the run of tails the less likely the sequence' causes the author to infer that it is more likely the individual interpreted the sequence element of the task in terms of the length of runs.

Examination of CLT responses not only provides insight into the interpretation used in an individual's completion of the task, but also provides the opportunity to determine whether or not an individual's interpretation matches the intended interpretation of the researcher implementing the task. The researcher, knowing the intended interpretation of the task, is able to determine—by the reading of responses made by subjects who have completed the task—whether or not the subject's interpretation aligns with the intended interpretation of the researcher.

## Participants

Participants in this study were 56 prospective elementary teachers enrolled in a methods for teaching elementary mathematics course, which is a course in the teacher certification program. The 56 participants consist of members from two different classes who were taught by two different instructors. In both classes the variation of the CLT was presented prior to the introduction of probability content in the course.

## Task

Participants were presented with the following iteration of the CLT seen in Figure 2.

> *Which of the following sequences is the least likely to occur from flipping a fair coin five times: a) HHTTH b) HHHHT c) THHHT d) HTHTH e) THHTH f) all five sequences are equally likely to occur*

*Figure 2.* CLT utilized.

## Results

While there were six choices available to the participants, responses fell into only four of the six categories. 27 of the 56 participants (approximately 48%) "correctly" chose that all sequences were equally likely to occur, and 29 of the 56 participants (approximately 52%) "incorrectly" chose that HHHHT was the least likely sequence to occur. Further, nine of the participants "incorrectly" chose the sequence HTHTH least likely to occur, and one individual "incorrectly" chose the sequence THHHT as least likely to occur. No participants chose sequences HHTTH or THHTH.

## Analysis of Results

*Likelihood Element Multivalence*

*Sample response justifications for multivalence of the likelihood element.*

> *Barney:* All five sequences are equally likely to occur because when you flip a coin it is random so you cannot predict whether it will turn heads or tails so all these sequences

have an equal chance of occurring.

*Catie:* F) because it is RANDOM!!!!!

Analysing Catie and Barney's responses via the TIF demonstrates an interesting relationship between randomness and likelihood: their responses evidence the likelihood element of the CLT is multivalent. As found in Lecoutre's (1992) research random events are considered equiprobable, yet here it is demonstrated that sequences are equiprobable because the process is random. As such, claims made regarding probability could be substituted for randomness and, similarly, claims for randomness could be substituted for probability without contestation. As a result, there exists a concurrency of interpretations with the term likelihood, thus the likelihood element.

*Question Element Multivalence*

Not necessarily evident from an examination of purely numerical results, the multivalence of the question element appears in many of the sample responses justifications presented. Further, different responses are aligned with different multivalent elements of the CLT, as earlier hypothesized. In fact, and according to the inter-elemental argument posed in the TIF presentation, each of the responses evidencing multivalence in the experiment, likelihood, and sequence elements are also evidencing the multivalence of the question element. Nevertheless, examples of the multivalence of the question element were evidenced.

*Sample response justification for multivalence of the question element.*

*Tawnie:* Letter T is the least likely to occur for A, B, C, and D. If you count them, T is the least. Because in A there's 5 letters, but T only has two or one so it is the least likely to occur.

Tawnie's response shows that the intended interpretation of the CLT was not achieved and, thus, it can be argued that the intended question was not answered. That said, the intended interpretation of the CLT was not achieved in other responses, yet the evidence of such is more nuanced because it plays on the other multivalent elements of the task, not the question element as seen with Tawnie's response. Nevertheless, Tawnie's response that the letter T is least likely to occur evidences the question element of the task is multivalent.

*Experiment Element Multivalence*

*Sample response justification for multivalence of the experiment element.*

*Monah:* …it all depends on the coin thrower. Coin throwing is really random

*Oliver:* You have to think of all the possible things that could happen like the wind could change how much it flips. How it lands also depends on it if it bounces.

*Ronald:* I think it really depends on how you flip the coin, or where it lands

*Nina:* because coins rarely flip h, then t, then h repeatedly

*Paloma:* I was flipping a coin earlier, and it always landed on tails

*Quentin:* I flipped a coin lots of times and I never got 4 heads in a row.

Monah, Oliver, and Ronald all allude to the experiment element of the CLT. The coin thrower, how the coin is flipped, and certain physical factors such as wind and where the coin lands, are taken into consideration in their answering the CLT. Given the classical interpretation is *a priori* probability, the intended interpretation of the task does not mean for any of the physical characteristics of the experiment to be take into consideration. As such, it can be argued that they too are not answering the intended interpretation of the task. In fact, given the intended interpretation of the CLT uses the classical interpretation of probability, the responses of Nina, which is indicative of a propensity interpretation of probability (i.e, probability is a physical propensity), does not meet the intended interpretation; and, as such, it can be argued that the subjects are not answering the intended question. Similarly, Paloma and Quentin adopt a different theoretical interpretation of probability—the frequentist perspective—than the intended classical interpretation; and, as such, are also not answering the intended interpretation of the task. The unintended interpretations presented all indicate that the experiment element of the TIF is multivalent.

*Sequence Element Multivalence*

From the analysis of results the majority of 'incorrect' responses were based on the multivalence of the sequence element. However, two particular interpretations (seen in prior research, e.g., Kahneman and Tversky, 1972) of the sequence element—the ratio of heads to tails and the perceived randomness determined by pattern or lack thereof—dominated the response justifications. As such, the response verifications for multivalence of the sequence element are further categorized into (1) the ratio interpretation, and (2) the perceived randomness interpretation.

*Sequence Element Multivalence: Ratio interpretation*

*Sample response justification for multivalence of the sequence element: Ratio.*

*Francine:* It's most unlikely to have four heads and one tail because there is a 50% chance.

*Gerard:* It has the "most uneven" amount of heads and tails.

*Hannah:* B, it's the only one with one T and four Hs, the rest have 3 Hs and 2 Ts.

Approximately 52% of the participants incorrectly answered that the sequence HHHHT was least likely to occur. That said, reasons for why HHHHT was considered least likely fell into two distinct subcategories. The responses of Francine, Gerard, and Hannah (and others) exemplify the ratio of heads to tails being used as a clue (e.g., B, it's the only one with one T and four Hs, the rest have 3 Hs and 2 Ts.), which has been seen in prior research (e.g., Kahneman & Tversky, 1974).

Batanero and Serrano's (1999) found "that students had a greater difficulty in recognizing run properties than frequency properties [which] indicates that the similarity between the observed and expected frequencies may be more important than the run lengths in students' deciding whether a sequence is random" (p. 562). As such, a second subcategory of reasons for HHHHT being the least likely sequence to occur used the population ratio, but, arguably, in a more subtle manner because runs involve ratio implicitly.

*Sample response justifications for multivalence of the sequence element: Runs.*

*Dianne:* the chances of getting the same one four times is least likely.

*Evan:* because it is hard to get one consecutive side to be flipped repeatedly.

*Uri:* B —because it is unlikely that you will flip heads 4 times and then one tails. But you could also say F because anything is possible. But my final answer is B

Dianne mentioned, "the chances of getting the same one four times is least likely." In this instance there is recognition of the ratio of head to tails, but ratio is not, necessarily, at the crux of the explanation. Dianne's comments are related to the length of the run of heads seen in the HHHHT sequence, four heads in a row. While the response is normatively incorrect, it can be inferred from Dianne and Evan's explanations that the interpretation of the sequence element of the task being used to answer the question posed are not the intended interpretation because the sequence is being interpreted in terms of runs. The interpretation of the sequence via runs and not the ratio of heads to tails demonstrates alternative unintended interpretations, and bolsters

Cox and Mouw's (1992) findings that disruption of one aspect of the representativeness heuristic, such as the population ratio, did not exclude the other, i.e., the appearance of randomness, being used as a clue, and, further, demonstrates the multivalence of the sequence element.

*Sequence Element Multivalence: Perceived Randomness Interpretation*

The use of length of run based on the ratio of heads to tails found in the responses of Dianne and Evan is connected to the perceived randomness of the sequence. Also, from the responses of Dianne and Evan (and others) it is inferred that interpreting the sequence element of the task by the appearance of randomness is also connected to likelihood element. More specifically, sequences that appear more random (i.e., had less of a pattern) are considered to be more likely to occur, and sequences that are considered less random (i.e., were more patterned) were considered less likely to occur, represented in prior research as not being equally representative yet equally likely (e.g., Tversky & Kahneman, 1974). While this connection between the appearance of randomness (via runs) and likelihood is seen in responses for HHHHT as least likely, the connection between randomness and likelihood is also seen in the responses for the sequence HTHTH being least likely to occur, as shown next.

*Sample response justifications for HTHTH.*

*Igor:* Usually, when you flip a coin, the answer won't usually be in a pattern. It would most likely be random.

*Justine:* Because its kind of odd for it to land in a pattern like that. Usually, it's a totally random sequence of heads or tails.

*Ken:* D because when something is random it doesn't usually go in a pattern.

Research has shown (e.g., Falk, 1981) that randomness was perceived through frequent switches and short runs. It appears from the justifications of Igor, Justine, and Ken, that a perfect alteration of heads and tails—the highest possible number of switches and the smallest possible runs—does not appear random. The sequence HTHTH did not appear random because it is patterned. The appearance of pattern implies a lack of randomness, and the lack of randomness implies that it is less likely to occur. As such, a perfect alteration of heads to tails demotes the likelihood of the pattern causing it to be the least likely to occur.

## Conclusion

In a general sense, the results evidence the multivalence of the CLT. More specifically, the results evidence the multivalence of the likelihood, experiment, question, and sequence elements of the CLT. Further, and also evidenced in the results, there is reason to suspect that certain individuals responses may be answering a question other than the one they were originally posed; and, thus, it becomes understandable that an individual may answer that one of the sequences presented in the CLT is least (or most) likely to occur.

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# MIDDLE AND HIGH SCHOOL STUDENTS' THINKING ABOUT EFFECTS OF SAMPLE SIZE: AN IN AND OUT OF SCHOOL PERSPECTIVE

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*This article examines middle and high school student thinking about an effect of sample size task related to coin tosses. The analysis identifies common justification categories for incorrect responses as well as correct responses. These results are then discussed using a lens of in and out of school mathematical thinking.*

## Objectives

This article presents empirical findings about student thinking about a mathematical task related to the concept of effects of sample size. I present results about student thinking and then discuss these results in terms of the relationship between in- and out- of school mathematical thinking.

## Perspectives

Over the last three decades, a variety of researchers have examined the ways that children solve problems differently in school or out of school contexts. The comparison has been framed as a contrast between formal and informal (Scribner & Cole, 1973), written or oral (Carraher, Carraher, & Schliemann, 1987), school and street (Nunes, Carraher, & Schliemann, 1993), or school and everyday (Brenner & Moschkovich, 2002; Saxe, 1988). The central tenet of these contrasts is that "school represents a specialized set of educational experiences which are discontinuous from those encountered in everyday life and…requires and promotes ways of learning and thinking which often run counter to those nurtured in practical daily activities" (Scribner & Cole, 1973, p553). For instance, Boaler (1999) describes how students used a high degree of precision in completing a classroom task about carpeting a space, alongside an admission that they would have done it differently in the "real world."

The notion of in and out of school mathematical thinking is tightly connected to a related notion of mathematical identity (Martin, 2000). Learning to do mathematics, in school, can be viewed as a process of being socialized to participate in that mathematics classroom (Cobb, Wood, & Yackel, 1993). The emerging literature on identity focuses on students' self-understandings, and the understandings that are assigned to them, about their relationships to mathematics (Horn, 2008). One implication of a focus on students' relationships to mathematics is the posing of problems that refer to contexts that are relevant to students' out of school experiences (Nasir, 2002). The mathematics identity literature typically investigates how students are self-positioned, or positioned by others, in relation to the mathematical task at hand (Cobb, Gresalfi, & Hodge, 2008; Horn, 2008, Martin, 2000). However, we should also examine the ways that students, or teachers, position the mathematical tasks, within a context of a mathematics classroom. This paper examines middle and high school student thinking about an effect of sample space task and discusses these results using a lens of in and out of school mathematical thinking.

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*Effect of Sample Size*

The empirical law of large numbers, in informal terms, states that if a probabilistic experiment is repeated over and over, a large number of times, the relative frequency of an event will approach its theoretical probability. Alternatively stated, as more trials are repeated, the experimental mean of a random variable approaches its theoretical mean. The earliest version of this law was proved by Jacob Bernoulli in the 17$^{th}$ century and proved in a more general form by Kolmogorov in the 20$^{th}$ century. For example, although a particular casino game might have only a very slight edge in the house's favor, like 51%, the more times that the game is played, the closer the casino's earnings will be to 51% of all of the bets placed. Similarly, the more times one tosses a coin, the closer the relative frequency of tails will be to ½.

Adults have been shown to maintain beliefs in a different law, the "law of small numbers" (Tversky & Kahneman, 1972), by which the empirical law of large numbers is applied to small samples of data. For instance, Kahneman and Tversky (1972), in what has become a classic task, posed the following problem to college students: "There are two hospitals, one small and one large, and over the period of a year, each hospital records the number of days on which more than 60% of the babies born are boys. Is either hospital more likely to yield such days, or should one expect the two hospitals to yield the same number of such days?" Results included only 20% of their undergraduate sample choosing the correct answer, the smaller hospital, while 56% said that the two hospitals would yield about the same number of such days.

Kahneman and Tversky explained these results as an instance of the representativeness heuristic, by which people determine the probability of an event based on how closely it resembles the same parameter in the parent population. Since births generate boys about half the time, people typically indicate that there should be roughly an equal number of boys and girls born in each hospital, regardless of the sample size of hospitals. Bar-Hillel (1982) replicated the hospital problem with college students and obtained similar results. However, when she increased the parameter of 60% frequency of born baby boys to higher percentages, like 70%, 80% or 100%, more students chose the smaller hospital as more likely to have such yields.

*Sampling or frequency distributions*. The hospital problem can be framed in two different ways, as a sampling distribution or as a frequency distribution (Sedlmeier & Gigrenzer, 1997). The sampling distribution format was used by Kahneman and Tversky (1972) as well as Bar-Hillel (1982), meaning that students consider the span of a year and to determine which of the hospitals would be more likely to have days in which 60% of the births were boys. In the frequency format, students are asked which hospital would be more likely, on a single day, to have 60% of its babies born boys. Sedlmeier and Gigerenzer (1997) aggregated results of studies that use alternate forms of this task and found that people are more likely to attend to the effect of sample size when the task is framed in terms of frequency distributions.

*School students' thinking about sampling*. School age children, even when facing the task in the frequency distribution format, also have difficulty with the concept that smaller samples are more likely to be non-representative than larger samples. Watson and Moritz (2000) adapted the hospital problem to a frequency distribution version involving schools, and still, most of the students (61%, n=41) said that both samples were as likely to occur. Fischbein and Schnarch (1997) posed a sampling distribution version of the hospital problem to middle and high school students and found that the students' seeming lack of attention to the significance of sample size was more prevalent among *older* students than younger students. More of the older students committed this error than did the younger students.

Fischbein and Schnarch (1997) also presented students with a second sampling task. They asked students to compare the likelihood of getting at least two heads on three tosses of a coin with the likelihood of getting at least two hundred heads on three hundred tosses of a coin. Again, their results showed that a higher proportion of the older students (11th graders and undergraduates) indicated that these would be equally likely.

Fischbein and Schnarch's counterintuitive results can be explained in terms of a more developed ratio schema among older school age students. Tirosh and Stavy (2000) elaborate this explanation by describing student thinking on this task in terms of the Same A Same B intuitive rule. If the two systems or objects are equal in terms of a quality A, students often indicate that those two objects will also be equal in terms of a second quality B. In this case, since there is an equal ratio of tails to total coin tosses, 2:3 and 200:300, (Same A), students reason that Same A implies equal probabilities (Same B). The hypothesis is that since older students are more adept proportional thinkers, they are more apt to use that strategy in tackling tasks such as these. In gathering the data described in this article, I posed an effect of sample size task in frequency format with the primary aim of further investigating Fischbein and Schnarch's surprising results.

## Methods

This task was part of a larger study (Rubel, 2002), whose main objectives were to explore student thinking about a variety of probabilistic constructs. In this article, I address one discrete subset of that study, captured by the following research question: When given frequency information, what reasoning do middle and high school students use in comparing the likelihoods of two events? Students were asked to respond to the following prompt:

Determine which is more likely, and explain your answer:

a) getting 7 tails on 10 tosses of a coin
b) getting 700 tails on 1000 tosses of a coin
c) they are equally likely

*Data Collection Procedures*

The sample was a convenience sample of 173 students in grades 5, 7, 9, and 11 at an academically rigorous, independent boys' school in New York City. I was a teacher at this school and was familiar to study participants. I visited each of the twelve represented mathematics classes and presented students with information about the process of educational research and the nature of this particular study. Students could choose to opt out of participating in the study, and were given an opportunity to ask questions about the research. Students completed the Probability Inventory during a regular mathematics period. Responses were initially aggregated according to the students' answers, and then were categorized a second time in terms of justification type.

The second phase of data gathering was the clinical interviewing of 33 of the 173 students, stratified to represent each of the age groups and classes (five 5th graders, seven 7th graders, twelve 9th graders, and nine 11th graders). These interviewees were selected primarily on the basis of their responses to the Probability Inventory, so as to ensure the representation of a variety of common answers and justification types to different tasks. Interviews were conducted by the researcher within a week of a student's completion of the Inventory; each interview lasted between twenty-five and forty-five minutes.

Interviews were semi-structured, with the primary goal of gaining greater detail about the specific student's thinking by using "How did you get this?" or "Why does this work?" types of questions.

## Results

Nearly all of the students' responses (95%) can be divided into two categories, that 7 tails out of 10 tosses is more likely, the correct answer, or that the two events are equally likely, the common misconception.

Table 1
*Distribution of Responses*

| | Grade 5 (n=36) | Grade 7 (n=45) | Grade 9 (n=50) | Grade 11 (n=42) | Total (n=173) |
|---|---|---|---|---|---|
| 7 tails out of 10 tosses is more likely | 17% (6) | 13% (6) | 22% (11) | 40% (17) | 23% (40) |
| Equally likely | 78% (28) | 87% (39) | 68% (34) | 55% (23) | 72% (124) |

Only 40 students (23% of the sample) indicated that 7 tails out of 10 tosses is more likely. Students used one of the three justifications for this response, described here in order of decreasing frequency. Many students explained their thinking with some form of the empirical law of large numbers. For instance, one student wrote, "It is easier to get a large number out of a small number than a large number out of a large number." The second most frequent justification was that 7 tails out of 10 is more likely since seven is many fewer tails than 700. Finally, the third justification category related to exactness. These students reasoned that 7 tails out of 10 is more likely since there are eleven possible numbers of tails on 10 tosses, whereas there are 1001 possible numbers of tails on 1000 tosses.

As shown in Table 1, nearly three-quarters of the sample, 124 of 173 students answered that 7 tails out of 10 tosses is as likely as 700 tails out of 1000 tosses. Unlike Fischbein and Schnarch's results, this response was not more common among the older students. Students' justifications to this incorrect answer belonged to two categories. Most of these students offered justifications for their choice based on equal ratios, equal fractions, or equal percentages, using Same A Same B thinking (Tirosh & Stavy, 1999). However, other students used the 50-50 approach (Rubel, 2007), by which they over-generalized the 50% likelihood of a specific outcome in the single event case to an outcome of a broader, compound case. In other words, these students indicated that 7 tails on 10 tosses is as likely as 700 tails out of 1000 tosses because both situations are "50-50."

## Discussion

In addition to the written component of my study, I also conducted cognitive clinical interviews with 33 of the students. In the course of conducting these interviews, I noticed that students often had multiple, contrasting approaches to the same problem and that they themselves characterized those approaches as in-school or "mathematical" versus out-of-school or "real world." In the next section, I give two examples of this phenomenon, using excerpts from interviews with two students about this particular task.

On the written component of the study, Kendall, a 7th grader, penned that 7 tails out of 10 tosses is as likely as 700 tails out of 1000 tosses. However, in his interview, Kendall revealed additional ideas about this task.

Kendall: "I have to say that even though both reduce to the same fraction, it would be more likely that you'd get 7 out of 10 because then your chances – actually, well, if you're going more likely, probably 7 out of 10 because the higher you go up, the less chance you have of being that precise. You know what I'm saying?"

Researcher: " Not exactly."

Kendall: "If I flipped a coin ten times, I'd think, again, not on paper, this is more real life, there's probably more of a chance than what you can do on paper. Ten times, there's more of a chance of getting 7. If you flip it 1000 times, getting 700 is, I mean it's possible. For some reason 7 out of 10 just jumps out at me."

Researcher: "Were you thinking about this since last week?"

Kendall: "That's not what I wrote. I've been thinking about it, what would be more likely. My strategy here is on paper." (Note: in the interview, Kendall had previously mentioned answering questions "the math way.")

Researcher: "The math way?"

Kendall: "If someone just came up to me and asked me this, I would definitely say 7 out of 10 because 700 out of 1000 just seems like more." (long pause)

Researcher: "You're a basketball player, right? Do you ever take free throws and keep track of how many you make?"

Kendall: "Yes. The larger number you get, the larger the amount of error. With the free throw analogy, it's not too hard to get 7 out of 10. In real life, it would be hard to keep up that, I want to say, ratio.

Later in the interview, with respect to a similar task about effect of sample space, Kendall explained, "It's easier to get 7 out of 10 in real life. But if you reduce them it's 7 out of 10. In real life, if I was betting, I would not bet on 700 out of 1000."

Researcher: "What if it were a question on a math test?"

Kendall: "The way I answered it on this. I'd say if you put it into a fraction, it's 7 out of 10. On paper, it makes more sense doing it this way. Usually in math, there's one answer. 7 out of 10 is seven tenths. 700 out of 1000 is seven tenths, and I know that's correct. That's how I would answer on a math test."

This Kendall example demonstrates the significance of a student's beliefs about mathematics and what it means to do mathematics in school. While Kendall explained in his interview the exactness justification as well as the empirical law of large numbers, he characterized those methods as being appropriate for outside of the classroom. Even in the face of these ways of thinking about the effects of sample size, Kendall maintained that the incorrect solution, the one that demonstrated that both fractions are equal, was the appropriate solution for a mathematics classroom.

In another interview, with 11th grader, Ned, we gain some additional insight to such categories of conflicting answers. Ned described the conflict as being between a concise, numerical way of answering the question with an imprecise comparison.

Ned: "The pure math is that they're the same thing. The probability of 7 out of 10 is the same as 70 out of 100. It's like 7 tenths. But if you actually do it 1000 times, it gets less likely to get 700. It's like the *Fight Club*; if you have enough stuff, it's harder to do it.

Researcher: "How's it like the *Fight Club*?"

Ned: "Because he always says, over a long enough timeline, everything goes to zero or something like that. But it's kind of the same thing. On a long enough span, it's going to be harder to get this percentage."

Researcher: "So, are you sticking with this answer?"

Ned: "There are two ways to look at it. I chose this way."

Researcher: "How do you choose if there are two answers?"

Ned: "The way where you say divide 7 by 10 or 700 by 1000. It's always gonna be this – there's no gray. It's less easy to prove the other way, even though I understand that there's another way to look at this."

Researcher: "Let me change the question. How about comparing getting 10 tails out of 10 tosses with 1000 tails out of 1000 tosses?"

Ned: (pause) "That's more like the other way."

Researcher: "What about your way: 10 out of 10 is the same as 1000 out of 1000.

Ned: "Obviously, they're not. Obviously, it's not the same: 1 out of 1 or 1000 out of 1000. So from thinking about it, it's not the same possibility. But on paper, it is. (pause) Ok, so 700 out of 1000 tails and 7 out of 10 tails. How much more likely is 7 out of 10?"

Researcher: "You said they were equally likely."

Ned: "I know, on paper, they're equally likely. But if you're going to say one is more likely, then how much more likely? I want to know how much more. If you can't figure it out, then the way you can figure it out is the way I did it."

Ned positioned the erroneous Same A Same B approach as being "pure math," and then continued to describe an analysis of the situation using the empirical law of large numbers. He seemed to accept the fact that he had indicated two conflicting answers to the same question. Then, in response to an alternate task, selected as way to initiate cognitive conflict (Borovcnik & Peard, 1996), similar to Kendall, he categorizes the Same A Same B approach as being the way to proceed "on paper." When I push him to reconcile this conflict, he persists that "on paper," the two frequencies are equally likely. The mathematical argument posed by the empirical law of large numbers did not allow him to quantify the likelihoods of either frequency, and quantification seems to be a necessary component of correct thinking "on paper," or in school.

## Conclusions

The interview segments dealing with effects of sample size items contained a common theme: there were instances of students at all grade levels who differentiated between a "math answer" and a "real world" answer to a single question. In the case of this particular task, the "math answer" was the incorrect answer, that 7 tails out of 10 tosses would be as likely as 700 tails out of 1000 tosses because of equal ratios. Even though these students' expressed "real world" answers that were, in fact, well aligned with normative mathematics, they favored the incorrect "math answer" because it involved an arithmetic operation.

On paper, it seemed that these students all had the common Same A Same B misconception, that equal ratios imply equivalent probabilities, as discussed by Tirosh and Stavy (2000) and Fischbein and Schnarch (1997). Yet when interviewed, these students revealed a great deal more thinking about the problem than appeared on the written page. This phenomenon has some clear assessment implications. Mathematics teachers, and certainly states and school districts, assess learning and understanding by evaluating students' written responses to pencil and paper tasks. Implicit in that process is the assumption that students will write what makes most sense as an answer to the given question. However, the results reported in this article demonstrate that it is not only possible, but likely, that students have multiple, conflicting responses to a given task, especially when that task is contextualized in terms of a real world phenomenon. This study has shown that students' beliefs about school mathematics, and their sense of what it means to do mathematics in school, can operate as a filtering mechanism by which they designate which ideas are appropriate for written, school responses.

Bringing relevant contexts into the teaching of mathematics is assumed to be useful in that it serves as a way for students to view mathematics as being pertinent to their lives and also that it enhances understanding by facilitating sense making of the various connections between the problem context and corresponding mathematical models or representations. However, the results reported in this article caution us to be careful about these assumptions.

Even more interesting, though, is the question as to why students view a distinction between thinking in the "real world" and thinking in a mathematics classroom. We typically categorize school mathematics tasks as being either context free or related to the "real world." But since the "real world" mathematics tasks that we pose in the classroom are typically stripped of most of their contextual features, perhaps we have inadvertently given students a clear message that the thinking that one does in mathematics classrooms is very different from the thinking one does outside of school.

## Notes

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# CONCEPTIONS OF STATISTICAL VARIATION

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*The need to think statistically stems from the presence of variation. Statistical thinking embodies understanding of how and why to engage in statistical problem-solving processes and understanding of the fundamental concepts that underlie these processes (Ben-Zvi & Garfield, 2004). Variation plays a crucial role throughout statistical problem solving. This paper describes experienced secondary statistics teachers' conceptions of statistical variation, articulated across design, data-centric and modeling perspectives. Results reveal that these advanced learners of statistics hold three distinctly different types of conceptions of variation: Expected but Explainable and Controllable, Noise in Signal and Noise, and Expectation and Deviation from Expectation.*

## Variation and Statistics

Statisticians view students' development of statistical thinking as fundamental to statistics education (e.g., Cobb & Moore, 1997). In general terms, statistical thinking includes knowing why statistical processes are needed and how to engage in the general process of formulating a statistical question, collecting and analyzing data to address the question, and interpreting results to answer the question (Ben-Zvi & Garfield, 2004). Throughout statistical investigation, variation plays a crucial role (Franklin et al., 2007). Failure to acknowledge variation or to anticipate possible sources of variation can render a statistical study meaningless before data collection begins. Identifying potential sources of variation allows some of those sources to be controlled through methods chosen to collect data and thereby increases the likelihood that effects of or relationships among variables of interest can be determined. Variation also plays a central role in the analysis and interpretation of data. Measuring variation and accounting for variability in the selection of distributions or models to fit the data allows assessment of whether independent factors affect dependent factors in ways beyond chance expectation. Variation precludes deterministic conclusions about relationships between independent and dependent factors from being made, leaving only probabilistically conditioned statements for interpreting results about populations of interest. The primacy of variation in statistics leads some statisticians to view statistics as the study of variation (e.g., Fisher, 1925).

The concept of statistical variation is central to the study of statistics and as such warrants attention by statistics education researchers. Reflective of the widespread role of variation in statistics, researchers have studied learners' reasoning about variation in a variety of statistical areas. Synthesizing this body of work, Shaughnessy (2007) identifies and describes eight variation conception types for how learners view variation. Some conception types are limited to particular contexts, such as time-series settings for "variability as change over time", and other types represent developing views of variation. For example, focus on individual data values, such as extremes and outliers, typifies the conception type of "variability in particular values". Students with this view of variation do not exhibit an aggregate view of data and distribution—a view fundamental to sophisticated statistical reasoning like making inferences from data (Konold & Higgins, 2002). The types of conceptions of variation identified by Shaughnessy are important for educators to recognize students' potentially limiting views of variation and students' views of variation in particular areas of statistics. Of equal importance are holistic images of advanced learners' conceptions of variation across multiple areas of statistics to inform developmental paths. This study provides an image of mature learners' conceptions by explicating the conceptions of statistical variation exhibited by experienced secondary statistics teachers.

## Conceptual Frameworks: Perspectives for Reasoning about Variation and SOLO

The breadth of learners' reasoning about variation can be captured by looking at their reasoning from three perspectives. In this study, Prodromou and Pratt's (2006) descriptions of data-centric and modeling perspectives for reasoning about distribution have been expanded and modified to describe perspectives for reasoning about variation. Addition of the design perspective is warranted by the types of thinking associated with reasoning about variation in consideration of design. Thinking associated with design includes strategic thinking to plan and anticipate problems within practical constraints and noticing and acknowledging variation, particularly during selection of investigative strategies (Wild & Pfannkuch, 1999).

Reasoning about variation from the design perspective entails using context to identify the nature of and potential sources of variation and considering design strategies to control variation from some of those sources. Reasoning from the data-centric perspective includes measuring, describing, and representing variation while exploring characteristics of distributions and using those representations to make informal comparisons about the relationships among data and variables. Reasoning about variation from the modeling perspective incorporates modeling patterns of variability in data or modeling patterns of variability in characteristics of data to reason about relationships among data and variables for the purposes of making predictions or inferences from data.

The perspectives provide a framework for considering the breadth of learners' reasoning about variation; the Structure of the Observed Learning Outcome (SOLO) (Biggs & Collis, 1982) provides a framework for considering the sophistication of that reasoning. SOLO is an empirically-derived, neo-Piagetian model of cognitive development that can both describe cognitive development and describe the complexity and cogency of knowledge that results from learning in response to tasks designed to assess understanding (Cantwell & Scevak, 2004). There exists a cycle of three levels of response to describe understanding and cognitive growth: unistructural, multistructural, and relational (Biggs & Collis, 1982). For this study, SOLO provided a useful lens to design tasks to elicit conceptions and complex reasoning indicative of relational reasoning about variation. In particular, the tasks were designed to reveal relational reasoning within each perspective and across perspectives. The relational level corresponds with responses that exhibit integrated reasoning about variation from a particular perspective and integrated reasoning about variation across the three perspectives. In contrast, the unistructural level corresponds with responses that focus on a single aspect of variation, and the multistructural level corresponds with responses that focus on two or more disconnected aspects.

## Data and Data Analysis

Sixteen teachers with a variety of statistical learning and teaching experiences participated in this study. Participants taught statistics for a median of 9.5 years and were selected for their leadership roles in statistics education. The primary source of data for determining teachers' conceptions of variation was a 90- to 120-minute semi-structured interview. During the interview, teachers responded to a set of tasks that required them to reason about variation. Each task statement was purposefully vague to allow teachers to approach the task from design, data-centric, or modeling perspectives—approaches that provided insights into aspects of variation that seemed to be most prominent for them. The collection of open-ended tasks contained tasks that many teachers are not likely to have encountered previously; however, the tasks are approachable with introductory-level statistics knowledge.

Discussion of conception types centers on teachers' responses to the Consultant Task, part of which is shown in Figure 1. The questions and order of questions used with any one teacher were determined by the direction taken by the teacher in response to the task statement. By providing no information about how administrators selected exams, teachers could respond that the samples might be biased, which would lead into reasoning from the design perspective. Because the only summary measures included in the statement are the average scores for each sample, teachers could request additional information about the data to form a conclusion, leading into reasoning from the data-centric perspective. By presenting information about means and asking for a comparison between consultants, teachers could respond by suggesting that they would conduct a test of inference to form a conclusion, which would lead into reasoning from the modeling perspective.

To improve students' test scores on state assessments, administrators from a large school district require students to take practice exams. Two outside consultants create and score the open-ended questions from these exams. Although both consultants use the same rubric to score student responses, the administrators suspect that the consultants do not interpret and apply the rubric in the same way, resulting in differences in scores between the exams scored by the two consultants. The consultants' contract with the district is up for renewal, and the administrators are trying to decide if they should renew the contract. They decide to use the most recent practice exam to compare the scores assigned from each consultant and to decide whether there is a difference in the way the exams were scored. The administrators select 50 exams scored by the first consultant and 50 exams scored by the second consultant. They find that the average score for the 50 exams scored by the first consultant was 9.7 (out of a possible 15 points), while the average score for the 50 exams scored by the second consultant was 10.3 (out of a possible 15 points). What should the administrators conclude about the scores assigned by these two consultants?

*Figure 1.* The consultant task.

After teachers described their strategies for analyzing the administrators' data, they were given standard deviations and dotplots of the data separately to elicit additional reasoning from the data-centric and modeling perspectives. From the summary measures, teachers were asked to describe the distributions they would expect to inform how they described variation and interpreted standard deviation. The task was designed with a discrepancy between summary measures and dotplots; one of Consultant Two's test scores was "misentered"—a value of 150 for a score of 15. The summary measures were calculated using the value of 150, but the dotplot only displayed scores on the interval from 0 to 15. Teachers were asked to estimate values for the mean and standard deviation of the data displayed in the dotplot and to explain how they estimated the values to inform how they used data to reason about variation. They also were asked to reason about what the administrators could conclude, which allowed them to reason about variation within each distribution and between distributions using summary measures and graphical representations of the data. Teachers had additional opportunities to reason about variation from the data-centric perspective in response to additional questions related to the corrected summary measures and dotplot for Consultant Two's scores and questions related to size-15 samples. Teachers' responses to two additional tasks, the Caliper Task and the Handwriting Task, further informed the way teachers viewed variation from the design, data-centric, and modeling perspectives.

Teachers' interviews were video-recorded, and recordings were transcribed and annotated and subsequently used in analysis. A table was created for each teacher, with columns labeled by perspective and containing a list of indicators of reasoning from each perspective. Any time a teacher exhibited reasoning about variation from one or more perspectives, the passage was copied and pasted from the transcript to the column with the corresponding perspective and indicator in the teacher's table. Using the constant comparative method articulated by Glaser and Strauss (1967), teachers' responses were coded using indicators of reasoning about variation from individual perspectives. Teachers' responses were reread and coded to consider the need for further refinement of the list of indicators. During the course of revisiting teachers' responses to tasks, different patterns of reasoning associated with different conceptions of variation began to emerge from the data. As continued comparisons were made through multiple additional passes through the coded data, distinguishing features of different conceptions were delineated. Analysis continued until there no longer existed any conflicts for describing teachers' conceptions of variation.

## Results: Types of Conceptions of Variation

Three types of conceptions of statistical variation emerged from data analysis: Expected but Explainable and Controllable (EEC), Noise in Signal and Noise (NSN), and Expectation and Deviation from Expectation (EDE).

*Conception: Expected but Explainable and Controllable (EEC)*

Individuals with EEC conceptions of variation see variation as omnipresent, explainable, and controllable. Their sense of variation's omnipresence leads them to expect variation in statistical settings, and their view of variation as controllable focuses on design strategies for both observational and experimental studies. Their view of variation as explainable aligns with their focus on context to identify factors that potentially contribute variability in data and with their attraction to experimental designs that allow them to determine causes for variation to establish cause-and-effect relationships.

A view of variation as explainable may be at the heart of privileging experiments over observational studies. A primary advantage of experiments is the capacity to establish cause-and-effect relationships, which provide stronger explanations for variability in data than association alone. Haley, one of the two teachers in the study who exhibited EEC conceptions, has a strong affinity for experiments. Haley seems to be dissatisfied with the limited inferences that she can make from data in the Consultant Task study, which is an observational study. She observes, "I don't understand, if you do a difference of two means, what's that going to prove?" (Haley, Lines 44-46). Haley seems to expect the administrators to want more information than a comparison of means allows—she may be looking for a potential cause for variability in improved scores or alternatively for a potential cause for variability in the form of changed scores. She notes the administrators' stated goal of improving scores and suggests that their design will yield little information towards achieving their goal: "They want—what—what is their goal…they're trying to get to improve students' test scores on the state assessment…The consultants' contract—see I'm not quite sure how that's going to…show improvement. There's no treatment there" (Haley, Lines 58-69). Haley notes that no treatment exists for determining how to improve scores—the administrators did not design an experiment. She seems to struggle with a design that appears to provide no explanatory power for how to improve scores. Haley's critique of the methods employed by others (in this case the administrators) and subsequent consideration of alternative designs that achieve greater explanatory power are characteristic of those with EEC conceptions.

Strategies to control variation are not limited to experimental design. Individuals with EEC conceptions also recommend control strategies for observational studies, including the analog to blocking in experimental design: stratified sampling. For example, when Isaac is asked how he would design the Consultant Task study, he suggests selecting a stratified sample in order to sample exams over the entire interval of scores from 0 to 15. He observes that one advantage of a stratified random sample over a simple random sample is precisely this dispersed effect. Using a stratified sample, he controls variation by imposing greater variation on each set of exams but (presumably) reduces variation overall by considering each stratum separately. Isaac even states that his goal is control: "If I could get a stratified sample, then I could in a sense control that [sample] distribution" (Isaac, Lines 287-288).

As the examples from Haley and Isaac suggest, individuals with EEC conceptions of variation seek to collect data in ways that allow them to discover patterns and relationships in data, with a preference for establishing cause-and-effect relationships. They use their knowledge of context to implement designs that allow them to control and to explain variation, and they tend to do these things naturally and without prompting. The totality of and tightly interwoven nature of reasoning about design issues is unique to those with EEC conceptions of variation.

When individuals with EEC conceptions reason from the data-centric perspective, they tend to view data through a lens of expectation—expectation for random variation if they properly control and explain variation. In the absence of apparent random variability, they seek explanations for aberrations in data. For example, when Isaac has the value of the standard deviation computed with the value of 150 for Consultant Two's scores, he immediately looks for an explanation for the magnitude of the value: "the explanation that leaps to mind is that somebody's just flipping a coin here" (Isaac, Lines 493-494). As Isaac's reaction may suggest, when individuals with EEC conceptions reason from the data-centric perspective, their reasoning often contains elements reminiscent of their reasoning from the design perspective. In working with data, they compare the characteristics and relationships they see with their expectation for randomness. When they reason about variation from the modeling perspective, they tend to view models through a relationship lens. They model patterns of variability to capture relationships among data or among variables and evaluate models according to the extent to which relationships are captured. They also use models to determine or confirm the strength or significance of the relationships among data or among variables.

*Conception: Noise in Signal and Noise (NSN)*

Central to NSN conceptions of variation are views of summary measures, data patterns, and relationships among variables as signals that are sometimes lost within noisy data. Everett and Cheyenne, the two teachers in this study with NSN conceptions of variation, see variation as the noise in data for data that does not precisely match underlying parameters, patterns, and relationships and thus interferes with identifying signals. Their view of variation as noise focuses their attention on exploring data; their desire to find patterns and relationships focuses their attention on aggregate features of data distributions while simultaneously considering individual datum that do not clearly fit the patterns and relationships.

Individuals with NSN conceptions of variation see data exploration as a necessary precursor to inference. They explore data to identify potential signals and to gauge the magnitude of noise in data before attempting to establish the significance of signals. In response to reading the Consultant Task, Cheyenne and Everett both reject making any decision from the means alone. Cheyenne indicates that:

> I would have liked to have taken a look at, um, I guess I'm a graphical person. I like to see the, the spread of the distribution to see what it is. Just looking at the means without knowing anything else about the distribution isn't gonna help an awful lot in making the decision. (Cheyenne, Lines 61-65)

Through stating a need to see the "spread of the distribution," Cheyenne seems to refer to a "distribution around" (Konold & Pollatsek, 2002) a signal for each distribution to determine the strength of each signal and to compare consultants' data distributions. Everett also notes that he would "need to know about the distribution of scores" (Everett, Line 105). Both Cheyenne and Everett mention that an observed difference of 0.6 in consultants' average scores does not seem to indicate a problem; without additional information, they hesitate to conclude any difference exists. What distinguishes Everett's and Cheyenne's reasoning is their desire to have information about the distributions and not just information about specific characteristics of the data, namely values for measures of variation like standard deviations.

Both Cheyenne and Everett exhibit revealing indicators of their conceptions as they reason while exploring data. They explicitly, thoroughly, and flexibly consider variation, in addition to center and shape, when they view data through the lens of distribution. They use the same distributional characteristics to compare distributions, considering variation within and between distributions while contemplating the relationship between data and the populations from which data are drawn. No single characteristic in reasoning about data appears to be exclusive to those with NSN conceptions, but the totality of their facility in reasoning about and from data and their continued focus on data to reason about variation is unique to those with NSN conceptions of variation.

Individuals with NSN conceptions reason about variation from the design perspective using a lens of control. They seek to control variability in data to strengthen signals and to increase the probability for identifying signals of interest. In their design considerations, Cheyenne and Everett do not focus on expectation or explanation. Their desire is to reduce noise in data to isolate a signal in data. Individuals with NSN conceptions tend to reason from the modeling perspective either in conjunction with or subsequent to reasoning from the data-centric perspective. They tend to use models as signals or in determining the significance of a signal, but they typically wait to do so until after they have thoroughly explored data. For example, in addition to considering summary measures, Everett suggests comparing size-15 samples by considering the likelihood of observing particular distributions of scores if the 30 scores were repeatedly combined and randomly divided into two groups. He proposes this data-based exploration before he considers conducting a formal parametric test of inference. Similar to how Everett sought to determine the relationship between consultants' scores, individuals with NSN conceptions tend to view models through the lens of relationships, searching for patterns and relationships among data values or among variables.

*Conception: Expectation and Deviation from Expectation (EDE)*

Views of variation as EDE were most prevalent among the teachers in this study. The most distinctive feature of EDE conceptions is variation juxtaposed with expectation. Those with EDE conceptions often approach statistical situations with some hypothesized expectation stemming from their statistical question or from the context in which their question is set—expectation for particular outcomes or measures (including variability), parameter values, patterns of variability, or relationships among variables. In addition to expected amounts of variation, they view variation as deviation from expected outcomes or measures, of statistics from parameters, of observed data from expected patterns, or of observed data from expected relationships.

Individuals with EDE conceptions of variation approach inferential settings with expectations for relationships among data and among variables, including relationships between statistics and parameters. They attempt to determine if observed statistics are probable given their stated expectations. Every teacher with an EDE view of variation suggested conducting a *t*-test in their initial analysis of the Consultant Task data. For example, Blake suggests using a t-test to compare means and to establish if one mean is significantly higher than the other. For Blake, the question is not whether a difference exists—he expects to see a difference—but whether the observed difference in means deviates from his expectation of zero with low probability.

> You can see that it's nine point seven versus a ten point three, we can obviously, uh, do some sort of *t*-test or something like that on it to, to see if that result is significant… We got the one score was ten point three. We could, everybody could see that that was higher. The issue and the statistic—from a person who's trained in statistics is it significantly higher. (Blake, Lines 56-66)

Although Blake does not explicitly acknowledge that he uses a theoretical *t*- distribution to model the situation, he later notes that "the *t*-test is a nice approximation to the model we're seeing" (Blake, Line 170). He clarifies that to him significance means "not reasonably attributed to chance" (Blake, Line 70). Focus on the difference in means and determining whether the difference deviates significantly from expectation seems to dominate Blake's initial considerations for analysis and the initial analysis considerations for others with EDE conceptions. They invoke comparisons of sample characteristics with theoretical models.

Characteristic of reasoning that distinguishes EDE conceptions of variation from other conceptions is the totality of reliance on expectation and use of models to develop a sense of expectation, to examine deviation from expectation, and to decide whether there is too much deviation from expectation. Individuals with EDE conceptions seem to naturally incorporate models into their reasoning, and they are most adept at reasoning about variation from the modeling perspective.

Individuals with EDE conceptions of variation view design through the lens of control, attempting to design studies that control variability to minimize deviation from expectation and to increase the probability for detecting significant deviations from expectation. They reason from the data-centric perspective using a lens of expectation as they explore data to gain a sense of expectation or to explore whether data conforms to expectation. For example, some sense of expectation for standard deviations can be formed from reading the Consultant Task description. The range is 15, which suggests that the standard deviation must be less than 15. Dustin's and Hudson's strong reactions to a standard deviation value of 20.2 for Consultant Two's scores certainly suggest that the value significantly deviated from their expectations. Hudson says, “holy moly!” (Hudson, Line 563), and Dustin reacts, “Yowza” (Dustin, Line 326). Their strong reactions and the reactions of several others with EDE conceptions were the strongest external reactions in response to the large standard deviation value. Characteristic of EDE conceptions, their affinity for expectation is revealed in their reasoning from multiple perspectives.

## Discussion

This study sought to answer the question of what conceptions of statistical variation experienced secondary statistics teachers, as mature learners, exhibit. Three types of conceptions were observed. Individuals with EEC conceptions see variation as something that needs to be controlled and explained and hence tend to focus their attention on issues of design. In contrast, individuals who harbor NSN conceptions see variation as something that needs to be explored, which manifests in strong consideration of variation during exploratory data analysis. Individuals who conceive of variation as EDE see variation as something that can be expected and modeled, and their reasoning is typified by a focus on models, particularly models related to inference. As their different foci of design, exploratory data analysis, and inference might suggest, individuals with EEC, NSN, and EDE conceptions view variation from primarily the design, data-centric, and modeling perspectives, respectively. The ways in which they reason about variation differ in relation to constructs associated with each perspective. These three types of conceptions reveal identifiably unique views of variation yet do not appear to form a hierarchy with regard to understanding. In a portion of the study not reported here, it was shown that at least one teacher with each conception exhibited reasoning consistent with robust understanding of variation. Several teachers exhibited what appears to be superficial and at times faulty reasoning about variation suggestive of one of the three types of conceptions of variation. Their conceptions appear to still be developing. This study presents mature learners’ views of variation. When considered in tandem with results from prior research, this study offers images of what learners’ developing conceptions might develop into and may offer insights into how instruction can be designed to facilitate that development.

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# THE SUBJECTIVE-SAMPLE-SPACE

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*This article will demonstrate that when probabilities are based on the perceived randomness of sequences of outcomes, the probabilities are in accord with, or model, a subjective-sample-space. Further, it will be demonstrated that an individuals' subjective-sample-space is partitioned according to the individuals' interpretation of the sequence of outcomes. Through the employment of a novel theoretical framework, which aligns individuals' verbal descriptions of events with more appropriate set descriptions of those events, it will be revealed that certain individuals' respond to the task of comparing sequences of outcomes via a subjective-sample-space partitioned according to switches. To achieve the above mentioned goals, respondents are presented with different sequences of heads and tails, derived from flipping a fair coin five times, and asked to consider the sequences chances of occurrence.*

## Introduction

A variety of research in psychology (e.g., Kahneman & Tversky; 1972, Tversky & Kahneman, 1974) and mathematics education (e.g., Batanero & Serrano, 1998; Borovcnik & Bentz, 1991; Chernoff, 2008; Falk, 1981; Green, 1983; Hirsch & O'Donnell, 2001; Konold, 1989, 1991, 1995; Konold, Pollatsek, Well, Hendrickson & Lipson, 1991; Konold, Pollatsek, Well, Lohmeier & Lipson, 1993; Lecoutre, 1992; Rubel 2006; Shaughnessy, 1977) is derived from presenting individuals with sequences of outcomes and asking said individuals to consider their chances of occurrence. More specifically, three (theoretical or cognitive) models—Tversky and Kahneman's (1972) representativeness heuristic, Konold's (1989) outcome approach, and Lecoutre's (1992) equiprobability bias—which were developed to account for responses to (what will be denoted here as) the comparative likelihood task, have subsequently dominated the research literature. In general, this article contributes to research in mathematics education (presented above) by introducing a new model to account for responses to the comparative likelihood task. In specific, the main objective of this article is to demonstrate that certain individuals answer the comparative likelihood task according to a subjective partition of the sample space (i.e., via a subjective-sample-space), which is based on an individuals' interpretation of the sequence of outcomes.

## Theoretical Considerations

Individuals, when responding to the comparative likelihood task, reason in (at least) one of three ways: normative, heuristic, and informal. Correct responses are associated with normative reasoning, while incorrect responses are associated with heuristic and informal reasoning. As mentioned, certain models have been developed to account for incorrect responses. More specifically, representativeness (Kahneman & Tversky, 1972) was developed to account incorrect responses derived from heuristic reasoning, the equiprobability bias (Lecoutre, 1992) was developed to account for correct responses also derived from heuristic reasoning, and the outcome approach (Konold, 1989) was developed to account for incorrect responses derived from informal probabilistic reasoning.

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*Heuristic Reasoning*

*The representativeness heuristic.* In examining how “people replace the laws of chance by heuristics” (Kahneman & Tversky, 1972, p. 430), the authors produced an initial investigation into the representativeness heuristic. According to their findings, an individual who follows the representativeness heuristic “evaluates the probability of an uncertain event, or a sample, by the degree to which it is: (*i*) similar in essential properties to its parent population; and (*ii*) reflects the salient features of the process by which it is generated” (p. 431). Alternatively stated, and more specifically related to the comparative likelihood task, the determinants of representativeness were broken down into two particular features: similarities between the sample and its parent population and apparent randomness. The authors theorized that events are considered more probable when appearing more representative; and, similarly, events are considered less probable when appearing less representative. Kahneman and Tversky (1972) presented individuals with birth sequences that were considered equally likely, but were hypothesized by the authors to not be “equally representative” (p. 432). Of the three sequences presented—GBGBBG, BGBBBB and BBBGGG—the sequence BGBBBB was considered less likely than GBGBBG because BGBBBB does not reflect the ratio of boys to girls found in the parent population. Further, BBBGGG was deemed less likely than GBGBBG because BBBGGG did not reflect the random nature associated with the birthing of boys and girls in a family.

*The equiprobability bias.* Lecoutre’s (1992) research led to another bias, which should, according to her, “be added to the list” (p. 558) of heuristics and biases from psychology. More specifically, Lecoutre’s research was based on interpreting comparative likelihood task responses that involved a relationship between randomness and equiprobability. Lecoutre (1992) declared, “random events are thought to be equiprobable ‘by nature’” (p. 557). For example, in the comparative likelihood task the two sequences of coin flips HHTTH and HHHTH would be considered equally likely, because flipping a coin is a random process and thus “the two results to compare are equiprobable because it is a matter of chance” (p. 561).

*Informal Reasoning*

*The outcome approach.* “A model of informal reasoning under conditions of uncertainty, the *outcome approach*, was developed to account for the nonnormative responses of a subset of 16 undergraduates who were interviewed” (Konold, 1989, p. 59). Application of the outcome approach demonstrated that incomprehensible statements could be accounted for with a new interpretation of how subjects were reasoning about probability (Konold, 1991). In essence, Konold recognized that normative interpretations and heuristic interpretations of probability did not capture the multivalence associated with the comparative likelihood task, and probability in general. Konold (1995) went further and claimed that multiple models—normative, heuristic, and informal—could conflict in responding to a question such as the comparative likelihood task. Konold et al. (1991), in examining for consistencies over different problems, found “switching among alternative perspective[s] of uncertainty” (p. 360). Further complicating the matter, “different perspectives can be employed almost simultaneously in the same situation” (p. 360) because “people use a variety of frameworks and beliefs concerning uncertainty” (p. 361). To explicate their point, Konold et al. (1993) gave students a most likely version of the comparative likelihood task followed by a least likely version. It was found that for the most likely version some subjects answered using the outcome approach, but for the least likely version subjects answered using the representativeness heuristic. Having demonstrated individuals’ ability to have different problems cue different knowledge Konold et al. (1993) concluded, “in one problem, a person may appear to reason correctly, but in another, this same person may reason in ways that are at variance with probabilistic and statistical theory" (p. 393).

**Abductive Reasoning Considerations**

Consider, for the moment, the following situation: If one studies hard, then one will get good grades. Just because one achieves good grades does not necessarily mean that one studied hard. For example, one may have gotten good grades because one cheated. Thus, it is more appropriate to declare that if one gets good grades the *most probable* or *best* explanation is that one has studied hard, which may or may not be the case. In other words, the observation of good grades does mean that one cannot declare with certainty that studying hard is the appropriate rule used for explanation. However, through abductive reasoning one can hypothesize that studying hard was the reason for the good grades; and if the hypothesis were true, then the achievement of good grades would follow suit. Consequently, there would then exist reason to suspect that studying hard, the hypothesis is true. Similar approaches have been used to develop the representativeness heuristic, outcome approach, and equiprobability bias.

Abduction, also widely known as inference to best explanation, can be characterized as developing a good or the best hypothesis in order to explain observations. In general, facts are used as a starting point, a particular hypothesis—derived from inferences and used to best explain the facts observed—is presented, and if it is the case that if the hypothesis were true it would best or most likely explain the observed facts, there exists reason to suspect the theory hypothesized is true (Lipton, 1991). However, one cannot declare for certain that an individual is in fact answering the comparative likelihood task with, for example, the representativeness heuristic. Further, it cannot be claimed with certainty that individuals, any individuals, employ any of the cognitive models known when answering the comparative likelihood task. Hypotheses, like the representativeness heuristic, the outcome approach, and the equiprobability bias can be seen as (1) models hypothesized to explain observed results, and (2) as new research created through the abduction process when analyzing comparative likelihood task responses. Nevertheless, the models discussed have garnered enough support that they saturate comparative likelihood task research literature; and, moreover, the saturation has occurred to such a degree that often the models are misconstrued as matter-of-fact in declarations such as, "the student was using the outcome approach."

**A New Model: The Subjective-sample-space**

The application of personal theories or informal conceptions is found in many areas of probability education research, including sample space. For example, "to justify the probabilities for the outcomes of dice games, learners construct informal sample spaces" (Speiser & Walter, 2001, p. 61). The inferred structure of personal sample spaces has been used to demonstrate particular anomalies found in probability education research. Speiser and Walter demonstrated that certain individuals found, for example, the outcome (5,6) for the experiment of rolling two dice to be as likely as the outcome (6,6). Further, researchers extrapolated the lack of discernment between pairs to all outcomes (e.g., (3,4) and (4,3)). Consequently, researchers hypothesized—and then concluded—that the sample space employed by certain individuals answering the question consisted of 21 possible outcomes and not 36 outcomes, because individuals treated the outcome (5,6) and (6,5) as one outcome. Alternatively stated, responses to the task explicated a certain structure of the personal sample space used when answering the task. Similarly, the subjective-sample-space models a more nuanced structure of sample space, partitioned according to an individuals' interpretation of the sequence of outcomes, used when answering the comparative likelihood task.

## Task and Participants

Participants in this study were 239 prospective teachers. More specifically, there were 163 prospective elementary teachers enrolled in a methods for teaching elementary mathematics course; and 76 prospective secondary teachers enrolled in a methods for teaching secondary mathematics course. The 163 elementary teachers consist of students in five different classes over two different years, taught by two different instructors. The 76 secondary teachers consist of two classes taught by the same instructor in two different years. In all instances, the task (seen in Figure 1) was presented prior to the introduction of probability to the course.

*Which of the following sequences is least likely to result from flipping a fair coin five times. Provide reasoning for your response. (A) H H T T H (B) H H H T T (C) T H H H T (D) H T H T H (E) T H H T H (F) All sequences are equally likely to occur*

*Figure 1.* Comparative likelihood task implemented in the study.

## Results

Of the 239 participants who took part in the study approximately 82 percent (197/239) correctly chose that each of the sequences presented were equally likely to occur. For elementary teachers the percentage was approximately 81 percent (132/163), and for secondary teachers the percentage was approximately 86 percent (65/76). Alternatively, the 42 participants (roughly eighteen percent) that chose a normatively incorrect answer to the comparative likelihood task were comprised of 8, 13, and 20 participants who chose HHHTT, THHHT, and HTHTH as least likely, respectively. For elementary teachers 8, 7, and 16 chose HHHTT, THHHT, and HTHTH as least likely, respectively. For secondary teachers 0, 6, and 4 chose HHHTT, THHHT, and HTHTH as least likely, respectively. Also of note, within the five normatively incorrect responses to the task, one participant chose HHTTH least likely and zero participants chose THHTH as least likely.

## Theoretical Framework

"Alternative set descriptions of the sample space can act as an investigative lens for research on the comparative likelihood task" (Chernoff, 2008, p. 313). More specifically, Chernoff introduced three alternative set partitions of the sample space—based on switches, runs, and switches and longest run organizations of the sample space's elements—as theoretical frameworks for comparative likelihood task responses. Chernoff's switches partitions of the sample space will be employed as the theoretical framework for analysis of results.

## Analysis of Results

The analysis of results is (recognizing contextual limitations) restricted to response justifications of the 20 respondents (*e* denotes elementary teacher, *s* denotes secondary teacher) who chose sequence HTHTH least likely to occur.

*Response Justifications for HTHTH as Least Likely*

*Response justifications (1): pattern v. randomness & likelihood.*

*(e) Claire*: 1st choice: (F) All have the same likelihood of occurring is what I think. It's random. 2nd choice: (D) The chances of *a nice tidy pattern like these seems unlikely*.

*(e) Michael*: *It's all pretty random but HTHTH seems too perfectly organised*.

*(e) Sayid*: It's *hard to find a pattern, so the ones that are the most random are most likely to happen*

*(e) Sun*: I think *it's not likely for it to follow a pattern*.

*(e) Aaron*: I think *it's [HTHTH] least likely because it has a pattern*.

*(e) Ana*: *random flipping does not produce neat patterns* like this.

*(s) Eko*: *c is least likely as it is patterned. Patterns are less likely to arise from random events,* but b would be the second less likely as there are 3H in a row, but it is not highly unlikely, just more so than a and e

*(s) Jin*: I would think *the odds of getting a perfect HTHTH pattern are slim* (at least one letter would be off most of the time)

The *italicized portions* of the response justifications presented above evidence a connection between pattern versus randomness (read: lack of pattern) and likelihood for all eight individuals. However, individuals' responses, while alluding to randomness, are in fact discussing the appearance or perception of randomness found in the sequences. For example, Aaron's response indicates that HTHTH is least likely because it has a pattern. Further evidenced from the responses and in accord with Kahneman and Tversky's (1972) assertion and conclusion, the more patterned (i.e., less random) the sequence the less likely its occurrence, and the less patterned (i.e., more random) the sequence the more likely its occurrence. For example, and according to Sayid, "the ones that are most random [read: least patterned] are most likely to happen." Whereas the other seven responses (e.g., Sun's response: "I think it's not likely for it to follow a pattern") demonstrate that a neater or tidier the pattern means the less likely the chances of occurrence.

*Response justifications (2): switches & likelihood.*

*(e) Kate*: I believe there is a 50/50 chance that the first flip will be a heads or a tails. Therefore, I believe that **D is least likely to occur b/c the odds of flip a coin from heads to tails is fairly slim**.

*(e) Ben*: I think **HTHTH is low percent because it appears alternately**.

*(e) Ethan*: In my opinion, **'D' is the least likely occur because it is hard that different sides continually appear**.

*(e) Penny*: **it's least likely that every flip will alternate between heads/tails**. However, I think all sequences are equally likely to occur.

The **emboldened portions** of the response justifications presented above evidence a connection between switches and likelihood. More specifically, each of the four responses evidence that the perfect alteration of heads and tails (e.g., HTHTH), that is the maximum number of switches possible for a sequence, would correspond to the least likely of the sequences to occur. According to Penny, "it's least likely that every flip will alternate between heads/tails." The connection between switches and perceived randomness seen in the justifications above aligns with previous research results (e.g., Falk, 1981), which demonstrated frequent switches were indicative of the appearance of randomness.

*Response justifications (3): pattern v. randomness & likelihood & switches.*

> *(e) John*: ***D is least likely to occur because the chances of having the coin*** *land on the opposite side each time* to **create a pattern of HTHTH** ***are very slim***, the longer the pattern the less likely it will be. Also, to get 3 H's in a row [sequence B] is probably next least likely.
>
> *(e) Hurley*: Although there is a 50% chance of getting a H or a T. ***It is very unlikely that you can get a sequence of alternating sides randomly***. The probability of this sequence happening would be the least likely.
>
> *(e) Jack*: With D, an alternating sequence could occur but not necessarily in this order, H + T are more likely to occur at a more random interval.
>
> *(e) Sawyer*: ***(D) is least likely to occur because with a 50/50 chance it is*** **unlikely that the results will be alternating H/T with each coin flip**. *It is more likely that the results would be random*.
>
> *(e) Juliet*: ***I believe that D is the least likely answer because it is too perfect of a pattern.*** Even though there is a 50/50 chance of the coin coming up heads or tails, ***it is very unlikely that it would rotate between the two each flip.***
>
> *(e) Tom*: *because the others are more random they are more likely*, ***but to alternate 1 and 1 each time no, that's like orchestrated fairness…doesn't happen it has to be guided***.
>
> *(s) Rose*: If a coin is flipped five times, **the chance of it going from head to tails, head to tails…is not likely**. ***Rather****, the coin will likely go from tails and head randomly*.
>
> *(s) Bernard*: Although not impossible, ***I think c (the alternating HTHTH) is least likely to occur*** *b'cuz flipping a coin is a random act and option c is not*.

Whereas the first set of response justifications evidenced a relationship between the appearance of randomness (determined through pattern or lack thereof) and likelihood, and whereas the second set of response justifications evidenced a relationship between switches and likelihood, the third set of response justifications evidence a relationship between (1) the appearance of randomness (derived from presence or absence of pattern), (2) likelihood, and (3) switches (or alterations). More specifically, the *italicized portions* above evidence a connection between randomness and likelihood, the **emboldened portions** evidence a connection between switches and likelihood, and the ***italicized and emboldened portions*** evidence the relationship between appearance of randomness, likelihood, and switches. For example, Hurley declares (in part), "It is very unlikely that you can get a sequence of alternating sides randomly." In Hurley's response, for example, the low likelihood, while connected to the perceived absence of randomness due to the pattern, is being determined by the alteration of the coin from heads to tails. In fact, and for all eight responses shown above, the likelihood of the sequence is derived from the absence of presence of pattern, also known as perceived randomness. However, the perceived randomness is derived from the alteration or switches of the sequence. As such, and syllogistically, it is contended that the switches attribute of the sequence element is being employed to determine the likelihood of the sequence. Alternatively stated, it is contended that a subjective-sample-space partitioned according to the switches attribute of the sequence, in this instance, is how subjects are interpreting the sequence element of the comparative likelihood task and subsequently responding to the task.

The justifications provided in participants' responses indicate that the subjective-sample-space they are describing corresponds to an entirely different partition of the sample space than responses are conventionally and traditionally pitted against. According to Chernoff (2008), based upon the verbal descriptions presented, a more appropriate or natural set description, corresponding to the verbal descriptions given, would be to partition the sample space according to switches, as shown in Table 1. When the response that HTHTH is least likely is pitted against the switches partition of the sample space, the response can be considered correct, in that HTHTH is the least likely sequence to occur, because n(4switches)<n(3switches)=n(1switch)< n(2switches). In other words, the event of alternating sides every time does have the least number of outcomes when compared to all of the other sequences, and, thus, would be least likely. As such, and through an alternative interpretation, all eight people represented in *Response justifications (3)* can be seen as 'correctly' answering the task. For example, Rose can be interpreted as correct in declaring that "if a coin is flipped five times, **the chance of it going from head to tails, head to tails…is not likely**." Further, the responses from *Response justifications (2)* when pitted against the switches partition of the sample space can also be considered as correct in their answering of the task. Thus, 12 out of 20—the claim for 20 out 20 cannot be made because what the 'pattern' is derived from was not able to be determined in *Response justifications (1)*—response justifications for HTHTH being least likely to occur can be considered correct when pitted against the switches partition of the sample space.

Table 1. *The Sample Space Partitioned According to Switches (Denoted S)*

| 0 Switch | 1Switch | 2Switches | 3Switches | 4Switches |
|---|---|---|---|---|
| HHHHH | **HHHTT** | HHTHH | HHTHT | ***HTHTH*** |
| TTTTT | TTTHH | TTHTT | TTHTH | THTHT |
| | TTHHH | **HHTTH** | HTHHT | |
| | HHTTT | TTHHT | THTTH | |
| | HHHHT | HTTHH | **THHTH** | |
| | TTTTH | THHTT | HTTHT | |
| | THHHH | HHHTH | THTHH | |
| | HTTTT | TTTHT | HTHTT | |
| | | HTHHH | | |
| | | THTTT | | |
| | | **THHHT** | | |
| | | HTTTH | | |
| n(0S) = 2 | n(1S) = 8 | n(2S) = 12 | n(3S) = 8 | n(4S) = 2 |

## Conclusion and Discussion

If it were the case that individuals are employing a subjective-sample-space partitioned according to the switches attribute of the sequence element, then individuals would think that a perfect alteration for the sequence (i.e., HTHTH) is least likely to occur, which was evidenced through Chernoff's (2008) switches partition of the sample space. As such, there exists reason to accept the claim that the employment of a subjective-sample-space under conditions of uncertainty, in this instance organized according to switches, is taking place when answering the comparative likelihood task, which lends support to the main objective of the article: To demonstrate that certain individuals answer the task according to a subjective-sample-space partitioned according to their interpretation of the sequence element of the comparative likelihood task. However, when abduction is used as a mode of reasoning, minor premises, such as the representativeness heuristic, outcome approach, equiprobability bias, and subjective-sample-space, cannot be declared with certainty. As such, despite conclusions presented, none of the assertions made in this research, nor in related prior research, can be declared with certainty…quite fitting for research in probability.

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# FIGURING IT OUT: MATHEMATICAL LEARNING AS GUIDED SEMIOTIC DISAMBIGUATION OF USEFUL YET INITIALLY ENTANGLED INTUITIONS

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*When participants in inquiry refer to an object, they may, unbeknown to them, construct the object differently. They thus tacitly attribute different idiosyncratic senses for their respective constructions and consequently draw different inferences regarding the phenomenon under investigation. A single person, too, may shift between alternative constructions of a mathematical object, assigning them different senses, thus arriving at apparently competing conclusions. Only upon acknowledging the different constructions can the person begin to explore whether and how the differing conclusions are in fact complementary. Building on empirical data of students engaged in interview-based tutorial activities targeting fundamental probability notions, we explicate breakdowns such false-contradiction introduces into learning processes yet suggest opportunities such ambiguity fosters.*

'Seeing as....' is not part of perception. And for that reason it is like seeing and again not like.
(Wittgenstein, 1953)

## Objectives

Objects per se do not carry any meaning—all meaning is mentally constructed. The same principle holds for classroom learning materials, be these plastic tokens, spatial–numerical diagrams, or symbolic inscriptions. Yet this fundamental tenet of phenomenology and constructivism—that meanings of objects are mediated by implicit mental structures and are anyhow transparent in the ongoing *Dasein* of goal-oriented activity—may be difficult for a teacher to bear in mind let alone apply successfully in the real-time contingencies of engaged mathematics discourse. Moreover, students are often unaware of the constructed nature of their own mathematical perception of objects and therefore do not differentiate between objects per se (the distal stimuli) and their personal constructions of these objects (the proximal stimuli) (Wittgenstein, 1953). Consequently, teachers and students may be explicitly speaking about the same object yet implicitly ascribing to it diverging meanings and related inferential implications, and therefore their communication fidelity is a priori compromised (e.g., Borovcnik & Bentz, 1991). Nevertheless, a teacher can be well aware that two or more students are seeing a mathematical object differently even though they are using similar lexical labels to index the same object, and a skilled teacher can capitalize on these covert ambiguities to orchestrate productive discursive negotiations (Moschkovich, 2008). Still, teachers cannot always interpret, monitor, foster, or amend students' idiosyncratic constructions so that they accord sufficiently with normative constructions (i.e., so that the meanings are taken-as-shared, Cobb, 2005). Thus, covert communication breakdowns in classroom discourse may be more ubiquitous than one might expect, with interlocutors bearing personal meanings that overlap just enough to preclude overt breakdown.

Yet is such covert polysemy and the communication breakdowns it engenders necessarily detrimental to learning? Here we wish to argue that some covert semiotic "fuzziness" may in fact ultimately support collaborative learning, because it enables interlocutors the ostensible intersubjectivity requisite of mutually supportive discourse, *even as they are seeing objects differently*. Specifically, when students construct differently a semiotic artifact under joint inquiry, they may contribute to a conversation different mathematically valid assertions precisely because they are not cognizant of their different constructions. For example, if you and I are gazing at an array of six dots, I may see it as two rows of three dots each even as you see it as three columns of two dots each. Referring to the array, I might say that, "It is two times three," but then you might disagree that, "It is three times two." Notably the "it" in each of our respective utterances does not refer to the "objective" array itself but to our respective mental constructions of the array. Sorting out disagreement over the meaning of objects thus becomes an opportunity to co-examine the semiotic elements implicit to the conversation, e.g., the unitized groups of dots in the array. Namely, the conversation may shift from arguing over some ill-defined *mis*taken-as-shared 'it' to speaking about how we are seeing 'it,' i.e., to *figuring it out.* So doing, we may discover the semiotic contingencies of our respective statements and formulate a mathematical assertion that reconciles their respective meanings in the form of the targeted mathematical content of the instructional activity, e.g., we may discover that "$2 \times 3 = 3 \times 2$."

This paper examines excerpts from one-to-one interview-based conversations between a researcher and three students, in which the students each sustained throughout a tutorial activity two different framings of a single iconic artifact, which they had been guided to construct so as to model a mathematical system under inquiry. Each framing of the object implied a different expectation for the behavior of this system, and the students' expectations shifted with their framing of the object. We argue that both mental constructions of the object were mathematically correct, if naively worded, and that the students were able to reconcile these constructions successfully only if they were aware of the contingency of their assertions on their implicit framings of the object. We further submit that the ambiguity of the object ultimately supported these students' learning, because it elicited the two key idea elements of the targeted mathematical notion and juxtaposed them for reflection. That is, embedding key idea elements of a targeted mathematical notion within a single semiotic artifact instantiated these elements as co-present in the problem space, thus honing a generative confusion that supported the conjoining of these idea elements into the targeted conceptual composite. Thus we support *embracing diverse perspectives*, in line with the conference theme.

## Theoretical Framework

In his *Philosophical Investigations,* Wittgenstein (1953) sets out by describing a Tower-of-Babel scene, in which construction workers are able to collaborate only because they share referents for their otherwise arbitrary verbal utterances. Thus, if I ask for a "brick" and you hand me a brick, we are capable of co-constructing an artifact, but if you instead handed me a bucket, the premise of our collaboration would be compromised. Yet along with my frustration resulting from this patent miscommunication, we maintain, I may gain a useful realization of the language game underlying human intersubjectivity. Namely, as language breaks down, its normatively obscure equipmentality is disclosed for scrutiny (Heidegger, 1962). To the extent that language, writ large, is the internalized vehicle of human reasoning (Vygotsky, 1934/1962), understanding its semiotic mediation of "objective" situations may be instrumental to reflecting on one's learning process, which necessarily requires the adoption of cultural forms of seeing and referring to aspects of one's personal, unreified phenomenology (Bamberger & diSessa, 2003; Goodwin, 1994; Stevens & Hall, 1998).

In the brick-vs.-bucket communication breakdown, above, one and the same verbal utterance, "brick," differentially referred to two objects in the joint perceptual field. Yet inherent to this miscommunication is that one and the same object was interpreted differentially—the object that you saw as a brick, I saw as a bucket. Such flagrantly conflicted constructions of distal stimuli, though reserved for rhetorical effect in philosophical discourse, may nevertheless underlie—if in a more nuanced caliber—challenges inherent to instructional discourse. In the case of the disciplines where unequivocal definitions are paramount to the production of texts (in the continental, multi-modal sense of 'text'), it thus becomes important to monitor for shared meanings of objects.

Note that the sense that interlocutors ascribe to objects are not necessarily personally available-it is not the case that students are consistently conscious of *how* they are seeing an object, even as they are capable of describing what the object *means* in the context of disciplinary discourse, such as problem solving. Indeed, idiosyncratic constructions of objects may be by-and-large inaccessible ('cognitively impenetrable,' Pylyshyn, 1973), unlike meanings, which may be verbally couched as rationalized inferences pertaining to a phenomenon under inquiry. Nevertheless, the very rationale of scholarly inquiry into students' understanding of instructional materials is the identification and articulation of their personal constructions of objects. This problematique of an analytic endeavor to name the ineffable psychological facets of human discourse has been treated before:

> We do not claim to make clear and explicit what the users of the unclear expression had unconsciously in mind all along. We do not expose hidden meanings, as the words 'analysis' and 'explication' would suggest; we supply lacks. We fix on the particular functions of the unclear expression that make it worthy troubling about, and then devise a substitute, clear and couched in terms of our liking, that fills those functions. (Quine, 1960, pp. 258-259)

Whereas ambiguity of discourse readily suggests *inter*subjective situations, Quine orients us toward *intra*subjective situations. Namely, by virtue of referring to an object by two different labels, one perforce brings out different meanings, demonstrating a phenomenon Quine called *intrasubjective stimulus synonymy*. For example, "For each speaker, 'Bachelor' and 'Unmarried man' are stimulus-synonymous without having the same meaning in any acceptable defined sense of 'meaning'" (Quine, 1960, p. 46).

In this paper, however, we present cases of intrasubjective stimulus *polysemy* and discuss their consequences for mathematical learning. Namely, we demonstrate how an individual student's competing perceptual constructions of a mathematical semiotic artifact initially create cognitive conflict between two inferences that are in fact both mathematically correct. These inferences appear to the student as conflicting, rather than complementary, because the student tacitly equates the objective artifact with its perceptual construction. We highlight the indispensable role of instructional designers and mathematics teachers in both eliciting from students each of the apparently conflicting inferences and facilitating discourse that aims at exposing the different perceptual constructions underlying each inference. The vocabulary, constructs, and definitions necessarily generated so as to achieve these disambiguations are pivotal aims of the instructional process, because these discursive tools help students *synthesize* (Schön, 1981) tacit and mathematical views of the instructional materials. Yet what are the implications of this thesis for teachers' practice?

Guiding students to construe mathematical objects in accord with disciplinary norms is generally an asymmetric process, in which a teacher enables a student to see things as she does and any alternative construction falls by the wayside (Goodwin, 1994; Stevens & Hall, 1998). Yet for some disciplinary content topics, multiple views of problems are intrinsic to mathematical discourse, so that fostering such ambiguity in classroom discourse may play a nurturing, rather than an obstructing role. For example, a sequence of coin tosses—Heads, Tails, Heads, Tails (HTHT)—may be construed as one of sixteen equiprobable elemental events in the sample space of the four-coin-flips experiment (1/16) or, alternatively, as the aggregate event "2 Heads and 2 Tails in any order" that has a 6/16 chance of occurring (on the contingency of mathematical definitions on social contract, see Barnes, Henry, & Bloor, 1996; Ernest, 2008; Weisstein, 2006). A student may sense that one must decide between these two mathematically valid constructions, thinking that HTHT cannot have both 1/16 and 6/16 chances of occurring. Namely, this student would experience a need to decide whether the object—the distal stimulus presented by the inscription "HTHT"—has this value or that value for the property of likelihood, where in fact the student is implicitly referring to different percepts but not articulating the implications of attending or not attending to the internal order of the four singleton events (Abrahamson, 2009).

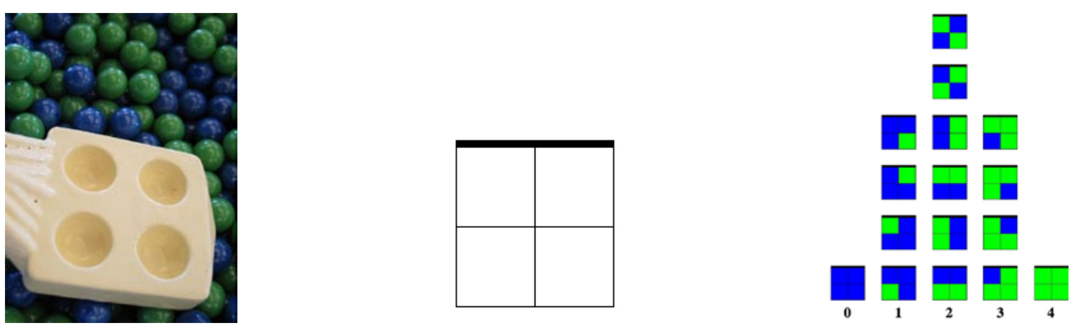
Indeed, in this paper, we present empirical data to argue that one challenge inherent to supporting students' sense making processes is that students are liable to implicitly equate mental constructions with objects per se and thus experience difficulty accepting, let alone reconciling, any competing meanings they may attribute to these objects. That is, when the students think they must make up their mind with respect to the assertions they express about a mathematical object, in fact these different assertions are not necessarily mutually exclusive but possibly complementary, because each assertion refers to a different mental construction of one and the same object. Differentiating these assertions on the basis of their underlying perpetual constructions is crucial for conceptual development in those cases where both assertions are conceptually pertinent. For example, acknowledging the ambiguity of HTHT may help a student understand that the probability of an aggregate event is the sum total of the probabilities of its elemental events (1/16 + 1/16 + 1/16 + 1/16 + 1/16 + 1/16 = 6/16).

## Background, Methods, and Research Focus

The episodes analyzed herein come from a larger corpus of data collected over a succession of cumulative studies conducted as part of the *Seeing Chance* project to understand and promote probability learning (Embodied Design Research Laboratory, UC Berkeley). Specifically, we examine the behavior of 3 out of 28 middle-school participants in Abrahamson (2009). The study took place in a private school in the SF East Bay area (33% on financial aid; 10% minority students), and all three focus students for this paper were ranked by their mathematics teachers as high achieving. The phenomenon of intrasubjective stimulus polysemy that we examine here was typical of all students, yet it elicited longer, richer, and more articulated deliberations from the older and higher-achieving students—perhaps because these students were more self-monitoring and self-exacting in their mathematical reasoning—and hence these study participants help us understand what may be a ubiquitous phenomenon characteristic of all students. Each student participated in a semi-structured clinical interview that lasted about one hour.

The project was conducted in the design-based research approach, which typically examines some conjecture as to an underlying mechanism inherent to a hypothetical learning phenomenon by creating empirical contexts in which to examine this conjecture (Confrey, 2005). Emerging from study cycles of design, empirical implementation, and analysis, in which the researchers tune the learning environment and, reciprocally, their emerging understandings of learning phenomena, are new instructional materials or principles as well as 'ontological innovations' (diSessa & Cobb, 2004), theoretical constructs that capture consistent patterns that the researchers discover in the empirical data. This paper is about intrasubjective stimulus polysemy, an ontological innovation that we are proposing.

Central to the interview was a set of instructional materials designed to elicit students' population-to-sample informal inferences, which are mathematically correct though only qualitative and unwarranted by mathematical argumentation. Students are then guided to construct the expanded sample space of this experiment as a means of creating a context for the dyad to discuss differences in how natural perceptual inclination and formal mathematical analysis couch inferences with regard to probabilistic behavior of random generators. Here we will introduce only those materials that feature in the data under inquiry. The interview begins by showing participants a tub containing many green and blue marbles of equal numbers as well as a *marbles scooper* (see Figure 1a), a utensil for drawing out of the box a sample with a precise number of marbles that are spatially arranged in a particular permutation. Strictly speaking, this is a hypergeometric (without replacement) problem, yet the large population-to-sample ratio enables us to treat it as an approximation for the binomial. Participants are asked to offer their guess for the distribution of outcomes in a hypothetical experiment with this random generator. Next, participants are given a set of blank cards with a 2-by-2 table structurally resembling the scooper (see Figure 1b) and are guided to construct the sample space of the experiment and assemble in the form of the *combinations tower* (see Figure 1c).




*a.* *b.* *c.*

*Figure 1.* (a) The marble scooper; (b) one of many cards for conducting combinatorial analysis of the experiment; and (c) the combinations tower—an assembly of the sample space in a format designed to resonate with students' inferences for the experiment.

Whereas students by-and-large guessed correctly that the plurality of experimental outcomes would be of type “2 green and 2 blue [in any order]” (hence 2g2b), they experienced difficulty in appreciating why analytic attention to the *order* of the four singleton events in each scoop may be advantageous to supporting their guess. Nevertheless, once they had completed constructing the combinations tower, participants appropriated this structure as a warrant for their guess by indexing the relatively greater number of 2g2b elemental events as compared to other aggregate events. In previous publications we attributed students’ reluctance to attend to the combinations’ internal order to a tension between tacit and mathematical constructions of the sample: whereas students naturally couch the experiment in terms of five (aggregate) events (no green, 1 green, 2 green, 3 green, and 4 green), combinatorial analysis requires attention also to the internal order of the four singleton events and therefore produces sixteen (elemental) events.

The current study focused on interview episodes in which participants switch between aggregate- and elemental-event constructions of a compound-event card containing four singleton events. We compared these episodes in an attempt to explore for relations between the participants’ awareness of their constructions and their success in coordinating the tacit and mathematical formulations of the anticipated experimental outcome distribution.

## Results and Analyses

Table 1, below, offers a preview of our results. For rhetorical clarity, we use the familiar duck-rabbit ambiguous figure. (Joseph Jastrow popularized it in the late 19th century so as to illustrate perceptual agency in constructing distal stimuli.) If a viewer is asked to infer the eating habits of this ambiguous creature, yet the viewer is unaware that his mental construction of the image keeps shifting (“duck…no, rabbit!”), then the viewer will not understand his vacillating inferences (“fish…no, carrots!”) and will take this inconsistency as marking confusion. If, however, the viewer can label each mental construction of this object as well as their critical disambiguating features (“beak…ears”), then the viewer will be equanimous with respect to his conflicting inferences (cf. Tsal & Kolbet, 1985).

Table 1. *Inferential Reasoning for an Overtly Ambiguous Figure*

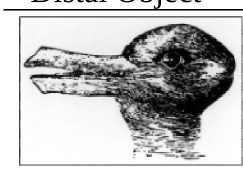
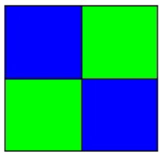
| Distal Object | Disambiguating Features | Proximal Object | Inference for Diet |
|---|---|---|---|
|  | Beak | Duck | Fish |
| | Ears | Rabbit | Carrots |

Table 2, below, presents the less familiar case, from probability studies, of a compound event as an ambiguous figure. A viewer who attends to the particular configuration of green and blue cells in this object may construct it as one of sixteen unique equiprobable elemental events in the sample space. However, a viewer who ignores the internal order of cells in this object and constructs it as 2g2b may interpret it as the aggregate event most likely to occur in the marbles-scooping experiment. If, however, the viewer is unaware of her shifting personal constructions, she will interpret her shifting inferences as marking confusion.

Table 2. *Inferential Reasoning for a Covertly Ambiguous Figure*

| Distal Object | Disambiguating Features | Proximal Object | Inference for Distribution |
|---|---|---|---|
|  | Order | Elemental event | Equiprobable |
| | Number | Aggregate event | Heteroprobable |

Each of the following three 6th-grade students, Lavi, Sima, and Razi, identified the completed combinations tower as resonating with their mathematically correct guesses for the outcome distribution of the marbles-scooping experiment. However, subsequent discussion suggested that their insight was unstably based on a global perception of relations among the combinations-tower columns and that they were still struggling to align their insight with the ambiguous construction of the combinations tower's constituent elements.

*Lavi: "My Mind's Going Back and Forth"*

The interviewer lifts out of the sample space two cards—one of the 3g1b cards and the 4g card—and asks Lavi to compare their likelihoods. The following conversation ensues:

Lavi: There's only four [ways] for getting three [3g1b]. I guess it wouldn't be chance…Oh, I guess it would be chance. And then there's only one way that there can be four [4g].

Res.: So, what do you mean [by] "It's not chance" and "It is chance?"

Lavi: Ah, I don't know, that thought just kind of [popped] into my mind and I just let it come out.

When Lavi says, "It wouldn't be chance," he is viewing the individual cards as representing heteroprobable aggregate events whose chance is indexed by the number of permutations in their respective columns. When he says, "It would be chance," he is viewing these same cards as equiprobable elemental events for which only chance, not logic, would cause greater frequency. The compound event is thus a physical object imbued with different mathematical constructions, and Lavi alternately refers to these competing constructions. However, he does not appear to realize that he is shifting his point of view, so he is confused.

The interviewer repeats the question for another pair of cards. Lavi asks whether he should take these cards to mean "a specific card or an amount of each color" and claims that all "specific cards" are equally likely. The interviewer asks Lavi to compare the cards on the basis of "the amount," i.e., to ignore placement. After some hesitation, Lavi nevertheless asserts, "It is chance," invoking the randomness of the sampling device (on the 'equiprobability bias,' see Falk & Lann, 2008; LeCoutre, 1992). Throughout a subsequent series of questions, Lavi vacillates between viewing individual cards as "ducks" or "rabbits," coming just short of reconciliation.

*Sima: Stuck on Rabbit*

Like Lavi, Sima begins by articulating the equiprobability of the sixteen compound events. She creates the term "color-wise" to refer to the groups and "place-wise" to refer to individual cards and states that "color-wise" the groups have different probabilities but place-wise "they're equivalent." Yet, once the interviewer asks her to compare two cards selected from different columns, she maintains that they have different likelihoods. Subsequently, she appears to experience difficulty in dislodging from the aggregate view and returning to the elemental view—she is "stuck on rabbit" and insists that any 2g2b card is more likely than any 3g1b card.

Only after the interviewer simulates random sampling from these sixteen cards and refers back to the initial experiment is Sima able to reassume equiprobability.

*Razi: Chooses Rabbit*

Like Lavi and Sima, Razy articulates that specific cards are equally likely yet that viewing them by the "number of each color" makes some cards more likely than others. She appears to command greater fluency than Lavi and Sima in shifting between the competing constructions of the events, yet she incurs greater difficulty in articulating the implications of each view for the outcome distribution.

Razi: The majority of the scoops would come out with two blues and two greens.

Res.: A moment ago you told me that each pattern has the same likelihood to show up. Is their a contradiction here?

Razi: Yes and no. Before I said "each specific pattern." Now I'm saying each pattern with two blues and two greens….

Res.: But do you still hold to the fact that each exact pattern has the same chance?

Razi: I am not sure.

Finally, when asked to compare another two cards, Razi becomes entrenched in the aggregate view. It appears that whereas Razi understands that there are two ways to see the object, she feels she must choose between "duck" and "rabbit."

## Conclusion

Students' awareness of their perceptual constructions of ambiguous mathematical objects—their intrasubjective stimulus polysemy—impacts their capacity to generate domain-specific constructs and, in turn, to coordinate tacit and analytic formulations of situated phenomena toward deep conceptual understanding. We have demonstrated this relation for the case of the binomial and will continue to pursue our conjecture as it plays out in the learning of other mathematical concepts.

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# EXPLORING STUDENTS' NOTIONS OF VARIABILITY IN CHANCE SETTINGS[1]

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*This paper revisits the authors' study of parts of the works of Watson, Kelly, Callingham and Shaughnessy (2003) in a different environment. It explores the notions that some Mexican students hold about statistics variability in chance setting. The study attempts to answer the question: Is it reasonable to classify students' answers according to the level in which they consider randomness, structure and variation? A questionnaire was administered to 327 middle school students, 214 high school students, and 74 college students. The analysis were carried out under; "Pre-structural", "Extreme values", "Structure", "Realist appearance", and "relational" categories. The results indicate a positive correlation between students' educational levels and the categories reported in some literatures.*

## Introduction and Background

One of the first empirical research studies on variation in chance setting was carried out by Shaughnessy, Watson, Moritz and Reading (1999), in which the following questions were asked: 1) what are students' understandings of variability or spread? 2) How can we begin to measure students' understanding of variability or spread? To answer the question, the Gumball task* was modified in such a way as to reveal thinking patterns of students in variability.

It was observed that students' understanding of centers increases with age while their understanding of variability oscillates through the grades. The results buttress the hypothesis that "there is considerable focus on 'centers' in the curriculum throughout school mathematics".

In an experiment carried out by Watson and Kelly (2003) to explore the thinking and progress of students of grades 3 to 9 after some studies on statistics variation, students were asked to predict the possible outcomes of repeated trials of a 50 – 50 spinner. The results among others show "a monotonic decline over the grades in the ability to provide reasonable variation" (p. 391); there is a strong tendency to conclude that the above was "in strict accordance with theoretical probability". As a result, a lesson for teaching of probability was drawn: Expectation must be balanced by variation (p. 393).

Shaughnessy, Canada and Ciancetta (2003) explored the thinking of 84 students of middle school with three tasks which involved repetitive tests. One of the tasks is similar to question 6 of the present work. It was observed that students tend to neglect variability in the context of chance and probability, in the tasks. The reason for this was attributed to the manner in which probability was introduced to the students in their school.

A questionnaire devised to assess students' understanding of variation by Watson et al. (2003) includes items in sampling variation, displaying variation, chance variation, measuring variation, and sources of variation. A coding scheme based on the SOLO taxonomy of Biggs and Collis (1991) was elaborated for that study, and a sample of 746, students in grades 3, 5, 7, and 9 were examined. One of their objectives was "to develop a scale to measure students' understanding of variation in the context of chance and data curriculum" (p. 15). They suggested,

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among others, for future research that "Trialing the questionnaire with other students, for example in other cultural settings" (p. 19). That suggestion accounts for this work which, in this case, studies Mexican students. However, we have followed another trend of analysis.

## Conceptual Framework

In this work, the concept of variation in chance settings is considered and elaborated within three important stages: the perception of the randomness, the consciousness of the probabilistic structure underlying a situation in a chance context and the understanding of the relation of this structure with the empirical data. A student is said to be *disorder, dispersion or randomness,* centered in if he believes that in a random situation "any data/thing can occur", that is to say, that any data can be obtained. In this case the fact that there is an underlying structure in the data generated is ignored. On the other hand a student is said to be *structure* centered if he expects the data generated to have a particular or regular pattern that conforms to a theoretical distribution (in this case- uniform or binomial), ignoring the randomness. Lastly a student is *variation* centered if he recognizes both the randomness or dispersion and the structure.

In a way of hypothesis a three stage model of evolution of the thinking of the intuitive notion of variation in random situations is supposed. In the first stage the answers are *disorder* centered, that is they show only the perception of the randomness in the data of a chance situation. Later they change to become *structure* centered, when they give evidence of the search expectation of a regular patterned data and, in the third stage and the last stage when answers appear the students consider both the randomness and structure. This last stage is divided in two sub-divisions; in the first, the answers do not give any evidence of establishing relationships between randomness and regularity, in the other, they do.

*Do students' answers pass through the stages of: disorder, structure, and variation while students construct their notions of statistical variation?*

Below, data are explored and elements are sought to support the hypothesis made.

## Methodology

*Participants*

Several groups of students of three levels of the Mexico City's educational systems were polled; 327 students from Middle School, 214 from High School and 74 from College.

*Instruments*

A questionnaire of 12 items was designed, most of which were picked from the questionnaires of Watson *et al.* (2003), while other were the slightly modified version of the same. In this report we will analyze three items, labeled 6, 10 and 11. We will present and describe them briefly.

Item 6. Imagine you threw the dice 60 times. Fill in the table below to show how many times each number might come up.

| **Number on Dice** | **Number of times it might come up** |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| **Total** | **60** |

Why do you think these numbers are reasonable?

*Figure 1.* Problem 6.

Figure 1 shows problem 6. Since this item asks students to suggest a distribution of frequencies of throwing a fair die 60 times, this is a prediction problem: what do you think it might happen?

When one solves this problem it is needed using statistic intuition as well as randomness, probability and variability notions. A disorder centered student will notice only the random nature of the experiment and would answer “any data can occur” or “we can not know”. This kind of answers is valid but not convenient, since ignore the probabilistic structure inherent to the experiment.

A structure centered student will tend to answer with the expectation; these answers are valid as well, but also inconvenient due to the low probability of such result.

“There is, in our brain, a constant tension between the general and specific” (Tal, 2001, p.69). Some variation centered students will notice this tension and solve it by filling out the table with specific numbers different to the mean. Nevertheless, this approach is still incomplete, in the same way that answering with the mean is: the low probability of particular outcomes. Some other variation centered students will be able to solve this tension trying to capture the highest probability possible by an interval around the mean.

Item 10: There is an urn containing 3 balls, each one marked with a letter (A, B, and C). John picks up randomly a ball, writes in a board its letter, and then he replaces it to the urn. John repeats 30 times the experiment. Which one of the following boards do you think John got? Put a mark (a tick) under the board you think is the correct one.

| Ball | Frequency |
|---|---|
| A | 12 |
| B | 7 |
| C | 11 |
| Total | 30 |

[]Board 1

| Ball | Frequency |
|---|---|
| A | 10 |
| B | 10 |
| C | 10 |
| Total | 30 |

[]Board 2

| Ball | Frequency |
|---|---|
| A | 11 |
| B | 18 |
| C | 1 |
| Total | 30 |

[]Board 3

Why did you choose the board?

*Figure 2.* Problem 10.

The problem presented in figure 2, poses to students an informal hypothesis test with three hypotheses to contrast: board 1 is right, board 2 is right and board 3 is right. To make a choice, students should discard options to keep the most adequate one. Intuition is valuable in this case

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in order to judge each board as a real outcome and decide how likely it is. Board 2 has the highest probability to occur, but this probability is still too low and do not consider the variation concerning the randomness. Board 3 shows a very atypical behavior, since ball C appeared only once and B 18 times. Thus, board 1 is the most reasonable choice.

This approach appeals to common sense as well as abstraction: board 3 represents the class of outcomes with extreme values, and classified as disorder center answers; board 2 represents the existence of structure in random experiences and considered structure centered answers; board 1 represents the class of real outcomes, in which it can be found a reasonable relation between structure and variability: it is realistic.

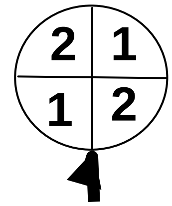
The item 11, (fig. 3) is analogous to item 10. In this case graph A is structure centered (corresponds to a theoretical binomial distribution and ignores randomness); graph B is disorder centered (does not reflect the binomial structure of the experience) and graph 3 is variation centered (shows an adequate balance between randomness and structure). The ideal answer is to identify graphs A and B as made up and C as real.

*Procedures*

Questionnaires were applied to the students by the teacher of each of the groups in the case of Middle and High School. One of the researchers applied the questionnaire to college students.

To classify the answers, the following codification, similar in some aspects to that used by Watson et al. (2003) and based on SOLO hierarchical cycles proposed by Biggs and Collis (1991), was employed: “Realist Appearance” (RA), is used to classify the answers to question 6 if an individual gives a distribution of frequencies, totaled 60 and whose elements fall in an interval of 4 to 16 (90% of confidence), a few college students proposed intervals around the mean and those answers were coded RA (See Trujillo, 2008 for details). Also RA is used to classify an answer of an individual if table one is chosen as answer to question 10. Lastly an answer is classified as RA if I-I-R sequence is chosen in question 11: where “I-I-R” means “Invented, Invented, Realist”. “Without variation” (WV), is used to classify an answer if a uniform distribution is given in question 6 and also if table 2 in question 10 is selected as an answer to that question.

Item 11. Three classes did 50 spins of the above spinner many times and the results for the number of times it landed on the part numbered 2 were recorded. In some cases, the results were just made up without actually doing the experiment. Can you identify what classes made up the results without doing the experiment?




a) Do you think class A's results are made up or really from the experiment? Explain
b) Do you think class A's results are made up or really from the experiment? Explain
c) Do you think class A's results are made up or really from the experiment? Explain

*Figure 3*. Problem 10.

Lastly an answer is classify as WV if R-I-I sequence is followed in question 11: where "R-I-I-" means "Realist, Invented, Invented"

In conclusion, "Extreme values" (EV) is used to classify the answers to question 6 if a distribution whose elements, at least one, is out of the range of 4 and 16, is given. Finally EV describes the answers if table 3 is selected as answer to question 10 and if the sequence I-R-I is followed in question 11. I-R-I means "Invented, Realist, Invented", while NR means "No Response or Inconsistent". Some Middle and High school students gave answers to item 6 totaling a number different to 60; those were coded as Pre-Structural answers (PS).

## Results

The data contained in the questionnaire have been organized to see if it is in agreement with the hypothesis. Simply put: Does statistics variation thinking begin with perception of disorderliness, then structure and finally an integration of both in a notion of statistical variation? If yes, majority of younger students are expected to incline towards choosing "extreme values". On the contrary, it is expected that majority of older students will choose "without variation" or "realistic appearance". The answers to questions 6 and 10 support to a greater extent these expectations.

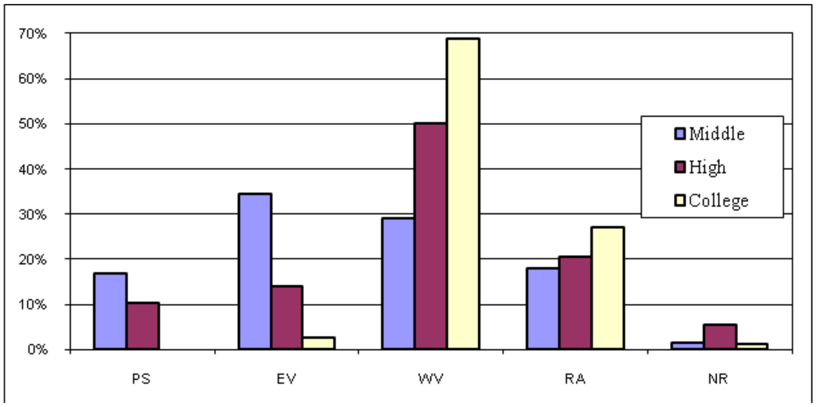



*Figure 4*. Relative frequencies of types of distribution proposed by students in response to question 6.

Notice in Figure 4 that some Middle and High School students gave answers coded ‘pre-structural’ (17 % and 10%, respectively). It could also be observed in graph 1, that the distributions, classified as “extreme values” were proposed with higher frequency by students of the Middle School (34%) than students of high school (14%) and only by a small proportion of College students (3%). On the other hand the distributions classified as ‘without variation’ were proposed with higher frequency as scholar level increases: 29% on High School, 50% on Middle School and 69% College students. The frequencies of distributions classified as “Realistic Appearance” follow an increasing trend: 18% Middle School, 20% High School and 27% College. A remarkable fact is the appearance of intervals as answers between College students (3%), for example, one subject answers:

*$O_1, O_2, O_3, O_4, O_5, O_6$ in such a way that $O_i \in \{0,\ldots,60\}$ y $\sum_{i=1}^{6} Oi = 60$ But I think that $O_i$ = 10 ± 1 or 2. We have a ‘big’ number of repetitions so the trend would begins to arise having in each line a number close to 10, supposing a fair die, it is equally probable any outcome*

It could be observed in Figure 5 that the distribution of the of answers coded as ‘extreme values’ follows a monotonic decreasing behavior (19% Middle School, 7% High School, 4% College). It is important clarify that the answers coded as EV among College students correspond to the choice of every board as a possible outcome. In the case of answers coded as ‘without variation’, lowest and highest scholar levels have similar proportions (47% Middle School and 49% College) while High School students propose answers of this kind in a proportion of 64%. This may be explained by the frequencies of the category ´realistic

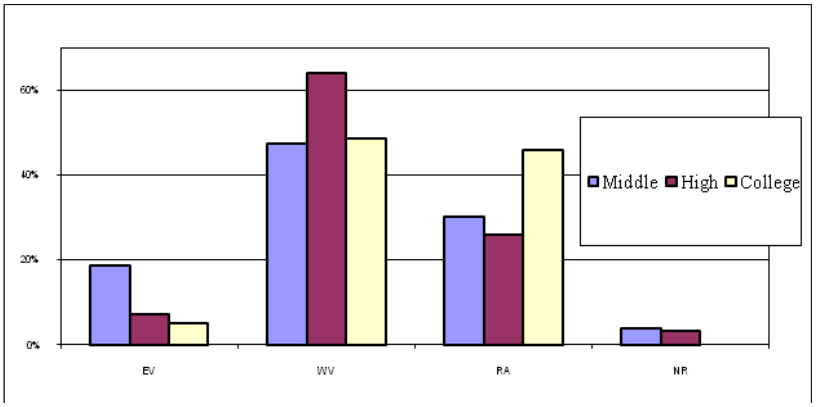



*Figure 5.* Relative frequencies of types of distribution proposed by students in response to question 10.

appearance': 30% Middle school, 26% High School and 46% College. These proportions agree with our hypothesis.

With reference to sub questions of the 11th question, the data were organized as follows: III – mean 'Invented, Invented, Invented' and IIR –'Invented, Invented, Realist', etc.

With reference to the variation evolution model, it is expected that the students of the middle school, accept as realistic the "extreme values". That is, greater frequencies of "*R*" was expected of the students of the middle school, where "*R*" = {IRI, IRR, RRI, RRR}. Also students of high school and college should tend to accept as realist the option "**R" = {IIR, IRR, RIR, RRR}. As can be observed, these expectations were fulfilled to a greater extent.

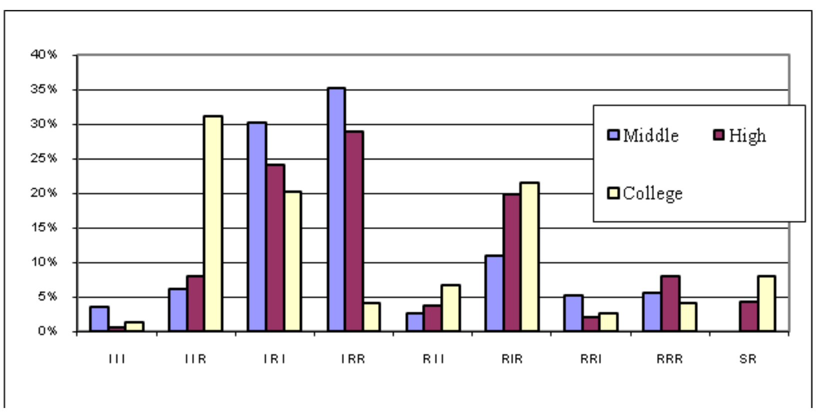



*Figure 6.* Relative frequencies of types of distribution proposed by students in response to question 11.

The options "*R*" and "**R" are focus of attention in the following. The response "*R*", except RRR, was chosen by greater number of students of Middle school than those of High school. On the other hand the response "**R", except IRR, was chosen by greater number of students of High school that those of Middle school. The IRI and IRR have a diminished frequency throughout the grades, which implied greater acceptability of *without variation* in the question 6, as realistic, among the students of Middle school.

The choice of RIR between the two levels is significant; this is possibly due to the fact that greater number of students of Middle school accept as realistic, the distribution with extreme values.

College students respond IIR, the ideal answer, in greater proportion than any other answer or group (31%).

**Discussions and Conclusions**

The results obtained in this work are consistent with the hypothesis that students' answers pass through the stages of: disorder, structure, and variation. Younger students (Middle school ones) tend to notice principally randomness in chance experiences, thus their answers reflect this trend. High school students are strongly anchored on centers may be due to statistics formal instruction, which –in Mexico- has a great emphasis on central measures and neglect variability. College students answers tend to be mostly of the category 'without variation', but ´realistic appearance' answers have an important proportion and some cases are classified as 'relational'

Students are observed to have the tendency of proposing definite numbers and distributions instead of a range of number and a class number of distributions. An important proportion of High school and college students attempt to mix the disorder and structure in question 6; except a few of them, they do this proposing a definite distribution, thereby giving a distribution less probable than the uniform distribution. They were unable to imagine an interval that would form a class of distributions. On items 10 and 11, which ask students identify plausible outcomes, a greater proportion of subjects are able to differentiate possible of reasonable outcomes and discard events with too low probability, than in item 6. This can be due to the fact that problems 10 and 11 do not demand students abandon their idea that prediction consists always in a single number result.

This tendency of thinking in isolated events, instead of a multiple ones could be also attributed to the manner in which courses of probability and statistics are developed. Even though the explanations above revealed the inadequacies in the courses of statistics, it is important to look for cognitive reasons that impede a better understanding of variability.

The three stages earlier discussed: disorder, structure and variation could be characterized through important stages as follow: Recognition of chance, which was observed by Piaget as a very important stage. The second state is the creation of instruments to deal with chance; this mean that all possible results are considered thereby determining the probability of a particular result. The third stage is a combination of disorder and structure through a refined knowledge of probability on: a) a general concepts of events; b) the use of instruments of probability in order.

The magical explanation for a phenomenon is discarded when an event is perceived as a chance. The game of chance is therefore seen as a product of interaction of multiple causes (Borel) or as an interaction of independent processes (Piaget). When a student admits the existence of chance and see a phenomenon as chance, it is natural that such a student will respond that any event can occur, predicting only disorderliness and irregularity. Probability therefore provides a tool that allows one to see structure behind disorderliness. This is witnessed at least in the games of chance. The number of possible results, favorable and its quotient and combinatorial are elements of structure. The students tend to think that the structures determined are not free for all (and indeed they are not) and they serve to make predictions. But the nature of those predictions is far from what the students can believe. The illusion therefore to a process of overcoming of the powers of the disorder, being created that can be adjusted and controlled with the aid of probability.

To be variation minded, one must be able to know the structure (the average, the uniform distribution, etc.) and take into consideration the randomness, dispersion or irregularity of the phenomenon. These two aspects are integrated in proposing an interval of the expected values whose length is determined according to the probability with which the prediction is desired, and that, therefore, it settles down beforehand. To achieve the aforementioned, knowledge of *confidence interval* is useful and desired. Despite college students do know the concept of *confidence interval* just a few of them were able to suitably integrate the dispersion with the structure.

John Tukey (1986) once asked: “What have statisticians been forgetting in principle?” And he answered himself: “That the history of statistics has involved –indeed, very nearly consisted of – successive enforced retreats from certainty. Each step of that retreat has brought further gains…” (p. 588). When a student pass from giving a determined single event to proposing a range of results in prediction problems in a certain way he or she retreats from certainty and started to have statistics thinking.

## Endnotes

[1] This work was sponsored through grant 45063-H by CONACYT (National Council of Science and Technology), México.

[*] The Gumball task asks students to predict the number of red candies in a sample of 10 candies from a population of 50 red, 30 blue, and 20 yellow candies. This task is one of the problems of the 1996 NAEP (National Assessment Educational Project) of USA.

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# EXPERIENCE AS A POWERFUL TOOL FOR MEANINGFUL LEARNING OF PROBABILITY

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Ordinary problems using standard algorithms do not enable students to understand probabilistic situations meaningfully. Like in other branches of mathematics, students learning probability need to be involved in authentic situations that motivate their way of thinking. Littlewood (1953) declared that a good mathematics riddle (or joke in his words) is worth more than a dozen fair exercises. Probability is very much connected to every day life, but the synthesis between determinism and uncertainty makes it difficult to understand. The theoretical models used in explaining probabilistic thinking sometimes contradict intuition, which is based on every day experiences. Our experiences are deterministic, not continuous, and usually not guided (Rokni, 2001). Using guided experiences concerning probabilistic situations which derive from authentic problems may result in meaningful understanding of probabilistic principles.

## The Research Question

What are the sources of mistakes in solving probabilistic problems and how does experience help lead to meaningful understanding of the correct answers.

*Subjects*

16 pre-service junior high school teachers**.**

*Instrument*

A questionnaire with three authentic situation probabilistic problems:

a. What is the probability of finding at least two people whose birthdays are on the same date, among 30 random participants?
b. There are three doors. Behind one door there is a prize. You are asked to guess where the prize is. After you guess, one of the other two doors is opened and you see that the space is empty. Now you are given the opportunity to change your guess to the remaining door. Will this increase your chance of winning? If so, what is the probability?
c. You have 2 discs: one is red on both sides and the other is red on one side and blue on the other. You choose one disc at random, put it on the table and you see red. What is the probability that the other side of this disc is also red?

*Procedure*

Step I

a. 15 of the 16 participants wrote that the probability of finding at least 2 people whose birthday is on the same date is very small and may be 30/365, because there are 365 days in a year. Only one gave the correct answer based on previous learning of such a situation.
b. All the participants wrote that changing the guess does not increase the probability. The only difference is that the probability changes from 1/3 to ½ in both cases.
c. 15 of the 16 participants wrote that the probability that the other side is red is ½, because there are only 2 discs, one with red on both sides. Only one participant (not the same one

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who answered item (a) correctly) gave the correct answer using an intuitive explanation based on 4 sides.

Step II

After sharing the results with the subjects, the researcher gave intuitive-logical explanations using demonstration and modeling for the 3 problems.

Step III

Some of the participants experienced cognitive dissonance after hearing the explanation because it did not fit their intuition and/or past experience.

The researcher then performed 3 experiments, one for each question above. The students participated directly and individually in all the experiments:

a. Collecting birthday data.
b. Simulating the 3 door game.
c. Playing a game with 2 discs.

*Results*

After step 3 (the direct experience), all 16 participants were convinced about the correct answer and changed their way of thinking about uncertainty.

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# JUSTIFICATION AND CONSENSUS: MATHEMATICAL REFERENTS AND PIVOTAL CUES IN DIVERGENT PROBLEM-SOLVING APPROACHES

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*Divergent solution approaches in problem-solving emerge in classrooms that support students' enactments of personal agency and small-group collaborative inquiry. University honors calculus students, engaged in building new mathematical understandings, chose specific mathematical referents to substantiate and justify divergent solution approaches to a task especially created to elicit student-generated, meaningful and essential mathematics for finding the volume of a solid of revolution. During class presentations of solution approaches, pivotal cues indicated a change in, or a support for, specific reasoning. Justification was grounded in personally meaningful mathematical referents and was a means for building mathematical consensus among students.*

## Introduction

Classrooms where student inquiry and question-posing are essential elements in mathematics learning present unique opportunities to evidence how mathematical meanings and consensus about those meanings develop when different solution approaches in problem-solving emerge. In the context of such a *thinking classroom,* we characterize four diverse, student-invented solution approaches for finding the volume of a solid of revolution. Our analyses feature the language and notations students choose in building two of those solution approaches, made evident in public presentations to the class, that support mathematical consensus in a thinking classroom. Our purpose is to understand how students reason and build convincing arguments and provide justifications that help others understand divergent solution approaches.

## Theoretical Perspectives

The mathematical work of students engaged in problem-solving may differ greatly between individuals as well as between groups of students. Indeed, Goffman (1959) recognized that "working consensus established in one interaction setting will be quite different in content from the working consensus established in a different type of setting" (p. 10). In particular, if construction of ideas in transitioning from concrete or empirical meanings toward abstraction (Pijls, Dekker & Van Hout-Wolters, 2007; Rivera, 2007), depends, in part, on explanation and critical examination of one's ideas, then retrospective traces of the "way points" (Dimond & Walter, 2006) in mathematics learning may be distinct not only for individuals, but for different learning groups as well. In other words, one may look back at the developmental work in different groups' solution paths and reasonably expect variance in those solution paths.

Individual and collaborative choices in problem-solving may be reflections of the inherent individual characteristic of personal agency. We view personal agency in terms of

the requirement, responsibility and freedom to choose based on prior experiences and imagination, with concern not only for one's own understandings of mathematics, but with mindful awareness of the impact one's actions and choices may have on others… Because people build understanding from experience, it is essential that they have opportunities to

> make personal choices that will foster learning in particular, perhaps unanticipated, ways as they explore mathematics and develop a sense of self as actor and participant. The exercise of agency is what makes mathematical thinking possible. (Walter & Gerson, 2007, p. 209)

We assert that the exercise of personal agency within a community of mathematics learners is made evident as students act and interact, for example, in pursuit of mathematical certainty during problem solving. Powell (2004) defined agency in mathematical learning as learners' individual initiative to define, redefine, build on or go beyond specificities of mathematical situations on which they have been invited to work. We agree with perspectives of personal agency which hold that in order to have the ability to initiate an act a person must be able to make choices based on perception and available information (Bandura, 1997; Holland, Skinner, Lachicotte Jr, Cain, & Delmouzou, 1998; Skovsmose, 2005a). Furthermore, the exercise of personal agency "is manifest in the direction of attention" (Dewey, 1913/2007, p. 8). Such perspectives of personal agency have implications for learning. Dewey noted that thinking is "something to be tried" and that learning is "an active, personally conducted affair" (1916/1944, p. 335) which occurs when a person's "established powers are redirected through intelligent effort...to arouse the person to clearer recognition of purpose and to a more thoughtful consideration of means of accomplishment" (1913/2007, p. 58-59). Acts of purposeful choice in learning may be seen as fundamental, intellectual *tryings* directed toward building understanding (Brown, 2005; Kohn, 1998; Rogers, 1969; Walter & Gerson, 2007). As such, the choices students make are creative acts that direct or redirect established and emerging powers toward the development of mathematical meaning, thinking and learning.

Skovsmose (2005b) noted that there are ethical demands associated with action and responsibility. Mathematics classroom conditions that constrain intellectual tryings by students constrain the exercise of personal agency as well as the exercise of temporally extended agential authority in groups. Synthesizing personal agency with socially constituted conditions, Bratman (2007) suggested that agency may include individual self-governance, intention, planning, and temporally extended agential authority. In our work, temporally extended agential authority emerges when individuals exercise personal agency over time in collaborative pursuit of collective goals. Our perspective on agency underscores social aspects of agency in learning, beyond individual choice and responsibility, and allows us to recognize the critical roles and effects of personal agency enacted by learners working in groups. Indeed, the exercise of personal agency is the genesis of creative acts that shape, and in turn are influenced within, the milieu of lived experience.

Goffman (1959) described performance as "all the activity of a given participant on a given occasion which serves to influence in any way any of the other participants" (p. 15). We use performance as Goffman did because we are interested in how students' mathematical tryings influence one another and the group effort. We suggest that performance in a learning community may be characterized as "an observable, flexible, synchronous process of reasoning, presenting and organizing one's thoughts…[and] begins with an individual choosing to act, which may influence and include actions of the group" (Walter & Gerson, 2007, p. 206).

Interaction was defined by Goffman (1959) as "the reciprocal influence of individuals upon one another's actions when in one another's immediate physical presence" (p. 15). A decade later, Blumer (1969) viewed interaction as "a flowing process in which each participant is guiding his action in the light of the action of the other suggest[ing] its many potentialities for divergent direction" (p. 110). Divergent direction, from our perspective, encompasses the potential for different solution approaches in mathematical problem solving.

In student-thinking centered, investigatory task-based classrooms, all three grounding attributes of our theoretical perspective (personal agency, performance, and interaction) are elemental in student endeavors. Against such a background, we arrived at our initial research question.

## Research Questions

How do university students collectively develop mathematical methods for solving problems? In particular, within the context of our study, we look at how university students collaboratively develop mathematical methods for finding the volume of a solid of revolution when no prior instruction on solution methods was given. How do these students reason to build convincing arguments and provide justifications that are meaningful for others who are trying to understand divergent solution approaches?

## Methodology

Eighteen second-semester honors calculus students, comprising four groups, worked collaboratively on tasks during two-hour class sessions three times per week. Tasks were designed to foster creativity while eliciting conceptually important mathematics as part of a 3-semester teaching experiment at a large private university in the mountain-west region of the United States. Pedagogical decisions were based on the progress and direction of inquiry by students, rather than by textbook organization. During these particular sessions, students worked on one task, the Gel-Pack Mug Task (Figure 1) without prior instruction on solution approaches. This task was especially created to elicit student-generated, meaningful and essential mathematics for finding the volume of a solid of revolution. Although these second-semester calculus students had prior experience with integration techniques and applications, finding volumes of solids of revolution was a new topic. Our analysis here focuses primarily on students' public performances of their small-group invented solutions for the Gel-Pack Mug Task.

*A cold mug consists of a gel pack sandwiched by two cylinders. For manufacturing reasons, we make the gel pack parabolic in cross-section. We are interested in knowing the volume of the gel. For your information, the height of the gel pack is 12 cm. The gel is filled up to the 11 cm mark. The top of the gel pack needs to be 2 cm wide at the top. The inside radius of the mug is 8 cm.*

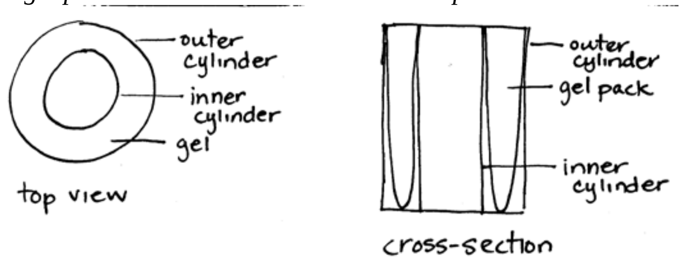


*Figure 1.* Gel-pack mug task.

Each class session was videotaped. Videotape captured and preserved a detailed chronology of interactions, discourse, and mathematical work of groups of 4 to 6 students seated at large hexagonal tables. In addition, whole class discussions and presentations by individual students or small groups of students were also videotaped. Transcripts of videotaped student discourse were created and verified by research team members including graduate and undergraduate research-mentored students. Transcripts are verbatim texts of student utterances. These transcripts were then memoed to record insights and relationships between events and further annotated to reflect gestures, intonations, pauses, and other student activity during student conversations. Interpretations and inferences were substantiated by constant comparison of all data sources, including the videotaped evolution of student written work and notations, homework assignments and completed task write-ups.

In a microlinguistic analysis of student discourse, open and axial coding (Strauss & Corbin, 1998) highlighted differences in students' solution approaches and supported the identification and characterization of the mathematical referents students chose to substantiate and justify their work. Open coding included repeated searches of the video and transcript for student language and notations that indicated student attempts at justification, argumentation, consensus, and proof. Axial coding of students' presentations revealed what we call *pivotal cues* in students' reasoning. Pivotal cues indicated a change in, or a support for, specific reasoning through the use of morphemes such as *because, but, is, like, mean, need, so, then* and *want*. Mathematical referents also surfaced during open and axial coding of students' problem-solving approaches. We defined these emergent mathematical referents as taken-as-shared mathematical objects to which students appealed during consensus building while problem-solving or during public performance presentations of their work. Personal and temporally extended agential authority was evident in the use of pronouns, such as "I" and "we" and language which indicated individual and collective decision-making.

Transcript excerpts presented here are from the three, two-hour class periods that students worked on the Gel-Pack Mug Task. Each excerpt includes, in columns from left to right, video timecodes for ease in referencing particular events and for a sense of temporal breadth (for example D1_1 29:36 would indicate the first day (D1) of student work on the task during the first hour of that class session at twenty-nine minutes thirty seconds into the video), speaker name, transcript, transcript annotations to clarify student discourse and performances, and coding from analysis to support interpretive narrative following each excerpt.

## Data and Analysis

Here, we present brief analyses of two of the four diverse, student-invented solution approaches for the gel-pack task. We begin by noting that all four groups of students came to the conclusion, via different approaches, that the volume of the gel-pack was $264\pi \text{ cm}^3$.

Retrospective traces of each group's work revealed that two groups, students at Table 1 (T1) and students at Table 3 (T3), chose to develop what might be identified as traditional calculus methods to find the desired volume. T1 chose to approach the problem with the shell method and T3 developed the washer method. If students had prior experiences with second-semester calculus content, the graphic presentation of the cross section of the mug may be seen by some as suggestive of these two approaches.

However, students seated at Table 2 (T2) and those at Table 4 (T4) invented distinct, rather unconventional methods for solving the task. Five students at T2, Eric, Justin, Daniel, Jamie, and Julie, developed what they termed the "ratios" method. During small-group problem-solving Justin responded (D1_1 29:36) to Eric by offering a speculative question (Walter, 2004).

| | | | | |
|---|---|---|---|---|
| D1_1 29:36 | Justin | What if you did this? What if you took like kind of saying what you're saying, if you took the ratio of the difference between [pause] of how these two apply to each other. I don't know what I'm saying. And then find the volume of the cylinder and apply the ratio to that. | ["ratio of the difference between"/ratio of parabolic area to circumscribed rectangular area] | [extended agential authority/mindful awareness of Eric's prior suggestion] |
| 29:54 | Eric | I wonder if that would work. | | [personal agency] |

In the ratios method, T2 found that the area of the rectangle circumscribing the parabola was 22 $cm^2$. They set up and evaluated $11\int_{-1}^{1}(1-x^2)\,dx = 14.66\overline{6}\ cm^2$ to determine the area of one of the parabolic faces represented in the cross-section of the gel-pack mug. and then calculated the ratio of the parabolic area to the area of the rectangle (see Figure 1), $\frac{14.66\overline{6}}{22} = .6\overline{6}$. The rectangle, if rotated around the *y*-axis at a distance from the *y*-axis equal to the interior radius of the mug, would form the 11 cm high, double-walled cylinder containing the gel-pack. The double-walled cylinder volume, 1244.07 $cm^3$, was found by determining the volume of the entire cup at a height of 11 cm and then subtracting the volume of the 11 cm high interior cylinder of the cup (D1_1 32:03). During their small group work, and again in public presentation of their solution, T2 stated that the ratio of the area of the parabola to the area of the rectangle would be equal to the ratio of the volume of the gel pack to the volume of the cylinder. Hence, T2 multiplied the area ratio by the volume of the cylinder or "shell containing the gel-pack" to find the volume of the gel-pack (Figure 1).

Meanwhile, students at T4, Danielle, Derrick, Paul and Nels, with remarkable creative insight, invented a method which, unknown to them, demonstrated the Second Theorem of Pappus. The students proceeded, in a compelling manner, to justify to the class their use of specific mathematical referents in their approach to solving the problem. The following is an excerpt from Derrick's portion of the group's performance.

| | | | | |
|---|---|---|---|---|
| D3_1 8:32 | Derrick | As we understand, by finding a volume, like, to find volume it's like length times width times height | [writes V=L x W x H] | [we/pronoun/ extended agential authority] [like/conjunction] [volume of right prism/mathematical referent] |
| 8:45 | | And for us, our length times width, I call it the base of our object, so that was the base of our object, the length times width | [draws a rectangle around L x W and writes "base" below] | [us/our/pronoun/ extended agential authority] [I/personal agency] [it/that/parabolic face of the right prism] |

| | | | |
|---|---|---|---|
| | | | [so/adverb] |
| 8:58 | So all we needed to do was take that, that fourteen point six six six six six seven and multiply it times a height | [writes $V = (14.666667)(H)$ on the board] | [so/adverb] [we/extended agential authority] [needed to do/mathematical necessity] [that/pronoun/ parabolic base area] |
| 9:59 | So we stretched that out | [points to the gel-pack parabolic prism he had drawn–see Figure 2] | [so/adverb] [kinesthetic deformation of gel-pack] |
| 10:01 | That's just like taking the mug and chopping it in half right there and stretching it out | [with a chopping motion draws a vertical line segment across the sketched washer, or top view, cross-section of the gel-pack] | [like/adjective] [process/what we have] |
| 10:06 | And so... so we took this middle radius, the average radius | [draws a circle representing an average circumference on the washer cross-section already drawn on the whiteboard] | [so/conjunction/ process] [we/extended agential authority] |
| 10:14 | ‘cause this is 8 | [draws and labels an inner radius 8] | [‘cause/conjunction] [backing for average radius] |
| 10:16 | and that’s 10 | [draws and labels an outer radius 10] | |
| 10:18 | So we took the radius at 9 | | [so/conjunction/process] |
| 10:21 | And we figured out why we did that. | [sketches an isosceles trapezoid with base lengths 8 and 10 units] | [we/pronoun/ extended agential authority] [nested mathematical referent] |
| 10:22 | Because if you really stretch it out, | [a longitudinal cross section of the gel-pack] | [because/conjunction] [kinesthetic deformation of gel-pack] |
| 10:24 | it looks like this | [points to isosceles trapezoid] | [nested mathematical referent] |
| 10:27 | and there's space unaccounted for | [points to a region outside the left leg of the | [identifies tensions between |

| | | | |
|---|---|---|---|
| | | trapezoid]. | mathematical referents] |
| 10:29 | But if we do it in the middle | [average circumference] [draws the median– a segment connecting the midpoints of the legs of the trapezoid] | [we/pronoun] [process] |
| 10:34 | then it makes up for that extra space | [superimposes on the trapezoid a rectangle with length dimension equivalent to the length of the median and height equivalent to the height of the trapezoid, Fig. 2 | [then/adverb] [resolves tension, reconciles rectangular and trapezoidal referents] |
| | Does that make sense? | | [checks for whole-class consensus] |

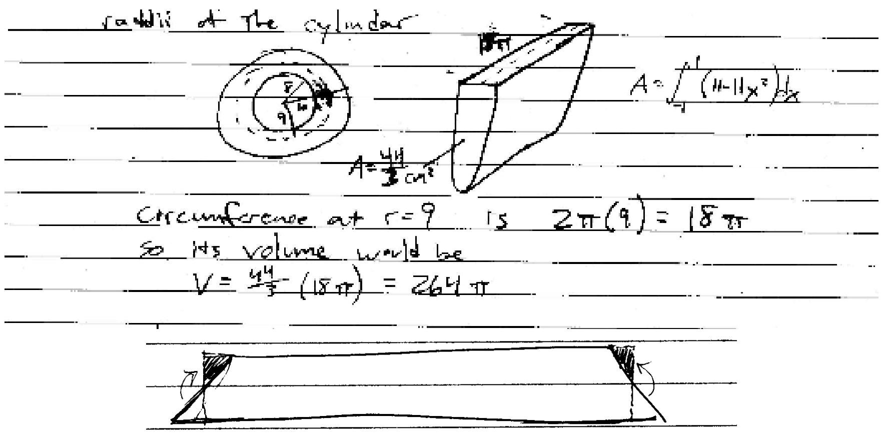


*Figure 2*. T4 mathematical referents and demonstration of second theorem of Pappus.

T4 recognized a parabolic right prism in the structural deformation of the gel-pack. They chose to use the formula for the volume of a prism as a mathematical referent in their invented solution demonstrating the Second Theorem of Pappus. The isosceles trapezoid was a nested mathematical referent to resolve tension between rectangular and parabolic right prisms and to justify the use of the average radius of the gel-pack for the height of the right-prism gel-pack.

## Discussion

These students flexibly used *pivotal cues*, mathematical referents, and collaboratively created notations to meaningfully relate prior experiences with developmental reasoning and mathematical inferences. Pivotal cues note emphases and direction of attention in problem-solving. Selected pivotal cues and purposes as used by these students are presented in Figure 3.

| Adverb | Adjective | Conjunction | Pronoun |
|---|---|---|---|
| So (in the way that follows; in the way that precedes)<br>Like (resembling closely, similar to; in the same way as; as if)<br>Then (immediately or soon afterward; next in order of time or place; as a consequence) | So (true as stated or reported)<br>Like (of the same form; similar, analogous; bearing resemblance; equivalent; possessing the same or almost the same characteristic) | So (in order that; with the result that; on the condition that)<br>Like (as if, in the same way as)<br>Because or 'cause<br>(for the reason that; due to the fact that; since) | I (self as agent)<br>We (nominative plural of I representing a collective viewpoint and temporally extended agential authority)<br>You (second person singular or plural; people being addressed) |

*Figure 3.* Selected pivotal cues in students' presentations of solution approaches.

In these students' work, justification was grounded in personally meaningful mathematical referents and was a means for building mathematical consensus. In subsequent problem solving, each student in the class chose to appropriately and efficiently utilize the Second Theorem of Pappus. We see enacted personal agency and extended agential authority as fundamental for creativity and meaning in students' mathematics. Our analysis expands our awareness of students' use of pivotal cues and mathematical referents in problem-solving. Additionally, we gain fine-grained insight into students' choices for personal instantiations and justifications that ground mathematical inferences. In turn, we begin to recognize more fully how students build convincing justifications for their understandings of mathematical concepts and processes.

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# LEARNING MATHEMATICS FROM CLASSROOM INSTRUCTION USING STANDARDS-BASED AND TRADITIONAL CURRICULA: AN ANALYSIS OF INSTRUCTIONAL TASKS

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*The LieCal Project longitudinally investigates the effects of the Connected Mathematics Program (CMP) and more traditional middle school curricula (non-CMP) on students' learning of algebra. To ascertain the curricular effects, we must attend to aspects of teaching that influence students' learning opportunities. In this paper, we particularly focused on the mathematical tasks to understand the instructional experiences provided when using CMP and Non-CMP curricula. We found that teachers in CMP classrooms implemented significantly more cognitively demanding tasks than teachers in Non-CMP classrooms. Also, teachers are much more likely to encourage multiple strategies in CMP classrooms than in Non-CMP classrooms.*

## Purpose

One of the major goals of educational research, curriculum development, and instructional improvement is to improve students' learning. Advocates of mathematics education reform often attempt to change classroom practice, and hence, students' learning, by means of changes in curricula (NCTM, 1989; Howson, Keitel, & Kilpatrick, 1981; Senk & Thompson, 2003). Historically, curriculum has been used as a means to convey what students should learn (NCTM, 1989) and it has also been used to serve as agents for instructional improvement (Ball & Cohen, 1996). However, curriculum does not always influence classroom instruction (Ball & Cohen, 1996; Fullan & Pomfret, 1977). One of the important factors is how teachers interpret and use the curriculum materials in classroom. The purpose of this study is to examine the kinds of learning provided by classroom instruction using Standards-based and and more traditional middle school curricula mathematics curricula.

## Background and Theoretical Considerations

*Standards-Based Mathematics Curriculum*

In the late 1980s and early 1990s, the National Council of Teachers of Mathematics (NCTM) published its *Standards* documents, which provided recommendations for reforming and improving K-12 school mathematics. In the *Standards* and related documents, the discussions of goals for mathematics education emphasize the importance of thinking, understanding, reasoning, and problem solving, with an emphasis on connections, applications, and communication (e.g., NCTM, 1989, 2000). This view stands in contrast to a more conventional view of the goals for mathematics education, which emphasizes the memorization and recitation of decontextualized facts, rules, and procedures, with the subsequent application of well-rehearsed procedures to solve routine problems.

With extensive support from the National Science Foundation, a number of *Standards*-based school mathematics curricula were developed in the United States and implemented to align with the recommendations in the *Standards* (see Senk and Thompson, 2003 or NRC, 2004 for details). The Connected Mathematics Program (CMP) is one of the *Standards*-based school mathematics curricula developed with the support of the U.S. National Science Foundation. The CMP curriculum is a complete middle-school mathematics program. The intent of CMP is to build students' understanding in the four mathematical strands of number and operation, geometry and measurement, data analysis and probability, and algebra through explorations of real-world situations and problems (Lappan et al., 2002). Because NSF-funded curricula like CMP claim to have different learning goals and also look very different from commercially developed mathematics curricula, a natural question is: What learning opportunities will a Standards-based curriculum like CMP provide that are different from the learning opportunities provided by more traditional middle school curricula?

*LieCal Project*

The study reported in this paper was conducted as part of a large project titled the Longitudinal Investigation of the Effect of Curriculum on Algebra Learning (LieCal Project). The LieCal Project is designed to longitudinally compare the effects of the Connected Mathematics Program (CMP) to the effects of more traditional middle school curricula (hereafter called Non-CMP curricula) on students' learning of algebra. The LieCal Project is being conducted in 16 middle schools and 10 high schools of an urban school district serving a diverse student population. At the start of the project, 27 of the 51 middle schools in the school district had adopted the CMP curriculum while the remaining 24 middle schools were using other curricula. Eight CMP schools were randomly selected from the 27 schools that had adopted the CMP curriculum. After the eight CMP schools were selected, eight Non-CMP schools were chosen based on comparable ethnicity, family incomes, accessibility of resources, and state and district test results. A total of 725 CMP students from 26 classes and a total of 698 Non-CMP students from 24 classes participated in the study, and these 1,423 students were followed for three years from grades 6 to 8 and into grade 9.

The goal of teaching is to help students learn. To understand the impact of *Standards*-based curricula, then, we must attend to aspects of teaching that appear to have potential to influence students' learning opportunities. Specifically, to help us understand the differences between the instructional experiences provided when teachers use CMP and Non-CMP curricula, in this paper we focused particularly on the instructional tasks posed and implemented by the teachers.

*Instructional Tasks*

Researchers have developed different paradigms and methods that can be used to identify important features of classroom instruction (e.g., see Koehler & Grouws, 1992; Porter & Brophy, 1988; Shulman, 1986). Instructional tasks have been identified as an important construct to study classroom instruction (Doyle, 1983; Stein et al., 1996). The term "instructional tasks" has been referred to by other researchers as "academic tasks," or as "mathematical tasks" (e.g., Cai & Lester, 2005; Doyle, 1983; Hiebert & Wearne, 1993; Stein et al., 1996). Mathematical tasks can be defined broadly as projects, questions, problems, constructions, applications, or exercises in which students engage. Mathematical tasks provide intellectual environments within which students can learn and develop mathematical thinking. Tasks help regulate not only students' attention to particular aspects of content, but also their ways of processing information. However, only "worthwhile problems" give students the chance to solidify and extend what they know and to stimulate mathematics learning (NCTM, 1991). In the classroom, students' actual opportunities to learn depend on the type of mathematical tasks presented and implemented. Regardless of the context, for a task to be worthwhile, it should be intriguing and it should provide a level of challenge that invites speculation and hard work. Most importantly, worthwhile mathematical tasks should direct students toward explicit learning goals by encouraging them to investigate important mathematical ideas and ways of thinking. The NCTM *Standards* (1991, 2000) recommend that students should be exposed to truly problematic tasks in classrooms so that they can practice mathematical sense making. Doyle (1988) argues that tasks with different cognitive demands are likely to induce different kinds of learning. Mathematical tasks that are truly problematic have the potential to provide the intellectual contexts for students' rich mathematical development. Such tasks can promote students' conceptual understanding, foster their ability to reason and communicate mathematically, and capture students' interests and curiosity (NCTM, 1991).

Worthwhile mathematical tasks alone do not guarantee students' learning. They are important, but not sufficient, for effective mathematics instruction because teachers may not implement worthwhile tasks as they were intended. Stein et al. (1996) found that only about 50% of the tasks that were set up to require students to apply procedures with meaningful connections were actually implemented that way. In our LieCal project, we analyzed three distinct categories of mathematical problems: those that appeared in the CMP and Non-CMP textbooks, those that were posed and implemented during classroom instruction, and those that were assigned as homework. In this paper, we report only the results from our analysis of the instructional tasks implemented in the classroom.

## Methodological Considerations

### *CMP and Non-CMP Curricula*

We have conducted detailed analyses of the CMP and Non-CMP curricula, with a focus on the algebra strand. Our preliminary analysis showed remarkable differences between the CMP and Non-CMP curricula. CMP can be characterized as a problem-based curriculum. Take the introduction to equation solving as an example. In one of the Non-CMP curricula, equation solving is introduced symbolically using the additive property (add or subtract the same quantity on both side of the equation, the equality holds) and the multiplicative property (multiple or divide a non-zero quantity on both sides of an equation, the equality holds). On the other hand, in the CMP curriculum, the introduction to equation solving is situated within real-life contexts that are used to help students understand the meaning of each step of the equation solving process (Nie, Cai, & Moyer, 2009).

The extent of the differences is also illustrated in Figure 1 below. Using a scheme developed by Stein et al. (1996), we classified the mathematical tasks in the CMP curriculum and one of the Non-CMP curricula into four increasingly demanding categories of cognition: memorization, procedures without connections, procedures with connections, and doing mathematics. As Figure 1 shows, significantly more tasks in the CMP curriculum than in the Non-CMP curriculum are higher-level tasks (procedures with connections and doing mathematics) ($\chi^2(3, \underline{N} = 3311) = 759.52$, $\underline{p} < .0001$).

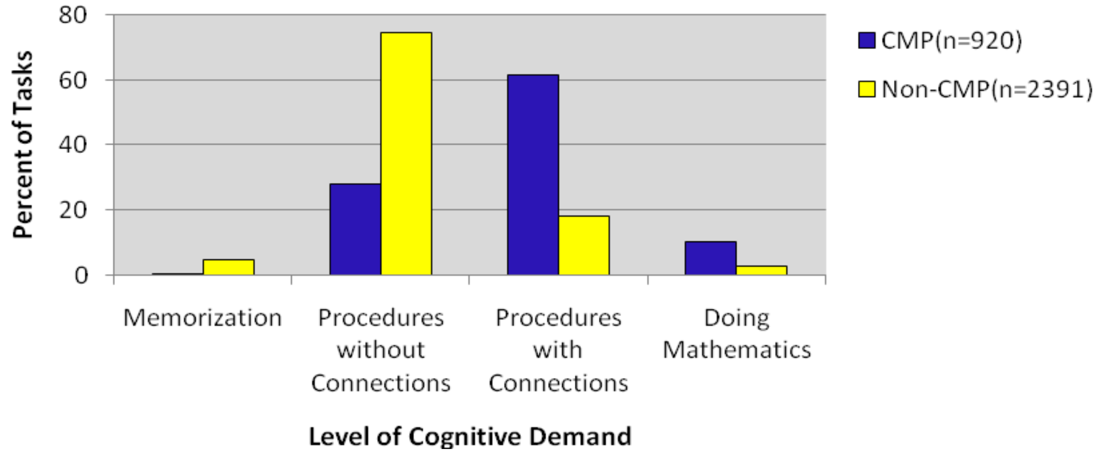


*Figure 1.* Percentages of various tasks in CMP and Non-CMP curricula.

*Classroom Observations*

As we indicated above, the research reported in this paper is part of a longitudinal study of the effect of curriculum on the algebraic thinking of approximately 1400 middle school students from 16 schools in a single urban district as they progressed from grades 6-8. The data was collected over a three-year period during 620 classroom observations. Approximately half of the observations were of teachers using the CMP curriculum. The other half were observations of teachers using Non-CMP curricula. Two retired mathematics teachers conducted and coded all the observations. The coders received extensive training that included frequent checks for reliability and validity throughout the three years. Over the course of the $6^{th}$-grade year, for example, we checked the reliability of the observers' coding three times. These three sessions revealed that the reliability of the coding done by the two specialists was quite high. The reliability achieved during the three sessions averaged 79% perfect agreement using the criterion that the observers' coded responses were considered equivalent only if they were identical (i.e., perfect match). The reliability averaged 95% using the following criteria: (a) If an item or sub-item was "scored" using an ordinal scale, then the specialists' coded responses were considered equivalent if they differed by at most one unit; (b) If an item or sub-item (e.g. representation) was "scored" by choosing from a list of alternatives all the words/phrases that characterize it, then the specialists' coded responses were considered equivalent if they had at least one choice in common (e.g. symbolic and pictorial vs. pictorial).

Each coder observed and coded about 100 algebra-related lessons each year: half in CMP classes and half in Non-CMP classes. Each class was observed four times, during two consecutive lessons in the fall and two in the spring. The coders recorded extensive information about each lesson in a 28-page project-developed observation instrument.

During each observation, the observer made a minute-by-minute record of the lessons on lined sheets. This record was used later to code the lesson. One section of the observation instrument is devoted to the analysis and coding of the mathematical tasks in the lessons. Instructional tasks were analyzed from three perspectives: (1) as intended by the author, (2) as set up by the teacher, and (3) as actually implemented by the teacher with students. The observers in the project coded each of the instructional tasks along four dimensions within each of the three perspectives: (1) Setting; (2) Solution Strategies; (3) Representations; and (4) Cognitive Demand. These dimensions are described in the results section.

## Results

In this paper, we only report the results from the analysis of the tasks actually implemented by the teacher with students. In addition, the difference patterns between the tasks from CMP and Non-CMP classrooms are similar across the three middle school grade levels ($6^{th}$- to $8^{th}$ grades). Therefore, we aggregated the tasks data from all three grades. A total of 646 instructional tasks from about 300 CMP lessons and 744 tasks from about 300 Non-CMP lessons were identified.

*Settings*

We classified the classroom settings in which teachers implemented instructional tasks into three types: whole classroom, small group work, or individual work. The same tasks in a lesson could be implemented in different settings. Nearly 80% of the tasks in CMP lessons and 80% of the tasks in Non-CMP lessons were implemented in a whole classroom setting. About 20% of the tasks in the CMP lessons and 8% of the tasks in the Non-CMP lessons were implemented in small group settings. These two percentages are significantly different ($z = 7.03$, $p < .001$). On the other hand, a significantly larger percentage of the tasks in Non-CMP lessons (57%) than in CMP lessons (41%) were implemented in an individual work setting ($z = 5.89$, $p < .001$).

*Solution Strategies*

We examined whether the instructional tasks implemented in the classrooms were solved using multiple approaches or a single approach. Figure 2 below shows the percentage of tasks that were solved using multiple solution strategies and the percentage of tasks that were solved using a single solution strategy in both CMP and Non-CMP classrooms. A Chi-square test shows that while a larger percentage of the tasks implemented in CMP classroom were solved using multiple solution strategies, a larger percentage of the tasks implemented in Non-CMP classrooms were solved using a single solution strategy ($\chi^2(1, N = 1390) = 122.49$, $p < .0001$).

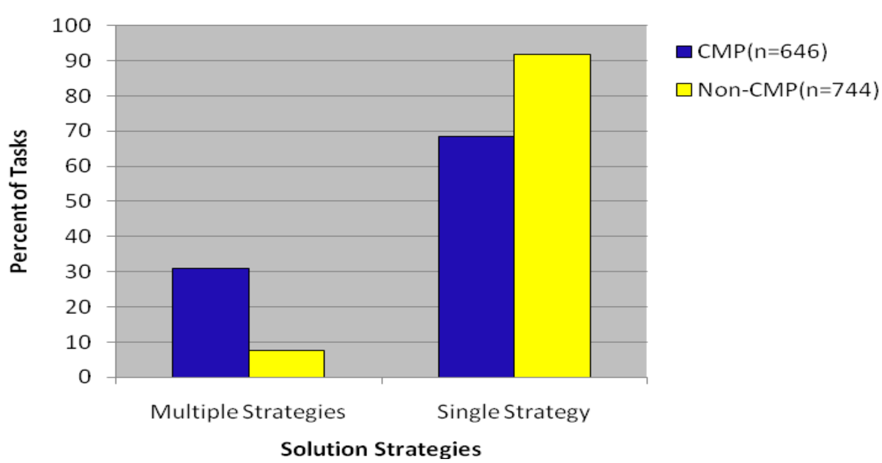


*Figure 2*. The distribution of solution strategies in CMP and Non-CMP classrooms.

*Representations*

The representations used to solve each problem were classified into 7 categories: (1) symbolic, (2) written words, (3) pictorial, (4) tabular, (5) graphical, (6) verbal, and (7) physical manipulatives. Table 1 below shows the percentage of tasks implemented in CMP and Non-CMP lessons using each of the representations. The solution to an implemented task can involve multiple representations. Only a small proportion of the tasks implemented in CMP and Non-CMP lessons were represented with physical manipulatives. The most frequently used representations of implemented tasks in both the CMP and Non-CMP lessons were symbolic, and the proportion of the tasks that were represented using symbolic representations in Non-CMP lessons was greater than that in CMP lessons ($z = 6.16$, $p < .001$). Compared to the use of symbolic representations, the proportion of using other representations is much smaller in both CMP and Non-CMP lessons. We compared the frequencies with which written words, pictorial, tabular, graphical, and verbal were used to represent implemented tasks in both CMP and Non-CMP lessons. We found that the proportion of the instructional tasks that were represented using each of these representations (written words, pictorial, tabular, graphical, or verbal) in CMP lessons was greater than that in Non-CMP lessons ($z = 3.80 – 8.78$, $p < .001$).

Table 1. *Percentages of Tasks With Each of the Representations*

| | Symbolic | Written Words | Pictorial | Tabular | Graph | Verbal | Physical Manipulatives |
|---|---|---|---|---|---|---|---|
| CMP (n=646) | 78 | 20 | 30 | 26 | 23 | 20 | 4 |
| Non-CMP (n=744) | 90 | 12 | 15 | 8 | 8 | 6 | 3 |

*Cognitive Demand*

Using a scheme developed by Stein et al. (1996), we also classified the instructional tasks from CMP and Non-CMP classrooms into four increasingly demanding categories of cognition: memorization, procedures without connections, procedures with connections, and doing mathematics.

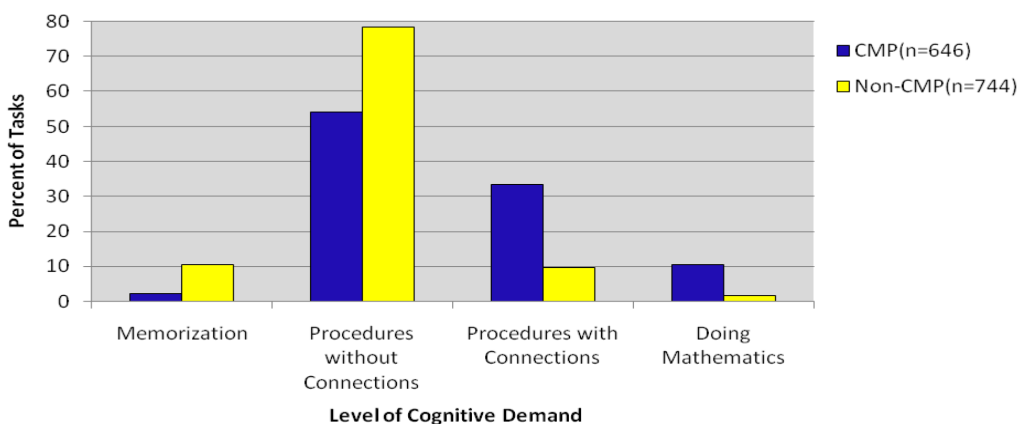


*Figure 3.* Instructional tasks implemented in CMP and Non-CMP classrooms.

Figure 3 illustrates the percentage distributions of the cognitive demand of the instructional tasks implemented in CMP and Non-CMP classrooms. A chi-square test shows that the CMP and

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

Non-CMP percentage distributions are significantly different ($\chi^2(3, \underline{N} = 1390) = 209.42$, $\underline{p} < .0001$). The difference is due to the fact that there was a larger percentage of high cognitive demand tasks (procedures with connection or doing mathematics) implemented in CMP classrooms than in Non-CMP classrooms ($\underline{z} = 13.79$, $\underline{p} < .001$). On the other hand, there was a larger percentage of low cognitive demand tasks (procedures without connection or memorization) implemented in Non-CMP classrooms than in CMP classrooms.

## Discussion

The research reported in this paper is part of a larger longitudinal study conducted in the LieCal Project. The LieCal project was designed to provide (1) A profile of the intended treatment of algebra in the CMP curriculum with a contrasting profile of the intended treatment of algebra in the Non-CMP curricula; (2) a profile of classroom experiences that CMP students and teachers undergo, with a contrasting profile of experiences in Non-CMP classrooms; and (3) a profile of student performance resulting from the use of the CMP curriculum, with a contrasting profile of student performance resulting from the use of Non-CMP curricula. In this paper, we analyzed a single aspect of the classroom experiences that CMP and Non-CMP students and teachers underwent, namely the instructional tasks implemented in the two types of classrooms. The initial analysis of the implemented instructional tasks clearly showed remarkable differences between CMP and Non-CMP classroom instruction. Instructional tasks are more likely to be implemented in small group settings in CMP classrooms than in Non-CMP classrooms, and vice versa for the tasks implemented in individual settings. The instructional tasks implemented in CMP classrooms were more than three times likely to be solved using multiple solution strategies than they were in Non-CMP classrooms. While solutions of instructional tasks in Non-CMP classrooms were more likely to be represented using symbols, solutions of instructional tasks in CMP classrooms were more likely to be represented using written words, pictorial representations, graphs, tables, or verbal representations. In addition, CMP teachers were more than three times as likely to implement high-level tasks during classroom instruction than Non-CMP teachers.

The findings of this study not only show the importance of examining the instructional experiences of students using CMP and Non-CMP curricula, but it also shows the power of focusing on instructional tasks to reveal the instructional differences. Recall that our analysis of the mathematical problems in the CMP and Non-CMP curricula showed that significantly more tasks in the CMP curriculum than in the Non-CMP curriculum are high level tasks (procedures with connections and doing mathematics). Thus it is reasonable to infer that the differences in setting, strategy, representation, and cognitive level of the tasks implemented in CMP and Non-CMP classrooms reflect the differences between the mathematical problems in the CMP and Non-CMP curricula.

The striking and clear differences between CMP and Non-CMP classrooms are of great interest and importance in our longitudinal investigation of the impact of curriculum on students' learning. As part of the parent study, we also collected large-scale, longitudinal student achievement data. In our presentation, we will identify and present important linkages between students' classroom experiences and their learning outcomes.

**Authors' Note**: The research reported in this paper is part of a large project, Longitudinal Investigation of the Effect of Curriculum on Algebra Learning (LieCal Project). LieCal Project is supported by a grant from the National Science Foundation (ESI-0454739). Any opinions expressed herein are those of the authors and do not necessarily represent the views of the National Science Foundation.

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# ASSESSING PROBLEM-SOLVING DISPOSITIONS: LIKELIHOOD-TO-ACT SURVEY


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*This paper reports an ongoing study that is aimed at developing an instrument for measuring two particular problem-solving dispositions: (a) impulsive disposition refers to students' proclivity to spontaneously proceed with an action that comes to mind, and (b) analytic disposition refers to the tendency to analyze the problem situation. The instrument is under development and consists of likelihood-to-act items in which participants indicate on a scale of 1 to 5 how likely they are to take a particular action in a given situation. The instrument was administered to 318 college students, mainly pre-service teachers. Statistical analysis indicates that likelihood-to-act items are reliable and that the current version of the instrument has room for further improvement.*

## Motivation for the Study

For many mathematics students, "doing mathematics means following rules laid down by the teacher, knowing mathematics means remembering and applying the correct rule when the teacher asks a question, and mathematical truth is determined when the answer is ratified by the teacher" (Lampert, 1990, p. 31). Students with such beliefs tend to exhibit dispositions such as "waiting to be told what to do," "doing whatever first comes to mind," and "diving into the first approach that comes to mind" (Watson & Mason, 2007, p. 207). In this paper, we use the term *impulsive disposition* to mean the tendency to "spontaneously proceed with an action that comes to mind without analyzing the problem situation and without considering the relevance of the anticipated action to the problem situation" (Lim, 2008a, p. 49).

Some problem-solving episodes found in mathematics education literature can be interpreted as instantiations of impulsive dispositions. For example, consider the following missing-value problem that was posed by Cramer, Post and Currier (1993) to pre-service teachers: *Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?* Thirty-two out of 33 pre-service teachers solved this problem by setting up a proportion such as $9/3 = x/15$. These pre-service teachers are considered *impulsive* if they had applied the proportion algorithm without analyzing the problem situation. In fact, Lim (2008b) found that after a course on rational numbers and algebraic reasoning pre-service teachers, on average, performed better on all four direct-proportional problems but worse on all three non-direct-proportional problems.

As mathematics educators, we are interested in helping students advance from impulsive disposition to *analytic disposition*, in which a student "attempts to understand the problem statement, studies the constraints, identifies a goal, imagines what-if scenarios, and/or considers alternatives" (Lim, 2008a, p. 45). To track this advancement, we need to identify where a student stands in terms of his or her disposition. In other words, we need an efficient and reliable instrument that can "measure" students' impulsive disposition and analytic disposition. In the field of mathematics education, it appears that no such instrument has been developed. In this paper we present a few theoretical constructs related to impulsive disposition, overview the literature associated with assessing cognitive constructs through the use of survey, report our research process, and discuss the results that we have obtained.

## Theoretical Constructs Related to Impulsive Disposition

*Psychological perspective*. In terms of cognitive tempo or response style, a person may be classified as either *impulsive* or *reflective*. Kagan, Rosman, Day, Albert, and Phillips (1964) constructed the Matching Familiar Figures Test to measure children's cognitive tempo. An impulsive is one whose response time is faster than the median and whose accuracy rate is below the median, whereas a reflective is one whose response time is slower than the median and whose accuracy rate is above the median. Nietfeld and Bosma (2003) describe impulsives as "individuals who act without much forethought, are spontaneous, and take more risks in everyday activities" (p. 119) whereas reflectives are "more cautious, intent upon correctness or accuracy, and take more time to ponder situations" (p. 119). In their study on consistency in cognitive responses among adults across academic tasks, Nietfeld and Bosma found moderate positive correlations for response styles among the three types of tasks they investigated: verbal, mathematical, and spatial. The mathematical tasks used in their study were two-digit addition or subtraction problems arranged in a traditional vertical format. Although such tasks are appropriate for measuring cognitive tempo along a speed-accuracy continuum, they are not appropriate for measuring disposition along an impulsive-analytic continuum. Whereas an impulsive tempo is characterized by a fast but inaccurate response, an impulsive disposition is characterized by "diving into the first approach that comes to mind" and not necessarily by how fast an approach comes to mind.

*Problem-solving perspective*. Schoenfeld (1985) has identified four categories of cognition that provide a framework for analyzing problem-solving behaviors: (a) mathematical knowledge base, (b) use of heuristics, (c) monitoring and control, and (d) beliefs about mathematics and doing mathematics. Impulsive disposition can be regarded as an externalization of certain beliefs such as "there is only one correct way to solve any mathematics problem—usually the rule the teacher has most recently demonstrated to the class" (Schoenfeld, 1992, p.359). Impulsive disposition can also be considered as a lack of metacognition—a term introduced by Flavell (1976) as "the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete goal or objective" (p. 232).

*Teaching-learning perspective*. According to Harel (2008), mathematics consists of two complementary sets: (a) *ways of understanding* refer to the products of mental acts while doing mathematics; they include definitions, theorems, proofs, problems, and solutions, and (b) *ways of thinking* refer to the characteristics of the mental acts while doing mathematics. Harel (2007) stipulates that "students develop ways of thinking only through the construction of ways of understanding, and the ways of understanding they produce are determined by the ways of thinking they possess" (p. 272). According to this principle, it is counter-productive to help students develop ways of understanding without helping them develop ways of thinking, and vice versa. Hence, students should be provided opportunities to engage in mental acts (e.g., generalizing, justifying, problem-solving, symbolizing, computing, generalizing, predicting, etc.) that can advance both their ways of understanding and ways of thinking. Lim (2008a) identifies *impulsive anticipation* and *analytic anticipation* as two ways of thinking in the context of problem solving. An important goal of mathematics education is to help students advance from undesirable ways of thinking (e.g., impulsive disposition, authoritative proof scheme) to desirable ways of thinking (e.g., analytic disposition, deductive proof scheme).

### Means for Assessing Impulsive Disposition and Analytic Disposition

One useful way to measure cognitive and psychological constructs is through the use of survey development. The use of surveys can be very informative, as they allow for the quantification of the constructs under study. With such quantification, we can investigate group differences on those constructs and assess how those constructs associate with other behavioral measures. Examples of the use of such measures in the psychological literature are vast, ranging from the measurement of social problem solving (D'Zurilla, Nezu, & Maydeu-Olivares, 2002) to the measurement of decision making styles (Nygren, 2000).

Nygren (2000) constructed the Decision Making Styles Inventory, which measures the degree to which a person makes everyday decisions using an analytical approach, an intuitive approach and an approach which minimizes regret. Analytical decision making involves considering every aspect of the problem before making a decision whereas intuitive decision making involves a reliance on one's gut feeling. These two constructs are analogous to analytic disposition and impulsive disposition.

The goal of this project was to develop a measure of mathematical disposition. We wanted to demonstrate the internal consistency reliability of the survey items. In addition, we wanted to see how well the items in each subscale are inter-correlated, and how well the two items in each impulsive-analytic pair are correlated. Finally, we wanted to assess how scores on such a measure are related to self-reported academic performance in mathematical classes and the participant's teacher training program.

### Research Process

Instrument design, testing, and refining are an elaborate process involving multiple cycles.

*Survey Development*

The initial instrument designed to assess impulsive disposition was a multiple-choice test on ratios and proportions. Students have a tendency to overuse proportional strategies for solving missing-value problems (Van Dooren, De Bock, Hessels, Janssens, & Vershaffel, 2005). The items were designed to determine whether students inappropriately use a proportion to solve missing-value problems that do not involve a direct-proportional situation (e.g., an additive situation, an inverse-proportional situation) or inappropriately use a ratio to compare "non-rate" quantities (e.g., the size of a person's palm, the magnitude of a project in terms of worker-hours).

In subsequent versions multiple-choice items were used for students to choose the action that they would most likely perform in a given scenario. The format was eventually changed from a multiple-choice test to a likelihood-to-act survey which takes less time for students to complete.

The likelihood-to-act survey developed for this ongoing study has undergone two revisions. The first two exploratory versions were administered to about 70 pre-service middle-school teachers and 14 graduate students (mainly in-service teachers) respectively in courses taught by the first author. The version reported in this paper consists of nine pairs of *likelihood-to-act* items, sequenced from A to R. Four pairs (A-J, N-E, O-F, and I-R) involve equation solving; two pairs (B-K and L-C) involve word problems; two pairs (D-M and Q-H) involve fraction division and fraction addition respectively; and one pair (G-P) involves geometry. Figure 1 shows 3 pair of such items.

Two versions of the likelihood-to-act survey were used in this study. All the items in Version

1 are considered “specific” items in that a specific scenario is provided. The nine impulsive-disposition items are based on specific mathematical rules, formulas, or procedures that are supposedly familiar to students.

Please indicate, as honestly as you can, how likely you are going to act in the manner specified in the statement using the following scale:

1. Extremely Unlikely 2. Unlikely 3. Not Sure 4. Likely 5. Extremely Likely

B. When asked to find the cost of 18 cans of specialty soda given that 6 cans of specialty soda cost \$4.10, how likely are you going to begin by setting up a proportion?

K. When asked to find the cost of 20 bottles of mineral water given that 4 bottles cost \$2.10, how likely are you going to study the values of the quantities and predict the answer?

D. When you are asked to find the answer for $\frac{55}{95} \div \frac{11}{95}$ without using a calculator, how likely are you going to use the invert-and-multiply rule?

M. When you are asked to find the answer for $\frac{44}{82} \div \frac{11}{82}$ without using a calculator, how likely are you going to study the two fractions and predict the answer?

I. When asked to simplify an equation (e.g., $10^{3x} \cdot 10^{2y} = 1000 \cdot 10^{3x}$), how likely are you going to begin by applying a formula (e.g., $B^m \cdot B^n = B^{m+n}$) or by following a procedure (e.g., taking $\log_{10}$ on both sides)?

R. When asked to solve $2^{5a} \cdot 2^{10b} = 8 \cdot 2^{5a}$ for $b$, how likely are you going to begin by inspecting the terms in the equation?

*Figure 1*. Three pairs of “specific” likelihood-to-act problems in Version 1.

Version 2 differs from Version 1 in that it contains 2 pairs of “general” items (see Figure 2); the remaining 7 pairs are identical to those in Version 1. “General” items were found to be less reliable in the pilot testing of an earlier version based on a small sample of 14 students. Version 2 was developed to verify this finding. Note that Pair I’-R’ is analogous to Pair I-R, but Pair G’-P’ is substantially different from Pair G-P.

G’. When asked to solve a word problem, how likely are you going to begin by applying a formula or using a procedure that comes to mind?

P’. When asked to solve a word problem, how likely are you going to begin by identifying the quantities in the problem and thinking about how the quantities are related?

I’. When asked to solve an equation, how likely are you going to begin by using a procedure (e.g. combining like terms) or searching for a formula (e.g. $B^m \cdot B^n = B^{m+n}$)?

R’. When asked to solve an equation, how likely are you going to inspect the terms in the equation before applying a standard procedure for solving the equation?

*Figure 2*. Two pairs of general likelihood-to-act problems in Version 2.

*Data Collection and Analysis*

A survey was administered in 13 mathematics classes in the final week of classes of the Fall 2008 semester. To encourage participation, a participant in each class was randomly selected to win a $10 gift voucher. The survey is comprised of two parts: (a) 18 likelihood-to-act items, and (b) either 18 *need-for-cognition* items or 18 *belief-attitude-confidence-in-algebra* items (these items are not discussed in this paper). 318 students were administered the survey, with 257 participants from 10 classes taking Version 1 and 61 participants from 3 classes taking Version 2 of the likelihood-to-act part.

Inter-item correlations were computed using Pearson correlations for the nine impulsive-disposition items and the nine analytic-disposition items. Items that are not significantly correlated with other items in the same category were analyzed to see if they could be improved or should be excluded from the next version of the instrument. Because the items were paired, the correlation between the impulsive-disposition item and analytic-disposition item in each pair was also determined. The reliability for each sub-scale of seven common items was determined using Cronbach's Alpha coefficient based on all 318 individuals. To assess the validity of each subscale, we also performed a 4×3 analysis of variance—four programs (Early Childhood to Grade 4 Generalist program, Grades 4-8 Generalist program, Grades 4-8 Math Specialist program, and B.S. Math program) by three self-reported grade-point-averages for mathematics courses (A, B, and C or below).

## Results and Discussion

*Reliability of the likelihood to act measure*. To compute the reliability of the likelihood-to-act subscales, we used the seven pairs of items common to both versions to create a larger sample—318 individuals. Missing data on any of these items for these 318 students were imputed. At most, one item had seven missing item responses. The Cronbach's Alpha estimate of internal consistency reliability for the seven impulsive-disposition items was 0.64, (95% Confidence Interval: 0.58, 0.70). The reliability of the seven analytical items was 0.63 (95% Confidence Interval: 0.56, 0.69). While these reliability estimates are not very high, it should be noted that each subscale has only seven items. It is well known from classical test theory that the addition of items tends to increase test score reliability. For the next phase, we will focus on developing and testing additional items, s well as improving existing items.

*Impulsive-disposition items*. The correlations among the nine general impulsive-disposition items in Version 1 are presented in Table 1 (ignore the two specific items, G' and I', for the time being). By excluding items L, N, and I, all the correlations among the six remaining items (A, B, D, O, Q, and G) are significant with $p < 0.01$. Items L, N, and I will be replaced in the next version of the instrument. The strong correlations among items A, B, D, O, Q, and G suggest that impulsive disposition is a trait that cuts across the four domains: equation-solving (A and O), word problem (D and Q), fractions (B), and geometry (G).

*Table 1*
Correlations among the Nine Impulsive-Disposition Items

| Item | A | B | L | D | N | O | Q | G | I | G’ | I’ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A | 1 | | | | | | | | | | |
| B | 0.32** | 1 | | | | | | | | | |
| L | 0.15** | 0.15** | 1 | | | | | | | | |
| D | 0.37** | 0.26** | 0.18** | 1 | | | | | | | |
| N | 0.13* | 0.17** | 0.21** | 0.10 | 1 | | | | | | |
| O | 0.30** | 0.30** | 0.14* | 0.29** | 0.22** | 1 | | | | | |
| Q | 0.25** | 0.17** | 0.09 | 0.15** | 0.12* | 0.21** | 1 | | | | |
| G | 0.24** | 0.31** | 0.13* | 0.19** | 0.17** | 0.28** | 0.23** | 1 | | | |
| I | 0.02 | 0.15* | 0.09 | 0.14* | 0.26** | 0.22** | -0.04 | 0.18** | 1 | | |
| G’ | 0.34** | 0.18 | 0.42** | 0.07 | 0.36** | 0.02 | 0.25 | - | - | 1 | |
| I’ | 0.47** | 0.42** | 0.28* | 0.29* | 0.09 | 0.49** | 0.33** | - | - | 0.19 | 1 |

*Note.* The *p*-values for the correlation of 0.29 for I’ and D is lower than that for the correlation of 0.29 for O and D because the sample size was 61 for I’ and D (Version 2) and 315 for O and D (Versions 1 and 2).
$^{*}p < .05$. $^{**}p < .01$.

As for Version 2 (G’ and I’ instead of G and I), we should retain items A, B, D, O, Q, and I’ and replace items L, N, and G’. Interestingly, the general item I’ appears to be better correlations than the specific item I. A probable explanation is that the participants might have difficulty interpreting Item I because of the equation $10^{3x}\cdot 10^{2y} = 1000\cdot 10^{3x}$ and the meaning of $\log_{10}$.

*Analytic-disposition items*. The correlations among the 11 analytic-disposition items shown in Table 2 are generally less significant when compared to those in Table 1. This finding suggests that the analytic-disposition items are not as effective as the impulsive-disposition items. Items C, E, P, R, P’ and R’ have to be excluded in order for the remaining correlations to be significant. However, items J, K, M, F and H can remain intact for the next version. Hence, the individual items in these five pairs, A-J, B-K, D-M, O-F, and Q-H, seem to be reliable.

*Table 2*
Correlations among the 9 Analytic-Disposition Items

| Item | J | K | C | M | E | F | H | P | R | P’ | R’ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| J | 1 | | | | | | | | | | |
| K | 0.28** | 1 | | | | | | | | | |
| C | 0.05 | 0.14* | 1 | | | | | | | | |
| M | 0.31** | 0.41** | 0.07 | 1 | | | | | | | |
| E | 0.17** | 0.05 | 0.08 | 0.08 | 1 | | | | | | |
| F | 0.22** | 0.16** | 0.21** | 0.17** | 0.19** | 1 | | | | | |
| H | 0.26** | 0.39** | 0.00 | 0.43** | 0.07 | 0.23** | 1 | | | | |
| P | 0.02 | 0.09 | 0.14* | 0.19** | 0.12* | 0.10 | 0.06 | 1 | | | |
| R | 0.21** | 0.18** | 0.10 | 0.14* | 0.23** | 0.08 | 0.08 | 0.34** | 1 | | |
| P’ | -0.01 | -0.14 | 0.21 | -0.05 | 0.35** | 0.06 | -0.33** | - | - | 1 | |
| R’ | 0.20 | 0.07 | 0.44** | 0.09 | 0.15 | 0.31* | -0.01 | - | - | 0.35** | 1 |

*Note*. *$p < .05$, **$p < .01$.

*Correlation between the two items in each pair*. The last column in Table 3 shows the correlation between the impulsive-disposition item and the analytic-disposition item in each pair. A significant negative correlation in Pair Q-H indicates that this pair of items differentiates impulsive disposition from analytic disposition. The significant positive correlations in Pair L-C and in Pair I-R mean that these two pairs of items should not be used in the next version. Note that the five good pairs (A-J, B-K, D-M, O-F, and Q-H) have either negative correlations or very small positive correlations. The lack of significant negative correlations suggests the possibility that impulsive disposition and analytic disposition are not necessarily mutually exclusive. In other words, a person may have two competing dispositions for the same problem situation.

*Table 3*
Comparing the Two Items in Each Pair

| | Mean for the impulsive item | Mean for the analytic item | Difference betw. the two means | Correlation betw. the two items |
|---|---|---|---|---|
| Pair A-J | 3.62 | 3.53 | 0.10 | 0.02 |
| Pair B-K | 3.66 | 3.53 | 0.12 | -0.10 |
| Pair L-C | 3.34 | 3.55 | -0.21 | 0.19* |
| Pair D-M | 4.19 | 3.21 | 0.99 | -0.00 |
| Pair N-E | 3.14 | 3.55 | -0.40 | 0.10 |
| Pair O-F | 3.77 | 3.53 | 0.24 | 0.04 |
| Pair Q-H | 4.00 | 3.33 | 0.67 | -0.16** |
| Pair G-P | 3.40 | 3.84 | -0.44 | 0.12 |
| Pair I-R | 3.77 | 4.09 | -0.32 | 0.14* |
| Pair G’-P’ | 3.86 | 4.07 | -0.21 | 0.24 |
| Pair I’-R’ | 3.37 | 4.28 | -0.91 | 0.16 |

*Note*. *$p < .05$, **$p < .01$.

*Association between training program, self-reported mathematics grade, and likelihood-to-act scores.* To assess the validity of the likelihood-to-act measure, we also performed a 4 (programs) by 3 (self-reported numerical grade) analysis of variance on the analytical and impulsive composite scores. For the analytical subscale, there was a main effect for program ($F(3,270) = 8.233$, $p < 0.001$, $\eta2 = 0.08$). Interestingly, elementary-school generalists (EC-4 program) had higher analytical scores than middle-school math specialists (4-8 Math program) and mathematics majors (B.S. Math program). Middle-school generalists (4-8 Generalists program) also had higher analytical scores than mathematics major. An interaction between self-reported grade and training program emerged only for the 4-8 Math program. Surprisingly, individuals in the 4-8 Math training program who reported a letter grade of B had higher analytical scores than those who self-reported a letter grade of A in their math coursework.

A similar analysis was performed for scores on the impulsive-disposition measure. There was a main effect for training program ($F(3,270) = 4.872$, $p = 0.003$, $\eta2 = 0.05$). Math majors had higher impulsive-disposition scores than did all other groups. There were no differences among the other groups. There was no main effect for self-reported grade and no interaction between self-reported grade and training program. In summary, these results seem to suggest that students with increased exposure to traditional math coursework are less analytical and more impulsive.

## Conclusion

The findings obtained in this study confirm the viability of using likelihood-to-act items to measure impulsive disposition and analytic disposition. Five out of eleven pairs of items have high inter-correlations and will be retained in the next version. The weaker items will be refined for the next version of the instrument. The reliability for the two subscales are 0.64 and 0.63. Our goal is to continue improving the instrument until a reliability of at least 0.75 is obtained.

Analytic-disposition items were found to be slightly less reliable than impulsive-disposition items due in large part to the former being typically less clear than the latter. The specific procedure or rule to which students are drawn can be stated explicitly in an impulsive-disposition item, but not in an analytic-disposition item. There is insufficient evidence in this study to support the claim that general items are not as reliable as specific item. Results from the 4-by-3 analysis of variance reveal an unexpected phenomenon. Students in more advanced but traditional math programs (i.e., B.S. program) were found to have lower analytic-disposition scores than those in the less advanced but reform-oriented mathematics programs (i.e., EC-4 program and 4-8 Generalists program). Future research is needed to account for this phenomenon.

One of the advantages of the likelihood-to-act survey is that it takes less time for students to complete than a mathematics test. Whereas students need to solve a problem in order to arrive at an answer choice in a test item, students only need to understand the problem statement and the action for consideration in a likelihood-to-act item to choose from a scale of 1 to 5 the likelihood level. Another advantage is that participants are less likely to feel threatened because the instrument, as a survey, is not perceived as an assessment of their mathematical knowledge.

However, like any survey, what participants say they will do may differ from what they actually do in a mathematics assessment or in a problem-solving situation. This raises the issue of the validity of the likelihood-to-act survey. Another limitation of the instrument is that an impulsive item is effective only if the students are familiar with the particular rule, formula or procedure that is mentioned in the item. For example, Item B will not be valid if it is administered to an elementary student who has not learned how to set up a proportion. Hence, the validity of a likelihood-to-act survey is limited to the group of students for which it is designed.

The likelihood-to-act items developed in this study were aimed at measuring impulsive disposition and analytic disposition. The idea of asking participants to indicate their likelihood to act may be extended to measure other dispositions such as waiting to be told what to do, relying on the teacher, consulting with peers, and so forth.

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# STUDENTS' MENTAL MODELS FOR COMPARISON WORD PROBLEMS

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*Comparison word problems have proven to be difficult for students to solve. Previous studies investigated students' solution strategies and described the mental models they used while working on these problems, although subsequent studies questioned the accuracy of some of these models. This study investigates the solution strategies that 210 undergraduate students used as they solved comparison problems. It also uses interviews with 27 students to describe the mental models that tend to lead to correct solutions and explores the role that equality plays in these models.*

## Introduction

Thirty years ago, Kaput and Clement (1979) noted that undergraduate students had difficulty solving the following problem:

> Write an equation using the variables *S* and *P* to represent the following statement: 'There are six times as many students as professors at this university. Use *S* for the number of students and *P* for the number of professors" (p. 288).

Frequently, college students wrote $6S = P$ as the solution instead of $S = 6P$, an example of a *reversal error*. Students had even more difficulty solving problems in which both coefficients in the algebraic expression were not 1. Furthermore, the reversal error was prevalent not only when students were translating words into an equation, but also when they constructed an equation based on a diagram or table of values.

We will refer to problems in which students are asked to translate these types of relational statements as *SP-type* problems. These can be seen as an extension of a class of word problems called *compare problems*, which have been classified (e.g. Riley, Greeno & Heller, 1983) and studied from a cognitive perspective; while *SP-type* problems involve writing an equation, compare problems involve computing a value.

In a pilot study, Weinberg (2007) found that students who were successful at solving SP-type problems seemed to use several strategies that had not been previously reported. The pilot data suggested that students' strategies depended on the specific task they were asked to complete and were associated with their conceptions of equality. The goal of the study reported here is to elaborate on the results from the pilot study and to describe students' problem-solving strategies and mental models as they solve SP-type problems.

## Theoretical Perspectives

Several researchers have attempted to understand why compare problems are so difficult for students. Lewis and Mayer (1987) describe a "consistency hypothesis," which posits that students' approach to solving problems is affected by the word order of the problem. Several researchers (e.g. Verschaffel, 1994) have verified that students have more difficulty and are more prone to make reversal errors when the problem is posed in a way similar to the language used in

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most SP-type problems.

Clement (1982) interviewed students as they solved SP-type problems and described three strategies/mental models. Some students appeared to syntactically translate the words into symbols, matching symbols to the order of the words. Another group of students appeared to think of the letter as a label (e.g. *S* stands for "students") and the coefficient as an adjective describing the number of objects. The third group of students viewed the multiplication as a "hypothetical operation," operating on the two variable quantities to make them equal to each other. Other researchers (e.g. Hegarty, Mayer & Monk, 1995) have described a *direct translation* approach and a *problem model* approach, with the former corresponding to Clement's syntactic model and the latter describing strategies in which the students formed a meaningful mental model. MacGregor and Stacey (1993) cast doubt on Clement's descriptions, concluding that using a letter as a label was a "post hoc explanation of an equation arising from a model formed without conscious intervention" (p. 230). In addition, they provided data that appeared to refute the hypothesis that students frequently used direct translation, proposing that all students were constructing some kind of mental model. They echoed Rosnick and Clement's (1980) suggestion that students' conceptions of equality may also play a role in the way they solve SP-type problems, although no subsequent research has explicitly addressed this perspective.

Several researchers have described the problem-solving process as involving multiple steps. For example, English and Halford (1995) describe three steps: constructing a problem-text model, a problem-situation model, and a mathematical model. In order to describe mental models and ideas of equality, this study focuses on students' problem-situation models (a student's mental representation of the situation) and the interaction with their corresponding mathematical models.

Previous research has suggested that a problem's word order affects students' responses. Similarly, MacGregor and Stacey (1993) noted that students construct mental models when words are ordered in a variety of ways in the problem. Although the present study does not focus on the problem-text model, it uses word problems in which the word order both matches and fails to match the order of the symbols in the corresponding algebraic representation.

Word problems are situated in a particular task, and different activities involving the same problem situation may cause students to use different strategies and form different mental models. While students are prone to making reversal errors when writing equations, it is possible that their underlying mental models may still allow them to complete other mathematical tasks. Consequently, this study presents word problem situations using three common activities: writing an equation, writing a function, and computing a value.

In order to investigate students' conceptions of equality, the study targets three misuses of the equals sign: the idea that equality is directional (e.g. not recognizing that factoring is equivalent to the distributive property), "run-on" equality (i.e. using the equals sign to connect a string of computations), and the equals sign as a directive to perform an operation (rather than indicating an equivalence).

## Research Questions

1. Are students more successful and less prone to making reversal errors on SP-type word problems when the specific task varies?
2. When solving SP-type word problems, do students' strategies vary depending on the specific task and the number of coefficients? Do students use these strategies consistently

across multiple problems or formats?
3. What strategies do students use *successfully* on each kind of task and problem, and what are the underlying mental models upon which these strategies are based?
4. Are students' conceptions of equality and the equals sign related to either their success on comparison problems or the strategies they use to solve them?

## Methodology

Students in nine sections of first- and second-semester Calculus classes ($n$ = 210) at a northeastern comprehensive college completed a written assessment with four word problems and ten equality problems.

Two of the word problems involved situations that could be described by an algebraic expression with one coefficient that isn't 1 (e.g. $6p = s$) while the others required two coefficients (e.g. $3p = 4c$); one of the latter situations was presented in a diagram rather than words. In one of the *1-coefficient* problems, the sentence described the coefficient before it described the two variable quantities, while in the other the coefficient was described between the quantities; this was done in order to increase the consistency between the word ordering and algebraic representation. Students were randomly assigned one of three formats for each problem: writing an equation to represent the statement (*R*-format), writing an equation that allows them to predict the value of one quantity if they knew the value of the other (*F*-format), or computing the value of one quantity if they knew the value of the other (*V*-format).

The ten equality problems presented students with equations and asked them to decide whether they were correct. Four of the equations were "run-on" expressions (e.g. 2+3 = 5+2 = 7), with two incorporating derivatives and two using arithmetic. Two of the equations expressed relationships that could be seen as "backwards" (e.g., $ac+ab = a(b+c)$), one equated an expression with an integer multiple of itself (i.e., $2x+12 = x+6$), and the remaining equations served as "filler" problems.

All students were invited to participate in an open-ended interview. Fifty-seven students volunteered, and 27 students were selected in random order; the interviews were videotaped and transcribed. In the interview, each student was asked to explain their reasoning on each of the problems. If students had originally seen an *R*- or *F*-format problem, they were asked to compute a value; students who had originally seen a *V*-format problem were asked to create an equation. If a student found the correct equation quickly, they were told that other students had arrived at a different answer and were presented with an equation that incorporated a reversal error.

## Results

Students' performance by format is shown in

Table ***2***. Students were more likely to find a correct value than write a correct equation or a function ($p = 8.58 \times 10^{-42}$) and were more likely to write a correct function than a correct equation ($p$ = .0076). Similarly, students were less likely to make a reversal error when finding a value than when writing an equation or a function ($p = 6.11 \times 10^{-22}$), although were not more likely to make this error on an *R*-form than an *F*-form ($p$ = .099). Students who had both a *V*-form and *R*- or *F*-form on their collection of problems (168 students) were more successful with the *V*-form problems: 69 of these students correctly computed a value but were unable to write an equation or find a function.

*Table 2*
**Performance by Form**

| | *R*-format | *F*-format | *V*-format | Overall |
|---|---|---|---|---|
| Percent of responses that were correct | 36% | 47% | 90% | 58% |
| Percent of responses with a reversal error | 37% | 31% | <1% | 23% |

Students' performance by problem is shown in
Table ***3***. As in previous studies, students were more likely to produce a correct answer on the 1-*coefficient* problems than the 2-*coefficient* problems ($p = 2.17 \times 10^{-10}$). In contrast to previous studies, students were *not* more likely to make reversal errors on 2-*coefficient* problems ($p = .08$). However, 8% of students' responses on the 2-*coefficient* problems were in an additive form with the coefficients paired with the incorrect variables (e.g. $4p+3c$), which may account for some of this difference.

*Table 3*
**Performance by Problem-Type**

| | 1-*coefficient* | 2-*coefficient* | Overall |
|---|---|---|---|
| Percent of responses that were correct | 69% | 48% | 58% |
| Percent of responses with a reversal error | 26% | 21% | 23% |

The strategies students used varied widely between the four problems and the three forms. While a complete discussion of the ways students used each strategy is beyond the scope of this paper, four strategies stood out for being associated with correct answers or suggesting previously undocumented mental models: *ratio, proportion, stepwise,* and *functional*.

*Ratio*

An answer was coded as a *ratio* if it only consisted of a formal ratio (e.g., $3c$:$4p$) or a fraction; typically if the ratio symbol were to be replaced with an equals sign, the student would have made a reversal error. This strategy was used in 24 responses, 21 of which were from an *R*-form, and 23 of which were 2-*coefficient* problems. In several interviews, students who had made a reversal error when writing an equation acknowledged that they were using the equals sign to represent a ratio, suggesting they were using the equals sign as indicating a comparison:

Interviewer: Okay. So then this [points to $3t = 5s$]—was this an equals sign or a ratio sign?

Student: This should be a ratio [writes a colon over the equals sign]—three trucks for every five sedans, 15 trucks for every 25 sedans.

*Proportion*

An answer was coded as a *proportion* if the student set up an explicit proportion (e.g. $c/3 = p/4$). Students used this strategy to solve 82 out of the 840 word problems. Most of these responses occurred when students were computing values (64 problems) and on the 2-*coefficient* problems (67 problems). This was a highly effective strategy, leading to correct answers on 95% of the problems with which it was used.

In interviews, proportional strategies suggested a mental model of *systematic comparison*. That is, the student used the colon (or equals sign) to show that two groups of objects were connected, and that changing one of the groups resulted in a corresponding change in the other group. The student from the previous example above explained this reasoning:

Student: I set *t* equal to trucks then *s* equals to sedans, then it says 3 times, or there are three trucks, 3*t*, for every five sedans, 5*s*. [points to 3*t* = 5*s*].

Interviewer: Okay, so what if there were, let's say, 15 trucks. How many sedans should there be?

Student: Um... that's three times five, five times five [writes x5 below the 3*t* and the 5*s*] so that would be 25? [writes 15t = 25s]

Interviewer: Okay, so 25 sedans? I see... and so—you plugged the 15 in here, and did three times 15? [points to 3*t* = 5*s*]

Student: Three times five. You said there were 15 trucks?

Interviewer: Yeah

Student: So you multiply that times five [points to 3*t*] and that times five [points to 5*s*] because you're doing the same... it's by the same amount you're multiplying this by.

*Stepwise*

An answer was coded as *stepwise* if a student performed two sequential arithmetic operations to produce an answer. This typically involved partitioning a group of objects into smaller equal-sized sub-groups. For example, here is one student's explanation for this strategy:

> What I wanted to do first is find the number of groups of five sedans, and divide that, or get that out of the total. So I divided [165] by five and got 33 groups of five in 165 cars. So if there's 33 groups of five sedans, and then three trucks for every five sedans, I got three times the 33 groups of five to get 99 trucks.

Out of the 840 word problems students solved, 56 were solved using this strategy. Most of these responses occurred when students were computing values (53 problems), and all were used on the 2-*coefficient* problems. As with the *proportional* strategy, the *stepwise* strategy also led to correct answers on 95% of the problems on which it was used.

In interviews, the stepwise strategy suggested a mental model of *partitioning and substituting*, which involves three mental steps. Students described partitioning a group of objects into smaller, equal-sized groups by one ratio term, reconceptualizing the group as a scaling constant, and then reversing the partitioning by multiplying the scaling constant by the other ratio term.

*Functional*

While the *stepwise* strategy involved transforming one quantity into another via two steps, an answer was coded as *functional* if the student transformed one variable quantity into another in a single step:

Student: Well... or would it be... three-fourths? Three-fourths the cows as there are pigs, so... yeah…

Interviewer: So you're saying this [4/3 $p = c$] might be three-fourths instead of four over three?

Student: Yeah, I think three-fourths, just cause there's four pigs, so you'd be four times three-fourths equals the three cows, so it would be... [writes ¾ $p = c$].

Although students used this strategy on all three forms, they used it most frequently when computing a value (75 out of 180 *functional* responses); they used it slightly more frequently on the 1-*coefficient* problems than the 2-*coefficient* problems (104 times vs. 76 times). In interviews, the *functional* strategy suggested a mental model of *transforming*. In contrast to manipulating two quantities so that they are equal in number (a strategy that Clement (1982) refers to as *operational*), the student typically described changing one quantity into another, sometimes describing a transformation of the actual objects (such as cows or pigs) themselves.

It is possible that students who used a single-step transformation in their written work were first writing an equation (such as $3p = 4c$) and then simply "pushing symbols" (yielding $c = ¾ p$; such written responses were coded as *functional-translation*). However, students were significantly more likely to make a reversal error on a *functional-translation* response than a *functional* response ($p = 1.2 \times 10^{-4}$), which suggests a different underlying mental model.

The *proportion*, *stepwise*, and *functional* strategies were the only three strategies that students tended to use repeatedly on written problems (primarily on the 2-*coefficient* problems). This suggests that these strategies may be based on a relatively well-defined mental model that these students viewed as useful. Other students used a wide variety of strategies, frequently using different strategies on problems that were presented in the same format or same number of coefficients. For example, only 57 out of 210 students used the same strategy on the two 1-*coefficient* problems. Out of the 210 students, 68 saw these problems in the same format, and only 30 used the same strategy on both.

*Equations*

Students routinely misidentified two types of expressions as correct: those involving several computations connected by equals signs ("run-on" expressions) and an equation in which the quantity on one side was a multiple of the quantity on the other side (see Table ***4***).

***Table 4***
**Student Responses to Equations**

| Expression | Percent identifying it as correct |
|---|---|
| $f(x) = 2x^3 = f'(x) = 6x^2$ | 88% |
| $g(x) = x^2 = g'(x) = 2x$ | 83% |
| $2 + 3 = 5 + 2 = 7$ | 66% |
| $5 \times 4 = 20 + 3 = 23$ | 36% |
| $2x + 12 = 6 + x$ | 37% |

In the interviews, several students reversed their decision, identifying some of the run-on expressions as incorrect. However, most of these students did not identify *all* of the expressions as incorrect; frequently students would identify all but the first derivative expression as incorrect. These students typically identified the two derivative expressions as "the power rule" and read the arithmetic expressions from left to right, simply asserting that they were correct. Several students expressed the idea that the run-on equations could be both true and untrue:

Student: If you just take it five doesn't equal seven and seven, but if you want to say two plus three equals five, plus two equals seven, take that separately, that's true, but I was just looking at it as five doesn't equal seven...

Interviewer: So in some readings it might be true, and in others...

Student: Yeah.

Students gave multiple reasons for interpreting the fifth expression as correct. Some students interpreted it as a problem to be solved instead of a statement indicating that the two sides were equal. For example, one student noted: "That's just a problem you would solve. It's not really right or wrong, it depends what *x* is.” Other students suggested that you could operate on one side to obtain the other, although you might need to take the multiplication into account. For example, one student asserted that the expression was correct because, “If I take the 2 out, then *x* plus 6 equals 6 plus *x*.”

Surprisingly, students who said that one of the run-on expressions was correct were not more likely to say that any of the other four were correct, suggesting that students may view the equals sign as serving different roles in each of these four expressions. Similarly, there was no association between students’ responses on the fifth expression and their answers on the four run-on expressions.

Students’ responses on the equality problems were compared with their performance on the word problems. There was a significant relationship between getting the run-on arithmetic questions correct and getting two of the word problems correct. However, there were no other significant associations, and there were no associations between performance on the equality questions and the word problems.

Taken together, these results suggest that students don’t have a single conception of equality or the equals sign. Rather, they view the equals sign as representing a variety of relationships and operations. Because their conception of equality is flexible and varies depending on the situation, there is no association between a specific conception (such as a connector for related mathematical expressions) and their response on another problem.

## Discussion

The results of this study provide additional support for the solution strategies previously suggested by Weinberg (2007): *proportional*, *stepwise*, and *functional*. In addition to providing quantitative evidence for the prevalence of these strategies, the study describes new mental models that appear to underlie the strategies. The study also explores the potential connection between problem solving and specific conceptions of equality as well as describing the ways students use the equals sign as part of their mental models.

The *proportional* and *stepwise* strategies led to successful solutions and prevented reversal errors. Both of these strategies involve forms of proportional reasoning. This suggests that it may be beneficial to further investigate students’ conceptions of proportions at the undergraduate level and to attend to the continued development of this type of reasoning.

In addition to describing these strategies, the results suggest several mental models that students may use when working on comparison word problems: *systematic comparison*, *partitioning and substituting*, and *transforming*. These add to Clement’s (1982) description of *operative* reasoning, in which students manipulate both sides of the equals sign to create a true equality. The data suggest that these models may be robust for some students in that they (successfully) used them on multiple problems.

There appeared to be no connection between problem solving strategies and the three conceptions of equality used in the equations on the written assessment. However, equality and the equals sign play a different role in each of these models. In the *transforming* model, it represents an action (much like the role of the equals button on a calculator). In the *partitioning* model, it doesn’t play a significant role, as it is not usually written as part of the solution. In the *operative* model, the equals sign represents a true equality. In contrast, the equals sign represents a comparison in the *proportional model*. However, this comparison is systematic and supports correct computations.

In addition, these models support Clement's (1982) conjecture that students were using literal symbols (i.e. letter representations of variables) as labels instead of true quantities; thus "6*s*" may represent "six students" instead of "six times the quantity *s*". Clement hypothesized that students were using the equals sign as an equivalence relationship. However, when we observe students working within their self-constructed representational systems, we see that the label conception of literal symbols can support productive solution strategies by using alternate interpretations of the equals sign.

While these conceptions of equality or literal symbols may not match the standard way mathematicians use them, difficulties with *SP-type* problems may originate when students attempt to use properties of their (non-standard) representational systems simultaneously with properties of our standard system. For example, they may use the literal symbol as a label and be able to compute a correct answer, but they arrive at erroneous conclusions if they attempt to substitute a value for the symbol or interpret the two sides of the equals sign as being equivalent.

In addition to helping students develop an understanding of proportional reasoning, this study suggests that we should help students understand the standard meaning of algebraic notation and reconcile their mental models and use of symbols with this notation. While there have been numerous studies of students' conceptions of these ideas at the K-12 level, it is important to further develop our understanding of undergraduate students' conceptions so that we can help them build robust conceptions of algebra and become successful problem solvers.

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# HOW DO CHANGES HAPPEN? TRANSITION FROM INTUITIVE TO ADVANCED STRATEGIES IN MULTIPLICATIVE REASONING FOR STUDENTS WITH MATH DISABILITIES

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*This study investigated how students with mathematics learning disabilities (MD) or at-risk for MD developed their multiplicative reasoning skills from intuitive strategies to advanced strategies through a teaching experiment. The participants consisted of two fifth graders with MD and one at-risk. A micro-genetic approach with a single subject design was used. Investigators coded and analyzed five strategies children used. Results showed that the participants had fewer strategies than normal-achieving students, but they improved their performance throughout the teaching experiment. The participants increased their use of double counting and direct retrieval, and decreased their use of unitary counting during the intervention.*

Approximately 5% to 8% of school aged children have math disabilities (MD), as defined by poor performance in class and poor standardized test scores (Geary, 1990). Individual variability is one of the most striking features of children's reasoning (Siegler, 2007). Low achieving students usually have fewer strategies than high achieving students, and use less advanced strategies more frequently than high achieving peers on a variety of reasoning tasks (Siegler, 2007). The purpose of this paper is to explore how students with MD develop their strategies of multiplicative reasoning through a teaching experiment.

## Framework

Studies on children's strategic development help people understand how students with MD gradually fall behind their peers by comparing the strategic development of children with and without MD (Geary, 1990). While developing additive reasoning, for example, children normally progress from "count all" (i.e., they simply count the two addends from 1) to "counting on" (i.e., they start counting from the first addend) and "count large" (i.e., they count from the larger addend), and eventually to verbal retrieval (Siegler & Shrager, 1984); however, although both students with and without MD equivalently develop various counting strategies at first and second grades, only students with MD continue to have difficulty in shifting to retrieving correct answers after third grade (Geary, 1990; Geary & Brown, 1991).

However, children's strategic development for multiplication is still much less understood than for addition (Lemaire & Sielger, 1985; Kouba, 1989), especially for students with MD. In multiplicative reasoning, one composite unit is distributed across the other, and children need to be able to coordinate the two quantities (Steffe, 1994). Although normal achieving children in kindergarten can solve some multiplicative problems by directly modeling the problem context and counting all the items one by one (Downton, 2008; Kouba, 1989; Mulligan & Mitchelmore, 1997), it takes time for them to establish a real conceptual understanding of multiplicative reasoning, which is an invariant relationship between two quantities (Piaget, 1965; Vergnaud, 1983).

Early studies (Anghileri, 1989; Brown, 1992; Kouba, 1989; Mulligan & Mitchelmore, 1997; Steffe, 1988) have identified a variety of strategies normal-achieving students use for multiplicative reasoning. These studies have also provided evidence that children's solution strategies begin generally with direct modeling and unitary counting; progress to skip counting, double counting, repeated addition or subtraction; and then, to the use of known multiplication or division facts (Downton, 2008). The general strategic developmental pattern is demonstrated in Figure 1 according to Kouba's data.

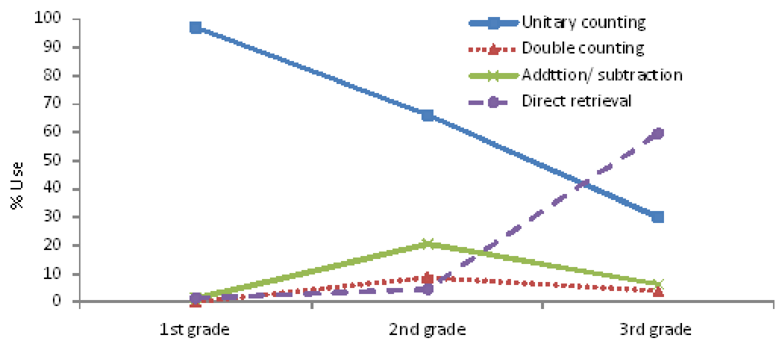


*Figure 1*. Normal achieving students' multiplicative strategic development (Kouba, 1989).

Direct representation and skip counting are two strategies of unitary counting, as they address only one number/counting sequence. Direct representation is at the most basic level (Anghileri, 1989). Kouba (1989) described direct representation as an activity where "children used physical materials to model the problem and some form of one-by-one counting in calculating the answer" (p.152, Kouba). Skip counting is a strategy in which children count by multiples, such as counting "five, ten, fifteen, twenty, twenty five, thirty" for solving "six groups of five" (Kouba). However, skip counting does not suggest a child is able to coordinates the two quantities by tracking *two counting sequences*. An indicator that students are not coordinating the two quantities is that children often do not know where to stop counting (Kouba).

Children's shifting to double counting is a milestone of their development of multiplicative reasoning. Double counting indicates the transition from a unitary counting stage to a binary counting stage (Vergnaud, 1983), where children explicitly keep track of two quantities while counting two number sequences. For example, double counting occurs when a child count "1, 2, 3, 4, 5" with one hand, then counts "1" with the second hand; and then the child continues count "6, 7, 8, 9, 10" with the first hand, then counts "2" with the second hand. Double counting is "an advance over the more basic direct representation because it requires more abstract processing and involves integrating two counting sequences" (p.152, Kouba, 1989). However, no study has investigated whether or not children with MD have double counting strategies.

Afterwards, additive or subtractive strategy occurs when the child exhibits use of repeated addition or subtraction to solve a problem (Kouba, 1989); for example, a child clearly states:

"three plus three plus three plus three" to solve "three times four." And eventually, normal achieving students shift to recalled number facts, which is the highest strategy people use. Kouba explained that this strategy is being used when "the child obtained the answer by remembering the appropriate multiplication or division combination" (p.153, Kouba). Unfortunately, little research has explored if children with MD have the same problems in direct retrieval in multiplication as in addition.

Little is known about how students with MD make their transition from less advanced strategies to advanced strategies in multiplicative reasoning. As such, the purpose of this paper was to explore how a teaching experiment effected on the multiplicative reasoning strategic development for students with MD. Specifically, (1) what strategies students with MD or at-risk used in pretests; (2) did the teaching experiment improve students' performance in solving multiplicative problems; (3) how did strategic development occur during the teaching experiment?

## Method

*Design*

Micro-genetic studies often employ single subject designs (Siegler, 2006). An adapted multiple probe design (Horner & Baer, 1978) across participants was employed in this study to establish a functional relationship between the teaching experiment and students' performance and strategic changes. Specifically, when a stable baseline was observed for one student, treatment was introduced. When improvement for Child A was observed, Child B was introduced to treatment. And when improvement for Child B was observed, Child C was introduced to treatment. In this design, replication of treatment effects is demonstrated if changes in performance occur only when treatment is introduced.

The independent variable was the sessions (the pretests and the teaching experiment). The primary dependent variable was students' strategy use across the sessions. Students' performance in solving multiplicative reasoning problems was also assessed before and during the last session of the teaching experiment.

*Procedure*

This study was conducted within the larger context of the NSF-funded, *Nurturing Multiplicative Reasoning in Students with Learning Disabilities* project (Xin, Si, & Tzur, 2008). Two fifth grade students with MD (Chad and Tina) and a student at-risk for MD (Megan) from an urban elementary school participated in this study. The pretest sessions involved five to six multiplicative problems such as "A platoon must have exactly 7 spaceships. The player received 21 spaceships to begin the first game. How many full platoons can be made?"

The third author, a professor in math education conducted the teaching experiment to the children. The activity was "Towers of Cubes." The goal was for students to figure out the relationships between three quantities: the number of towers, the number of cubes in each tower, and the total number of cubes. Tasks involved multiplicative, partitive, and quotitive division questions such as "Please go and bring me 5 towers of 6 cubes. How many cubes do you have in all?" and "I have 12 cubes in 3 towers. How many cubes are there in each tower?" Although they varied from session to session based on an on-going assessment of students' performance, all tasks shared the common nature of multiplicative reasoning. The instructor explicitly demonstrated double counting to students during the teaching experiment. During the last session of the teaching experiment, the instructor asked students to solve questions similar to those in the pretests, such as “I have 30 cookies in 6 boxes. How many cookies do I have in each box?” Each session lasted 30-50 minutes. The discourse of the teaching experiment was videotaped and transcribed.

*Data Coding & Analysis*

Students’ strategies were coded according to a coding scheme investigators developed based on existing theoretical and empirical literature on solving multiplicative problems and the nature of students’ activities. Five general types of strategies were involved: unitary counting strategies, repetitive addition or subtraction, double counting, direct retrieval, and DK (don’t know). Inter-rater reliability was checked by a team of graduate students who were unaware of the purpose of this study recoding 33% of the transcripts. The inter-rater reliability was 92%.

Graphical presentation of data is important for micro-genetic studies (Sielger, 2006). By visually analyzing data, five dimensions of change were investigated: (1) source of change (what leads children to adopt new strategies); (2) path of change (the sequence of strategies children use while gaining competence); (3) rate of change (the amount of time or experience from the initial use of a strategy to consistent use of it); (4) breadth of change (how widely the new strategy is generalized to other problems); as well as (5) the variability of change (differences among children in the previous four dimensions).

## Results

*Students’ Multiplicative Problem Solving Performance*

All the participants increased their percent of accuracy in solving multiplicative problems from baseline to the last session of the intervention. Specifically, Chad improved his percentage correct from 44% on average of the pretests to 83.33 % during the last session of the teaching experiment; Megan improved her percentage correct from 55% to 87.5 %; and Tina improved from 40 % to 83.33%.

*Students’ Strategic Development*

Figure 2 represented the three students’ multiplicative strategies used during baseline and intervention. Generally, participants increased the types of strategies they used, and they also increased the frequency of advanced strategies used during intervention.

*Baseline.* The baseline data across three students consistently demonstrated the students with MD used very few strategies, and the most frequently used strategy was unitary counting. Chad only used unitary counting strategies for 91.93% of the trials during pretest sessions, and he replied with not knowing how to solve the problem or what strategy to use for 8.19% of all trials. Similarly, unitary counting was the dominant strategy Tina used during the pretest sessions (77.77%); Tina also used repetitive addition strategy (11.11%) and of direct retrieval strategy (11.11%). Megan used more types of strategies than Tina and Chad, but unitary counting was also the dominant strategy for her. She used unitary counting strategies for 41.67% of all trials; she also used repeated addition/subtraction strategies (33.33%), direct retrieval strategies (16.67%), and replied with “don’t know” for 8.33% of the problems. Double counting strategy was not found in any of the three students. Compared to Figure 1 for normal-achieving students (Kouba, 1989), the participants had fewer strategies and more heavily rely on unitary counting.

*Teaching Experiment.* The data from the teaching experiment suggested that students increased the variety of strategies they used; in particular, they increasingly used more advanced strategies. Specifically, double counting strategies appeared in the first session of the teaching experiment for Chad (14.28%) and Megan (7.14%), and in the third session for Tina with

40.91%. Double counting strategies consistently increased and became the most frequently used strategies in the last session across the three students (50% for Megan, 60% for Chad, and 33% for Tina). The appearance and increasing frequency of double counting strategies indicated that children explicitly mastered how to coordinate two quantities in multiplicative reasoning.

In addition, all three children increased their frequency of the direct retrieval strategy used. Chad did not use this strategy during the pretests, but he began to employ it during the second session in teaching experiment (11.11%) and used it with 20% of the tasks during the last session. Similarly, Megan's usage of the direct retrieval strategy increased to 21.43% in the last session of intervention and Tina increased to 16.67%.

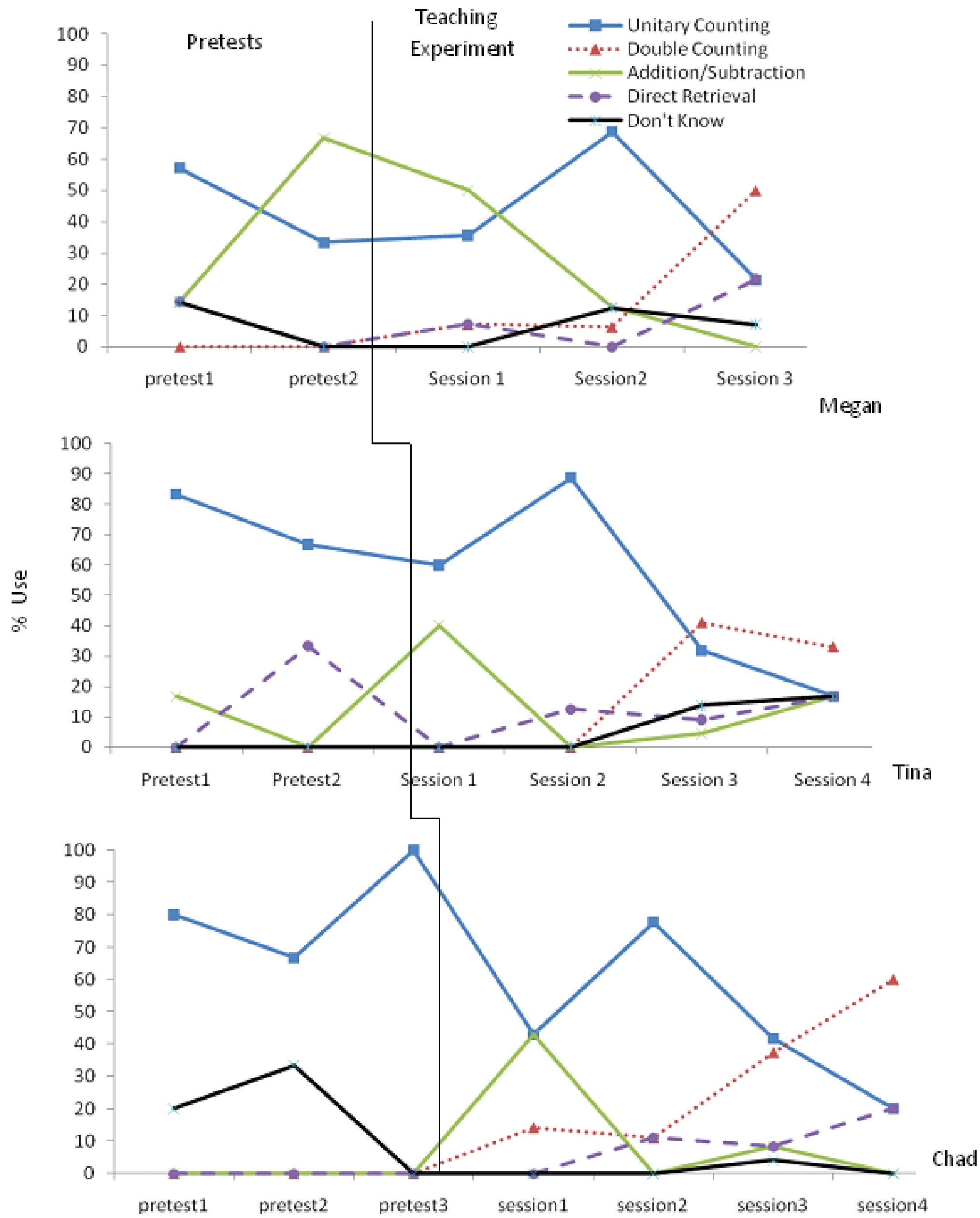


*Figure 2.* Three participants' multiplicative strategic development

On the other hand, the graph suggested that the participants decreased their frequency of using the unitary counting strategies. Chad consistently decreased from 91.93% on the pretest sessions to 20% during the last session. Tina and Megan increased their frequencies of using unitary counting strategies at the beginning of the intervention (Tina increased to 90% in Session 2, Megan increased to 68.75% in Session 2), but decreased the frequency of their usage of unitary counting strategies during the later sessions of the intervention (Tina decreased to 16.67% and Megan decreased to 21.42% in the last session). Similarly, the students also decreased their frequency of using repeated addition/subtraction strategy during the teaching experiment.

## Discussion

The purpose of this study was to explore how students with MD or at risk for failure in mathematics differed from normal achieving students in multiplicative strategy choice, and how these students shift from intuitive strategies to multiplicative reasoning strategies while receiving instruction to improve their multiplicative reasoning. Based on the limited data from the three students, the results suggested the students with MD and those at risk seemed to have fewer strategies and their strategies were less advanced than normal achieving students on the pretest sessions. Nevertheless, after the intervention, students improved their percentage correct in solving multiplicative problems; the frequencies of strategies the participants used changed as well. Five dimensions of change were discussed below according to the framework of micro-genetic studies (Siegler, 2006).

Regarding the *path of change*, the three participants demonstrated the similar pattern with which normal achieving students go through during the multiplicative reasoning strategic development; that is, beginning generally with counting by ones; then, transitioning to double counting, repeated addition or subtraction; and then, to recall of math facts. A short-term increase of their use of unitary counting was found before the participants consistently faded it out. Double counting increased robustly; direct retrieval also increased on a limited basis.

Specifically, unitary counting was the most dominant strategy of all three participants during the baseline sessions (97.93% for Chad, 71.77% for Tina, and 41.67% for Megan). They did not use double counting at all, and used direct retrieval very rarely. These results suggest that the three participants could only keep track of one number sequence while in the fifth grade and that these students did not use double counting to keep track of two number sequences. Whereas early studies (Geary, 1990; Geary & Brown, 1991) found that the major problem for children with MD in additive reasoning was direct retrieval, the current results indicated that the participants with MD seem to have problems both in conceptual understanding and retrieval in multiplicative reasoning.

The baseline data indicated a significant gap between students with and without MD. According to the data in Kouba’s (1989) study, normal-achieving students dominantly use unitary counting strategies at only grade one (97.23%); while they gradually decrease their frequency of employing the unitary counting to 66.16% of all strategies used at grade two and 30.11% at the grade three (Figure 1). The pretest data in this study showed that the three participants used an extremely high percentage of unitary counting strategies. It seemed like that Chad was equivalent to the first-grade level normal students in strategic development; Tina seemed to be equivalent to the second-grade level and Megan was equivalent to the third-grade level. The differential strategy choice may explain why students with MD or at-risk for failure in mathematics have lower performance in solving multiplicative problems than normal achieving students.

The appearance of double counting strategy is an indicator that children conceptually understand the nature of multiplicative reasoning. Children coordinate two quantities by keeping track of two number sequences with double counting. The three participants consistently increased their use of double counting throughout the teaching experiment (i.e., up to 50% for Megan, 60% for Chad, and 33% for Tina). It is noteworthy that normal-achieving students usually increase the frequency of using double counting to 8.74% at Grade 2 and then decrease the use to 3.98% at Grade 3 (Figure 1, Kouba, 1989). This study suggested that double counting appears to be especially useful for children with MD. The explicit demonstrations of keeping track of two quantities strengthened the students' conceptual understanding of multiplication and reduce their cognitive load when processing the problems. In addition, as the students with MD have difficulties in direct retrieving (Geary, 1990; Geary & Brown, 1991); double counting makes it especially suitable for them to explicitly demonstrate the coordination of two quantities, which is the core meaning of multiplication.

Although the three participants increased their use of direct retrieval strategy, they did not get to the level at which normal-achieving fifth grader students perform. Normal-achieving third graders use direct retrieval strategy for 59.66% of all trials (data from Kouba, 1989), but Chad, Megan and Tina only used direct retrieval strategy for 20%, 21.43% and 16.67% of the last session of intervention, respectively. The limited increase indicated that students with MD may need special interventions to help them shift from counting into verbal retrieval

For the *source of change*, an adapted multiple probe design across participants established a functional relationship between the teaching experiment and students' strategic development. Thus it is the "towers and cubes" activity that helps students with MD to adopt new strategies. This activity provides students with manipulatives to solve multiplicative problems. The teacher's demonstration of double counting seems to be effective in teaching students how to coordinate two number sequences, and it may explain the appearance of double counting strategy. During the teaching experiment, the instructor explicitly demonstrated the double counting in finding out the total number of cubes across a specific number of towers. The intervention also emphasized making distinctions between the unit of one (1's) (e.g., total number of cubes) and the composite unit (e.g., a tower of 6). In all, through the "cubes and towers" game, children seem to conceptually develop multiplicative reasoning.

As for the *rate of change*, the participants' use of double counting strategy increased saliently as soon as the occurrence of first use of this strategy during the teaching experiment. The participants' gaining of direct retrieval was slower than the change of double counting.

For the *breadth of change,* the participants could solve problems with various semantic structures; the students were also able to successfully associate the real life problem context in the last session with the multiplicative scheme they learned in the "cubes and towers" problems, and use the advanced strategies (e.g., double counting) to solve problems within new contexts.

And for the *variability of change,* the participants demonstrated some differential patterns from normal achieving students regarding strategic developmental level and transition pattern. Results also demonstrated individual differences among the three participants.

In sum, this study revealed how three students with MD or at-risk for MD in mathematics progress in strategic choices during multiplicative reasoning instruction. Due to the limited generalization of single subject design, a group design study is underway.

## Endnotes

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# PRECALCULUS STUDENT UNDERSTANDINGS OF FUNCTION COMPOSITION

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*Existing research on function composition has focused on students' ability to solve function composition problems relative to the student's conception of function. However, little research has examined the mental actions and understandings needed to understand and use function composition meaningfully when solving novel problems. This research addresses this gap, presenting a conceptual analysis of a typical function composition problem, along with the results of a study that investigated three precalculus students' understanding and ability to use function composition to solve novel problems.*

## Background

Educational policymakers often agree on the importance of function composition in high school mathematics, advocating that students in grades 9 through 12 should be able to "understand and perform transformations such as arithmetically combining, composing, and inverting commonly used functions" (NCTM, 1989; NCTM, 2000). However, despite agreement on the importance and desirability of students building a robust understanding of, and fluency using, function composition, little research has addressed students' understanding of this idea. This study begins to close this research gap, analyzing student products and behaviors from task-based clinical interviews in an attempt to gain insights into students' understanding of and ability to use function composition in novel contexts.

Engelke, Oehrtman, & Carlson (2005) found that, in the case of composition problems that require students to find the value of a composition of two functions at a point, most students in their study were able to do so, provided the functions in question were defined algebraically. Engelke *et al.* (2005) conjectured that the relatively high levels of success in such problems were related to the fact that correctly solving these problems required only an action view of function (Dubinski & Harel, 1992). They further conjectured that the much lower student success rate (less than 50%) for problems presented using other representations (tabular, graphical, context) was the result of such representations requiring students to engage in process-level thinking about functions. In general, Engelke *et al.* found that students were most successful in solving function composition problems that could be solved using memorized algebraic procedures.

Carlson, Oehrtman, & Engelke (in review) describe the importance of covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) and a process view of function in understanding function composition. For a student with an action view of functions, "composition is substituting a formula or expression for *x*." This is quite different from the mathematical reality of a student with a process view of function, for whom "composition is a *coordination* of two (or more) input-output processes; input is processed by one function and its output is processed by a second function."

The present study is part of a larger study that is investigating how precalculus students know and learn function composition, and the impact of students' quantitative reasoning, covariational reasoning, problem solving abilities, and understanding of function on their ability to use function composition to solve novel problems. This report describes three students' ability to use function composition to help them solve static and dynamic problems defined using contextual, algebraic, tabular, and graphical representations.

## Brief Conceptual Analysis of Function Composition

The theoretical perspective for this study emerged from conceptual analysis (von Glasersfeld, 1995; Thompson, 2000) of function composition problems. This approach serves to highlight the mental actions and ways of thinking that are propitious to a robust understanding of function composition. Consider the following problem:

> A rock is thrown into a pond, creating a circular ripple that travels outward from the point of impact at 9 cm/second. Express the area enclosed by the ripple as a function of the elapsed time since the rock hit the water.

To provide an ideal response to this problem, a student must perform a series of mental operations. First, the student must develop a mental image of the situation described in the problem. This mental image must then be made amenable to further mathematical activity. A key component of the process of reconceptualizing a problem in a way that makes it amenable to mathematical activity is what Thompson (1989) describes as *quantification*. This involves identifying an attribute of a situation, conceiving of that attribute in a way that admits a measurement process (which may or may not be explicit), and conceiving of the *quantity*, with appropriate units, that is the result of that measurement process. In this problem, the student must mentally construct quantities corresponding to the elapsed time since the rock hit the water, the radius of the circle formed by the ripple, and the area enclosed by the ripple. As a basis for quantification, the importance of the student's initial mental image of the problem cannot be overstated. The attributes that the student uses to create quantities are not attributes of a problem external to the student, but rather are attributes of the problem as it exists in the mind of the student.

A precalculus student attempting to solve this problem is likely to remember a formula for the area of a circle. As a result, a student's first attempt to write a formula for the area enclosed by the ripple is likely to be $A=\pi r^2$, where *A* represents the area enclosed by the ripple and *r* represents the radius of the circle formed by the ripple. However, the student has been asked to relate the area enclosed by the ripple with an elapsed time, rather than with a radius. A key to the student's advancement toward a solution is the realization that his goal is to relate area and elapsed time. To progress further toward a correct solution, the student must construct a relationship between elapsed time and the radius of the circle formed by the ripple, a relationship similar to $r=9t$, where *r* is the radius (in centimeters) of the circle formed by the ripple and *t* is the elapsed time (in seconds) since the rock hit the surface of the water.

Construction of the algebraic relationships $A=\pi r^2$ and $r=9t$ does not necessarily imply that the student has constructed a dynamic mental model of the problem situation. In fact the student might still possess only a static mental model; such a student might be able to answer a question like "What is the area of the circle 5 seconds after the rock hits the water?" while remaining unable to describe how the area of the circle varies as the elapsed time varies. The student may also still not be able to express a relationship between *A* and *t* using a single formula. The ability to describe how the area and elapsed time vary together requires that students reason about the values of two quantities and how they change in tandem, also referred to as *covariational reasoning* (Carlson, Jacobs, Coe, Larson, & Hsu, 2002).

The problem asked the student to relate the area of the circle and the elapsed time, and to use a *function* to express the relationship. To do so, the student must possess an understanding of the concept of function. Specifically, to provide an ideal solution to this problem, the student must possess a *process conception of function* (Dubinsky & Harel, 1992). A key notion in building a process view of function is that a function is something that accepts input values and produces output values. Prior to constructing the composite function $A = g(f(t))$, the student must conceive of the output of *f* as being a suitable input for *g*. Only then can the student think about "connecting together" the two functions he previously constructed. After the student has mentally connected the functions *f* and *g*, the next step in the "ideal solution" is to think of the composite function as a single function (say *h*). A student who has conceptualized the composite function, *h*, as a single function is able to imagine two processes. The student can imagine the covariation of time and radius, and the covariation of radius and area. He/she can imagine an amount of time being mapped to a radius and that resulting radius being mapped to an area. Typically the student speaks about time being converted to area through a series of two processes.

Certain composition problems present added complexities that warrant further discussion. For example, consider a problem that prompts students to "express the area of a circle as a function of its circumference." Many of the mental actions required for the student to solve this problem are similar to those discussed above. However, this problem incorporates an additional source of complexity: it requires the student to invert a function prior to composing two functions. Inversion is necessary because a student is likely to remember the formulas $A = \pi r^2$ and $C = 2\pi r$, where *A*, *r*, and *C* are the area, radius, and circumference of a circle, respectively. Written in this way, both formulas lend themselves to an interpretation of the radius as the input quantity, with area and circumference the output quantities. To conceive of a function that takes circumference as input and produces area as output, the student must first conceive of the inverse of the second relationship, before composing that inverse with the first formula to create a formula that gives area as a function of circumference.

## Methods and Setting

This study occurred in the context of a precalculus course at a large public university in the southwestern United States. The subjects of this study were three volunteers from a section of the precalculus course that emphasized quantitative reasoning and covariational reasoning. This study consisted of pre-interviews with each student, a single teaching session focused on function composition, and post-interviews. All interviews and the teaching session were videotaped, and all written student products were retained. The videotapes of all interviews and the teaching session were reviewed and transcribed. The transcripts were analyzed in an attempt to gain insights into what the students might have been thinking. Conceptual analysis (von Glasersfeld, 1995; Thompson, 2000) was used in an attempt to answer the question, "What mental actions in the person I'm observing would explain the behavior I seem to be observing?"

Clinical, semi-structured task-based interviews (Clement, 2000; Goldin, 2000) were a key source of data for this research. The pre-interview was designed to gain insights about student knowledge of function composition and ability to work novel function composition problems *before* participating in the teaching session. The interview included tasks related to functions defined by formulas, tables, and graphs.

The teaching session was designed to build on students' knowledge of functions. Students were presented with a problem in which they were asked to develop a function relating two quantities that were not trivial to relate directly. Through interactions with each other and with the instructor, I intended that students build a conception of linking two processes together to create a composite function, which could then be explored, allowing students to reconceptualize this composite function as a single function.

The final sections of the teaching session were intended to allow students to explore function composition using functions defined by tables and functions defined using graphs. The activities using functions defined by graphs included questions requiring students to consider the variation of the output of the composite function in response to changes over an interval of input values.

The post-interview was designed to gain insights about student knowledge of function composition and ability to work novel function composition problems *after* participating in the teaching session. As in the pre-interview, the interview tasks included functions defined by formulas, tables, and graphs. The post-interviews included activities that asked a student to reason about dynamic variation of output quantities in response to changes in the input quantity.

## Results from Pre-Interviews

In her pre-interview, Rachel did not make use of function composition to help her solve problems. For example, when asked to find the area of a square as a function of the perimeter, Rachel stated that she was "not that good at expressing things as functions yet", but that she knew how to find the area of a square. She eventually drew a square, labeled each side of the square as *x*, and wrote $p = 4x$. When asked about a formula for the area, she stated that it would be "*x* squared". However, even after probing, she was unable to construct a formula that related the area of a square with its perimeter.

Rachel was, however, able to meaningfully interpret function composition notation, despite claiming to have not seen it before and having been only briefly introduced to function notation. In a problem involving two functions defined using formulas, Rachel was able to find $g(h(2))$, by finding $h(2) = 5$ and substituting 5 as the input to *g*, but she did not spontaneously use input/output language. When asked to use input/output language to explain what she had done, she stated:

> To do this problem, I was given… I needed to find *g* of *h* of 2, so the input of $h(x) = 3x - 1$ is 2, so I would replace the *x* with 2 as the input, and then whatever I got would be the output, so 5 would be the output of the function *h* with the input of 2… um, and then basically once I got that, I was able to make the output of $h(2)$ the input of $g(x)$.

This pattern of reasoning appeared to continue when solving problems using other representations. When asked to use graphs of *f* and *g* to find $g(f(2))$, Rachel coordinated input and output values, first finding $f(2) = -2$, and then finding $g(-2) = 1$. As she explained it,

> OK, well I know this line is *f*, so to find $f(2)$ I would find 2 on the *x*-axis, which is right here, so I know that $f(2)$ has an output of -2. So now it's asking me to use the output of – $f(2)$ as the input to *g*. So for this line *g* right here, I would find the input -2, which is right here, so I know that $g(f(2))$ has an output of 1.

When questioned about how she was able to find output values that corresponded to given input values, Rachel indicated that she found the appropriate value on the *x*-axis, moving vertically until she intersected the graph of the function, and then moving horizontally until she intersected the *y*-axis to determine the appropriate output value. This suggested that Rachel understood graphs as coordinating values of input quantities with values of output quantities.

Rachel was also able to solve problems involving composition using two functions defined in a table. When asked how she determined which function she evaluated first, Rachel indicated that it was because it was the part "in parentheses", and clarified that she was using "order of operations" rules to make this decision.

The second student, Alicia, solved all of the problems in the interview, but she was often not able to articulate the input and output quantities of composite functions. For example, when asked to express the area of a square as a function of the perimeter, Alicia immediately wrote the equations $A = s^2$ and $P = 4s$. She then solved for *s* in the second equation, "used substitution" (her own words), and wrote $A = \left(\frac{P}{4}\right)^2$. Her solution to this problem did not use function notation, or the word *function*, or input/output language. When asked to write this relationship as a function, she wrote $A\left(\frac{P}{4}\right) = \left(\frac{P}{4}\right)^2$. Asked to explain why she had written this function with $\frac{P}{4}$ as the input, she stated this was because "to get the side by itself, I'd have to divide 4 into the perimeter". This suggests that Alicia was not thinking of a composite function but was still thinking of the function $A(s)$, where in this case *s* was replaced by an expression equivalent to *s*.

Even though she did not spontaneously use function composition in solving these problems, Alicia was able to make sense of problems that used function composition notation. For example, in a problem that required evaluation of composite functions defined by algebraic formulas, Alicia was able to quickly find the correct answer. Her explanation suggests that she solved this problem by thinking of $h(2)$ as the input to *g*: "This just means that you have to do $h(2)$ first, because $h(2)$ is the input to $g(x)$". When asked to explain why she chose to evaluate the functions in the order she did, Alicia stated that this was because of the "order of operations", just as Rachel had stated in the previous interview.

Alicia was also able to solve problems that required composition of functions defined by graphs or tables, using specific input values. For example, Alicia was able to explain how to find $g(f(2))$ using the given graphs. She was also able to solve a series of three evaluations of function composition using functions defined by tables quickly and correctly.

The third student, Hayley, demonstrated weaknesses in her function knowledge during her pre-interview. She was unable to make progress on problems asking her to express the area of a square as a function of its perimeter, or to express the diameter of a circle as a function of its area. Unlike Rachel and Alicia, Hayley was not able to use an "order of operations" interpretation to complete tasks that required her to interpret function composition notation. Instead, she demonstrated a detailed and consistent interpretation of function composition notation. She correctly interpreted the inner function as evaluating a function with a given input, but she interpreted the outer function as a command to multiply. Exactly what she was supposed to multiply varied. For example, one problem presented Hayley with two algebraically defined functions, and required her to find $g(h(2))$. She read this expression aloud as "*g* of *h* of 2", but described the requested operations as "you're multiplying the function *g* times the function *h* of 2". She then proceeded to find $h(2)$ correctly, and then she multiplied the result by the rule for function *g*, giving her an answer of $g(h(2)) = 5x^2$.

When asked to "use the graphs of *f* and *g* to evaluate $g(f(2))$", Hayley's first question was "What point do you want me to use?" Her subsequent response to the task suggested a lack of understanding of what information the graph of a function is intended to convey. The graph of *g* had one point labeled with its coordinates, (-2,1), and the graph of *f* had two points labeled with their coordinates, (2,-2) and (4,3). Hayley gave the written response $g(f(2)) = (-2,1)[(2)(2,-2)(4,3)]$, and gave the following explanation of her response:

> *g* is -2 and 1. Those are the points on the graph, so that's right there, multiplied by *f* of 2, so you're gonna multiply these two points by the number 2, and that's why they're in brackets.

Presented with a table that defined two functions and asked to evaluate composite functions at specific input values, Hayley correctly evaluated the inner (first) function, but her second step was to multiply each of these numbers by the name (*f* or *g*) of the outer (second) function. This suggested a fragile understanding of function notation and function inputs and outputs.

## Results from Teaching Session

The teaching session focused largely on the ripple problem described earlier. Working as a group, the students settled on quantities that were meaningful and measurable attributes of the problem. Having identified quantities, the students created useful formulas relating the values of the quantities. However, students' descriptions of the formulas sometimes suggested that they had constructed vaguely defined quantities and incorrect quantitative relationships. For example, when explaining the correct formula, $f(t) = 0.7t$, Hayley explained that using this formula "you take 0.7, because that's the radius, and multiply it by the number of seconds".

The group members also voiced no concerns with the suggestion that to express the area of the circle as a function of the circle's radius, they could write $A = g(r) = \pi(0.7t)^2$. When probed, Alicia suggested, "You could just do $\pi r^2$". Hearing this, Hayley first wrote $A = g(r) = \pi(0.7)^2$, suggesting that she thought of 0.7 as being the radius. This response suggests that Hayley, and perhaps other group members, were not visualizing the radius as a function of time, and hence they were not visualizing the quantity radius as varying with variations in time.

When using the correct formulas to find the area inside the ripple 6 seconds after the rock hit the water, Alicia correctly described the input and output quantities of both functions, including a description of why she used the output of *f* as the input to *g*. Alicia consistently exhibited a better ability to talk about function input and output quantities than Hayley or Rachel did.

When asked to use graphs of the functions *f* and *g* to find the area inside the ripple 6 seconds after the rock hit the water, Rachel did not refer to inputs or outputs; however, she did indicate with her finger how she would use the graph of radius as a function of time, locating $t = 6$ seconds on the input axis, moving up vertically until she reached the graph of the function, and then moving horizontally until she reached the output axis, where she could determine the radius.

When asked to evaluate composite functions defined by tables, at a given input value, Alicia and Rachel were able to explain how they used the output value from the first function as the input value to the second function. However, Hayley exhibited difficulty similar to her pre-interview. Asked to evaluate $g(g(1))$, Hayley correctly found that $g(1)=2$, but used the word "multiply" to describe what she should do next, and was unsure whether she should multiply 2 by *g*, or perhaps square 2, since she wanted to find $g(g(1))$. With help from her groupmates, Hayley discovered the correct way to solve this problem, and commented that she was seeing the problem differently than she had before.

The group was next asked to use the graphs of two functions to find the output of a composite function at various given input values. All three students were able to describe the process of using two graphs defined on the same set of axes to find specific outputs of a composite function, explaining how to find the output of the first function and use that value as the input to the second function. The students were much less successful in an activity in which they were asked to use graphs to evaluate the output of a composite function over an interval of input values. Since the students had all experienced success in evaluating composite functions at a specific input value, their failure to describe variations over an interval of input values suggests a weakness in their covariational reasoning ability, i.e., they were unable to attend to how the quantities changed together while imagining changes in the input variable.

**Results from Post-Interview Session**

In her post-interview, Alicia was able to describe the input and output quantities for all of the functions used in the post-interview tasks. Alicia was also able to describe the output of the first function as becoming the input to the second function. This suggests that she had developed a more robust conception of functions as accepting inputs and producing outputs.

Just as they had done during the pre-interview, both Alicia and Hayley described the input and output quantities of a composite function in a way that suggests they were still having difficulty conceptualizing the composite function as a single function. For example, when $h(t)$ was defined as $g(f(t))$, both Alicia and Hayley stated that the input to *h* was $f(t)$.

Hayley showed a clear improvement in her ability to use input/output language to talk about functions and function composition over the instructional sequence. In her pre-interview she typically evaluated what she referred to as "the first function" and followed this with a "multiplication" by the second function. However, in her post-interview, Hayley consistently described using the output of the first function as the input to the second, suggesting that she had emerged with a process view of function and had acquired the ability to articulate what it means to compose two functions.

Hayley also showed a significant improvement in her ability to make sense of a function defined by a graph. In the post-interview, she was able to perform evaluations of composite functions, including composite functions that required using the inverse of one of the graphically defined functions. She was able to use input/output language to describe what she was doing, and why she was doing it. However, she was still unable to determine how the output of a composite function varied over an interval of input values, suggesting a continuing weakness in Hayley's covariational reasoning.

In her post-interview, Rachel was successful on problems with real-life contexts or clear procedural solutions. However, she had difficulty completing tasks in situations that had neither a real-life context nor a clear procedural solution. For example, she remained unable to make meaningful progress when asked to "express the area of a square as a function of its perimeter", a problem that had also appeared in the pre-interview. She indicated that she didn't understand why a person would want to do that, since you could just express it based on the length of the side of the square.

## Discussion

All three students exhibited behavior that suggested weaknesses in their covariational reasoning, their quantitative reasoning, and their views of function. In many instances they exhibited a weak understanding of inputs and outputs of functions, and in some cases, the students acted in procedural ways suggestive of an action view of function, rather than a process view. All three students exhibited difficulty reasoning about the behavior of functions over an interval of input values. This suggests impoverished covariational reasoning, as students showed an inability to attend to the relationships between changing input and output quantities in function composition problems.

In many cases, the students in this study also showed a tendency to define quantities based on attributes that were either not measurable or were not well defined. This suggests that these students would benefit from a curriculum that emphasizes conceiving and reasoning about quantities.

Hayley exhibited behavior that suggests she had little understanding of what information the graph of a function conveys. During her pre-interview, her solutions for problems involving graphs were highly procedural, and suggested that her only understanding of graphs involved manipulating numbers that were used to label points on the graph. However, during her post-interview, Hayley demonstrated dramatic improvement in her ability to make meaning of a graph. This suggests that for her and perhaps other students, it would be beneficial to engage students in tasks to support their understanding of the meaning of a function's graph.

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# DIFFERENT MOMENTS IN THE PARTICIPATORY STAGE OF THE SECONDARY SCHOOL STUDENTS' ABSTRACTION OF MATHEMATICAL CONCEPTIONS

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*This study reports characteristics of participatory and anticipatory stages in the abstraction of mathematical conceptions. We carried out clinical task-based interviews with 71 secondary school students to obtain evidence of constructed mathematical conceptions and how they were used. We could distinguish both stages in different mathematical conceptions and, furthermore, two cognitive moments in the participatory stage. We argue that (a) the capacity of perceiving regularities in sets of particular cases is characteristic of reflection on activity-effect, and (b) the coordination of information provides the opportunity for changing the attention focus from the particular results to the structure of properties.*

## Introduction

Understanding how mathematical conceptions are constructed can help in thinking about teaching with the aim of encouraging learning. In this sense it is essential to have accurate descriptions of the processes by which mathematical knowledge is developed. This situation generates issues about what it means to know something about mathematical objects, and how the learner develops or constructs that knowledge (Dörfler, 2002). Cognitive theories based on Piagetian stances assume that mathematical conceptions reflect regularities from human actions and mental operations. In this perspective is generated the question of how to explain the way in which learners cognitively construct their mathematical conceptions. For our purposes and henceforth, "construction" refers to the emergence of a new structure through constructing actions (Monaghan, & Ozmantar, 2006; Simon, Tzur, Heinz, & Kinzel, 2004). Simon and his colleagues (Simon et al., 2004) postulate the existence of a mechanism that they call Reflection on Activity-Effect Relationship to explain this construction process. Taking into account the two phases of reflective abstraction (projection and reflection) described by Piaget (2001), Tzur & Simon (2004) point out that in the projection phase, where the actions become the objects of the reflection, learners sort activity-effect records in terms of an established goal distinguishing between records that get closer to their goal and those that do not. In the reflection phase, where a reorganization of knowledge takes place, learners reflect on the relationship between the activity and its effects.

During the resolution of a problem, the student may call-up a mathematical conception already constructed (anticipatory stage), but in the case in which this conception there isn't, student trigger some actions guided by a goal to obtain information to solve the problem (participatory stage). In this context, we adopt Simon et al.'s (2004) account of a construction process trying to provide empirical support to (i) the distinction between participatory stage and anticipatory stage in the abstraction of mathematical conceptions and (ii) a finer description of how proceeds the participatory stage.

## Methodology

### *Participants*

511 students in the last year of compulsory education (15-16 years old) solved a questionnaire with five mathematical problems in the domains of variability, divisibility and generalization. The analysis of the replies to the problems displayed students' diverse behaviours while solving the problems from the perspective of how they used the different mathematical conceptions. These behaviours may be considered evidence of anticipatory and participatory stages in the construction of mathematical notions involved in the mathematical problems posed. To obtain further information about this phenomenon we conducted 40-minute task-based clinical interviews with 71 of these secondary students. The interviews were focused on how the mathematical conceptions were used during problem solving as a manifestation of the conception constructed. Data come from of audio-records and transcriptions of students' justifications and their written replies to the five problems. Figure 1 shows an example of the problems used.

*Job offers for pizza delivery workers have appeared in a local newspaper.*
*Pizza takeaway A pays each delivery worker 0.6 euros for each pizza delivered and a fixed sum of 60 euros a month. Pizza takeaway B pays 0.9 euros for each pizza delivered and a fixed sum of 24 euros a month.*
*Which do you think is the better-paid job?*
*Make a decision and explain why your choice is the better one.*

*Figure 1.* The job offer.

The interviews were carried out after the students completed the questionnaire and the researchers undertook a first analysis of their replies. The aim of the clinical interview was to get the pupils to verbalise their thought-processes used in solving the problems (Goldin, 2000) in order to obtain evidence of how they generated some abstraction processes of mathematical conceptions or used them. The interviewer had a prior interview script constructed considering the characteristics of each problem and the type of answer given by the pupils. In any case, the interviewer could modify her questions in view of the pupil's behaviour, in order to clarify or investigate more deeply the reasoning processes followed.

*Data Analysis*

The students' responses to the problems and the interviews were analysed from a descriptive point of view using a constant-comparative methodology (Strauss & Corbin, 1994) and taking into account the way in which each pupil set up and used elements of mathematics knowledge as tools in order to interpret the situation and then make a decision (Llinares, & Roig, 2007). Characteristics of the abstraction process generated by the students were identified through the way in which they considered the variability of the quantities, the conditions that had to be fulfilled by these quantities and the way in which discerned generalities from the registers of particular data. We interpreted these characteristics from the process involving students' goal-directed activity and the reflection process (Clement, 2000). Next, we considered the characteristics and the interpretations generated according to the stage distinction from the effect of reflection on activity-effect relationship as a coordination of the available conceptions and identified two moments in the participatory stage with similar characteristics in the different mathematical conceptions taking into account how students created records of experience, sorted and compared the records, and identified patterns in those records.

## Results

Table 1 shows the results obtained from the combined analysis of the interviews and the answers of the questionnaire.

Table 1. *Percentages in Different Stages of the Abstraction Process*

| | P1 | P2 | P3 | ***P4*** | ***P5*** | 71x5=355 |
|---|---|---|---|---|---|---|
| | n (%) | n (%) | n (%) | n (%) | n (%) | n (%) |
| *Participatory stage* | 30 (42.2) | 23 (32.4) | 68 (95.8) | 13 (18.3) | 55 (77.4) | 189 (53.2) |
| *Anticipatory stage* | 7 (9.9) | 1 (1.4) | 3 (4.2) | 15 (21.1) | 12 (16.9) | 38 (10.7) |
| Others | 34 (47.9) | 47 (66.2) | 0 (0) | 43 (60.6) | 4 (5.7) | 128 (36.1) |
| Total | 71 (100) | 71 (100) | 71 (100) | 71 (100) | 71 (100) | 355 (100) |

More than 10% of students had anticipated the mathematical conception in the situation (anticipatory stage). On the other hand, 10.7% of students generated particular cases in order to obtain information about the situation (participatory stage). We identified during the interviews how some students coordinated information from particular cases and generated an answer which reflected a certain degree of generalisation which had not been present in their original written answers. This behaviour indicated a change of focus during interview lending to the generation of an abstraction that fits the reflection on activity-effect relationship mechanism, and revealed the existence of two cognitive moments in the participatory stage. We use some answers to problem 1 to explain these two moments: projection (generating a set of registers) and local anticipation (Reorganization, Identification of Regularities and Acceptance of the Generality).

*Projection: Generating a Set of Registers*

In nearly 20% of the total of 355 answers, the students created from the situation some type of set of registers, but had difficulty in coordinating the information available. In "The job offer" problem, 5.6% of pupils used particular cases to obtain information that might help in making a decision. A typical example of the procedure employed to create a set of registers was the following:

- For 10 pizzas delivered, Earnings A = 66€ > Earnings B = 33€ → A is better.
- For 20 pizzas delivered, Earnings A = 72€ > Earnings B = 42€ → A is better.
- …

Here the pupils centred their attention exclusively on the information provided by the set of particular cases. This kind of behaviour, using very low numbers of pizzas delivered, or focusing the attention on only some of the account in the situation, prevents the more or less explicit appearance of the existence of a change in the profitability of the offers as the number of pizzas increases. The following protocol shows an example of this kind of procedure.

E19: *What else did you do? In the end, what conclusion did you come to?*

A: *Well, I saw that in pizza takeaway A they pay better because you are guaranteed the 60 euros, so you don't have to worry about delivering one pizza more or one less.*

The consequence of using very low quantities is that in all cases job-offer A is considerably better than job-offer B. Student E19's attention was centred on the six particular cases considered instead of on the information that could have been obtained by comparing the difference in earnings as the number of pizzas delivered increased.

*Local Anticipation: Identifying and Using a Regularity*

In the course of the interview some of the pupils coordinated the information derived from particular cases in response to prompts from the interviewer which allow them to identify a regularity. Sometimes they made inferences of a general kind from the situation, with no written trace of the activity carried out. On other occasions however the pupils wrote down registers which enabled them to investigate how to compare and relate the particular data, or generated a search for new information. In both cases they were coordinating the information.

An example of this approaching is the way in which E11 perceived during the interview the change of profitability in total earnings, basing the conclusion on a single particular case he had constructed on the written answer paper. On paper, E11 calculated the monthly earnings at each of the pizza takeaways in the case of *"20 pizzas delivered*", concluding that the better-paid job *"is the one at pizza takeaway A because you earn just over twice as much as at B"*. We had considered this kind of answer a manifestation of the Projection moment. During the interview, however, he indicated the following:

E11: *OK. Let's start with the first one. Do you remember what it was about?*

A: *Yes, here it is … you have two job offers, in one it's 6 cents for each pizza, and a fixed amount every month. In the other, the amount for … what they pay for each pizza you deliver, and then the fixed amount every month. And the other, the amount they pay for each pizza delivered is quite high, but the amount they pay every month is lower. I've given an example. I mean, imagine you have to deliver about 20 pizzas a month. So you multiply the 20 pizzas, the pizzas by 6 cents, which is the same as 12 plus 12 and then the 60 euros you get every month, that's 72 altogether. In the other case 20 by 0.9 [by 9 cents] is 18, plus 24, that's 42. So the difference is bigger. So my better offer was A. A was much better.*

E11: *You'd take A, then?*

A: *Yes.*

As the interview continued, the researcher asked him what would happen if a greater number of pizzas were delivered.

E11: *And what do you think would happen if more pizzas were sold?*

A: *Yeah, that's what I was going to tell you, that probably as the number of pizzas increased you would earn more with option B. But with the example I've given you the better offer is A. Maybe with 200 pizzas B is a better offer.*

This reply seems to show that E11 perceives the existence of a change of profitability in the offers as the number of pizzas delivered increases. To find out how he managed to perceive this change, i.e. how the abstraction was produced, the interviewer asked him to explain why he thought it might be possible to earn more in job B.

E11: *Why do you think, then, why do you think you might be able to earn more in job B?*

A: *Because … because for each pizza, eh, you get 3 cents more than at the end of that … as you deliver more and more pizzas, you get, like, 3 cents for each* *pizza. I mean, after a lot, that's more, more money […] In the end, in the end … the more pizzas you deliver you get back the difference you've got here.*

In his answer E11 refers to the difference in the money paid by each pizza takeaway for each pizza delivered, saying *"because for each pizza, eh, you get 3 cents more than at the end of that … as you deliver more and more pizzas, you get, like, 3 cents for each pizza. I mean, after a lot, that's more, more money"*. He therefore perceives that the difference between the fixed amounts offered by pizza takeaways A and B can be compensated by selling a large number of pizzas. This is possible due to the difference in payment for each pizza delivered, and E11 comes to this conclusion via a qualitative analysis of the data without having to carry out calculations for particular cases. The regularity lies in the fact that the difference between the two offers diminishes as the number of pizzas delivered increases (the earnings in A get closer and closer to those in B) and therefore there comes a point at which B is better than A (there has been a change of tendency in the profitability of the two offers). Another relevant aspect of this procedure is the way in which the identification of the regularity is triggered by the researcher's prompt *"What do you think would happen if more pizzas were sold?"*. From a theoretical viewpoint the question functioned as a prompt which moved the pupil's focus of attention from a single case of what a pizza-deliverer might earn towards a consideration of "how the difference between the two amounts earned might vary" depending on the number of pizzas delivered. We have called this change of attention-focus reflection, which makes it possible to identify the regularity by coordinating certain types of information as a consequence of the interviewer's prompts.

On the other hand, once a regularity (change of profitability) has been identified it can enable the students to look for the exact number of delivered pizzas that equals both offers. In this problem the characteristic of local anticipation lies in the "adjustment" of the decision and is revealed when the pupil considers particular cases approaching 120 (which is the number of pizzas delivered that makes the two offers the same in earnings). In his written answer, E22 drew up a table showing various particular cases and the earnings corresponding to each one for both job-offers. In the interview he explains the process he followed.

A: *Look, in the first one they say there are two pizza takeaways, right? A and B, so in takeaway A they give you 60 euros a month, a fixed sum every month, and in B they give you 24, right? So if they give you more in one than in the other, but in … in the first one they give you 0.6 for every pizza you deliver, and in the second one 0.9, right? So that means that for every 10 pizzas you sell it'll be 0.6 times 10, six euros, you move the decimal point, and here it's 9 euros. So for every 10 pizzas you sell … I mean, look, it's here. From 20 to 40 that's 20, right? Well, you go on adding on, and here it says which will pay you better, right? Well, in the first one as it's 60 euros, in the first one if you don't sell many pizzas the chance is you'll get quite a bit of money, right? I mean it's quite a lot, a lot, a lot of money every month. But not in the second one. But in the second one you take more of a risk because you have to sell more pizzas. In the second one they give you more, less money every month, but they give you more money for every pizza you sell.*

E22: *Yes.*

A: *So when you get to 120 pizzas …*

E22: *What did you do? Did you keep trying it, going up and up, seeing how many deliveries…*

A: *Sure, I went 1, 2, 3, 4, 5, right? I kept on multiplying it.*

E22: *Is that the number of pizzas? [pointing to the first row in the table]*

A: *The number of pizzas sold. 5, right? But I saw it was not enough, so I went on adding more and more.*

E22: *Fine.*

A: *I went on multiplying, and here I wrote an equation, right?*

E22: *Yes.*

A: *Say x is the number of pizzas you sell at 0.60, at 0.60 cents plus the money they give you every month, then you multiply, it might only be two pizzas. Two times 0.60, 1.20 plus 60 euros maybe, and so on.*

The particular cases used are organised in a table beginning with the case of “1 pizza delivered”, and increasing by one pizza at a time for the subsequent cases up to the case of “5 pizzas delivered”. From 10 pizzas onwards, he uses the relation *“for every 10 pizzas you sell it’ll be 0.6 times 10, six euros [Job-offer A], you move the decimal point, and here it’s 9 euros [Job-offer B]”*. This regularity is perceived from the comparison between the amounts paid for each pizza delivered. As he states in his written answer:

- *“Every ten pizzas sold in A mean 6€”*
- *“Every ten pizzas sold in B mean 9€”*

The coordination of the information is revealed in the way he looks at the amounts earned for pizzas delivered (going up in tens of pizzas), together with the comparison between the fixed monthly amounts, which lead E22 to realise that job-offer B can be better than job-offer A (i.e. the regularity in the situation seen as a change of tendency). He is searching for the number of pizzas which will make the two offers the same by setting up new registers of particular cases, ten by ten. This “directed” search for the number that will indicate the change of tendency is a manifestation of the coordination of information, in which the particular cases are used as an iterative activity towards a pre-established goal. After calculating the case of 120 pizzas, E22 states that *“If you sell 120 pizzas you earn the same in both places, but if you are going to sell fewer pizzas you should choose A and if you think you will sell more you should choose B”*.

*18A: And in the end I went on doing that and with 120 pizzas you earn the same in both. So if 120 pizzas are sold you would earn the same in both. So you could take either. But from 120 onwards you’d earn more in B. So …*

*19E: So which of the two would you choose?*

*20A: Personally, I’d take A because it’s difficult to sell 120 pizzas. The thing is … but if you want to take a risk and you think you’ll sell more, you’d take B.*

E22 therefore discerns the change of tendency which occurs as the number of pizzas delivered increases, and is able to use it to discover at what number of pizzas the two job-offers pay the same. At the end of the interview he states that “Personally, I’d take A because it’s difficult to sell 120 pizzas. The thing is…but if you want to take a risk and you think you’ll sell more, you’d take B” (line 20). The perception of the change of tendency and the use of this insight into the structure of the situation to find the number of pizzas at which the change occurs enables the pupil to make a decision and justify it appropriately.

## Discussion

The written answers and the interviews provided us with detailed information regarding different manifestations of the abstraction process and the use of mathematical conceptions in secondary-school pupils. The results obtained enabled us to zoom in describing the distinction between the participatory and the anticipatory stages as proposed by Simon et al. (2004), observing a wide range of behaviours in connection with the mechanism that Piaget called "transposing knowledge to a higher level" and "the reorganisation-reconstruction of the knowledge at this level. We identified two different moments in the participatory stage and highlighted the importance of the prompts given during the interviews to students accede to anticipation. The use of different kinds of problems in the same study, together with a broad sample of pupils and the combination of questionnaire and post-reflection interviews made possible to amplify and complement previous characterisations of the abstraction process (Ellis, 2007a; Hershkowitz, Schwartz, & Dreyfus, 2001; Sriraman, 2004). Our findings have enabled us to generate two ideas which may help to explain some aspects of the abstraction process. In the first place, the way in which activity-effect reflection reveals what route is followed from projection to local anticipation and, secondly, the two manifestations of reflective abstraction in the process of problem solving.

Progress from projection to local anticipation stage is based on the capacity to observe regularities (the effect of the activity) and coordinate information in the set of particular cases. The way in which learners use particular cases is evidence of the steps they take when they have not identified a previously-constructed mathematical structure (participatory stage). The use of particular cases is linked to the performance of cognitive actions such as comparing, relating or searching. This kind of actions leads the student to notice the effect of his/her activities and coordinate the information which in turn leads to a change in the learner's attention-focus. Such prompted attention-changes, linked to cognitive actions, are what reflection consists of. A process of this nature has also been identified by Ellis (2007a, 2007b) via different kinds of generalisation tasks in which learners related and associated two situations or properties discernible in two situations, or used repeated acts to search for a relation. In these cases, the prompts proceed from the design of the task or from the interviewer. Our data have shown that in certain cases the existence of some kind of prompt or stimulus (made by the teacher/researcher or the task design) allow to student change through reflection and accede to anticipation (mathematical conception). These prompts favour the change of focus which is itself the beginning of the recognition of some kind of regularity in the set of data (effect of activity).

We argue that it is possible to identify different aspects of the abstraction process using problems from different mathematical domains all of which provides evidence of the general nature of this model. The relationship between the participatory and anticipatory stages in the abstraction process (Piaget, 2001) and the actions of generalisation and the characteristics of what has been generalised (Ellis, 2007b), give greater strength to this way of understanding the abstraction process when learners think mathematically, and locate the focus of attention on the relation between the learner's mental actions while abstracting, the outcome of these acts and their subsequent use. The results obtained therefore have implications with regard to the design of tasks to encourage the construction of an abstraction and the consolidation of the construction. In the first place, the role played by prompts (in the task itself or as made by the researcher/teacher during the interview) would seem to indicate that when abstraction-centred tasks are designed they should take into account the nature of the prompts which will help the learners to coordinate the information and thus go on to local anticipation. This recommendation is compatible with that made by Tzur (2007) following a whole-class teaching experiment. Secondly, in order to give learners the opportunity change their attention-focus and begin to see a set of activity-effect registers as a unified object (the identification of the regularity and/or the general aspect) (Dörfler, 2002) it will be necessary to create opportunities for the development of language-items for the new construction. This characteristic of the task has also been considered relevant in designing tasks to consolidate a new construction (Monaghan, & Ozmantar, 2006). In any event, more research is evidently required to provide information that will be useful in reaching a clearer theoretical understanding of task-design, with all the obvious implications for the improvement of teaching methods.

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# ROLE OF PERTURBATIONS IN MAKING SENSE OF FRACTIONS

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This study examines the role of perturbations in students making sense of fractions. Steffe and Tzur (1994) define *making sense* as "(a) to construct ways and means of operating in a medium, based on current knowledge, in order to neutralize perturbations induced through social interactions and (b) to become explicitly aware of those potential ways and means of operating through a process of symbolization"(p.111). Therefore, in students' mathematical sense-making, perturbations, social interactions and symbolization are important. In this presentation, I will discuss those three important contributors to learning, using a teaching segment taken from semester-long interactions (17 meetings) conducted with a pair of US $8^{th}$ grade students. I will mainly focus on one of the students, Jasmine, and her actions and operations to discuss her perturbations, the ongoing social interactions, and her use of symbols.

The particular problem that Jasmine attempted to solve was this: "A half-inch candy bar is cut into two parts. If one part is 13/3 as long as the other part, how long are the parts?" It took a total of 15 minutes for Jasmine and her partner to solve this problem. After analyzing the video-recorded interactions among Jasmine, her partner, her teacher (me) and two observers as well as her written work, I inferred that Jasmine was perturbed three times: 1) At the start, she forgot the length of the whole candy bar but operated and produced results without using that information. A possible reason for not using 1/2 inch might be that until this problem, she had solved problems when the length of the candy bar was given as a whole number of inches. When the length of the candy bar was given as a fractional part of an inch, she unintentionally forgot this information and solved the problem with whatever was meaningful to her. She took 16/3 as the length of one of the 16 pieces of the whole candy bar and multiplied it by 13 and 3 to find the lengths of the two parts. She was in a state of perturbation after I (the teacher) reminded her that the length of the candy bar was 1/2 inch. 2) Throughout the interactions, it became clear that she had the question of "how can the same quantity (the whole candy bar) be 16/3 of the smaller part and at the same time sixteen pieces?" The way Jasmine produced 16/3 was notational in that she utilized the denominator of 13/3 to label the other part as 3/3. She then added the number of equal pieces in the two parts, 13 and 3 respectively, and said that the whole quantity would be 16 thirds. It was not because 16/3 was 16 thirds of the 3/3 quantity but it was because all the equal pieces were given in thirds, e.g., 13/3 was given in thirds. She eliminated this perturbation by focusing her attention on the number of equal pieces (instead of using a multiplicative relationship of 16/3 of something is 1/2 an inch, what is the length of that quantity?) and wrote 16 pieces = 1/2 inch. 3) Towards the end of the interactions, she did not realize that when she wrote 1/32*16 the result would be 1/2 inch, the length of the whole candy bar. In addition, when I analyzed the interactions, it seemed that she did not have a goal of finding the lengths of two parts when she wrote 1/32 16/1.

In the poster session, I will present the details of these perturbations and show how Jasmine proceeded in the context of the social interactions that occurred during teaching.

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# COLLABORATION AND PROBLEM SOLVING IN AN ONLINE, SYNCHRONOUS ENVIRONMENT

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*In this study, we observe students working in small teams as they engage in problem solving in a synchronous online environment to understand how they relate to one another, work collaboratively, and communicate their emergent understanding of the problem task without contemporaneous teacher intervention. We present evidence that the students, using the computer as a cultural tool, actively engage in high forms of thinking and reasoning as well as engage in scaffolding as they co-construct mathematical interpretations and ideas.*

The data analyzed for this poster is a subset of a larger data corpus from the *e*Math Project. We are currently conducting our *e*Math Project with two groups of seniors from two different high schools, who meet in a virtual communication environment to solve open-ended mathematical tasks. Our research question emerges from a grounded study in how students collaborate in a synchronous environment carried out without direct teacher intervention. By studying how students communicate and ultimately collaborate within an online environment without direct teacher intervention, we can gain knowledge on how to structure lessons, and more particularly, problem tasks, so as to optimize learning. The theoretical perspective that guides our collection and analysis of data includes sociocultural theory (Vygotsky, 1978), communication (Sfard, 2000), and notions of socially emergent cognition (Powell, 2006) and group cognition (Stahl, 2006). This poster focuses on evidence of how the students communicate their emergent understanding of the problem tasks and coalesce as collaborative learners.

The poster will display evidence of how the students coalesced into a team and collaborated in a virtual environment by presenting clips from the chat window and screen shots of the white board. The screen shots display both edited text boxes and representations the students used to convince one another of their reasoning. Over the course of 13 sessions it is evident that the students improved upon their collaborative skills within the chat window excerpts and whiteboard summaries.

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# A COMPARISON OF PART-WHOLE AND PARTITIVE REASONING WITH UNIT AND NON-UNIT PROPER FRACTIONS

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*We report on quantitative methods applied to a pair of hypotheses formed through teaching experiments. In particular, our research affirms conceptual distinctions between part-whole and partitive reasoning with fractions, as theorized in previous literature (e.g., Steffe, 2002). These distinctions include a developmental hurdle in moving from partitive reasoning with unit fractions to partitive reasoning with non-unit proper fractions. Whereas we concentrate on hypotheses arising from scheme theory and teaching experiments, findings relate to fractions concepts identified by researchers employing other methods and frameworks as well (e.g., Kieren, 1980; Mack, 2001).*

## Objectives

Teaching experiments with pairs of students provide opportunities for teachers to closely analyze students' problem solving activities while taking into account student-student interaction (Steffe & Thompson, 2000). Based on students' actions (including verbalizations) teachers can build models of students' reasoning using schemes: hypothetical ways of operating that explain the students' actions. Such models have produced numerous hypotheses with regard to students' learning of fractions (e.g., Olive, 1999; Tzur, 1999; Steffe, 2002; Hackenberg, 2007). We examined some of these hypotheses in a quantitative study that afforded a much larger sample. We recognize that assessing student knowledge with written tests always comes with limitations, but Kilpatrick (2001) reminded us that, as mathematics education researchers, we are also obliged to quantitatively test hypotheses.

Here, we report on results from our examination of one pair of hypotheses, which relate to three particular schemes: the part-whole fractional scheme, the partitive unit fractional scheme, and the partitive fractional scheme. We describe the schemes in more detail in a subsequent section. For now, we note that Steffe (2002) and Olive (1999) learned about these ways of operating from their work with children in the Fractions Project. They theorized that the schemes form a hierarchy (respective to the order listed above) in which each preceding scheme is reorganized to form the succeeding scheme. Based on this idea and our own work with children, we posit the following research hypotheses.

1. In part-whole reasoning there are no operational differences between situations involving unit and non-unit proper fractions.
2. In partitive reasoning there are operational differences between situations involving unit and non unit proper fractions; success with situations involving unit fractions precedes success in situations involving non-unit proper fractions.

In reporting our results concerning these hypotheses, we can also report on the construction of fractional schemes in general, during grades 5 and 6.

## Theoretical Framework

Following Glasersfeld (1995), we define schemes as three-part structures, which include a set of perceived situations that activate the scheme (a trigger, or recognition template), a set of operations to act on the situation, and an expected result from operating. Glasersfeld defined operations—the active and therefore most important component of schemes—as mental actions abstracted through reflection on previous experience. For example, students might abstract partitioning operations from experiences in forming equal shares, folding paper, or otherwise creating equal parts from an existing whole. The partitioning operation results from reflection on the cognitive processes involved in completing such physical activities. Once a student has abstracted the operation, she can apply it to new situations, such as determining how to break a whole candy bar into five equal parts. Iteration—another operation important to working with fractions—involves creating copies of a unit. For example, a student might create three-fifths from one-fifth by iterating a one-fifth piece three times.

### *Part-Whole Fractional Scheme*

Students who have constructed a part-whole fractional scheme conceive of fractions as so many pieces in the partitioned fraction out of so many pieces in the partitioned whole. This scheme relies upon operations of identifying (unitizing) a whole, partitioning the whole into equal pieces, and disembedding some number of pieces from the partitioned whole. However, “a child who has constructed [only] a part-whole fractional scheme is yet to construct unit fractions as iterable fractional units” (Steffe, 2003, p. 242). In other words, a fraction such as three-fifths means three pieces out of five equal pieces in the whole, but it is not yet understood as three iterations of one of the fifths. Consider the tasks illustrated in Figures 1a and 1b.


*Figures 1a and 1b.* Task responses providing indication/counter-indication of part-whole fractional scheme.

Note that one of the tasks involves a unit fraction, whereas the other involves a non-unit proper fraction. The part-whole fractional scheme, as abstracted from researcher interactions with children, theoretically operates on both cases in the same way. The response to the first task correctly indicates one out of six equal parts. On the other hand, the response to the second task (by a different student) serves as counter-indication of a part-whole fractional scheme because the parts in the fraction are not equal in size to the unshaded parts of the bar.

*Partitive Unit Fractional Scheme*

Both a part-whole fractional scheme and a partitive unit fractional scheme generate fractional language, but the difference between the powers of the schemes is evident in resolving the task illustrated in Figure 2. Students with only a part-whole fractional scheme cannot determine the fraction because the whole is unpartitioned.

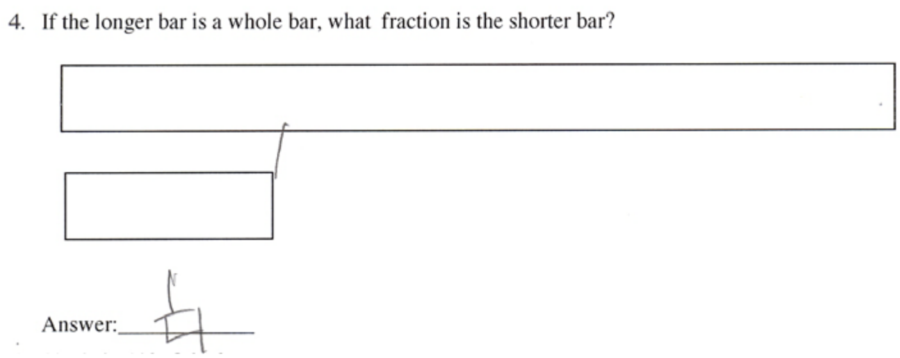


*Figure 2*. Task response providing indication of a partitive unit fractional scheme.

The response in Figure 2 indicates a partitive unit fractional scheme because the student seems to understand (note the mark on the left side of the large bar) that the unit fractional part (small bar) could be iterated four times to re-produce the whole (large bar) and that this number of iterations (four) determines the *size* of the unit fraction relative to the whole (one-fourth). The partitive unit fractional scheme, "establishes a one-to-many relation between the part and the partitioned whole" and involves "explicit use of fractional language to refer to that relation" (Steffe, 2002, p. 292). However, a partitive unit fractional scheme cannot be used to determine the fractional size of a non-unit fraction, because the iterations will not reproduce the whole (unless, of course, the fraction in question simplifies to a unit fraction; e.g., two-sixths).

*Partitive Fractional Scheme*

The *partitive fractional scheme* is a generalization of the partitive unit fractional scheme. Students can use the more general scheme to conceive of a proper fraction, such as three-fourths, as three of one-fourth of the whole. This involves producing composite fractions from unit fractions through iteration, while maintaining the relation between the unit fraction and the whole. It also involves *units coordination* at two levels (Steffe, 2002; Hackenberg, 2007), because the student must coordinate three-fourths as three iterations of the fractional unit and the whole as four iterations of the fractional unit. In order words, three-fourths is a unit of three fractional units, and the whole is a unit of four fractional units. Consider the coordinations indicated by the student response in Figure 3.

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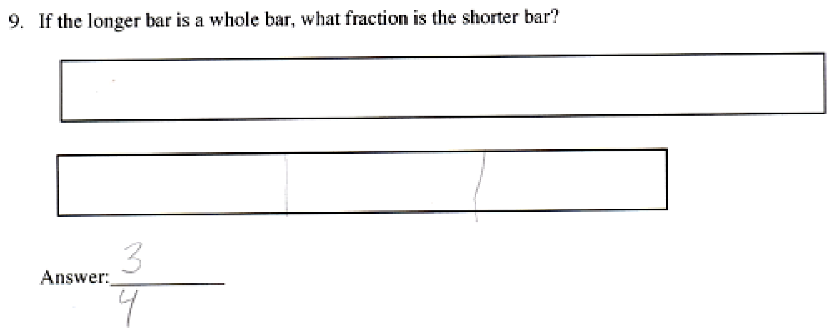


*Figure 3*. Task response providing indication of a partitive fractional scheme.

The response provides indication of a partitive fractional scheme because the student identified the smaller bar as three iterations of a part that, when iterated four times, would reproduce the larger bar (the whole).

*Research on the Progression of Fractions Concepts*

Several teaching experiments contributed to the hierarchy of schemes presented above (Norton, 2008; Olive, 1999; Olive & Vomvoridi, 2006; Saenz-Ludlow, 1994; Tzur, 1999). Collectively, student progress reported in these studies affirms the theoretical hierarchy of schemes and provides some indication of development by grade level. Most students seem to develop part-whole fractional schemes by fifth grade, but other schemes often lag years behind or do not develop at all.

Kieren (1980) identified part-whole relations as one of five subconstructs vital to the understanding of rational number. Although researchers agree that part-whole conceptions are fundamental to understanding fractions (Pitkethly & Hunting, 1996), "teaching efforts have focused almost exclusively on the part-whole construct of a fraction" (Streefland, 1991, p. 191) and this can lead to misconceptions. For example, Olive & Vomvoridi (2006) worked with a student named Tim who conceived of 1/n and n/n as the same thing because he could not consider 1/n apart from the whole before constructing a partitive unit fractional scheme (Olive & Vomvoridi, 2006). "Sparse conceptual structures limit students' understanding; once these conceptual structures had been modified and enriched, Tim was able to function within the context of classroom instruction" (p. 44).

Saenz-Ludlow (1994) emphasized the importance of developing a partitive conception of fractions when she referred to the need for students to understand fractions as quantities. This conception aligns with Mack's description of fractions as "multiplicative size transformations" (2001, p. 269). It also aligns with the *measurement* and *operation* subconstructs identified by Kieren (1980), in which fractions are understood in terms of sizes relative to a specified whole, rather than a simple comparison of numbers of parts in the fraction and the whole.

## Methods

*Data and Assessment*

We administered pre-tests during each fall and post-tests during each spring as part of a two-year professional development study, which took place from 2005 to 2007 in a low- to middle-income small-town school in the mid-western United States. The study involved one fifth-grade classroom and one sixth-grade classroom for each of the two years, with no students involved in both years. Students with missing data were removed for the present analysis, leaving 84 students for the pre-test and 86 students for the post-test. Each test contained nine randomly-ordered items: including items like those illustrated in Figures 1, 2, and 3. We paired the nine items across the two forms of the test so that only the numbers and respective sizes of the fractions changed (e.g., using a one-fifth bar instead of a one-fourth bar for the partitive unit item). We randomly assigned the tests so that each student took each form of the test once, either in the fall or spring.

We designed the items as indicators of particular schemes. That is, responses to each item provided opportunities for students to enact particular ways of operating. When assessing student responses, two scorers looked at the students' work on each single item, and inferred from all markings (e.g., written answers, drawn partitions, shading, calculations) whether there was indication that the student had operated in a way that is compatible with the particular scheme. Responses to each item were scored in the following way:

0: There was counter-indication that the student could operate in a manner compatible with the theorized scheme or operation. Counter indication might include incorrect responses and markings that are incompatible with actions that would fit the scheme. For example, the response illustrated in Figure 2b indicated that the student did not understand the importance of creating equal parts in the fraction and the whole.

1: There was strong indication that the student operated in a manner compatible with the theorized scheme or operation. Indications might include correct responses and partitions. For example, the mark and final answer in Figure 2 indicate the student iterated the smaller bar four times within the larger bar to correctly determine the size of the unit fraction, ¼.

Some items were initially scored as 0.5 and required further inference on the part of the scorers to select a final score of 0 or 1. For example, in response to the task illustrated in Figure 2, a student might have estimated the size of the fraction to be 1/5, with supporting marks indicating he had iterated the fractional stick 5 times. The two scorers reexamined such responses on a case-by-case basis to come to a consensus, and then used that consensus to inform decisions on similar, subsequent cases.

*Measures*

*Part-whole fractional scheme (PWFS).* Two items were used as indicators of students' part-whole fractional schemes. One item involved unit fractions (e.g., see Figure 1a) and the other item involved non-unit fractions (e.g., see Figure 1b). These items were used to test for students' operational differences between situations involving unit and proper fractions (Hypothesis 1).

*Partitive fractional scheme.* Two items were used as indicators of students' partitive fractional schemes. One item involved unit fractions (e.g., see Figure 2) and was used as an indicator of a partitive unit fractional scheme (PUFS) and the other involved non-unit fractions (e.g., see Figure 3) and was used as an indicator of a partitive fractional scheme (PFS). The two items were used separately to test for students' operational differences between situations involving unit and proper fractions (Hypothesis 2).

*Analysis*

Frequencies of student scores were entered into contingency tables and were analyzed using appropriate measures of association for ordinally scaled variables. Our hypotheses test both symmetrical and asymmetrical associations.

Hypothesis 1 was analyzed using a Binomial Test. In this case, a test for no operational differences is concerned with the distribution of students who did not get both items correct. In other words, if there are no operational differences then students should get both items correct or both items incorrect, and the distribution of those students missing one item should not reflect a bias toward either the unit or non-unit proper fraction items.

Hypothesis 2 describes an asymmetrical situation in which partitive reasoning with unit fractions is hypothesized to precede partitive reasoning with non-unit proper fractions. In other words, one variable is an independent variable (e.g., partitive reasoning with non-unit fractions) and the other is a dependent variable (e.g., partitive reasoning with unit fractions). Somer's *D* is an appropriate statistic for testing asymmetrical associations between two ordered variables (Siegel & Castellan, 1988) and was used for Hypothesis 2. Somer's *D* is a measure comparing the number of agreements in order between two variables with the number of disagreements in order. Its value ranges from -1 to 1 with values closer to -1 and 1 indicating a stronger asymmetrical relationship. In the perfect relationship there would be no disagreements. Visually this would be seen in a contingency table as a staircase.

## Results

### *Scheme Hierarchy*

We investigated the scheme hierarchy by examining the overall percent correct for each variable representing the different schemes (see Table 1). In this case, higher percentages were considered to be indicative of those schemes that are, in general, developmentally more stable for the particular grade level students considered; that is, the scheme develops earlier in the hierarchy. Based on the percentages, the schemes align as hypothesized, with the part-whole fractional scheme developing earlier, followed by the partitive unit fractional scheme, and then the partitive fractional scheme.

*Table 1*
Percent Correct for Each of the Items Associated with the Schemes

| Grade 5 | | | | |
|---|---|---|---|---|
| | Pre-test (N = 44) | | Post-test (N = 43) | |
| Scheme | *M* | *SD* | *M* | *SD* |
| Part-whole | .58 | .39 | .82 | .28 |
| Partitive unit fractional | .52 | .50 | .56 | .50 |
| Partitive fractional | .27 | .45 | .58 | .50 |
| **Grade 6** | | | | |
| | Pre-test (N = 40) | | Post-test (N = 43) | |
| Scheme | *M* | *SD* | *M* | *SD* |
| Part-whole | .78 | .22 | .79 | .30 |
| Partitive unit fractional | .70 | .46 | .65 | .48 |
| Partitive fractional | .48 | .51 | .56 | .50 |

### *Part-whole Reasoning*

We hypothesized that when working in part-whole situations there are no operational differences for unit and proper fractions (Hypothesis 1). In other words, the PWFS develops for both situations at the same time. The number of students missing the unit fractional item compared to the non-unit item was found to be statistically the same for the pre-test (Binomial Test, $p = .71$) and the post-test (Binomial Test, $p = .80$) indicating that there is not sufficient evidence to reject the null hypothesis that there are no operational differences between situations involving unit and proper fractions (see Table 2). Because most students were successful with the part-whole items, it would be good to replicate this investigation with a younger sample of students in which an increased numbered of students would not have developed part-whole reasoning.

*Table 2*

Success in Non-Unit Part-Whole Situations by Success in Unit Part-Whole Situations

| | Part-whole | | | | | |
|---|---|---|---|---|---|---|
| | Pre | | | Post | | |
| | 0-Non-Unit | 1-Non-Unit | Total | 0-Non-Unit | 1-Non-Unit | Total |
| 0-Unit | 11 | 13 | 24 | 7 | 7 | 14 |
| 1-Unit | 16 | 44 | 60 | 9 | 63 | 72 |
| Totals | 27 | 57 | 84 | 16 | 70 | 86 |

*Note*. Binomial $\text{Test}_{\text{PRE}}$, $p = .71$; Binomial $\text{Test}_{\text{POST}}$, $p = .80$.

*Partitive Reasoning*

We present frequencies for student success on the PUFS and PFS items in Table 3 for the pre- and post-test. We hypothesized that, for partitive reasoning, there would be operational differences between situations involving unit and proper fractions, and further, we would expect successful operations on unit fractions to precede operations on proper fractions (Hypothesis 2). We found a statistically significant direct relationship between PUFS and PFS for both the pre-test (Somer's $D = .37$, $p < .001$, one-tailed) and post-test (Somer's $D = .40$. $p < .001$, one-tailed) indicating that, in general, for partitive reasoning there are operational differences between situations involving unit and proper fractions; and moreover, in general a PUFS tends to develop prior to a general PFS.

*Table 3*

Success in Non-Unit Partitive Situations by Success in Unit Partitive Situations

| | Partitive | | | | | |
|---|---|---|---|---|---|---|
| | Pre | | | Post | | |
| | 0-Non-Unit | 1-Non-Unit | Total | 0-Non-Unit | 1-Non-Unit | Total |
| 0-Unit | 28 | 5 | 33 | 23 | 11 | 34 |
| 1-Unit | 25 | 26 | 51 | 14 | 38 | 52 |
| Totals | 53 | 31 | 84 | 37 | 49 | 86 |

*Note*. $D_{\text{PRE}} = .37$ ($p < .001$, one-tailed); $D_{\text{POST}} = .40$ ($p < .001$, one-tailed).

## Conclusions

We based our hypotheses on theoretical distinctions between part-whole and partitive reasoning with fractions. These distinctions arose from and are supported by several teaching experiments with children (Norton, 2008; Olive & Vomvoridi, 2006; Saenz-Ludlow, 1994), and they are indicated by the two names given to the partitive schemes. The present study affirms those distinctions by measuring significant differences between students' performance on unit and non-unit *partitive* items, and by indicating no significant difference between students' performance on unit and non-unit *part-whole* items.

Our research findings imply that teachers should recognize that progressing from partitive reasoning with unit fractions to partitive reasoning with non-unit fractions may pose a developmental hurdle on par with the more obvious one of developing improper fractions. To the degree that our results generalize to students at other schools, our findings also indicate grade-level development of fractional schemes, which might inform curricular design.

Percentages of successful performance by students on the items support the theoretical hierarchy of schemes proposed by Steffe (2002) and Olive (1999) and affirmed by numerous teaching experiments. These percentages also indicate that most students have developed part-whole reasoning before entering fifth grade and that most students develop partitive reasoning during fifth or sixth grade. This finding aligns with those of the teaching experiments described in the theoretical framework for our study. Indeed, comparing pre-test and post-test mean averages in Table 1, we find the largest increase on the PFS item, in fifth and sixth grades.

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# IMPROVING SEVENTH GRADE STUDENTS' LEARNING OF RATIO AND PROPORTION USING SCHEMA-BASED INSTRUCTION AND SELF-MONITORING

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*The present study evaluated the effectiveness of an instructional intervention (schema-based instruction with self-monitoring, SBI-SM) which emphasizes the mathematical structure of problems and also provides students with a heuristic to aid problem solving. One hundred forty eight seventh-grade students and their teachers participated in a 10-day intervention on learning to solve ratio and proportion word problems, with classrooms randomly assigned to SBI-SM or a control condition. Results indicated that students in SBI-SM treatment classes made greater gains than students in control classes on a problem solving measure, both at posttest and on a delayed posttest administered four months later.*

## Introduction

Reasoning with ratios and proportions is widely regarded as a critical bridge between the numerical, concrete mathematics of arithmetic and the abstraction that follows in algebra and higher mathematics. Recent mathematics education policy documents echo this sentiment by identifying proportional reasoning as a "capstone" of elementary mathematics (National Research Council, 2001) and as a foundational topic for further success in mathematics (National Mathematics Advisory Panel, 2008). US students' difficulties in working with ratio and proportion are seen in both national and international assessments. On the 2003 TIMSS assessment, only 55% of US 8th graders were able to solve a routine proportion problem.

The topics of ratio and proportion are frequently encountered in elementary and middle schools in the form of word problems. Word problem solving has proved to be a significant challenge for students, in part because it requires students to understand the language and factual information in the problem, identify relevant information in the problem to create an adequate mental representation, and generate, execute, and monitor a solution strategy. Yet despite their difficulty, word problems are critical in helping children connect different meanings, interpretations, and relationships to the mathematical operations.

There is a rich history in the field of mathematics education of interventions designed to help students become more successful at understanding and being able to solve ratio and proportion word problems (Lesh, Post, & Behr, 1988; Behr, Harel, Post, & Lesh, 1992; Litwiller & Bright, 2002; Lamon, 2007). Although research on ratio and proportion word problem solving was particularly prominent in the 1980s and early 1990s, scholars in mathematics education continue to explore ways to improve students' learning of this important, yet challenging topic. All told, the mathematics education literature provides many compelling examples of methods and curricula that have the potential to enhance students' performance and understanding of word problem solving in the domain of ratio and proportion (Behr, Harel, Post, & Lesh, 1992; Lamon, 2007).

Despite the large literature on word problem solving with ratios and proportions, research on how to address low achieving students' difficulties with word problem solving is somewhat conflicting. In the field of mathematics education, the approach advocated by the NCTM *Standards* (and that is used in most of the NSF-funded reform curricula) is a student-centered, guided discovery approach for teaching students problem solving (NRC, 2001). However, recommendations for this kind of instructional approach are somewhat at odds with the literature on problem solving instruction for low achieving students in the field of special education, which has found that low achieving students benefit far more from direct instruction and practice at problem solving than competent problem solvers (National Mathematics Advisory Panel, 2008). In fact, research conducted in reform-oriented classrooms suggests that many low achieving students (particularly those with learning disabilities) may assume passive roles and may encounter difficulties with the cognitive load of the discovery-oriented activities and curricular materials (e.g., Baxter, Woodward, Voorhies, & Wong, 2002). Yet, despite the robust literature from special education in support of more direct instruction for low achieving students and those with learning disabilities, many in the mathematics education community have strong negative reactions to this instructional approach, in part because of perceived associations and historical links between direct instruction and the development of rote, inflexible knowledge.

The goal of the present study was to design an instructional intervention to meet the diverse needs of students in classrooms using the research literatures from both special education and mathematics education. Our instructional intervention uses a type of direct instruction that involves explicit strategy instruction, which has strong support in the special education literature for increasing the performance of at-risk populations. However, our approach is carefully designed to address three critical concerns with the ways that direct instruction has sometimes been (mis)applied in mathematics instruction.

First, one concern about some direct instructional approaches is that the same procedure (e.g., cross-multiplication) is used to solve all problems on a page. As such, students do not have the opportunity to compare and contrast (and thus learn to discriminate) among different types of problems and approaches, perhaps leading to exclusive reliance (and perhaps rote memorization, without understanding) on a small set of problem solving strategies. Our instructional approach (described below) addresses this concern by exposing students to multiple problem types and strategies and by encouraging reflection on the similarities and differences between problems types and strategies. There is growing evidence from both the education and psychological literature that exposure to multiple strategies facilitates students' learning of mathematics (e.g., Rittle-Johnson & Star, 2007).

A second concern about some direct problem solving instructional approaches is the use of superficial cues such as key words (e.g., *in all* suggests addition, *left* suggests subtraction, *share* suggest division; Lester, Garofalo, & Kroll, 1989) that students are encouraged to use to select an operation or a solution procedure (e.g., "cross multiply"). The use of keyword methods, focusing on surface level features, does not emphasize the meaning and structure of the problem and thus may not help students to reason and make sense of story situations to be able to successfully solve novel problems (e.g., Ben-Zeev & Star, 2001). Our approach moves away from keywords and superficial problem features and more explicitly focuses on helping students see the underlying mathematical structure of problems.

A third concern about some direct instructional approaches is the reliance on a general problem solution method that involves the use of a heuristic and multiple strategies based on George Pólya's seminal principles for problem solving. Pólya's four-step problem solving model includes the following steps: *understand the problem*, *devise a plan*, *carry out the plan*, and *look back and reflect*. However, this method has come under scrutiny for several reasons, including the failure of general heuristics to reliably lead to improvements in students' word problem solving performance (Lesh & Zawojewski, 2007).

Our instructional approach, schema-based instruction (SBI), which is intermediate in generality between key word approaches and general heuristic methods described above, addresses the above-noted concerns with some aspects of direct instruction in that it entails specific problem solving strategies that are linked to particular types or classes of problems (e.g., ratio, proportion). Specifically, SBI in this study includes the following three features.

First, our instructional model uses schema training to help students see the underlying mathematical structure of word problems, which is critical to effectively deploy content knowledge. Schema theory suggests that cognizance of the role of the mathematical structure (semantic structure) of a problem is critical to successful problem solution. Schemas are domain or context specific knowledge structures that organize knowledge and help the learner categorize various problem types to determine the most appropriate actions needed to solve the problem (Marshall, 1995). For example, organizing problems on the basis of structural features (e.g., *rate* problem, *compare* problem) rather than surface features (e.g., the problem's cover story) can evoke the appropriate solution strategy.

One way that problem solvers can access schema knowledge is through the use of schematic diagrams, which have been found to be particularly useful in highlighting underlying problem structure and is deemed by many to be central to mathematical problem solving (e.g., Stylianou & Silver, 2004). It is important to note that a schematic diagram is not merely a pictorial representation of the problem storyline that may focus on concrete, irrelevant details, but rather depicts the relationships between critical elements of the problem structure necessary for facilitating problem solution. Research on the effectiveness of schema training in isolation or combined with schematic representations has shown that it is effective for students of different ability levels (e.g., Jitendra, Griffin, Haria, Leh, Adams, & Kaduvetoor, 2007; Xin, Jitendra, & Deatline-Buchman, 2005).

A second feature of our instructional approach is our focus on multiple solution strategies. Comparing and contrasting multiple strategies is a central feature of mathematics reform efforts (Silver et al., 2005) and is advocated in the National Council of Teachers of Mathematics (NCTM) Standards (2000). An emphasis on having students actively compare, reflect on, and discuss multiple solution methods is also identified as a key feature of expert mathematics instruction (e.g., Silver et al., 2005) and considered to be an important differentiating feature of teachers in countries that have performed well on international assessments such as TIMSS (Richland, Zur, & Holyoak, 2007). Further, two recent studies by Rittle-Johnson and Star (2007; Star & Rittle-Johnson, in press) provide empirical evidence for improving student learning when instruction emphasizes and supports comparing and contrasting solutions. Students who learned to solve equations or to compute estimates by comparing and contrasting multiple solution methods outperformed students who were exposed to the same solution methods but presented sequentially.

Finally, an additional feature of our instructional model is the use of student "think-alouds" to help in the development of self-monitoring skills, a critical component of metacognitive ability (e.g., Kramarski, Mevarech, & Arami, 2002). Teachers model how and when to use each problem solving strategy and work with students to reflect on the problem before solving it. Recent research suggests that instruction that includes a focus on metacognitive skills has an added positive effect on students' mathematical problem solving performance (e.g., Kramarski, Mevarech, & Arami, 2002). As such, our approach emphasizes self-monitoring, an important aspect of metacognitive processes, by having students direct their problem solving behavior to focus on comprehending the problem, representing the problem, planning to solve the problem using appropriate strategies, and reflecting on the solution via the use of "think-alouds."

*Purposes of the Present Study*

The primary purpose of the current study was to evaluate the effectiveness of schema based instruction on ratio and proportion word problem solving. The present study extended prior work of Jitendra and colleagues (e.g., Xin et al., 2005) on SBI-SM in the multiplicative domain in several ways. First, while Xin et al. exclusively focused on students with disabilities and low achieving students, we targeted students of diverse ability levels in general education classrooms. Second, we extended the focus of Xin et al. beyond ratio and proportion word problem solving to also include foundational concepts (ratios, equivalent fractions, rates, fraction and percents) involved in ratio and proportion problem solving. Third, in addition to the schematic diagrams for organizing information (as was used in Xin et al.), we also incorporated multiple solution strategies and flexible application of those strategies. Fourth, instruction was provided by classroom teachers, rather than by research assistants (as was done in Xin et al.).

We hypothesized that students receiving SBI-SM instruction would make greater gains in problem solving performance than their peers receiving "business-as-usual" mathematics instruction (control condition). The following research questions were addressed in this study: (1) What are the differential effects of SBI-SM and control treatment on the acquisition of seventh grade students' ratio and proportion word problem solving ability? (2) Is there a differential effect of the treatment (SBI-SM and control) on the maintenance of problem solving performance four months following the end of intervention?

## Method

*Participants*

Seventh grade students from eight classrooms and their teachers in a public, urban school participated in the study. For mathematics instruction, students in the school were grouped into classes based on three ability levels: Academic (high), Applied (average), and Essential (low). In the present study, each treatment group (SBI-SM and control) included two sections of average and one each of high and low ability classrooms to adequately represent the different levels in the school. The sample of 148 students (79 girls, 69 boys) included those who were present for both the pretest and posttest. The mean chronological age of students was 153 months. The sample was primarily Caucasian (54%), and minority students comprised 22% Hispanic, 22% African American, and 3% American Indian and Asian. Approximately 42% of students received free or subsidized lunch and 3% were English language learners.

All six teachers at the participating school were responsible for teaching mathematics in the different ability level classrooms. The teachers (3 females and 3 males) were all Caucasian, with a mean of 8.58 years of experience teaching mathematics (range 2 to 28 years). Three of the teachers held secondary education certification, four had a master's degree, and only three had a degree in mathematics.

*Design*

A pretest-intervention-posttest-retention test design was used. After matching classrooms in pairs on the basis of ability level, one classroom from each pair was randomly assigned to the SBI-SM or control treatment.

*Materials and Measures*

The SBI-SM intervention unit content provided the basis for solving problems involving ratios and proportions. We identified specific concepts and problem-solving skills by reviewing the textbook used in the 7th grade classrooms and appropriately mapping the relevant topics to the ratio and proportion unit. The unit included exercises to build an understanding of the concepts of ratios and rates that are critical to understanding proportions and for engaging in proportional reasoning as well as to solving ratio and proportion word problems.

To assess mathematics competence on ratio and proportion problems, students completed a researcher-designed mathematical PS test prior to instruction (pretest), immediately following instruction (posttest), and four months following instruction (delayed posttest). The PS test consisted of 18 items derived from the 8th grade TIMSS, NAEP, and state assessments and assessed ratio and proportion concepts and word problem solving knowledge similar to the instructed content. Students had 40 minutes to complete the same 18-item test at pretest, posttest, and delayed posttest.

*Professional Development.* Teachers assigned to the SBI-SM condition attended a 1-day session that described the goals of the study and how to mediate instruction and facilitate discussions and group activities. Teachers in the control condition attended one half-day training session that focused on the goals of the study and the importance of implementing the standard "business-as-usual" curriculum faithfully.

*Procedure*

Students in both conditions received instruction on ratio and proportion and were introduced to the same topics (i.e., ratios, rates, solving proportions, scale drawings, fractions, decimals, and percents) during the regularly scheduled mathematics instructional period for 40 minutes daily, five days per week, across 10 school days, delivered by their classroom teachers in their intact math classes. Lessons in both intervention and control classrooms were structured as follows: (a) students working individually to complete a review problem followed by the teacher reviewing it in a whole class format, (b) the teacher introducing the key concepts/skills using a series of examples, and then (c) assigning homework. Further, students in both conditions were allowed to use calculators.

*SBI-SM*. For the SBI-SM condition, the researcher-designed unit replaced the students' regular instruction on ratios and proportions. Lessons were scripted to provide a detailed teaching procedure (i.e., questions to ask, examples to present) for the purpose of ensuring consistency in implementing the critical content. However, rather than read the scripts verbatim, teachers were encouraged to be familiar with them and use their own explanations and elaborations to implement SBI-SM. To solve ratio and proportion problems, students were taught to identify the problem schema (*ratio* or *proportion*) and represent the features of the problem situation using schematic diagrams. Students first learned to interpret and elaborate on the main features of the problem situation. Next, they mapped the details of the problem onto the schema diagram. Finally, they solved ratio and proportion problems by applying an appropriate solution strategy (e.g., unit rate, equivalent fraction, or cross multiplication). The instructional approach encouraged students' "think-alouds" to monitor and direct their problem-solving behavior along the following dimensions: (a) problem comprehension (e.g., Did I read and retell the problem to understand what is given and what must be solved?, Why is this a *ratio* problem?, How is this problem similar to or different from one I already solved?), (b) problem representation (e.g., What schematic diagram can help me adequately represent information in the problem to show the relation between quantities?), (c) planning (e.g., How can I set up the math equation? What solution strategy can I use to solve this problem?), and (d) problem solution (e.g., Does the answer make sense? How can I verify the solution?)

*Control*. Students in the control group received instruction from their teachers who used procedures outlined in the district-adopted mathematics textbook (Bailey et al., 2004). Each lesson in the chapter on Ratios and Proportions begins with a real-life application of the mathematics to introduce and motivate the day's topic. The text then suggests a direct instruction approach for defining key concepts. Subsequently, several worked out examples are presented to expose students to the target problem types of the lesson, following by a period of guided practice.

## Results

The mathematics problem solving data were first examined for initial group comparability on the problem solving pretest measure by contrasting the SBI-SM with the control treatment, using a 2 group (SBI-SM, control) x 3 ability level (high, average, low) analysis of variance (ANOVA). Next, we assessed the acquisition and maintenance effects of the problem solving skill by conducting separate two-factor analysis of covariance (ANCOVA) with the problem solving pretest serving as a covariate for both the posttest and delayed posttest.

### *Student Learning*

Results of the ANCOVA applied to the posttest scores demonstrated statistically significant main effects for group, $F$ (1, 141) = 6.30, $p$ = .01, and ability level, $F$ (2, 141) = 16.53, $p$ < .001. The pretest was found to be a significant covariate, $F$ (1, 142) = 32.16, $p$ < .001. The adjusted mean scores indicated that the SBI-SM group significantly outperformed the control group. A low medium effect size of .45 was found for SBI-SM when compared with control. Post-hoc analyses using the Bonferroni post hoc criterion for significance indicated that the mean problem solving scores for the ability levels were significantly different (High > Average > Low). No significant interaction between group and ability level was found, $F$ (2, 141) = 2.01, $p$ = 0.14.

In addition, results from the delayed posttest administered four months following the completion of the intervention indicated statistically significant effects for group, $F$ (1, 135) = 8.99, $p$ < .01, and ability level, $F$ (2, 135) = 24.16, $p$ < .001. The pretest was found to be a significant covariate, $F$ (1, 135) = 34.06, $p$ < .001. The adjusted mean scores indicated that the SBI-SM group significantly outperformed the control group. A medium effect size of .56 was found for SBI-SM when compared with control. Post-hoc analyses using the Bonferroni post hoc criterion for significance indicated that the mean problem solving scores for the ability levels were significantly different (High > Average > Low). No significant interaction between group and ability level was found, $F$ (2, 135) = 2.04, $p$ = 0.13.

## Discussion

This study replicates and extends prior work by Jitendra and colleagues and others on the effects of schema-based instruction on students' learning of mathematics. The focus here was on ratio and proportion, a critically important but quite challenging content area for students. Our SBI-SM approach is relatively unique in its synthesis of best practices from the at-times conflicting special education and mathematics education literatures. SBI-SM uses explicit strategy instruction, which has been shown by special education researchers to be effective with low achievers, but with an emphasis on multiple strategies and the underlying mathematical structure of word problems—two features with strong foundation in the mathematics education and cognitive science research literatures.

Our research questions addressed the differential effects of SBI-SM and control treatment on the acquisition and maintenance of seventh grade students' ratio and proportion word problem solving performance. Consistent with our prediction, we found a statistically significant difference in students' problem-solving skills favoring the SBI-SM condition, suggesting that SBI-SM represents one promising approach to teaching ratio and proportion word problem solving skills. In addition, our results indicated that the benefits of SBI-SM persisted four months after the intervention. The effect sizes comparing the SBI-SM treatment with the control group were 0.45 on the immediate posttest and 0.56 on the delayed posttest. It is important to note that our control group received instruction on the same topics and for the same duration of time as the SBI group. At the same time, the effects for SBI-SM were not mediated by ability level, suggesting that it may benefit a wide range of seventh grade students. In sum, our use of direct instruction, but modified to move students beyond rote memorization to developing deeper understanding of the mathematical problem structure and fostering flexible solution strategies, helped students in the SBI-SM group improve their problem solving performance and maintain it over time.

A noteworthy feature of SBI-SM that future experimental research should investigate is its focus on multiple strategies. Prior work in special education has not encouraged the use of such an approach, particularly for low achieving students, because many special educators are skeptical of its benefits. One possible explanation for this skepticism among special educators is the inability of many low achieving students to meet the cognitive overload involved in learning multiple strategies (e.g., Baxter, Woodward, Voorhies, & Wong, 2002). Although ability level did not mediate the effects of SBI-SM in this study, visual inspection of the data suggests that, contrary to prior work in special education, the progress of students in the low ability classrooms was comparable to the performance of low achieving students in the control condition. This finding suggests that the cognitive overload of learning multiple strategies in SBI-SM may not necessarily be a concern, because it did not impede the learning of low achieving students in the SBI-SM condition.

In conclusion, the focus on ratio and proportion problems in the present study extends into middle school and to students with diverse needs the prior work on word problem solving with students with disabilities or low achieving students in other mathematical domains from the elementary and middle school curriculum. The present findings suggest that students can benefit from instruction that emphasizes the underlying mathematical structure of word problems, an important feature of SBI-SM.

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# MEXICAN SIXTH GRADE STUDENTS' UNDERSTANDINGS OF FRACTION NOTATIONS AS NUMBERS THAT EXPRESS QUANTITY[1]

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*We report on a study that consisted of administering tests to 297 sixth grade students from 13 different schools in Mexico. Pupils were asked to identify the quantity represented by common fractions (e.g., 1/2, 1/4, 1/3, 3/4). Results suggest that many students are finishing elementary school with a deficient understanding of fractions: some lagging behind so significantly that they have not developed understandings that allow them to readily and correctly interpret the quantitative meaning of the most common fraction notations, including "1/2." We discuss the implications of these results for students' opportunities to learn mathematics in middle school.*

## Background

Mathematics educators have long been concerned with the ways in which fractions are learned. According to several researchers, students must come to develop relatively sophisticated understandings of fractions in order to have access to important mathematical ideas, particularly those encompassed by the multiplicative field (Lamon, 2007; Thompson and Saldanha, 2003). In addition, studies based on written assessments administered to relatively large samples of students have identified the shortcomings of educational systems around the globe with respect to fraction instruction. For instance, several studies have documented the difficulties experienced by numerous students in grades 5 through 9 when identifying fractions on the number line (e.g., Gould, 2005; Hannula, 2003; Hart, 1989). In the case of the Mexican National Assessment of Educational Quality and Achievement (Backhoff, Andrade, Sánchez, Peon, & Bouzas, 2006) it was reported that 76.9% of Mexican sixth-grade students did not meet the probability criteria $P \geq .67$ of being correct when answering items that involved identifying a fraction such as 3/5 on the number line.

Studies based on assessing relatively large samples of students have typically been useful for gauging the existing limitations of fraction instruction in an educational system, and for measuring longitudinal change. However, these studies have rarely been helpful for developing strategies for improvement, particularly in terms of specifying instructional design and professional development challenges.

The study that we report in this paper was conducted for the purpose of identifying how ready Mexican students finishing elementary school (educación primaria; sixth grade) were to learn the mathematical ideas prescribed in the National Curriculum for Middle Schools (grades 7 to 9; Secretaría de Educación Pública, 2006). In particular, we were interested in documenting differences in how students made sense of a fraction notation of the kind "$\frac{a}{b}$", as a number that expresses quantity. We considered that the way in which a student made quantitative sense of these notations would be consequential in his or hers opportunities to learn the relatively sophisticated mathematical ideas encompassed by the multiplicative field as prescribed in the Middle School Curriculum (e.g., fractions, percents, decimals, ratios, rates, averages and quartiles). It is worth clarifying that such a consideration is consistent with the instructional tenet

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

that students who can attribute quantitatively sound meanings to the numbers that they encounter in instruction will have better opportunities to understand increasingly sophisticated mathematical ideas than those who do not (cf., National Council of Teachers of Mathematics, 2000; Thompson, Philipp, Thompson, & Boyd, 1994).

## Methodology

The study was conducted in the spring of 2006. It consisted of administering a test to 297 sixth grade students from 13 classrooms in different schools. The schools were selected from a pool of about 60, to which the research team had access. Some of those schools were in the highlands of the Mexican State of Chiapas and the rest on the south side of Mexico City. An effort was made to select classrooms with diverse student populations. As a consequence, the sample included classrooms that were in different kinds of schools (Table 1).

Table 1. *Classification of the 13 Classrooms that Formed the Sample by School Characteristics and by Classroom Size*

| | Chiapas N=6 | Mexico City N=7 |
|---|---|---|
| Urban Public Schools | 3 | 6 |
| Private Schools | 1 | 1 |
| Rural Schools | 2 | 0 |
| SES | | |
| High | 1 | 1 |
| Middle | 1 | 3 |
| Low | 4 | 3 |

| | Chiapas N=6 | Mexico City N=7 |
|---|---|---|
| Bilingual Schools | 1 (Spanish-Tzotzil) | 0 |
| Classroom size | | |
| 11-16 students | 3 | 0 |
| 20-25 students | 2 | 4 |
| 28-33 students | 1 | 3 |

The great majority of the students (86.2%) were eleven or twelve years old at the time of the study. There were some students that were ten years old, and others that were older than twelve (Table 2). It is worth clarifying that, in Mexico, the presence of significantly over-aged students is common in elementary schools that serve children living in poverty.

Table 2. *Distribution of the 297 Students in the Sample by Age*

| **Age** | **10** | **11** | **12** | **13** | **14** | **15** | **16** | **18** | **19** |
|---|---|---|---|---|---|---|---|---|---|
| **N** | 3 (1%) | 110 (37%) | 146 (49.2%) | 28 (9.4%) | 2 (0.7%) | 4 (1.3%) | 2 (0.7%) | 1 (0.3%) | 1 (0.3%) |

Prior to designing the test, the National Mathematics Curriculum for Seventh Grade (*primer año de secundaria*; Secretaría de Educación Pública, 2006) was examined (cf. Cardoso, 2008). The purpose of the examination was to identify the kinds of quantitative understandings about fractions that students would have had to develop in elementary school in order to be ready to engage meaningfully with the mathematical ideas prescribed for middle school.

It became evident that the designers of the National Mathematics Curriculum for Seventh Grade did not assume that students who enter middle school would have mastered all the content prescribed in the Elementary School Curriculum. Instead, they chose to review parts of the elementary school curriculum content. For instance, the Middle School Curriculum indicates that teachers should start instruction by helping students identify fractions on the number line, despite the fact that, according to the Mexican Elementary School Curriculum (Secretaría de Educación Pública, 1993), students would have already been taught to do so in fifth and sixth grades.

Based on our examination of the Middle School Curriculum, we conjectured that students would have to enter middle school understanding, at least, the basic rationale of fraction notation; namely, that there is a numerator and a denominator, that each represents something in particular, and that together they express quantity. We also conjectured that students would have to be capable of correctly imagining the quantities represented by relatively common fraction inscriptions as expressing something that would be smaller than, as big as, or bigger than 1/2 and 1. We concluded that without those understandings it would be unreasonable to expect students to meaningfully engage in instructional activities that involved identifying fractions on the number line, as well as others that would come latter in the curriculum.

The test included 19 items. The first 6 involved comparing the fractional amount of milk contained in two milk cartons, in terms of one having more milk than the other, or of the two having the same amount. Students were shown the drawing of two milk cartons, each with a fraction inscription on the bottom (Figure 1). They were asked to mark the level of the milk in each carton, accordingly to the fraction shown, and to indicate which was fuller or if both had the same amount. The fractions that students compared were: 1/3 vs. 1/2; 3/4 vs. 1/4; 1/3 vs. 2/3; 2/4 vs. 1/2; 4/9 vs. 3/4; 5/10 vs. 1/2. All the fractions were relatively common, and all the comparisons could be solved correctly by assessing each fraction inscription as representing a quantity that is smaller than, as big as, or bigger than 1/2. In addition, the items involved a familiar context for students (containers from which liquids are served), although it was probably a context in which pupils were not used to dealing with fraction inscriptions.

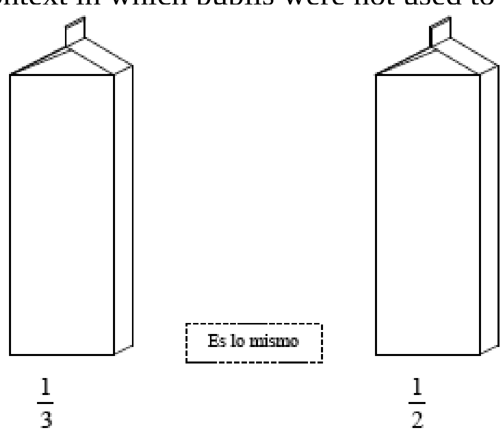



*Figure 1.* One of the milk carton items.

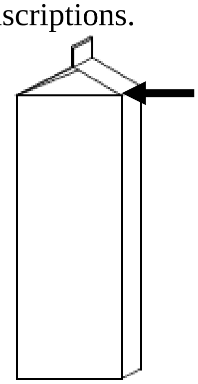
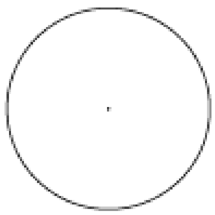


*Figure 2.* The level of the milk when the carton is full.

Students were given a detailed explanation of what they had to do before solving the milk carton items. A real milk carton was used to give this explanation. Among other things, students were told about where the level of the milk would be when a milk carton was full (Figure 2). In addition, there were at least three adults[2] present in every classroom in which the test was administered. These adults aided individual students who had doubts about what they were expected to do.

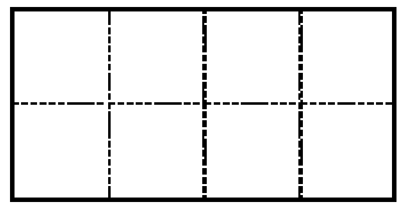
The seventh item involved comparing the amount of milk that was used to make each of three cakes. Students were shown a picture of three identical drawings of cakes, each with one of the following inscriptions on the bottom: $\frac{5}{4}$ *of a liter*, $\frac{8}{9}$ *of a liter*, *liter*. Students were told that the inscriptions showed the amount of milk that was used to make each of the cakes, and were asked to mark the one in which they thought that the most milk was used.

The remaining 12 items were more similar to typical school exercises. Six consisted of circles marked with a fraction inscription on the bottom (Figure 3). Students were asked to shade the area corresponding to the fraction shown. Those fractions were: 1/2, 2/4, 3/4, 1/3, 2/3, and 3/3. The other six items consisted of rectangles marked with a fraction inscription on the bottom (Figure 4). Again students were asked to shade the area corresponding to the fraction shown. Those fractions were: 1/2, 1/4, 3/4, 1/8, 4/8, 8/8.



$\frac{1}{2}$

Figure 3. One of the circle items.



$\frac{1}{4}$

Figure 4. One of rectangle items.

The circle and rectangle items were included for the purpose of identifying possible differences in students' performance when expressing fractional values using representations that are common in schools than when using representations that are not (such as milk cartons). We considered that if a student's performance were to be very different in the two kinds of items, it could be reasonable to conjecture that a poor performance in the *Milk Carton Items* would be the result of the student's difficulties to make sense of the activity. In contrast, if a student's performance in both kinds of items were consistent—in general terms—it would be reasonable to consider that a student's performance in the *Milk Carton Items* reflected his ways of making sense of fraction inscriptions as numbers that express quantity.

## Results

A coding scheme composed of four categories was developed for classifying the tests. The scheme emerged from the analysis of the first 38 tests that were administered. The analysis involved searching out similarities and differences among the test responses that reflected possible similarities and differences in students' quantitative understanding of fractions—similarities and differences that would be important to account for in instruction. The four categories turned out to be useful for classifying 292 of the 297 tests that were administered. Each of these 292 tests was classified, without difficulty, as belonging to one and only one category. The 5 remaining tests (1.7%) were discarded because of inconsistencies (e.g., the test was not completed).

*Category A*

The tests that were classified into Category A (N=59; 20%) were those in which the students responded correctly all the *Milk Carton Items* (or all but one of them), as well as the *Three Cakes Item* and the *Circle* and *Rectangle Item.* These students' responses suggested that they correctly imagined the quantities represented by relatively common fraction inscriptions as expressing something that would be smaller than, as big as, or bigger than 1/2 and 1. Generally speaking, these students seemed to be ready to engage in instruction involving identifying fractions on the number line.

*Category B*

The tests that were classified into Category B (N=57; 19.5%) were those in which the students responded correctly to all the *Milk Carton Items* (or all but one of them), as well as the *Circle* and the *Rectangle Items*, but that responded to the *Three Cakes Item* incorrectly. Specifically, all of them identified the cake with the inscription “1 liter” as the one in which the most milk was used.

The responses on the tests that were classified into this category suggest that the students that answered them properly construed relatively common fraction inscriptions as expressing quantities smaller than, as big as, or bigger than 1/2. However, the responses also suggest that the students might have conceived of fraction inscriptions as expressions that always represent quantities that are smaller than or equal to 1. It is reasonable to conjecture that the students whose tests were classified into Category B may not have been prepared to engage, readily and meaningfully, in instruction involving the mathematical ideas prescribed for seventh grade in the Mexican Curriculum. In order to be prepared, they would have to be supported in understanding why and how a fraction inscription can, legitimately and soundly, express a quantity that is bigger than 1.

*Category C*

The tests that were classified into Category C (N=87; 29.7%) were those in which, throughout the *Milk Carton*, the *Circle*, and the *Rectangle Items*, the students consistently represented the fraction “$\frac{1}{2}$” as “one half”, but misrepresented most of the other fraction inscriptions on the test (Figure 5). These students seemed to have had developed sound quantitative imagery about a limited number of fractional inscriptions (i.e., 1/2 and, in some cases, 1/4 and 3/4).

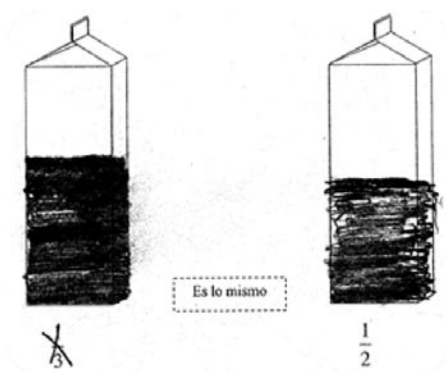



*Figure 5.* Representation of 1/3 as more than 1/2 in a test classified into Category C.

The responses on the tests that were classified into this category (Category C) suggest that the students that provided them might have not yet developed an understanding of the system of fractional notation that would allow them to correctly construe fraction notations whose quantitative meaning they did not previously know. An example of such an understanding would be to consistently conceptualize the denominator of a fraction as expressing a number of equal parts in which a whole was divided, and the numerator as a certain number of those parts. This was not something that the students whose tests were classified into Category C seemed to do. Instead, when encountering fraction notations whose meaning they did not previously know, these students seemed to conceptualize them following the same quantitative rationale they used for interpreting natural numbers: the bigger the number the bigger the amount it represents. For example, in the *Three Cakes Item,* many of these students chose the “8/9 Liter” cake as the one in which the most milk was used, probably because—in the natural number system—8 and 9 represent larger quantities than 5 and 4, and than 1.

It is reasonable to conjecture that the students whose tests were classified into Category C may not have been prepared to engage, readily and meaningfully, in instruction involving the mathematical ideas prescribed for seventh grade in the Mexican Curriculum. To be prepared, they would have to be supported in developing understandings about the quantitative rational of fraction notation that would allow them to, at least, correctly interpret relatively common fractions—whose meaning they did not previously know—as representing quantities that are smaller than, as big as, or bigger than 1/2 and 1.

*Category D*

The tests that were classified into Category D (N=88; 30.1%) were those in which, throughout the assessment, the students did not consistently represent any fraction correctly, including “$\frac{1}{2}$” (Figure 6). The responses on the tests that were classified into this category suggest that the students that provided them might have not yet developed imagery about the quantitative meaning of fraction notations that would allow them to correctly construe them as symbols that express quantity—not even with respect to a limited number of inscriptions. Many of these students seemed to approach all fractions as symbols that followed the same quantitative rationale as natural numbers: the bigger the number bigger the amount it represents.

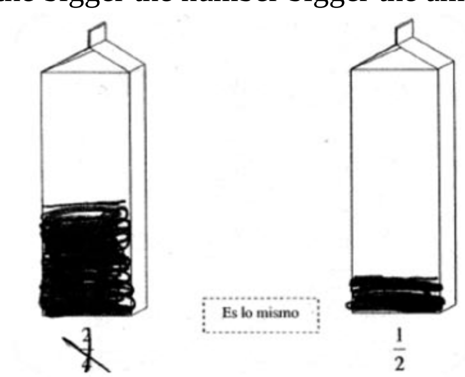



*Figure 6.* Representation of 2/4 as more than 1/2 in a test classified into Category D.

It is reasonable to conjecture that the students whose tests were classified into Category D may not have been prepared to engage, readily and meaningfully, in instruction involving the mathematical ideas prescribed for seventh grade in the Mexican Curriculum. To be prepared, they would have to be supported in developing understandings about the quantitative rationale of

fraction notation that would allow them to, at least, correctly interpret relatively common fractions as representing quantities that are smaller than, as big as, or bigger than 1/2 and 1.

Once the tests were coded, the distribution of students in each classroom—according to the four categories—was identified (Figure 7). It became apparent that there were important differences across classrooms, some with more than half of tests classified into Categories A and B, and others with only a few tests (or no) tests in these Categories. Six classrooms fell into the former group, of which five were in schools attended by students with high or middle SES. Seven fell in to the latter group, of which six were in schools attended by students with low SES (Figure 7). It also became apparent that, despite these differences, all the classrooms had students whose tests were classified into Categories C and D. These observations suggest that the problem of sixth grade students lagging behind in their quantitative understanding of fractions might be widespread across the Mexican Educational System, but more acute in schools attended by children of low income families.

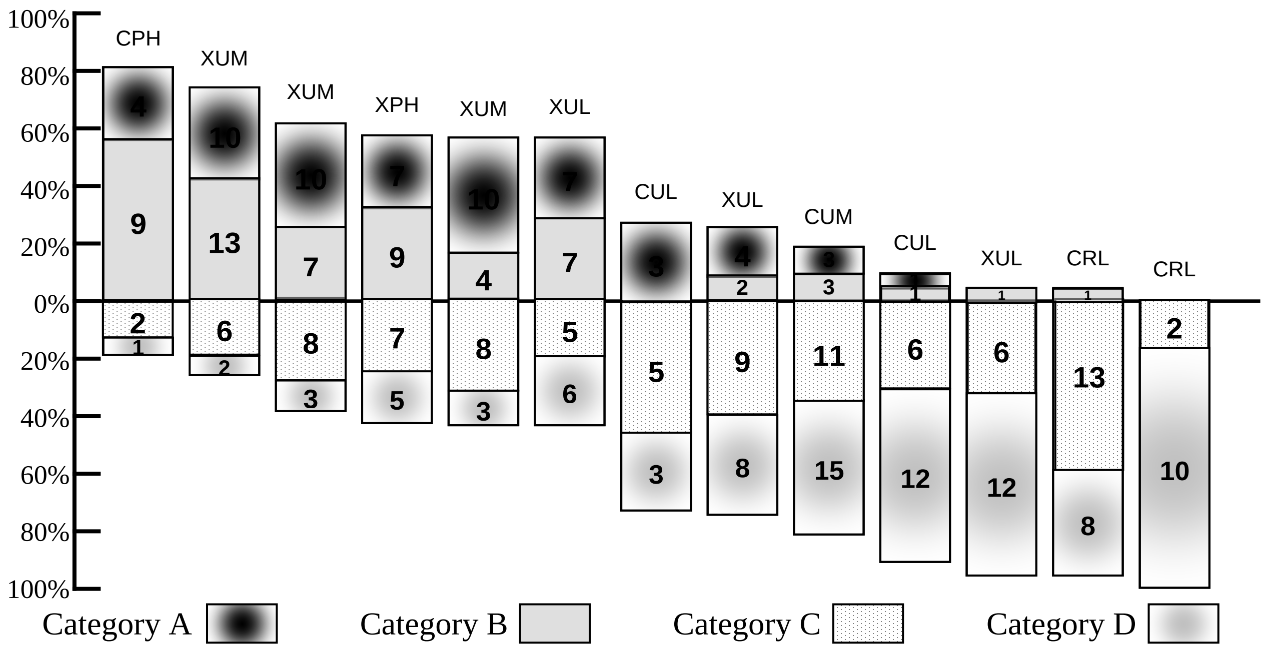



*Figure 7.* The thirteen classrooms by how the tests of its students were classified into the four categories, both in absolute terms (numbers inside the boxes) and proportionally (size of the boxes). The first letter on the label of each classroom indicates the school's location (C: Chiapas; X: Mexico City); the second letter, the type of School (P: private; U: urban public; R: rural public); and the third letter, the typical SES of the students (H: high; M: middle; L: low).

**Conclusions**

Our results suggest that many students in the Mexican Educational System are entering middle school lagging behind in their understanding of fractions as numbers that express quantity. Some students are lagging behind significantly, to the point that they have not developed understandings that allow them to interpret, readily and correctly, the quantitative meaning of the most common fraction inscriptions; in many cases, including “$\frac{1}{2}$”. The results also suggest that the problem is widespread across the educational system, but more acute in socially and economically deprived contexts.

Based on our results, it is reasonable to conjecture that the Mexican Educational System is facing a serious problem regarding mathematics instruction, at least instruction concerning the ideas encompassed by the multiplicative field. It seems that the system is being ineffective in supporting many pupils in developing the mathematical understandings that are necessary if they are to fulfill the learning goals that the system itself has established. As a consequence, many students are being asked to engage in instruction involving mathematical ideas that they are not ready to make sense of. This situation is likely to create frustrating learning experiences in mathematics classrooms for many students. In addition, it can limit students’ opportunities not only to learn the mathematical ideas prescribed for their grade level, but also to make sense of more basic notions.

We are uncertain about how this problem could be solved at the systemic level. However, we believe that it would be worthwhile to develop instructional resources that help Mexican sixth grade teachers identify and respond to the particular challenges they face in their classrooms, regarding fraction instruction. In addition, it would be important to implement professional development programs that help Mexican sixth grade teachers learn about the challenges they face with respect to fraction instruction, and about the resources they can use to meet those challenges.

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**Notes**

[1] The Mexican CONACYT supported the study reported here, under project No. 53448. The opinions expressed do not necessarily reflect the views of the CONACYT.

[2] We are grateful to Claudia Zúñiga, Luz Pérez, and Filiberto Méndez for their help in conducting this study. We are also grateful to the authorities that granted us access to the schools in which the study was conducted.

# THE IMPACT OF PROFESSIONAL DEVELOPMENT ON TWO TEACHERS' UNDERSTANDING AND USE OF REPRESENTATIONS

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*In this study, we consider the experience of two alternatively-certified teachers' experiences in a professional development course. We looked at their abilities to interpret drawn representations and the impact they reported those abilities had on their teaching. We relied on interviews with the participants and videotapes of their professional development experience. Findings indicate that the teachers both made significant strides in their abilities to interpret drawn representations of fraction operations and in their reported use of them in their classrooms.*

## Background

A nationwide shortage of teachers combined with high turnover rates in the areas of special education, mathematics and science (Ingersoll, 2001), have resulted in a practice of recruitment of teachers from non-traditional backgrounds, especially in urban areas. While the interest in research on these teachers has risen, the research in this area is still minimal with questionable results. Nonetheless, results of studies suggest alternatively certified teachers are more willing to work in the critical needs areas with low-achieving students (Zeichner & Conklin, 2005).

This influx of alternatively-certified teachers into mathematics classrooms creates particular challenges given that the vision of classroom mathematics has changed (NCTM, 2000) and these teachers are often sent into classrooms with little more than classroom management guidance. The mandates placed on schools through various national, state, and local accountability efforts have driven the need for teacher professional development (Hill & Ball, 2004), but that professional development is generally designed for teachers with traditional certification and backgrounds. Clearly, the need for professional development of alternatively certified teachers is also great, but little is known about the kinds of support they may need – particularly in their first few years – and whether professional development for regularly certified teachers is appropriate when the alternatively-certified teachers have not had previous opportunities to develop knowledge for teaching (Shulman, 1986; Ball, Lubienski, & Mewborn, 2001). Fortunately, much modern professional development has moved away from the traditional "make and take" model of professional development that focused on development of a single lesson to use in the classroom. Modern professional learning instead often seeks to develop the mathematical knowledge for teaching (Ball, Lubienski, & Mewborn, 2001) of the participating teachers – starting from wherever the teachers are in their current development. This means that these efforts seek to address teacher content knowledge, develop their teaching strategies, and explore student thinking.

Built into this new vision of professional learning and relying on guidelines about high-quality professional development (e.g., Elmore, 2002; Hawley & Valli, 1999; Hill, 2004), the InterMath approach provides teachers with a full semester-long hands-on learning experience designed to engage them in development of content knowledge and pedagogical knowledge simultaneously. To meet the professional development goals, InterMath engages teachers in mathematical problem solving and exploration through the use of software including Fraction

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Bars (Orrill, undated), wikis, and spreadsheets with which teachers can model and analyze a variety of mathematical situations. In the InterMath class sessions, teachers begin to experience the learning environments advocated in reform documents (e.g., NCTM, 2000). For example, InterMath participants engage in problem-solving by using technology to model and solve a variety of open-ended problems.

For this study, we developed an InterMath course focused on rational numbers. This topic was chosen because of the notorious need for professional development in this content area (e.g., Ball, Lubienski, & Mewborn, 2001; Ma, 1999), the necessity for students to develop strong rational number sense (Lamon, 2007), and its appropriateness for middle grades teachers, which was our population of interest. The InterMath – Rational Numbers course (IM) had three explicit goals: to raise participant awareness of referent units; to provide opportunities for participants to develop understandings of drawn representations for solving rational numbers problems; and development of proportional reasoning skills as they relate to ratio and fraction situations. To meet these goals, we designed the IM course to last 14 weeks with approximately 40 instructional hours.

The data reported here were collected as part of the NSF-funded *Does it Work* (DiW) project which was investigating three critical research questions: (1) What do teachers learn in professional development?; (2) How did teachers' practice change as a result of the professional development?; and (3) How does teacher change impact student performance on assessments? The current study is concerned with the first research question – the impact of the professional development on teacher learning as it relates to this alternatively certified population. Specifically, we wanted to understand the impact of IM (http://intermath.coe.uga.edu) on teachers with emergent mathematical knowledge for teaching who have not taken traditional mathematics education courses. We considered to what extent a single professional development course can impact alternatively certified teachers' learning and their reported practices.

## Framework

### *Learning Environments*

For teachers who have never experienced learning standards-based ways, providing these opportunities to students is difficult (Goldsmith & Shifter, 1997). After all, these teachers "may not have useful images from their personal experiences to guide the creation of a focused and productive classroom culture that emphasizes inquiry and the exchange of ideas" (Goldsmith & Shifter, 1997, p. 25). However, certain learning experiences appear to impact teacher change. For example, teachers consistently report developing the most effective strategies and practices through experiences and collegial interactions (Wilson, Cooney, & Stinson, 2005).

Research has shown that the learner-centered approach of the InterMath course allowed teacher participants to take "ownership of the content and investigations" (Polly, 2006, p. 15). While InterMath courses do not provide a prescription for teaching, the type of learning environment that InterMath offers teacher participants may affect the type of learning environments the teachers provide for their own students. These environments include student-centered mathematical investigations with different learning media such as technology.

### *Understanding and use of Drawn Representations*

Lesh, Post and Behr (1987) discussed understanding as the ability to "recognize the idea embedded in a variety of qualitatively different representational systems", the ability to "flexibly manipulate the idea within given representational systems", and whether one "can accurately translate the idea from one system to another" (p. 36). The authors suggest that to support understanding, teachers need to be able to "diagnose a student's learning difficulties, or to identify instructional opportunities, teachers can generate a variety of useful kinds of questions by presenting an idea in one representational mode and asking the student to illustrate, describe, or represent the idea in another mode" (p. 37).

Lesh, Post and Behr (1987) identified manipulative models, static pictures, written symbols, spoken language, and real scripts as "five distinct types of representation systems that occur in mathematics learning and problem solving" (p. 34). The authors claim that while knowledge and facility with these representations are important, "translations among them, and transformations within them, also are important" (p. 34). Furthermore, the authors discuss that "good problem solvers tend to be sufficiently flexible in their use of a variety of relevant representational systems" (p. 38). This is critical because using mathematical representations may open the classroom to more novel student approaches to the problems, thus, creating a situation in which the teacher has to be able to reflexively interpret and respond to novel student thinking.

## Methodology

Data for this study included videotapes of each session of IM as well as participant "write-ups" for each of 10 investigations. These write-ups outlined not only the solutions to the investigations, but also documented the approach(es) the participants took in working them. The teachers were also expected to create two lesson plans that incorporated InterMath-like approaches into their own classrooms.

In addition to the InterMath assignments, the teachers were also asked to complete a pretest and a posttest. The assessments both drew from a bank of items comprised of those from the "Learning Mathematics for Teaching" middle grades instruments (LMT; SII/LMT, 2004) and items developed by the DiW team. The items developed by the DiW team had been validated through a nationwide effort that included 201 teachers that was further augmented by a set of over 25 videotaped cognitive interviews in which teachers described their approaches to solving the items developed by the DiW team. The DiW assessment reports performance using *z*-scores and forms are equated to allow measurement of growth over time.

To further enhance our understanding of teacher learning in IM, we conducted cognitive interviews with a subset of the InterMath participants about items on the assessments. In this study, we present data from two teachers who participated in the InterMath course. Using mixture Rasch models, we analyzed our participants' performance on the pretest and determined in which of two latent classes each participant belonged. These classes were statistically-identified groups of teachers who shared some aspect of their approach to the items on the assessment. Both teachers were in the same latent class.

Both teachers were interviewed after the pretest and the posttest. Each interview was videotaped using two cameras, one focused on the participant and the other focused on their hands to capture gestures and writing. The interviews were used to capture each participant's discussion about a subset of the assessment items that were of particular interest to the researchers as they aligned with the explicit goals of the course and, for the most part, they relied on drawn representations.

In addition to the pretest and posttest interviews, members of the research team also conducted weekly phone interviews with the InterMath participants. The purpose of the interviews was to gauge teacher perception of the content and teaching of the course. The interviews also gained qualitative information about the teachers' stated beliefs in regards to the teaching and learning of mathematics with respect to the three main foci of the course.

The data were analyzed by a member of the research team to search for evidence of teacher learning – specifically in regards to the three main foci of the Rational Numbers Course: understanding of the referent unit, understanding of drawn representations of fraction and decimal operations, and proportional reasoning. Instances of participant discussion in either the professional development sessions or the interviews were analyzed in which the participants made direct comments to their learning and use of the referent unit, drawn representations, or proportional reasoning. The participants' understanding and use of drawn representations are presented next.

*Participants*

The present study considers two IM participants. The two participants, whose pseudonyms are Brian and Corey, were African-American males teaching middle grades mathematics at the time of the study. Both teachers had alternative certifications and preparations in fields other than middle grades education. Brian was in his second year of teaching sixth grade while Corey was in his fifth year of teaching sixth, seventh, and eighth grades to low-achieving and special needs students.

## Findings

*Pre-InterMath Experience with Drawn Representations*

In our initial interviews with them, both Corey and Brian reported having limited prior experience teaching and learning with drawn representations. Corey, with more years of teaching experience, stated that although he had four years of teaching experience, this was his first year of "actually getting into drawing the lines and actually understanding numbers and operations." Brian, a second-year teacher, discussed his experience of teaching with drawn representations in his first year of teaching as working with "a lot of plane figures, so rectangular prisms, triangular prisms, things like that." Neither provided evidence that they considered the use of drawn representations as a means of modeling mathematical operations.

Items on the pretest asked the participants to consider various models of rational number operations either in the context of student solutions or in the context of finding a solution to a given problem. When presented with number line representations for fraction subtraction, multiplication and division both Corey and Brian displayed confusion and (for Corey) an outright dislike for the representation. When asked why he did not like the number line to model fraction operations, Corey replied "Because it's confusing." He felt that using them to teach with would be "too confusing to [the students], because it doesn't give them a real representation." As he selected his answers on the items, it was apparent that he first found the answer using an algorithm and then found a representation to match that solution. Brian expressed that he was not familiar with modeling fraction operations on the number line and that modeling fraction division with any representation was difficult for him.

Both Corey and Brian expressed general familiarity with array models and had used them for teaching. Corey, in particular, made reference to a "math seminar" he took in the previous school year on learning how to teach with the array model. However, even with this experience, Corey displayed limited ability to interpret array models. For example, in a problem presenting different correct array representations of the same fraction multiplication, Corey chose only two of the three drawings. He based his choice on the one representation that he understood best and whether or not the other two could be rearranged to look like the representation with which he felt most comfortable. In contrast, Brian identified all of the array models as correct and even appeared to understand how the operation was modeled in each representation.

Modeling fraction division with array models proved difficult for both participants. We saw this difficulty in a problem that asked the participants to select the representation that correctly modeled the quotient 2 as it related to the division situation. While Brian initially identified the correct representation, he talked himself away from this choice as he tried to make sense of the representation. Corey, on the other hand, did not find any of the possible representations correct because none of the possibilities represented two wholes – they only showed "parts of a whole." Neither of the teachers understood what the answer, 2, was referring to – they both had a sense that it was two wholes rather than two parts of a divisor.

*InterMath Experiences*

Corey and Brian both reported that they chose to participate in the InterMath course to increase their repertoire of teaching strategies. Corey stated, "I'm always looking for different strategies and different ways to present material. So I'm always trying to learn. I just figured this is a great way to learn more strategies." Brian focused on his students' apparent trouble with rational number understanding stating, "I was actually most interested in the Fraction Bar software because I wanted to be able to have another resource . . . And I knew from last year that fractions was one of the most challenging subjects that my students had."

Throughout the InterMath course, both Corey and Brian were active participants, regularly sharing their thoughts and mathematical ideas with the class. Corey, through the InterMath investigations, started to have a new experience with number lines. "I usually use area models but just looking at some of the stuff that we're doing, I'm starting to feel more comfortable with number lines, as far as fractions." By the eighth week of the course, he shared that "number lines and pictures and everything else have actually helped me expand upon my instruction for my students." Surprisingly, both participants expressed an interest in sharing one particular investigation that involved number line representations with their classes.

While both participants reported finding the IM representations helpful for teaching their students, Brian appeared to be particularly affected by his experiences in the course with modeling operations through drawn representations. In the case of fraction division, an area he expressed having difficulty modeling prior to the course, he noted: "When we went over dividing fractions, I actually the next day went and modeled it for my kids the same way." Unsurprisingly, he explained his pre-IM experience as "just looking at the numbers and, you know, flipping and doing that type of stuff, but never actually seen it modeled visually." Like Brian, Corey reported that he was "taught the algorithm, and you go straight through it and I was comfortable." However, the structure of IM, which included considerable group discussion, provided Corey with access to other teacher participants' mathematical views. "It's more than a few [problems] where somebody else has said or seen it a different way and thought it out and given a darn good explanation and those are the ones that I write down for later."

Both participants provided insight into their teaching through our interviews. Both stated that they were providing more discovery time for their students' mathematical investigations and using questioning strategies. For example, in the eighth week of the InterMath course, Brian discussed his transition:

> "I think at the beginning [my teaching approach] was completely different and now I've really tried to adopt the same kind of approach in terms of asking

> questions, especially last year, when they asked the questions, I'd answer them. I'd give them a straight up answer. . . . So I'm asking them questions to get them thinking about to access where that knowledge is. So I've really changed how I go about the execution of my lesson."

Corey also stated a similar desire to change his teaching approach: "Well, [the InterMath instructor] has a facilitating approach. . . .I'm kind of trying to continuing to emulate that in my classroom."

*Post InterMath Experiences*

At the end of IM, Brian and Corey took the posttest and participated in another interview about that assessment. Both showed qualitative and quantitative improvement on their posttest after having completed IM. During the cognitive interviews, the participants demonstrated improvement in their discussion of the fraction operations problems with drawn representations. In his interview, Brian repeatedly referred back to his experience in IM as he discussed his solutions to different problems. Corey, who was particularly vocal about his dislike of the use of the number line representation before IM, demonstrated fluency and comfort with the representation on the posttest.

Further, both Corey and Brian changed their latent class membership between the pretest and the posttest. Based on our earlier analyses of the meanings of the latent classes, this movement indicates a movement toward more reliance on referent unit knowledge in reasoning about the items on the assessment (Izsák, Orrill, Cohen, & Brown, in review). We interpreted these changes in their assessment performance as indication that the course increased their mathematical knowledge for teaching. Some of the posttest items on drawn representations were identical to the pretest items.

As noted above, the two participants spoke qualitatively differently about those particular problems, indicating a different mathematical point of view and understanding of the representation and mathematical operation used. For example, Corey, who had shared his dislike of the number line model initially, was more confident and comfortable in the posttest noting only that he wished the number line had been labeled. He tied this to his teaching saying, "I'm a little – I'm a stickler on labels also because my children, they will draw something like this, when I actually want to see all the labels." This statement also indicates a possible change in Corey's pedagogical strategies. Before IM, he mentioned he would not use number lines to teach fraction multiplication to his students, however, his comment suggests that he not only taught with number lines during IM, but also required certain features of the number line model to be clear as he assessed his students' work. Corey also explained that he used the array model to help him reason with the number line model stating that thinking about the array model "just gave me a concept of what I needed to be looking for on a number line." Research suggests that while a teacher may be very proficient with one particular model of fraction operations, he/she have difficulty transforming (Lesh, Post & Behr, 1987) that understanding to a different model of the same operation (Orrill, Sexton, Lee, & Gerde, 2008). Corey's statements suggest that he was at least beginning to find connections between the models that were important to his understanding. He also indicated that his work with representations was great and that, "it's showing some dividends in my class."

Brian also provided evidence of increased flexibility in interpreting drawn representations and he attributed that flexibility to his experience in IM. For example when discussing representing fractions he remarked, "And then I remembered after learning it in the class I focused more so on … having one whole and then breaking it into thirds and that third, breaking it into fourths." Brian, like Corey, reported using the representations in his classroom. He stressed that he used them to help the students understand ideas rather than numbers, a focus he had before IM as well.

## Discussion

The data reported here show that the IM experience had a positive impact on these two teachers both in terms of their mathematical knowledge for teaching and their teaching practice. While we would not claim that these results were typical for all InterMath participants, they are encouraging as a sign that a single course can have significant impact on teacher understanding of critical aspects of the content they teach.

While IM did not provide a prescription for teaching, both Corey and Brian remarked on a number of occasions that they had used variations of the problems investigated in the course with their own students. Despite differences in the middle school populations they taught, both teachers reported finding the drawn representations practical and useful for supporting their students. Further, seeing representations modeled by the instructor in the InterMath course supported the teachers in making changes to their own teaching strategies.

In a time of teacher shortages, especially in the area of mathematics, more and more administrators are turning to alternatively certified teachers. Such teachers may lack adequate background in their content knowledge or knowledge for teaching. Professional development is one way in which to help these teachers fill in some of these gaps, as is evidenced by this study. Here, two participants, provided evidence that they not only learned new teaching strategies but also deepened their mathematical content knowledge even though the professional development had not been differentiated for the alternatively certified teachers. To us, this suggests that using a hands-on approach with the instructor modeling desired teaching actions may provide one effective model for supporting alternatively certified teachers in building their vision of what school mathematics should be – one of the critical purposes professional development can serve (Sowder, 2007).

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# PROSPECTIVE K-8 TEACHERS' FRACTION MULTIPLICATION SCHEMA

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*Interviews with 12 prospective elementary and middle school teachers focused on computation, problem-posing, visualization for problem-solving, and eliciting from participants explanations of the connections among these, particularly explanations that would be accessible to children learning about multiplication. We analyze the nature of participants' conceptions and explanations and discuss implications for teacher preparation.*

## Background

In elementary schools, multiplication of two values is taught as a number of equivalent groups (*multiplier*) times the size of each group (*multiplicand*), with *multiplier* x *multiplicand* as the default order. (Harel, Behr, Post, & Lesh, 1994). Research with children up through middle school age, working with word problems, suggests that their choice-of-operation is strongly affected by the nature of the multiplier (Bell, Fischbein, & Taylor, 1984). For example, given two word problems with the same context but different number types may result in different choices: *Suppose peanuts cost $2 per pound. (a) What is the cost for 3 pounds of peanuts? (b) What is the cost for ½ pound of peanuts?* Students will often identify multiplication as the operation to use for (a) and division for use in (b) (Af Ekenstam & Greger, 1983). Fischbein and colleagues (1985) proposed an intuitive model to provide a theoretical account for this "non conservation of operations" (Greer, 1988): When the constraints of the underlying model are incongruent – for the learner – with the numerical data given in the problem, the choice of an inopportune arithmetic operation may occur. Though Marshall and colleagues (1989) proposed an instructional intervention based on semantic analysis to assist students in learning to match a situation to a useful schematic representation, researchers have expressed concern that such intervention may foster superficial strategies in solving world problems without helping learners to construct conceptual representations situated in the problems (Verschaffel & De Corte, 1993).

Some studies and teaching experiments have approached the learning of multiplication through *problem-posing* rather than problem-solving tasks (Fischbein et al., 1985; Lowrie, 2002). This work supposes that a generative connection to the task might support conceptual engagement during subsequent problem-solving. However, in some cases children's performance on such tasks improved only when the numbers were whole (not fractions). In other studies, researchers have examined how learners connect their solving of word problems to acting on manipulatives to solve problems, in terms of units of quantity (Behr et al., 1997). Such studies have challenged the dominance in school curriculum of the use of a context-independent interpretation of 1-unit (Steffe, 1988). To generate a mental construct, a learner needs to re-present the concept even in the absence of perceptual input (von Glaserfeld, 1982). Thus, neither students' capability in algorithmic calculation nor their competence in acting on manipulative aids cued by problem context reaches the utmost goal of constructing conceptually rich mental schema. Several researchers have worked to describe and explain the process of formation of mental constructions into object-like cognitive entities, such as *encapsulation* (Asiala et al., 1991, after Piaget), *reification* (Sfard, 1989), and *proceptual* thinking (Gray & Tall, 1994). In each case, the role of making connections among object-like cognitive entities is central.

### Theoretical Perspective and Research Question

*Action-Process-Object-Scheme* (APOS) theory (Asiala et al., 1991) describes a hierarchical relationship among types of mental structuring (action, process, object-like entity, and schema) in which learner awareness, perception of totality, and coordination of aspects of a concept (identified by researchers through a genetic decomposition of the concept) are salient features. A style of teaching associated with APOS theory aims to assist students to move from one level to another and to gradually stabilize a developing mental construct. Reification and proceptual theories pay significant attention to the use of symbolic representation. According to Sfard (2000), in the process of reification, naming and symbolizing (creating a signifier) is no less important than the cognitive entity (signified). Gray and Tall (1994) have asserted that a flexible use of mathematical symbolism may compress action/process into object/concept, which in turn may liberate more capacity for cognitive activity and advanced mathematical thinking. Both perspectives, reification and proceptual thinking, value procedural skills in which manipulation on symbolic representations plays a central role in the process of stabilizing a mental construct into a conceptually rich understanding.

For a prospective teacher, the mathematical knowledge needed to *teach* is more than the knowledge needed to *do* mathematics (Shulman, 1986; Ball & Bass, 2000). Also important is a facility in packing and unpacking object-like understandings in order to supply explanation, and make sense of students' thinking, both in planning for instruction and in-the-moment-of-teaching (Hill, Ball, & Schilling, 2008). In particular, we started from the hypothesis that understandings of multiplication by prospective K-8 teachers would be foregrounded if they were asked to describe a particular kind of connection: that between doing multiplication in response to: (a) a symbolic statement (decontextualized) and to (b) a word problem (contextualized).

The study reported here sought to gain insight into the following questions: (1) What are the ways in which prospective grades K-8 teachers may perceive the isomorphic relationship between abstract structures (decontextualized mathematics problems) and concrete structures (contextualized or story problems) for fractions in simple multiplication? (2) What roles in problem-posing and problem-solving might fraction as APOS-object, and fraction multiplication as APOS-object (Asiala et al., 1991) play in understanding multiplication?

### Design and Setting

The 12 women in this study were prospective grades K-8 teachers who had completed the first 2 of 3 semesters of teacher-preparatory mathematics at a comprehensive U.S. university. According to the instructors and textbook authors (Bennett & Nelson, 2000), the courses aimed to teach mathematics with conceptual understanding. One task-based interview with each participant (60 to 100 minutes each) formed the primary data for the study. The interview was framed in a preparing-for-mathematical-teaching context and was designed to bring to the surface participants' understandings of multiplication. Specifically, each participant worked with four numerical prompts,

$$\text{(a) } 4\times 3 \quad \text{(b) } 4\times\frac{5}{6} \quad \text{(c) } \frac{3}{4}\times 6 \quad \text{(d) } \frac{3}{4}\times\frac{2}{5}$$

in interviews that followed five steps: (1) computation, (2) problem-posing, (3) visualization of problem-solving, (4) sketch for visualization, and (5) comparison of ideas and material generated in Steps 1 and 4. All interviews were audio and video recorded and transcribed. Analysis was phenomenological, using constant comparative methods. In recording researcher observations about participants' interactions with tasks and in analyzing their responses we relied on Pirie and Kieren's (1994) method for diagramming a person's progressions through, and folding back among, layers of understanding (e.g., facets of action, process, object, and schema activity). We used these Pirie-Kieren models for participants' problem-posing and associated problem-solving interactions so that the dynamic patterns emergent from the interviewees' efforts could be classified into categories based on the nature of object-like entity understandings.

## Results

During Step 1 of the interview, 8 of the 12 participants made no errors in computing numerical prompts. However, 4 of 12 confounded "multiply across" (e.g., $a/b \times x = a/b \times x/1 = ax/b$), with "cross multiply" (e.g., $a/b \times x = a/b \times x/1 = bx/a$, or $a/b \times x = a/b \times x/x = ax/bx$). Note that at the time of the interviews, all were enrolled in the third semester of their mathematics sequence and studying proportions, including "cross multiplying" to find the unknown value $x$, in a proportional equation like $a/b = x/d$. Though analysis of computational error was not the purpose of this study, participants' procedural skills with decontextualized symbolic prompts in Step 1 may have mediated their efforts to identify isomorphisms in Step 5 of the interview. Subsequently, in Steps 2 through 5 of the interviews, five categories of object-like entities in concept building appeared to be problematic for participants: multiplier-as-operator, fraction-as-multiplicand, fraction as only a part-whole-relation, fraction-as-multiplier, and fraction multiplier acting on a fraction multiplicand.

*Multiplier-as-operator*

All 12 participants posed a complete story and described in words or using a sketch the process of solving for prompt (a) 4 x 3. However, the nature of their understandings varied. In the following excerpt, the interviewer (denoted *Int*) and Ann (all names are pseudonyms) negotiated the personality of multiplier to act on the multiplicand.

> *Ann*: [4×3] probably means that I have four pieces of candy and I have three friends. If I were to say pretend you had three candies, or three piles of candy with one in each pile, so you would have three candies and I want to add four candies to each pile, then I would add 1,2,3,4, 1, 2, 3, 4…then I would end up with the same answer [as 12 candies] and they would represent the same thing both three candies and four candies they are. I am representing candies all the way across. But in the problem that I gave I said that you have three friends and you are giving them each four pieces of candy so the numbers represent different things.
>
> *Int*: So in this case, what does the number three represent?
>
> *Ann*: kids, friends, people.
>
> *Int*: Four is four pieces of candy, and four candies times three people is?
>
> *Ann*: Twelve.
>
> *Int*: Candy or people? Have you ever thought about it?
>
> *Ann*: Yeah, I have never thought of that. Your answer is candy, you end up with 12 pieces of candy. I guess your three doesn't matter like I thought it would. Okay then, the three would represent where you are putting them, like how you are separating them

so how many times you need to use them like when you, often times when you use multiplication you would say four candies and I need to give them to three people, it is just like adding four three times, in relation to addition.

*Int*: Three groups of four?

*Ann*: Exactly. So your three is your groups, your four is your number within those groups.

The participant's perception of multiplier 3 went from "three friends" to "three candies" and then to "three groups" (of four candies). Her struggle in identifying the nature of multiplier concurs with Steffe's (1988) observation about the nature of unit in such contexts. The multiplier 3 is not just for 3 one-units as a number of groups, but also for 3 units (groups) of 4 one-units (candies). In the interview excerpt, Ann's understanding of multiplier-as-operator with natural number may be seen as moving from shaky *process* toward *object*.

*Fraction-as-multiplicand*

In working with (b) 4 x 5/6, two of the participants did not recall an appropriate property of positive integer as multiplier. Beth's story for the prompt 4 x 3 was, "John has grouped four groups of three marbles in each group." Here, the multiplier 4 played an explicit role as operator. However, Beth did not *conserve* the operation (Greer, 1988) in going from 4 x 3 to prompt (b) 4 x 5/6. In Step 3, visualizing her problem-solving, for (b), Beth changed representations and rewrote the whole number multiplier as a fraction, 4 x 5/6 = 4/1x5/6 = 20/6 = 3 1/3, and altered its personality as multiplicative

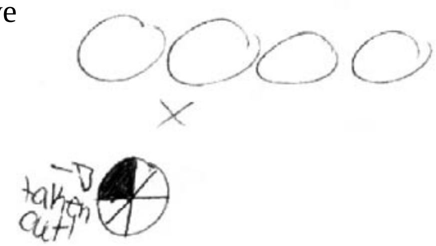


*Figure* 1. Beth's Step 3, visualizing problem solving of 4 x 5/6.

*Beth*: It helps to think about the four as a fraction, so like four over one. That helps because it puts it in, both in the same context.

*Int*: So now we have, an integer times a fraction, any idea where to start?

*Beth*: Um, well you have four wholes. So I am just going to go ahead and draw them here, four wholes. Then we have five [out of six], almost a whole.

*Int*: Okay, now I would like you to compare your sketch to its numerical calculation. The number one and the number five-sixths, which one is bigger?

*Beth*: One. So five-sixths is less than a whole, duh. Let's try that one again. So this [sketch] is not good, we are just going to forget about that.

*Beth*: Okay. We still have four wholes, correct?

The interviewer drew Beth's attention to her story for 4 x 3 and asked her to pose a story for (b) in an analogous way. When the multiplier is a whole number, the multiplication-as-repeated-addition model can also work for a fractional multiplicand.

*Beth*: Oh, could you have four groups of five-sixths, does that work? You would have four of these, so these are all like five-sixths?

*Int*: Does it make more sense?

*Beth*: I think so but I don't know how to explain it…

*Int*: I have four groups. What is inside each group?
*Beth*: Not even one, a part of one.
*Int*: A part of one. How many ones? How many sixths…
*Beth*: Hey, that works.

As proposed by Fischbein et al. (1985), one of the values in 4×5/6 is incongruent to 4×3. In terms of unit types, the numerical prompt 4×5/6, thought of as 4(5(1/6(1))), is one more layer than 4×3, or 4(3(1)). We suspect Beth may not have chosen a useful arithmetic operation because of this incongruence. She may have had, at the time of the interview, a schema of fraction multiplication that was repeated *action*-based, essentially additive, and was the only schema she recalled in the moment.

*Fraction as Exclusively a Part-whole-relationship*

Most of the participants contextualized and visualized a fraction numerical prompt. However, 7 of the 12 participants' understandings of fraction seemed to be confined to the part-whole-relationship personality. Kieren (1980) differentiated part-whole-relationship personality from measure personality for the rational number *x/y*. In the former, some whole is split up into *y* parts and *x* of these parts are taken. The latter sees 1/*y* as a unit to be used repeatedly to determine an *x/y* quantity. In the following excerpt, Cher's conception of 5/6 included five out of six pieces but the idea of 5(1/6), that is of five units of size one-sixth did not appear to come to mind for her.

*Cher*: 5/6… [means] something is divided into six portions, there is five remaining [of 6]
*Int*: How about 6/5?
*Cher*: 6/5, um there were two something that were divided into six [five] pieces, the remainder of what is left is one full one and one of the six [five]…

An understanding of fractions that is exclusively part-whole could be a challenge for participants in tracing the connections between transformation of units in their problem-posing and problem-solving visualization efforts – these multiplication algorithms are mainly based on measure personality. We saw some additional evidence to support this result in Daisy's interview.

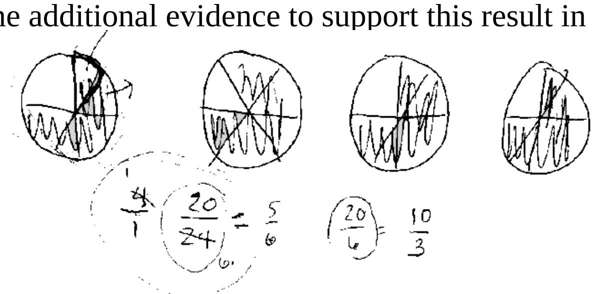


*Figure 2.* Daisy's visualizing problem solving of 4 x 5/6.

*Daisy*: [For 5/6,] there are six pieces so five of them are colored in because it is five of the six pieces and that is where the 5/6 comes in, and then I did four of them for the four times the 5/6. So I know the 20 of the 24 pieces, are colored which would equal 5/6, which doesn't help me out though because I am stuck at – Oh, what if I did four of them, one, 5/6 then I am left with 20/6 which could be reduced to 10/3 but then I don't know that leads me to so...

Daisy's understanding of fraction as part-whole-relation led her to see 4x5/6 as (4x5)/(4x6), or 20 out of 24, which was inconsistent with her computation of 4x(5/6)=20/6=10/3. Mathematically, her computational procedure and sketch matched perfectly. Psychologically, she did not perceive the personality of 6 in 5/6 and in 20/6 as one type of measuring unit, 1/6, derived from partitioning one into six equal parts, or as units of measure 1/6.

*Fraction-as-multiplier*

Several participants called on the symmetric property (axb=bxa) and said prompts (b) and (c) were "exactly the same." The request to create a story where a fraction was a multiplier acting on a whole number, challenged participants' belief in the symmetric property and was, ultimately, not fruitful during interviews. However, prompt (d) 3/4x2/5 involved only fractions, so fraction-as-multiplier was required in some way. In posing a story and/or visualizing problem-solving for (d), 5 of the 12 participants used addition. Elda immediately posed a story for the prompt 4x5/6, but had five unsuccessful attempts on 3/4x5/6. Elda's first attempt was: "If Megan was making a pasta dish and it asked for 3/4 cup of milk and 2/5 cup of salt. How many cups are needed to make the pasta dish?" She immediately realized what she posed involved the operation of addition rather than multiplication. She tried again:

> *Elda*: Megan's pasta dish called for 3/4 cup of milk but she put in 2/5. How much did she forget to put in?
>
> *Int*: Did you mean to put 2/5 *of the 3/4 cup*?
>
> *Elda*: Like, she put in 3 tablespoons instead, how much did she put in?

Later, she contextualized 2/5 as "2/5 cups," though Elda may have tried to express 2/5 *of* the 3/4 cup, in which "3 tablespoons" was about 2/5 of 3/4 cup in her sense. What Elda said might mean a mental structure where 2/5 acted on 3/4 to get "3 tablespoons," or it could be her second attempt was another additive one (in this case, subtraction). In drawing the visualization of her problem solving, Elda tried three times, but each time her strategy involved addition only and Elda seemed to be aware that her attempts did not use multiplication.

*Fraction Multiplier Acting on a Fraction Multiplicand*

Flora created a story for prompt (d) 3/4x2/5 involving the concept of group size and number of groups where 3/4 was group size. However, instead of 2/5 as two groups of measure one-fifth, of something else (i.e., 2((1/5(1)) where the inner 1 represents one whole group of three-fourths), she used "two out of the five" groups. In her drawing, she had five groups of 3/4. She circled two of them and added them together to get 3/2 and said,

> *Flora:* So here are my five groups of three-fourth. And – I want to add these two [groups of 3/4] together. So is, that's the same as three and a half [3/4 x 2/1 =6/4 = 3/2] – No. – Yeah. So that is it. So I need to say two-out-of-the-five somewhere.

Flora saw that 2/5 was not an operator that acted on three-fourths but also clearly articulated understanding of fraction as part-whole: "I need to say two-out-of-the-five somewhere."

Gina also used part-whole relationship to assign context to the multiplier 2/5.

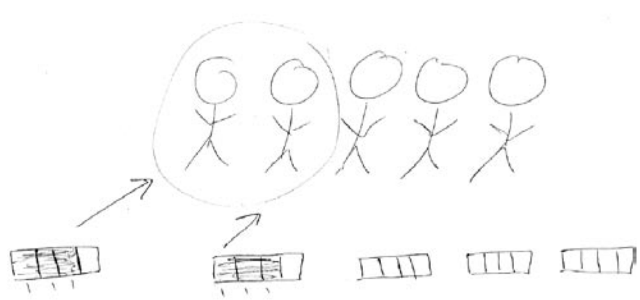

*Figure 3.* Gina's visualizing problem solving of 3/4 x 2/5.

Gina had five candy bars with four pieces in each candy bar and 20 pieces total. Instead of 2/5 acting on 3/4, she had 2/5 act on 5 and 3/4 act on 4 1-units, i.e., 2/5(5(3/4(4(1)))), to have 6 out of 20 pieces as the contextualization for 3/4x2/5. By doing so, there was no need to conceptualize a transformation of units like 1/5 (or 1/4), 1/20, or to form different unit types.

## Conclusion

One can perform actions on an object, physically or mentally, as in Daisy's sketch for visualizing problem solving of 4x5/6, and adopt symbols to represent it like "4x5/6=20/6=10/3," without bringing to mind some properties in the process. An encapsulation of the incomplete process into a sort of *pseudo object* may lead to a "pseudostructural" conception (Sfard, 2000). We hypothesize that all of the 12 participants, like Elda, had experienced *action* of fraction multiplication. But they may not have had awareness of all key properties in the *process*, and therefore did not perceive the process as a totality. A pseudostructural object of fraction multiplication appeared to be sufficient for many to choose an appropriate operation for solving a given word problem. However, the visualization task called for de-encapsulation or unpacking of both *number* and *operation*, from object back to process and action.

De Corte's (1988) empirical study supported the claim of Fischbein et al. (1985) that children's difficulties in solving multiplication word problems may arise when their underlying models are incongruent with the numerical data given in the problem. This study suggests that for adult prospective teachers, a complex version of incongruity is at work. In unpacking both understanding of number and of operations, several sites for incongruity emerge. Cognitively, for the same operation (multiplication) the fraction multiplicative structure is not congruent with whole number multiplicative structure. Teaching with emphasis either on identifying key features from word problems and procedural skills or on concrete experience is, although necessary, not sufficient for learners to construct complexly connected cognitive objects that can be untangled from multiple potential incongruities. This suggests that a richly connected and "unpackable" understanding of multiplication for positive rational numbers may require an equally complex constellation of ways to identify and respond to incongruity. That is, we suggest that this study offers empirical support for the assertion of many that mathematical discourse incorporating procedural skills, problem posing, visualization, and identifying isomorphic corresponding relationships can all play valuable roles in arousing learners' awareness of actions and process, in reifying and encapsulating mental constructs into object-like entities, unpacking or de-encapsulating the same, and using symbolism flexibly to advance mathematical thinking.

Finally, our experience in interviews with prompt (c) and its challenge to participants' belief in the symmetric property leads us to the following suggestion for teacher-educators. In working with prospective teachers, consider working with the abovementioned constellation of activities in the context of multiplication of two fractions (as in prompt (d)) before situations with one fraction; and then address a similar constellation of activities in connecting and unpacking the ideas of *fraction of* and *out of* to move into the context of fraction as multiplier acting on whole number multiplicand (e.g., problems like prompt (c)).

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# EQUIPARTITIONING A CONTINUOUS WHOLE AMONG THREE PEOPLE: STUDENTS' ATTEMPTS TO CREATE FAIR SHARES


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*Although many children can share a continuous whole with two or four people at a young age, students find sharing for three to be problematic. We present results from eight clinical interviews with students age four to eleven. Results show that students use various strategies as they attempt to share a continuous whole among three people. Many students relied on halving, quartering, or parallel cutting strategies. Older students created fair shares for three, often by referring to the "peace sign." We conclude with a discussion of why students may find constructing thirds to be challenging and call for equipartitioning in the early grades.*

## Introduction

The purpose of this paper is to present findings from clinical interviews conducted with the Diagnostic E-Learning Trajectories Approach (DELTA) project. The goal of the DELTA project is to build diagnostic assessments for rational number reasoning concepts for grades K-8. While the larger project involves the study of seven strands of rational number reasoning (multiplication and division; area and volume; decimals and percents; fractions; ratio and rate; similarity and scaling; and equipartitioning), this paper focuses specifically on the equipartitioning strand. Based on extensive literature reviews and syntheses a progress variable and learning trajectory for equipartitioning were developed. The DELTA team has conducted a number of clinical interviews to test and refine our constructs using students from diverse racial and socio-economic backgrounds. This paper will report on a few students as they attempt to partition a circular region into thirds.

## Related Literature

A learning trajectories view was chosen for our synthesis work of the partitioning research. Confrey et al. (2008) have defined a learning trajectory as:

> A researcher-conjectured, empirically-supported description of the ordered network of experiences a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time.

Confrey et al. (in preparation) note that learning trajectories are a useful construct for this work, "because of its potential to unpack complexity by revealing characteristics of gradual student learning over time" (p. 1-2). While all students do not progress along the same path, there are certain landmarks and obstacles that we consistently observe students encounter in equipartitioning

Equipartitioning is defined as the cognitive behaviors that lead to the creation of equal sized groups from a collection, or equal sized pieces from a continuous whole, and which result in fair shares (Confrey, 2008). Many students come to school with ideas about how to share a continuous whole (e.g., pizza, cookies, cake) and are able to transfer those ideas to equipartitioning tasks. These informal or out-of-school experiences contribute to students' prior knowledge. One of the first landmarks students reach is the two-split. Previous research has shown that very young children, even pre-schoolers can use a two-split on a collection of items or on a continuous whole (Ball, 1993; Pothier and Sawada, 1983, 1990). Pothier and Sawada (1983) also identify a five-level framework of student's partitioning behaviors, where level one, sharing, is the most primitive behavior and level five, composition, is a more complex behavior. Level four in their framework (oddness) is achieved when students realize that a two-split cannot be used to create fair shares for odd numbers. We have also found that students have trouble with odd splits, especially on a circular region.

While our use of learning trajectories is a more global view of how equipartitioning is situated within the larger framework of rational number reasoning, progress variables show a progression of knowledge from less sophisticated to more sophisticated in specific cases (Wilson & Sloane, 2000). Based on Confrey's synthesis work, the DELTA team has built a progress variable for equipartitioning (Table 1). Based on our review of the literature and empirical work with students, sharing a continuous whole among three and other odd numbers is placed at level 1.6. Despite knowing that students have difficulty in creating thirds, few studies have focused specifically on the tensions students encounter as they attempt to create thirds.

Table 1. *Equipartitioning Progress Variable*

| Case | Description |
|---|---|
| D | 1.8 $m$ objects shared among $p$ people, $m > p$ |
| C | 1.7 $m$ objects shared among $p$ people, $p > m$ |
| B | 1.6 Splitting a continuous whole object into odd # of parts ($n > 3$) |
| B | 1.5 Splitting a continuous whole object among $2n$ people, $n > 2$, and $2n \neq 2^i$ |
| B | 1.4 Splitting continuous whole objects into three parts |
| B | 1.3 Splitting continuous whole objects into $2^n$ shares, with $n > 1$ |
| A | 1.2 Dealing discrete items among $p = 3 - 5$ people, with no remainder; $mn$ objects, $n$ = 3, 4, or 5 |
| A, B | 1.1 Partitioning using 2-split (continuous and discrete quantities) |

## Method

### *Participants*

Clinical interviews were conducted with eight elementary and middle school students. The sample consisted of one pre-school student (Lara), one 1st grade student (Wilson), two 2nd grade students (Ethan and Rhea), and four 6th grade students (Casey, Dora, Keisha, and Bobby). The pre-school and elementary participants were sampled using a convenience sampling technique (McMillan, 2004). The three middle school students attend a magnet school that participates in a university connections program with a local university.

*Task*

In this interview, participants were asked to share circular birthday cakes (made out of play dough) fairly, for two, four, three and six pirates in succession. This order of partitions was suggested by previous research indicating that a) students first share by creating a two-split, and b) even splits are easier for children than odd splits because students can use a half as a benchmark split.

At the beginning of the interview, participants were told that some pirates were having a birthday party and they needed help to share the birthday cakes fairly. Participants were then presented with a circular birthday cake (measuring 6 inches in diameter). A plastic knife and a straightedge were available for use during the interview. Each participant was asked the following questions:

- Can you share the birthday cake fairly for two (four, three, or six depending on which partition the participant was working on) pirates? Show me how you would do it.
- How do you know that each pirate got a fair share?
- Could you show me one $n^{th}$ (half, fourth, third, or sixth depending on which partition the participant was working on) of the cake?
- Is there another way to share the cake?

Note that prior to sharing the circular birthday cake, participants had been asked to share 24 gold coins with two, four, and three pirates. They were also asked to share a rectangular birthday cake with two, four, three, and six pirates.

*Data Collection and Analysis*

The participant was interviewed using a clinical interview method (Opper, 1977; Ginsburg et al, 1983). A specified interview protocol was utilized, with additional questions and probes based on the behavior and responses of the participant. Data collection consisted of videotaping the interview and saving artifacts from the session. Relevant segments from the interview were transcribed for further analysis.

## Results

*Initial Attempts to Create Thirds*

In our work with students, they often times create fair shares for two or four pirates on their first attempt. In these interviews, two students made initial attempts to share for three pirates and realized that strategies they previously used would not be sufficient in this case. Lara began by doing a 2-split on the circle, as shown in the left side of Figure 1. After doing the 2-split, she hesitated about what to do next.

L: There’s not enough play dough.

I: There’s not enough play dough…hmm…Why is there not enough play dough?

L: I can’t cut this one because this one…because this one won’t have a fair half.

I: Ok. You can’t cut this one because… [Lara cuts the half piece in half again]

L: This will be three, but this one has more [Lara places her hand over the larger piece]

Lara knew that her goal was to create three equal-sized pieces, but was unable to meet that goal by starting with a 2-split. Ethan also knew that he needed to create equal-sized pieces. Earlier in the interview, when he shared a rectangle for three, he made two vertical (parallel) cuts.

Although Ethan displayed very strong conceptions for sharing a rectangle for two, three, and four pirates, the same did not hold true for the circle

I: Can you share the birthday cake fairly with three pirates?

E: No. [Shaking his head]

I: You cannot…Why can't you?

E: Because the only way I would really think of it…because an odd number of people…umm…wouldn't be equal. On the others I thought I could do it three ways and I did [Ethan makes the hash marks at the top of the circle—See Figure 2, right side] this, but then this would be a big one, this would be a small one, and this would be a small one [noting that if he were to cut it all of the pieces would not be the same size].

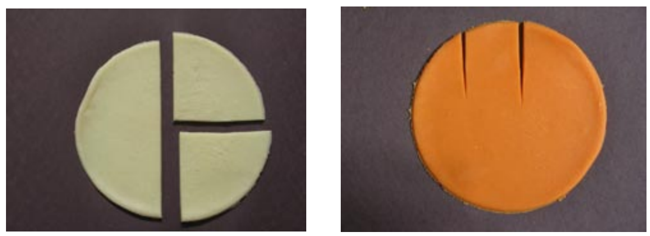

*Figure 1.* Initial attempts to share for three pirates (Lara, left; Ethan, right).

*Sharing for Three After Creating Fourths*

Three students shared the birthday cake by creating quarters and then sharing the remaining piece. Lara and Keisha both created quarters, deal out three pieces and note that the pirates can share the remaining piece. When asked if the pirates have a fair share, Lara notes that they do because they each have the same number of pieces.

[Lara cuts the birthday cake into fourths and puts one piece to the side]

L: We don't need this piece.

I: But what if we want to share the entire birthday cake?

L: The first person can get this one [points to a piece], the second person can get this one [points to a piece], the next person can get this one [points to a piece] and they can all share this one.

I: How can they all share that one?

[Lara cuts the fourth piece into three unequal pieces]

I: Does each pirate have a fair share?

L: Yes

I: How do you know?

L: One, one, one [pointing to the three smaller pieces]

I: What about these? [pointing to the three quarters]

L: One, one, one. One, one, one.

Keisha also began by cutting the cake in fourths, and then hesitated and stated,

K: Three is different. Like you would probably have some left over but then you could cut that.

I: Oh ok. Show me how you would do it because you've got to use up all the pizza. I don't have anywhere I could store it so the three of you would have to use it all up.

K: So these right here will be for that person, that person, and that person [Keisha deals out three of the quarters]

I: Ok

K: You have this left over [referring to the remaining quarter] so you would just cut this into three different pieces but they're kind of not the same shape.

I: Not the same, uh oh

K: Yeah

I: Can we figure out a way to make them the same so they would get the same amount?

K: Yeah you can probably just…

I: How would you do that?

K: I would think of something that's this shape…then…like…how would you put that?

Keisha then thinks about how she could possibly get three equal pieces of the remaining quarter. She notes that she thinks she could get two of the pieces to be the same but that one piece would look different. She concludes this segment by noting that three is hard because it is an odd number.

Wilson also quarters the cake and deals out three of the quarters. When asked if he had used all of the cake, he notes that he has one piece left over. When asked if there was a way he could share the last piece so that no cake is left, Wilson cuts the last piece into four more pieces and deals out those pieces. Since he had four pieces, one pirate received two of the smaller pieces. When asked if each pirate got a fair share, he responded:

W: No. Because this one has four.

I: Can you think of any other way to share for three pirates?

[Wilson takes the extra piece that he had given to one pirate and cuts it into three pieces and deals out each of the pieces.]

Each pirate now has three pieces of cake (one quarter of the cake, plus two smaller pieces).

*Figure 2.* Sharing for three pirates after quartering the cake
(Lara, Keisha, and Wilson respectively).

*Sharing for Three Using Parallel Cuts*

Although equipartitioning by making parallel cuts is sufficient for the rectangular birthday cake, students find that it does not work for the circular region. Rhea recognized that by making two parallel cuts she would have three pieces. When she was asked if each pirate had a fair share, Rhea responds:

R: Yes.

I: They do.

R: Yes.

I: How do you know that each pirate has a fair share?

R: Wait, because this piece [referring to the largest piece] is bigger than this piece [she lays the larger piece on the top piece of the circle] and this piece [referring to the middle piece]. This piece has a big hump like a circle, but this piece [the middle piece] doesn't, it's flat.

I: Can you think of another way to share the cake to make it fair?

*Figure 3.* Parallel cuts (Rhea).

As Rhea thinks out loud of how she may share the cake fairly, she notes that she could cut it in half but that would not work because it would give her two halves.

*Sharing for Three by Creating Equal-Sized Pieces*

Three of the four 6th grade students successfully created thirds on the circle. Two of the students (Bobby and Casey) specifically referred to using the "peace" sign to create thirds on a circle and immediately cut the birthday cake. Although Dora also created fair shares on the circle, her thinking and dialogue about how to create thirds was different from the other two students in this category.

D: It's a little harder to measure on this, but you sort of have to get to the center. Then you have to sort of…it's a little harder to measure the angles.

[Dora uses a straight edge to find the center of the circle and then cuts a straight line from the center to the bottom of the circle]

I: Tell me what you are thinking about the angles.

D: Well…umm…cause you have to get like this…or something like that [Dora makes a radial cut on the circle to create the first third] and it's a little harder to get that right…umm…because there's no other marks on the circle and you have to kind of guess at the angle.

I: So harder than what though?

D: Harder than dividing it in half or getting fourths…Because for the half you just have to find halfway up and then cut across. For the fourth you just…umm…you make sure that you…then you just kind of turn it to make sure that the half is horizontal or vertical, whichever way you cut it. Then cut…like if it's horizontal, cut vertically.

*Figure 4.* Equipartitioning a circle for three pirates (Dora, Bobby, and Casey).

For Bobby and Casey, the peace sign was a tool that they could use to create thirds. Dora however provides some interesting insights about constructing thirds. She notes that the angles are more difficult to construct because essentially she has no benchmark or reference point.

## Discussion

A number of interesting issues arise from students' work on this task. Throughout most of our interviews, students note that three is tricky, or that it will be difficult. But why is it that so many students find three so challenging? Students often state that three is an odd number. So the real question here is how does oddness affect equipartitioning behaviors. Students think of odd numbers as having "one left over." So in the case of three, students recognize that cutting in half does not create enough pieces, but that cutting in half again leaves them with an extra piece. Even when three of the four pieces are dealt, students have difficulty handling the remaining piece. Students are limited in this task because they do not know how to construct a radial cut. When students construct halves or other powers of two, they can usually estimate the midpoint of a side or half of a circle. For thirds, these references are not as fruitful, they have to now locate the center of the circle and use it as a reference point.

Because schools emphasize counting and number properties (e.g., even/oddness, part-part-whole) extensively in the early grades and do not address equipartitioning, many students are unable to use rational number reasoning to justify their behaviors. For example, Wilson noted that each pirate had a fair share because they each had the same number of pieces, although the smaller pieces he cut were different sizes. A tension exists for students between equipartitioning and counting. If counting continues to dominate students reasoning, they will be at a disadvantage when dealing with topics in rational number reasoning.

We also found evidence that for students unfamiliar with the peace sign, sharing among six may lead to the invention of the radial cut earlier than among three. After splitting the circle in half, students are able to use the straight side of the half, locate the center (or the midpoint of that line) and then construct a linear cut. After using this strategy to successfully share among six, Keisha was able to return to her work on three and share the remaining fourth in three equal pieces.

Each of the students in this study provided us with valuable information about how they think about thirds. Although creating thirds is an obstacle that most students encounter, we argue that they should have more experiences with equipartitioning early in schooling so that tensions between counting and rational number reasoning can be negotiated. Providing students with opportunities to engage in equipartitioning early in schools will better prepare them for multiplication, divisions, and fractions.

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# WHAT DO ELEMENTARY STUDENTS' SOLUTIONS TELL US ABOUT THEIR STRATEGIES FOR SOLVING A DIVISION WITH REMAINDER

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## Introduction and Theoretical Framework

Student solutions to mathematical problems reveal strategies they use as well as error patterns that reveal conceptions and misconceptions they have about a particular topic. We present an analysis of student solutions to a division with remainder problem (27÷4). We were particularly intrigued by student performance on this particular computation problem because many students in grades 3-5 missed this problem in a pre and post test given as part of a larger professional development project. According to researchers division problem involving remainders is difficult because students do not fully understand the division concept or wrongly interpret computational results (Anghileri, 1996, 1999; Spinillo & Lautert, 2004; Li, 2001; Spinillo & Lautert, 2002; Silver, Shapiro & Deutsch, 1993; Squire, 2002).

## Method

The data presented here is part of a larger three year professional development study. A pre and post test was administered to129 children in grades 3-5 in a western State. The post test results analysis on students' solutions to the division problem with a remainder is presented here. The response samples were made, and analyzed in the following ways:

- Performance was recorded and analyzed across the grades
- Types of errors were identified, coded, and examined
- Types of strategies were identified, coded, and studied for emergent trends
- Different treatments of the remainder were recorded and examined.

## Findings

In summary, students used seven different strategies, made five common errors, and treated the remainder in four ways as outlined below:

| STRATEGIES | | ERRORS | | REMAINDER | |
|---|---|---|---|---|---|
| **Code** | **Explanation** | **Code** | **Explanation** | **Answer** | **Treatment** |
| LD | Traditional long division. | O1 | Computational error. | 6.75 | As decimal |
| DD | Dividend is multiple of divisor. | O2 | No remainder indicated. | $6\frac{3}{4}$ | As fraction |
| EST | Estimation | O3 | Wrong division. | 6r3 | As integer |
| MF | Result is multiple factor of divisor | O4 | Misread division symbol. | Integer | Increased quotient or dropped remainder |
| RAS | Repeatedly add or subtract | O5 | Order of answer reversed. | | |
| DLG | Dividend is dealt or dished out. | | | | |
| PT | Dividend is partitioned. | | | | |

The analysis reveals that 4$^{th}$ and 5$^{th}$ grade students used the standard algorithm, whereas 3$^{rd}$ graders used more pictorial representations to solve problems. This indicates the differentiated approach to the bare division problem, where the upper grades may treat it abstractly and lower grades embed it within some context. In addition, the types of strategies and errors that students made give us insight into different thinking patterns of students.

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# REPRESENTING DECIMAL NOTATIONS AS SHADED PARTS OF AREA

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*This poster reports results from a study of fifth grade students' understanding of representing decimals as shaded parts of area. Evidence from students' responses indicates that students may interpret and represent decimal notations in ways that reflect limited understandings of decimal notation. Educational implications are addressed.*

Rational number has long been described as hard to learn and hard to teach. Area model representations such as rectangular regions are often used to support students' understanding of fraction notation. While area model representations can also be used to represent decimals, little is known about how upper elementary school students understand decimal notations as shaded parts of area. The goal of the present study was to systematically examine students' representation of decimals as shaded parts of area.

## Methods

Thirty-one fifth grade students drawn from 5 elementary schools in an urban area in Northern California were interviewed individually. Two of the interview tasks are the focus of this analysis. In each task, the student was presented with a rectangular region with a decimal notation written above it. The interviewer pointed the decimal notation and explained that the students' task was to shade that part of the rectangle. Students were asked to explain their thinking. In task 1, the decimal notation was '0.3'. In task 2, the decimal notation was '0.7'.

## Results

Across tasks, four patterns of responses emerged. As shown in Figure 1a, some students interpreted the decimal notation as representing an extremely small quantity and shaded a very small portion of the rectangular region. As shown in Figure 1b, other students interpreted the tenths digit of the decimal notation as representing the number of parts into which the whole should be divided (three in the case of '0.3') and the ones digit as representing the number of parts that should be shaded (zero on the case of '0.3'). As shown in Figure 1c, still other students respected the relative magnitude of the decimal notation in shading; however, they did not partition the rectangular region into tenths and then consider how many parts to shade. Finally, as shown in Figure 1d, still other students partitioned the rectangular region into ten parts of approximately equal size and then shaded three of those parts.

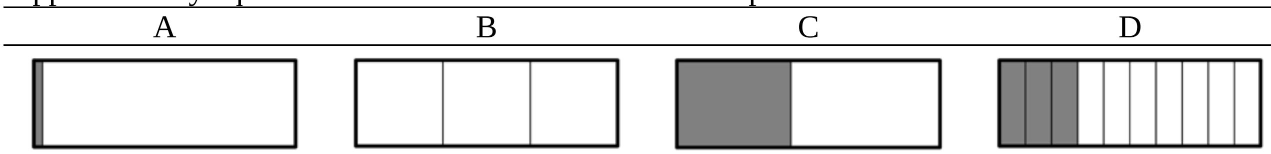


*Figure 1.* Patterns of responses on the '0.3' task

## Discussion

Finding from this student revealed the diverse ways that students may represent decimal notation as shaded parts of area. In my poster, I address implications for instruction.

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# FACTORS INFLUENCING A PROSPECTIVE MIDDLE SCHOOL TEACHER'S VALIDATION OF PROOFS: THE CASE OF MARY

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*This case-study examined a prospective middle school teacher's conceptions of proof and the arguments that she accepted as proofs. The factors influencing the prospective teacher's decisions of whether an argument is a proof or not are highlighted in this paper. Some of these factors are familiarity with the statement and certainty about its truth, the closeness of the argument to the participant's own proof and the context of the argument.*

## Introduction

Several mathematicians and mathematics educators advocate proof to be central to mathematics education (Ball et al., 2002; Carpenter, Franke, & Levi, 2003; Hanna, 1995; Knuth, 2002a; NCTM, 2000). To accomplish this goal it is crucial for teachers to be well equipped to teach mathematical reasoning and proof. As Peressini et al. point out "the extent to which mathematical ideas such as proof and justification appear in classroom discourse will be influenced by both the teacher's choice of task and the questions and comments she makes during class, which are, in turn, influenced by the teacher's knowledge of proof" (Peressini et al., 2004, p. 81). However, the results of studies concerning teachers' conceptions of proof are not very promising. The goal of this study was to examine a pre-service middle school teacher's conceptions of proof and analyze her process of validation of proofs (Selden & Selden, 2003) to provide insights into her understanding of a proof and what constitues a proof.

## Theoretical Perspective

Previous research on teachers' conceptions of proof has focused on pre-service elementary school teachers (e.g., Martin & Harel, 1989; Simon & Blume, 1996), in-service elementary school teachers (e.g., Ma, 1999), pre-service secondary school teachers (e.g., Jones, 1997), and in-service secondary school teachers (e.g., Knuth, 2002a, 2002b), as well as undergraduate mathematics majors (e.g., Harel & Sowder, 1998). A common conclusion of these studies is that teachers tend to accept empirical arguments as proofs (Knuth, 2002a; Ma, 1999; Martin & Harel, 1989, Simon & Blume, 1996). In other words they rely on either evidence from examples (sometimes just one example) of direct measurements of quantities and numerical computations or perceptions to justify a claim. To evaluate arguments some teachers focus on the correctness of the algebraic manipulations or form of the argument as opposed to the nature of the argument (Knuth, 2002a), and others accept false proofs based on their ritualistic aspects (Martin & Harel, 1989). Although teachers rate deductive arguments as valid proofs, they still may not find them convincing (Knuth, 2002a). Treating the proof a particular case as the proof for the general case is also common among teachers (Knuth, 2002a; Martin & Harel, 1989). Although familiarity of a statement influence the degree that secondary teachers are convinced of an argument (Knuth, 2002a) this was not found to be a factor in Martin & Harel's (1989) study. However, familiarity might be very subjective. Finally, Jones (2000) concluded that technical fluency does not necessarily mean richly connected subject knowledge.

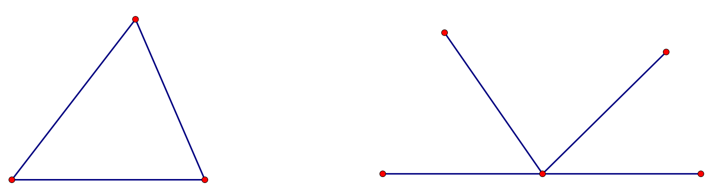
Although past research informs us about teachers' conceptions of proof and agrees that teachers tend to accept empirical arguments as proofs, there is still not a clear picture of the underlying reasons. Furthermore, as Harel and Sowder (1998) point out "despite its dominance, the inductive proof scheme phenomenon is not entirely understood" (p. 252). This paper uses the case of a pre-service middle school teacher to identify some factors that interfere with her acceptance of an invalid argument as a proof.

## Methods

The case analyzed in this paper is one of three participants who had volunteered to be a part of a study investigating pre-service middle school teachers' conceptions of proof. At the time of the study the participants were enrolled in a collegiate level mathematics content course designed for pre-service middle school teachers with a focus on geometry. Two of the objectives of the class as listed on the class website were to strengthen the understanding of and the ability to explain why various procedures and formulas in mathematics work and to promote the exploration and explanation of mathematical phenomena.

One semi-structured interview (Bernard, 2002) was conducted with each participant and their written work was also kept as data source. Each interview started with questions trying to elicit participants' knowledge and beliefs about the nature of proof and the role of proof in mathematics and mathematics education such as "What is your experience with proof?", "What does it mean to prove something?" and "What is the role of proof in mathematics?" Next the participants were asked to prove that the interior angles of a triangle add up to 180 degrees. After that they were given arguments for the same statement and asked to evaluate those arguments

(a) I tore up the angles of a triangle and put them together (as shown below).



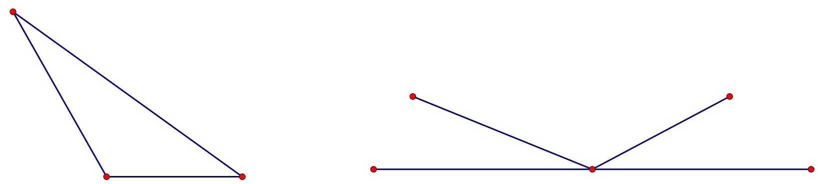
The angles came together as a straight line, which is 180 degrees. Therefore, the sum of the measures of the interior angles of a triangle is 180 degrees.

(b) I tore up the angles of the obtuse triangle and put them together (as shown below).

The angles came together as a straight line, which is 180 degrees. I also tried it for an acute triangle as well as a right triangle and the same thing happened. Therefore, the sum of the measures of the interior angles of a triangle is 180 degrees.

*Figure 1.* Two arguments about the sum of the interior angles of a triangle.
and decide whether they were proofs or not. Two of the three arguments were analyzed for this study and they are given in Figure 1 (adopted from Knuth, 2002b). The following task was to ask the participants what they knew about the sum of the interior angles of other polygons and depending on their answer ask them to figure out an answer and/or prove their answer. The final task was to evaluate arguments for an algebraic statement in order to see if the different nature of the arguments would influence the way that the participants would evaluate them. The task which was taken from a study by Healy and Hoyles (2000) is given in Figure 2.

To analyze the data I went through the entire transcript line by line and the corresponding written work and tried to make conjectures regarding what the participant was thinking. I identified the places where a particular theme was re-occurring and made sure that each conjecture was supported by other relevant things that the participant said throughout the interview. Eventually, I connected these conjectures to a general account of the participant's thinking.



**A1.** Arthur, Bonnie, Ceri, Duncan, Eric, and Yvonne were trying to prove whether the following statement is true or false:

**When you add any 2 even numbers, your answer is always even.**

*Arthur's answer*

$a$ is any whole number
$b$ is any whole number
$2a$ and $2b$ are any two even numbers
$2a + 2b = 2(a + b)$

*So Arthur says it's true.*

*Bonnie's answer*

$2 + 2 = 4$ $\quad 4 + 2 = 6$
$2 + 4 = 6$ $\quad 4 + 4 = 8$
$2 + 6 = 8$ $\quad 4 + 6 = 10$

*So Bonnie says it's true.*

*Ceri's answer*

Even numbers are numbers that can be divided by 2. When you add numbers with a common factor, 2 in this case, the answer will have the same common factor.

*So Ceri says it's true.*

*Duncan's answer*

Even numbers end in 0, 2, 4, 6, or 8. When you add any two of these, the answer will still end in 0, 2, 4, 6, or 8.

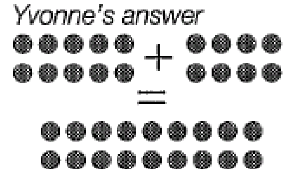
*So Duncan says it's true.*

*Eric's answer*

Let $x$ = any whole number,
$y$ = any whole number
$x + y = z$
$z - x = y$
$z - y = x$
$z + z - (x + y) = x + y = 2z$

*So Eric says it's true.*

*Yvonne's answer*



*So Yvonne says it's true.*

From the above answers, choose **one** that would be closest to what you would do if you were asked to answer this question.

From the above answers, chose the **one** to which your teacher would give the best mark.

*Figure 2.* Seven attempts at proving that the sum of two even numbers is always even.

## Results

This paper reports results for one of the three participants in the overall study. Mary (pseudonym) was chosen because the interview with her seemed to offer richer data in terms of providing insights into and understanding complexities of a pre-service teacher's conceptions of proof. Mary, with regards to her experience with proofs, said that she remembered learning them in high school geometry classes and also mentioned that they have done some simple proofs in their content class. She also mentioned her discomfort with proofs.

*Mary's Conception of Proof*

According to Mary to prove something meant "to show it as a fact and true that it is that way for, in all situations that you say." She explained in response to a follow-up question that "if you are working in general terms, you can, you don't have to test every situation because every situation will follow within those boundaries." Mary seemed to believe that a proof establishes the truth of a statement and to have an understanding of the generality aspect of a proof. She indicated that proofs are important because once something in proven, in other words is shown to be true, then it can be used to solve other problems.

During the interview Mary made a distinction between formal and informal proofs. Although she had a hard time articulating what she meant by formal proof, she referred to textbook proofs as being formal and having "a lot of formal and geometric terms." The way she thought of informal was "kids putting it kind of in their own words and being able to explain it but not necessarily using all the lingo and all the higher knowledge aspect." According to Mary, the ritual aspects of a proof and the level of the knowledge and the language used in a proof determined its being formal or informal. As it will be explained later, this distinction influenced her decisions when she evaluated an argument.

*Mary's Construction of Proofs*

When asked to prove that the sum of the interior angles of a triangle is 180 degrees, Mary was able to construct a valid mathematical proof using the parallel postulate and she concluded that her argument "proves that there is 180 degrees in a triangle." When asked if her proof holds for all triangles she said “yes, because you can draw like, I mean if you wanted to, you could even draw like a right triangle (drew a new picture where the triangle formed was a right triangle), you know what I’m saying, so that one covers right triangles. That (pointed her first picture) is kind of like an almost an equilateral so you could even draw it to where you had like obtuse kinda stuff (drew another picture where the triangle formed was obtuse), it covers anything.”

Without any further information about Mary it could be argued that since she drew extra triangles – right and obtuse – she didn’t understand the generality of her argument or that she needed to check her proof for different cases. However, since she said “it covers anything” it is also possible that she drew those extra triangles to illustrate the generality of her argument and convince the other person about this. In other words, she might be trying to say that it doesn’t matter which type of triangle you draw, the argument holds in any case. This interpretation is also supported by what she said later in the interview "you will still get the same thing because you’ll still get the same transversals just like that (moved her pen along the two transversals in her own second and third drawings) and the same alternate interior angles no matter what (pointed to the right and obtuse triangles that she had drawn) type of triangle you use."

The next proof construction task was about the sum of the interior angles of other polygons. Mary said that they had derived the formula in one of her classes and correctly stated it as “180(n-2)” with n being the number of sides. When asked to prove it she talked about “patterns” and “testing it out on something that you already know like the triangle.” She tested it for the triangle and the pentagon. I tried to challenge her by asking how she would be sure that the formula gives the correct answer for a 12gon, she again referred to patterns. However, all of a sudden, she thought about drawing a pentagon and dividing it into three triangles and wrote 180 into each triangle. She also demonstrated it for a polygon. When I challenged her about how she knew the relationship between the number of the triangles that she created and the number of the sides of a polygon, she said “just by following patterns” and explained that “at some point numbers get too big that you just can’t … you develop a pattern, a consistent pattern, then you can count on it (laughs).” Although Mary could remember a way to prove this statement, she didn't see the generality of that argument, hence relied on her knowledge of patterns for a proof and at the time seemed to be satisfied with this argument.

*Mary's Validation of Proofs*

When Mary was presented with the argument in Figure 1(a) her initial reaction was "I think that's pretty similar to kind of what I did up there. I just didn't cut anything. I just used the parallel postulate to say that." She further stated that it was "accurate" and accepted it as a proof. At the beginning of the interview Mary had said that "when you are trying to prove it or whatever you have to test more than one scenario" however she accepted this argument as a proof. It seems like this piece of knowledge was insufficient to help her see that the argument was "testing only one scenario." One reason might be that she might not have really internalized what she said at the beginning or this argument might not have invoked that knowledge for her. On the other hand, she had also said that "if you are working in general terms, you can, you don't have to test every situation" so she might not have refuted this argument as a proof because she might have viewed it as a "general argument." This is plausible given that she thought that the argument was similar to what she had done. Nevertheless, she labeled it as informal since it didn't involve "any higher math."

Although Mary thought that the first argument was a proof, after examining the argument in Figure 1(b) she seemed to change her mind about it. I think there are two important issues here. Although she thought that the second one was more complete she still did not realize why both of the arguments were empirical. On the other hand, she was aware of the fact that these arguments were not the same as the argument she provided herself (mathematically valid proof) however since she didn't see those as invalid she named them as informal and as not having the "higher knowledge aspects" and as representing "a different level of thinking."

One of the factors that was influencing Mary's decision about the first argument was the relationship she saw between the given argument and her own argument. As it became clear later in the interview another important factor which influenced her decision was her familiarity with this statement. Since she already was familiar with the statement and was confident about its truth, seeing just one case might have been enough for her: "As a mathematician for me or as a math teacher for me this (pointed the first argument) is plenty … I don't think testing one thing when you are trying to figure something out is sufficient I guess, but once you already know like I already know it's a proof or that it's true then this (pointed the first argument) is enough cause I can be ok with that, but for a student like, if I was just trying to figure it out then I would say this (pointed the second argument) would help me a little more because I can actually see that for the different other types of triangles that it works also. … once you know that it's true, I don't think you need all the different examples to back it up but for the initial go at it I think I would want my student to be a little more thorough (pointed the second argument)." In other words, she already knew that the sum of the interior angles of a triangle is 180, so the presented argument was correct or "accurate" for her and being accurate might have been enough for it to be a proof.

Next I presented her the problem about the even numbers accompanied with all the arguments. She first told that she liked Duncan's answer and that it would be closest to what she would do. She also liked Yvonne's "visually because even numbers are always gonna have a pair" and thought that this was explained in words by Ceri. Consistent with the way she identified formal proofs before, she identified Arthur's as the formal one.

An important part of her discussion about these arguments was when she talked about Bonnie's answer: "This (Bonnie's), even though I used the same excuse of patterns, it's, to me it doesn't hold as much water. It's not as, that, like not as valuable to me, just to show patterns (pointed the examples in Bonnie's answer), but I mean cause it works, it proves it, but I guess to a kid they are always gonna say, I mean there's a million ifs, what if this number and this number." The first time during the interview she said "I wouldn't call Bonnie necessarily a proof" and added that "too many what ifs with this one but I agree with it that I agree with what she's thinking that yeah anything any two even number that you add together is gonna be even. That's what she is stating. She needs to go a little further, same as I did, but I couldn't figure it out with those polygons."

Contrasting this episode with earlier episodes of her analysis of empirical geometrical arguments, it can be hypothesized that the nature of the algebraic argument made it possible for her to realize that it was empirical. Although it was not evident in the geometric case that the argument depended on the properties of a particular triangle, it was easier to realize for the algebraic case that the argument only involved numerical computations for a certain number of cases. Hence, Mary could conclude that it was not a proof.

## Discussion and Implications

The case I analyzed in this paper shows there are criteria that one pre-service teacher used to evaluate arguments. Although she did not refute the empirical argument in Figure 1(a) as a proof, I don't think that it can be concluded that she simply believed that showing that something is true for one case constitutes proof. According to Ma (1999) the teachers in her study "ignored the fact that a mathematical statement concerning an infinite number of cases cannot be proved by finitely many examples – no matter how many. It should be proved by a mathematical argument" (pp. 86-87). While this might be a reason for some teachers to accept empirical arguments as proofs, it doesn't necessarily explain why the participant in this study did that in some cases. Whether Mary accepted empirical arguments or proofs or whether she provided an empirical argument when she was asked to prove something depended on other factors as well - factors other than whether she was aware of the fact that empirical evidence doesn't constitute proof. She evaluated arguments according to her criteria of formal vs. informal and the level of thinking used in the argument. It also seemed like she treated arguments differently depending on whether she already knew that the statement was true or not. Furthermore, it was easier for her to realize that an algebraic empirical argument was not a proof as opposed to a similar geometric argument. This study shows that a teacher's decision of whether an argument is a proof or not is not only influenced by whether s/he knows that the argument needs to show that the statement is true for all cases. There seems to be other factors which determine the evaluation of an argument by a pre-service middle school teacher.

An implication of this study for research is that it provides a lens to look through when analyzing teachers' validations of proofs. It points out several factors that researchers need to be aware of and take into account as they look into teacher's conceptions of proof. There are also implications for teaching. Based on the results of this study, it is suggested that rather than trying to prove statements that they are already familiar with students might benefit more from working on statements the truths of which they're not certain about. Furthermore, algebra might provide a better context to introduce proofs where students are more likely to differentiate non-proofs from proofs.

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# ELICITING STUDENT REASONING THROUGH PROBLEM SOLVING

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*Researchers have found that students as young as elementary school can engage in mathematical reasoning. However, particular tasks tend to encourage this reasoning. This paper provides insight into the characteristics of tasks that lead to arguments that represent certain forms of reasoning. In this paper we report on arguments built by diverse student groups, of different ages, that were used to justify their solutions to problems from the fraction and counting strands of longitudinal and cross-sectional studies.*

## Purposes of Study

The National Council of Teachers of Mathematics (NCTM, 2000) Principles and Standards document suggests that a primary goal of mathematics education in grades K-12 is the development of reasoning and proof. The document calls for exposing students at all ages to different forms of reasoning, facilitating their ability to choose and use appropriate forms of reasoning, and encouraging them to develop and evaluate their own and others' mathematical arguments and proofs. The focus on reasoning and proof is alighting to the elementary and middle-grades for good reason; reasoning and proof are the foundation of mathematical understanding and necessary for acquiring and communicating mathematical knowledge (Hanna & Jahnke, 1996; Polya, 1981; Stylianides, 2007; Hanna, 2000). While researchers have shown that children as young as eight and nine years old make conjectures, and justify their claims with sound arguments, we are only beginning to understand how students' mathematical reasoning develops and what environments can best support the development of student reasoning (Yackel and Hanna , 2003). Through a combination of cross-cultural and longitudinal studies, we have observed that a mixture of environment/sociomathematical norms, teacher questioning that evokes meaningful support of conjectures, and well-designed tasks contribute to students' success in building convincing arguments. In this paper we report on arguments that represent certain forms of reasoning across ages and from diverse student groups that were used in justifying solutions to problems from the fraction and counting strands of longitudinal and cross-sectional studies. We found that certain tasks tended to elicit particular forms of reasoning across all age groups and populations.

## Theoretical Framework

Stylianides (2007) defines proof as a *mathematical argument* that that builds upon statements or facts that are accepted by the community, utilizes various forms of reasoning shared by the community, and is communicated by a shared meaning of discourse. Other researchers stress the role of discourse in the mathematics classroom in reasoning and proof (Balacheff, 1991; Hanna, 1991; Maher, 1995, in press). Thus the notion of proof is dependent upon the community in which it emerges. As students engage in reasoning and justify this reasoning to the community they begin to develop proofs. A well-defined, well-written task is the impetus from which reasoning emerges and therefore, task design is crucial (Doerr & English, 2006; Francisco and Maher, 2005; Henningsen & Stein, 1997; Maher, 2002; Maher & Martino, 1996; Stein, Grover,

& Henningsen, 1996). Further these tasks can elicit students building multiple representations, as well as, multiple strategies for solutions (Maher, 2002; Fransisco & Maher, 2005; Henningsen and Stein, 1997). Tasks that are open-ended and complex encourage students to rely on their own mathematical resources and make possible the building of new knowledge. In order to promote justifications and reasoning we recommend that students be given sufficient time to work with each other with minimal teacher interventions. We suggest, also, that students later revisit tasks and reflect on their prior work and the explanations of others (Maher, 2002; Maher & Martino, 1996). When tasks are presented to students as strands of related problems that can be revisited over time and in different contexts, opportunities emerge to extend one's ideas, build on the ideas of others, and construct convincing arguments.

## Method of Inquiry

The episodes presented in this paper come from three data sets.[1] The first is a year long study of students' mathematical thinking that was conducted by researchers in a fourth grade classroom in a rural school in New Jersey. The second source of data is an informal after-school math program consisting of twenty-four sixth grade students that was conducted by researchers in a low socioeconomic urban community in New Jersey, drawn from a school consisting of 99% Latino and African American students[2]. The third source is a longitudinal study now completing its 20th year, in which students from a suburban community engaged in strands of mathematical investigations, as a context for research on the development of students' reasoning and constructing of mathematical knowledge and understanding. The three series of sessions were videotaped with at least two cameras. This study uses data from the first seven 60 minute sessions from the fourth grade study and the first five 60-75 minute sessions from the sixth grade study. Data from the third study includes segments from sessions as fourth grade students investigated problems in counting and combinatorics. Because of space limitations, we give examples of two tasks, one from a strand on fractions and the other from cominatorics. The students in the first two studies worked collaboratively on tasks involving fraction relationships. Cuisenaire rods (see figure 1) were available and students were encouraged to build models. Many of the tasks were identical in both studies. The students in the third study worked on

*Figure 1*. Staircase model of rods.

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

building towers using plastic cubes of two different colors (see figure 2). In all three studies students were encouraged to provide justification for their solutions and to challenge and question the explanations of others.

The video data were transcribed and coded for forms of reasoning. Then, the data sets were compared, and similarities and differences were noted.

## Results

For the purpose of this paper we focus on two tasks. Task 1, Rods, was posed to fourth and sixth graders during a session involving fractions: *If I call the blue rod one, which rod can I call one half?* Task 2, Towers, involved finding all possible towers of a particular height selecting from cubes available in two colors and was posed to students in grades 3 through high school. Students built arguments whose reasoning was both direct and indirect. In particular, for the Rods task, student reasoned using cases, contradiction and upper/lower bounds. For the Towers task, reasoning took the form of cases, induction, contradiction, and recursion. Numerous examples of the above forms of reasoning have been documented (Alston and Maher, 1993; Maher and Martino, 1996, 2000: Francisco and Maher, 2005; Maher, in press; Mueller and Maher, 2007, 2008). Here we offer representative examples that we regularly observe with a wide range of students from a variety of communities.

*Reasoning by Contradiction*

Students used reasoning by contradiction (also known as the indirect method; based on the agreement that whenever a statement is true, its contrapositive is also true) to convince their classmates that there was not a rod whose length was half of the blue rod and that they had built all of the towers of a given (n) height.

*Rods, Grade Four.* While working on the task of finding half of the blue rod, a student used faulty, direct reasoning to name the yellow rod and purple rod one-half. David used a *contradiction* to show that the yellow rod and purple rod were not the same length and therefore could not be named one-half. He used a model of a purple rod and a yellow rod placed next to a blue rod and argued using the definition of one-half, explaining that in order to be called half of the blue rod the two rods would need to be the same length. Alan and Jessica built on David's argument and together formulated a contradiction.

Alan: When you're dividing things into halves, both halves have to be equal—in order to be consider half

Jessica: This isn't half. Those two aren't both even halves.

*Rods, Grade Six*. In the sixth grade Chris reasoned using a contradiction by lining up a train of nine white rods next to a blue rod and explaining that nine is an odd number and therefore it cannot be halved. As Chris' group members offered different arguments, he refined his contradiction five times.

Chris: There is not a rod that is half of the blue rod because there's nine little white rods, you can't really divide that into a half, so you can't really divide by two because you get a decimal or a remainder…"

*Towers, Grade Four*. Stephanie approached the task by applying the procedure of constructing a tower and it's "opposite" to find the 32 unique towers five cubes tall. When explaining her procedure to the class, Stephanie used a proof of *contradiction* to justify her thinking.

Stephanie: With the two [red cubes] together you can make four [towers]. With one [yellow cube] in between you can make three [towers]. With two [yellow cubes] in between you can make two [towers]. With … three [yellow cubes] in between you can make one [tower], but you can't make four in between or five in between [four or five yellow cubes between the two red cubes] …or anything else because you don't have enough … because you can only use five blocks [towers of height five].

*Reasoning by Cases*

For the purpose of this study, critical events were coded as reasoning by cases when students defended an argument by defending separate instances.

*Rods, Grade Four*. David offered an argument by *cases* to show that all of the rods could be organized as either odd or even based on whether or not they could be divided in half. He explained that the white, light green, yellow, black and blue rods were all "odd" since there was not a rod equal to half of their length. He then showed that the red, purple, dark green, brown, and orange rods were "even", using a model to show that two purple rods are equivalent to the length of the brown rod and two yellow rods are equal to the length of the orange rod (in order to demonstrate that these rods had a half).

*Rods, Grade Six.* Justina explained that her strategy of showing that the blue rod does not have a rod that is equivalent to half of its length was to instead find all of the rods that do have a rod equal to half of their length. She drew all of the rods that have a half next to the two rods that make up the half, for example, two yellow rods lined up next to an orange rod. Justina explained that all of the rods in her diagram had a rod that was equivalent to half of their length. She listed all of the cases of these rod combinations and named them "singles". Justina explained, "I was just making half of the color rods, I just made this picture, so like um, half of the orange was yellow, half of the brown was purple, half of dark green was light green, and the same for those two."

*Towers, Grade Five*. After constructing all possible towers four cubes tall when selecting from black and white plastic cubes, Stephanie was interviewed. She explained how she found patterns of towers and searched for duplicates. She then organized her groups of towers according to color categories (e.g., exactly one of a color and exactly two of a color adjacent to each other) in order to justify her count of 16 towers, thus she organized the towers by *cases* (see figure 2). Stephanie used this organization by cases to find all possible towers of heights three cubes tall, two cubes tall and one cube tall when selecting form two colors.

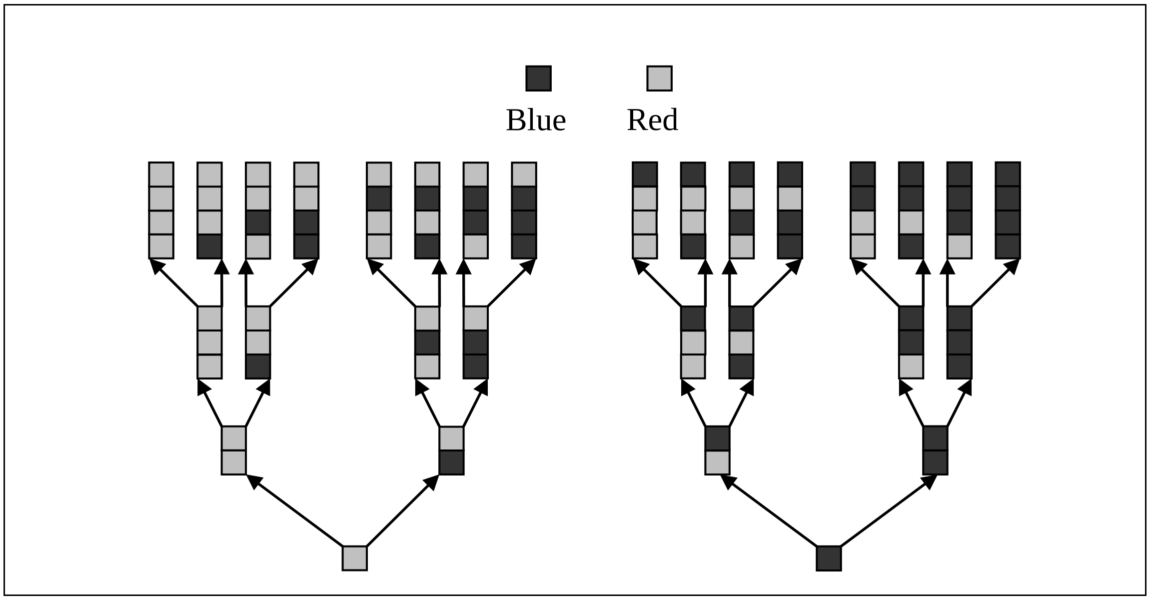


*Figure 2*. Stephanie's extension of the tree diagram introduced during the interview.

*Reasoning using Upper and Lower Bounds*

When reasoning using upper and lower bounds, students displayed the part of the set that was greater than or equal to every element in the set (upper bound) and the part of the set that was less than or equal to every element (the lower bound) and established that there was not an element in the middle.

*Rods, Grade Four*. David began the task of convincing his classmates that there was not a rod whose length was half of blue by offering an argument using upper and lower bounds.

David: I don't think that you can do that because if you put two yellows that'd be too big, but then if you put two purples that's uh, that's uh, that'd be too short.

The researcher asked David if there was any rod between the purple and the yellow, David replied, "I don't think there is anything." When asked to explain further, David showed that the purple rod was one white rod shorter than the yellow rod, and lined up the rods in a staircase pattern in order to illustrate that each rod was one white rod longer than the previous rod (see figure xx). He used this model to show that there is no rod that is shorter than the yellow rod or longer than the purple rod.

*Rods, Grade Six*. In a whole class presentation, Dante explained that instead of using the model of nine white rods lined up next to the blue rod he used a model of a purple rod and a yellow rod. He used the model to show that the purple rod could not be considered to be half of the blue rod because the combination of two purple rods was not equivalent to the length of the blue rod (they were too short). Likewise, the yellow rod could not be named half of the blue rod because the combination of two yellow rods was not equivalent in length to the blue rod. He explained that the yellow rod was one white rod too long to be a half the length of the blue rod and the purple rod was one white rod too short. When asked why this persuaded him that there was not another rod whose length was half of the blue rod, Dante responded, "Because we tried all we can because if usually for the blue piece, it would usually be purple or yellow but yellow would be one um one white piece over it and the pink would be, I mean purple would be one white piece under it."

Chanel backed up Dante's justification using a model that showed the discrepancy of one white rod, using two yellow rods as an upper bound and two purple rods as a lower bound., indicating: "this is blue and the yellow is a little, the yellow is a little bit more than a half and the purple is shorter than a half."

*Reasoning Using an Inductive Argument*

Inductive reasoning was noted when students made generalizations based on individual instances or when the premises of an argument were believed to support the conclusion but did not ensure its truth.

*Towers, Grade Four*. Milin also used cases to organize towers five-tall. He then used simpler problems of towers 4 high and 3 high to build on to towers five high. Next, he organized the towers by "families", using the term "family" to explain the relationship from shorter to taller towers. While his partners based their arguments on number patterns and cases, Milin explained using an *inductive argument*. Milin's explanation in each instance was based on building from a shorter tower exactly two towers that were one cube taller. For example, when asked to explain why from two towers he created four:

Milin: [pointing to his towers that were one cube high] Because – for each one of them, you could add one – No – two more – because there's a black, I mean a blue, and a red- See for that you just put one more – for red you put a black on top and a red on top – I mean a blue on top instead of a black. And blue – you put a blue on top and a red on top – and you keep doing that

Milin then explained, "… and for each one you keep on doing that and for 6 you'd get 64".

*Rods, Grade Four.* David extended the task of finding a rod whose length is equivalent to half of the blue rod by showing that some rods do have a rod that is equivalent to half of their length and some do not. Using the rod staircase he identified all of the rods as "even" or "odd" where odd rods do not have a rod equivalent to half of their length. Thus, David used inductive reasoning by generalizing his solution to show that all of the rods are either even (can be divisible by two) or odd (cannot be divisible by two). The class then discussed the possibility of "cutting" a rod in half to create a new rod and therefore finding half of "odd" rods. Michael used inductive reasoning to broaden the concept of cutting a rod in half to encompass the entire set of rods:

Michael: If you're going to make a new rod, then you'd have to make a whole new set because there'd have to be a half of that rod, too.

David reinforced Michael's argument (also reasoning inductively) by explaining that each time a smaller rod was cut in half, it's half would have to be cut in half and therefore a new set would emerge:

David: Well, what I told you. I thought that, uh, to cut it in half, too, but then I realized that, uh, that you would have to make a whole set…… And make a half for every one.

## Discussion

Our results indicate that both tasks, one dealing with fraction ideas and the other with combinatorics, elicited similar forms of reasoning at multiple grades levels, across different socioeconomic communities. While attending to both tasks students reasoned by contradiction, cases, using upper and lower bounds, and inductively. Although one of the highlighted tasks focuses on fractional relationships and the other combinatorics, they share many characteristics.

In both tasks, students were invited to use manipulative materials but could also approach the tasks with other representations. The building of models naturally led to student collaboration as students were eager to understand each other's models. For both rods and tower tasks, students built models, providing for them a meaningful understanding of the problem. The varieties of representations that the students used to express their ideas were shared in open discussion with others.

Both tasks were open-ended, challenging, and allowed for multiple entry points. They were novel such that a solution was not readily available and therefore students were encouraged to rely on their own resources. Students at all levels of mathematics could engage and realize success. Students had opportunities to be successful in building understanding and communicating that understanding in the arguments they built to support their solutions. Both tasks were open to multiple representations and multiple strategies for solutions. These multiple strategies elicited various forms of reasoning, as in the case of David, who justified his reasoning using three different forms of reasoning (cases, upper/lower bounds, and contradiction).

In addition, the tasks were revisited which allowed students time to reflect on their previous justifications and those offered by their classmates. Thus, when revisiting the tasks students had a schema upon which to build and often revised their justifications and/or used different methods/forms of reasoning.

We suggest that problems, such as these, be integrated into regular mathematics instruction and that students be asked to revisit the same or similar tasks, so that they can build on and extend their approaches, offering opportunities to experience a variety of ways of reasoning. Tasks such as these can serve to engage students in doing mathematics and building arguments. We suggest that strands of open ended tasks that elicit reasoning be integrated in the curriculum at all grade levels.

## Endnotes

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2. See Mueller (2007) for more detailed data analysis.

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# CONCRETE METAPHORS IN THE UNDERGRADUATE REAL ANALYSIS CLASSROOM

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*As part of a study on classroom communication and study habits in undergraduate real analysis, we investigate one professor's use of concrete metaphors as an instructional tool. We offer a classification of the metaphors according to their function and properties as well as evidence regarding how students understood and used the metaphors in constructing their concept definitions and in constructing proofs. The present findings support these metaphors as an effective communicative and cognitive tool which the students applied appropriately to gain understanding. We also address the biological structural metaphor which these concrete metaphors introduced into the study of real analysis.*

Over time there has been a growing awareness among mathematicians and mathematics educators alike of the dynamic role intuitive understanding plays in mathematical understanding (Fischbein, 1987; Burton, 1999; Oehrtman, 2002). The definition of "intuition" in the context of mathematics is not well defined (Burton, 1999), but most agree that metaphor belongs in this category instead of in the realm of formal knowledge which is centered upon proof. Whatever role metaphor might play in formal mathematics, it is certainly integral to human communication. The real analysis classroom represents a primary intersection between the realm of formal mathematics and communication, so one might naturally ask what role metaphor can or should play in that context.

Oehrtman (2002, 2003) conducted some of the only in depth research on metaphor and communication in the undergraduate classroom. He studied calculus students' work on conceptually complex tasks and observed what metaphors the students employed most commonly. Students' use of metaphor proved to be formative in problem solving, but at the same time problematic by often introducing misconceptions. His framework for understanding the role of metaphor in understanding was primarily informed by the work of Max Black who argues that metaphors can generate understanding by a dynamic interaction of different domains of knowledge. He calls this theory "interactionism" which posits that in metaphor the whole can be greater than the sum of its parts. Black argues that strong metaphors have two key attributes: emphasis and resonance (Black, 1962, as cited in Oehrtman, 2002). A metaphor has emphasis if the metaphorical context couldn't be easily replaced with another without losing meaning. A metaphor has resonance if the parallels between the two contexts are strong enough to allow deep elaboration. Oehrtman references John Dewey's theory of instrumentalism where students use "relevant tools are applied technologically against problematic aspects of situations" (Hickman, 1990, as cited in Oehrtman 2003) to establish the dual role of metaphors both for comprehension and for solving specific problems.

## Present Investigation

During a larger study on classroom communication and study habits in an undergraduate Real Analysis (senior level, proof-based) course, the professor repeatedly used concrete metaphors in her classroom explanations. She not only introduced these metaphors to explain concepts or proofs, but integrated them into her classroom language by repeatedly referring to them in later instruction. The full effect of such metaphors cannot be known because we cannot directly observe student thinking during class lectures nor can we understand how the class would be without these metaphorical explanations since they are integrated into this professor's teaching style. However, in light of their repeated use as a tool of explanation, we investigate the students understanding of the metaphors themselves and their ability to relate them to the course material. In this way we hope to shed light on how students organized their understanding about the mathematics using these metaphors as a tool both to construct understanding and to solve problems which in this context meant to construct proofs.

In order to gain insight into the students' understanding and use of the metaphors, we pursued the following questions:

- How well can the students recall the metaphors and understand how they apply to the concepts? Accordingly, how well do the students understand when the professor would reference the metaphors repeatedly over time to remind students of a given idea or point?
- To what extent do students integrate these metaphors into their concept image or concept definition and use them to articulate or reason about a given concept?
- What, if any, are the misconceptions that arise in student conceptions as a result of the metaphors provided?
- What, if any, are the affective influences of the metaphors upon the students' classroom experience and overall approach to the course?

We borrow our theoretical framework from that of Oehrtman (2002) analyzing the metaphors in terms of Black's classification of metaphors and seeking to gain some view into how students use metaphors as tools to construct understanding and build proofs according to Dewey's theory.

## Methods

The study observed the professor teaching first-semester undergraduate Analysis during two 15-week semesters at a mid-sized (about 25,000 students) university in the southwest. The professor had taught for about 10 years and had taught undergraduate Analysis 2 times prior to the study. She has received several teaching awards and has a reputation among the students as a good but challenging teacher. The course content consisted of analysis on the Real number line covering the cardinality of infinite sets, limits of sequences, limits of functions, and continuity. Each course began with between 20 and 30 students and ended with between 15 and 20 students. Math majors made up most of the cohort because only they are required to take Analysis.

We observed every class meeting keeping record of the written and verbal explanations and discussions. Each semester, we conducted weekly interviews with a set of 4-6 volunteers asking them to:

- recall and explain portions of the classroom discussion,
- give explanations of their understanding of specific concepts, definitions, or theorems,
- write specifically chosen proofs to see their functional understanding of the concepts, and

- report their homework, study, and test preparation habits.

All of the students interviewed were math majors. We also interviewed the professor regularly and she explained her plans for the classroom instruction, provided reasoning behind how she structured her explanations and her emphases, and related any interactions she had with the students which were particularly interesting or meaningful to her.

To understand how students' used the metaphors to construct their concept images, the interviews sometimes invited students to articulate their understanding of those concepts which had been explained using a concrete metaphor. Specifically we observed whether any extraneous aspects of the metaphor were carried over into the mathematics or whether the students would use the metaphor to articulate or recall the ideas. After that, usually at least one week after the metaphor was introduced, they were asked to explain the metaphor, its purpose, and the mathematical concept to which it was applied. This was intended to discover how well-formed the student's understanding of the metaphor was and how effective the professor's repeated reference to the metaphor might be.

## The Metaphors

The extent to which the students would be expected to articulate the metaphor or use it significantly for sense-making depends upon the role the metaphor plays in the explanation and how closely linked it is to the concepts being presented. There were two major roles the metaphors played in the discussion. Some metaphors were used to either clarify the logical structure of the definitions or theorems or to guide the development of proofs regarding their logical structure. These metaphors generally lack emphasis, and have strong resonance in their logical structure and nothing else. This first category shall be referred to as *logical metaphors.* Other metaphors were actually used to demonstrate aspects of the concepts themselves or the structure of definitions. These metaphors on the whole had emphasis, but lacked deep resonance. This type shall be referred to as *mathematical metaphors*. The second category contains those metaphors most likely to be taken into the students' concept images of specific concepts. Thus these are most likely to be used in recall, employed for sense-making, integrated into student language, and introduce misconceptions.

Another form of metaphor appears in the classroom which has minimal conceptual influence. In describing the definition of a cluster point, the professor compared the points of a set to fire ants and said you do not want to be a cluster point for ants. This metaphor falls into the third category which merely introduced concrete language to make a compelling image, but based on which no conclusions were drawn. In this way, the metaphor may influence recall, but is less likely to be used for reasoning purposes. This use of metaphor is the source of the common historical perception which asserts that metaphors are primarily stylistic devices rather than reasoning devices. These could be called *visual metaphors*.

### *Logical Metaphors*

One of the most prominent and well-explored examples of a logical metaphor from the classroom was that of the orange and white tigers. Two sets are defined to have the same cardinality if there exists a bijective (one-to-one and onto) function between them. However the professor reported finding that students would often think two sets did not have the same cardinality if they were presented with a function between them which was not bijective. This misconception hinges upon the students' understanding of the quantifier "there exists" and the orange and white tiger metaphor seeks to clarify the role of this quantifier. The professor defines that a forest is called "special" if there exists a white tiger in it. She then tells the students that someone goes into a forest and finds an orange tiger. She asks if this is enough evidence to say that the forest is not special. Similarly she explains that a function which is not bijective proves nothing about the relative cardinality of two sets

Almost two weeks later on a homework problem, the students were asked to find a function between two sets of the same cardinality (the natural numbers and the even natural numbers) which is injective but not surjective. The students knew the two sets were the same size, so they thought no such function existed. She responded to the students' belief by asking whether her finding a white tiger in a forest proved there were no orange tigers in the forest. One and a half months later, another student was working on a problem trying to disprove a "there exists" statement and the professor asked her, "How do I show there does not exist a white tiger?" The student then replied that, "You look at every single one." The teacher repeatedly referred to the metaphor in the context of "there exists" proofs and it became integrated into her classroom language.

During the second semester, the professor used the white tiger metaphor to explain a sequence convergence proof. She showed how each requirement of the definition was addressed in the proof using the language of the metaphor. The definition requires that for any epsilon, an index be found such that all terms of the sequence after that index are closer to the limit value than that epsilon. The professor expressed this by saying that whatever cage she gives the students, they must find a white tiger which fits in that cage. In this way, the analogy was extended to provide the students with a more global understanding of the structure of the proof.

The professor also used logical metaphors to give clarification to the requirements placed upon definitions. The definition of the limit of a function at a point requires that the point be a cluster point or accumulation point. In her interviews, the professor emphasized how much she wanted the students to know the difference between a limit not existing at a point and when the limit definition simply doesn't apply because the point in question is not a cluster point to begin with. In the classroom, she compared considering the limit at a non-cluster point to finding a tiger and trying to measure its RPM (revolutions per minute). She pointed out that since tigers have no wheels, then the question is irrelevant, just like the question of the limit of a function at a point which is not a cluster point. She later created a parallel metaphor to explain why the function must be defined at a point for it to be continuous at that point. She said that someone asked her if her dog likes her cat when in fact she has a dog, but no cat. Similarly one cannot determine if the limit of a function equals the value at the point if there is no value at the point. Here the logical metaphor works to motivate the need for a condition rather than explain the structure of the situation in question.

*Mathematical Metaphors*

There were several key metaphors which fit into this category. One example compared functions between two sets as arrows being shot by the elements of the first at the elements in the second. Even the language and diagrams used early in the course regarding functions call upon this metaphor by calling the co-domain the "target" and expressing f(a)=b by drawing an arrow from a to b. This language was then used to understand the concepts of one-to-one and onto. One-to-one was thus described to mean no one gets hit with two arrows while onto means that "everyone gets hit." This portion of the metaphor gives an intuitive understanding of the two definitions which is relatively consistent, but the professor then employed the metaphor to explain how to prove that a function is onto. She points out that if onto means "everybody gets hit," then the proof must begin in the target, it must pick any element of the target and show that it "gets hit." The professor follows up by asking the students what it means mathematically for that element to "get hit," namely that f(a)=b. In this way, the metaphor was not only explanatory and concrete, but it was also made functional as it guided the construction of a proof.

During the section of the course about sequences, the professor introduced two key sequences to help the students decide what a function tending to infinity should mean. One function was unbounded, but did not tend to infinity (1,2,1,3,1,4,1,5,…) while the other did tend to infinity, but was not monotonic (2,1,4,3,6,5,…). Both pairs of qualities were chosen by the professor because she had observed students falsely equating the two, namely that unbounded sequences tended to infinity and that sequences tending to infinity were monotonic. The professor built the metaphor around equating sequences that tended to infinity with mammals. The class came to refer to the former as the penguin sequence it looked like it had fur like a mammal, but in fact it was a bird. The second was deemed the platypus sequence because it had qualities most mammals don't such as laying eggs and a bill, but in fact it was a mammal.

A third mathematical metaphor was developed from a student's comment during the second semester of the study. The professor introduced the definition of convergence of a sequence. This definition requires that for any epsilon greater than zero, there must exist a term in the sequence after which every term is closer to the limit than epsilon. As the class was discussing this definition on the first day it was introduced, one student compared the epsilon neighborhood of the limit to a party. He explained that it is not a party unless eventually everyone goes to the party. The professor seized upon this language and began to explain the definition in these terms elaborating the metaphor to include all of the key aspects of the mathematical definition. For two weeks thereafter, her classroom explanations regarding sequences always included some verbal reference to the party analogy.

The final form of party metaphor was as follows: no matter how big your room is, it is only a party if after some time, everyone is at the party. We don't care if there are a few stragglers at the beginning, but only finitely many people can be left outside of the party. Like the arrows metaphor, the professor used the metaphor to explain the general logic of a convergence proof both to motivate each next step and to correspond the pieces of the proof and the definition.

*Structural Metaphor*

The professor's use of specific metaphors had a more universal role in the class beyond the specific concepts which they were used to explain. The most common arena from which the metaphors were built was biology. Biology was the structural metaphor which she used to guide her explanation of the concepts. The professor built her classroom instruction around key examples. Very often before a theorem was introduced or suggested, the professor invited the students to see if they could come up with an example of a convergent but unbounded sequence, a monotone and bounded sequence which is not convergent, a bounded sequence without a convergent subsequence, etc. all of which are impossible. From the students' observations that they cannot in fact manufacture such an example, they discovered the theorems and went on to prove them. To the professor, classifying different sequences and functions according to their properties parallels classifying animals by their properties into species and families. This was also reflected in the classroom language which discussed the "behavior" of mathematical objects.

## Student Responses

There were five major aspects of students' understanding of the metaphors which were of particular interest: recall, application, reasoning or articulating with the metaphor, misapplication, and affective response. Of the 7 students directly interviewed regarding the white and orange tiger metaphor, 5 students could correctly recall it. Some took a minute to reconstruct the logic, but once they knew the context, they could recall the metaphor and explain how it applied. Others could not remember to which specific concept it was originally applied, but could come up with another situation in which it would apply. As part of an ongoing study, one student was asked to recall the metaphor eight months after the class ended and once he remembered it was applied to bijections and cardinality, he could correctly explain it.

One student in each semester spontaneously used "white tiger" language to talk about proof. The first referenced the metaphor to explain the relative difficulty of disproving a there exists statement (showing for every). "When you make a claim about for every, it is usually very difficult to say, you know: the white tiger thing. All the tigers in this forest are orange. Oh, really? You know, find them all and bring 'em here." The fact that the student references this metaphor in the context of a "for every" proof constitutes an extension of the metaphor beyond its original application.

The next semester, another student used the tiger metaphor when explaining rules for certain types of proofs:

> Student: "For these kind of proofs specifically I remember she said you use your scratch work and you look for the white tiger and then you show the white tiger in the proof."
> Interviewer: "Okay and what kind of proofs are those?"
> Student: "Existence proofs."

Of the five students who were asked to articulate their understanding of one-to-one and onto, three used the arrow metaphor's terminology, primarily to talk about onto. When directly asked what metaphor she used, all of them recalled the arrow language. Two students pointed out that because onto means that "everything in the target gets hit," proofs that a function is onto begin with an arbitrary element of the target rather than beginning with a point in the domain. In this way, the students not only had integrated the metaphor into their concept image of onto, but used the metaphor to guide the construction of proofs.

All of the students interviewed referred to the penguin and platypus sequences under those names and understood that they were named for their curious properties that made them atypical examples. It was less common for students to directly be able to explain the logic which paralleled "mammals" to sequences which tends to infinity. All but one of the students recovered the metaphor either from direct memory or reconstruction from their knowledge of the two sequences. None of them gave evidence of using the metaphor to remember which sequence tended to infinity. However the names appeared integral to their memory of the sequences since some would recall a slightly different sequence which had parallel properties such as (2,1,3,2,4,3,…) for platypus or (0,1,0,2,0,3…) for penguin. Two of them were asked about these two sequences by name 8 months after their class ended, and they both could recall the basic behavior of the sequences, explain the metaphor, and explain why they were interesting examples (platypus shows that not every sequence tending to infinity is monotone and penguin shows not every unbounded sequence tends to infinity).

Of the four students interviewed during the second semester about the definition of a sequence converging, three of them used "party" language to explain the definition and the other one could recall the metaphor when asked directly about it. One of them even extended the metaphor. He deemed the term after which the sequence was in the given epsilon neighborhood as the “popular guy” because after he comes, everyone else goes to the “party.” He reported having explained the definition to several other people as they studied together which might mean he extended the metaphor for explanatory purposes. However none of the students were observed to use this terminology in proving the convergence of a specific sequence. One student used this language to explain the proof that limits are unique because we form two separate parties around the two limits. Also, only one student persisted in using this language after the test. The other students began to speak almost exclusively in more mathematical terms and the one who still used party language merely interchanged the terms “party” and “neighborhood,” but did not seem to use the metaphor for sense-making.

Though the professor explained the party metaphor pointing out that it doesn’t matter how big the room is, most of the students during class discussion and during interviews either failed to address the arbitrarity of epsilon or only at the end of the definition that “it works for any epsilon you pick.” It could be that the party metaphor, being primarily concrete, fails to capture this aspect of the definition which led to this weakness in the students’ concept definitions. However any pictorial representation of the limit definition usually includes at most 3 specific epsilon neighborhoods. Thus it remains unclear whether the metaphor was a primary cause of this misunderstanding.

Several students reported that the presence of the metaphors made the class much more enjoyable. There was a uniformly positive affective response to this mode of instruction, sometimes quite enthusiastic. One of the students, despite consistently not being able to recall the metaphors throughout the interviews, at the end of the class reported that they were one of his favorite parts of the course. When asked whether he ever actually referred to the metaphors when solving problems, he said he did not. “[I don’t use the metaphors] really on the test, just as I’m learning it. Like, if I understand I don’t have to think about the analogies, I guess, once I know what is going on.”

Several students recognized the parallels between their study of analysis and biological classification. One student noted the appropriateness of biological metaphors because, “in a sense, what are we trying to do but categorize things... and so basically this is coming down to organization and classification, you know?” Another student similarly described their study of sequences and functions saying they, “classify and group [various examples] into like groups.”

## Discussion

Alibert & Thomas (1991) explain that the sequential structure and presentation of proof often obscure the central ideas. Students may verify a proof to themselves by checking the validity of each line without gaining any global understanding. The professor used the tiger metaphor to point out the global structure of a convergence proof. At least one student then used this metaphor to develop his own proof scheme of “there exist” proofs and another was able to transfer the metaphor to disproving “for every” statements.

The interviews provided evidence that the students did understand the metaphors which were presented in class as they applied to the mathematics. Moreover, even if they couldn’t recall the metaphor by itself, they could recall it in from context in which it was introduced. This implies that the professor’s integration of such metaphors into classroom language was an effective form of communication. Some of the metaphors proved very powerful for recall as evidenced by students being able to recall the penguin and platypus sequences months after the class was over, one without the names of the sequences being mentioned. There was evidence of students using the metaphors for sense-making, namely that they extended the metaphors (tigers and party) or used them to organize their proof approaches (tigers and arrows). Most prominently the students reported a positive affective response to this mode of communication.

The metaphors were primarily used as tools for the students to construct their understanding. None of the students were limited to explaining their understanding by the metaphor and would often acknowledge the less-formal nature of these explanations. The metaphors were not central to the student's concept images because some of them only referenced the metaphor when specifically asked to. They used the metaphors less over time as their concept images were more fully developed. One student even reported directly that he only used the metaphors to gain initial understanding, and then left them behind. No direct evidence arose of students misapplying metaphors or bringing extraneous aspects of the metaphor into their mathematical understanding. All of this supports the characterization of these metaphors according to Dewey's instrumentalism as tools the students used to mediate difficult situations, in this case building a strong concept definition or learning how to prove in the context of real analysis.

These findings stand in contrast to those of Oehrtman (2002) who found that calculus students often used metaphors to reason about problems to the extent that they became mathematically incorrect. It is unclear however whether these differences can be attributed to the relative difference in mathematical sophistication between calculus and analysis students, the difference between a computational versus a proof-based course, or whether the fact that the metaphors in the present study were obviously non-mathematical helped the students to apply them properly. The logical metaphors in the present study were also much less susceptible to misapplication since they weren't applied to the primary concepts being studied, and because they were primarily logical they had a very high degree of resonance.

The present study thus affirms the possibility of effectively employing concrete metaphors in classroom discussion to help students construct their understanding of undergraduate analysis. Moreover, the use of biological metaphors appeared to implicitly provide students with a structural metaphor by which to think about analysis. This is one aspect of the overall model this professor used of which a more full description is forthcoming.

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# A JOURNEY WITH JUSTIFICATION: ISSUES ARISING FROM THE IMPLEMENTATION AND EVALUATION OF THE MATH ACCESS PROJECT

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*Justification is a practice central to mathematics and mathematics learning. In this paper, we report on results and issues related to justification that arose during the course of a professional development program, the Math ACCESS Project. Justification was one of the three themes central to the project. Data sources included records of project activities, pre-post assessments, and documentation of analyses and the issues that arose while conducting the analyses. Results suggest that the Math ACCESS Project advanced teachers' understanding of justification, but that some important issues were left in an unsatisfactory state. Implications are discussed.*

## Objectives and Purposes

Justification is a central to doing and learning mathematics. This practice requires the articulation of reasoning and is facilitated significantly by the use of mathematical terminology and forms of expression. Students' participation in practices of justification and argumentation expands their mathematical knowledge, as they identify and articulate their own reasoning and gain exposure to others' arguments (Boaler, 1997; Hiebert et al., 1997; National Research Council, 2001; Wood, Williams & McNeal, 2006). In this sense, the practice of justification is a *learning practice* (Cohen & Ball, 2001). Justification is also a core activity in classrooms that support more equitable outcomes in mathematics (Hiebert et al., 1997; Boaler & Staples, 2008).

Supporting student justification in classrooms, however, is challenging (Chazan, 2000; Stylianides, 2007) and researchers have documented that this important practice is absent from many classrooms (Jacobs et al., 2006). Fostering student justification requires a broad repertoire of pedagogical moves (Staples, 2008) supported by a commitment to this practice, appropriate classroom norms, and content knowledge (Knuth, 2002). A further difficulty is that what counts as a justification varies across settings (Stylianides, 2007). It depends on the community's shared background and its purposes. Currently the field of mathematics education has an underdeveloped notion of justification (Stylianides, 2007).

In this paper, we document the design and impact of a professional development program, the Math ACCESS (Academic Content and Communication Equals Student Success) Project. This on-going project has a unique focus – working with teachers in urban settings to understand the language demands of student participation in justification and higher-order thinking in mathematics classes. While important for all mathematics learners, the need for teachers to be able to organize instruction to foster student participation in justification is perhaps heightened in urban areas where students are more likely to be linguistically diverse and have received lower quality instruction at some point in their mathematics careers.

Although there is no standard definition for justification, for the purposes of our work and this paper, we used as a working definition for justification the process of removing doubt about a claim using logical reasoning. We also draw on Toulmin's (1958) framework of argumentation and see justifications as comprised of claims, warrants and evidence, either explicit or implicit.

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**Context**

The Math ACCESS Project was designed as a professional development experience for teachers of mathematics working in urban settings. The work was supported by a Teacher Quality Partnership Grant from the Connecticut State Department of Higher Education. The conceptual model guiding the professional development comprises three "pillars": *Appropriate/ Effective Use of Academic Language, Student Justification and Collective Building of Arguments,* and *Access by all Students.* Each pillar addresses a core component of instruction that has a strong research base documenting its value for student engagement and learning, and promoting more equitable outcomes. In this paper, we focus on the second pillar, *Student Justification,* and document and discuss the relevant activities, outcomes and issues from our work with a group of teachers during the ACCESS Summer Institute. See Truxaw, Staples and Ewart (under review) for discussion of the first pillar, *Academic Language*.

Given our goals and the requirements of the grant program, we organized the PD into two main components. The "instruction" comprised a one-week intensive summer institute (40 hours) during July and a half-day (5 hours) session in September. The "follow up" comprised a modified form of lesson study where teachers, organized in grade-band teams, collaborated to develop, implement, and debrief lessons that used pedagogical strategies related to each pillar. We briefly describe some of the institute activities, emphasizing those related to justification.

Our work with respect to *Justification and Collective Building of Argument* focused on the need to provide students with opportunities to engage in *sense-making* (Hiebert et al., 1997) through explanation and justification. Teachers analyzed the cognitive demands of tasks (Stein, Smith, Henningsen, & Silver, 2000), modified existing tasks to prompt higher-order thinking, discussed the validity and completeness of student justifications (Toulmin,1958), examined the language demands of offering justifications (e.g., expressing a general statement), developed strategies for providing formative feedback on student written justifications, and worked in teams to write and implement lessons that promoted higher-order thinking and justification. For example, one activity engaged teachers in examining student work on a test item (from the State tets) that requested an explanation. Teachers analyzed the responses for the claims, warrants and evidence and discussed the validity, completeness, and merits of different responses. Note that this was not easy work. A prerequisite for this work was a clear understanding of the central mathematical ideas needed to solve the problem.

The other two pillars were also critical to the justification work, as these pillars are mutually supportive. Related activities during the Summer Institute included attention to establishing productive classroom norms, providing multiple entry-points into justification tasks, and developing strategies to expand students' math vocabulary and develop student command of certain phrases (e.g., at least, the least, for each person, for every person). For example, a series of activities involved unpacking language demands within mathematics curriculum materials, state testing materials, and student work samples. Participants subsequently learned to write language objectives for mathematics lesson plans (Echevarria, Vogt, & Short, 2004) that focused not only on vocabulary, but also functional language (Schleppegrell, 2007) (e.g., *Students will continue to build an idea of what makes a good explanation by using a language frame: "____ is correct/incorrect because ____."*).

To document the impact of the project and further our understanding of the practice of justification, we address the following research questions:

1. What are the outcomes and effects of the ACCESS professional development activities

for participating teachers?
2. What issues arise as teachers work on understanding the nature of justification and practices related to supporting student participation in justification?

Twenty-four grades 4-9 teachers from one urban area participated in the ACCESS Summer Institute; 20 teachers continued the program during the academic year. Of the 24 teachers, 11 teachers taught the single subject of mathematics (grades 7-9), 10 taught multiple subjects, including math (grades 4-7), and 3 teachers were special educations who worked in mathematics classrooms. Teaching experience ranged from 0 years to over 21 years. The first two authors did the majority of the instruction; all authors are supporting school-based follow-up activities.

Data collection and analysis are ongoing as we are currently engaged in the academic year "follow up." The remainder of the paper focuses on the results from the Summer Institute.

## Methods of Inquiry

This study employed a mixed-methods design to investigate outcomes and effects of the ACCESS Summer Institute activities. Participants completed two pre-post assessments, one primarily targeted content knowledge and the other primarily targeted issues related to academic language (e.g., teachers' awareness of the language demands of mathematics (specifically open-ended prompts) and ability to generate strategies to promote academic language). Across these two assessments, five questions focused on justification, specifically teachers' production of justifications and/or their analyses and evaluations of student justification.

*Instrument Development*

*Content knowledge assessment.* Items for the content knowledge survey were drawn from previously validated sources (e.g., CT State Department of Education; Healy & Hoyles, 2000; *Learning Mathematics for Teaching Project;* Hill, Ball, & Schilling, 2004)) and were selected by a team of mathematics educators to fit the content themes of the project, algebraic and proportional reasoning. The items were field tested for appropriateness and timing. The final instrument included 7 multiple-choice questions and 2 open-ended questions.

*Language assessment.* The second assessment was developed specifically for this investigation to uncover participants' growth related to supporting the development of students' mathematics academic language, especially with attention to ELLs and higher order thinking. Following recommendations of Gable and Wolf (2001), content-validity was sought through the use of research literature and experts' content validation (i.e., mathematics educators, linguistic experts, and methodological experts) that noted the adequacy of the items as representative of the specified key constructs. The final instrument included 6 open-ended questions related to language use, challenges, and strategies. The assessment also included 7 Likert-type questions asking participants to self-report on their perceived knowledge about and confidence in addressing various issues related to language and justification addressed during the PD. The content validity questionnaires and the final instruments are available from authors upon request.

*Data Collection and Analysis*

The data relevant for justification comprised five assessment items and two Likert-scale items. Multiple choice and scaled items were performed using standard statistical techniques (Green, Salkind, & Akey, 2000). The open-ended responses were analyzed using rubrics and additional qualitative analyses were conducted using standard techniques (Glaser & Strauss, 1967). For example, for questions that asked teachers to select from a set of three which student justification they thought was *best*, researchers employed open coding of emergent themes, discussed the themes, and then employed axial coding to make connections among the categories and to refine the coding schemes (Strauss & Corbin, 1990). Coding categories and definitions for each question were developed and tested by at least two researchers. Disagreements were resolved in discussion with a third research as needed. Applying these codes allowed us to look for change on both the group level and the individual level.

## Results and Discussion

Results from the pre-post assessments indicate that the Math ACCESS Project did have a positive impact on teachers' knowledge of content and language-related issues and strategies. We will describe content results briefly and then focus on the justification items.

*Evidence of Overall Impact*

Twenty-four participants completed the pre- and post- content assessments related to content. The results demonstrate increases in content knowledge overall. Scores of 15 participants (63%) increased; scores of 7 participants (29%) decreased, scores of 2 participants (8%) remained the same. Overall content knowledge showed a statistically significant difference in mean scores ($p < .05$). (See Table 1.)

Table 1. *Assessment of Content Knowledge of PD Participants*, n = 24

| Administration | Mean | SD | *t* | df | *P* |
|---|---|---|---|---|---|
| Pre | 9.33 | 3.67 | | | |
| Post | 10.63 | 3.63 | 2.24 | 23 | .035 |

Teachers' self-report on the Likert items also demonstrated significant gains. Nineteen participants completed all questions on both pre and post administrations. The mean score for all 7 items increased from pre to post (between 0.50 and 0.95 points per item), with all items except Item 1 being a statistically significant difference ($p < .05$). A one-tailed t-test showed a significant difference between the pre- and post-results. Overall, these data demonstrate a positive and relatively strong impact of the professional development work with respect to language.

*Evidence of Impact Specific to Justification*

Qualitative analyses of the various assessment items that targeted justification offered additional insights into what the teachers had learned, and not learned, and raised some important issues for our consideration. We focus our discussion on two main issues that emerged as we worked on the idea of justification during the ACCESS Institute and as we analyzed participant data from the Institute. The first pertains to teachers' (and mathematicians') thinking about the *qualities* of a good student justification. The second pertains to the various *types* of justifications that can be offered and evaluating the validity of students' work.

Two assessment items in particular offered insights into how teachers were thinking about the qualities of a good justification, including validity. We describe and discuss each below. These two items demonstrated quite a bit of change between the pre- and post-assessments, but not necessarily towards what we would consider deeper understanding of justification. As will be discussed, we ultimately interpreted our results as indicating that we had raised issues related to justification enough to prompt awareness, new questions, and reflection, but we did not work on the ideas intensively enough to solidify new understandings of what comprises a mathematically valid justification and a high quality justification.

Question 7, adapted from Healy & Hoyles (2000), asked teachers to evaluate two student justifications of a particular claim (see Figure 1). Teachers determined whether each student's response showed that the claim was (always) true and whether it demonstrated *why* the statement was true.

Bonnie and Duncan were trying to prove whether the following statement is true or false: When you add any 2 even numbers, your answer is always even.

| *Bonnie's answer* | | *Duncan's answer* |
|---|---|---|
| 2 + 2 = 4<br>2 + 4 = 6<br>2 + 6 = 8 | 4 + 2 = 6<br>4 + 4 = 8<br>4 + 6 = 10 | Even numbers end in 0, 2, 4, 6, or 8. When you add any two of these, the answer will still end in 0, 2, 4, 6 or 8. |
| *So Bonnie says it's true.* | | *So Duncan says it's true.* |

*Figure 1.* Prompt from assessment item, question 7, adapted from Healy and Hoyles, 2000.

On the pre-assessment, 43% of the teachers (incorrectly) thought Bonnie's response showed that the claim was always true and 61% thought it showed *why* the statement was always true. On the post-assessment, these values decreased significantly to 22% and 35%, respectively. Indeed, we had an extended discussion about a similar problem during the institute. Initially, it was not clear to the teachers why a response such as Bonnie's was inadequate. Several felt that, by showing many examples, the student had demonstrated that the result was always even. Part of the difficulty seemed to be gaining distance from the truth of the result to evaluate the *validity* of the student's *method*. As the teachers knew the result was true, they did not authentically need to be convinced by the argument. In the Institute, we addressed the idea of "proof by example" (commonly known as an empirical proof) and its limitations. We also discussed using examples that exhausted all possible cases as a valid approach. This latter approach seemed akin to proof by example for some of the teachers.

For Duncan's argument, 74% of the teachers on the pre-assessment and 70% on the post-assessment agreed (correctly) that his response showed the statement was always true. (See Table 2.) Despite the similar overall values, we found that many teachers had changed their response to this question.

Table 2. *Percent of Teachers Identifying the Correct Response for Question 7*

| Question | Correct response | Overall N=23 | | Multiple Subject N =12 | | Single Subject N =11 | |
|---|---|---|---|---|---|---|---|
| | | Pre | Post | Pre | Post | Pre | Post |
| 7a. Bonnie's work | | | | | | | |
| Has a mistake | Disagree | 91 | 83 | 92 | 75 | 91 | 91 |
| Shows that it's always true | Disagree | 57 | 78 | 58 | 67 | 55 | 91 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Shows why it's true | Disagree | 39 | 65 | 33 | 67 | 46 | 64 |
| 7b. Duncan's work | | | | | | | |
| Has a mistake | Disagree | 96 | 87 | 92 | 83 | 100 | 91 |
| Shows that it's always true | Agree | 74 | 70 | 75 | 58 | 73 | 82 |
| Shows why it's true | Disagree | 39 | 26 | 58 | 50 | 18 | 0 |

Furthermore, the elementary and secondary teachers responded quite differently. Whereas 75% of the multiple-subject (elementary) teachers agreed that Duncan's work showed it was always true on the pre-assessment, only 58% identified it as always true on the post. The single-subject teachers showed the opposite trend, increasing from 73% to 82%. We suspect that there were lingering questions about the distinction between "proof by example" and proof by exhaustion, which perhaps were less clear to the elementary grade teachers. It's interesting to note that the idea of proof by exhaustion may have eluded many secondary teachers initially, and that they better understood this idea at the end of the Institute.

In evaluating whether Duncan's work shows *why* the statement is true (it does not), most teachers, on both the pre- and post-assessment, felt that Duncan's response did demonstrate this. In fact, there was an increase in the number of participants who felt the response did address *why*. Clearly, the participants found this student response compelling. In particular, all single subject teachers agreed that Duncan's work demonstrated why the result held.

Supporting these "mixed bag" results are analyses from the second question (language assessment, Q5). This question included a state test item (open ended problem) and asked teachers to identify whether each of three student work samples a) offered an *adequate* response (which requested an explanation) and b) which of the three they felt was the *best* justification and why. Each of the 3 student explanations had merit and none was complete. We used this question to infer, on the group level, what the teachers seemed to identify as characteristics of a good justification. On the individual level, we analyzed teacher responses for any changes in their thinking about what comprised a good justification.

There is no absolute correct answer to the question of which of the three student responses was the best justification. Indeed, among a group of 12 mathematicians with whom we discussed this problem, at least one mathematician selected each of the three student work samples. To analyze this problem, we coded teachers' responses to identify the characteristics the teachers felt were important for a good justification. There were 4 thematic categories of reasons offered for why a particular response was "best" included the following:

- the detail/explicitness of the response (e.g., I could follow their thinking)
- the student used, or stated that s/he used, a particular mathematical method (e.g., student A used proportional reasoning; he explained he saw the pattern)
- the student understood/had deduced the key mathematical relationships needed to solve the problem (e.g., the student knew there were the same number of red and blue blocks, and half as many green blocks)
- a relative argument (e.g., it made more sense than the other ones)

The second and third bullets perhaps need an additional comment. The second bullet reflects that teachers placed high value on recognizable and potentially named methods or mathematical techniques. Note that the mathematical tools used do not inherently make one justification better than another. Nevertheless, this was valued by many teachers. (It is possible that this was a variation of the first category, being detailed/explicit, namely with respect to the method used.) For the third bullet, we interpreted this to mean that the teachers valued that the student was able to identify the core mathematical relationship that was needed to solve the problem.

As we reflected on these reasons offered, we realized that participants generally did not identify qualities such as the student response being convincing or demonstrating why the result *must be* so. We also noticed the preponderance of responses focused on detail and explicitness, although it was not clear what the teachers wanted the students to be explicit about. It was interesting to note that the attributes of clear and concise only came up once each. This led us to wonder whether teachers generally value explicitness and detail, or the obviousness of student method, more than whether the justification "does its job" as a justification and removes doubt about the validity of the claim. Although it is ultimately a judgment call whether an offered justification serves this function, the lack of attention to any associated attribute by the teachers was concerning to us. We also realized that the prompt might not have elicited attributes of a good justification because the prompt, which asked for which response was best, potentially focused teachers on comparing responses and explaining why one was *better* than the others and not what was absolutely good about the one they selected.

In addition to what the teachers' noticed, we queried the data for evidence of growth between the pre- and post-assessments. We coded individual teachers' responses for whether they evinced some change in their perspective about what comprised a good justification. We coded each as either increased sophistication, stayed the same, or decreased sophistication. A response that was more sophisticated was more specific about the components of the response that were valued and why. It included fewer general claims (it made sense to me) and more specific claims (it made sense because the student explained the pattern he found). We found that 9 respondents increased in their sophistication; 5 stayed same; and 5 decreased.

Consequently, we conclude that the ACCESS Summer Institute heightened teachers' awareness of justification, some important qualities of justifications, and the language issues that might arise when asking students to justify, but the evidence suggests that we were not successful in expanding participants' notions of what a good justification is. Indeed, our assessments show that we made them question their original ideas, but did not do enough work to solidify their understandings in a manner that aligns with the mathematics community.

## Implications and Conclusions

Our work with participants in the Math ACCESS Project confirms earlier findings about the challenges teachers face as they work with justification. Justification is a complex practice the requires content knowledge as well as an understanding of how ideas are logically connected, the limitations of certain modes of argument, and how we determine– with a high level of certainty – whether a claim is true. During the ACCESS Institute, we were successful in raising some issues related to justification, but many more discussions are needed to more fully examine this important learning and disciplinary practice. In particular, sustained discussions about the qualities of a justification, and what one might expect from different students at different grade levels, would be beneficial.

As we reflect on the challenges in organizing professional development activities on justification, we have come to appreciate the power of a system that does not encourage teachers to reflect on or think about justification on a regular basis. On mandated assessments, clear explication of steps is as acceptable as a well reasoned justification (both are considered explanations). Most textbook tasks do not request justification. Planning time is tight, precluding deep thinking about issues of justification and formative feedback to students on justifications they produce. It is nearly impossible for teachers to engage this kind of work and thinking on their own given the incentives and constraints of the system. Without sustained efforts, however, findings that suggest abysmally low participation among students in justification will remain unchanged, as teachers are the linchpins in our system for promoting this important practice.

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# HEURISTICS AND REASONING IN AN ONLINE, SYNCHRONOUS ENVIRONMENT

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*In this report, we analyze the heuristics and reasoning that students exhibit as they solve an open-ended combinatorics problem in an online, synchronous environment. While significant research exists on students' mathematical reasoning and heuristics in problem solving, there is little work in these areas as students collaborate to solve problems online. In our analysis, we find that as the students make sense of interpretations of the problem and jointly develop solutions, they display emergent heuristics and reasoning that evolve over the course of their online sessions.*

## Objective

Information-based economies such as in the United States are experiencing an increased demand for individuals who know how to harness information and communication technologies (ICT) to collaborate and to identify and solve complex problems (Reich, 1992). It is important for mathematics educators to acquaint students to environments and learning activities through which they develop abilities and dispositions that allow them to transition to ICT-intensive careers. While much research has been conducted on students' mathematical problem solving (Francisco & Maher, 2005; Schoenfeld, 1985; Silver, 1994) and on Internet-based instruction (e.g., Pena-Shaff, Altman, & Stephenson, 2005; Wallace & Krajcik, 2000), with the exception of a few studies in the learning sciences (e.g., Chernobilsky, Nagarajan, & Hmelo-Silver, 2005; Hiltz & Goldman, 2005; Powell & Lai, in press; Stahl, 2006), little research has been done on Internet-based, collaborative problem solving in mathematics. In light of this gap, it is crucial for mathematics education researchers to investigate how students jointly construct mathematical ideas within online environments.

We have therefore engaged in a longitudinal study to investigate (a) how students use ICT tools to collaborate in mathematical problem solving and (b) what resulting mathematical ideas and reasoning they develop. This study focuses on the latter component of the larger study, and investigates the heuristics and lines of reasoning that students evidence in their interactions as they collaboratively solve open-ended combinatorics problems using an online communications environment.

## Conceptual Framework

The key conceptual terms in this study include heuristics and reasoning. By *heuristics*, we speak of actions that human problem solvers perform that serve as a means to advance their understanding and resolution of a problem task. We do not imply that when problem solvers implement heuristics that they will necessarily advance toward a solution but only that their intent is to do so. Our sense of heuristics includes explicit and implicit general strategies such as categories outlined by Pólya (1945/1973, pp. xvi-xvii, 112-114).

We distinguish heuristics from *reasoning*, which we view as a broad cognitive process of building explanations for the outcome of relations, conclusions, beliefs, actions, and feelings. In agreement with English (2004), mathematical reasoning encompasses such processes as analyzing data, conjecturing, argumentation, forming and justifying logical conclusions, and proving claims. As she describes, these processes make possible such types of thinking as conditional, proportional, spatial, and critical thinking, and deductive and inductive reasoning (p. 13). There is evidence that imagery is deeply tied to mathematical reasoning (Presmeg, 1997; Thompson, 1996), and can affect students' abilities to solve problems, and ultimately, to form generalizations.

## Methods

*Participants and Context*

The fourteen students in this study come from two high schools. Eight of the students, whose mathematics teacher is the fourth author, are from the urban, public high school in Long Branch, NJ (LBHS). The remaining 6 students, whose mathematics teacher is the third author, are from a suburban, private high school in Somerset, NJ (RPS). All of these students are in their fourth year of high school and, in their respective schools, have average mathematical performance.

The students use an ICT tool called the Virtual Math Teams Chat (VMT Chat). Developed by researchers at Drexel University (Stahl, 2006), with support from the National Science Foundation, it is an Internet-based, dual-interaction space with whiteboard and chat windows, which allows users, whether co- or remotely-located, to communicate about mathematics.

For our study during the 2008-2009 school year, we planned for the students to interact online for approximately 18 sessions, each 45 minutes long. During the first half of the year, students would work collaboratively on open-ended combinatorics problems. Students would then switch to solving social choice problems for the second half of the year.

To neither overcrowd nor under-populate each chat room and to facilitate discussion of the problems, we assigned three to four students to a virtual room. Of the four chat rooms, two rooms contained two students from each school, and the other two rooms contained two LBHS students and one RPS student.

To ensure minimal face-to-face communication and that communication among students would occur in the VMT environment, students who were co-located and in the same chat room were placed at desks at opposite ends of their classroom. We also discouraged paper and pencil, as we wanted students to place all of their inscriptions into either the whiteboard or chat windows. These and other procedures were made clear to the students at the beginning of each session through a script that the third and fourth authors read to their students before they began working on the problems.

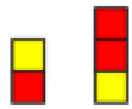
*Data Sources*

Our data sources are the mathematical problem and the persistent archives of the VMT interactions in each of the two interaction spaces.

*Mathematical problem*. During two sessions in October 2008, the students worked on the towers problem, the text of which we present in Figure 1. We chose this mathematical problem for two reasons: (1) its context is familiar to students from urban and suburban communities; and (2) mathematically it affords different solution approaches, ranging from simple listing procedures to more advanced methods involving combinatorial analysis.

Towers Problem

Your team has an inexhaustible set of two colors of cubes. You will use these cubes to build towers. Pictured below are two types of towers: a tower 2-cubes tall and a tower 3-cubes tall.



1. Work together with your teammates to make as many different towers 4-cubes tall as possible when selecting from two colors. Find a way to convince each other that you have accounted for all possible towers 4-cubes tall.
2. How many different towers do you think are possible when selecting from two colors for each of the following situations: 5-cubes tall, 3-cubes tall and n-cubes tall? Work together to determine how many there are for each situation.
3. Work together to write a report of your findings for questions 1 and 2 above. It should include a justification that in each situation you have accounted for all possible towers. You want your report to convince others who are not in your team that your findings are correct. Enter your report into the Summary tab.

*Figure 1*. The towers problem.

*Persistent archives*. VMT Chat records and archives the chat text and whiteboard inscriptions that comprise a chat-room session. These archived sessions are available as Java files (with extension .jno) that can be replayed with a player application. These archived, viewable VMT Chat sessions are one part of the data upon which analyses can be performed. For more detailed analytic purposes, we also produce transcripts of all the actions in the environment. To do so, we use an automated transcriber that translates all the actions in the environment into text. These transcription files can be viewed in a web browser, in a Microsoft Excel spreadsheet, or in a Microsoft Word document as a table. For this study, we used the third method for ease of coding.

*Analysis*

For this report, we analyze two sessions of data from one chat room. This particular chat room contains two students from LBHS and two students from RPS. X lil pit 21 x and johnc250 are from LBHS, and we will refer to them as LP and JC. Cammalleri and 16oncebabyjesus are from RPS, and we will refer to them as CM and SO. During the two sessions, the four students worked on the towers problem. We found this chat room to be particularly interesting because of the way the students interpreted the problem and the solutions that resulted as a result of their interpretation.

To investigate the online, problem-solving actions of learners so as to understand how they build mathematical ideas, heuristics and reasoning, we code for instances in the data of different types of heuristics and evidence of mathematical reasoning (Table 1). These codes were developed inductively for the data.

Table 1

*Codes Used in Chat Room Transcriptions*

| | | | | |
|---|---|---|---|---|
| **H**euristic | **L**isting **S**equentially (HLS) | Using **S**ymbolic **N**otation (HSN) | Using **I**conic **N**otation (HIN) | **L**isting by **C**ases (HLC) |
| Evidence of **R**easoning | **S**tructure of **T**ower **L**isting (RSTL) | **A**rgumentation and **J**ustification (RAJ) | **C**ontributing a **M**issing Solution to Another **P**articipant's work (RCMP) | |

In Table 2, we present an example of one coded time interval from a VMT Chat transcription. In this example, which takes place at 2:49:01, JC (johnc250) has created in the whiteboard a textbox listing a tower. This information can be seen in the first four columns of Table 2, which lists characteristics of an action at a particular time. Note that "wb" under the Type column stands for whiteboard, indicating that this particular action took place in the whiteboard.

In the fifth column, we have coded this moment for the heuristic of using symbolic notation (HSN), as well as for the structure of the tower listing (RSTL) and, as evidence of mathematical reasoning, contributing a missing solution to another participant's work (RCMP).

Table 2

*Sample Coding of Transcription*

| Time | Author | Type | Content | Code and explanation of code |
|---|---|---|---|---|
| 14:49:01 | johnc250 | wb | [johnc250 created a textbox : yellow red red yellow red] | HSN, RSTL, RCMP<br>EC: (HSN) JC uses symbolic notation to list a five cube tall tower. He uses a string of five colors to represent a five-tall tower, with each color being either red or yellow.<br>EC: (RSTL, RCMP) JC creates a textbox containing a tower that CM missed and places it among CM's five-tall towers. JC uses CM's notation to write this tower, and seemingly would have had to enter into CM's logic to not only use CM's notation, but also to discover that CM had not listed this particular tower. |

## Results

With regards to our inquiry into the cognition of the team of participants, our investigation concerns the following guiding question: What heuristics and lines of reasoning are evidenced in students' interactions as they collaboratively solve open-ended combinatorics problems using the VMT Chat? We present our results first with regards to heuristics and second with regards to mathematical reasoning.

*Heuristics*

The VMT Chat environment seemed to afford many of the students' heuristics. During these two sessions, the students seemed to have propensities to use the textbox tool, which is one means to list towers on the whiteboard space. To represent the three-, four-, and five-tall towers in these lists, students used symbolic notation. While CM (cammalleri) used strings of the words red and yellow to represent his towers, LP (x lil pit 21 x) and JC (johnc250) used strings of the letters r and y. In the former case, red and yellow stand for red and yellow cubes, respectively. In the latter case, r and y stand for red and yellow cubes, respectively.

To list their towers, students used three methods: listing sequentially, listing by cases, or both. Listing sequentially specifies instances where each successive tower is related to the preceding tower and where the list contains sequentially two or more instances. At 2:22:55, LP lists six four-tall towers: 1) yrry, 2) yyrr, 3) rryy, 4) ryry, 5) yryr, and 6) ryyr. To get the second tower, yyrr, from the first tower, yrry, it seems that LP shifted rr from the second and third position to the third and fourth position. To get the third tower, rryy, from the second tower, yyrr, it seems that LP switched the positions of yy and rr. To get the fourth tower, ryry, from the third tower, rryy, it seems that LP interchanged the positions of the second r and first y. To get the fifth tower, yryr, from the fourth tower, ryry, it seems that LP changed both ry's to yr's. Finally, to get the sixth tower, ryyr, from the fifth tower, yryr, it seems that LP changed the first yr to ry.

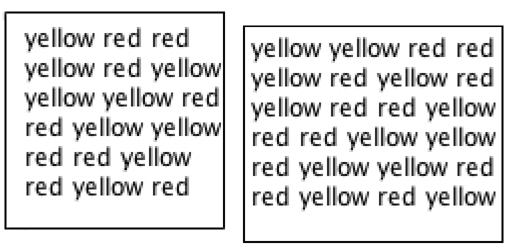
CM lists by cases as well as sequentially. In listing four-tall towers, he used three cases: 1) Towers containing three red's and one yellow; 2) Towers containing two red's and two yellow's, with red in the first position; and 3) Towers containing two red’s and two yellow’s, with yellow in the first position. Within each case, CM lists towers sequentially in a similar manner to how LP listed his towers sequentially.

The heuristic of listing towers sequentially seems to be effective since it allows students to create towers methodically, one at a time. Once a tower is listed, it needs only to be slightly modified to create a next tower. The same process can then be used with this second tower to create a third tower. In this way, students can create a succession of towers in a manner that is not daunting, as the student needs only to make small changes to the previous tower to create a new tower. When used in conjunction with listing by cases, as CM does, it becomes an effective way of listing all of the towers within a case. As we will explain in the next section on reasoning, through his interpretation of the problem, CM was actually able to list all three-, four-, and five-tall towers, save for one.

*Reasoning*

Throughout the two sessions, there is evidence in the students’ whiteboard inscriptions and chat text that they reason mathematically. This is evident in how students structure their lists of towers. When CM lists three-, four-, and five-tall towers, he seems to use knowledge gained from one listing to help with a subsequent listing.

CM lists three- and four-tall towers in two different ways (Figures 2 and 3). Initially, CM starts out by listing three-tall towers, all of which contain either two red cubes and one yellow cube, or two yellow cubes and one red cube. He first lists towers containing yellow in the first position, and then lists towers containing red in the first position. CM follows a similar tactic in listing four-tall towers, all of which contain two red cubes and two yellow cubes. He again first lists towers containing yellow in the first position, and then lists towers containing red in the first position (Figure 2).


yellow red red
yellow red yellow
yellow yellow red
red yellow yellow
red red yellow
red yellow red

yellow yellow red red
yellow red yellow red
yellow red red yellow
red red yellow yellow
red yellow yellow red
red yellow red yellow

*Figure 2*. CM’s initial listing of three- and four-tall towers.

During his second time of listing three- and four-tall towers, CM starts again with three-tall towers. He lists towers containing either two red cubes and one yellow cube, or two yellow cubes and one red cube, but first lists towers containing red in the first position, and then lists towers containing yellow in the second position (Figure 3).

red red red yellow
red red yellow red
red yellow red red
yellow red red red
red yellow red yellow
red yellow yellow red
red red yellow yellow
yellow red yellow red
yellow yellow red red
yellow red red yellow

5 cubes
red red red yellow yellow
red red yellow yellow red
red yellow yellow red red
red yellow red yellow red
yellow red yellow red red
yellow yellow red red red
red red yellow red yellow
red yellow red red yellow
yellow red red red yellow

3 cubes
red red yellow
red yellow red
red yellow yellow
yellow yellow red
yellow red yellow
yellow red red

*Figure 3*. CM's final listing of three-, four-, and five-tall towers.

CM's method of sequential listing evolves as he adds an additional case to his listing of four-tall towers: the case of towers with three red's and one yellow. To represent towers in this case, he lists them sequentially and shifts the position of the yellow cube from the last position to a position one place to the left for each new tower, creating the towers red red red yellow, red red yellow red, red yellow red red, and yellow red red red. After listing these towers, using the same cases he used in listing three-tall towers, CM lists four-tall towers containing two red cubes and two yellow cubes. He starts with towers with red in the first position, and then lists towers with yellow in the first position (Figure 2).

CM adapts this method of sequential listing to construct five tall towers, starting with the case of towers with three red's and two yellow's. The first tower he lists is red red red yellow yellow. To construct subsequent towers, he shifts the yellow yellow from its position at the end of sequence to a position one place to the left for each subsequent tower, creating the towers red red yellow yellow red and red yellow yellow red red. Interestingly, CM starts to list towers sequentially after the last tower in that case, and so does not create the tower yellow yellow red red red until later. This can be seen in Figure 3 in the textbox labeled "5 cubes."

In these two instances represented in Figures 2 and 3 of building three-, four-, and five-tall towers, CM seems to reason by analogy. He first builds three-tall towers sequentially and by cases, and then seems to transfer this method to constructing four-tall towers. He then revises his method of sequential listing for constructing three-tall towers, and after doing so, comes up with a new case and method of sequential listing when building four-tall towers, while retaining the older method as well. He then uses these methods to produce five-tall towers.

Reasoning is also evidenced in the way that students make sense of the problem and come to a solution based on their interpretations of the problem. Early in the session, CM seems to infer from the problem statement that it may be possible to construct towers with a single color. He is unsure and asks LP whether this is possible. LP replies, based on his own interpretation of the problem statement, that towers must be four-cubes tall, each containing two colors. From this point onwards, CM continues only to construct towers with two colors and no longer evidences entertaining one-color towers. The other two members of the team who are present at this session also seem to agree with this interpretation, as they also only construct towers with two colors.

Later in the session, LP asks how to answer the question involving *n*-tall towers. CM states that five is the largest number of cubes available and that there are only three red cubes and two yellow cubes. This interpretation helps us to determine exactly which towers CM constructs as well as which towers he omits. We believe that his interpretation stems from our diagram of the two sample towers (Figure 1). The first tower is a two-tall tower containing one red and one yellow cube, and the second tower is a three-tall tower containing two red cubes and one yellow cube. This accounts for the three red cubes and two yellow cubes that CM deems only to be available for constructing towers. Taking into account these restrictions, CM lists all possible towers with the exception of one five-tall tower that JC contributes (Table 2) that also fits within the CM's interpretation of the problem. CM ultimately comes up with six three-tall towers, 10 four-tall towers, and nine five-tall towers. Indeed, with these restrictions, the number of three-tall towers is $C(3,1)+C(3,2)=6$, the number of four-tall towers is $C(4,2)+C(4,3)=10$, and the number of five-tall towers is $C(5,3)=10$.

## Discussion

The results of this preliminary study raise interesting questions about the use of online communication environments as well as the role of teacher or researcher intervention and the development of students' problem-solving heuristics and mathematical reasoning. Indirect teacher or researcher intervention occurs in the design of the online environment and the mathematical tasks as well as in decisions about the composition of the work teams. However, the problem-solving sessions in which CM, LP, SO, and JC participate are mediated by an Internet-based environment and, importantly, without direct intervention of researchers or teachers. Nevertheless, both in the chat and whiteboard spaces, the students engaged in thoughtful discussions and displayed emergent heuristics and mathematical reasoning.

Some of the reasoning in which the students engage is corroborated in the literature. When CM constructs 3-, 4-, and 5-tall towers during the towers problem sessions, he seems to be engaging in reasoning by analogy. English (1997) defines this type of reasoning as "the transfer of structural information from one system, *the base*, to another system, *the target*" (p. 5, emphasis in the original). Indeed, once CM has listed possibilities for three-tall towers, he seems to transfer his strategy from how he lists three-tall towers to how he lists four-tall towers.

CM then starts over and creates a revised version of his original three-tall towers. Afterwards, CM then comes up with an additional case and method for his four-tall towers before. This additional case seems to be influenced by CM's new list of three-tall towers, as well as by how CM's interprets the problem statement. Indeed, the case of four-tall towers containing three red cubes and one yellow cube fits within the constraints that CM imposes on the problem, that there are a maximum of three red and two yellow cubes with which to construct towers. CM then seems to use this new method as well as the method from three-tall towers to list five-tall towers.

Throughout the VMT Chat sessions, there are multiple transfers of structural information. CM's strategy used in his original three-tall tower listing is transferred to his original listing of four-tall towers. CM's strategy used in his revised three-tall tower listing is not only transferred, but also evolved in his new listing of four-tall towers. Finally, CM's strategy for listing four-tall towers is transferred to his listing of five-tall towers.

Some of CM's reasoning goes beyond reasoning by analogy. He not only transfers structural information but also evolves his strategies as he successively constructs three-tall towers, four-tall towers, and five-tall towers. When CM constructs his new listing of four-tall towers, he not only uses his strategies from his revised list of three-tall towers but also evidences an awareness of another pattern. Consider the first two towers listed in his new three-tall towers listing: red red yellow and red yellow red. To get the second tower from the first tower, CM seems to use the strategy of listing sequentially, and switches the positions of the first red and first yellow in the first tower. Note that it is also possible to shift the yellow in the first tower one position to the left to get the second tower. CM may have seen this shifting pattern and could have used it to create the first four towers in his new listing of four-tall towers, where the yellow cube is shifted one position to the left three times (red red red yellow, red red yellow red, red yellow red red, and yellow red red red).

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# PROPERTIES OF REPRESENTATIONAL ACTIVITY IN CALCULUS STUDENTS' CONSTRUCTION OF PROOF

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The goal of this study is to examine the construction and use of representations in mathematical problem solving, and, in particular, how representations arise and are deployed in their interactional, mathematical, and social contexts. Specifically, we examine students' representational activity while working on proof-based problems in undergraduate Calculus. This poster will demonstrate how reasoning with and through mathematical representations in this setting is interactionally accomplished and identify and characterize qualities of representations that shape those processes of reasoning and proving.

Data include audio-video recordings of a focus group in an introductory calculus course where discussion sections were organized as "workshops" in which students worked in small groups and instructors emphasized the need for explanation and justification. Recordings were content-logged and initial strips of interaction were selected for close interactional analysis. Using these, we developed grounded theoretical categories in an iterative fashion, integrating constructs from the literature, then extended our analysis systematically across the data corpus.

We draw upon two theoretical lenses to examine the students' construction and use of representations: first, we borrow from Speiser, Walter, & Sullivan (2007), who identify a representation as a presentation used to facilitate communication with oneself or others (p. 15); second, we view a representation as a cultural artifact with a local history of production as well as a history embedded in mathematical and pedagogical meanings and conventions (Latour, 1999).

Our analysis reveals three properties of representational activity—complementarity, prospectivity, and generativity—that we find important in understanding the constraints and affordances of representations in their interactional construction and use in mathematical explanation and justification. Complementarity refers to the coordination of multiple forms of representation to support reasoning beyond the affordances of a single representation. Prospectivity describes how representations, due to specific properties embedded in the structure and customary use of a certain representation, can afford prospective attention to and yet also bound the nature of immediate next steps in the problem-solving process. Generativity refers to the productive quality of representational activity for fostering new insights into mathematical situations, in particular across representational systems. This poster presentation will display selected examples of different representations—possessing one or more of these three properties—from the students' work, along with a discussion of the constraints and affordances of such representational activity for students' as they work together to produce proof and justification.

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# DOCUMENTING CURRICULUM IMPLEMENTATION: A CASE STUDY FROM UCSMP GEOMETRY

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*For studies of curriculum effectiveness to be meaningful and useful, information is needed about classroom implementation of the curriculum. One challenge for researchers is how to obtain data about implementation that are informative, reliable, and relatively inexpensive to collect and analyze. This paper describes instruments for assessing variations in both content taught and instructional practices and illustrates their use in a study of the field trial of the Third Edition of the UCSMP Geometry curriculum.*

## Introduction

The National Research Council's report, *On Evaluating Curricular Effectiveness* (NRC 2004), recommends that evaluations of curriculum "present evidence that provides reliable and valid indicators of the extent, quality, and type of the implementation of the materials. At a minimum, there should be documentation of the extent of coverage of curricular material" (p. 194). Educators have long recognized that classroom instruction, even for the same course, can vary from teacher to teacher and school to school. Differences often occur because teachers implement curriculum based on their personal teaching philosophy, including what they believe to be the most important content, what expectations they believe are reasonable for their students, and what district and state frameworks form the basis of accountability measures. As Hiebert and Grouws (2007) note,

> The emphasis teachers place on different learning goals and different topics, the expectations for learning that they set, the time they allocate for particular topics, the kinds of tasks they pose … all are part of teaching and all influence the opportunities students have to learn. (p. 379)

The challenge for curriculum researchers is how to collect and analyze implementation data. Clearly, one approach would be for someone to observe a classroom for an extended period of time. However, lengthy observations are expensive. Although such observations might be possible in a study of a few teachers in close proximity, extensive observations are not feasible in moderate or large scale research. Alternatively, a video camera could be placed in a classroom to record lessons. But videos must still be watched and transcribed to be useful – again, an impractical approach except when studying a small number of cases.

The purpose of this paper is to provide insights into two research questions. The first is methodological; the second relates to a particular curriculum evaluation project.

1. What instruments can be used to document teacher implementation of curriculum in a low-cost and reliable manner?
2. What differences exist in how teachers implement and use a new geometry curriculum?

Data to address these research questions are drawn from an evaluation study of the field-trial version of *Geometry* (Third Edition, Benson et al., 2007), developed by the University of Chicago School Mathematics Project (UCSMP). The field trial was conducted in 12 schools

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

from 9 states during the 2006–2007 school year. The schools represented a mix of urban, suburban, and rural environments across the U.S.A. Although comparison classes using the curriculum already in place at the school were present in 9 of the 12 schools, the results reported here are based solely on data from the teachers using the UCSMP *Geometry* (Third Edition) curriculum. Consequently, the results are controlled for curriculum, permitting a discussion of how the instruments and analysis provide insights into variability in implementation.

## Documenting the Implemented Curriculum: Content Coverage and Opportunities for Practice

Some researchers (e.g., Tarr et al., 2006) have used textbook diaries to collect data about the lessons that teachers teach from a particular textbook. This often involves having teachers make notes on a Table of Contents as they use a textbook. Because we were collecting data as part of a formative evaluation, we had teachers record slightly more information for each chapter they taught. For each chapter in the book, teachers completed Chapter Evaluation forms on which they indicated the lessons taught, the number of days spent per lesson, and the questions assigned; they also rated each lesson and set of questions on a scale of 1 (low) to 5 (high), and made comments about various aspects of the lessons. The ratings and comments provided insight to the curriculum developers when revising the materials. The other information provides insights into classroom implementation, and is the main source of data reported in this paper.

*Lesson Coverage: Opportunity to Learn Content*

Benson et al. (2007) contains a total of 114 lessons in 14 chapters. The chapter titles are listed in Table 1.

*Table 1*

Chapter Titles for UCSMP *Geometry* (Third Edition, Field-Trial Version)

| Ch | Title | Ch | Title |
|---|---|---|---|
| 1 | Points and Lines | 8 | Lengths and Areas |
| 2 | The Language and Logic of Geometry | 9 | Three-Dimensional Figures |
| 3 | Angles and Lines | 10 | Formulas for Volume |
| 4 | Transformations and Congruence | 11 | Indirect Proofs and Coordinate Proofs |
| 5 | Proofs Using Congruence | 12 | Similarity |
| 6 | Polygons and Symmetry | 13 | Consequences of Similarity |
| 7 | Congruent Triangles | 14 | Further Work with Circles |

Figure 1 displays the percent of lessons taught as reported by the 12 teachers teaching from Benson et al. (2007) by thirds of the text and overall. Note that the number of lessons is 39 for Chapters 1–5, 45 for Chapters 6–10, and 30 for Chapters 11–14.

Figure 1 demonstrates how implementation of curriculum by teachers using the same text can vary considerably. The percent of lessons taught was relatively similar for Chapters 1-5, with 7 of 12 teachers each teaching at least 90% of the lessons. However, greater variability in implementation was observed in Chapters 6-10, with half of the teachers teaching less than 78% of the lessons. In the final third of the book, three teachers taught none of the lessons; and only three taught more than 40% of the lessons. Overall, the percent of lessons taught ranged from 52% to 92% with a median of 67%.

Although some teachers taught comparable percentages of the text, an analysis of the actual lessons taught reveals differences in students' opportunities to learn specific content. Figure 2 indicates the actual lessons reported taught by all teachers, called Teachers A through L, in the Field Trial study.

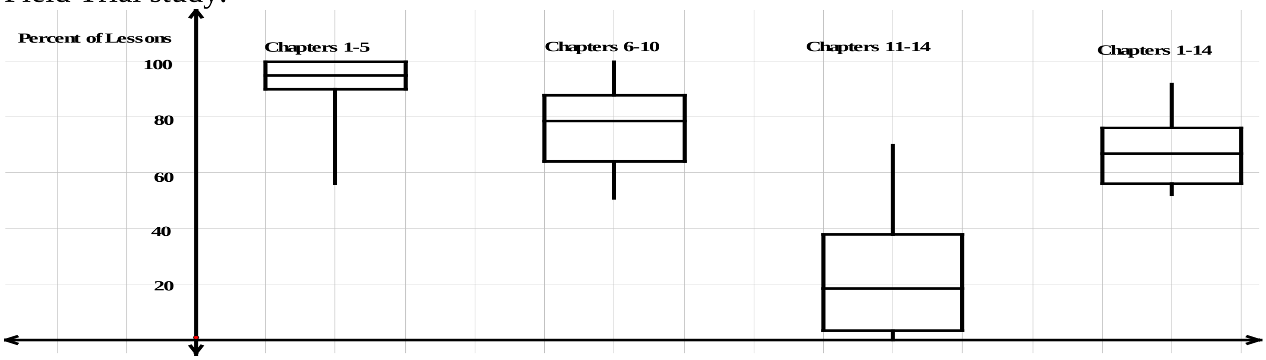


*Figure 1*. Percent of lessons in UCSMP *Geometry* textbook taught by 12 teachers.

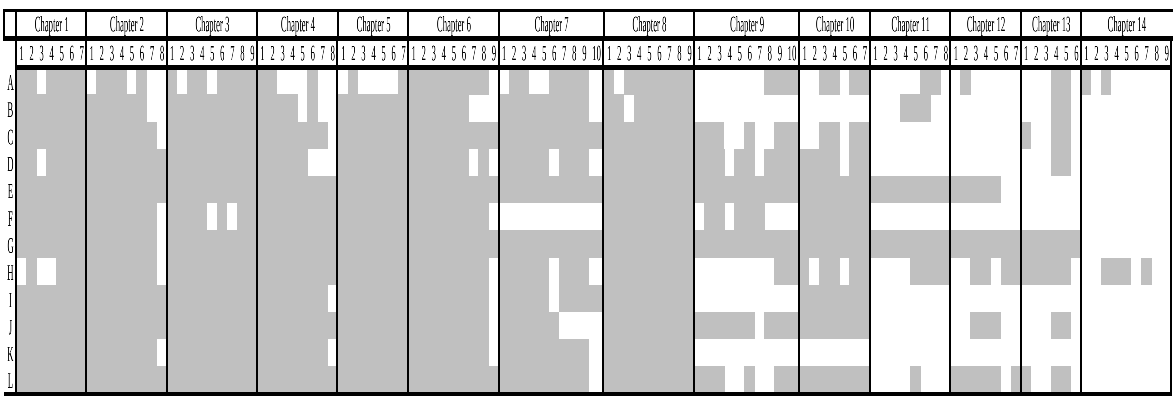


*Figure 2*. Coverage of lessons taught by teachers A-L in UCSMP *Geometry*. A dark rectangle indicates that the lesson was reported as having been taught. A light rectangle indicates that the teacher reported not teaching that lesson.

Although Teachers A, B, F, and K taught comparable percentages of the text (between 52% and 57% of the lessons), each pattern of coverage is different and leads to different opportunities for students to learn geometry content. Teacher A skipped lessons in many chapters but covered some lessons from every chapter in the text. In contrast, Teacher K skipped few lessons but only taught through Chapter 8. So Teacher K's students had little to no opportunities to study three-dimensional figures, volume, or coordinate proofs; even though Teacher A's students studied some lessons from every chapter, they studied fewer lessons in each of the chapters studied by Teacher K's students. Teacher B also taught most of the first eight chapters but then taught some lessons related to proofs with coordinates and trigonometric ratios. Likewise, although Teachers C, D, and I all covered between 60% and 70% of the text, each of their patterns of coverage was different leading to different opportunities for students to learn geometry content.

Examining specific lesson coverage may identify other implementation patterns. For instance, all teachers taught Lesson 1–7, a lesson that focused on using a dynamic geometry drawing tool (DGS). However, although each had regular access to technology, either graphing software such as Sketchpad, or calculators with either Cabri or Cabri Jr, many found the use of the DGS to be difficult and confusing. When another lesson on DGS appeared in Chapter 2, seven of the twelve teachers omitted it. Some teachers came back to the use of DGS later in the year, but an early difficult experience influenced lesson coverage negatively.

*Exercise Coverage: Opportunity for Practice*

Curriculum implementation also includes making decisions about providing opportunities for students to practice mathematics, typically through homework. In the UCSMP curriculum, question sets are constructed with four types of questions. *Covering the Ideas* questions focus on the essential aspects of the lesson; students who complete these questions can use the basic skills and concepts. *Applying the Mathematics* questions require students to use the essential aspects in new ways or in new contexts; students need to take the basic ideas to a slightly higher level. UCSMP uses a modified mastery approach so *Review* questions in each lesson provide an opportunity for students to continue working on concepts throughout a chapter and throughout the text. Finally, *Exploration* questions provide an opportunity for extension and discovery. With the exception of the Exploration problems, the curriculum developers recommend that students generally be assigned almost all the problems.

On the Chapter Evaluation form, teachers listed the questions assigned to students. Figure 3 provides a visual of the variability in implementation of homework assignments.

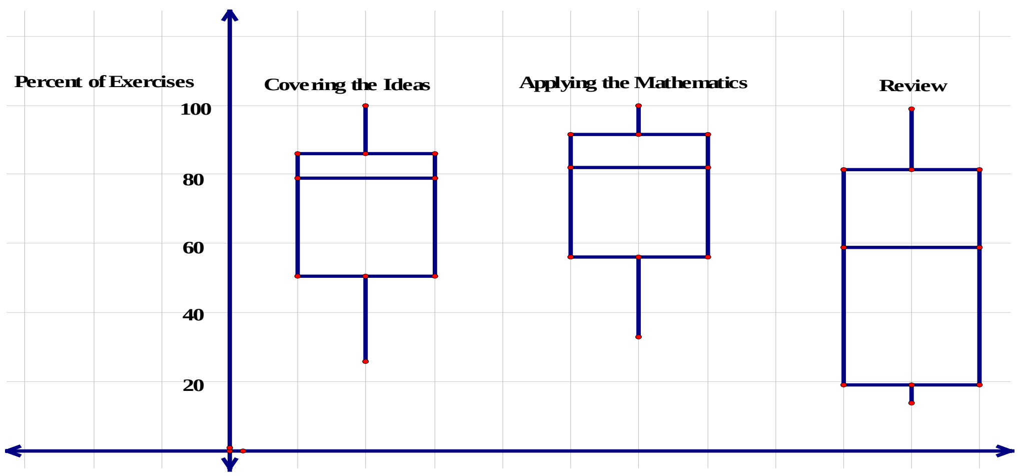


*Figure 3*. Percent of questions assigned by 12 teachers using UCSMP *Geometry*, based only on lessons taught.

Although the median percentage of questions assigned from Covering the Ideas and Applying the Mathematics is roughly 80%, there are major differences in expectations between teachers assigning the maximum percentage of exercises and those assigning the minimum. For instance, two teachers (Teachers D and F) assigned only about a third of the Covering the Ideas questions; eight teachers assigned more than 75% of these questions. Likewise, eight assigned at least 75% of the Applying questions. However, only five of the twelve assigned at least 75% of the reviews; in fact, five teachers assigned less than a fourth of the reviews. Given that review problems are an essential feature of the modified mastery approach used in the UCSMP secondary curriculum, the lack of attention to the review questions raises concerns about the fidelity of implementation of the curriculum in those classrooms. The limited assignment of review questions by some teachers also means that students in their classes may have had less opportunity to achieve mastery than the developers intended.

### Documenting Teachers' Expectations Relative to the Assessed Curriculum

Teachers' expectations relative to the assessments given to their students also provide information about curricular implementation. On all posttests administered as part of evaluations of the UCSMP curriculum, teachers are asked to indicate for *every* item whether they taught or reviewed the content needed for their students to answer the item. This measure provides insight into whether teachers perceive that their lesson coverage provided students with opportunities to master content at a level sufficient for assessment. Even when teachers have taught the same content, based on reported coverage of lessons or assignment of exercises, they sometimes respond differently to these opportunity-to-learn questions.

Two multiple-choice posttests were administered as part of the formative evaluation of the *Geometry* curriculum. The posttests were constructed based on content in the first ten chapters, which the developers expected most teachers to teach. Of the 60 items on these tests, 20 (items 36-55) are from the first four levels of the van Hiele Geometry test (Usiskin 1982). Among the other 40 items, 19 deal with figures and their properties, 4 with transformations, 9 with measurement, 6 with reasoning, and 2 with graphing of lines. Although the 12 teachers in the field trial were using the same curriculum, Figure 4 illustrates that their expectations about the extent to which students had an opportunity to learn the content for the items varied.

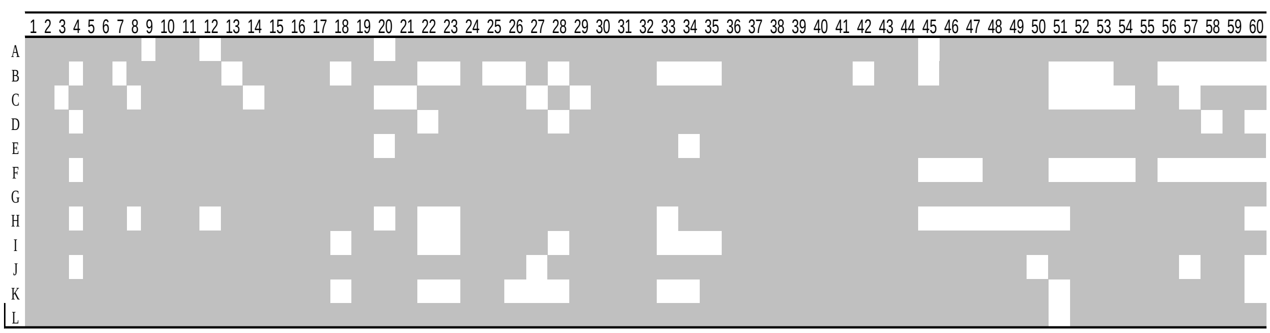

*Figure 4*. *Geometry* teachers reported opportunity-to-learn (OTL) for the UCSMP-constructed posttests. A dark rectangle indicates that the teacher responded "yes" to covering the content on the OTL form; a light rectangle indicates that the teacher responded "no."

Only on 23 of the 60 posttest items (38%) did all teachers report having taught the content needed to answer the items. The number of assessment items for which teachers reported they had taught or reviewed the content needed for students to answer the items ranged from 38 to 60, with a median of 54 items. The documented variability represented in Figure 4 suggests that student achievement data that does not consider teachers' OTL responses on assessment items has the potential to lead to incorrect conclusions. Thus, documenting teachers' views relative to the assessed curriculum is an essential part of documenting curriculum implementation.

**Documenting Instructional Practices Related to Curriculum Implementation**

Data about instructional practices can also be collected from self-reports. At the end of the school year, teachers completed two documents: a questionnaire about instructional practices and a supplement designed to obtain details about curriculum features that might influence revisions to the instructional materials. On the questionnaire, teachers generally checked or circled responses from a set of options, so it required only a few minutes to complete; the supplement required short responses to open-ended questions. Here, we describe questions and present results related to assigning homework and two instructional practices emphasized in the entire UCSMP secondary curriculum—reading and writing mathematics.

Teachers were asked, "On the average, how many minutes of homework did you expect the typical student to do each day?" Choices were 0–15 minutes per day, 16–30 minutes per day, 31–45 minutes per day, 46–60 minutes per day, and more than 60 minutes per day. Five teachers (Teachers A, B, D, F, and H) expected students to complete 16–30 minutes of homework per day, five (Teachers C, I, J, K, and L) expected 31–45 minutes of homework per day, and two (Teachers E and G) expected 46–60 minutes of homework per day. In general, the teachers who expected the least amount of time per day on homework were also the teachers who assigned the fewest percent of the exercises. Hence, the responses to the questionnaire provide a check on the reliability of the data reported about homework on the Chapter Evaluation Forms.

Because reading and writing in mathematics are both prominent features of the UCSMP curriculum, teachers were asked several questions about these practices. Table 2 contains the questions we focus on in this paper and the coding used to create an emphasis index for each. The coded responses were summed to obtain an index of reading emphasis and writing emphasis for each teacher as reported in Table 3.

Again, considerable variability among teachers as well as an individual teacher's relative emphasis between reading and writing in mathematics is apparent. For instance, Teacher A places more emphasis on writing mathematics than reading; for Teacher B, the reverse is true. Several teachers (Teachers D, E, I, and J) have relatively high indices for both reading and writing.

Teachers' responses on the supplement validated their questionnaire responses. When asked to describe how reading was handled, Teacher D replied, "Throughout the year, I gave reading quizzes and the students who read the sections did fairly well. …I expect them to have read the section before we go over the material. I expect them to do a lot of the learning by reading."

*Table 2*
Reading and Writing Questions Asked of Teachers and Coding of Responses

| Reading | Writing | Code |
|---|---|---|
| How often did you expect students to read their mathematics textbook? | How often did you expect students to write explanations to show what they were thinking when solving mathematics problems? | Almost every day = 4<br>2-3 times per week = 3<br>2-3 times per month = 2<br>Less than once a month = 1<br>Never = 0 |
| How often did students read silently in class? | How often did students write complete solutions when they solved problems? | Daily = 3<br>Frequently = 2<br>Seldom = 1<br>Never = 0 |

| | | |
|---|---|---|
| How often did students discuss the reading in class? | How often did students explain or justify their work? | Daily = 3<br>Frequently = 2<br>Seldom = 1<br>Never = 0 |

*Table 3*
Index for Reading and Writing in Mathematics for Teachers Using UCSMP *Geometry*

| Teacher | A | B | C | D | E | F | G | H | I | J | K | L | Median |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Reading Emphasis | 4 | 6 | 7 | 8 | 8 | 3 | 4 | 4 | 10 | 8 | 6 | 8 | 6.5 |
| Writing Emphasis | 7 | 3 | 5 | 10 | 9 | 8 | 8 | 4 | 7 | 9 | 7 | 7 | 7 |

*Note*. For both reading and writing, the maximum = 10 and the minimum = 0.

## Discussion

To address the first research question, "What instruments can be used to document teacher implementation of curriculum in a low-cost and reliable manner?", we have described three survey instruments: Chapter Evaluation forms, Posttest Opportunity-to-Learn Forms, and a Teacher Questionnaire about instructional practices. Each is relatively short and easily completed by teachers. Unlike classroom observation data that may require extensive coding and high inference, all UCSMP survey data are low-inference and can be analyzed and displayed using relatively low-cost software.

In some cases, questions asked on different survey forms can be used to check reliability or to expand perspectives on responses to questions on other forms. Although not discussed here, an end-of-year survey of geometry students also had some questions about content covered and instructional practices, so that responses by teachers and students can be used to validate each others' perspectives. UCSMP researchers also conducted a 2–3 day visit to each field trial school to observe classes and interview the teachers. Data from the school visits generally confirmed the data collected from the surveys. We found that when differences occurred, teachers were often a bit more conservative on the questionnaires than in the interview. For instance, only for Teacher A did the questionnaire responses seem to suggest more frequent use of reading and writing than would have been expected based on the interview responses. Such reliability checks across surveys and between surveys and observations confirm that self-reported data from teachers of the type presented here are quite robust.

To address the second question, "What differences exist in how teachers implement the UCSMP *Geometry* curriculum?", data were provided from a year long field trial conducted with twelve teachers from nine states. Teachers' responses to questions on the survey forms provide a picture of many differing kinds of curriculum implementation. In particular, there is considerable variability from teacher to teacher on the following dimensions:

- Percent of lessons taught, ranging from 52% to 92% with a median of 67%;
- Content studied during the year, including classes where students had little to no opportunity to study three-dimensional figures, volume, or coordinate proofs and classes where all these topics were studied;
- Nature and extent of assigned homework questions, including some classes where teachers regularly skipped the review questions designed to develop and maintain mastery.

- Extent to which they implemented other instructional practices recommended by the curriculum developers, such as emphasizing reading and writing mathematics.

In this paper, we have focused on curricular implementation, not on achievement. However, variations in implementation of the geometry curriculum are accompanied by variations in achievement levels (Thompson & Senk, in preparation). The nature of the data reported here permits detailed profiles of implementation of UCSMP *Geometry* to be constructed. Although space does not permit such profiles to be included here, they can be used to understand class-to-class and school-to-school differences in end-of-year achievement.

As recommended by the National Research Council (2004) and reinforced by the data reported here, achievement results should be accompanied by information about curriculum implementation. As indicated here, students who have studied the same curriculum often have very different opportunities to learn. Without information about curricular implementation, it is difficult to make conclusions about differences in achievement.

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# DIPCING A DESIGNING METHODOLOGY FOR UNIVERSITY MATHEMATICS STUDY PROGRAMS

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*In this paper a research is reported where the research problem is how to design a methodology for the designing of mathematics study programs in engineering careers. In such way that the professor has a clear idea of why each theme included in the program has to be taught. With this basis he can motivate the student, showing him the link between mathematics and engineering subjects and the professor can know what kind of abilities he can develop in the students.*

## Introduction

Mathematics are subjects with a high percentage of reprobation. But this is only a symptom, in this educational problem there are several factors that are of curricular type, caused by the teaching and learning, inferred by the study theme by itself, caused by the cognitive infrastructure of the students, due to social, emotional and economical factors, etc.

Among the great range of problems there are some specific that have something to do with the curricular situations that are mentioned below. When for the first time a professor is going to teach mathematics courses, he finds a list of themes that he understands the best he can, which leads to make different courses in the same study program, one for each professor. If a professor is asked about why these themes are included in the study program, it is common that he can not answer precisely and correctly. As regards to the students, different exclamations are heard from them in the classroom as follows: What is the purpose of studying mathematics? Why do we have to study them?. These questions, at best, are answered by the professor who says that in engineering courses that they will take later they will use them (Camarena, 1984; Camarena, 1988).

From the percentage of students that does not approve mathematics courses and from the teaching experience, it can be said that the few interest that the students have for these science is because they do not see their immediate application, neither the object to use them. An element that affects is the fact that there is not an appropriate curriculum where these mathematics courses are taught, consequently, the professors who teach them do not know why the contents are included in the study programs (Camarena, 1984).

*Objective*

With the background described, the objective research problem is to build a methodology to design mathematics study programs in engineering; in such way that the professor has a clear idea of why each theme that is included in the program has to be taught. With this basis he can motivate the student, showing him the link between mathematics and engineering subjects. Knowing what kind of abilities he can develop in the students.

## Perspectives and Theoretical Framework

The theory in which this research is based is Mathematics in the Sciences Context (Camarena, 1984; Camarena, 1990; Camarena, 1995; Camarena, 2001; Camarena, 2003). This theory has been developed since 1982 in the National Institute Politechnical of Mexico. It takes mathematics learning and teaching in engineering careers as a system, which includes the student, the teacher and the mathematical knowledge. It takes into consideration the interactions among the student, the teacher and the mathematics knowledge, all included in the learning environment where there are social, economical, political and human relations aspects. This systemic look makes five phases of the Mathematics in the Sciences Context theory: Curricular developed since 1984. Didactic started since 1987. Epistemological tackled in 1988. Teachers Training defined in 1990. Cognitive studied since 1992.

Mathematics in the Sciences Context is based in three paradigms: Mathematics are supporting tools and educational subjects. Mathematics have a specific function in each educational level. Knowledge is born integrated.

The educational philosophic assumption of this theory is that the student is trained to transfer mathematics knowledge to the areas which require it, so that he develops competences for his working and professional life. We want mathematics for life.

This research presents incidence in the curricular phase, where Mathematics in the Sciences Context thinks about the link among mathematics and other knowledge areas of the student, the future professional and working activities and daily life.

In other hand, it is necessary to take into account particular characteristics of mathematics in engineering by the specific methodology to be constructed. Engineering has mathematics as a foundation, because mathematics are engineering language and mathematics make sense to engineering, also mathematics are a supporting tool. Besides having a formative character for the students, that is to say, mathematics develop thinking abilities, critical judgment, scientific heart, etc. Even more, mathematics curriculum in engineering schools requires special attention, since mathematics are not a goal by themselves; in other words, we are not going to form mathematicians (Camarena, 1988).

*Educational Paradigm and Premise*

The methodology for the curricular designing of mathematics study programs in engineering careers is founded in the following educational paradigm: With mathematics courses students will have cognitive elements and tools he will use in the specific subjects of his career. Also, the premise of the methodology is that: Mathematics curriculum should be objective, that is to say, it should be a curriculum based on objective basis (Camarena, 1988).

## Work Method

The work method to follow requires establishing a net between the problematic of mathematics in engineering with the possible ways to tackle them (see first and third columns in fig. No. 1). For such purpose, the problems of mathematics were re-written in terms of research questions in order to clear the working method (see second column in fig. No. 1).

The fig. No. 2 shows, respectively, in column two and three the research questions selection and the way to tackle them, corresponding to the objectives of engineering graduate, as well as the objectives of searched methodology (see first column in fig. 2).

With the co-relation net between research questions and possible ways to tackle them (fig. No. 1 and No. 2), the tendency that minimizes the paths was established and the maximum activities to meet the objectives stated in the research project were taken.

The different activities described were the following:

1. Analyze engineering textbooks.
2. Investigate how mathematics contribute to engineering through the history.
3. Analyze the bibliography of the elements which motivate the high level student.

4. Interviews to engineers in practice.
5. Diagnosis of the students when they go into engineering career.

| Problematic | Research questions | Ways to tackle the questions |
|---|---|---|
| Program as a thematic list of mathematics. | Why mathematics themes are included? | Identify what is needed from mathematics in each engineering theme (textbooks analysis) and determine if it is enough. |
| Where will mathematics be used. | Where do they apply? | Identify what is needed from mathematics in each engineering theme (textbooks analysis). |
| Which is the purpose of mathematics. | Which is mathematics function in engineering? | Determine how mathematics themes in engineering are used (textbooks analysis). |
| Why to study mathematics. | Which is the benefit of mathematics to engineering? | Investigate through history how do mathematics contribute to engineering. |
| There is few interest for mathematics. | How to motivate the student? | Analyze bibliography about which elements motivate the high level student. |

*Figure 1.* Relation among mathematical problems, research questions and how to tackle them.

From these activities and what is expected from them we have two elements groups which are: general for any type of engineering, that is to say, that are independent from engineering; and the ones which depend from engineering which is worked for. In the first category are activities number 2 and 3, while the ones that really depend on engineering are 1, 4 and 5.

| Objectives | Research questions | Ways to tackle the questions |
|---|---|---|
| A graduate efficient to solve problems is searched. **OG** | How mathematics are connected with engineering subjects? | Analyze engineering textbooks. |
| A graduate competent to design is required. **OG** | What is required from mathematics in the professional work of an engineer? | Interviews to engineers in practice about mathematics. |

| | | |
|---|---|---|
| Putting into practice engineering. **OSM** | Which applications of mathematics to engineering? | Analyze engineering textbooks. |
| An objective curriculum is wanted. **OSM** | Where and how mathematics are applied?<br>With which focus and notation? | Analyze engineering textbooks. |
| How to construct knowledge. **OSM** | What is it required so that mathematics are significant for the student? How to give sense to mathematics?<br>How to tie up the cognitive structures of the student? | Present mathematics in engineering context. Determine the knowledge level of the students when they go into the superior level. |

*Figure 2.* Relation among objectives of searched methodology (OSM), objectives of graduate (OG), research questions and ways to tackle them.

*General and Independent Elements of Engineering*

a) The activity which describes: Investigate how mathematics contributes to engineering through the history, lead to make a study consisting of surveys made to historians and books revisions about the historical development of engineering and mathematics. From this there were found results which respond to the research question: which are the benefits of mathematics for engineering?. We founded that mathematics are a supporting tool for engineering and educational forming subject for the ones who study them. At the same time it was tackled the questioning of why to study mathematics, because it supports the foundation of engineering. Notice that the result is consistent with the particular characteristics of mathematics in engineering, and the educational paradigm described for the research development, about mathematics are engineering supporting tools, not leaving behind the formative character offered by them.

b) The second independent element from engineering says that the bibliography about the elements that motivate the superior level students have to be analyzed. For this work several theories were found about the motivation. Among the most relevant references are: Ausubel David P., Novak Joseph D. y Hanesian Helen (1990); Nickerson Raymond S., Perkins David N. y Smith Edward E. (1994) and De Bono Edward (1997). One that makes reference to the age of the student is the cognitive psychology of Ausubel David P., Novak Joseph D. y Hanesian Helen (1990). Although this theory does not specify the educational levels, these can be inferred according to the pupils ages and associate with their corresponding motivators. For the superior level case the facts that contribute to the motivation are mainly the ones which surround the interests of the selected career. Of course this is true as long as the superior level studies have been selected by the student. Several interviews were held with professors, students and engineers, about how the students could be motivated so that they have interest in mathematics, taking into account that they like their career. After analyzing the suggestions it was considered that the most frequent proposal was that applications to engineering have to be given for mathematics themes studied. So, it is necessary to show the link between mathematics and engineering subjects. Notice that it is consistent with the research problem about the motivation of the student through showing him mathematics linked and applications.

It is good to mention that in these two partial studies only the results were presented due to the length of the investigation.

*Dependent Elements of Engineering*

It is evident that the activities: analyze engineering textbooks, interviews to engineers in practice and diagnosis of the students when they go into engineering career, are elements that depend of the engineering entered upon. These will be the ones that determine the methodology looked for.

## Results: The Proposal Methodology Design

In order to fulfill the premise into the educational paradigm frame, a research strategy is proposed in three stages: the central, the preceding and the consequent (Camarena, 1984; Camarena, 1988).

CENTRAL STAGE. Make a textbooks analysis of the specific engineering courses in order to detect mathematics contents, both explicit as well as implicit.

PRECEDING STAGE. Diagnose the knowledge level of mathematics that the students have when they go into the career.

CONSEQUENT STAGE. Interview engineers in practice and researchers engineers about the use of mathematics in their professional work.

*Proving the Proposal*

To establish the feasibility of the proposal we selected electronic engineering area. The experience is reported below.

*Central stage*. In order to realize the first stage, that is, the analysis of mathematics contents, it is necessary to know the profile of engineering graduate that the institution wants, the study plan and the programs (themes and bibliography) of engineering subjects that the student will course. There are three blocks: basic sciences, engineering basic sciences and engineering specialization sciences, as it is established by Asociación Nacional de Universidades e Instituciones de Educación Superior, of Mexico (ANUIES).

With this information, groups of mathematics teachers will be formed, which will analyze engineering subjects. For such activity textbooks will be used directly, in order to get the required mathematics themes, including of course, the focus and depth of the themes, the notation with which they are described and their applications. With the described activity it is established the curricular link between mathematics and basic sciences, also the link between mathematics and subjects of engineering basic sciences, as well as, between mathematics and engineering specialization sciences.

These teacher groups should make a written report of the themes they detected, jointly with the focus required by said concepts in order to take them into account in the didactic proposal.

These teacher groups, who chose one or more subjects to analyze, after having revised engineering textbooks will be competent to teach courses to the rest of the members of the corresponding Teachers Academy, with the purpose of spreading their knowledge about the focus, notation, depth and applications they found in the elected subjects. In the same way, these teachers could elaborate books of mathematics problems and notes about the concepts they detected.

From textbooks analysis, we had detected that from all mathematics used in engineering carriers, a percentage between 70 and 90 percent supports basic sciences and engineering basic sciences, this percent depends of which engineering. The remaining percentage, between 30 and 10 percent, supports engineering specialization sciences.

Also, we detected that engineering basic sciences and engineering specialization sciences have two focus, one theoretical and the other applied. From this, we have mathematics that support theoretical courses and applied courses. In mathematics included in applied courses the professor has to develop, in the student, abilities for modeling and algorithmic handle. While in mathematics included in theoretical courses it is not necessary, but if the professor has curricular time we propose that he develops abilities too. That is to say, whit this analysis the professor can know what kind of abilities he can develop in the students.

*Preceding stage.* Once the contents of mathematics needed in engineering are determined, we will tackle the second stage. Based on the knowledge of this discipline and teaching experience we can determine the necessary requirements for the detected mathematics contents.

In other words, the mathematics link between superior level and middle superior level is established.

From these requirements, the ones supposed to be received by the student in the middle level courses will be selected and the rest should be included in the curriculum of the first years of the career or as preparatory courses.

With this analysis, mathematics entrance profile of the student in superior level is defined. Also, with this selection, a diagnostic evaluation method is elaborated to determine the level of knowledge and abilities in mathematics of the new student in the career.

From the concepts in which most of the students are deficient the following classification will be made:

a) Mathematics themes that should be known by the student and that he can to study by himself with a simple bibliographic orientation.

b) Mathematics themes that the students should be known and to have abilities, should be taken into account to be included as a preparatory part in the elaboration of the curriculum of the first courses of mathematics.

*Consequent stage.* With the end of the second stage the third will be tackled, in which interviews should be held to engineers being in the industry’s practice, that is to say, engineers that are designing. We ask them about the use of mathematics in their work activity.

It is good to make clear that from this information there would be the contents detected from the textbooks analysis and the ones that are not included. The results obtained from this study offer for the curriculum a better hierarchization of the importance that should be given to mathematics themes. While the ones that are not included in the textbooks of engineering careers are the themes and concepts of mathematics that should be considered for post grade engineering studies, establishing the mathematics link between superior level and post grade level. With this stage the link between mathematics of engineering and industry is also established.

After making the mentioned activities and with the reported information by the different working groups, the person who coordinates the elaboration and/or reorganization of mathematics study programs could to group mathematics themes founded in the subjects of the three engineering blocks, as well as, the necessary themes obtained in the second stage.

That is to say, as up to now there are only mathematics themes that are going to be used in their engineering courses, to this should be added mathematics contents necessary to build the logical structure of the knowledge, so that the teaching of the themes make sense.

*Resume*. With the end of the third stage, it can be deduced: the number of subjects to be taught from mathematics. The location of these courses and the link with the other subjects of the curricular map of engineer in question. For each course, the thematic content with its extension, time and depth to be devoted should be determined, as well as, the focus, notation and applications that should be given to them.

Also, it is important to point out that this methodology takes into account the internal and external mathematics link of engineering. In fact, the internal link is established between mathematics and basic sciences, mathematics and engineering basic sciences, as well as, mathematics and engineering specialization sciences, originated from the first stage. In other words, the interdisciplinary subjects of the curricular map now are in fact explicit and known by the professors. While the external link is established between middle superior level and engineering careers, which came out from the second stage of the methodology, as well as, among the latter with post grade and industry, determined by the third stage.

The methodology to design mathematics study programs in engineering has been denominated Dipcing methodology (Metodología de Diseño de Programas de Estudio de Matemáticas en Ingeniería).

In conclusion, we have established the feasibility of the proposal methodology design, Dipcing, because with this methodology, as it can be seen, study programs can be designed where the professor knows why each one of the themes that constitute the program are included. That is, whether they are of direct application to engineering or because they are added to form the logical structure of knowledge. So, the professor knows exactly how linked mathematics with engineering subjects to motive the student. With this, the professor knows what kind of mathematics abilities develops in the student.

## Discussion

From the Dipcing methodology arise support elements for the execution of the study programs, as well as, an updating program for teachers, where among others are included mathematics courses integrated with engineering. Also, arise a didactic strategy to follow in engineering careers, where mathematics should be presented to the students in an integral way with engineering.

*Didactic Aspects*

From the methodology come off in natural way didactic guidelines to be followed by mathematics in engineering careers, among them is the context or link.

The didactic strategy called *mathematics in context* is an ideal way for teaching mathematics subjects, since it offers applications that are not artificial but all contrary they are interesting for the student and they can be easily motivated with this, mathematics in engineering context is not so dry, and it is not out of the student reality and in fact it is ease the teaching learning process (Camarena, 1995; Camarena, 2003).

If the student really likes his career he finds not only a need in mathematics in context but also likes them very much and has a great interest to control them. He can tackle any problem in his future work life. More information about didactic strategy can be find in Camarena (1987; 1993; 1995; 2003) references.

Some supporting didactic materials are found in books of mathematics problems which are guides for the course both for the students as well as for the professors. For more information see Camarena (1998) reference.

Another important element is the use of electronic technology as a didactic support. Nowadays this aspect cannot be dismissed since the students are in direct contact with the computers and on the other side, in their work life they will use this tool.

*Teachers Updating*

Due to the way the study programs are designed and the didactic lead from them, it is observed that the teachers with a mathematical formation should be more prepared in engineering areas they work for. Engineering teachers should receive a stronger preparation than the one they have when they graduate from their career, in mathematics area they teach.

Obviously from this methodology the teachers updating courses should be courses that have mathematics in engineering context, as well as, courses about the teaching and learning process. For more information see Camarena (1990; 2004) references.

## Conclusions

With this methodology, objective mathematics study programs are obtained and they are linked with the subjects of engineering they support, as well as, linked with a middle superior level, post grade and industry. Also, mathematics in engineering context are generated easily and the themes on which the professors must be updated are obtained.

DIPCING methodology helps to make a better professional quality. When *mathematics in context* is presented the significant learning of the student is helped, which will influence durable and motivating learning.

Is good to mention that the Dipcing methodology has been applied in schools of the National Politechnical Institute of Mexico successfully.

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# IMPLEMENTATION FIDELITY IN THE MATHEMATICS CLASSROOM

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The purpose of this poster is to present the results of an analysis of an extensive literature review of studies and to consider the methodologies used to measure implementation fidelity in mathematics classrooms. This review is part of a long term project to implement a new high school mathematics curriculum and will serve as a basis for its methodology for measuring implementation fidelity. Implementation fidelity is defined as the extent to which a teacher implements a curriculum as the authors intended. If teachers do not implement the curriculum as the authors intend, they could undermine the effectiveness of the new research-based curriculum. Due to the varying conditions across schools and teachers, the need for measuring implementation fidelity is strong. Curriculum evaluators must consider whether the curriculum could "survive or thrive" across sites (NRC, 2004, p. 114).

Curriculum evaluators use various instruments to measure implementation fidelity. However, there are three instruments which are frequently utilized. The first and the most common methodology used to measure implementation fidelity is classroom observation (e.g. Tarr, Reys, Reys, Chavez, Shih, & Osterlind, 2008). Researchers also use surveys (e.g. Schoen, Cebulla, Finn, & Fi, 2003) and teacher interviews (e.g. Remillard & Bryans, 2004) to gather information on a teacher's implementation of a curriculum. In addition, a few researchers consider additional measurements such as homework assignments, chapter evaluation forms, textbook-use diaries, and table-of-contents records. Researchers use these instruments to measure several categories of implementation fidelity, including use of curriculum, teacher instructional practices, verification of teacher-reported data, teacher background information, and teacher beliefs

Through the curriculum evaluations, researchers conclude that professional development, students' comments and questions, and teachers' beliefs and backgrounds greatly influence teachers' level of implementation fidelity. Thus, researchers must be concerned with the effectiveness of professional development, the need for teachers to learn and adapt prior to and during instruction, and the correlation between beliefs and backgrounds and teachers' implementation of curricula. Future research in this project will build on and adapt what is learned from this current research.

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# DELPHI METHOD: A DIFFERENT APPROACH OF CREATING MEASURES OF PROFESSIONAL KNOWLEDGE FOR TEACHING

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*In this paper, we describe a research project in which a new assessment tool was developed and operationalized to measure teachers' pedagogical content knowledge (PCK) of geometry and measurement at the middle school level using the Delpi method. This method, often used in the field of economics, allows a panel of experts to come to a consensus about a given set of tenants or beliefs about knowledge. Delphi methodology provides an opportunity for experts to receive feedback and to modify and refine their judgments based upon their reaction to the collective views of the group. As a result of this study an instrument with an evaluation rubric for assessing middle school teachers PCK in geometry and measurement were developed.*

In selecting the participants for this project several factors were taken into consideration: a) the number of participants, b) their expertise, and c) the difference in their perspectives. When using Delphi method for research it is generally recommended to identify between twelve and twenty participants (Altschuld, 1993; Dalkey, Rourke, Lewis, & Snyder, 1972; Debecq, Van de Ven, & Gustafson, 1975; Edwards, 2003). Researchers picked twenty Delphi study participants based on their expertise. They will be referred as experts from this point on. The experts were selected from the four categories: a) researcher experts, b) mathematics educator experts, c) teacher experts, d) mathematics education leader experts. A detailed description of each group and the selection process will be provided during the presentation. In this study the development and administration of this survey was interconnected. The researchers' role in the data collection process was a) gathering the data from the research literature and creating the initial measures, b) identifying a panel of experts, c) corresponding with experts, collecting their ratings of the measures, and feedback on each measure, and d) analyzing collected data and reporting the results. The qualitative analysis included: a) the review of literature, b) the content analysis of the data, c) the identification of emerging categories of the data, and d) the operationalization of the instrument. The quantitative analysis included: a) calculating reported rating means for each item of the instrument, b) identifying outliers in the reported data, c) recalculating reported rating means of the items, d) conducting factor analysis, and e) establishing reliability, such as test-retest, etc. The researchers decided to use three rounds to elicit experts' suggestions for developing appropriate measures of PCK. The data analysis and data collection were done parallel to each other. The instrument was modified based on experts' feedback, and analyzed according to the categories of the table of specifications developed by the researchers. New categories in the table of specifications emerged from the data, and were used to complete the analysis. More detailed explanation on the development of the table of specifications reliability and validity considerations and other aspects of the Delphi method in the context of developing instruments of teacher knowledge, as well as strengths and limitations of this method will be presented during the talk. The Delphi method used in this project may be further adapted in the context of the broader instrument development process.

# THE VALIDATION OF AN OBSERVATIONAL MEASURE OF MATHEMATICS INSTRUCTION

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In the field of mathematics education, there is a need for viable and valid observational measures of mathematics instructional quality. Examining constructs central to mathematics teaching and learning is essential in order to gain a deeper understanding of these phenomena. Currently, such measures for use in large-scale studies do not exist. The goals of this poster presentation are: to introduce a new measure, the Mathematics SCAN (M-SCAN), and to explain the process of validating the measure for its reliable use.

The M-SCAN focuses on eight dimensions of mathematics instructional quality: structure of the lesson; multiple representations; students' use of mathematical tools; cognitive depth; mathematical discourse community; explanation and justification; problem solving; and connections and applications. The developers of the M-SCAN extended the work conducted by Borko, Stecher, et al. (2005) to create the measure and coding guide.

The validity study was conducted in a large, suburban, mid-Atlantic school district. Sixty third and fourth grade classrooms were randomly selected from a larger sample of classrooms from a federally funded large-scale study. A thirty-minute segment from each classroom was coded using the M-SCAN and three other observational measures: the Reformed Teaching Observation Protocol (Sawada, Piburn, et al., 2000), the Classroom Assessment Scoring System (Pianta, La Paro, & Hamre, 2008), and a time sampling measure of instructional format and content type.

To validate the use of the M-SCAN, the integrated conception of validity proposed by Messick (1995) was utilized. Mathematics education experts provided feedback about the constructs and content of the coding guide. To examine the substantive aspect of validity, the coders' rationales for scores were analyzed in regards to the dimension descriptors. Analyses were conducted to investigate convergent and discriminant evidence with the other observational measures. The development and validation of the M-SCAN aided in establishing reliability. Analyses were conducted to determine generalizability of the reliability findings. Specific results regarding evidence of validity and reliability will be shared at the poster presentation.

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# IMPLEMENTATION OF INTEGRATED MATHEMATICS TEXTBOOKS IN SECONDARY SCHOOL CLASSROOMS

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*Teachers are the ultimate decision makers with regard to mathematics content taught in secondary school classrooms and research has shown that they rely on textbooks to help with those decisions. The purpose of the study reported here was to gain an in-depth understanding of students' opportunity to learn mathematics from an integrated textbook with respect to the specific content strands contained within the textbook. The results reveal that although teachers provide some attention to each content strand contained within the textbook, the decisions they make about what specific objectives to teach and what to omit results in a different emphasis on each strand than is represented in the composition of the written curriculum.*

## Objectives

The purpose of this paper is to report findings related to curriculum implementation of integrated mathematics textbooks in secondary school classrooms and to discuss implications of these findings. As part of curricular reform, new content has been added to textbooks that bring topics such as statistics, probability, and discrete mathematics to a more central position in the school mathematics curriculum. Moreover, established content such as algebra and geometry is presented in a more integrated fashion than embodied in predecessors to standards-based textbooks. Although we know textbooks are the centerpiece of mathematics instruction in U.S. schools (Grouws & Smith, 2000; Weiss, Banilower, McMahon, & Smith, 2001; Weiss, Pasley, Smith, Banilower, & Heck, 2003), we know little about the relative emphasis teachers place on content embodied in these standards-based textbooks. Teachers may choose to move through the textbook sequentially or not; they may choose to cover most of the chapters of the textbook or not; they may supplement the textbook with materials from other resources or not. All of these decisions affect the extent to which textbooks are implemented and therefore influence the mathematics that students have the opportunity to learn.

This investigation is an extension of a larger study known as the *Comparing Options in Secondary Mathematics: Investigating Curriculum* (COSMIC) project. The COSMIC project is a three-year longitudinal comparative study that addresses questions regarding the impact of two distinct organizations of high school mathematics curricular materials on student learning. During the course of the COSMIC project, the extent to which the textbook is implemented in each classroom is carefully assessed and will subsequently be used in interpreting data analyses that examine student learning under each organization of content. In my investigation reported here, the classroom implementation of an integrated textbook is examined in relation to four content strands: (1) algebra and functions; (2) geometry and trigonometry; (3) statistics and probability; and (4) discrete mathematics.

## Perspective

Although textbooks play a prominent role in the teaching of mathematics in K-12 schools, prior research suggests that different teachers implement the same materials in different ways

(Bowzer, 2008; Chávez, 2003; Tarr, Chávez, Reys, & Reys, 2006). Thus, a difference exists between curriculum as represented in textbooks or other instructional materials and the curriculum that students experience in the classroom (Stein, Remillard, & Smith, 2007). Figure 1 illustrates how the use of the curricular materials and textbooks are only one part of a larger conceptual frame of curricular influences on student learning. Developed by Stein, Remillard, and Smith (2007), this figure models curriculum as "unfolding in a series of temporal phases from the printed page (*the written curriculum),* to the teachers' plans for instruction *(the intended curriculum),* to the actual implementation of curricular-based tasks in the classroom *(the enacted curriculum)"* (p. 321). The oval represents possible factors that influence the transition from one phase of the curriculum to another. For example, teacher beliefs and knowledge mediate the way they interpret the written curriculum and make decisions about what to use from the textbook and what to omit. Within this conceptual framework, I examine the written curriculum in relation to the enacted curriculum.

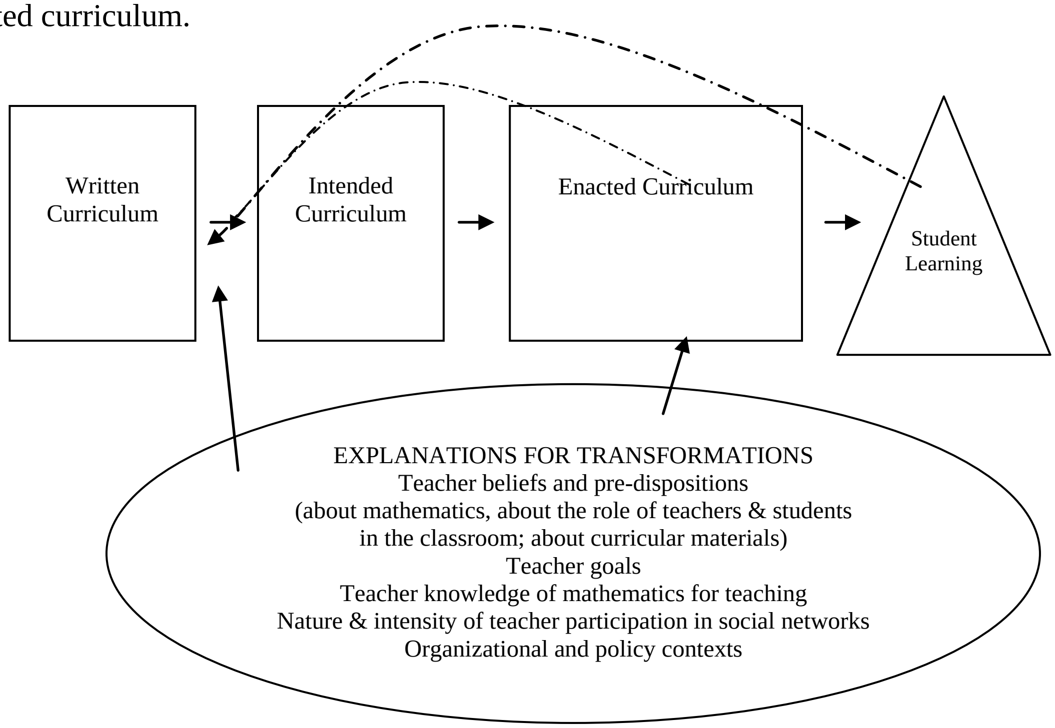


***Figure 3.*****Temporal phases of curriculum (Stein, Remillard, & Smith, 2007).**

## Methods

The purpose of the study was to gain an in-depth understanding of students' opportunity to learn mathematics from an integrated textbook with respect to the specific content strands contained within the textbook. Specifically, the following research questions were investigated.

To what extent do high school mathematics teachers provide students the opportunity to learn the *mathematics content embodied* in an integrated mathematics textbook?

a. What are students' opportunities to learn with respect to the Algebra and Functions strand?
b. What are students' opportunities to learn with respect to the Geometry and Trigonometry strand?

c. What are students' opportunities to learn with respect to the Statistics and Probability strand?
d. What are students' opportunities to learn with respect to the Discrete Mathematics strand?

*Participants*

Teacher participants for the study were 44 teachers from sites included in the COSMIC study, 21 in year 1 and 26 in year 2 with 3 teachers teaching in both years. At each site, all teachers teaching *Core-Plus Course 1* during year 1 and *Core-Plus Course 2* during year 2 were asked to participate. Six school districts in five states participated in the study, all employed a dual curricular option program (an integrated approach and subject-specific approach) allowing students to choose freely between the two options rather than be assigned via past achievement. The school locales varied from rural to urban settings and the student bodies ranged from high-middle socioeconomic (SES) backgrounds to schools where many students were from low SES families.

*Data Sources and Analysis*

*Table of contents record.* On the Table of Contents [TOC] Record, the teacher provided information regarding each lesson of the textbook being used and whether the textbook lessons were altered or adjusted. To gather this information, the teacher was provided an instrument that included a copy of the table of contents of the integrated textbook and was asked to indicate the level of textbook use for each section by choosing one of the following options: (1) content taught primarily from textbook; (2) content taught from the textbook with some supplementation; (3) content taught primarily from an alternative source; and (4) content not taught. Although every unit of each course is integrated around all the strands, the authors have identified each unit as having a primary strand within which connections are made across strands.

The extent to which the textbook was used during instruction and the manner in which it was used is reported utilizing two indices developed from the Table of Contents Records: (1) Opportunity to Learn index; and (2) Textbook Content Taught index. These two indices were calculated for each of the four content strands in the *Core-Plus* textbook: (1) Algebra and Functions; (2) Statistics and Probability; (3) Geometry and Trigonometry; and (4) Discrete Mathematics.

*Opportunity to learn (OTL) index.* The OTL index indicates whether the textbook content is being taught or not taught. Content here refers to the content objectives included in the textbook lessons and includes content taught primarily from the textbook or from alternative sources. The OTL index was computed by summing the frequency of occurrence of the first three options reported across all textbook lessons on a table of contents record divided by the total number of lessons included in the particular textbook. The OTL index essentially represents the percentage of the content in the textbook that students were provided the opportunity to learn for the academic year and aggregated by content strand. The formula below provides a specific example of how the OTL index was calculated:

$$OTL = \frac{6+6+29}{75} * 100 = 54.67$$

For this example, the teacher chose option 1 (content taught primarily from textbook) on the table of contents record 6 times, option 2 (content taught from the textbook with some supplementation) 6 times, and option 3 (content taught primarily from an alternative source) 29 times out of the 75 investigations contained within the textbook. The remaining 34 investigations were marked as option 4 (content not taught) and thus represent content untaught.

*Textbook content taught index (TCT).* The TCT index considers only those lessons whose content was taught in some manner and ignores content students were not given the opportunity to learn. It was determined by weighting each of the first three options provided to the teachers on the table of contents records. The largest weight was given when the first option was identified for a section, *i.e.*, content was taught primarily from the textbook. This was given a weight of 1. The lowest weight was given to the third option, *i.e.*, content taught primarily from an alternative source and it was given a weight of . The second option, taught with supplementation, was given a weight of and omitted sections were assigned a 0. The index was then calculated by summing the weights across textbook lessons and dividing by the number of lessons reported as content being taught in any manner and again multiplied by 100.

$$TCT = \frac{1(6) + \frac{2}{3}(6) + \frac{1}{3}(29)}{75 - 34} * 100 = 47.97$$

Again, this index was reported as a scale ranging from 0 to 100. An index of 100 would represent that every lesson taught was taught neither without supplementation nor from alternative sources. In contrast, an index of 0 would represent that every lesson taught was taught utilizing alternative resources with similar content that substituted for that in the intended textbook. All indices in between would roughly indicate the extent to which lessons were supplemented or replaced. Ultimately, this index reports the extent to which teachers, *when teaching textbook content,* followed their textbook, supplemented their textbook lessons, or used altogether alternative curricular materials.

## Results

Using the Table of Content Records, indices were computed for each year for the entire textbook and in respect to the four content strands. These indices reveal teachers' tendencies to select particular *strands* of mathematics contained within the textbook.

*Overall.* The overall mean opportunity to learn (OTL) index across the 21 teachers participating in Year 1 was 66.60 with standard deviation of 13.23 as represented in the first graph of Figure 2. Across the 26 teachers participating during year 2, the mean OTL index was 57.44 with a standard deviation of 8.57 as represented in the second graph of Figure 2. These indices indicate the percent of content taught *as defined by the textbook* that students were provided the opportunity to learn during each year. Interestingly, although during year 1 the OTL is larger than in year 2, there was substantially more variation among these teachers in terms of the OTL index. For example, during year 1 the lowest OTL index was 42.86 while the highest was 90.91.

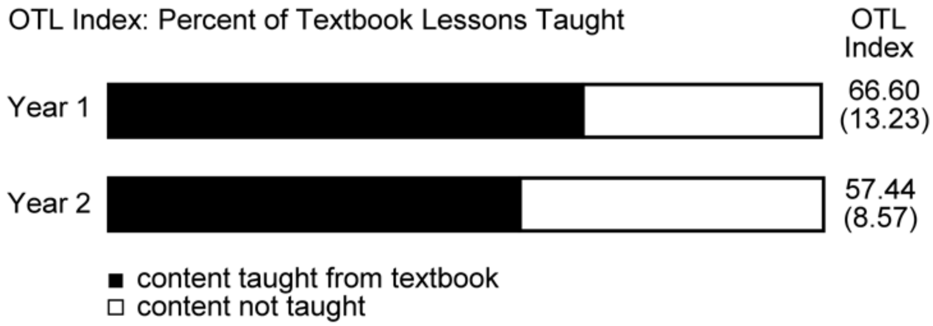


*Figure* 4. Percent of textbook lessons taught disaggregated by year.

Whereas the OTL indices provide information regarding the use of the textbook, the TCT indices narrow the focus to just the content embodied in the textbook that was taught. Figure 3 displays the *Textbook Content Taught* indices. The results show that the content taught across year 1 teachers, approximately 65% of that content was taught directly from the textbook. As shown visually in the second bar, during year 2, less content was taught directly from the textbook (56%) as compared to year 1. Note that the difference was smaller with regard to content supplementation with teachers during year 2 supplementing 29% of the content as compared to 25% during year 1. In general, little content was taught from alternative sources (9% during year 1, and 15% during year 2).

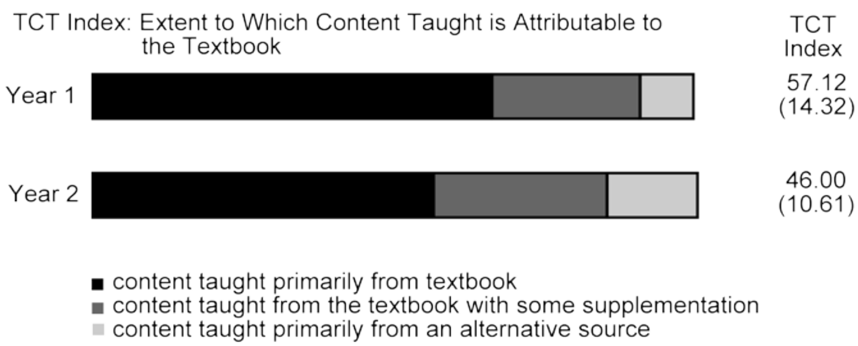


*Figure* 5. Extent to which content taught is attributable to the textbook.

As the overall indices indicate, year 1 teachers using Course 1 materials on average covered approximately 67% of the content contained within the textbook, whether utilizing the *Core-Plus* textbook to do so or not; year 2 teachers using Course 2 materials covered approximately 57% of the content. Given these results, teachers are making decisions regarding what content to teach and what to omit. Using the same Table of Content Records, the OTL index was computed with respect to each content strand contained within the *Core-Plus* integrated textbook.

*Content strand indices.* The results indicate that during year 1, the Algebra and Functions, Statistics and Probability, and Geometry and Trigonometry strands received similar attention. Students were provided the opportunity to learn approximately three-fourths of the content contained within the textbook for each of these three strands. However, within these three strands, the greatest variance existed within the Geometry and Trigonometry strand (Figure 4).

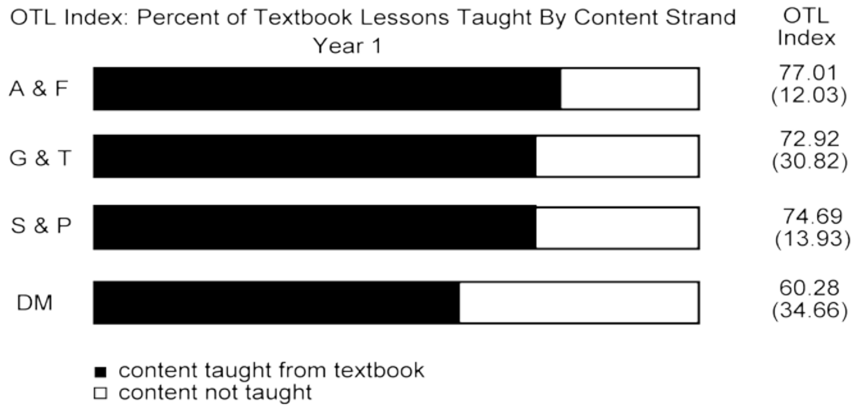


*Figure* 6. Percent of textbook lessons taught during year 1 disaggregated by content strand.

For example, two teachers omitted all units in the Geometry and Trigonometry strand whereas four teachers completed every geometry lesson contained in *Course 1*. The Discrete Mathematics strand received the least attention with only 60% of the material contained in the textbook being taught. Moreover, it is also the strand with the greatest variation. For example, three teachers omitted all units pertaining to discrete mathematics whereas only two teachers completed every discrete mathematics lesson.

Despite the relatively uniform attention given to the various content strands by teachers in year 1, the picture is strikingly different in year 2 (Figure 5). In particular, the Algebra and Functions strand not only received the greatest emphasis but was also implemented in the most consistent manner. Over 90% of the Algebra and Functions content was taught. In comparison, the remaining three strands received much less attention. Only 27% of the Statistics and Probability content was taught. Furthermore, the remaining three strands were implemented with considerable more variation than the Algebra and Functions strand. Similar to year 1, the Discrete Mathematics strand was the least consistently implemented.

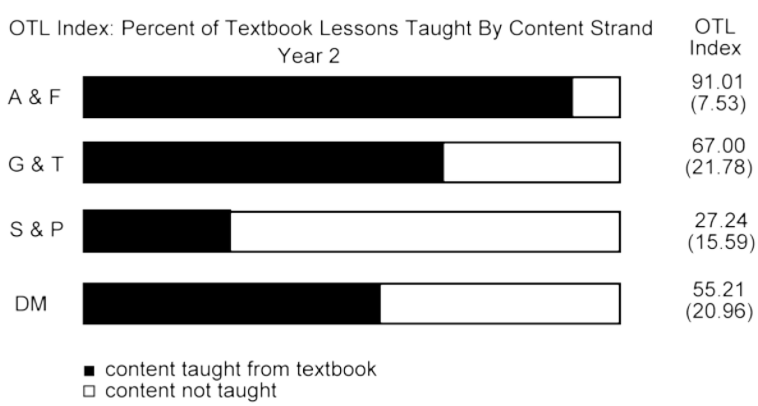


*Figure* 7. Percent of textbook lessons taught during year 2 disaggregated by strand.

*Textbook content taught indices.* Figure 6 displays the TCT indices for year 1 participants. The results show that when teaching content contained in the textbook, teachers relied mostly on

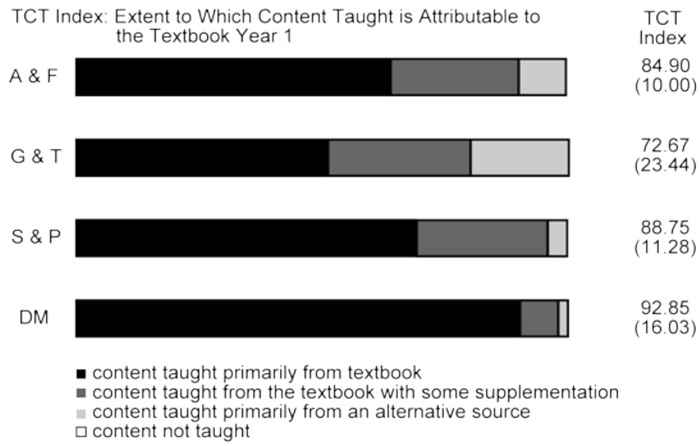


*Figure* 8. Extent to which content taught is attributable to the textbook disaggregated by content strand, year 1.

the textbook when teaching the discrete mathematics content, 90% when compared to 64% (A & F), 51% (G & T), and 69% (S & P). The use of supplementation was similar among the first three strands (26%, 29%, and 27% respectively) although it was greatest within the Geometry and Trigonometry strand. Moreover, teachers' use of alternative materials was also greatest when teaching geometry and trigonometry content (20%) when compared to the other three strands (A& F: 10%, S&P: 4%, and DM: 2%).

During year 2 (Figure 7), the results show that when teaching content contained in the textbook, teachers again relied heavily on the textbook when teaching the discrete mathematics content (71%) and little on alternative resources (2%) but the pattern was similar for the statistics and probability content. However, the discrete mathematics content was implemented more consistently than the statistics and probability content. The pattern of supplementation among all the strands was similar but again was the greatest for the Geometry and Trigonometry strand, 34% as compared to 30%, 21%, and 27%. The greatest use of alternative materials when teaching geometry and trigonometry content was repeated during year 2, 26% as compared to 14%, 7% and 2%.

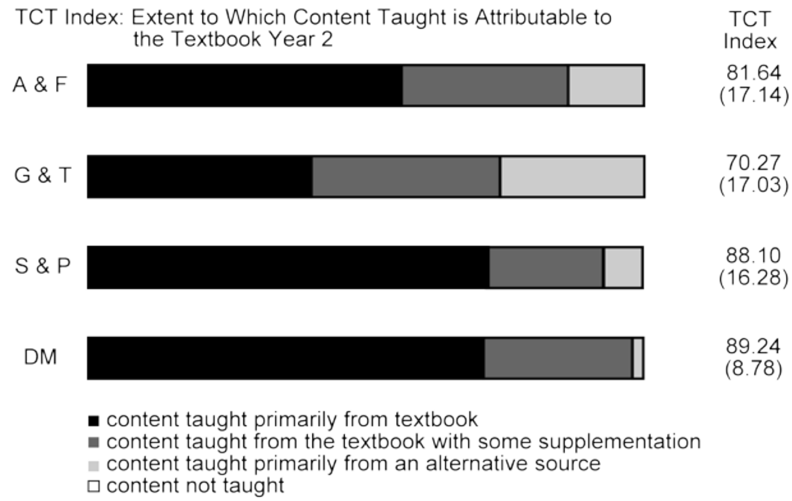


*Figure* 9. Extent to which content taught is attributable to the textbook disaggregated by content strand, year 2.

## Discussion

Teachers do attend to each major content strand included in high school textbooks where the mathematics is organized in an integrated manner, but their daily decision making results in student opportunity to learn being different than that intended in the textbook design. Overall, teachers on average taught less than two-thirds of the content embodied within the integrated textbook during a school year. Interestingly, of the textbook content taught, only 67% in year 1 and 56% in year 2 can be attributed solely to the textbook, with as much as 9% and 15% (year 1 and year 2 respectively) on average being taught from other sources.

Two primary implications of this data are worthy of attention here, one with respect to research and the other with respect to practice. In terms of research, given that during the course of two years, as much as one-third of the content taught includes use of supplementary materials, measures of the impact of this particular curriculum on student learning should be called into question, if this supplementation is not accounted for during data analysis of studies designed to assess its effectiveness in promoting student learning. In terms of practical implications, if textbooks are selected on the basis of content included and emphasized within the textbooks, then those concerned with what mathematics students learn need to be vigilant about how teacher decision-making may not be consistent with the intentions association with the textbook selections. Teachers are making decisions regarding what content to teach and what to emphasize and these decisions impact students' opportunity to learn content. For example, if teachers were utilizing an algebra textbook, the content omitted or deemphasized would most likely be algebraic content. However, when teachers are utilizing an integrated textbook, the content omitted may be from any of the four major content strands contained within the textbook. The results during year 1 revealed that, on average, three of the four strands (Algebra and Functions, Geometry and Trigonometry, and Statistics and Probability) received similar attention while the Discrete Mathematics received the least attention. However, during year 2, the Algebra and Functions strand received considerable attention and it came at the expense of the other strands. Reasons for differential attention may be attributable to the timing of this particular course within a student's education. In other words, this course is the second in a series of four and would most likely be taken during a student's 10th grade year, a targeted testing grade within the requirements of mathematics testing outlined in No Child Left Behind. Consequently, teachers may decide to focus on Algebra and Functions content because they perceive it to be the mostly likely content to be included on the test. This cause of the wide range of attention to strands across the two years suggests an interesting research question to be pursued further.

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# MATHEMATICS AS A TOOL: HIGH SCHOOL GIRLS' PERCEPTIONS

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*Nineteen high achieving young women in a 10-year longitudinal study were interviewed in high school about their experiences and interests in mathematics. Their perceptions of mathematics included that it was a tool for problem solving, a tool for their education, a tool to increase their self-confidence, a tool for future careers and other real world experiences. Noting that their teachers were instrumental in changing their perceptions of mathematics positively and negatively, none of these young women plans a mathematics major.*

## Background

A longitudinal study of young women selected by their teachers to take Algebra 1 in seventh or eighth grade is now in its tenth year. While some of the young women in the study may be talented in mathematics, we choose the term "high-achieving" to describe their scholarship, motivation, and work ethic. An historical perspective of Girls on Track provides a context to understand the scope of the larger, longitudinal study. Therefore, some of the findings that were previously reported are also presented here. In the first few years of the study, the purpose was to encourage these young women to continue on the advanced mathematics track, taking calculus no later than their senior year in high school. Some of girls continued beyond high school calculus, taking other college mathematics courses during their junior and senior years. Although, many of the young women in our study did continue on the advanced mathematics track, only 2 out of more than 250 young women, planned to or choose to major in mathematics as undergraduates, neither of these girls were in this current study.

In the first three years of the project, 45 of 174 girls were interviewed to investigate how they and their parents perceived their school achievement. Howe and Berenson (2001) reported that there were four characteristics that emerged from these interviews. First, every girl reported strong support from her family in terms of academics. A second factor to emerge was the girls' desire to do well in mathematics. These girls all want to understand what they are doing in math class. The girls discussed the desire to understand and their opinions of their teachers in relation to their wishes to do well. The interviewer did not directly ask about their teachers but the girls gave an opinion that their success and interest in mathematics, in almost every case, depended on whether the teacher helped them understand the topic or concept being studied. The third characteristic to emerge was assertiveness. These are not the type of girls who are passive in class, remaining quietly in the background while the teacher asks the boys the hard questions. Their desire to understand, as well as their desire to get a good grade, prompts them to ask questions in class and to seek help offered by teachers before and after class. When they get lower grades than they want they go to the teacher and ask why they got the low grade and what they can do to bring it up. Finally, these girls believe in hard work. What they all do to bring up a grade or maintain an A, is to work hard or harder. A good math student is *someone who tries hard and does their best*.

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In 2004, 39 telephone interviews were conducted with high school juniors and seniors who were members of cohorts 1 and 2 as middle school students. Berenson, Vouk, Michael, Greenspon and Person (2004) reported results on the girls' generally positive attitudes towards high school mathematics (See Table 1). Of the eight themes, the last three denoted areas in which the girls were not comfortable. These girls' attitudes concerning computer science changed from positive attitudes in middle school to negative attitudes in high school. Using data obtained from school records, we reported the fact that among high school girls in the longitudinal study 119 out of 141 girls was on track to take calculus by the senior year (Berenson, et al., 2004). The 20 girls who took algebra 1 in the seventh grade in 1999-2000 were all on track to take calculus by their junior year in high school.

Table 1. *Cohorts 1 and 2 Attitudes towards Math in High School**

| Attitude | Number |
|---|---|
| Expressed Confidence in Math Ability | 21 |
| Had a Great Math Teacher | 8 |
| Enjoyed the Challenge in Math | 7 |
| Liked Math | 5 |
| Used "fun" to Describe Math | 4 |
| Felt Nervous about Next Math Class | 5 |
| Thought Math was Difficult/Hard | 3 |
| Didn't Like Math Because of the Teacher | 2 |

*N=39

In 2005, we interviewed by telephone 86 girls, aged 16-20, who were juniors and seniors in high school, and freshmen and sophomores in college. Two papers reported at PME in 2006 indicated the mathematics pathways that these high achieving girls took in high school and the relationship between their math course selections and their career choices (Berenson, Michael, & Vouk, 2006). Approximately half of these 86 girls are planning to or have committed to major in STEM careers, and 60% have taken calculus 1 or beyond, four were undecided as to their career choices. We recognize that through high school and the early years of college these career choices are subject to change.

Collapsing categories we conducted a Chi Square test [À$^{2}_{\tilde{5}1}$= 3.897] to compare frequencies of advanced mathematics courses taken and choosing STEM (science, technology, engineering, and mathematics) or non STEM careers. Results indicate that high school girls' taking advanced calculus courses in high school were more likely to choose a STEM career. (Berenson, Michael, & Vouk, 2006). One of the 86 intended to major in mathematics and one in teaching mathematics. When asked to list strengths they will bring to the workplace, this sample of 86 describe themselves as hardworking, determined, organized leaders who are good with people. Disappointingly, only one out of 86 girls described herself as "smart.", when asked what attributes they would bring to the workplace. A quantitative study conducted by Wilson et al., (2006) found that decisions among girls in this longitudinal study to take mathematics beyond Algebra 2 correlated with measurements of mathematics success that include standardized tests, course selection, and course grades.

The results of this ten-year study reveal that high achieving girls in mathematics are confident, hard working, assertive, and have strong parental support for their academic and career choices. Most set high expectations of themselves academically. Half of these girls enjoy mathematics from middle grades through high school, indicating that they enjoy being challenged. Approximately half of this sample of 86 girls is planning to prepare for STEM careers with a majority of choices in the medical field. These high achieving girls are not interested in engineering (with the exception of bio-medical engineering), physics, or computing careers. The girls in this study from cohorts 1-5 are now in college, have graduated from college, or gone on to graduate school.

The current study takes a closer look at the girls' perceptions of mathematics when they were in high school. The design of this study is one of grounded theory and aims to develop a framework of study from the data. According to Creswell (2007), no theoretical foundations are given before the analysis begins. Interviews conducted in late 2004 and during the winter break of 2006/2007 are being used as sources of data. The remaining sections of this paper will discuss the participants, methodology, and discussion of the findings. The study hopes to answer how the high-achieving high school girls in this study perceive the role of mathematics in their lives.

## Participants

For the analysis conducted in this study, we looked at 19 of the high school interviews. The first 6 of these interviews were conducted 2004, these girls are currently juniors and seniors in college. The other 13 interviews were conducted during the winter of 2006/2007. These girls are currently freshman and sophomores in college. These girls were sophomores, juniors, or seniors in high-school. Table 3 outlines the participants, their grade at the time of the interviews, what type of high school they attended, and what their career interests were. The public high school, which is an academic magnet, has a strong focus on preparing students for college. This high school also offers the widest variety in advanced placement courses of all the schools.

Table 2. *Description of the Participants*

| Pseudonym | Grade at time of interview | Career interest | High school at time of interview |
|---|---|---|---|
| Ray | $10^{th}$ | Psychiatrist, Medicine | Public High School Academic Magnet |
| Amy | $11^{th}$ | Veterinarian | Private High School |
| Katya | $11^{th}$ | International Policy | Private High School |
| Zena | $11^{th}$ | Medicine | Public High School Academic Magnet |
| Wendy | $11^{th}$ | Undecided | Public High School Academic Magnet |
| Crystal | $11^{th}$ | High School Math Teacher | Public High School |
| Cara | $11^{th}$ | Nursing | Public High School |
| Ginger | $12^{th}$ | Television or Foreign Affairs | Public High School Academic Magnet |
| Amber | $11^{th}$ | Biotechnology, Science Industry | Public High School Academic Magnet |

| | | | |
|---|---|---|---|
| Kelly | $12^{th}$ | Engineering | Public High School |
| Kim | $11^{th}$ | Paleontologist, Engineer, Graphics Design | Public High School |
| Mia | $11^{th}$ | Music Engineer | Public High School |
| Alison | $12^{th}$ | Biomedical Engineering | Private High School |
| Lisa | $12^{th}$ | Lawyer, History Professor | Public High School |
| Marie | $11^{th}$ | Undecided | Public High School Academic Magnet |
| Rita | $11^{th}$ | Elementary School Teachers, Speech Pathology, Missionary | Public High School |
| Tina | Graduated | Biotechnology Forensics, Pharmacy | Home-Schooled |
| Annette | $11^{th}$ | English, Spanish or Math Teacher | Public High School |
| Vera | $11^{th}$ | Undecided | Public High School |

**Methodology**

During the history of the Girls on Track program researchers conducted several interviews of girls who participated in the summer camp held for middle school girls. Researchers contacted possible participants via e-mail and/or by phone, therefore these interviews represent a small group of girls willing to contribute to the research several years after their camp experiences. The first round of interviews analyzed for this study occurred in 2004, the girls from the first two cohorts were in high school, and a second set of interviews were held in late 2006 and early 2007 when the girls from later cohorts had become juniors and seniors in high school. This research presents the results from 19 different girls. The question asked during the analysis was: how did high-achieving girls perceive mathematics?

All interviews conducted were semi-structured and conducted in person. During the semi-structured interviews, a series of structured questions were asked in each interview with researchers also using open-ended questions to probe more deeply (Gall, Gall, & Borg, 2003). Different interview protocols were used during the two times interviews were conducted. However, all protocols explored participants' experiences, interests, and activities. All interviews were audio-taped. Interviews lasted from 30 to 90 minutes.

The interviews were transcribed verbatim and analyzed using NVivo qualitative software (Weitzman, 2003). Much of the initial analysis of the interview transcripts took place in regular team meetings over the course of four months. Researchers discussed codes and definitions of these codes. They also coded several interviews together. For first and level coding processes, both serial tagging, analyzing each transcript one at a time, and parallel tagging, reading and comparing each response to the same question, were employed by researchers (Baptiste, 2001). Initially, data were labeled, tagged, and coded based on emerging themes in the data (Baptiste, 2001). The codes were created during a separate analysis of data using Wenger's (1998) community of practice framework. The research team was analyzing 12 college girls' interviews, during this coding discussion turned to how the students were discussing their views of mathematics and science in relation to their chosen careers and education. The team then created a series of codes about the girls' beliefs and perceptions of mathematics (Table 4).

To contribute to the validity, reliability and veracity of the study, strategies of verification included using incremental evidence (Morse, Swanson, & Kuzel, 2001), triangulation between researchers (Merriam, 2002), and adequately engaging in data collection so that the data become saturated (Merriam, 2002). In order to address issues relating to reliability, multiple coders were used, coding was checked and refined at both the first and second levels of analysis, and inter-rater reliability was established (Morse et al., 2001).

Table 3. *Codes for Describing Girls' Beliefs and Perceptions of Mathematics*

| Codes | Explanation |
|---|---|
| Mathematics as a tool for thinking and problem solving | Girls like to be challenged, math increases their thinking skills, also some talked about the problem solving skills they developed in higher mathematics courses |
| Mathematics as an educational tool | Advance study in other areas such as Chemistry or Physics, business degrees, as well as in future mathematics classes. Following curriculum requirements and educational goals |
| Mathematics as a tool for STEM career pursuit | Mathematics will help them in their pursuit of STEM related careers. |
| Mathematics as a tool to build confidence in abilities, enjoyment, and satisfaction | The girls are good at mathematics, and therefore, it helps them build confidence in their academic abilities. They also express that they have natural ability and genuinely enjoy mathematics. |
| Mathematics as a tool for connecting to real-life and other fields | The girls talk about liking classes that are connected to other areas. Prefer applied mathematics classes that are relevant for their future majors and career interests. |
| Mathematics as tool for teacher influence on the young women's perceptions of mathematics | The young women describe classroom experiences and teachers in which their perceptions of mathematics was changed. Including their enjoyment and satisfaction in studying mathematics. |

## Discussion

This section will describe each of the six codes in more detail with evidence from the interviews. Because these codes were originally created from college interviews conducted in the winter of 2006 and spring of 2007, they had to be reevaluated for the high school transcripts. For example, the first code, mathematics as a tool for thinking and problem solving, was much more prominent among the college aged girls (Lambertus, Berenson, & Bracken, 2009), but the high school girls discussed their desire to be challenged. The research team saw this desire as being connected with increasing thinking skills. Therefore, the high school interview transcripts have a slightly different focus.

The first code Mathematics as a tool for thinking and problem solving was created to capture the young women's beliefs about how mathematics increases their thinking and problem solving skills. Tina, who was home-schooled and graduated from high school at the age of 15 was taking a year off to do community service, because her mom felt she was too young for college. She continued her mathematical studies, taking calculus for no credit and because she "wanted to keep her mind sharp". The girls liked not only to progress through the mathematics curriculum, but to take challenging courses that improving their thinking skills and prepared them for college. This code also encompasses the idea that the girls like to be challenged and what to be engaged in their learning. Amy talked about the difficulties she has if she is not being challenged. "[I]f you put me in an easy class, I'll just sit there and not do anything…It's got to be challenging enough for me." Ginger, a high school senior, stated that the challenge of a mathematics or science course is what draws her to those courses. This is further demonstrated by the fact that most of the girls, 16, took at least one advanced placement course, and 14 took an advanced placement course in either science, mathematics or a combination of both.

The second code, mathematics as an educational tool has two facets. The first facet being that the girls take mathematics classes that are required and in line with their educational goals. This code is closely tied with the first. The girls are preparing themselves to go to college and are following the curriculum path that will put them in a position to possibly gain admissions to the universities of their choice. For example, Kim stated "I'm in AP Calculus. Last semester I had AB and then this year, I have BC Calculus." A second girl, Lisa, described how she took all the mathematics and science courses available at her school, so that she could "test out" of courses in college, because she wants to focus on history and political science in college. Lisa later stated that she felt taking calculus would "look good to colleges". The girls appear to have spent quite a bit of time thinking about what types of courses and activities would get them into the colleges of their choice. The second aspect focuses on the fact that the girls see mathematics as a way to advance their study in other areas such as physics or chemistry. Kelly expressed that she took calculus before physics so she could take a higher level course than general physics. However, she did not want to take AP physics. Therefore, she was placed into honors and felt that it would be fun and "hard" at the same time.

Mathematics as a tool for STEM career pursuit is the third code. The girls explained the different ways in which the felt their mathematics have helped them or will help them in their pursuit of STEM careers. Because the girls are in high school, we know that their career interests are not solidified. They are probably going to change and several of the girls listed multiple unrelated careers or simply states that they did not know. Amy stated "I'm not really sure, exactly, what I'm interested in. But, I know that I'll need math. It'll be…[math] will definitely be important." Other girls were more confident in their career choices and knew that mathematics would be important in some way. Kelly knew that she wanted to be an engineer. She also had taken steps to prepare for that route. Kelly articulated that she needed calculus in order to pursue an engineering degree; also, she had visited a couple of chemical engineering classes, and observed how mathematics was used. Most of the girls expressed that they would need to use math in their future careers.

The girls are confident in their abilities in mathematics classes; therefore, they talk about mathematics as being a tool for building their self-confidence, enjoyment in math. They gain a level of satisfaction through being able to perform well in challenging mathematics classes. Everyone of the girls in this sample are good at mathematics, and had taken Algebra I in either the 7th or 8th grade. The girls continued down the advanced mathematics track. To date, we know that most the girls either started the Calculus sequence, or completed advanced placement in statistics. In terms of expressing their confidence and enjoyment, the girls gave comments such as:

> I'm taking calculus because I like math (Ginger)
> I'm doing pretty well in them [math classes], so – I like classes that I do well in (Mia)
> I'm really good at calculus and math-based stuff (Lisa)
> I like the way that math works (Annette)

The girls also see mathematics as a tool for connecting to real-life and other fields. They talk about the relevance of mathematics to real problems. They express their enjoyment of being able to apply mathematics to situations that have a direct impact or application. For example, Alison stated that she liked "calculus … better than any of the math that I'd ever done before just because you could see how it applied to real life and how it's important." When the girls were asked about the Girls' on Track camp they stated that they like the mathematics problems because they were applied to real-life problems and situations.

Finally, mathematics as a tool for teacher influence on their perceptions, enjoyment, and continued study was the final code. All of the girls talk about the influences, both positive and negative, of teachers. From examining these particular interviews, we can see that teachers have a large impact on the girls' course choices and their enjoyment of the subject matter. Kim expressed "I love it [BC Calculus] because our teacher, she's an absolutely amazing teacher – she really knows how to teach Calculus." While Katya's enjoyment in Calculus class is because she feels that she has a "really good teacher this year." Her teacher is open to questions and is very good at explaining different concepts.

The girls discuss that a particular course was difficult because the teacher did not explain material well, or that a course was exciting and fun, because the teacher was motivated and excited about the topic. Cara was the girl who expressed that she did not really like math. She verbalized that she did not really enjoy her current math class and that "math is not one of my strong points and I don't really have a good teacher so I don't really like math." When asked what it was that Cara did not like about her teacher, she said that the teacher covered too much material at once, and does not spend enough time explaining the material.

In some instances, a teacher has influenced future mathematics choices of the girls. Mia stated that her teachers influenced what classes she took "they see how well I'm doing in a class and they're like, 'you'd probably do well in this class' or 'you'd probably do better in this class'." Another participant, Marie, stated that she did not register for pre-calculus because her teacher told her there was a lot of geometry involved "I didn't do very good in Geometry, I'm better at Algebra stuff. So I took AFM [Advanced Functions and Modeling] because she [the teacher] said it was a lot of algebra."

## Conclusions

The high school girls expressed the opinions that their teachers were an important factor in their education. The teachers' willingness to help students, answer questions, and explain material all contribute to how the girls feel about their mathematics abilities and their enjoyment of the subject. It also seems to influence the courses the girls take in high school. While teachers may influence their enjoyment of mathematics, these girls perceive the role of mathematics not as a career option but as a tool for their future education and careers. In our study of the high achieving young women, we reported earlier on the college women's perceptions of mathematics and their career interests (Lambertus et. al, 2009). The college girls showed similar enjoyment in mathematics and the use of mathematics as a tool. However, of all the young women interviewed, none of them is studying mathematics. To date, we know of only one participant that has chosen to study mathematics. This leads us to question why these high achieving girls in our longitudinal study do not wish to study mathematics in college.

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# SHIFTS IN PROSPECTIVE SECONDARY TEACHERS' CONCEPTIONS OF MATHEMATICS, TEACHING, AND PROOF

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*Prospective teachers' views of mathematics, proof, and teaching mathematics were examined in a content-focused mathematics education course. Both changes claimed by the participants and small shifts observed by the researchers were noted, along with possibly influential class activities. In addition, an examination of how the participants held their beliefs revealed that they could be tentatively described as isolationist, naïve idealist, and reflective connectionist, similarly to the participants in Cooney, Shealy, and Arvold's (1998) study.*

## Introduction

Previous research suggests that teachers' beliefs about mathematics influence their classroom practices, although not in a way that is straightforward or easily measured (Philipp, 2007). In a previous study, Conner (2007) found that student teachers' conceptions of the role of proof in mathematics aligned with a particular aspect of classroom practice: support for collective argumentation. This paper reports preliminary results about shifts in three prospective secondary mathematics teachers' (PSMTs') conceptions of mathematics, proof, and teaching mathematics over the course of one semester. These participants are part of a cohort involved in a larger study, which will examine these conceptions and changes therein over two semesters and examine their connection to student teachers' classroom practice with respect to support for argumentation. This paper reports on the relevant parts of the following research questions:

- What do PSMTs believe about the nature of mathematics, the role of proof in mathematics, and the teaching of mathematics at the beginning of a two-semester sequence of mathematics education courses (content-focused followed by methods)?
- How do PSMTs' beliefs about mathematics, proof, and teaching change during a content-focused mathematics education course and to what do they attribute those changes?

## Theoretical Perspective

Cooney, Shealy, and Arvold (1998) suggest that we should expect little understanding of the connection between teacher education and teacher practice until we "understand the linkages between our activities in teacher education and the impact of activities on teachers' belief systems" (p. 331). They conceptualize a belief structure as encompassing the ways in which a teacher holds beliefs, providing insights into how consistent teachers' practices may be with their beliefs and how changeable these beliefs may be. At least as important as what PSMTs believe is how they hold these beliefs and how they deal with challenges to these beliefs. According to Cooney, Shealy, and Arvold, a teacher may uncritically incorporate newly encountered ideas into existing beliefs (naïve idealist); may completely reject new ideas that conflict with already held beliefs (isolationist); or may critically analyze ideas, merging aspects of both new and old into a compatible system (connectionist). Considerable research has been conducted on the link between teacher beliefs and practices. A teacher's practice may be connected to his or her view of mathematics, view of teaching mathematics, or some combination of these views. It may also be necessary to examine other related factors to understand the relationships between beliefs and practice. The elementary teacher in Raymond's (1997) study held beliefs about teaching math that differed from her practice, but her practice aligned with her views of mathematics. Borko, Eisenhart, Brown, Underhill, Jones, and Agard (1992) also found inconsistencies between beliefs and practice in their study of a preservice elementary teacher. Philipp (2007) suggests apparent inconsistencies between teachers' beliefs and their practice may be explained by examining contextual factors and the teachers' belief systems. Teachers' belief structures may influence how readily their beliefs change, and previous research has established that teachers' beliefs change slowly, over time, and are not readily changed in one semester (Thompson, 1992).

## Methods

This research was conducted in the context of a mathematics education course, which was designed to help PSMTs think deeply about the middle and high school mathematics they will be teaching. The focus of this class was on mathematical concepts such as complex numbers and trigonometry and mathematical processes such as proof and representation. The course was taught by engaging small groups of students in mathematical investigations followed by whole class discussions. One of the authors of this paper taught the course, and all students in the class ($n$ = 10) agreed to participate in the research. Participants completed a survey (adapted from Yoo, 2008) and participated in semi-structured interviews at the beginning and end of the course to provide a means for us to understand their views of mathematics, proof, and teaching mathematics. Other data included video and audio recordings and field notes of each class session, copies of students' written work, and their responses to weekly reflection questions.

The survey asked students to identify, along a continuum from 1 to 8, their views of various statements about mathematics, proof, and teaching mathematics. Figure 1 shows one of the items on the survey. Each item involved two statements, labeled (a) and (b), and students were asked to select a number between 1 and 8 that reflected their level of agreement with the statements. The first interview involved questions about mathematics, teaching, and proof such as "What does it take to be a good math teacher?" The second interview involved follow-up questions relating to participants' answers to the first interview questions, direct questions about whether participants believed their views had changed, and several tasks that asked participants to validate proofs.

2. The main purpose of proof in mathematics is (a) to show the truth of a mathematical proposition or (b) to explain why the statement is true.

| *Mostly a* | ← *Toward a* | | *Both a and b equally* | | *Toward b* → | *Mostly b* | *Neither a nor b* |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

Comments:____________________

*Figure 1*. Sample survey item.

Data analysis is ongoing and has involved several passes across the various data sources. For the three focus cases (chosen to be representative of the cohort's initial views of math, teaching, and proof), the surveys and interviews were examined to ascertain individual students' initial and ending views of teaching, mathematics, and proof. Analysis involved identifying pertinent passages from interview transcripts, summarizing what the participants said about each of the relevant ideas, considering alternate interpretations of the participants' statements, and examining the transcripts, written work, and surveys for confirming or disconfirming evidence. The goal was to characterize each participant's views of mathematics, proof, and teaching mathematics.

After each focus participant's initial and final views of mathematics, proof, and teaching mathematics were characterized, changes in views were noted. We noted both changes that were apparent to the researchers and areas in which the participants claimed to have changed their thinking. When possible, we asked participants to identify class activities that influenced their thinking, and we examined those activities to derive characteristics that may have been influential. Future analysis will involve examining other classroom activities for evidence of when the changes may have occurred and what particular activities or conversations may have led to the changes noted by researchers.

## Results

The results reported here are for a subset of the participants in the overall study and are for only one of the two semesters for which data will be collected. David, Kylee, and Helen (pseudonyms) were chosen to be as representative as possible of the range of conceptions of mathematics, proof, and teaching mathematics observed in the participants at the beginning of the semester. All three had completed at least two mathematics classes past the calculus sequence, had taken at least two previous mathematics education classes, and were enrolled in at least two upper-division mathematics classes in addition to this course.

### *David*

David's views of some aspects related to mathematics shifted over the course of the semester, but no evidence was found to suggest that David's views regarding the nature of mathematics changed during the course. He referred to mathematics itself as "independent of human invention" but saw the ways people interact with it as new and innovative; mathematics is already present, waiting to be discovered (David survey 2, question 1). David seemed to utilize the lens of 'practicality' or 'applicability' to determine which mathematical topics are most important. Through the semester, this lens appeared to have been refined from what is applicable in school to what is applicable in life. At the beginning of the semester, he indicated that algebra was the most practical because it "is something that all high school students are going to have to go through at some point" (David interview 1, lines 38–39). David appeared to alter this stance as a result of his study of the concept of area for a class project. This project required him to examine area in depth and consider alternative definitions and possible generalizations of the concept. At the end of the semester David claimed that measurement is the most practical because it is "really easy to relate to real world situations" (David interview 2, line 286).

A second possible modification in David's thinking might be termed a *comprehension of unexpected complexity*. This comprehension is most clear when David recalled a class activity where each student attempted to represent arithmetic fraction operations visually, with the goal of gaining insight into how future students, who are not fluent with a standard algorithm, may think about performing such operations. "Fractions is one of the basic things you learn in math…. That was something I probably would have never… noticed or thought, you know, to see the concept behind" (David interview 2, lines 16, 144–145). This new comprehension seems to be at least partly due to the recognition that teaching a concept involves a level of understanding greater than that typically achieved by students. In addition to possibly gaining a deeper conceptual understanding of fraction operations, there is evidence that David began to generalize the idea of looking deeper in order to “see the concept behind.” He stated, “It’s not just going to be with fractions, obviously, it’s going to be with other concepts that… we just kind of overlook” (David interview 2, lines 148–152).

In both interviews, David emphasized that his reason for becoming a mathematics teacher was first about working with students and secondly about the mathematics. What appeared to differ, however, were his views on what and how mathematics should be taught. David’s comprehension of the unexpected complexity inherent in mathematical topics he understood procedurally developed concurrently with a belief that conceptual understanding should be a primary goal of teaching. Again referencing the work with fractions, he said, “Teaching… is really going into depth about like how to… divide and multiply fractions and what that actually means” (David interview 2, lines 139–141). Additionally, David’s explanations regarding what good teaching looks like became more specific over the course of the semester. In his first interview, David described a good math class as one with “examples on the board, do things like that, but in the end, or maybe the majority of the class time can be spent with students working in groups” (David interview 1, lines 171–173). In this view of teaching, student-led discussion is noticeably absent. In contrast, David referred to the final few weeks of the course as mirroring his ideal classroom: students working on problems in groups and participating in whole class discussion where “what happens is you can get the group that does know it to explain it, and that always helps” (David interview 2, lines 212–213).

There is strong evidence to suggest that David’s views of mathematical proof underwent some revision. At the end of the course he stated, “I used to think of like proof like as a more, like, step-by-step, uh, kind of recipe to follow to prove different things. And now, I kind of see it a little bit more as, um, as using, like, your mathematical knowledge to show something’s true” (David interview 2, lines 462–465). This minor movement in his thinking does not preclude him from later noting that some types of proof have expected structural elements, and, if employing a particular type of proof, one should know and use the traditional elements of that type. Aligning with his current views about mathematics teaching, David stressed that proofs can help students understand mathematical concepts. He pointed to a class activity where several proofs of the same statement were analyzed as one instance that helped to change his thinking in this regard.

*Kylee*

Kylee demonstrated subtle changes in her thinking about doing mathematics, although her view of the nature of mathematics did not change. In both interviews, Kylee stressed her objective view of mathematics, calling it unambiguous in August and, in December, stating that her favorite thing about mathematics was that it can only be interpreted in one way. She does believe that there are multiple ways to solve a problem, but each problem has only one correct answer. The transition with respect to doing mathematics is reflected in her survey responses. Initially, Kylee’s responses indicated that natural mathematical talent and procedures were more important than hard work or using one’s own knowledge to make sense of mathematics, respectively. For example, when asked if she would give up or keep working if a problem took a long time to complete, she leaned toward not finishing the problem (Kylee survey 1, question 16). At the end of the semester she expressed a belief in the value of perseverance: “if you wanna figure it out you can” (Kylee interview 2, lines 215–216). One might hypothesize that the class structure of group investigations and class discussions challenged her thinking and contributed to Kylee's strong belief in hard work. When reflecting on the class she wrote, "I love the open discussions that we have; the thought-provoking questions provided by our instructor(s) are a big part of that" (Kylee week 4 reflection).

Kylee thought her view of teaching changed, although interview and survey evidence provide little support for major changes. At the beginning of the semester, she stated that she knew what to expect in teaching because her parents are both teachers. Student characteristics seemed to play a big part in how she would run her classroom, but she generally remained committed to a more teacher-centered class, expressing a belief that some students need content taught by direct instruction: "I know the 'new-aged' idea of math is to help students understand it on their own, but there are so many things in math that I believe need to be just taught" (Kylee survey 2, question 7). Her comments about what teaching mathematics entails, however, did become more specific over the course of the semester. In her first interview Kylee seemed to focus on general teaching issues such as sharing her love of mathematics, explaining ideas in multiple ways, and helping students stay interested and focused during class. In the second interview, when asked if her thinking had changed, Kylee stated, "it might be a little bit harder than I thought, but not harder in a bad way just a lot more things to think about, … like having to explain things different ways and [being] more prepared for how students may think" (Kylee interview 2, lines 76–80). She pointed to the class activities of fraction operations and generalizing problems as experiences that contributed to changes in her thoughts about teaching. The generalizing activity involved working a number of similar problems in different contexts and determining how all of the problems could be examples of a general problem type. We hypothesize that Kylee's learning experiences during these group activities provided an opportunity for her to think about her views of teaching.

Although Kylee believed that her thinking about proof has changed, she attributed that change to her concurrently taken abstract algebra course. She demonstrated consistency in her overall idea about proof as a means for connecting and building up mathematics: "Proof is like taking ideas you already know and putting them into something, building another idea to make you believe something you already know" (Kylee interview 2, lines 284–287). Despite her emphasis on proof reinforcing facts that are already known, Kylee, over the semester, expanded her view of the purpose of proof to include explaining *why* statements, such as the four color theorem, are true (Kylee survey 2, question 2).

*Helen*

Helen viewed math as a body of knowledge that is waiting to be discovered. This body of knowledge has the characteristic of building upon itself. She also believed that math is not about memorization; it can be applied to anything and that there are many ways to do things in math. Her conception of math with respect to these characteristics seemed to be stable and was expressed in both interviews. Helen believed that her thinking about math changed over the course of the semester. In particular, she stated that she realized that she can have her own way of doing things in math. Previously she had realized that people can do things in math in different ways, but she may not have internalized that she might do things differently from others and still be doing them correctly. She said, "Kylee might do it this way and Bridgett might do it that way but Helen has her own way, too…. their ways are not the only ways that I can do things" (Helen interview 2, lines 245–248). We conclude that although Helen's view of the nature of mathematics did not change, her view of herself in relation to mathematics did.

Helen believed her view of teaching changed over the course of the semester. Precisely how her thinking changed is difficult to determine, but it seems to relate to her conception of group activities. Helen's view of teaching at the beginning of the semester seemed to be that a teacher should be the one presenting the information to the students with the students having the responsibility to ask questions and participate in class. In an attempt to explain the change in her thinking, Helen said, "I've been accustomed to like a teacher standing at the board and lecturing whereas this semester…. it was more of like, you know, let's work as a group, let's figure this out as a group or let's figure this out as a class…. I think this class has really helped because it makes me like maybe want to teach the class … like half a little bit of lecture, half a little bit of like let's do group activities" (Helen interview 2, lines 142–150). Experiencing a class in which students were expected to construct their understandings by interacting with other students around carefully chosen tasks seemed to have influenced Helen's view of the activities that are included in teaching. Helen's survey answers also reflect this change in her view of teaching. Originally, she said the goal of instruction is equally (a) to transmit established mathematical facts and procedures to students and (b) to guide students to construct mathematical knowledge and understanding on their own. At the end of the class, she indicated that the goal of instruction is mostly (b), explaining, "The teacher has to allow the students to try on their own and to guide the students to the information, not tell them the information" (Helen survey 2, question 7).

Helen's view of proof seemed to stay relatively consistent over the course of the semester. She saw the importance of proofs in mathematics but did not like them or feel that she was good at proving. For Helen, a large part of proving was related to understanding – understanding why a mathematical statement is true and understanding what the mathematical statement means. Her response to the survey question 'The main purpose of proof in mathematics is (a) to show the truth of a mathematical proposition or (b) to explain why the statement is true' was 4, (a) and (b) equally, both at the beginning and end of the semester. From her statements and work with proof tasks in the second interview, Helen's clear emphasis was on understanding each step of a proof when validating it. She had difficulty understanding one of the statements within an argument she was examining in the second interview and stated, "if I understand that one step it will probably be really convincing and I'll probably like that proof the most" (lines 790–792). In fact, she did find that argument most convincing and labeled it a proof, but only after the interviewer provided more information that helped to explain that particular part of the argument to her.

## Discussion

As we examined the activity of our three focus participants, we found some similarities in the conceptions held by the students, the kinds of changes or shifts we observed in their conceptions, and the class activities that the students reported as influencing their thinking. Our analysis considered mathematics, proof, and teaching mathematics separately. Our initial hypothesis was that this course, which focused on mathematics, might be influential in relation to mathematics and proof, while the next course, which focuses on pedagogy, might be more influential in terms of teaching. However, all three participants reported changes in their thinking about teaching, while only one suggested that this course influenced his thinking about proof.

David, Kylee, and Helen believe that math is 'out there' and either has already been discovered or is waiting to be discovered. They talked about math as a known entity, although David and Kylee both referred to mathematicians developing new ideas or new mathematics. Helen suggested that each person re-discovers math for himself or herself. David's perspective on what was important about math seemed to broaden, and we found shifts in all three of their perspectives on math in relation to teaching and learning. Helen's view of herself in relation to math shifted from a belief that mathematics problems can be done in multiple ways to a belief that she may personally have a valid method that differs from those of her peers. Kylee shifted from a view that espoused ability as more important in doing math to one that emphasized more sense making and hard work. David saw more complexity in what he had before considered to be simple mathematical ideas. Essentially, David's view of mathematics in relation to the rest of the world changed; Helen's view of herself in relation to mathematics changed; and Kylee revised her view of what it means to do and learn mathematics.

All three focus participants were concurrently enrolled in an abstract algebra course. This proof-intensive course may have influenced their thinking about proof more than our mathematics education course. Kylee directly stated that her thinking about proof changed during the semester, but in response to the abstract algebra course rather than the mathematics education course. David described a change in his thinking about proof over the course of the semester, although we might describe it as a maturing in his views of proof from an initial view of proof as step-by-step to a current view as a more global process dependent on understanding. He conceived of the purpose of proof as to show why a statement is true, although he also talked about proofs as showing that a statement is true. Both Kylee and Helen also spoke about the importance of understanding when writing proofs.

The most interesting shifts during the semester occurred within the three participants' views of teaching mathematics. All three felt that their thinking about teaching changed over the course of the semester, and we believe it was the nature of the class activities that most clearly influenced those changes. David and Helen both changed in their view of group activities. David saw more potential for learning to occur through group activities. Helen professed more willingness to use group activities, and perhaps saw more of an opportunity for learning to occur in them. Even though we do not believe Kylee changed from her view of the job of the teacher as explaining and presenting information to students, Kylee and David both attended more to the complexities of teaching mathematics during their second interviews. Their initial comments about teaching could be applied to teaching almost any subject, but they specifically commented on the complex nature of the knowledge needed for teaching math in their second interviews. Kylee and David both pointed to the fractions activity as influential in their thinking about teaching, while Helen attributed changes in her perspective to her experience as a learner in a class that was, for her, unique. Previous studies have pointed to the notion that teachers are influenced by their own learning experiences as students (Thompson, 1992). This, along with Helen's, and to some extent Kylee's and David's, report of experience suggests that even one course taught in a way that is significantly different from the rest of their experience may allow them to begin to think differently about teaching.

Despite some similarities in views of mathematics, proof, and teaching mathematics, we believe these three participants hold their beliefs in very different manners. Choosing Cooney, Shealy, and Arvold's (1998) vocabulary, we believe we can tentatively describe David's belief structure as reflective connectionist, Kylee's as isolationist, and Helen as a naïve idealist. Though his core beliefs, such as his metric of practicality, inform his subsequent beliefs and actions, David showed a willingness to refine these subsequent beliefs based on reflecting on his experiences and interactions, such as his concept analysis of area and work with the fractions activity. Isolationists generally believe that there is always a right or wrong, a belief Kylee repeatedly expressed in reference to mathematics, and that truth comes from authority figures rather than rationality. Because her parents are both teachers, it is likely that many of Kylee's views on teaching derive from her parents. She does not seem willing to challenge what her parents, as her authority figures, may have told her. Generally, her responses to the survey questions seeking to understand her beliefs about mathematics, teaching, and proof were quite consistent from August to December. Her only major shift concerned her own perseverance in completing mathematics problems. This possible shift in her thinking about her own interactions with mathematics does not seem to have influenced her thinking about teaching others, providing more evidence that she holds her beliefs as an isolationist. Helen seems to be a naïve idealist, wanting to bring the new ideas that arose from her experiences in class into agreement with her original ideas based on her classes with her favorite high school teacher, leaving her with the conclusion that she would teach half with lecture, half with group activities. Rather than making choices between teaching methods or reconciling differences between her original views and those arising from new experiences, Helen embraced all as good.

## Implications

The small shifts in conceptions that we saw over the course of only one semester suggest that experiencing a course taught differently may influence students' beliefs about teaching and learning. Although their beliefs about the nature of mathematics and the role of proof in mathematics did not change, each participant's views of some aspect of mathematics and teaching changed. Carefully designed mathematics education courses that provide PSMTs with experiences that mirror the ways that they will be expected to teach may allow them to reconsider or extend their initial beliefs about teaching mathematics.

We initially hypothesized that this course would provoke changes in the PSMTs' beliefs about proof based on comments made by students who had completed the course in previous semesters. Since our participants did not seem to experience comparable change, we examined our course activities, comparing them to previous semesters. We concluded that we did not address proof and proving as explicitly as in previous semesters. This leads us to suggest that more explicit attention to proof might be necessary to provoke students to reconsider their conceptions of proof. Given the small shifts in conceptions of proof in this semester, it may be necessary to attend to proof much earlier in teacher preparation programs.

The results reported here tell only part of the story. At this writing, data are still being collected about the PSMTs' conceptions of math, proof, and teaching. We describe the changes that we saw as shifts in thinking, and we will continue to examine how stable these shifts are, given their individual belief structures, during another mathematics education class and their student teaching practicum.

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# PROMOTING EFFECTIVE GRAPHING CALCULATOR USE: REVEALING UNINTENTIONAL PRIVILEGING

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*This paper reports on an exploratory case study aimed to identify the ways in which an algebra one teacher privileged the graphing calculator and the ways in which her goals were interpreted and ultimately practiced by her students. The study employed an adaptation of an existing framework (Pierce & Stacey, 2002, 2004) highlighting the mathematical, technical, and personal aspects of graphing calculator use. The authors suggest the consideration of each of these aspects as important for future research on the complex issue of privileging in the promotion of graphing calculator use.*

## Background

Graphing calculators are a mainstay in the U.S. high school mathematics curriculum and because of that considerable research has been done on the effect of graphing calculators in the math classroom (Ellington, 2003; Burrill et al., 2003). In a national survey completed in 2000, it was revealed that over 80% of high school mathematics teachers in the United States reported that they used graphing calculators in their classrooms (Weiss, Banilower, & Smith, 2001) and due to the current state of standardized tests in our country that percentage has likely risen in the last nine years. Given the pervasiveness of graphing calculators in the culture of our high school mathematics classrooms, it is necessary to understand the ways in which and the reasons why students incorporate them as tools in their mathematics learning and problem solving.

Studies have shown that the ways in which technology is used in the context of a mathematics classroom are formed by a shared understanding about the appropriate modes of use that are developed over time by the members of the classroom community, the teacher and the students (Doerr & Zangor, 2000; Goos et al, 2003; Kendal & Stacey, 2001; McCulloch, in press; Pierce & Stacey, 2004). Of these studies, those that aimed to understand and describe effective graphing calculator use have largely been set solely in the context of the classroom, failing to reach beyond and examine how promotion in the classroom might impact student decision making with regards to their graphing calculator use in independent situations (Burrill et al., 2003). Furthermore, the classrooms in which these studies are typically situated are the equivalent of pre-calculus or calculus classes (Burrill et al., 2003). However, since most statewide algebra exams require the use of graphing calculators many students first encounter them in algebra one. As such, it is important to examine how effective use is promoted, understood, and then applied in independent situations in these classrooms.

The study described here was set in a year long algebra one class in a large urban high school. It is different than previous work in that we look at an algebra one teacher and her students, both in and out of the classroom. This particular teacher incorporates graphing calculators regularly because she wants her students to become “effective graphing calculator users”. We aim to identify the aspects of graphing calculator use that this teacher deems consistent with effective graphing calculator use, the aspects that she actually promotes in her classroom practice, and the ways in which her goals are interpreted and ultimately practiced by her students in independent situations.

## Framework

Pierce and Stacey (2002, 2004) have offered a series of frameworks for analyzing both the cognitive and affective aspects of effective use of calculators with computer algebra systems (CAS). They have identified four aspects of CAS use: mechanical, technical, personal, and mathematical. The mechanical aspect refers to knowledge of the actual calculator hardware. The mathematical aspect refers to the mathematical knowledge that is drawn upon during problem solving. These two aspects act as the extremes in a continuum of knowledge that students must draw upon when using CAS. The interaction between the mechanical and mathematical aspects is what Pierce and Stacey (2004) refer to as the technical aspect. The technical aspect is synonymous with knowledge of the machine software, meaning it is the knowledge of how to get the machine to complete the mathematical actions you want it to carry out. For example, to solve an equation using a table requires both the mathematical knowledge of how a table could be used to determine a solution and the mechanical knowledge of how to create a table on the CAS. Finally, the personal aspect refers to attitudes toward CAS use and judicious decisions regarding its use. It is through the personal aspect that decisions regarding CAS use are made. Since the tools available on graphing calculators are a subset of the tools available on CAS, it is appropriate to draw upon Pierce and Stacey’s work to frame this study.

The purpose of this study was to begin to understand the intricate ways in which the promotion of graphing calculator use in the classroom might impact students’ use in independent situations. We aim to go beyond the generalities of attitudes toward graphing calculator use and look carefully at the actions and words associated with its use by both the teacher and the students, in and out of the classroom. Specifically we aim to examine interaction between the mathematical, technical, and personal aspects of graphing calculator promotion and usage. As such, Pierce and Stacey’s (2004) framework for effective use of CAS was adapted for graphing calculator use. This framework captures both the technical and personal aspects of graphing calculator use (see table 1). Pierce and Stacey (2002) have also explicated a framework for the mathematical aspect of CAS and graphing calculator use, which they refer to as *algebraic insight*. The algebraic insight framework identifies the “part of symbol sense that is most affected by the availability of having CAS” (p. 622) (i.e. algebraic expectation and ability to link representations). Given the context of this study, the algebraic insight framework is an appropriate lens through which we can view the mathematical aspect of graphing calculator promotion and usage.

Table 1. *Framework for aspects of effective use of graphing calculators**

| Aspects | Elements | Common Instances |
|---|---|---|
| 1.Mathematical (Algebraic Insight) | 1.1 Recognition of conventions and basic properties | 1.1.1 Know meaning of symbols |
| | | 1.1.2 Know order of operations |
| | | 1.1.3 Know properties of operations |
| | 1.2 Identification of structure | 1.2.1 Identify objects |
| | | 1.2.2 Identify strategic groups of components |
| | | 1.2.3 Recognize simple factors |
| | 1.3 Identification of key features | 1.3.1 Identify form |
| | | 1.3.2 Identify dominant term |
| | | 1.3.3 Link form to solution type |
| | 1.4 Linking of symbolic and graphic representations | 1.4.1 Link form to shape |
| | | 1.4.2 Link key features to likely position |
| | | 1.4.3 Link key features to intercepts and asymptotes |
| | 1.5 Linking of symbolic and numeric representations | 1.5.1 Link number patterns or type to form |
| | | 1.5.2 Link key features to suitable increment for table |
| | | 1.5.3 Link key features to critical intervals of table |
| 2. Technical | 2.1 Fluent use of program syntax | 2.1.1 Enter syntax correctly |
| | | 2.1.2 Use a sequence of commands and menus proficiently |
| | 2.2 Ability to systematically change representations | 2.2.1 Plot a graph from a rule and vise versa |
| | | 2.2.2 Plot a graph from a table and vise versa |
| | | 2.2.3 Create table from a rule or vise versa |
| | 2.3 Ability to interpret GC output | 2.3.1 Locate required results |
| | | 2.3.2 Interpret symbolic GC output as conventional mathematics |
| | | 2.3.3 Sketch graphs from GC plots |
| | | 2.3.4 Interpret GC non-response |
| 3. Personal | 3.1 Positive Attitude | 3.1.1 Value GC availability for doing mathematics |
| | | 3.1.2 Value GC availability for learning mathematics |
| | 3.2 Judicious Use of GC | 3.2.1 Use GC in a strategic manner |
| | | 3.2.2 Discriminate in functional use of GC |
| | | 3.2.3 Undertake pedagogical use of GC |

*Adapted from Pierce and Stacey (2002, 2004)

## Methodology

The focus of this study was a single high school algebra one class, both the teacher and the students. Since the purpose was to gain insight from all members of the classroom community about how graphing calculator use was being promoted and actually used, data was collected from both the teacher and the students before, during and after the unit of study.

*Data Collection*

The classroom teacher, Ms. Kersee (a pseudonym) identified a unit of study (solving equations) that she would be teaching during the fall semester and in which she planned to integrate the graphing calculator. The class was video taped every day during the unit (13 days). In addition to classroom video, both survey and interview data were collected from Ms. Kersee and each of her students. Ms. Kersee was interviewed both prior to and upon completion of the instructional unit. The purpose of these interviews was to gain an understanding of the ways in which she planned to promote graphing calculator use (prior) and believes she did promote its use. Students' perceptions of her promotion of the graphing calculator and their actual use were captured using both survey and interview data. A survey designed to identify how each student typically used the graphing calculator, their comfort level with different modes of the graphing calculator, and their perceptions of the teachers promotion of graphing calculator use was administered both prior to and after the instructional unit. Finally, the students participated in an interview at the completion of the unit, a portion of which was task-based. The student interviews were intended to provide insight into how and why the students used their graphing calculators in private situations and how consistent their use was to the ways in which they perceived use was promoted in their class. All interviews were both video and audio recorded. Video of the graphing calculator screen was also captured when calculators were used during the interview process.

*Participants*

Ms. Kersee is in her 14th year of teaching high school mathematics. She has spent her entire career at one high school, a large urban high school in the south eastern United States. During her 14 years she has taught at least one section of some version of algebra one every year. She was asked to participate in this study based on the recommendation of school administrators as a “very good teacher” that is known for “using technology well” in her classes (She was honored as teacher of the year during the 2007-2008 school year). Ms. Kersee has a B.A. in secondary mathematics education and is currently pursuing a master’s degree in educational leadership.

The class at the focus of this study is an algebra one course that meets 90 minutes each day for an entire school year. The class is comprised of 21 students, all of whom had been unsuccessful in a previous algebra course. Due to excessive absences only 17 of the students (8 female, 9 male; 15 Black American, 1 Mexican American, 1 White American) completed this study. These include 2 freshman, 7 sophomores, 6 juniors, and 2 seniors. A special programs teacher was in the classroom full-time due to the number of students who have instructional and testing modifications related to learning disabilities or other exceptionalities. Ms. Kersee expresses very firm beliefs on how students in such a class need to learn mathematics. Beyond the development of an environment of high standards and accountability, Ms. Kersee looks to incorporate new teaching methods that incorporate different learning styles and highlight different methods of solution.

## Observations and Preliminary Inferences

Ms. Kersee labels herself as being committed to professional development, a claim that is verified through her actions as a professional. She has been self-motivated to learn new methods of using calculators, specifically graphing calculators, in the classroom. While she admittedly does not know all of the instructional capabilities of the graphing calculator, she continues to search for new ideas. Without any technical, instructional, or financial support from her school or district, she explores Internet tools and other materials offered by Texas Instruments (TI) or other publishers. In the place of costly workshops and conferences, Ms. Kersee has taught herself how to use the graphing calculator through what she refers to as “playing around.” In order for her students to benefit from what she feels is a powerful tool and to address a shortage of available calculators in the school, Ms. Kersee wrote two grants to obtain a class set of TI-84 calculators, which students use in class and are allowed to check out to bring home in certain situations.

Ms. Kersee incorporates the graphing calculator into her algebra one course because of the value she places on effective calculator use as a benefit on state and in-class tests. She has also observed the positive impact of the calculator’s availability on students’ confidence. In the instructional unit in which this study was situated (solving equations), Ms. Kersee identified the use of tables to solve equations as a skill that she intended to promote as a solution method that was as legitimate as using “by hand” methods. Indeed, she stated that she hoped her students that struggled with computational errors would use tables instead of “by hand” methods on both their classroom based and standardized tests.

For the purposes of this paper we have chosen to focus on a very small sliver of Pierce and Stacey’s (2002, 2004) adapted framework, one particular technical aspect (constructing a table from a rule) and the mathematical and personal aspects related to it from both the teacher's and the students’ points of view. Specifically, we focus on Ms. Kersee’s goal of having her students learn to use graphing calculator produced tables to solve equations, how she actually carried out this goal in her classroom, and some preliminary observations with respect to how her words and actions influenced student decision making regarding graphing calculator use in independent situations.

*Graphing Calculator Promotion in the Classroom*

Ms. Kersee introduced the use of tables to solve equations on the tenth day of the instructional unit. The previous nine days had been focused on solving one and two step equations, special equations (e.g. identity and no solution), solving literal equations, and word problems. This, the tenth day, was the day before the unit test and Ms. Kersee’s goal was to provide her students with another method for solving equations. She opened the class by stating, “Today I’m going to show you another way to solve equations.” She began by using a large poster that displays the TI-84 plus calculator, on which she pointed to the buttons that would be used in the lesson. After identifying these keys on the graphing calculator Ms. Kersee put an example problem on the board, 2x - 3 = x + 4, and solved it "algebraically." Next, while using the overhead display of her own graphing calculator, she instructed the students to enter the left and right side of the equation into $Y_1$ and $Y_2$, respectively. She reminded the students that the solution of the equation will be "the X that makes it a true statement." To find the solution, Ms. Kersee told the students to "search until our $Y_1$ and our $Y_2$ is the same." Then she identified x = 7 as the solution to this equation.

On the board, Ms. Kersee checked the solution found on the calculator by substitution, and reiterated that the table process is "another way to solve an equation." The students were then given four equations to try to solve on their own ($7-2x=x-14$ , $2(4-2r)=-2(r+5)$ , $(2/3)(6x+3)=4x+2$ , and $3.2h=9.3-3.2h$ ). After the students worked the examples independently, she returned to the board and "checked" solutions given to her by the students "algebraically" by hand, without using or referring to the table. After finishing the second equation, a student asked Ms. Kersee why she did not "plug in" the solution to check. She responded, "To check your answer, you plug it in. I was doing it algebraically to show if you work the steps, you still get the same answer." After they had gone over all four examples, Ms. Kersee assigned a review for the test the following day. No mention of the graphing calculator was made when discussing the review assignment.

The following day, as they went over the review assignment all problems were worked out on the board and no mention was made of the new solving method. However, as Ms. Kersee passed out the tests she reminded the students that “when you are solving equations you can use any method. And don’t forget you can do that to check.” Further, she explained, “If you use your calculator to solve a problem on your test, you don't have to show any work, just make sure you write down that you used your calculator so I know where you're getting your answer from."

*Student Graphing Calculator Use*

Though Ms. Kersee stated that the students were free to use any method on any problem, preliminary analysis indicates that the students did not all interpret her goals as she intended. In some cases, the students did not understand how the new method of solving, using the tables, could even be used to meet those goals. Though there are many, we have chosen to highlight two examples of this misinterpretation of goals here: (1) the interpretation that tables are to be used to check, not to solve, and (2) the interpretation that tables can only be used when solving an equation with variables on both sides.

*Tables used to check, not solve.* Though Ms. Kersee introduced the table solving process as another method to solve, she only used it to solve one equation with the class. She then gave the students four examples to try on their own, when they came back as a whole group to share their solutions, she checked them by working the problems out by hand on the board. These actions have been interpreted by some students as a message that this process should be used to check, not solve equations. For example, when she was asked how she thinks Ms. Kersee wanted her to use her graphing calculator in this chapter one young lady responded, "Actually, I mean, she would have wanted us to use it. But, I think she ... I mean, she would want us to use it a lot but like maybe on problems that she went over with us to tell us to use it. So, if it wasn't the problem she went over that said use calculator, I guess she wouldn't really expect us to use the calculator." She went on to share a problem on her test for which she used a table. She explained, "I did it by hand and then by the tables, to check…how she showed us." Similarly, a young man shared that he used the tables to check his work as well. When discussing his test he said, "I used these (points to table) to see if it is right." Later when he was asked to solve a problem he first did it by hand and then used his graphing calculator to check his solution (see figure 1 below). The conversation follows:

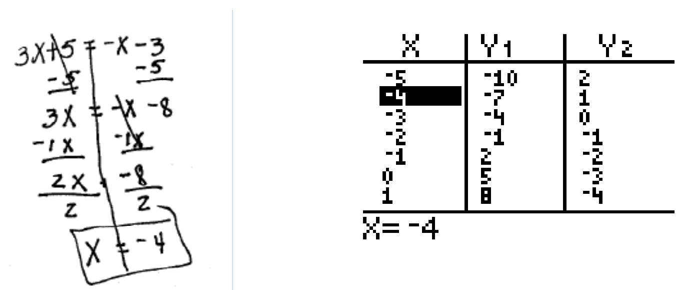


*Figure 1.* Student's "by hand" work and corresponding table used to "check".

| | |
|---|---|
| Interviewer: | Okay. So is your answer correct? |
| Student: | It [the graphing calculator] says it wrong. |
| Interviewer: | It says it's wrong, huh? Can you tell me what the answer should be? |
| Student: | Negative seven, negative seven. |
| Interviewer: | But, so that tells you that it's wrong, since it says negative seven and one, right? |
| Student: | Yeah. |

Interviewer: So when you look through here, can you tell what the answer is supposed to be? It's not negative four.
Student: Yeah.
Interviewer: Can you tell what it is?
Student: Uh, no.

This particular student not only had interpreted that the table was intended to check his work, but even though the solution was in his viewing screen he did not know how to use the table to determine the solution to the equation.

*Tables can only be used when solving equations with variables on both sides*. As described above, in her introduction of using tables to solve equations Ms. Kersee worked one example with the students using the overhead graphing calculator, which was followed by having them try four examples on their own. All five of these examples were of similar form; each had variables on both sides of the equal sign. In more than one case we saw that this choice of examples influenced students' beliefs about the types of problems for which using graphing calculator produced tables are either appropriate or that Ms. Kersee would deem valuable. For example, one student was explaining how he used his graphing calculator to solve a set of tasks. The interviewer noticed that he did not use the table on an equation that had a constant on the right hand side, but started to use his graphing calculator when their were variables on both sides. The interviewer inquired about this decision. Their conversation follows:

Interviewer: Okay, how come, this one you decided to use the table? What's different about these two problems?
Student: Oh, because this one here, when it looks like that, I can't put it in the table
Interviewer: When it's like that?
Student: Yeah.
Interviewer: What do you mean…
Student: Like, for, like, two equate--one equation's one side, one equation one side.
Interviewer: Oh, so there's a variable on both sides...
Student: Yeah.
Interviewer: ...and that's why you used the table. Okay. I understand. Can you use the table on a problem like this one, too? [pointing to an equation with a constant on the right]
Student: No you can't.

Due to the examples used in class this particular student either didn't know how enter an equation with a constant on one side into the graphing calculator to use a table, didn't think it was possible to do, or didn't think his teacher wanted him to use it on those types of problems. Regardless, his misinterpretation about when it is appropriate or even possible to use tables to solve equations was influenced by the examples presented in class.

## Discussion

Previous research has shown that students' graphing calculator use is influenced by the ways in which it is privileged in the classroom (e.g. Kendal and Stacey, 2001; McCulloch, in press). The preliminary results of this study support these claims, but also point to the complexity of the issue of privileging. Ms. Kersee's language, both in interviews and in classroom observations, indicates that she values the graphing calculator as a tool with which one can solve equations. Her intention was to send a message to her students that graphing calculator solving methods are as valued in her class as by hand methods. However, it is clear that the message that she intended to send to her students regarding her goals for them was not the message they received. It appears that the examples she chose and the way in which she discussed them might have influenced the ways in which her students interpreted her goals.

Though the focus of this paper was narrow, one particular technical aspect and the mathematical and personal aspects related to it, it is apparent that in order to build an understanding of how teachers actions influence the development of effective graphing calculator users, we must look at all three of these aspects together. If we had focused on just the technical aspect of how to actually create at table we would have missed how Ms. Kersee's value statements regarding using the graphing calculator to check might have been interpreted by her students. Furthermore, had we not focused on the mathematical aspect of interpreting the tables we might have missed the fact that some students interpreted that the tables were only appropriate to use on equations with variables on both sides. These findings bring about an important feature of the methodology for this study, data collection both in and out of the classroom for both the teacher and the students. Future research that aims to describe student decisions regarding technology use should include data from all angles in order to construct a complete picture.

It appears that the personal aspect of graphing calculator use is an important one. Its role was evident in every decision made by both Ms. Kersee and her students. Pierce and Stacey describe the personal aspect of graphing calculator use to include attitude toward graphing calculator availability and judicious use of the graphing calculator. However, based on both observations here and previous research (e.g. McCulloch, in press) it appears that there is more at work within the personal aspect. It seems as if the personal aspect includes not only attitudes, but also values, beliefs and even emotions and not only of one's self, but those perceived of others as well (e.g. students perceived Ms. Kersee's values). To better understand student decision-making regarding technology use future research should be designed to further operationalize the personal aspect and its relationship to the technical and mathematical aspects.

The overarching purpose of this study was to identify the aspects of graphing calculator use that this teacher deems consistent with effective graphing calculator use, the aspects that she actually promotes in her classroom practice, and the ways in which her goals are interpreted and ultimately practiced by her students in independent situations. This case provides evidence that issues of privilege with regards to technology use are complex, often misinterpreted, and in need of further research.

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# STUDENT BELIEFS WITHIN THE CONTEXT OF ELEMENTARY SCHOOL MATHEMATICS CLASSES

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*The objective of this research is to gain knowledge about some of the beliefs activated within the framework of the mathematical-argumentation processes that arise in the classroom setting. We are interested in analyzing the evolution of those beliefs and explaining them in view of the possible reasons subjects have for holding those beliefs, their personal motives and the contexts within which the beliefs take shape. In this research our interest lays in discovering the beliefs that arise spontaneously in class, which is the reason why the research was undertaken in the natural classroom scenario, where researchers limited their participation solely to observation.*

## Interpretative Framework

For purposes of this paper, the researchers considered that:

- The beliefs of a subject S are representations (Goldin, 2002) that possess an apophantical function (Duval, p. 98, 1999) to which S associates:
  A degree of probability (0,1] to their truth (Villoro, 2002)
  And a degree of personal relevance (that can start at zero), which relates to the importance the subject attaches to the belief, in turn producing in the subject a conative, affective, interest or expectations-related charge (Villoro, 2002; Schoenfeld, 1992, p. 358).
- The beliefs lead the individual holding them to respond consistently in favor of the belief held under different circumstances (cf. Villoro, 2002. Schoenfeld, 1992.). This condition enables assuming, with fair well-founded basis, that behind a regular or uniform behavior practice lies a guiding belief –or system of beliefs.

The truth or verisimilitude of a belief may be based on reasons—and in this case the subject is said to be convinced to a certain extent of that truth, yet it can also or only be based on affective aspects or on the interests of the subject holding the beliefs. In the latter case, the subject’s belief is said to be based on his/her ‘motives’ (Villoro, 2002).

Unlike formal logic, in the domain of beliefs the veracity load is not based on dichotomic principles. The subject professing a belief can associate to that belief a truth likelihood that ranges within a continuum from complete certainty of its veracity (“belief in the strong sense”) to uncertainty (“belief in the weak sense”) (Villoro, 2002).

The types of beliefs present in mathematics classes are (Schoenfeld, 1992; Thompson, 1992): (a) beliefs and value judgments of students and teachers concerning their participation in classroom mathematics activities, with the how and why they participate; foremost here are those related to teaching and learning; (b) Meta-mathematical beliefs; foremost among which are those that deal with the ways of justifying in mathematics, validation criteria and the semiotic systems involved; (c) ‘Mathematical beliefs’; foremost are beliefs in the truth or verisimilitude of the mathematical statements and those dealing with the validity or plausibility of a mathematical argument (in the broadest sense of the terms) (Goldin, 2002).

## Methodology and Data Collection

This is an ethnographic case study of the instrumental type (Stake, 1995) and of a longitudinal nature, undertaken of three primary education centers (two public schools and one private school) in Mexico City. Data collection and interpretation consisted of the actions listed below, although such actions were not necessarily carried out sequentially:

1. Design and application, within the context of a pilot study, of a school-type test for third and sixth grade students, dealing with proportionality problems and of a questionnaire for their teachers;
2. Interviews of the teachers from the three schools, dealing with their beliefs concerning mathematics and its teaching;
3. Observation of sixth grade classes on the subject of proportionality (ten observations of one teacher from each school, undertaken throughout the school year). The classes were video-taped using two video-cameras, one of which focused on the teacher and the other on the students;
4. Transcription of all video tapes taken; and
5. Analysis of the data collected in the video tapes as compared with the data from written records. Given the nature of the study, project researchers limited their classroom involvement solely to observation. During the classes observed, the teacher usually uses the official mathematics textbook as a guide. The didactic proposal of such official mathematics textbooks focuses on solving exercises and problems. Consequently in this study, a classroom argumentation consists of a process of social interactivity among teacher and students, in which reasons are presented in order to sustain the solution of problems raised in the classroom.

## Interpretation and Results

For this paper, the authors chose a single episode of one of the classes observed (Lesson 80). The episode corresponds to the participation of a student (Mar) who stands out, not just because of the mathematical quality of her interventions, but because of the strength of her beliefs and the determination with which she attempts to convince the teacher and her classmates of her ideas. Below readers will find a chart analysis that highlights the most significant steps in Mar's participation and in her interaction with her teacher (T). In italics, readers will find the possible beliefs and the epistemic states (certainty, presumption, conviction) that are eventually associated with those beliefs. The Annex to this paper contains the text of the exercise and several lines (L.) that are representative of the extract transcribed.

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- *T:* "How can we prove who swam the fastest?" *Mar*: "By proportionality" (L.14-20)

The T asks about the strategy and the student (Mar) answers.

Mar's belief that proportionality is the method that must be used to solve the problem.

- Preparation of Table 2 (L. 23)

Mar constructs a proportionality table. She uses the distances covered by the competitors as her point of departure and calculates the times that Be would have registered 'had he swum the same as' –or at the same speed as- when he swam the 50 meters.

Table 2 provides Mar with the elements needed for her to know that she stands on more solid ground.

The logic behind Mar's reasoning is unknown, but several feasible options exist:
i. Solution and confirmation argument. In this case Mar has a clear strategy, but initially she does not know what the solution will be. To find the solution, she goes to the blackboard and begins to analyze the simplest case –that of Be, because the distance he swam is divisible by the rest of the distances, so constructing the proportionality table is easy. This is a feasible option given that in the video she does not initially propose a result. Also the teacher has asked for a strategy, which is what Mar provided.
With her participation, she seeks to convince the T and the group of the method used.
ii. Presumption and confirmation argument. Presupposes a conclusion based on a few quick calculations that Mar carried out prior to her presentation at the blackboard, where she proves that her presumption is correct.
She has confidence in the method and is to a degree certain of the answer.
iii. Argument dealing with a hunch or intuition. This appears to be the least feasible option because in the video Mar is doing operations while the T is posing the initial questions.
In this possible case, Mar is persuaded of the solution and has confidence in the method.

- *Mar*: "… in order to find … the amount of time it took them". (L. 29-33).

Mar compares pairs of reasons possessed by a common term (distance), which reduces the problem to a linear comparison of two amounts (times). Her idea of speed consists of 'the one who swam the same distance in the least amount of time'. There does not seem to be any relative thought, rather it is absolute based on additive procedures.
Mar appears convinced of her procedure and of her interpretation of speed in the register. It is likely that her conviction and decision to externalize it increased after having drawn the table.
Mar conveys her beliefs about her role as a student and about self-confidence.

- *Teacher*: "I think the amount is wrong …" (L. 34-52).

Some of the possible explanations for the T's behavior include:
i. The T had another type of strategy in mind. This option is fairly likely given that the T showed signs throughout the entire course of her preference for general, parsimonious and symbolic strategies. The manner in which she closes the solution (L 65) and the way in which she later solves the exercise (in Solution 2) are also evidence in favor of the assumption. (L. 64-68).
ii. The T did not understand Mar's strategy. Possibly because she was distracted thinking about another way of approaching the problem (in the video, she is seen to do calculations in her book while Mar is at the blackboard) or because Mar did not explicitly state her conclusion.
Nonetheless one must admit that the T truly masters this type of strategy, in addition to the fact that it is precisely the type of strategy recommended by the textbook for the exercise. Hence her distraction was probably the result of her having other plans and expecting another type of intervention from her students.
The T makes it possible for observers to perceive some of her meta-mathematical beliefs ('the more general it is, the more mathematical it is' (Hersh, 1993)) and some of her ideas about the teaching of mathematics. Her intervention also reveals the techniques of the maieutics art that she resorts to and the underlying credences.

- *Mar*: "It's just that … you didn't understand me" (L. 58).

Perhaps the T's attitude in L (34-56) served to spur Mar on to reinforce her arguments, explain her conclusion (L 62) and increase the certainty of her beliefs.
Mar gives indications—by way of persuasive rhetorical resources—of how sure she is of her

belief in the result found and the method used. Her ideas about her role as a student and the role of her T can be perceived. She proceeds in response to her own interpretation of the T's behavior: if the strategy is not understood, then it has to be clarified.

- *Mar*: "I think Be was faster, so if we have Be taking 50 seconds for 50 meters, then he would have swum differently …" (L. 58-62).

Mar allows observers to more clearly see the logical structure of her procedure: She makes an assumption ('Be is the fastest, based on certain mathematical elements) and verifies that it is correct on the blackboard. This is a presumption and confirmation argument (possibility ii, previously alluded to). Mar's argument is to a certain extent logically complex, given that she applies –implicitly- an exhaustive reasoning of the type:

(dC) If B(dC) then B>C,

which can be translated as

For all distances (d) covered by Competitors (C), if B had swum them all at the same speed as he did for the 50 meters, he would have swum faster than anyone else.

In solving school-type problems, Mar is both intuitive and talented which is why she knows—perhaps implicitly—that the simplest assumptions generally work, as is the case in this exercise. Her reasoning appears to enhance her certainty that her conclusion is indeed correct, that the method used is valid and pertinent, and she attempts to share her conviction with the T and her classmates.

- *T*: "¿… how many seconds did it take A…?" (L. 64).

The T does not institutionalize Mar's solution and her question suggests a change of strategy (her question cannot be answered using the strategy proposed by Mar). She thus prompts for a new manner of solving the problem (based on quotient d/t) (L. 64-66).

The T calls on her authority and the obligations she feels she must honor as a teacher: to lead her students to 'discover' methods and processes they would be unable to arrive at alone. Once again she demonstrates her disposition for the pedagogical resources offered in maieutics.

*Chart 1*. Comments on student's and teacher's interactions.

Chart 1 identifies several of the student's beliefs, as follows: inter alia, her confidence in the conclusion of the exercise and in the validity of the solution (mathematical beliefs), and her firm conviction that the 'table' strategy is the most suitable for solving the problem (meta-mathematical belief). Observers can also distinguish her ideas regarding her role as a student, the role of the teacher and self-referred beliefs.

As regards the teacher, one can perceive her certainty about the teaching of mathematics based on usage of general formulae, a certainty which she underscored throughout the course and which appears to orient her standardized didactic practices. Also of note to observers was her confidence in the maieutics of Socrates, as revealed in her carrying out of daily routine pedagogical duties.

The beliefs highlighted in the extract chosen can be explained from the standpoint of different levels. One takes into account the possible reasons that led the subject to believe. Another considers a person's own motives, interests and preferences. While yet another looks to the socio-cultural context within which such credences were generated, in other words the background and circumstances that led a person to believe (cf. Villoro, 2002). The foregoing concepts are used below in analyzing the beliefs identified.

*Mar's Beliefs*

Toward the end of her intervention, Mar gave indications that she had confidence in the conclusion and in the value of her procedure. It is plausible to think that her belief underwent a change as the solution process evolved. She may have had a hunch at the beginning that then became a presumption as a result of some of the calculations she did in her head. The student ended up providing a high degree of probable truth to the result and her certainty was substantiated by firm and conclusive—for her—reasons, although her reasons did not receive the inter-subjective backing of her Teacher. The fact that she was so very sure of the veracity of her conclusion could also have been derived from the treatment and objectification that she undertook in one of the registers (tabular) that she is so certain of (personal motives) and from the fact that the procedures she used are usually recommended in the Textbook (context).

Mar showed that she was sure of the truth of the conclusion. However, it is very much possible that that was not the vector of her argument, rather her certainty was about the pertinence of the strategy she chose ('table'). What convinces her of the strategy and why does it convince her?

Several reasons can be cited, to wit: that she has proof of reliability or that she has evidence that the exercises in the textbook can usually be correctly solved using the same type of strategy; or those relative to the characteristics of the method per se in which, as in informal procedures, it is possible to signify each step and operation involved, which makes it possible to objectify the mathematical notions and obtain treatments imbued with meaning and sense.

Surely Mar bases her certainties on personal motives and intentions as well. Throughout the course her special disposition for and liking of tabular registers was apparent, and this denotes the importance and appreciation she attaches to that method.

Mar's belief in the strategy could also have been induced by the classroom context. In that context the textbook affords a prominent role to that particular solution method, as did Mar's own teacher in the previous course (a teacher well respected by Mar, her classmates and the entire school). Moreover tables and tabular registers used as a means of problem solving are well known by all her classmates, thus Mar can share the method with them.

*The Teacher's Beliefs*

The teacher's scarce acceptance of Mar's proposal and the emphatic support she attached to the introduction of quotient (s/t) as the means to solve the problem analyzed (L.64-66) does not seem to be either circumstantial or casual. As previously stated the Teacher showed (at every opportunity that we observed) a marked interest in use of general rules and mathematics laws, even taking her own initiative to introduce them. Through her attitude during Mar's participation—where Mar resorts to what is in the teacher's opinion a particular, ad hoc, hypothetical and inconclusive argument—the Teacher deliberately or involuntarily conveys to her students that she is firmly convinced of her own didactics and of her conception of mathematics focused on use and mastery of general mathematics rules. Surely, the teacher has her own reasons that support this didactic position (symbolic formulae are efficient and universal; they work in practice and are easy to apply and learn). It is also very likely that her reasons are driven by her motives, values and intentions (that her students go beyond that called for in the official curriculum, that they get better grades in official evaluations or that algebra pose less of a problem for them), and by aspects having to do with the context (the school's plans may place a high priority on this type of strategy, or that may be the way she learned mathematics).

It is quite clear from that previously stated that the disagreements between the student and her teacher have to do with the strategy chosen and the beliefs underlying that choice. Teacher and student appear to have their own reasons and motives for supporting the ideas of which they are convinced, and what was witnessed in class was the asymmetrical struggle between them, each aiming to implement her own ideas.

During the confrontation, meta-mathematical meanings and beliefs were constructed concerning how to proceed in mathematics, how to go about expressing a justification and what type of register should be used. Another belief that was built has to do with who makes the decisions in class. There is little doubt that the students took note of all of this and constructed their personal interpretations based on their own belief nodes and networks. As part of an accommodation process, the reference frameworks will be changed by the experiences filtered through the frameworks themselves.

## Final Remarks

The interpretative belief model introduced in this paper is supported by the epistemological position that considers beliefs to be a part of a subject's stock of truths. That is to say, that knowledge is a proper sub-set of a subject's beliefs. The model is furthermore supported by a didactic position according to which in a successful learning model as students progress they associate meaningful mathematical deeds with levels of confidence and certainty that increase in line with the generality and conclusiveness of the reasons to which they have access (epistemically).

The interpretative framework used makes it possible to identify different types of beliefs that arise in the classroom setting, focusing on mathematical and meta-mathematical beliefs. The framework also made it possible to analyze changes of epistemic states and values as they are conveyed by the actors with regard to those beliefs. In the class being studied, in particular terms, the model enabled interpreting the evolution of a single mathematical belief of one student, and her confrontation with the teacher's position.

Of interest in the study are questions such as precisely what are students convinced of in mathematics class? How are they convinced of it? And what roles are played by conviction and certainty? In the case analyzed, one student seeks with great determination to persuade her teacher of the pertinence of a particular strategy for solving an exercise, showing that not all students can be dissuaded easily, contrary to that stated by Hersh (1993). Her case also provides evidence in favor of another point. Although arguments must be constructed in the mathematics classroom to convince people of the truth of conclusions and to explain them (consistent with that reported by De Villiers (1991) and as occurs with the experts proof (Hanna, 1996)), there is also a desire to convince people of other issues, such as the suitability of a particular type of register or of a method to solve problems (as also occurs with expert rigor criteria (Tymoczko, 1986).

## Annex. Lesson 80, Episode 2, Solution 1

1. *T:* Who swam the fastest? (repeats several times)
2. *A:* Beto … Catalina … Beto …
(*The Ss, and Mar* in particular, do operations in their notebooks)

| | Distancia | Tiempo | | |
|---|---|---|---|---|
| | | Horas | min | seg |
| Am | 100 m. | 0 | 2 | 0 |
| Be | 50 m. | 0 | 0 | 50 |
| Ca | 150 m. | 0 | 2 | 51 |
| Da | 1500 m. | 0 | 40 | 0 |

(Table 1. Textbook)

10. *T:* Can we know just like that?
14. How can we prove who swam the fastest?
17. *Mar*: By proportionality …
19. *Mar*: To see if, for example, 50 m. were covered in 50 sec., for 100 they would have had to take (pause) one minute and forty seconds.
20. *T*: You're relating distances to times, right?

| m. | min | seg |
|---|---|---|
| 50 metros | | 50 seg |
| 100 | 1 min | 40 seg |
| 150 metros | 2 min. | 30 seg. |
| 1500 metros | 25 min. | |

23. (Table 2. Drawn by *Mar* at the blackboard; does the operations in the book that she has in her hand, erases them and writes the results in the table on the blackboard).
29. *T:* How can we compare those amounts?
31. *Mar:* (asks to participate) seeing this (points to table 2, that she drew on the blackboard) with what they did, to find what's missing … and their times.

33. *Mar*: "Amalia swam 100 m in 2 min., but here she's
got 1 min. 40 sec. (points to the amount in table 2 on
the blackboard)
34. *T:* One hundred meters in how long? I think the
amount is wrong. Check it ….
36. *T:* Can someone tell her what's going on with
42. Amalia?
48. *T:* There's a mistake there sweetheart, erase it!
*Ale*: The thing is that where it says 150 meters it's 2
52. minutes 51 seconds…
(Upon the teacher's instructions, Mar erases the
entire table she had drawn on the blackboard. The
teacher then dictates, with the help of the Ss, the
56. table in the Textbook).
*T:* How can we compare this time and these
58. distances? … Who swam the fastest?
*Mar:* It's just that, I had already done the table but
59. you didn't understand me …
60. *T:* Ah ha! Now I understand you! (astounded)
61. *Mar:* I based it on what Beto did …
62. *T.* In other words … the table was just for Beto!
*Mar:* It was to see, … because I think that Beto was
faster, so if Beto took 50 seconds for 50 meters, then
63. he would have swum differently …"
64. *T:* Very good.
*T:* What can we do to find out how many seconds it
65. took Amalia, Catalina and Darío?
66. *Di*: Divide the distance by the time.
*T*: Divide the distance by the time! Of course!
(emphatically).

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# U.S. AND CHINESE TEACHERS' PRACTICES AND THINKING IN CONSTRUCTING CURRICULUM FOR TEACHING

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*By focusing on teachers' preparation for teaching fraction division, this study examined eight US and Chinese mathematics teachers' practices and thinking in lesson plan development. Both lesson plans and interview data were collected and analyzed. The results presented a contrast picture between US and Chinese teachers' practices and thinking in constructing curriculum for teaching. While Chinese teachers' lesson plans presented a well-structured and detailed picture, US teachers' lesson plans were brief and more like a reminder for what to teach and relevant teaching procedure. Although both Chinese and US teachers thought that lesson planning is needed, the nature of their thinking differed and helped explain cross-national differences in their lesson plans.*

## Background

Results from several cross-national comparative studies suggest that students from East Asia outperform U.S. students in school mathematics. High quality classroom instruction has been taken as one important factor contributing to Asian students' achievement, including China and Japan (e.g., Stigler & Hiebert, 1999; Watkins & Biggs, 2001). Researchers tend to seek factors that may contribute to the quality of teaching in these countries, including curriculum materials, teachers' knowledge, and teachers' work outside of their classrooms (e.g., Li, Chen, & An, 2009; Ma, 1999). One key factor might be teachers' lesson planning and interactions that happen before and after their classroom instruction (e.g., McCutcheon, 1980; Stigler & Hiebert, 1999). If teachers can design well-thought-out and high quality lesson plans, as a process of curriculum planning at the micro level, they build a solid base for classroom implementation. Quality instruction is, therefore, more likely to occur. However, not every teacher tends to agree on this idea, especially in the United States (see O'Donnell & Taylor, 2006). The value of lesson planning and its role in developing high-quality lessons are still contested in teachers' views and practices. Further studies are needed to explore teachers' practices in lesson planning and their thinking behind their practices.

As part of a larger research project, this study was designed to investigate Chinese and US teachers' practices and thinking in lesson preparation. In particular, this study focused on teachers' curriculum construction for teaching the content topic of "division of fractions". As addressed by Li, Chen and An (2009), division of fractions (DoF) is a complex content topic that is included in elementary school mathematics for sixth grade in China, sixth and/or seventh graders in the U.S. Although fraction division is procedurally straight forward, it is a conceptually rich and difficult topic. In textbooks from many education systems including China and the United States, fraction division is a content topic that has been given various conceptual treatments (e.g., Li, 2008; Li, Chen, & An, 2009). The topic presents a rich context to explore possible variations in teachers' thinking and lesson planning.

By focusing on this content topic of DoF, a case study approach was used to focus on eight mathematics teachers' lesson planning from different schools in China and the U.S. Through collecting rich data around these eight teachers' lesson planning on the topic of fraction division, this study was designed to address the following two questions:

1. What are the characteristics of Chinese and US teachers' daily lesson plans on the topic of fraction division?
2. What may Chinese and US teachers normally do and think when developing lesson plans?

## Theoretical Perspectives

Studying teachers' lesson planning is not a new endeavor. In the United States, the importance of investigating teachers' lesson planning and their thinking was recognized more than two decades ago (e.g., McCutcheon, 1980; Peterson, Marx, & Clark, 1978). The results from studies on U.S. expert and novice teachers' cognition indicated that variations in teacher's planning relate to their classroom teaching behavior (e.g., Hogan, Rabinowitz, & Craven, 2003). Cross-nationally, it is reported that Chinese teachers' lesson plans differed from their counterparts in the United States (e.g., Cai & Wang, 2006). Yet, much remains to be understood about teachers' practice and thinking in constructing curriculum for classroom instruction. In particular, this study aimed to examine teachers' planning practices through analyzing teachers' daily lesson plans. Interviews with participating teachers were also used to explore teachers' thinking behind their practices and to triangulate the lesson plan analysis.

In analyzing individual lesson plans, both content and process are important aspects of lesson plans (Cai & Wang, 2006). While the content aspect focuses on what to teach, it translates into a teacher's interpretation and specification of instructional objectives and content treatment for teaching. Likewise, the process aspect focuses on how a teacher plans to teach, and it translates into the teacher's planning of lesson activity and its structure with the use of different strategies and problems. Moreover, students should be an integral part of a teacher's consideration in lesson planning. It is also important to examine how a teacher may anticipate possible students' learning difficulties and progress in the process of lesson planning (e.g., Shimizu, 2008). Thus, a three-dimension framework was developed for analyzing teachers' plans of individual lessons as outlined below:

Content aspect: the content scope to be covered in one lesson, instructional objectives, important content points of teaching, difficult content points of teaching, and materials/tools to be used.

Process aspect: activity segments and structure, use of instructional strategies, use of problems and representations.

Student aspect: the places in lesson plans where the teacher considers students, the nature of teacher's considerations about students in lesson plans.

## Method

*Participants and Context of the Study*

This study focused on four US schools and four (out of seven) Chinese schools participating in a larger research project that aimed to investigate Chinese and US mathematics classroom teaching. With local mathematics education experts' help, the eight schools were selected with comparable variations in school quality and reputations in these two education systems. While the four US schools were located in one state, the four schools in China were in two different provinces. With one teacher from each school, a total of eight teachers participated in this study.

The schools and participating teachers were all informed that the data collection was only for the research purpose. As part of the research project, the teachers' lesson plans on the topic of fraction division were requested and collected. All eight teachers were interviewed about their practices and thinking in constructing curriculum for mathematics classroom instruction.

*Types of Data Collected*

In the larger research project, all participating schools were site-visited and participating teachers were given semi-structured interviews about their lesson preparation and teaching of the content topic of DoF. As the topic of DoF was treated differently in Chinese and US textbooks (Li, Chen, & An, 2009), the complete instruction of the topic presented different arrangements with different amount of time required in these two education systems. Thus, all the data collection was focused on the first several lessons that participating teachers prepared and taught the content topic of DoF. However, video-taped teachers' lesson instruction was not included for analysis in this study. The data used for this study included the following:

1. Eight participating teachers' daily lesson plans on the topic of fraction division.
2. Semi-structured interviews with the eight teachers about their practices and thinking in constructing curriculum for mathematics classroom instruction. In particular, teachers were asked such questions as instructional goals for planning the lessons on DoF, difficult content points for students in teaching this topic, their perceptions of lesson planning, and normal procedure in lesson planning.
3. Field observations of these eight elementary schools and these teachers' daily working environment.

*Data Analysis*

All the data for this study were analyzed in the original Chinese or English languages, and then translated the Chinese into English if needed. To address our first research question directly, we analyzed these eight teachers' lesson plans. Because there were some variations among these teachers in terms of their teaching pace and what was planned after the first lesson, the variations made teachers' plans for the first lesson on the topic a default choice for comparison.

Our analysis of teachers' lesson plans then followed the framework outlined above. The data analysis was a process that integrates iterative lesson plan examination and code development, together with extensive discussions between two coders. A consensus in data coding was reached after discussions. By referring to the codes used in Cai and Wang's study (2006), teachers' first lesson plans were categorized in terms of their content and process features. Teachers' considerations of students' possible responses and difficulties were also examined.

To support the above data analysis and address the second research question, we analyzed the interviews with all these eight teachers to reveal their lesson planning routines, and to capture the ways in which these teachers came up their ideas in constructing curriculum for lesson instruction on this content topic as well as their thinking about lesson planning. In particular, we focused on teachers' perceptions of the process of lesson plan development, their perceptions of important factors influencing their development of lesson plans, their instructional objectives, and the difficult and important points of teaching this content topic. The interview data can allow us not only to triangulate our analysis of teachers' lesson plans on the content topic, but also to examine teachers' perceptions and beliefs of the role of lesson planning in developing effective classroom instruction. The field observation was also incorporated to support our analysis of the interview data in terms of teachers' lesson planning practices in the school.

## Results

Information about these eight teachers and their schools were obtained through interviews and school site visits. Table 1 shows that participating teachers were all experienced teachers, with at least eight years' teaching experiences. However, all participating US schools let teachers use their own classrooms for lesson preparation and teaching. In contrast, all schools in China provided teachers with office space for lesson preparation that is separate from classrooms.

*Table 1*
Information about Eight Participating Teachers and Their Working Organization

| Teacher | Years of teaching | Teachers' office arrangement | Normal practices in lesson planning |
|---|---|---|---|
| CH-T1 | 10 | Subject-based teacher office | Mainly individual preparation, periodically group lesson preparation |
| CH-T2 | 10 | Grade level-based teacher office | Use common lesson plan prototype, then modify by individual teachers |
| CH-T3 | 19 | Grade level-based teacher office | Used to be group preparation, now mainly individual preparation |
| CH-T4 | 11 | Grade level-based teacher office | Use common lesson plan prototype, then modify by individual teachers |
| US-Ta | 9 | The teacher's Classroom | Individual preparation |
| US-Tb | 12 | The teacher's classroom | Individual preparation |
| US-Tc | 8 | The teacher's classroom | Individual preparation |
| US-Td | 8 | The teacher's classroom | Individual preparation |

Related to the teachers' working office arrangement, all four US teachers had their lesson planning on their own, whereas all Chinese teachers reported the combination of individual and group works for lesson planning activities. Although variations existed across these Chinese schools in terms of how group efforts were formally utilized, individual work did not prevent teachers from having informal discussions about teaching and planning from time to time.

In general, although the four Chinese teachers were from four different schools in two provinces, their lesson plans shared many similarities. Based on the unified curriculum standards in China, all their lesson plans contained clear and same instructional objectives taken from teachers' instruction reference book. The complete instruction of DoF was arranged in the Chinese curriculum for the sixth grade and would take at least two weeks (Li et al., 2009). All four teachers' first lesson plans focused on the conceptual understanding of why the computation of fraction division works. Although the format of lesson plans varied across these two provinces, all lesson plans had a clear teaching process that includes reviewing the previous knowledge, introducing new content, and summarizing and practicing.

In contrast, the US teachers' lesson plans presented a diverse picture. Although these teachers were from four different schools, their lesson plans were surprisingly brief and varied in content and instruction arrangements. In fact, only three teachers had some kinds of lesson plans available to share. And one teacher (Tc) actually made her lesson plan as filling in a small box for each day on a monthly planner. All these teachers were teaching DoF to seventh graders and planned to use one lesson period (about 40 to 50 minutes per lesson period) to cover the topic.

In analyzing features of these teachers' lesson plans, we thus focused on their first lesson plans on this topic. Similarities and differences among these participating teachers' lesson plans were evidenced along the three dimensions specified in the framework, i.e., content features, process features, and knowing about students. The following three sub-sections are structured to present findings in detail.

*Content Features of Teachers' Plans for the First Lesson*

In analyzing seven (there was no lesson plan provided by one US teacher - Td) teachers' first lesson plans, five content features were identified in their lesson plans: content topic specification for the lesson, instructional objectives, teaching emphasis, difficult points of teaching, and instructional materials or tools. Table 2 summarizes content features included in these seven teachers' first lesson plans.

*Table 2*

Content Features of Seven Teachers' First Lesson Plans

| | Content specification | Instructional objectives | Teaching emphases | Difficult points | Materials |
|---|---|---|---|---|---|
| CH-T1 | + | + | | | |
| CH-T2 | + | + | + | | + |
| CH-T3 | + | + | + | + | + |
| CH-T4 | + | + | + | | |
| US-Ta | + | + | | | |
| US-Tb | + | + | | | |
| US-Tc | + | + | | | + |

*Note*. "+" means that the item was presented clearly in that teacher's lesson plan.

Table 2 shows that all seven teachers' first lesson plans included both content topic specification for the lesson and its instructional objectives. However, while Chinese teachers tended to include more information such as teaching content emphasis, US teachers' lesson plans basically did not contain further information except one teacher (Tc) specified what materials need to be prepared for instruction. The following sub-sections will provide further detailed information.

*Content specification.* In specifying the content topic, three Chinese teachers (T1, T2, and T3) provided an item (e.g., a fraction divided by a whole number) at the beginning of their lesson plans to explicitly state the teaching content, while one (T4) placed the content topic directly as the lesson plan's title. In contrast, all three US teachers used the content topic simply as the title of their lesson plans.

*Instructional objectives.* Next, all four Chinese teachers provided instructional objectives in their lesson plans. They all thought that understanding the meaning of DoF and correctly doing the algorithm of a fraction divided by a whole number are major objectives in the first lesson. Moreover, they included both understanding (why it works) and mastering the computation of a fraction divided by a whole number (how to do the computation) as for correctly doing the algorithm. The four teachers included three same instructional objectives, but expressed varied ideas in their specifications of other instructional objectives. For example, T1 specified the objective of understanding the relationship between fraction multiplication and fraction division based on the meaning of division. T4 indicated the need to develop students' inquiry and enhance their confidence. Two teachers (T1 and T3) included as an instructional objective to develop students' ability to compare, analyze, and generalize results.

In contrast, the three US teachers included simple objective statement. Two of them simply used the title as the lesson's objective, while the remaining one used "dividing fractions" as content topic and "to divide fractions and mixed numbers" as objective.

*Teaching emphases and difficulties.* Three Chinese teachers (T2, T3, and T4) from the same province further specified teaching emphases and difficulties in their lesson plans. They used the same form of lesson plan, which was designed to place teaching emphases and difficulties together in the same item. While T2 and T4 addressed teaching emphasis explicitly, T3 did not separate teaching emphasis from difficulties. For those plans including teaching emphases and difficulties, they were similar in their specifications of teaching emphases and difficulties. For example, T2 indicated "teaching emphases" as the computation of a fraction divided by a whole number, T4 indicated as the meaning of fraction division and the computation of a fraction divided by a whole numb, while T3 mentioned the meaning of fraction division and the computation of a fraction divided by a whole number as both "teaching emphases" and "teaching difficulties". However, such content specifications were absent in the US teachers' lesson plans.

*Teaching materials.* Table 1 shows that two Chinese teachers (T2 and T3) and one US teacher (Tc) mentioned "teaching materials" besides the textbook. T2 planned to use concrete materials (e.g., rope), and T3 planned others (e.g., small blackboard). The US teacher (Tc) listed several materials as supplies, including ruler, scissors, tape, and colored pencils.

*Process Features of Teachers' Plans for the First Lesson*

In general, lesson plans were considered as scripts for what teachers will do in classrooms (e.g., Cai & Wang, 2006). Although these teachers' lesson plans varied in many ways, the four Chinese teachers' lesson plans shared a similar structure of instructional process. In particular, all lesson plans except one (T3's) were outlined as containing four steps: (1) reviewing previous knowledge, (2) introducing new knowledge, (3) exercises and practicing, and (4) summary and assigning homework. In contrast, the three US teachers' lesson plans were brief and varied dramatically. In fact, Tc' lesson plans contained no process information at all, while Ta and Tb filled in blanks on a pre-set one-page lesson plan. However, Ta wrote one sentence for each of four activities in a list, like "Recall any previous knowledge of dividing fractions" as activity 1. The lesson plan looked more like a reminder for a sequence of lesson activities. Tb listed six fraction division computations and corresponding word problems, together with two exercise sheets for students. Given such limited information available on US teachers' lesson plans, the following sub-sections will focus on process features presented in Chinese teachers' lesson plans.

*Reviewing previous knowledge.* As explained in the last section, all four Chinese teachers' lesson plans contained two instructional objectives: to understand the meaning of fraction division and to master the computation of a fraction divided by a whole number. Subsequently, these teachers, except T3 whose lesson plan did not have a reviewing step, planned relevant knowledge review. In particular, two teachers (T2 and T4) planned the review of the meaning of whole number division, while T1 planned to review multiplication of fractions.

*Introducing the new knowledge.* All four teachers planned the part of "introducing new knowledge" in detail. A difference was observed as T2 and T4 placed the meaning of fraction division as part of reviewing and then introduced the computational rule as "new knowledge". Others (T1 and T3) included both the meaning and the computational rule of DoF in the step of "introducing new knowledge". For introducing the meaning of DoF, T1 used the word problems of whole number division provided in the textbook, whereas T3 provided a fraction division expression and used the equation of multiplication to introduce the meaning of fraction division.

Pedagogically, all four teachers were consistent in planning to ask students to discuss the computational rule in groups and to report their findings. In order to help students' learning, three teachers (T1, T2, and T4) planned to provide students some hints first by showing concrete manipulative or pictorial representation. With the use of concrete or pictorial representation, each teacher expected students to predict the answer for the problem. Although these four teachers' lesson plans varied in terms of the degree of their details, they all showed the use of pictorial representation in proving the computational rule and the use of multiple ways for coming up the algorithm.

*Exercises and practicing.* After introducing the new knowledge, all four Chinese teachers provided exercises for practicing. T1 included exercises for understanding the meaning and mastering the algorithm, others mainly focused on the computation. All four teachers provided exercises for practicing the algorithm. With the exception of T2 with one type of exercise provided for practicing the algorithm, others planned to provide multiple types of exercises for doing the algorithm.

*Summary and assigning homework.* All four teachers ended their lesson plans with a summary and assigning the homework. In particular, these teachers designed specific questions in their lesson plans to ask students in order to check whether or not students' learning achieved the lesson's instructional objectives.

*Knowing and Predicting Students' Responses and Difficulties*

Three teachers (except T3) planned to organize and use students' group discussion about the fraction division algorithm. Moreover, T4 explicitly predicted possible solutions that students would generate through group discussion. Based on the solutions that students would make, the teachers planned to generalize for developing characteristics of each algorithm.

*Teachers' Thinking about Lesson Planning*

To my surprise, all eight teachers said that they always had lesson planning and considered that lesson planning is important and useful to them. These teachers' responses presented a picture different from cross-national differences in teachers' lesson plans reported above. Further analyses of teacher interviews were carried out to reveal details.

For US teachers, lesson planning mainly referred to "getting ready for a lesson". It involved knowing what to teach and how to teach a lesson. Although these teachers would use textbooks, they would use textbooks mainly as resources and for getting exercise problems. It was important for them to know students' learning style, what may make their learning difficult, and to think about how to connect mathematics with real world. While all four US teachers thought that they had enough time to plan for their lessons, they were not sure whether they were satisfied with their lesson preparation until they taught students.

For Chinese teachers, lesson planning was a process necessary for students and teachers. Lesson planning for students meant to plan lessons from the student's perspective. Teachers should think what students have already learned and how the teachers provide students a good learning approach. Lesson planning for teachers meant that the teachers can deepen their own understanding of content through the lesson planning process and develop their teaching. In this regard, studying the textbook is very important to develop a deep understanding of the content topic in terms of instructional objectives, emphases, and difficulties. Studying textbooks can also help teachers make knowledge connections between the current content topic and others. Moreover, lesson planning is for teachers to think about how to teach cleverly and use teaching strategies. Teachers can thus develop teaching from the process of lesson planning.

## Discussion

Overall, the study presented contrast results between US and Chinese participating teachers in their practices and thinking in constructing curriculum for teaching. Chinese teachers' lesson plans presented a coherent and detailed picture, not only in terms of what these teachers considered as important for their students to learn among these teachers, but also in structuring these contents together for classroom instruction in individual lessons. Their lesson plans resulted from these teachers' intensive study of textbook content in relation to their students' situation. In contrast, US participating teachers' lesson plans were very brief, if they made one. Lesson plans were developed more like a reminder for what to teach and its teaching procedure. Because these US teachers tended to pay more attention to students and their learning but not a lesson's content treatment, their lesson plans presented a picture that is consistent with these US teachers' thinking about lesson planning. Although this study is limited with a few participating teachers, the findings provided a valuable glimpse of what may help contribute to the quality of classroom instruction in China. To Chinese teachers, the value of lesson planning is beyond what it is for classroom instruction. It is also a valuable professional activity for teachers themselves.

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# BECOMING A TRIPLE A GEOMETRY TEACHER: THE CASE OF ROSE

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*Factor analysis was used to analyze data that was collected from 520 high school mathematics teachers' questionnaire responses to 48 Likert type statements. The analysis revealed a three factor solution: disposition towards activities, a disposition towards appreciation of geometry and its applications, and a disposition towards abstraction. These results allowed for the classification of teachers into one of eight groups depending on whether their score was negative or positive on the three factors. Knowing which group a teacher belongs to would allow for appropriate professional development activities to be undertaken as was done in the following case study where techniques for scaffolding proofs were used as an intervention for a teacher who was in Group 2 with a positive disposition towards activities and appreciation of geometry and its applications but with a negative disposition towards abstraction.*

## Theoretical Framework

Beliefs play a central role in shaping the practice of teaching (Ernest, 1989, 1991; Raymond, 1997, Thompson, 1984, 1992). Questionnaires that included both statements that required responses on a Likert scale and open-ended questions are used to measure beliefs (Leder and Forgasz, 2002). Thompson (1984) used the method of case studies to report on teachers' beliefs about mathematics, mathematics teaching, and their criteria for judging effectiveness of instruction. Case studies give deeper insights into beliefs.

Students take their "cues" from their teachers. Classroom experience affects students' beliefs about mathematics. Teachers need to examine their own beliefs about proofs, in particular in order to understand how they may influence their students (McCrone, Martin, Dindyal, & Wallace 2002; Schoenfeld, 1988; Senk, 1985). If teachers strongly convey the idea that proofs are necessary to fully understand and appreciate the fundamental geometrical principles being taught, students may become more interested and involved in learning about and doing proofs. Otherwise, doing proofs becomes a dry, rote classroom drill. As earlier researchers have reported, doing formal proofs should come after students have made some sense of the underlying geometrical and mathematical ideas through hands-on explorations (Battista & Clements 1995; Freudenthal, 1971).

## The Study

One of the questions that I wanted to follow up after the preliminary questionnaire was: What happens in a class where a teacher is required to teach geometric proof but has scored negatively on factor 3: a disposition towards abstraction? Could something be done to help a teacher overcome a negative disposition towards abstraction?

In my position as a mathematics specialist-consultant, I was carrying out professional development in Rose's school and the principal suggested that I observe Rose's class in which she was about to start teaching geometric proof. This provided an opportunity for me to delve further into these questions, using Rose as an 'opportunistic sample'. It was in the position of observer participant that I was present in and observed Rose's class seven times taking extensive field notes.

Rose who has an undergraduate degree in mathematics education had taught ninth and tenth grade mathematics in a small urban high school for two years. She was in her late twenties, when I started to work with her. She is enthusiastic in the classroom and she exhibits good classroom management skills.

I met with Rose each morning to discuss her lesson plan for the day and after each of these classes to conduct a debriefing with her. During these sessions I made several suggestions, such as always listing all six corresponding parts of congruent triangles on the board when referring to them and marking them on the diagrams. She implemented these suggestions and others described below in her class almost immediately. This study was presented to Rose so that she could concur or refute any inferences made.

I observed Rose's class several times. I also had the results of the factor analysis and my intention was to find a way to make her comfortable teaching students about proof. The intervention is described below, along with her responses to a further follow-up questionnaire. When she taught with the movable cards containing statements and reasons which were part of the intervention she felt more at ease.

Rose was at the end of her third year of teaching. During the previous year, she had been one of the respondents to the questionnaire. Her scores on the three extracted factors were positive on factor 1: *a disposition towards activities*, positive on factor 2: *a disposition towards appreciation of geometry and its applications*, and negative on factor 3: *a disposition towards abstraction*.

## Rose in Her Second Year of Teaching

Based on Rose's factor scores I went back to look at her actual responses to a number of statements on the questionnaire.

She responded *disagree slightly more than agree* to the following statements:

4. Learning to construct proofs is important for high school students.
6. Geometry should be included in the curriculum for **all** students.
13. High school students should be able to write rigorous proofs in geometry.

This indicated to me that Rose was concerned about teaching average or below average students how to do proofs.

Rose responded *agree slightly more than disagree* to these statements:

1. I enjoy teaching geometry.
2. Learning geometry is valuable for high school students.
9. Geometry should occupy a significant place in the curriculum.
10. High school geometry should not contain proofs.
21. I enjoy doing geometric proofs.

These responses appear to show that Rose believed that geometry is worth learning and that she did enjoy teaching geometry as long as she did not have to teach students how to do proof. She herself likes doing proofs.

Rose responded *strongly disagree* to the statements:

16. My students enjoy doing geometric proofs.
44. I enjoy teaching my students how to do geometric proofs.

Rose responded *moderately agree* to the statement:

48. I enjoy proving theorems for my students.

These responses and the conversations that I had with her led me to conclude that she was uncomfortable about teaching students how to do proofs. The fact that she enjoyed proving theorems for students and doing proofs gave me a glimmer of hope that she might reconsider teaching proofs if she was armed with the appropriate tools and therefore more confident.

**Rose in Her Third Year of Teaching**

By her third year of teaching, Rose had a desire to teach mathematics to upper grade students and so she sought and accepted a position at another small urban high school whose students were supposedly "more academic" than at Rose's first school. I was doing short-term professional development at the school where I worked with three of the four mathematics teachers. The principal asked me to work with both Rose and another teacher who were both starting a unit on proof in geometry. Although I observed both teachers and suggested similar interventions this study focuses on Rose because she was an identified respondent to my questionnaire.

Teaching students how to prove theorems involves all of the problem-solving that teachers do in their 'work of teaching' (Ball, Bass, & Hill, 2004; Kazima & Adler, 2006). These include: Using mathematically appropriate and comprehensible definitions; designing mathematically accurate explanations that are comprehensible and useful for students; working with students' ideas; making judgements about the mathematical quality of instructional materials and modify as necessary and assessing students' mathematics learning and take next steps. Rose exhibited these problem solving skills in her teaching of other aspects of geometry. Could Rose incorporate these skills when teaching proof? I believed I could share a method of teaching students how to do proofs that would be appealing to Rose. The method is described below.

**Congruent Triangles**

The students in the class were learning how to prove geometrical results. They were mostly tenth graders who had already learned definitions and properties of triangles and quadrilaterals in the ninth grade or the beginning of the tenth grade.
Rose used the concept of congruent triangles as a vehicle for introducing students to proving conjectures.

The following is a snapshot of the type of questions Rose asked on the first day of the unit: "What makes triangles congruent?" Students respond that the triangles have to be exactly the same. Rose then drew a picture of two triangles on the board. How can I show that triangle *ABC* is congruent to triangle *DEF* (See Figure 1) based on the information given?

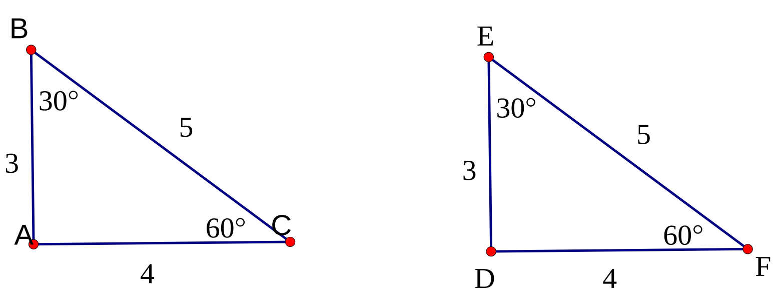



*Figure 1.* Rose's example of congruent triangles.

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

The students recognised that the triangles were congruent from the given information. No student noticed that these triangles couldn't really exist because in a 30-60 triangle the length of the side opposite the 30 degree angle is equal to half the length of the hypotenuse. Bills, Dreyfus, Mason, Tsamir, Watson, and Zaslavsky (2006) asserted that when selecting instructional examples the teacher should take into account 'learners' preconceptions and prior experience'. Zaslavsky and Zodik (in press) studied what considerations went into teachers' choices of examples. They found there was a tension between the desire to construct real-life examples and mathematical accuracy. A random choice of example could lead to an impossibility. When Rose and I discussed her example she was surprised at what she had done. She expressed a desire to be more careful about her choice of examples in the future.

Another question that Rose posed was whether the information given was sufficient to prove triangles congruent: She drew the diagram shown in Figure 2 and asked students, "In the square ABCD, is ΔABC congruent to ΔADC?"

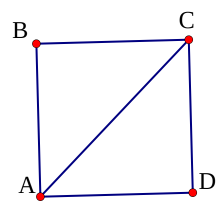
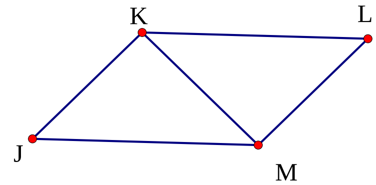



*Figure 2.* Rose's second example.

Her students had to remember the properties of a square in order to answer this question. They knew the sides were all congruent. Rose wanted the students to focus on SSS congruent to SSS. Rose had the students rely heavily on the visual aspects of the problem. I suggested that she have the students investigate the other congruence relationships. I loaned her Michael Serra's book *Discovering Geometry: an Investigative Approach* (2003). She prepared a hands-on lesson for the investigation: Is ASA a Congruence Shortcut? Rose gave each group of students a work sheet with a line segment and two angles drawn on it and asked them to construct a triangle. The students used scissors and tape to cut out the segment and angles and paste them together to form a triangle. She had the groups compare their results. Rose placed the results up on the bulletin board.

Rose kept telling me that she was anxious about having the students do actual proofs. I gave her three worksheets from a set of worksheets I had received from Sandra Gundlach, a teacher, who had presented them at a conference. The first one had six statements to prove along with a diagram for each (See Figure 3). The next two sheets had mixed up answers to each of the proofs from the first sheet. I brought in envelopes with the given, the "to prove", and the diagram for each of the six proofs taped onto the outside and the cut up statements and reasons inside.

5. Given: $\overline{KL} \cong \overline{JM}$
$\overline{JK} \cong \overline{LM}$
Prove: $\angle J \cong \angle L$




6. Given: $\overline{JK} \parallel \overline{LM}$
$\overline{JK} \cong \overline{LM}$
Prove: $\overline{KL} \cong \overline{JM}$

*Figure 3.* Last 2 examples from sheet 1: Proving triangles congruent.

Rose took proof #1 (See Figure 4) and enlarged the cut up statements and reasons. She taped them to the blackboard, wrote the given and to prove statements, and drew the accompanying diagram. Some of the students had difficulty with how to use the definition of midpoint. Rose used coloured chalk effectively to illustrate. My suggestion was to use Geometer's Sketchpad to demonstrate angle bisectors in proof #2, but the technician was not available to bring a laptop to Rose's classroom. (I mention this here to make the point that even if a teacher wants to use technology it is not always readily available.) Students complained that one angle looked bigger than rather than equal to the other angle. (Sometimes such arguments are productive but in this case time was wasted).

D is the midpoint of $\overline{AB}$ $\quad$ $\overline{AD} \cong \overline{DB}$ $\quad$ $\angle A \cong \angle B$ $\quad$ Given

$\angle ADC \cong \angle BDC$ $\quad$ $\overline{CD} \cong \overline{CD}$ $\quad$ $ACD \cong \Delta BCD$ $\quad$ Given

Def of midpoint $\quad$ SAS

*Figure 4.* Proof #1 mixed up answers.

Rose used a metaphor of identical twins to help the students understand that corresponding parts of congruent triangles are congruent. "If the twins are identical, what can you say about their eye colour, their height etc.?" The students responded, "They are the same." "So if the triangles are congruent by SSS, SAS or ASA, what can you say about the other parts of the triangles?" The students were able to understand this concept. In the United States some teachers abbreviate the statement *corresponding parts of congruent triangles are congruent* – CPCTC. Unfortunately many students use the abbreviation but fail to remember what it represents.

Some students struggled with the logical sequencing of the steps. In proof #4 they placed the statement G is a midpoint of EI in the middle of the proof. One student, Gary said, "You have to look at cause and effect." This was a useful insight.

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

Eventually Rose used the same format for proofs that she found in the text. She assessed how the students were doing by giving them a quiz where all the statements and reasons were written in mixed-up order on the page and the students had to put the proof together correctly. She was pleased with the results.

**Some of Rose's Response to the Follow-Up Questionnaire**

Rose's responses to the follow up questionnaire were:

1. What do you most love about geometry and why?

*I love geometry proofs. I feel they help students think logically. A proof is like a jigsaw puzzle where everything must fit and when it is complete it's a nice accomplishment. Proofs make students realize that nothing in geometry can be taken for granted there always has to be a reason.*

Rose's response indicates a positive experience with proofs, but I knew from conversations with her that she was worried about teaching proofs. Her next response gave me a glimpse into why she was anxious about teaching students how to do proofs.

2. What is your most memorable experience or experiences as a student in a geometry class?

*My teacher explained the topics very thoroughly. However eliminated geometry proofs from the curriculum. I feel this turned me off from proofs for quite some time.*

4. Is there any topic or topics that are in the current geometry curriculum that you believe should be eliminated? Please explain why.

*I believe constructions should be eliminated from the curriculum, time does not allow for it.*

This question was included to try to find out what teachers do not value in geometry. The way the curriculum is arranged in Rose's state, geometry is part of integrated courses. Constructions are taught in the first course and proofs are taught in the second course. There is no context for the unit on construction. It is left to the last lessons of the course. Rose cannot do justice to the topic and therefore wanted to see it eliminated.

5. Do you include real world applications in your geometry course? What are these and why are they included?

*Geometry is a topic in mathematics that lends itself to real world application. I tell my students geometry is something that is used in every field in the working world. Construction works as well as carpentry works need to know geometry. Individuals who work in advertising need to think about space when they make up an advertisement. Police officers need to use geometry when they are on a chase or when a shooting occurs. This year I took my students outside in the courtyard and we went around looking at the building and trying to find quadrilaterals and explain their properties and purpose by looking at them as well as their purpose in the building.*

I was able to understand Rose better from her responses to this short open-ended questionnaire. Her own experiences with proof in high school (Raymond, 1997) influenced her belief that it would not be easy to teach students how to prove. Rose did not understand the relationship between constructions and proof (Schoenfeld, 1988) and felt that teaching constructions should be eliminated from the curriculum. The curriculum emphasises the procedure for constructions. Since Rose's high school teacher did not teach proof to the class she may have had the students working aimlessly at constructions which is what Rose did in her own class and felt it was a waste of time. In Rose's responses, the *formal, intuitive,* and *utilitarian reasons* for studying geometry can be found.

## Case Study Conclusions

Rose's factor scores on the questionnaire placed her in group 2. Rose had a fear of teaching students how to do proofs. From her response to question 2 above we find that because Rose's teacher did not teach her how to do proofs when she herself took a geometry class, she was reluctant to now teach her own students how to do proofs. From another perspective, Rose left her first high school teaching job in order to teach at a school with more academic students. Not all of her students at the second high school were as academic as she expected. She might have believed that many of them were not capable of doing proofs. I created an intervention by showing her an approach to teaching proofs that fitted well to her disposition to work in a hands-on manner and use manipulatives since she had a positive score on factor 1. She used the intervention successfully in her class and has now requested to teach two sections of this course in the coming year. She has also taken an intermediate level training course in Geometer's Sketchpad during the summer in order to become more adept at using it in her class when she is teaching geometry (Cinco & Eyshinskiy, 2006).

In this one case, by looking at the factor scores I was able to find an appropriate intervention for the teacher. Can one look at the factor scores of other respondents and introduce them to interventions that would help them in their teaching of geometry? We can't generalise Rose's success to others since Rose was already implementing most of the aspects of problem solving in her work as a teacher (Ball, Bass, & Hill, 2004; Kazima & Adler, 2006).

Rose believed that she has a professional responsibility to continue learning and perfecting her craft. Beswick (2007) refers to this belief as "*commitment to seeking out 'second voices' and is related to a propensity to reflect on one's practice with a view to continual improvement*" (p. 115). She attributed the notion of "second voices" to Lerman (1997). Rose was willing to incorporate suggestions made to help improve her practice. Teachers who are unwilling to listen to "second voices" may not be able practice their espoused beliefs.

## Follow-Up: Rose in Her Fourth Year of Teaching

During Rose's fourth year of teaching I observed her class at the beginning and towards the end of her unit on proof. She again used investigations to verify conjectures about when triangles are congruent (Serra, 2007). She displayed the results of these investigations on the classroom walls. She also used the cut out statements and reasons that I had shown her the previous year. She increased the number of proofs that her students did using this method. Her questioning had improved. She had the students planning out their proofs. She asked, "Why does this belong here? Why can't it be placed earlier in the proof?"

On examinations she included matching up statements and reasons instead of cutting them out. She then had the students put the matched up pairs into a formal proof. Some of her students were finally able to complete proofs on their own.

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# INVESTIGATING IN-SERVICE TEACHERS' BELIEF MODELS

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Over the last few decades more emphasis has been placed on the role teachers play in the learning process. Teachers organize and shape the learning context and so have enormous influence on what is being taught and learned. With this recognition, the mathematics education community began to invest more time and resources into teacher research. Specifically, mathematics education researchers, educational psychologists, and those involved in teacher education have become increasingly aware of the influence of teachers' beliefs on their pedagogical decisions and classroom practices. This collective case study reports on an investigation into the relationship between mathematics teachers' beliefs and their classroom practice, namely, how they organized their classroom activities, interacted with their students, and assessed their students' learning. It also examined the pervasiveness of their beliefs in the face of efforts to incorporate reform-oriented classroom materials and instructional strategies. The participants were five high school teachers of ninth-grade algebra at different stages in their teaching career. This study contributes to the body of literature by illuminating the clustered organization of these teachers' beliefs into an interdependent belief network. This network is presented as hypothesized models reflecting the derivative nature of the teachers' sets of mathematical beliefs. The researcher sought to understand the teachers' beliefs from their own descriptions and experiences to identify dimensions of the phenomenon not covered by preexisting theory (Ezzy, 2002).

The qualitative analysis of the data revealed the teachers' beliefs about the nature of mathematics served as a primary antecedent for their beliefs about pedagogy and student learning. Findings from the analysis concur with previous studies in this area that reveal a clear relationship between these constructs. In addition, the results provide useful insights for the mathematics education community as it shows the diversity among the in-service teachers' beliefs (presented as hypothesized belief models), the role and influence of beliefs about the nature of mathematics on the belief structure and how the teachers designed their instructional practices to reflect these beliefs. Implications for teacher education will also be presented.

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# CONFRONTING PRACTICE: CRITICAL COLLEAGUESHIP IN A MATHEMATICS TEACHER STUDY GROUP

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*This study investigates the development of "critical colleagueship" (Lord, 1994) by eight middle-grades mathematics teachers participating in a teacher study group as part of a project focused on improving mathematics classroom discourse. Analysis of the action research phase of this project indicated that aspects of critical colleagueship, such as self-reflection, openness to new ideas, the capacity for empathetic understanding, and the ability to reject flimsy reasoning were exhibited by the teachers. These aspects were manifested in three interaction patterns ranging from the common patterns of praise and advice-giving to the uncommon pattern of teachers engaging as challenging colleagues.*

## Background

With the current climate of educational reform in the United States and with teacher quality seen as critical to success (Wilson, Duffy, Fiori, Halladay, & Mapuranga, 2006), understanding how teachers learn through professional development and how contexts promote this learning is crucial. Although professional development has been labeled as "the ticket to reform" (Wilson & Berne, 1999, p. 173), there is little in terms of empirical evidence of the effects of professional development on practice or on student learning (Elmore, 2002).

According to the "consensus on effective professional development" learning is a collaborative activity and "educators learn more powerfully in concert with others who are struggling with the same problems" (Elmore, 2002, p. 8). Therefore, professional development should be designed to include the development of teachers' ability "to work collectively on problems of practice within their own schools and with practitioners in other settings, as much as to support the knowledge and skill development of individual educators" (Elmore, 2002, p. 8). Similarly, Wilson and Berne (1999) determined that a common thread in "highly regarded" projects was the "privileging of teachers' interaction with one another" (p. 195). These projects all had similar conceptions of professional development and were "aiming for the development of something akin to Lord's (1994) '*critical colleagueship*'" (p. 195). All of the professional development projects studied had difficulty building "trust and community while aiming for a professional discourse that includes and does not avoid critique" (p. 195). Unfortunately, this is contrary to the culture of teaching, in which teachers have a great deal of autonomy and are not often asked to explain their actions (Wilson, Miller, & Yerkes, 1993).

## Purpose of Study

Because of the lack of empirical evidence documenting the benefits of collegiality, there is a need to explore the concept of collegiality. There is also need for research that examines groups of mathematics teachers in order to gain insight into collegiality as it applies to mathematics teachers. This study has two purposes, one of which is methodological. It aims to answer the following questions: a) What are some of the ways that critical colleagueship is exhibited by mathematics teachers participating in a teacher study group as part of a project focused on teachers engaging in action research to improve mathematics classroom discourse? and b) Is it possible to identify the aspects of critical colleagueship exhibited by mathematics teachers by observing a group of mathematics teachers?

## Theoretical Framework

For a broader transformation, collegiality will need to support a critical stance toward teaching. This means more than simply sharing ideas or supporting one's colleagues in the change process. It means confronting traditional practice – the teacher's own and that of his or her colleagues – with an eye toward wholesale revision (Lord, 1994, p. 192).

In an effort to explain how teachers learn, Lord (1994) proposed his idea of *critical colleagueship* based on research about teacher collegiality. Critical colleagueship involves not only working together, sharing ideas and supporting each other, but also confronting unproductive practices and pushing one another to confront these practices. According to Lord, the elements of critical colleagueship are:

1. Creating and sustaining productive disequilibrium through self reflection, collegial dialogue, and on-going critique.
2. Embracing fundamental intellectual virtues. Among these are openness to new ideas, willingness to reject weak practices or flimsy reasoning when faced with countervailing evidence and sound arguments, accepting responsibility for acquiring and using relevant information in the construction of technical arguments, willingness to seek out the best ideas or the best knowledge from within the subject-matter communities, greater reliance on organized and deliberate investigations rather than learning by accident, and assuming collective responsibility for creating a professional record of teachers' research and experimentation.
3. Increasing the capacity for empathetic understanding (placing oneself in a colleague's shoes). That is, understanding a colleague's dilemma in the terms he or she understands it.
4. Developing and honing the skills and attributes associated with negotiation, improved communication, and the resolution of competing interests.
5. Increasing teachers' comfort with high levels of ambiguity and uncertainty, which will be regular features of teaching for understanding.
6. Achieving collective generativity – "knowing how to go on" (Wittgenstein, 1958) as a goal of successful inquiry and practice. (p. 192-193)

## Methods

*Participants and Context*

The participants included two university researchers and eight middle-grades (grades 6 – 10) mathematics teacher-researchers (TRs) from seven different schools in the Midwest United States. These middle and high schools were in a variety of communities (rural, suburban, urban) and included students from a range of socioeconomic levels. The teachers were also "purposefully selected to vary gender, context of teaching situation, certification level, years of teaching experience, extent of involvement in professional development, and reasons for entering the teaching profession" (Herbel-Eisenmann, Drake, & Cirillo, 2009). For more about the participants, see Herbel-Eisenmann, Drake, & Cirillo (2009). All of the TRs volunteered to be a part of this project[1]. The participants were involved in regular project meetings ranging from three-hour meetings to full day and overnight retreats. The data for this study is transcripts from 10 of the project meetings, all of which came from the action research phase of the project.

*Data Analysis*

This study draws on discourse analytic methods (Fairclough, 1992; Chouliaraki & Fairclough, 1999). First, the corpus was reviewed and summarized in broad terms (Fairclough, 1992), using codes that reflected the topic of the discourse. These topic codes were then used to choose transcripts that contained discourse in which the teachers shared with one another. Once the transcripts where chosen each transcript was broken up into episodes by topic or theme. These episodes were then broken up into question/advice blocks. Each of which was coded for who was doing the questioning, and who was being questioned, and the type of question(s) asked (clarification, elaboration, probing, challenging). Additionally, notes were taken on how these question/advice blocks related to the components of critical colleagueship (i.e., pushing someone to reject flimsy reasoning, evidence of openness).

## Findings

In the analysis of the 10 transcripts, three patterns of interaction surfaced: praising colleague, advising colleague, and challenging colleague. Due to limited space, I have chosen to describe the praising and advising colleague using a single transcript that exhibits both of these interaction patterns. I will then discuss the challenging colleague by exploring a different transcript. It should be noted that these interaction patterns did not always emerge in isolation; however in most cases one interaction pattern lasted for multiple teacher turns. All teacher names are pseudonyms. Since the purpose of this study was to identify the teachers exhibiting the aspects of critical colleagueship, I have chosen to indicate the researchers by using initials.

*Praising and Advising Colleague*

The praising colleague interactions were those in which teachers' engaged in dialogue that included a multitude of praise, usually directed at the teacher who was presenting their work. This interaction rarely involved questioning and the questions that were posed were mostly clarification questions. Similarly, the advising colleague interaction involved little questioning. However in addition to questioning in this interaction, the teachers offered advice (solicited and unsolicited) to the presenting teacher. This transcript is a representation of the types of praising (bold) and advising (italics) interactions in the data. The underlined text indicates utterances that include aspects of critical colleagueship.

In the following transcript one of the TRs, Robert, shared his action research progress. He showed videos from two lessons (one from the beginning and one from the end of the project).

BHE: So what did other people notice in what you heard happen in that period of time if Robert's goal is to be having kids talk more, right?

Robert: Contribute more of their ideas in classroom discussion.

Mike: I **liked** how you didn't give any cues when you were, when you were writing down answers. I mean there was no evaluation going on.

Cara: I **really liked** the idea that you gave it to them the day before and had them if they didn't understand it or needed something, they had to write down a question. And **I like the fact**, I mean I have so many kids that come up to me and say, you know three of them just put it down in front of you and just stand there. And then it's, I don't get it. And so by your asking them to write down a question ahead of time really made them get into the thinking of it. I thought that was **a really neat strategy** and one I'm gonna take.

Robert: It's stolen from...

Cara: That's what you got to do and **I like that a lot**. And **I like** the problem because, and I guess I relate to Robert, well, because we're both teaching sixth grade and that's the exact concepts that I work with all the time, is that it depends on the size of the whole on what the fraction is.

Stacey: **You're successfully getting them more engaged** in **meaningful ways**.

Robert: I mean honestly things that you've enjoyed in class that, more opportunities to provide kids with activities like this I could always use. Cause I'm willing to try them now where before I wasn't going to try them.

We can see that in lines 04, 06, 14, and 18, different teachers praise Robert for the good things they see in his teaching using words such as “like” and commenting on the “neat” things he is doing. In addition, in line 18 Stacey uses the word “successful” to describe Robert’s progress.

Although this interaction between the teachers did not seem to be what one might define as critical, many of the aspects of critical colleagueship were exhibited. Throughout this excerpt the aspects of empathetic understanding and openness are exhibited. Cara indicated that she had similar situations to Robert and not only is she placing herself in his shoes (lines 08) she explicitly aligns herself with Robert in line 15 where she stated “I relate to Robert.” Openness to new ideas is reflected in both what Robert says and in what other teachers say during this interaction. In line 12 Cara explained that Robert had a “really neat strategy” and she claimed that she was “gonna take” it. An even more explicit instance of openness to new ideas is exhibited by Robert. In line 20, Robert asked for more ideas from the other TRs because “I’m willing to try them now where before I wasn't going to try them.”

We continue the transcript form the same meeting where we left off.

Cara: Do you like what you’re doing better?

Robert: Yeah. Like I said, it was more fun. I don’t know if, I mean I think they get, like I said I'm asking them to do something that's different than the norm. So they ask. Like I said the one girl did ask, why does each problem that we do have to have a discussion or a lot of discussion.

Gwen: I've got students that ask that too. Why do we have to talk so much?

Kate: *It might be interesting to ask them too what their reaction is to talking about them more.*

Cara: *I have done that. It’s very interesting. For example, I've gone to that no hands policy and I just said, why do you think I've gone to this policy? And it was very interesting for them to say different reasons.*

MC: Robert, you said in the past I wasn't willing to try this. And I was wondering is, do you think that's accurate that you weren't willing to or do you think it's more that you didn't know what other possibilities there were?

Robert: I guess I don't know. At that time, even if these were posed to me I'm pretty, you know I might think they're pretty cool or whatever, but if you haven't known I was pretty passive in my behavior. But I was at a transition point where I didn't want to teach anymore. So I was pretty frustrated in my job that I was doing. You know I was seriously looking at other things to do. And then you know I just kind of had a change of heart, changed grade level, and then

42 this kind of came up too.
43 MC: So then it was that you were more willing it sounds like. So what about the
44 participation here made you more willing?
45 Robert: Gosh, I don't know. Maybe. Probably accountability to everyone else. Just
46 change. Seriously. And that we're all going through the same stuff too. We're
47 all in the same, same problems, same issues. Yeah and just having materials
48 to use that were pretty user friendly and I was able to understand you know
49 what the objectives were and stuff.
50 Holly: *I would say Robert, even just since we started the project I think there's more*
51 *resources on the NCTM's illuminations website that are more by topic so*
53 *that's a richer resource even in the last few years. And I find that like two or*
53 *three times a week to see if there's an open ended thing.*

We see in this excerpt that the TRs also provided Robert with advice. In line 28, Kate advised Robert to talk with his students about discussing problems, and in line 30, Cara offered him an example of how he might be able to do this. Later in the discussion in line 50, Holly also gave Robert some advice on where he might be able to find rich resources.

This excerpt also illustrated the aspects of critical colleagueship. Cara (line 30) continued to exhibit empathetic understanding and so did Gwen (line 27) by commenting that she also had students ask similar questions. We also see Robert engaging in self-reflection (lines 23, 36, 45) and in lines 36 and 45, discussing his openness. Robert shared his progress with the other teachers and seems to be engaging with the difficulties he experienced and the ways in which he was able to change is practice. He also talked candidly about how his willingness to try new things changed. Interesting is the fact that when MC asked Robert about his willingness to try new things prior to the project, one can see that the project encouraged his openness. He stated in line 36 that he was the type of person that “was pretty passive in my behavior.” Then, in line 45 Robert talked about the project and how it had encouraged his openness because of “the accountability to everyone else” and the fact that “we’re all going through the same stuff.”

*Challenging Colleague*

The least frequent type of interaction was that of the challenging colleague. In these interactions teachers asked elaboration, probing and challenging questions. This interaction happened only when one particular TR presented to the group. Bold is used to identify utterances that were challenged or challenging and the underlined text indicates utterances that include aspects of critical colleagueship.

In the following excerpt a different TR, Owen, discussed his action research project, which was to identify his students as procedural or conceptual by giving them an assessment.

01 Owen: So to me, using the Pythagorean theorem to find a distance **is more of a**
02 **conceptual response,** because it connects back into something they already
03 know. So whether or not they learn the Pythagorean theorem in a procedural
04 fashion, or a conceptual fashion, right. **Okay, are they relying on their**
05 **procedural understanding of the Pythagorean theorem**?
06 Stacey: **How are you going to determine that?**
07 Owen: Well **I do** the Pythagorean theorem when I’m supposed to be doing
08 distance formula, I do the Pythagorean theorem.
09 Holly: **What do you do?**
10 Gwen: **What do you mean do?**

Owen: We ah, they get a grid paper and I have them draw a right triangle, and they make the squares off to the sides.
Gwen: The question would be, when you are teaching them distance, you just taught as, a triangle Pythagorean theorem. **So, you never actually gave them a problem where they did this?**
Owen: Oh **no, we did**, there are homework problems like this.
Gwen: In class, **did you show them using Pythagorean theorem to solve the problem**?
Owen: **Yes. That's the way we did them**.
Gwen: So **you couldn't say**, that a kid said, oh this is how you did it, so that's how I'm supposed to do it. **So how is that different than, I know the distance formula**, so that's how I'm going to do it?
Owen: Because the distance formula **is** an exterior entity which **they have no actual understanding of**. All they have is their memorization of what the distance formula is as opposed to having them draw a triangle, which connects a problem they are presented with, back to something else they are already familiar with. Well no kid is going to come out, with a one hundred percent we would assume.
Kate: **Well they might, but you might have a lot of people who are**,
BHE: **They have a lot of variation.**
Owen: Right. Yeah.
Stacey: **I don't think that's the issue**, I think **the issue is whether or not you can really make the initial statement, this is procedural, this is**, and I'm one of the, I'm a big, big advocate of conceptual, so you know that about me. You are trying to talk about conceptual and **you're not giving conceptual tasks, it's hard to justify the claim** that this is procedural or this is conceptual.

In this transcript we see that Owen is asked questions that require him to elaborate, justify his reasoning, and reflect on his own thinking about his action research project. Owen is asked fairly quickly in his presentation (line 06) how he plans to determine whether students have conceptual or procedural understanding, and then in lines 09 and 10 both Holly and Gwen asked Owen to explain what he means by doing the Pythagorean theorem. As Owen continued to present his ideas, Gwen asked many questions (lines 13, 17, 20). It is important that these questions be looked at together because as individual questions they may seem to be only asking Owen to clarify how he taught the concept of the distance formula. However, these questions are challenging Owen to consider alternate explanations for the results he may get from his students. Gwen and others are attempting to push Owen to think about his reasoning, which they believe to be faulty. This is evident by phrases such as "So you never actually…", "So you couldn't say…", and "So how is that different than…" In lines 29 and 30, both Kate and BHE posed issues for Owen to think about that challenge his plan to identify students. In line 33, Stacey not only challenged Owen but also Kate and BHE. In response to the issues they were discussing with Owen, Stacey stated "I don't think that's the issue." Stacey then continued to challenge Owen's action research project and stated that "it is hard to justify the claim that this is procedural or conceptual."

The challenges that the teachers posed for Owen brought about other aspects of critical colleagueship, although perhaps not fully realized, that were not exhibited in the praising or

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advising colleague interactions above. In this challenging colleague interaction the teachers pushed Owen to reject what they believed to be flimsy reasoning. They provided him with countervailing evidence and sound arguments to encourage him to reconsider his project objectives. However, Owen did not seem to be open to these ideas as evidenced by how he continued to explain his reasoning in the same way without incorporating ways to address the issues posed by the teachers.

## Discussion

This study is only the first step in unpacking critical colleagueship. According to Wilson and Berne (1999), critical colleagueship may help to explain how teachers learn. Although this study does not claim to draw a link between the aspects of critical colleagueship and teacher learning, it has provided us with what some of the aspects of critical colleagueship sound like in a particular mathematics teacher study group. This study is a necessary step in determining whether this type of collegial development does in fact explain teacher learning since it is first necessary to determine whether these aspects can be observed in a teacher study group. From a methodological standpoint, this study did in fact illustrate that it is possible from observing a mathematics teacher study group to identify some of the aspects of critical colleagueship. The teachers in this study exhibited openness to new ideas, capacity for empathic understanding, self-reflection, and were working towards aspects of rejecting flimsy reasoning by providing their colleagues with sound arguments and countervailing evidence. However, there are many aspects that were not identified in this study. The question remains as to whether these aspects (e.g., increasing teachers' comfort with high levels of ambiguity and uncertainty and sustaining productive disequilibrium) were simply not exhibited by the teachers in this group or if these aspects are not observable. Since this analysis included only 10 project meetings, it is possible that observing more meetings across a longer period of time might illuminate more than was possible here.

This study also raises more questions about how collegiality develops and specifically how it develops within a mathematics teacher study group. First, why was it that the teachers engaged as challenging colleagues around only one teacher's sharing? Although we may like to believe that personal relationships, status, and personality do not play a role in the development of critical colleagueship, it appears that they do. Even though the teachers in this study seemed to feel comfortable sharing and collaborating with each other, it is possible that the existing relationships and issues of status (teaching experience, certification level, mathematical knowledge, and experience with reform curriculum) may have had an impact on how critical colleagueship developed. Further research is needed to examine how status in a teacher study group affects collegiality. Also, the question remains as to how the context of mathematics allows for the development of critical colleagueship. Future research should examine the mathematical aspects of the teacher talk within study groups.

## Implications

This study has implications for further professional development work. According to Wilson and Berne (1999), effective professional development opportunities were those that involved something akin to critical colleagueship. This study group was one of those opportunities. Although the teachers in this group have not mastered all the aspects of critical colleagueship, they were well on their way to developing these types of relationships. Needless to say, critical colleagueship does not develop over night. This project was a four year project and the meetings analyzed for this study were from the fourth year of the project. Special care was taken in this project to promote trust and to help the teachers feel comfortable sharing. The types of interactions discussed above are less likely to develop in traditional professional development settings where teachers are not given opportunities to collaborate with the members of their group on a regular basis.

Regarding action research, the context of the study groups examined in this study, Atweh (2004) made a compelling argument for the use of action research in mathematics classrooms. Action research can be a catalyst for teacher learning since the learning process involves prior knowledge, experience, and reflection, all of which are a part of the action research process. This type of research also affords teachers the professional status and autonomy they deserve. "If teachers are to enjoy the status of autonomous professionals they should feel that they are in control of the processes of knowledge generation as well as knowledge application" (Atweh, 2004, p. 203). This study adds to the argument for doing action research. Teachers, just like students, need good tasks and environments in order to be able to engage in good discussions. The action research project proved to be a good venue for the types of discussions that promoted the teachers to engage as critical colleagues. It was through these projects that the teachers were able to share their classroom practices, including their successes and their struggles, and give and receive constructive feedback.

## Endnotes

1. This data was collected as part of an NSF grant (#0347906) focusing on mathematics classroom discourse (Herbel-Eisenmann, PI). Any opinions, findings, and conclusions or recommendations expressed in this article are those of the authors and do not necessarily reflect the views of NSF. We would like to thank the teachers for allowing us to work in their classrooms.

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# TEACHERS' CONSTRUCTION OF DYNAMIC MATHEMATICAL MODELS BASED ON THE USE OF COMPUTATIONAL TOOLS

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*To what extent does the use of computational tools offer teachers the possibility of constructing dynamic models to identify and explore diverse mathematical relations? What ways of reasoning or thinking about the problems emerge during the models construction with the use of the tools? These research questions guided the development of the study that led us to document the process exhibited by high school teachers to model mathematical situations dynamically. In particular, there is evidence that the use of computational tools helped them identify and explore a set of mathematical relations dynamically. In this process, the participants had opportunity of fostering an inquisitive approach to models construction that values ways of formulating conjectures or mathematical relations and ways to support them.*

## Introduction

Models construction plays a fundamental role during the development of mathematical knowledge. In particular, the modeling cycle that involves examining the phenomenon to be modeled, identifying and discussing assumptions and elements to construct the model, and exploring and validating the model provides useful information to frame an instructional approach to foster teachers' practices and students' mathematical thinking. In this context, we argue that a central activity in students' process of developing mathematical concepts and solving problems is the construction of models that are used to identify, explore and support mathematical relations. Goldin (2008, p. 184) states that "…A model is a specific structure of some kind that embodies features of an object, a situation or a class of situations or phenomena – that which the model represents". How a model is constructed? How can one evaluate the pertinence of a model? What is the role of the use of computational tools in the construction of models? To respond to and discuss these questions, we identify an inquisitive or inquiry approach as a crucial activity associated with the modeling process.

> Mathematical modelling is the process of encountering an indeterminate situation, problematizing it, and bringing inquiry, reasoning, and mathematical structures to bear to transform the situation. The modelling produces an outcome –a model– which is a description or a representation of the situation, drawn from the mathematical disciplines, in relation to the person's experience, which itself had changed through the modelling process. (Confrey & Maloney, 2007, p. 60)

Teachers need to problematize their instructional practice in order to construct instructional routes. In this process, it is crucial that they get engaged into an inquisitive approach to examine the situation (formulation and discussion of questions) in terms of mathematical resources and strategies that lead them to the construction of models. A model then, is the vehicle for teachers to identify mathematical relations and to solve problems. We contend that the development and availability of computational tools offer teachers and students the possibility of enhancing their repertoire of heuristics strategies to deal with mathematical relations embedded in models. It is also important to recognize that different tools may offer distinct opportunities for them to represent and approach mathematical problems. For example, with the use of dynamic software, such as Cabri-Geometry or Sketchpad, some tasks can be modeled dynamically as a mean to identify and explore diverse mathematical relations or conjectures. Thus, tasks or problems are seen as opportunities for teachers and students to engage in the construction of models. In this process they pose and pursue relevant questions as a mean to identify and represent relevant information that guides that construction. In this study, high school teachers worked on a series of mathematical tasks in which they had the opportunity to construct and explore mathematical models. Those models provided them relevant information to think of and design their instructional routes. They were encouraged to use dynamic software during the process of constructing and refining the models.

## Conceptual Framework

Kelly and Lesh (2000) have recognized that researchers, teachers, and students rely on models to represent, organize, examine, and explain situations. For instance, researchers construct models to analyze and interpret teachers and students' activities. Teachers use models to describe, examine, and predict students' mathematical behaviors, while students use models to describe, explain, justify, and refine their ways of thinking. Thus, a model is conceived of as a conceptual unit or entity to foster and document both the teachers' construction of instructional routes and the students' development of mathematical knowledge.

In this context, it becomes important to identify not only the basic ingredients or elements of a model; but also to characterize the process involved in the construction of models. Doerr and English (2003, p. 112) define models as "systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behavior of some other familiar system". That is, the models construction involves examining the situation or problem to be modeled in order to identify essential elements that are represented and scrutinized through operations and rules with the aim of identifying and exploring mathematical relations. Here, we are interested in documenting cognitive behaviors that the problem solver (teacher or student) exhibits during the interaction with the task. Thus, it is important to distinguish phases or cycles that explain relevant moments around the teachers or students' process of models construction. In particular, ways in which teachers or students refine or transform initial models of the situation or phenomenon into more robust or improved models to deal with the situation.

In order to identify the essential elements embedded in a task or phenomenon it is important to comprehend initially the situation or problem (Polya, 1945). Understanding phenomena or situations that involve real contexts demands not only the recognition of the key elements around the problem or dilemma; but also ways to represent them mathematically. This phase is crucial to construct the model that will be explored through mathematical resources and strategies. The exploration stage leads us to search for different approaches and media to examine the model and eventually to solve the problem. The next stage is to interpret and validate the solution in terms of the original statement or problem conditions. In this process, it is important to analyze and discuss whether the model used to solve the problem or situation represents a tool to approach a family of problems or situations.

Is the model of the situation appropriate? Is the solution reasonable and consistent with the problem statement? Can the model be improved? Can the model be extended? What are the mathematical resources, concepts and strategies that were relevant during the construction and exploration of the model? These types of questions are crucial to evaluate the strengths and limitations of the model and to extend the model scope (Niss, Blum, & Galbraith, 2007).

Figure 1 represents a modeling cycle that shows relations and operations that allow transferring features of the phenomenon into the model construction. How can the use of computational tools influence and help the problem solver construct and explore mathematical models? Zbiek, Heid, and Blume, (2007, p. 1170) suggest that in experimental mathematics, computational tools can be used for:

(a) Gaining insight and intuition, (b) discovering new patterns and relationships, (c) graphing to expose mathematical principles, (d) testing and especially falsifying conjectures, (e) exploring a possible result to see whether it merits formal proof, (f) suggesting approaches for formal proof, (g) replacing lengthy hand derivations with tool computations, and (h) confirming analytically derived results.

In particular, the use of dynamic software could play an important role in constructing dynamic models of situations or tasks. The models represent configurations made of simple mathematical objects (points, segments, lines, triangles, squares, etc.) in which, some elements of the models can be moved within the configuration in order to identify and explore mathematical relations. As a consequence, the same process of model construction and exploration incorporates new ways to represent, formulate, and explore mathematical relations.

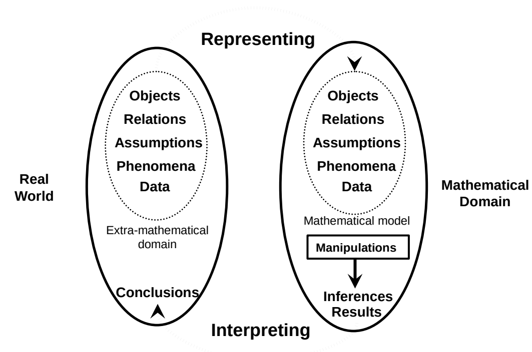



*Figure 1*. Modeling cycle.

inquisitive approach to the tasks. In this process, the teachers worked as a part of the community not only to solve the problems; but also to have opportunities to review mathematical contents that emerged while solving the tasks. The problem solving sessions were recorded and each team handed in a report that included the software files. The researchers took notes and discussed the advantages in using the tool during the diverse problem solving phases. In this report, we focus on analyzing the work shown by the community while dealing with a problem embedded in a real context. Thus, the unit of analysis is the work shown by the six participants as a group during the sessions. The task discussed throughout this report is representative of the type of problems that the community addressed during the development of the sessions.
In general, the participants had experience in using computational tools and they were encouraged to use them during their interaction with the tasks.

*The task.* Figure 2 shows a car going on a straight highway. Aside there is a palace and the driver wants to stop so that his friend (the passenger) can appreciate the facade of the palace. At what position of the highway should the driver stop the car, so that his friend can have the best view? (Adjusted from Vasíliev & Gutenmájer, 1980).

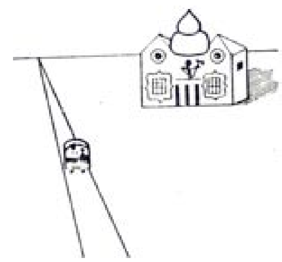


*Figure 2.* The palace.

**Presentation of Results**

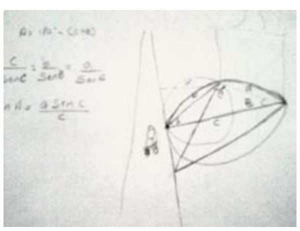
We organize and structure the results in terms of identifying essential phases around the process of model construction that the participants exhibited during the interaction with the task. These phases include the initial comprehension of the statement of the task; the identification of basic elements to construct a model of the problem; the exploration of the model and the formulation of conjectures; ways to support mathematical relations and conjectures, interpretation of results and validation and extension of the model.

*A. Understanding the task statement: An inquiry approach.* This phase was important to comprehend the task and to identify and discuss a set of assumptions that led the community to identify the elements to be considered in the model construction. To this end, the community posed and discussed the following questions: How can we identify that for a distinct position of the car, the passenger has different views of the palace? What does it mean to have the best view? Is it sufficient to consider the position of the passenger on the highway as a reference instead of the car to determine the best position? It was observed that the figure provided in the statement of the task helped them to visualize and eventually represent the problem. They relied on the figure to assume that the observer could be identified as one point that is moved along a line (the highway). Here, the community also discussed if the provided information was sufficient to solve the task since there was no quantitative data involved in the statement.

Initially, the community identified two ways to characterize the best view of the facade: One that related the distance between the observer and the facade (less distance better view) and the other that focused on relating the best view to the angle that is formed between two points on the facade and the observer. The former interpretation was chosen by the community and became a source or a departure point to construct the model of the situation. At this moment, the task was thought of in terms of the basic elements (line, angles, segments, etc) as a way to construct a mathematical model.

*B. Model construction.* The initial analysis of the statement led the community to construct a dynamic model of the situation by representing elements of the task (highway, facade, and observer) through geometric objects (lines, segments, points, angles). In this context, Sophia proposed to represent the highway with a straight line and the passenger a point of that line, and a segment as the base of the palace's facade. Why can the facade of the palace be represented through a segment? Some of the participants argued that the best view means to compare angles that relate the wide of the palace (represented by a segment) and the point that represents the observer. In general, the participants agreed that Sophia's representation included the relevant information of the task. It is important to mention that in order to explain what happens to the angle for various position of the point (the observer) they recognized that it was necessary to identify and notate explicitly the main objects embedded in the problem (points, angles, line, segment). Here, some teachers initially used paper and pencil to sketch a problem representation but later, the use of the tool (dynamic software) became important to visualize the angle variation for different positions of the observer.

*C. Model exploration and conjectures.* At this stage, there appeared two ways to represent the statement: One in which some participants used paper and pencil and relied on trigonometric relations to construct and explore the model (Figure 3, left); and the second approach in which the use of the tool guided the model exploration. Thus, the participants who decided to use the software, started to observe the behavior of some attributes of triangle APB (area, perimeter, and angles) when point P was moved along line L. Here, by assigning measurements to those attributes, both the areas and perimeters of the family of triangles did not reach a maximum value. In particular, Sophia and Jacob noticed that when point P was situated on a position that was collinear with point A and B, then the angle APB measured zero degrees; but when point P was moved to the right of the collinear position the measurement of angle APB increased for some positions and then the angle value decreased.



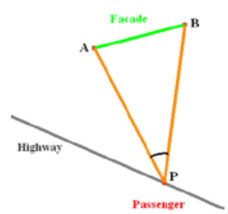



*Figure 3.* Paper and pencil representation (left) and a dynamic model (right).

Thus, they focused on determining the position for point P on L in which the angle APB reaches its maximum value. There appeared two ways to identify the maximum value of the angle. One in which some of the participants directly visualize the numbers displayed while moving point P along the line L (Figure 4); and another in which the participants construct a graphic representation that involves the distance AP and the corresponding value of angle APB (Figure 5).

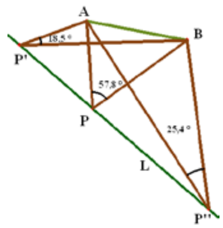



*Figure 4*. Identifying various positions for point P and the angles formed with end points of segment AB.

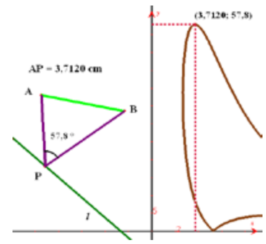



*Figure 5*. Graphic representation of the variation of the value of angle ABP when point P is moved along line L.

The participants were aware of the need of looking for an algebraic or geometric argument to justify the position of point P where the angle reaches its maximum value. In this process, Daniel and Emily decided to draw the circle that passes through points P, A, and B. Based on this construction, they realized that when point P is moved along line L, then the circle that passes through points P, A, and B seems to be tangent to line L at the position where angle APB reaches its maximum value (Figure 6).

Based on this information a conjecture emerged: *To identify the point where angle APB reaches its maximum value is sufficient to draw a tangent circle to line L that passes through points A and B. That is, the tangency point of the circle and line L is the place where the observer gets the best view of the palace*. How can we construct the circle that passes through A and B and is tangent to line L? Emily posed this question to the rest of the participants during the class discussion. Sophia and Jacob suggested that it was relevant to identify relevant properties of the tangent circle assuming its existence. That is, if the tangent circle exists, what properties should it have? Here, it was recognized that the circle must lie on the perpendicular bisector of segment AB and also that its centre must also be on the perpendicular to line L that passes through the tangent point. Thus, David drew a perpendicular line to L that passes through point P and the perpendicular bisector of segment AP. These lines get intersected at point D. What is the locus of point D when point P is moved along line L? With the use of the software the locus was determined (Figure 7). Thus, the intersection point (C) of the locus and the perpendicular bisector of AB was the centre of the tangent circle. Here, to draw the circle they drew the perpendicular from point C to line L, and the distance from point C to line L was the radius of the tangent circle (Figure 7). During the class, it was also argued that the locus of point D when point P is moved along line L is a parabola, since point D is on the perpendicular bisector of segment AP and it holds that d(P,D) is always the same as d(D,A) (definition of perpendicular bisector). Here, the focus of the parabola is point A and the directrix is the line L.

Sophia and Jacob constructed the tangent circle to L that passes through point A and B by drawing initially the perpendicular bisector of segment AB. Later, they situated a point C on that perpendicular bisector and drew a circle with center point C and radius the segment CA. They also drew a perpendicular to line L that passes through point C. This perpendicular and the circle get intersected at point D. What is the locus of point D when point C is moved along the perpendicular bisector of AB?

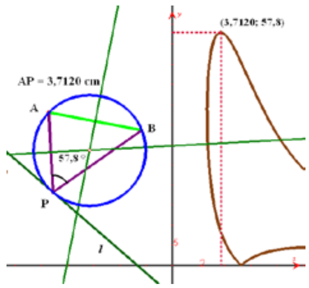



*Figure 6*. The circle that passes through point P, A, and B seems to be tangent to line L when angle APB gets the maximum value.

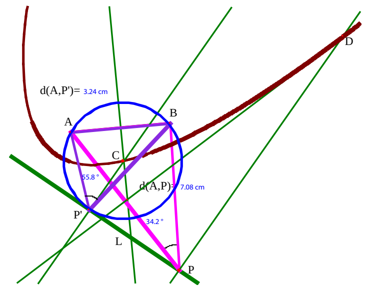



*Figure 7*. Drawing a tangent circle to line L that passes through points A and B.

Again, the use of the software showed that such locus was one branch of a hyperbola. The locus intersects line L at point P. The perpendicular line to L that passes through point P intersects the perpendicular bisector of segment AB at point C'. Thus, to draw the tangent circle to L that passes through points A and B it was sufficient to draw the circle with center point C' and radius segment C'P (Figure 8).

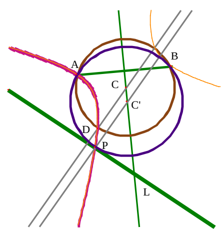



*Figure 8*. Using a hyperbola to construct a tangent circle to line L that passes through points A and B.

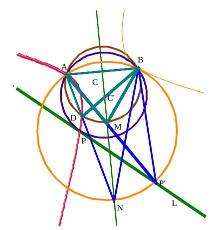



*Figure 9*. Providing an argument to show that angle APB reaches the maximum value.

*D. Interpretation and model validation.* During the class discussion, the participants recognized that the problem of finding the best view was reduced to construct a tangent circle to a line that passes through two given points; however, it was important to provide a mathematical argument to validate that the tangency point was the position where the angle gets its maximum value. To present the argument they relied on figure 9: Point M and N are the intersection points of the perpendicular bisector of segment AB and circles that pass through points ABD and ABP' respectively.

Thus, to compare the values of angle ADB and angle AP'B is the same as comparing angles AMB and ANB. This is because angle ADB is congruent with angle AMB and angle ANB is congruent with angle AP'B. It is also observed that d(A, N) becomes equal to AP when D coincides with point P (tangency point), otherwise d(A, N) is always larger than AM. Therefore, the tangency angle is the angle with maximum value. To evaluate the appropriateness and feasibility of the model the participants changed the original position of the essential elements (highway and facade) and they observed that the model also allowed them to identify the best view of the facade. Including the case in which the facade (segment AB) and the highway (line

L) are parallel, here the best view appears at the intersection of the perpendicular bisector of segment AB and line L.

The participants observed that the domain of the solutions lies on the interval between the intersection of the perpendicular from the extreme of the facade that is closest to the line (highway) and the intersection point of the perpendicular bisector of segment AB and the highway (Figure 10).

*Figure 10*. The model's domain.

## Discussion and Remarks

The model approach used to guide the development of the problem solving sessions helped the participants to focus on key aspects associated with the development of mathematical thinking and practice. For example, firstly, the participants, working as a part of a community, realized that the process of initially comprehending the problem statements is crucial not only to identify essential aspects of the situation, but also to recognize a series of assumptions needed to construct a model of the task or problem. Secondly, they recognized that the model exploration phase represents a departure point for the problem solver to examine the model from distinct perspectives with the aim of identifying a set of relations or conjectures. Later, the conjectures that emerge, during the model exploration phase, need to be supported with mathematical arguments. Finally, the model used to solve the problem needs to be examined in order to evaluate and contrast its pertinence and possible extension to be used in isomorphic or related tasks.

In this context, there is evidence that the use of the tool helped the participants to initially construct a dynamic model of the task. Thus, moving a point (P) on a line (highway) led them to identify and relate the "best view" with the angle formed between the ends of a segment (palace) and that point. How can we measure the angle for distinct positions of point P? How can we identify the angle with a largest value? The teachers used the software to measure the angle for various positions of point P to observe that there was a position where the angle's value was the largest. This visual and empirical approach became important to think of other ways to represent the angle variation. The graphic solution involved a functional approach in which the use of a Cartesian system helped them relate the distance from one end of the segment (AB, the palace) and its corresponding angle. Thus, the graphic approach became relevant to visually identify the point where the angle reaches its maximum value. In addition, moving the point P on the line helped the participants to observe particular behaviors of other objects (circles, segments) within the representations. For example, the teachers observed that the circle that passes through the three points (A, B, and P) becomes tangent to the line when the angle APB reaches its maximum value. Thus, the solution of the task was reduced to draw a circle tangent to the line and the tangency point was the desired point. Again, analyzing relevant properties of the possible solution led them to construct the perpendicular bisector of segment AB and the perpendicular line to L that passes through point P. The locus of the intersection point of those lines when P is moved through line L generated a parabola. Here, the parabola was the key to find the solution of the task. The participants were surprised that a conic section was used to find the point where the observer gets the best view of the palace. They also recognized that the dynamic representation of the problem became important to identify two mathematical results: (a) given a line L and a segment AB that is not parallel to line L, then there is a point P' on the line where the circle that passes through points A, B and P' is tangent to line L. Here the angle AP'B is the angle with the maximum value, (b) given a line L, a point P on that line, and a segment AB that is not parallel to L, then the locus of the intersection point of the perpendicular bisector of segment AP (or BP) and the perpendicular line to L that passes through point P when point P is moved along line L is a parabola. The modeling phases, described in this report, provided useful information to identify a potential route for students to approach mathematical tasks with the use of the tool. Thus, the construction of the dynamic representation, the quantification of attributes (measures of segments, angles, etc.), the identification of loci and the graphic representations are key activities that can help teachers and students to identify and explore interesting mathematical relations. In addition, the use of the tools is also relevant to search for arguments to support those results.

In short, during the modeling processes the participants had an opportunity of identifying and discussing assumptions and essential components that were relevant to construct a dynamic model of the task. The exploration of the model, through an inquisitive approach, led them to formulate a set of conjectures and relations that were important during the solution process. To reach the solution, they relied on empirical, numeric, visual, and algebraic approaches to support and validate conjectures. In this context, the use of the tool seems to help the teachers to experience themselves diverse routes to reconstruct basic mathematical results. These routes are key ingredients for teachers to identify instructional strategies that can foster their students' development of mathematical thinking.

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# TWO FOURTH-GRADE TEACHERS' DIFFERENT USE OF MATHEMATICS TEXTBOOKS: COGNITIVE DEMANDS

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*This study explored how two fourth-grade teachers transformed problems and teacher questions in terms of cognitive demands in teaching. This study also examined factors influencing their textbook use. Analysis results revealed that although two teachers used the same textbook, they used it differently. One teacher closely followed the textbook and thereby maintained the higher level of student thinking. However, the other teacher lowered the cognitive level by using teacher questions that focus on procedure and finding the answer. Teachers' different teaching goals for learning were identified as a significant factor that leads teachers to use the same textbooks differently.*

## Introduction

Historically, curriculum materials or textbooks have been a key agent of policies to regulate mathematics practice in ways that align instruction with the reformers' ideas. Unlike objectives, assessments, and other mechanisms that seek to guide curriculum, textbooks are concrete, and provide the daily information of lessons and units: what teachers and students do. Textbooks are, therefore, often used as a means to shape what students learn (Dow, 1991). Accordingly, research on teachers' textbook use and influential factors has been done over the course of two decades and has provided a substantial number of categories of teachers' textbook use patterns and factors that influence them (e.g., Freeman & Porter, 1989). However, most of the previous studies on textbook use focused on the maximal extent of coverage, such as to what extent teachers use textbooks in planning and teaching school subjects. Therefore, they provided three or four different textbook use patterns such as a textbook-follower, a textbook-adaptor, and a textbook-ignorer. However, these findings do not help us understand how teachers use their textbooks to provide different students' learning opportunities.

According to the *Professional Standards for Teaching Mathematics* (NCTM, 1991), opportunities for student learning are not created simply by putting students into groups, by placing manipulatives in front of them, or by handing them a calculator. Rather, the level and kind of thinking in which students engage with mathematical problems, what Stein and Smith (2000) called, "cognitive demands" of mathematical problems *determines* what students will learn (p. 19). Yet, there are a few studies looking at how teachers use their textbooks in terms of cognitive aspects. Although several studies have examined teachers' practices in terms cognitive demands (e.g., Stein, Grover & Henningsen, 1996), they did not consider the teacher-text relationship (i.e. how the cognitive demands of mathematical tasks presented in *textbooks* were changed when teachers planned and implemented these task during instruction).

This study therefore explored whether and how the cognitive demands of the textbook versions of problems and questions were changed when teachers moved content from text to teaching and what factors influenced their textbook use. In particular, this study examined whether and how two teachers used the same textbook differently in terms of cognitive demands.

## Research Questions

The purpose of this study is to examine two fourth-grader teachers' textbook use in terms of cognitive demands of problems and questions and influential factors. The detailed research questions are as follows:

1. How do two teachers use their textbooks in terms of the cognitive demands of problems and questions?
2. What factors influence teachers' use of the same textbook?

## Conceptual Framework

In order to examine teachers' textbook use in terms of cognitive demands, this study refers to Stein and Smith (2000)'s study. A problem in this study means a mathematical object to be solved or answered that requires logical thought, whereas a question is a pedagogical object suggested in the textbook or used by teachers during instruction that directs the student to think in certain ways, reflect on their math work (i.e., teacher questions). According to Stein and Smith, *cognitive demands of problems* mean the kind and level of student thinking required when (students) engage in problems; *cognitive demands of questions* means the kind and level of student thinking required when (students) engage in teacher questions (Stein, Grover & Henningsen, 1996).

Problems and questions presented in textbooks and those used by teachers during instruction can be categorized into two different levels of cognitive demands—problems (and questions) requiring high-level cognitive demands on students and those requiring low-level cognitive demands on students as Table 1 shows.

*Table 1*

Two Types of Problems and Questions in Terms of Cognitive Demands of Student Thinking

| | High-level | Low-level |
|---|---|---|
| Characteristics | • Require complex and non-algorithm thinking.<br>• Require students to explore and understand mathematical concepts<br>• Require students to analyze the task and possible solution strategies<br>• Usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols)<br>• Require making connections among multiple representations.<br>• Require engagement with the conceptual ideas that underlie the procedures. | • Involve either reproducing previously learned facts, rules, formula, or definitions<br>• Are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience<br>• Do not require students to make connections to the concepts or meanings that underlie the procedure being used.<br>• Are focused on producing correct answers<br>• Do not require students to give explanations,<br>• Focus solely on describing the procedure that was used. |
| Example | Use a diagram to illustrate how the fraction 3/3 represents the same quantity as the decimal 0.6 or 60% | Find an equivalent fraction. ½= ( )/6<br>Completing multiplication tables |

A mathematical problem demanding high-level cognitive processes requires students to recognize transformed versions of a formula they have already learned. The focus for high-level problems is on comprehension, interpretation, flexible application of knowledge and skills, and assembly of information from several different sources to accomplish work. In contrast, a problem involving lower cognitive demands requires students to use memory: Students are required to reproduce or recognize information they have already seen or they have to use algorithms to generate answers to a set of problems. The focus for low-level problems is on memory, formulas, or algorithms to accomplish work. This study employed this framework in analyzing problems and teacher questions in textbooks and those used by teachers in teaching.

## Methods

*Participants and Textbooks*

Two fourth-grade teachers participated in this study. Brad had 7 years of teaching experience at the elementary school while the other teacher, Karen had four years of teaching experience. They worked together at the same elementary school and in the same grade in the U.S. (all names are pseudonym).

*Math Trailblazers* (Wagreich et al., 2004) was used by the teachers. *Math Trailblazers* is one of many commercially published texts revised to reflect changes called for by the NCTM *standards*. Problems and questions in this textbook were analyzed based on Stein and Smith's framework above before conducting the study. The vast majority of the problems and questions presented in the textbook were categorized as high level problems and questions that require students to use *procedures with connections* to meaning, concepts, or understanding and *doing mathematics* (Hiebert, 1999). For example, a typical problem presented in *Math Trailblazers* (Grade 4, Wagreich, et al, 2004, p. 922) is: "Look for patterns in the number sentences and find another equivalent fractions." This problem requires students to focus on the relationship between numerators and denominators and use the patterns, as opposed to simply following the rule, in order to find other fractions equivalent to one half. Questions presented in the teacher's manual of *Math Trailblazers* were also categorized as requiring students to engage in thinking, reasoning, problem-solving, justifying, and communicating about mathematics. These questions include the language like "look for", "explain"," justify".

*Data Collection and Analysis*

In order to examine how two fourth-grade teachers interact with the textbook and what factors influence their decision-making, I observed and interviewed each teacher while they were teaching a fraction unit across two semesters. During the observation, I took field notes focusing on problems and questions the teachers or students worked on. Post-observation interviews were tape-recorded. In the analysis of two teachers' textbook use, audio-taped interview data, the transcripts of interviews, documents related to teachers' use of textbooks (teachers' lesson plan from the teacher's manual and my observation notes), and my observation were used. Based on Stein and Smith's (2000) framework, each problem used by teachers was classified into either high-level or low-level. The interview data were analyzed in order to explore possible reasons why teachers used the same textbook differently. First, interviews were transcribed into printed text. I then read the transcripts carefully. While I obtained a general sense of the information through that reading, I tried to make a list of emergent ideas from the reading of the data (e.g., Miles & Huberman, 1994).

## Results

Analyses of the two teachers' use of the textbook revealed that there were similarities between these teachers on the dimension of the mathematical problems they set up. The same

mathematical problem was used in both classes and was set up in essentially the same manner. However, different from Brad, Karen lowered the cognitive demand level of the textbook problems by using teacher questions that focus on procedure and finding *the* answer the ways in which students actually went about working on the problems differed in the two classes. During instruction, Karen shifted the emphasis from meaning, concepts, or understanding to using procedures. She took over students' thinking and reasoning and specified explicit procedures for finding equivalent fractions. As class went over, Karen's questions became much narrower, asking students to fill in the blank rather than construct an answer (e.g., "two times two is?). Karen attempted, and rarely managed, to surface students' mathematical thinking using her questions. In this paper, I will describe how Karen transformed the cognitive demand of textbook problems when teaching the topic of equivalent fractions using *Math Trailblazers*. In particular, I will use the lesson "equivalent fractions" in Karen's class, which I observed in Brad's class.
*1. How do two teachers use their textbooks in terms of the cognitive demands of problems and questions?*

Before observing her class, Karen gave me a copy of her lesson plan, reproduced as Figure 1, which came directly from the teachers' manual of her textbook. Brad closely followed suggestions presented in Figure 1.

1. Ask students to use their fraction chart from Lesson 3 to find all of the fractions that are equivalent to ½. List these on the board or overhead.
2. Ask students to compare the numerators and the denominators of the equivalent fractions in order to look for patterns.
3. Ask students to suggest other factions that are equivalent to 1/2
4. Write number sentences on the board or overhead showing the equivalencies
5. Students look for patterns in the number sentences
6. Students use the patterns (multiplying or dividing the numerator and the denominator by the same number) to find fractions equivalent to 3/4, 1/3, and 2/5.
7. Students use the patterns to complete number sentences involving equivalent fractions
8. Students complete *Questions 1-5* on the *Equivalent Fractions* Activity Pages in the Student Guide as independent practice
9. Assign Homework *Questions 1-15* on the *Equivalent Fractions* Activity Pages in the Student Guide. Students will need their fraction charts to complete this assignment

Source: From *Math Trailblazers Fourth Grade Teacher's Lesson Guide* (p. 924), by Wagreich, P., Goldberg, H., Bieler, J.L., Beissinger, J.S., Cirulis, A., Gartzman, M., Inzerillo, C., Isaacs, A., Kelso, C.R., & Peters, L(2004), TIMS Project, Dubuque, IA:Kendall/Hunt.

*Figure 1*. Summary of lesson activities in lesson.

Like Brad, Karen used activities suggested in the lesson guide. Karen asked students to look at the fraction chart they used in the previous lessons and find all of the fractions that were equivalent to one half. Like Brad's class, students in Karen's class easily came up with the equivalent fractions. As suggested in the lesson guide, Karen asked students to arrange these equivalent fractions in an order such that a denominator gets bigger. As in Brad's class, students arranged them and Karen wrote them on the board, such as. Karen again asked students to write this arrangement of these equivalent fractions down underneath the word "equivalent fractions" in students' notebook. In the meantime, she told them that all of these fractions are the same size.

As students finished writing down equivalent fraction sentence, Karen asked students to find patterns by comparing the denominators. Students noticed two patterns, "counting by twos" and “even”, but they did not come up with the answer that Karen expected, which is multiples of two. In this instance, Karen interacted differently from Brad. While Brad gave hints, waited until students volunteered to answer, and had students discuss the incorrect answer, Karen told her students the expected answer. This interaction with students exemplifies how teachers decrease the cognitive demand of problems during instruction. Henningsen and Stein (1997) reported that teachers often decrease the cognitive demand of student thinking by taking over student reasoning and telling students how to do the problems.

For the rest of the lesson, I repeatedly observed this pattern. For example, after having students find the relationship between the numerators and between the denominators of fractions equivalent to one half, Karen asked students to look at the numerators and denominators together of each of equivalent fractions and find patterns. Several students responded, but none provided her expected answer. Karen again said to students “the numerator is half of the denominator, isn’t it?” Indeed, Karen’s telling was more obvious when students did not figure out patterns in the number sentences. She wrote four number sentences on the board and said:

> I have on the board four number sentences. Four number sentences. The first one, one half equals two fourths. The second one, one half equals four eighths. The third one, one half equals five tenths and the fourth one is one half equals six twelfths. Look at that pattern in these number sentences, first by looking at the numerators and then by looking at the denominators. What do you see?

This problem requires students to focus on the relationship between numerators and denominators and use the patterns, as opposed to simply following the rule, in order to find other fractions equivalent to one half. One student answered but his answer did not show the relationship between numbers and denominators. The same mistake occurred in Karen’s class as in Brad’s class. In both Karen’s and Brad’s classes, when students were asked to find the relationship in the number sentences as above, students tended to compare the denominator of the first fraction and the numerator of the second fraction, and find the relationship. For example, in Karen’s class, one student found the relationship for the first number sentence, such as “If you take two and times it two you get four”. Karen responded to this study as follows: “Is this what you were saying? One times two is?” Karen paused and students said “two”. Karen again said, “two times two is?” and paused. Students said “four.”

Although Karen set up problems finding the relationship and used it to find other equivalent fractions, as students became confused, Karen gave the pattern as a rule that students needed to follow, which requires low level of student thinking. Karen said, “Here is the rule. Get your pens ready. If you multiply both the numerator and the denominator of a fraction by the same number, the result will be an equivalent fraction”. Karen asked students to copy out the rule in their notebook, and repeatedly stated the rule while students were writing the rule down. During an interview, Karen mentioned that “students need to memorize it to apply it”; contrast with Brad who said that “students need to make a discovery for themselves.” By using the rule, Karen tried to have students see patterns in the rest of number sentences. For example, Karen asked students to take a look at the third number sentence and see whether the rule works. Some students nodded their heads to say yes and some kids shook their heads for no. She said:

> That’s what the rule says. Multiply both the numerator, that’s my numerator and the denominator by the same number. The result will be an equivalent fraction and I said over here that one half equals five tenths (interview, 04/27/2007).

Karen led the students through four number sentences involving equivalent fractions, all the while focusing on the procedure. For example, for the fourth number sentence, Karen repeatedly asked, “Three times what equals six?” Karen paused and students said “two.” Karen asked again, “Four times what equals twelfths?” and paused. Students said “two.” Karen’s questions became procedure-oriented, required only multiplication facts. The conversations that Karen had with students during this portion of class revealed what her goal for the lesson was: writing equivalent fractions by applying the rule.

Like Brad, Karen asked students to explain and justify their answers. For example, Karen also asked students to see if it is true that one third equals two sixths. However, in her class, it was sufficient and acceptable for students to rely on the rule. Consider the following Karen’ remark: “We said it’s true if we could multiply both the numerator and the denominator by the same number to come up with the answer. What number do I multiply one times two is equal?”

Karen ended up the lesson by asking students to create the number sentences involving equivalent fractions such as “one times two is two and six times two is twelve”, which is expressed as in a numerical form. Karen called out one student. But the selected student said, “I don’t know how to do this”. Karen called out other students. A few students could make the number sentences described above. Karen used the same mathematical problems as those used in Brad’s class for an introductory teacher activity and student exercise. Therefore, the problems set up in Karen’s class focus students’ attention on the discovery of patterns and the use of the patterns for finding equivalent fractions, which demands complex thinking and a considerable amount of cognitive effort. However, the questions most frequently used in Karen’s class were “what number do we multiply?” which require only multiplication facts. In Karen’s class, the procedure of “how to do it” was stressed above all else. In an interview, she confirmed her typical questions as below:

> I think probably the question that I asked over and over again and maybe not in this exact word is “What number should I multiply both the numerator and the denominator by to find the equivalent fraction?” I probably said that a hundred times (interview, 04/27/2007).

Her remark “a hundred times” shows not only how frequently she used this type of questions during the class but also how she was aware of her frequent use of this type of questions in her class. Students in Karen’s classroom were rarely pushed to elaborate on their answers. If students’ responses reflected the correct answer, the teacher did not raise follow-up questions to make students’ mathematical thinking explicit. If students’ responses reflected the incorrect answer, teachers paraphrased the answer, changing it to make it more “accurate.” Although *Trailblazers* provide a lot of suggestions in the lesson guide for how teachers should approach questioning or approach discussing concepts, Karen did not use those questions. She shifted the emphasis of their work from meaning or understanding to the use of procedure without connections, and decreased the cognitive demands of teacher questions in ways that require lower level of student thinking.

*2. What factors influence teachers’ use of a textbook?*

Karen supplements the lessons from *Trailblazers* with practice problems. While Brad claimed to be a “follower” of *Math Trailblazers*, Karen claimed to be “modifier” of the textbook. She reported that, in total, 50% of lessons came directly from *Math Trailblazers* and the rest 50% came from other resources such as other textbooks. She said that she supplements the textbook with more practice problems from other textbooks, in particular, an old textbook, *Houghton Mifflin Mathematics*. During an interview, Karen said:

> I’ve used old textbooks. I find that in the *Trailblazers* series there is a lot of introducing the concepts but not a lot of mastering the concepts… We have some old texts [*Houghton Mifflin*] like that has more practice than the *Trailblazers* does....*Houghton Mifflin* has some from when I was in school so 1980’s. The Trailblazers series provide a lot of explore, look at, manipulate but it’s very shallow in the practice areas as you can see from the lesson that there’s I don’t know ten problems or seven problems (interview, 04/27/2007).

Together with her use of lower level teacher questions, frequent supplementing of the standards-based textbook with more practice problems reduces the cognitive demand of student thinking. Why did Karen transform her textbook in that way? What factors account for Karen’s transformation? During the interview, Karen provided the rationales for why she added practice problems out of her obligation to meet the Grade Level Content Expectations (GLCE) for Michigan. In addition, Karen’s notion of how students learn also is a factor that influences her use of the textbook and her teaching practice.

*Conflict between Teachers’ Goal and Perceived Goals of the Textbook*

Karen believes that she should cover all the content presented in the GLCEs. Karen described her use of the textbook as follows:

> When I start out at the beginning of the year, I lay out the content expectations that the state has mandated then I try to match up the text with the GLCEs. So when I get to each unit I have to first make sure that I’ve covered all of those content expectations. Because that’s what the state says we have to do. The district has said, this is the text we’re using and that the Trailblazers series. But the district also has said that the Trailblazers series does not cover all of the Michigan grade level content expectations. So therefore you must supplement. So when I’m planning I take a look at the lessons that are in the Trailblazer series and I see which ones of those cover the grade level content expectations and then I look at all the holes that are left over and start using other resources (interview, 04/27/2007).

Karen’s obligation to the contents of the GLCEs influences her pedagogy. During an interview, Karen said:

> When I was at a math meeting a year and half ago we were talking about the new content expectations, what 4th graders are required to know….The curriculum director at the time said, these things are suppose to be taught to mastery and here was this huge group of experts saying, we’re introducing the concepts but the children are not practicing it enough to say that we’ve mastered it….So I’m working really hard on trying to make sure that I can pull from anywhere that I can find practice stuff to make sure my kids are going home and

> working on it on their own, working on it with friends here, that their parents are informed about what we're doing and also working with them at on it (interview, 04/27/2007).

Indeed, "mastery of the concepts" can be interpreted in various ways. Some may think it only from procedural aspects, such as proficiency in computation, whereas others consider it from both conceptual aspects and procedural aspects of the mathematics contents. Considering the recommendations in the GLCEs, definition of mastery requires both conceptual understanding and computational skill (Mcthighe & Wiggins, 2005; Wormeli, 2006). However, Karen seems to recognize the definition of mastery from the procedural aspects. This understanding may cause her to lead instruction more procedure-oriented and emphasize application of the rule in her class.

*Procedural fluency as a Teaching Goal*

Indeed, Karen's obligation to meet the GLCEs is in keeping with her notion of how students learn, as evidenced by the focus of her questions on procedures—on *how* rather than *why*. Karen put more emphasis on application of the rule than on sense-making or meaning in learning mathematics. Karen articulated her goals as "proficiency in writing equivalent fractions" and "mastering the concepts and applying it" in general. This differs from the evidenced by teachers in the first pattern which matches the category "understanding-oriented". Together with Karen's view on the emphasis of Grade Level Content Expectations, her learning goal seems to push her to supplement the textbook with more practice problems and change the emphases from understanding to following rules to solve problems in her classroom.

## Discussion and Implications

Although Brad and Karen used the same textbook, Brad maintained the cognitive demand of the textbook problems by using higher level teacher questions, whereas Karen decreased the level by using lower level teacher questions. The reason behind this different use of textbook is their different teaching goals for student learning. While Brad wanted students to develop the underlying ideas between equivalent fractions, generate the rule and apply the rule to find equivalent fractions, Karen wanted students to follow the rule to find equivalent fractions. She believed that practice makes perfect. Different teaching goals led them to use the same textbook in different ways, which in turn provided different notions of knowing mathematics and doing mathematics. This study suggests that the extent to which teachers' ideas about how mathematics is learned matches the teaching and learning philosophies of the textbook contributes significantly to their use of curriculum.

This study provides implications for policy makers, curriculum developers, professional developers, and teacher educators. For example, for professional developers, this study suggests that they should provide opportunities for teachers to learn and participate with their textbooks in professional development and should provide teachers with opportunities to change their notions of learning and teaching mathematics. Research has documented the challenges that many teachers face when they try to conduct lessons that take into account and productively build on student responses (e.g., Ball, 2001). All teachers need support. In particular, teachers like Karen need to teach about mathematics as a field of inquiry, not as a body of procedures. They need to learn to think about the goals of learning mathematics as greater than the mastery of computational skills. Teachers cannot make fundamental changes in their teaching without several kinds of support, such as time and assistance in examining and evaluating their own assumptions about how children learn mathematics and comparing their assumptions to those represented in standards-based curriculum. Reformers and policy makers must find ways to communicate about change in a way that makes sense and respects where teachers are, while still helping them realize that they are being asked to rethink what they do, and in a way that provides guidance for that change.

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# MATHEMATICS TEACHERS AND PROFESSIONAL LEARNING COMMUNITIES: UNDERSTANDING PROFESSIONAL DEVELOPMENT IN COLLABORATIVE SETTINGS

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*For professional learning communities (PLC) to be used as a form of mathematics professional development (MPD), more work needs to be done to identify the extent to which mathematics teachers in PLCs engage in the activities that address their content and pedagogical needs. This paper reports on a study that investigated mathematics teachers' attempts to implement principles of PLCs. Based on the findings, the role of PLCs as the sole source of professional development for mathematics teachers is questioned. Suggestions for future research on collaborative MPD and its benefits for the design and replication of professional development interventions are offered.*

## Background

The rapid growth of the field of literature on mathematics professional development (MPD) is due to the realization that teachers should be better prepared to be able to improve their own instructional practices (Sowder, 2007). As a result, recent studies have attempted to identify effective MPD initiatives and find a set of features that are commonly part of these successful programs (e.g., Yoon et al, 2007; Garet et al, 2001). This search has led to findings that professional development experiences that are sustained (Garet et al, 2001), practice-based (Ball & Cohen, 1999), and allow for teacher involvement in the decision making process (Yoon et al, 2007) are successful in terms of the high value that teachers place on the experience as well as resulting increases in student achievement. Additionally, research has shown that the opportunity for collaboration is both valued by teachers (Garet et al, 2001; Arbaugh, 2003) and plays a role in supporting inquiry and problem solving (Loucks-Horsley et al, 1998). Given the recent focus on professional development, especially in collaborative settings, it is important to define the work of teachers in groups in order to identify what influences teacher learning and aid in the design of effective MPD.

Professional learning communities (PLC) provide both an organizational framework and a set of requisite dispositions and activities for teacher learning. PLCs are defined as sustained collaborative opportunities where teachers focus on student learning and critically reflect on their shared practice (e.g., McLaughlin & Talbert, 2006; DuFour & Eaker, 1998). In such communities, teachers are empowered to make changes to their practice by inquiring into the best methods of instruction and developing and testing new hypotheses (Louis, Marks & Kruse, 1996). Additionally, the decisions to make changes are based on data from the classroom to determine new and appropriate teaching strategies (DuFour & Eaker, 1998). Studies have shown positive changes in the practice of teachers (i.e., student-centered instruction, high expectations for student learning) in PLCs (McLaughlin & Talbert, 2001; Vescio, Ross & Adams, 2008).

However, not all collaborative groups of teachers engage in the type of examination of practice found with PLCs. Research studies have shown communities of teachers that share values and set expectations for their work with students but avoid the conflicts that can arise during the critical reflection on practice (e.g., Wells & Feun, 2007). As a result, teachers tend to stick with the teacher-centered, traditional methods that are prevalent in schools (McLaughlin & Talbert, 2001). While structural changes such as the availability of meeting time for teachers are important for the work of a collaborative team (Louis, Marks & Kruse, 1996), such changes are relatively easy and are less effective in impacting instructional practices. As a result, the attributes of PLCs are special in the bigger picture of teacher collaboration.

Many MPD interventions claim to rely on collaboration, specifically the shared values and norms for critical reflection that are part of PLCs (e.g., Borko et al, 2008; Lachance & Confrey, 2003; Arbaugh, 2003; Kazemi & Franke, 2004). However, a team's inclination to engage in critical reflection is not the only component of most collaborative MPD interventions. Many programs aim to promote the use of instructional practices such as cognitively demanding tasks, mathematical technology, and *Standards*-based curricula. As a result, certain features are commonly incorporated into collaborative MPD such as video study, task analysis, student work analysis, and engaging with mathematical content and technology. These activities are important as Kennedy (1998) found that a strong content focus in professional development programs had a positive impact on student learning. Additionally, research has linked increases in student achievement to teachers' mathematical and pedagogical knowledge (e.g., Hill, Rowan & Ball, 2005) and their attention to student reasoning (e.g., Carpenter et al, 1999). Ultimately, mathematics teachers are faced with unique challenges that must be addressed with specific forms of professional development activities.

PLCs only provide a general guideline for the organization, activities, and individual and group dispositions necessary to make meaningful change to instruction, regardless of subject. Additionally, there is little documentation of the nature of the work that teachers do while working in PLCs (Vescio, Ross & Adams, 2008). In order for PLCs to serve as an effective form of MPD, more work needs to be done to identify the extent to which mathematics teachers in PLCs engage in the activities that improve their content and pedagogical knowledge and, ultimately, improve student achievement.

The study described here investigated two teams of teachers attempting to implement principles of PLCs as part of a district-wide intervention. The goal of this study was to discover both teams' success in implementing these principles and to what extent the presence of features that are commonly found in effective MPD was evident in their work. From those findings, the author questions the role of PLCs in the professional development of mathematics teachers and highlights other factors that could be attributed to a group of teachers' inclination to engage in activities found in effective MPD. As a result, the author offers suggestions for future research on collaborative MPD and how findings from such research could be used to inform the design and replication of MPD.

## Framework

When comparing the literature on professional communities, PLCs, and collaborative MPD, there are differences in terms of the activities teachers do, their content focus, and what teachers ultimately take away from the experience. Further, some types of collaborative settings rely on many of the characteristics of another type. For instance, PLCs rely on a focus on student learning, experimentation, and inquiry but also rely on the shared values that comprise more traditional collaborative groups. Collaborative MPD interventions often consist of specific activities that focus teachers' attention to students' mathematical thinking. However, the productive use of these types of classroom artifacts relies on the shared practice, results orientation, and comfort with critical reflection, traits which are attributed to PLCs.

As a result, a conceptual framework has been developed to illustrate both the hierarchy and the links among different types of collaborative work (see Figure 1). The framework refers to three stages: Collaboration, Teacher Learning, and Specialized Growth. By using the term stages, it is implied that the features of each type of collaborative setting serve as a necessary foundation for subsequent types of work.

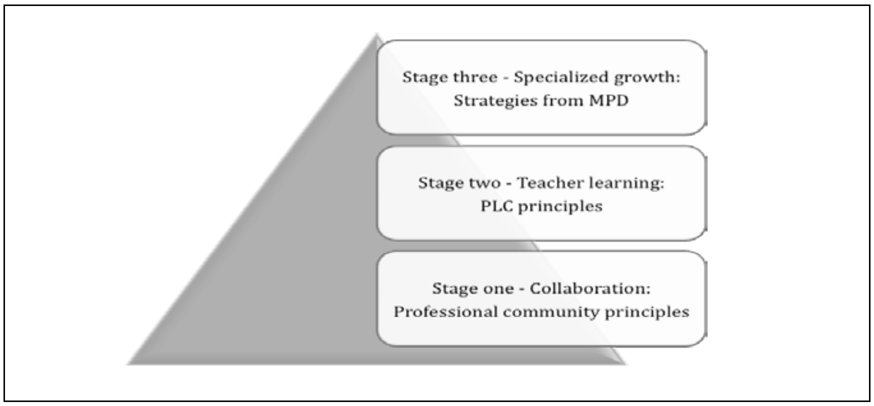


*Figure 1.* Conceptual framework: Stages of mathematics teachers' collaborative work.

To better describe the attributes and features of these three stages, elements have been identified for each. The Collaboration stage consists of three elements: beliefs on collaboration, shared values and goals, and shared role. The Teacher Learning stage consists of three elements in addition to those in the previous stage: collective inquiry, assessment of practice on basis of results, and focus on student learning. Finally, the Specialized Growth stage has three elements: content focus, use of artifacts of practice, and planning and implementing reform-inspired instructional practices.

Each stage also refers to the type of growth that teachers experience at each stage. At the Collaboration stage, teachers worry about logistical and other non-instructional issues, resulting in unchanged practice and, thus, no growth. Teachers at the Teacher Learning stage function as a PLC and, in turn, focus on issues of curriculum, instruction, and assessment resulting in a general pedagogical growth. At the Specialized Growth stage, teachers are focused not only on teaching but are also focused on content, fostering specialized growth in both mathematics content and pedagogy.

Previous work has documented the factors attributed to fostering a group of teachers' movement toward becoming a PLC (stage one to stage two). While structural features such as the availability of meeting time are beneficial toward the growth of a group, other factors such as the empowerment of teachers to be involved in the decision-making process (Louis, Marks, & Kruse, 1996), a focus on issues of curriculum and instruction instead of issues of behavior and policy (McLaughlin & Talbert, 2006), and the use of classroom data to drive decision-making (DuFour & Eaker, 1998) help develop a group of teachers into a PLC.

The transition of a PLC consisting of mathematics teachers to a group engaging in the types of activities found in effective MPD (stage two to stage three) is not as clear. While many MPD interventions claim to value and rely on collaboration and the traits of PLCs, researchers or facilitators drive many of the decisions for activities and content. In order for true PLCs to serve as a source of professional development for mathematics teachers, the role that PLCs play in the specialized growth of teachers must be better defined.

## Methods

*Context*

The study reported in this paper investigated the collaborative interactions of two teams of high school mathematics teachers. The study took place in a large, urban school district in the southeastern United States. A goal of improving high school graduation rates as well as a mission to stay current with movements in the field of education resulted in the district adopting the idea of PLCs to be implemented throughout its schools. Specifically, the district promoted the principles for PLCs described by Rick DuFour and his colleagues (DuFour & Eaker, 1998). Workshops for district teachers were held to present these principles of collaborative learning. Knowing the benefits of teacher collaboration, district administrators hope PLCs eventually serve as the arena for the professional development for all teachers. As a result, attempts have been made to ensure that teachers are given adequate time to work collaboratively and that they develop the skills needed to function as member of a PLC.

*Participants*

Both teams' involvement in the study was the result of their positive response to participating after interest was gauged from schools across the district. Additionally, both teams were focused on issues in Algebra I classes, which provided some control in the study. The two teams were housed at different schools in the same district, Brantley High School and Elmwood High School (both pseudonyms). Both teams met on Tuesday mornings for 45 minutes each week as part of a scheduling change at the school level. Each team was in their first year together, though PLCs had been in the district for two years prior and some teachers from each team had previously worked together in that capacity.

The team at Brantley High School consisted of five teachers, three female and two male. Four of the teachers had five or fewer years of teaching experience and all five teachers had between two and four years of experience teaching Algebra I. The team at Elmwood High School included four teachers, three female and one male. There was generally more teaching experience on this team, with three of the four teachers having taught for at least ten years. Two of the teachers had taught Algebra I for 13 years each. However, the other teachers are in their first year of teaching the course.

*Data Sources*

Data was collected for this study using three sources: team meeting observations, individual surveys, and individual interviews. While the team meeting observations provided the most information about the team's interactions, the interviews and surveys allowed for triangulation of data to ensure consistency of observed phenomena across different environments.

The researcher observed, audio recorded, and took notes on four of each team's weekly meetings over a span of two months. The researcher did not provide any input regarding the agenda for each meeting nor did he provide any feedback after meetings. The teachers participating in the study were also given two surveys consisting of demographic information, goals for their collaborative interactions, and a set of Likert scale questions regarding teachers' experiences with their team and the activities in which they engage with their teams (adapted from McLaughlin & Talbert (2001) and Wells & Feun (2007)). The teachers also participated individually in two interviews over the course of data collection. The first interview was designed to allow teachers to elaborate on their experiences with their team as well as other forms of professional development. The second interview was designed for a different goal as teachers interacted with mathematical tasks and corresponding student work to explicate their dispositions when analyzing tasks and student work.

*Analysis*

For each data source, coding sheets were developed to pair positive indicators of effective group interactions with a corresponding element. For example, the mention or use of common lesson plans, activities, or assessments would be positive indicators for the collective inquiry element of the Teacher Learning stage and working through mathematical tasks or discussing algebraic concepts would be positive indicators for the content focus element of the Specialized Growth stage. By then combining the results across all data sources, consistencies could be noted in order to identify the strengths and weaknesses of a team at each element and, thus, each stage of the framework. These strengths and weaknesses, as well as any inconsistencies across the three data sources, were further examined as possible factors associated with fostering or inhibiting mathematics teacher learning.

## Results

The analysis of data from both teams yielded results regarding each team’s implementation of the principles of PLCs and the extent to which they engaged in activities commonly found in effective MPD. Since, in this context, PLCs had been setup to serve as a form of professional development, this section will highlight each team’s performance as well as factors that can be attributed to each team’s success or difficulties at each stage of the framework.

*Brantley High School*

Based on teachers’ responses regarding collaboration as well as the full use of their available time to meet as a group, it is clear that the team at Brantley High School valued the benefits of collaboration. A common goal for all of the team members was the sharing of ideas and sharing the responsibility on items such as test creation and other materials. However, the roles and responsibilities were not evenly distributed amongst team members, as two teachers were new to the team, which could have impacted their willingness to take a more prominent role.

Despite uneven participation, the team was able to thoroughly engage in activities that are recommended for PLCs. The team made significant changes to their curriculum in response to changes in the course’s cumulative state exam and its effect on student achievement. Among other commonly used materials, the team administered common chapter tests, which were used for data analysis and decision-making. In all, the team focused much of their time on issues of curriculum and assessment and, in turn, student learning. The team’s success has even been used as a model for other teams in their school attempting to implement the same PLC principles.

Despite the team’s success in implementing the principles of PLCs, there was little to no evidence of any of the elements of the Specialized Growth stage in their team meetings or individual responses. Aside from an occasional discussion of algebraic topics with respect to their curricular changes, there was no time spent with mathematical content, artifacts of practice such as video or student work to supplement the assessment data on which they relied, or planning the use of mathematical tasks or technology. In this case, while the team at Brantley was very effective at implementing the PLC principles, they did not engage in the materials and activities based in mathematical content and pedagogy as recommended in MPD literature.

The team at Brantley's success at implementing the principles of PLCs could be explained by the fact that four of the five team members attended at least one of the PLC workshops offered by the district. As a result, it seemed that the team implemented those principles literally, assuring that activities such as common planning, data analysis, and refining curricular materials were a fixture in their collaborative work. However, such a determined focus plus the fact that most of their other professional development experiences were focused on general issues such as classroom management might explain the group's inability to engage in the types of activities that are common in MPD. Without much previous professional development based in their content area, it is not too surprising that these teachers were not self-motivated to incorporate mathematical content and pedagogy into their collaborative interactions.

*Elmwood High School*

For the team of Algebra I teachers at Elmwood High School, while the teachers did seem to share goals for student achievement, there was less evidence that they valued their collaborative interactions. The members of the team seldom cited the value they placed on collaboration and the team also often cut meetings short, even though the meeting time caused no conflicts with the school day. Despite these problems, there was an expectation that all four members of the team participate in the group discussions and bring ideas and materials to share.

At the same time, however, the teachers on the team at Elmwood did not engage in a shared practice. While the team followed a common curriculum and pacing guide, they would only occasionally use common chapter tests and classroom materials. Additionally, the team spent very little time reflecting on their practice or drawing conclusions from classroom data. Much of the team's time was focused on issues not pertaining to curriculum, instruction, or assessment, instead resorting to discussions on class size and student behavior. As a result, this team was not successful in implementing PLC principles at the time of this study. In essence, the team at Elmwood High School was functioning as a traditional collaborative team by sharing values and materials but not taking a critical look at their instructional practice.

Even though much of the team's time was unfocused, when the team did discuss issues of curriculum and instruction, there were many instances that served as indicators for elements in the Specialized Growth stage. During team meetings, teachers worked through mathematical tasks that teachers were going to use in class. The team also referenced curricular resources and other literature on mathematics content and pedagogy. Some of the team's time focused on the use of graphing calculators in the classroom. As a result, the team at Elmwood showed more evidence of engaging in the types of activities that are common features of effective MPD despite their deficiencies as a PLC.

The Elmwood team's performance was far different than that of the Brantley team. However, like the Brantley team, some factors were identified that may be attributed to their effectiveness in implementing the PLC practices and the extent to which they engaged in activities found in effective MPD programs. Only one of the teachers on the Elmwood team attended a PLC workshop offered by the district, which could explain the team's lack of emphasis on principles such as shared practice and data analysis. Moreover, the lesser value that this team placed on collaboration in comparison to the Brantley team could make it difficult for the team to accomplish much at the teacher learning level. Such a claim is consistent with literature on professional communities and PLCs and is in line with the idea of stages found in the framework. However, the team's ability to begin to implement features of professional development commonly found in mathematics-focused programs seems to defy the idea of stages. Upon closer examination, several factors could be attributed to their use of these specialized activities. First, three of the four members had graduate degrees in mathematics education or educational leadership. Given this fact, the professional development of these teachers was much more focused on mathematics content and pedagogy. The individuals on the team also shared a willingness to change the way they teach but did not seem to have the norms to address these issues as a group.

## Discussion

The goal of this study was to evaluate the effectiveness of PLCs in terms of the specialized growth of mathematics teachers and offer recommendations for future research on and design of collaborative experiences for mathematics teachers. The grounds for this study are based in literature on collaborative professional development, specifically research on collaborative MPD. In such work, teachers benefit from the activities and dispositions that come from working as a PLC. However, facilitators provide the direction for these groups toward the specialized growth that impacts mathematical instructional practices and, in turn, student achievement. In order for collaborative MPD to be effective and replicable, the role of PLCs in the professional development of mathematics teachers must be better understood.

The implementation of PLC principles as a professional development intervention should be questioned based on the results of this study. One team was not successful in implementing the principles, which could lead to doubts about the methods used in such a widespread implementation. The other team was successful at implementing the PLC principles but showed no evidence of engaging in the activities or discussions that are commonly found in successful MPD programs, leading to doubts about the real benefits that mathematics teachers can take away from these collaborative experiences.

The results of this study also raise questions about the conceptual framework considered in this paper. The team at Elmwood was not effective in implementing the principles of PLCs but did show evidence of some of the activities that are commonly found in MPD, which would be characteristic of teams at the Specialized Growth stage. Given this team’s ability to incorporate a content focus while struggling to incorporate collaborative norms, such a framework could be reconsidered across two dimensions (collaborative norms and content focus) instead of the one-dimensional approach taken in the original framework. Ultimately, this framework could be used to inform the design of collaborative MPD as an emphasis is needed on both the collaborative norms and the content focus of a group’s interactions in order to lead to the specialized growth of mathematics teachers’ instructional practices.

This study looked at a small number of teams for a relatively short amount of time. However, the findings from this study can inform future research on the collaborative work of mathematics teachers. Future research should look at more teams for a longer period of time to get a more representative account of the growth of teachers in collaborative settings across subjects, schools, and districts. Based on the results of this study, it is also recommended that future research studies take measures of teachers’ content knowledge, pedagogical knowledge, and beliefs on content, collaboration, and reform-inspired instructional practices using the tools, instruments, and assessments available in the field. The results of this study suggest that individual teachers’ knowledge and dispositions could affect the collaborative work of a group. Similarly, researchers should also take into account any ambient factors that surround a collaborative team, such as existing professional development programs, graduate coursework, or the implementation of new curricula. The results of this study suggest that these experiences can impact the work of a collaborative team. By better explicating these factors, the design of MPD interventions can take a broader scope by considering all of the attributes of effective programs and allowing for more efficient replication.

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# DEVELOPING TEACHERS' FLEXIBILITY IN ALGEBRA THROUGH COMPARISON

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*We describe a one-day professional development activity for mathematics teachers that promoted the use of comparison as an instructional tool to develop students' flexibility in algebra. Our analysis indicates that when teachers were presented with techniques for effective use of comparison, their own understanding of multiple solution methods was reinforced. In addition, teachers questioned their reliance on one familiar method over others that are equally effective and drew new connections between problem solving methods. Finally, as a result of experiencing instructional use of comparison, teachers began to see value in teaching for flexibility and reported changing their own teaching practices.*

## Introduction

Recently, Star (2005, 2007) proposed a new conceptualization of procedural knowledge, highlighting the critical importance of strategic flexibility as an instructional outcome for school mathematics. Star (2005; Star & Seifert, 2006; Star & Rittle-Johnson, 2008) defines strategic flexibility as knowledge of multiple approaches for solving mathematics problems and the ability to select the most appropriate strategy for a given problem.

Consider the domain of linear equation solving. What does it mean to be a flexible solver within this domain? A standard algorithm exists for solving linear equations; this algorithm is often explicitly taught as the optimal approach. For an equation such as $3(x+1)=9$, the standard algorithm would involve first distributing the 3, then collecting and isolating like variable and constant terms to opposite sides of the equation, and finally dividing both sides by 3 to solve for *x*. However, this is not the only strategy for solving linear equations; for this particular equation, it may in fact be more efficient to divide both sides by 3 as a first step. A flexible solver not only knows both strategies but also chooses to use the more efficient approach on this problem. This choice reflects expanding knowledge of *when* the divide step is appropriate to use. In addition, if the problem were altered slightly to $3(x+1)=10$, a flexible solver (particularly a middle school student) might realize that dividing by 3 on both sides, though possible, might not be the optimal solution method, since 10 is not evenly divisible by 3. Flexible strategy knowledge reflects better *conditionalized* knowledge of when to use strategies.

Flexibility as an important outcome is alluded to in several recent policy documents, including the National Research Council's [NRC] "*Adding It Up*" report (2001), the *Curriculum Focal Points* from the National Council of Teachers of Mathematics [NCTM] (2006), and the recently issued report from the National Mathematics Advisory Panel (2008). Flexibility also appears to have a strong metacognitive component. Flexible solvers engage in metacognition when they think critically about a problem and choose to use a more efficient or effective solution strategy to solve it, when they compare multiple solution methods to problems and note why one seems better than another, when they tap into their repertoire of multiple solution procedures, and when they even realize an opportunity or need for efficiency.

*The Development of Flexibility*

Drawing on the literature in cognitive science and mathematics education, Star and colleagues have identified comparison as a particularly effective means for promoting the development of flexibility. Rittle-Johnson and Star (2007) found that middle school students who learn to solve equations by studying multiple worked examples, presented side by side, become more flexible than students who see the same examples but presented one per page. Similarly, Star and Seifert (2006) found that students who were asked to solve previously completed equations using a different ordering of solving steps become more flexible in their knowledge of equation solving strategies. These results from Star and colleagues on the benefits of comparison can be summarized by the following three instructional practices that have been found to positively impact students' strategy flexibility in mathematics.

First, research on comparison indicates to-be-compared solution strategies should be presented to students side-by-side, rather than sequentially. Side-by-side placement allows for more direct comparison of solution strategies and facilitates the identification of similarities and differences between strategies. A side-by-side comparison helps students notice and remember the features that are important to each or both solution strategies (Rittle-Johnson and Star, 2007).

The second practice is for teachers to engage students in comparison conversations. Discussion of and comparison of multiple strategies helps students justify why a particular solution strategy or solution step is acceptable and helps students make sense of why certain strategies are more efficient than others for particular problems (Silver et al., 2005). Teachers can help guide comparison conversations to ensure that students are able to make connections among strategies that they would not always be able to make on their own.

The final recommended practice is to provide students with the opportunity to generate multiple solution methods to the same problem, either by investigating multiple solution methods of the same equation or by creating new equations to solve by a given method. In general, knowledge of multiple solution strategies seems to help students more readily consider efficiency and accuracy when solving problems. Additionally, by generating multiple solution methods, students are encouraged to move away from using a single strategy and, rather, other, possibly better strategies that work for the problem (Star & Seifert, 2006; Star & Rittle-Johnson, 2008).

*The Present Study*

The goal of the present study was to design and pilot a professional development activity for inservice secondary mathematics teachers, focusing on improving teachers' flexibility via the three comparison practices described above. Our professional development activity had two goals. First, we sought to make teachers aware of the three comparison practices described above. Research has linked these practices to students' flexibility, so our primary goal was for participating teachers to learn how to implement comparison in their own classrooms.

However, the success of our efforts to change teachers' practices would clearly be dependent on the flexibility of the teachers themselves. For example, teachers attempting to orchestrate a comparison conversation with a group of students would be better able to direct the conversation along a fruitful path if they understood the nuances of different solution methods and problems, e.g., if they were flexible. Furthermore, a flexible teacher has the ability to spot a potentially interesting and innovative approach to a problem during a class, and will offer it for classroom discussion to highlight the specific aspects of this unique approach. Conversely, less flexible teachers, those with only a superficial knowledge of procedures, may have the tendency to teach only one method for solving particular problems, and disregard students' innovative solution methods as unimportant digressions. Thus, a secondary goal of our professional development activity was to impact participating teachers' flexibility, by implementing the comparison practices in the professional development. Teachers must themselves see value in flexibility before they will regard it as an important instructional outcome for their students—and in order for flexibility to be valued, teacher participants must be consistently presented with problem solving situations in which they can develop and exercise their own flexibility.

## Method

In June of 2007, a two-week professional development institute for 24 middle and high school algebra teachers was held at California State University, Chico. The teachers were participants in a 5-year project; the results reported here are from the first year of the project. The focus of the professional development institute was algebraic reasoning and pedagogical strategies for use in algebra classrooms. The professional development activity focusing on comparison was implemented during one eight-hour day.

The comparison practices were introduced to the teachers by giving a brief presentation on the notion of flexibility and the comparison practices. This introduction was followed by a series of problem solving activities where groups of 3-4 teachers were given two similar math problems (P1, P2) and two suggested strategies (S1, S2) for solving the problems and asked to solve both problems using both strategies. Teachers were then asked to create a poster with all four combinations of problem and strategy (P1S1, P1S2, P2S1, P2S2) that they would then present to the remaining participants. The purpose of the presentations was to model the first two comparison practices -- that is, to present different solution methods side-by-side and to facilitate comparison conversations among the other participants. This activity format was used several times over the course of the two-week institute. Below we show the various problems and strategies that groups of teachers were asked to present. They will be referred to hereafter as Topics 1-7.

1. Systems of Equations: Solve the following systems by (S1) substitution, and (S2) linear combinations:
   (P1) $\begin{array}{l} 4x-3y=2 \\ 2x+5y=8 \end{array}$ (P2) $\begin{array}{l} y=3x-2 \\ 5x+2y=8 \end{array}$
2. Linear Inequalities: Find the solution sets for the inequalities by (S1) moving the variable to the right-hand side of the inequality and (S2) moving the variable to the left-hand side of the inequality:
   (P1) $3-5x>10$ (P2) $x \geq 3x-2$
3. Solving Proportions 1: Solve for x by (S1) using cross multiplication, and (S2) multiplying both sides of the equation by a single value:
   (P1) $\frac{2}{x}=\frac{16}{5}$ (P2) $\frac{x}{6}=\frac{5}{9}$
4. Solving Proportions 2: Solve for x by (S1) using cross multiplication, and (S2) comparing the ratio of the numerator to he denominator:
   (P1) $\frac{x}{14}=\frac{16}{8}$ (P2) $\frac{x}{14}=\frac{3}{8}$
5. Finding Linear Equations: Determine the equation of the line passing through the two points by (S1) using the slope-intercept form of the linear equation, and (S2) using the point-slope form of the linear equation:
   (P1) (0, 4) and (5, –2) (P2) (–2, –2) and (6, 1)

6. Simplifying Fractions: Simplify the expressions by (S1) dividing numerator and denominator by successive common divisors, and (S2) writing out the prime factorization of the numerator and denominator:
   (P1) $\frac{38}{98}$ (P2) $\frac{2a^2b}{6ab^3}$

7. Finding the Least Common Multiple: Find the LCM by (S1) generating a table of multiples, and (S2) writing out and using the prime factorization:
(P1) 18, 30 (P2) $4x^2$, $x^2 + x$

The data reported below was taken from recollections of the facilitators but also from teachers' responses to a written open-ended survey administered at the conclusion of the comparison activity and discussion. This survey contained two prompts: "Reflect on the comparison activity in regards to your teaching," and, "Reflect on the comparison activity in regards to your own mathematical ability and understandings."

**Results**

We describe the discussions that took place during the professional development comparison activities in order to illustrate that using comparison with teachers gave insight into teachers' own flexibility, and moreover helped to develop an appreciation for flexibility as an instructional outcome. As we point out, these discussions were both mathematical and pedagogical in nature.

*Observing the Activity*

*Topic 1: Systems of equations*. This topic was chosen since it is a very familiar topic within which teachers would see value in using comparison techniques to teach flexibility. Many indicated they had already used modified comparison techniques when teaching solution methods for systems of equations, noting that the typical textbook teaches solution methods as totally separate, rarely indicating a connection between any two. However, many teachers noted while they present different solution methods, they do not allow their students to have conversations about the solution methods, nor do they let them have a choice as to which method to use to solve a given problem.

In this specific example, teachers found that the set of equations that featured an equation in slope-intercept form is easier to solve by substitution then the other system, and more cumbersome to use for elimination due to the re-arranging of the equation that had to come first. Many remarked that this example addressed a difficulty for students; if a student does not properly rearrange an equation for setting up the elimination method (i.e. having all variables to one side and lined up in order of *x* terms and *y* terms, for example), or for setting up the substitution (placing the equation into slope-intercept form), then they will incorrectly solve the problem. Many teachers noted that literally having side-by-side comparisons of solution methods might help students see that elimination is often more efficient when the least common multiple of either the coefficients of the *x* terms (or the *y* terms respectively) is easy to find.

*Topic 2: Linear inequalities*. One of the most fruitful of the comparison discussions concerned solution methods for solving linear inequalities. The intent of the two problems and the two solution methods in Topic 2 was to emphasize that problems involving inequalities can be solved by moving the variables to either side of the inequality, as opposed to the more common way students are taught to solve equations, which involves isolating the variable on the *left* side. We had hoped to show that the potential for error when dividing by a negative in working with inequalities and of forgetting to 'flip' the inequality could be taken care of by being flexible in working with inequalities. Indeed, this is what the discussions indicated.

During the discussion of the solution methods in Topic 2, several teachers commented on the fact that they always ask students to move the variable to the left side of the inequality. In fact, many of the same teachers admitted their own difficulties in trying to solve the problems by moving the variable to the right side first. Such comments clearly showed that throughout the activity our teachers were forced to consider their own flexibility in solving problems. A significant portion of this discussion was focused on the difficulty with the negative sign depending on which side of the inequality the variable was on. For instance, when solving the inequality, $3-5x \geq 10$ , by moving the variable to the right, $3-5x \geq 10 \Rightarrow 3 \geq 5x+10$ , the coefficient of the variable is positive, and so there is no need to worry about making mistakes in dividing by negatives and forgetting to flip the inequality. Of course, the ability to do this requires flexibility in understanding how to read inequalities in both directions.

*Topic 3: Solving proportions*. Our intent in Topic 3 was to allow our teachers to discuss the strategy of cross-multiplication for solving proportions and to compare it with other strategies. Many teachers report that cross-multiplication, although a useful strategy for solving proportion problems, is often incorrectly used by students who lack an understanding of why it is a valid strategy. Moreover, the overreliance of students on this strategy indicates a lack of flexibility in their understanding of what a proportion can represent, and subsequently of the lack of tools for solving such problems. One possible strategy for solving Topic 3 Problem 1 $\frac{2}{x} = \frac{16}{5}$ by multiplying both sides of the equation by a single value would be to multiply both sides of the equation by $(5x/16)$. Our teachers needed some instruction in this solution strategy, which they indicated was not completely intuitive to them. In fact, teachers reacted quite negatively to this strategy, both in terms of how students might view this approach ("We don't expect our students to be able to do that"), and also in terms of their own comfort with and willingness to use this strategy ("that seems like too much work"). More generally, in the discussions of both Topics 3 and 4, our participants revealed their own reliance on cross-multiplication as the 'best' strategy to use when solving proportion problems, and perhaps a little reluctance for investigating other means of solving them.

*Topic 5: Finding linear equations*. In presenting this topic, we wanted to learn about our participants' flexibility in finding linear equations. In particular, finding the equation in P1 (see above) is perhaps easiest by using the slope-intercept form (S1), since one of the given points is the *y*-intercept. Our teachers picked up on this and indicated so in the discussions. Many of our teachers seemed more in favor of using the slope-intercept form to solve either case, by first finding the slope, then substituting the coordinates of one of the points for *x* and *y*, and lastly solving for *b*. In using the point-slope formula strategy (S2: using $y - y_1 = m(x - x_1)$ ) to solve P1 and P2 of Topic 5, many teachers commented that it takes just as much work and just as many steps to use it to solve Problems 1 and 2. In other words, in their view, the fact that the point $(0,4)$ is given in P1 does not offer any advantages when using the point-slope formula. But perhaps surprisingly, some participants were more in favor of the point-slope formula in both cases. For example, one person remarked, "Many students like the point-slope form because there's no *y*-intercept. They don't need to solve an equation [referring to solving for *b*]. It takes less steps. [Students often wonder,] "What does it really mean that I'm solving for *b*?" This comment suggests that teachers feel that students can use the slope-intercept form of the equation of a line without understanding what solving for the variable *b* means in the context of the problem.

*Topic 6: Simplifying fractions*. The discussion of Topic 6 served as an exemplar of how solving problems side-by-side and directly comparing the solution methods can lead to a better understanding of the mathematics involved and when different methods are appropriate. In this topic the relationship between factoring and then cancelling common factors in the numerator and denominator, and "successively dividing" numerator and denominator by the same values are examined as two different methods for simplifying fractions. The impetus for such an exploration comes from seeing the 'cancelling factors' strategy appear in the text when solving fraction simplification problems, with no real explanation as to why it is a valid solution strategy.

In the case of simplifying the fraction found in P1, $\frac{38}{98}$, it was more or less obvious to teachers that each number is even, so a common factor of 2 may be 'divided out' from numerator and

factors of the numerator and denominator share. When participants saw the differing solution methods to P1 side-by-side,

$\frac{38}{98} = \frac{19}{49}$ and $\frac{\cancelto{19}{38}}{\cancelto{49}{98}} = \frac{2 \cdot 19}{2 \cdot 49} = \frac{2}{2} \cdot \frac{19}{49} = 1 \cdot \frac{19}{49} = \frac{19}{49}$,

they were better able to draw the connection between "canceling out" common factors and factoring. The observation was made that the process of canceling out a common factor is covering up several steps involving factoring and division. But when shown side-by-side, connections can be drawn between the two. If armed with the flexibility to rewrite the expression in P2 as $\frac{2 \cdot a \cdot a \cdot b}{2 \cdot 3 \cdot a \cdot b \cdot b \cdot b}$, and then to divide out common factors, a student may have better success in solving such a problem. In this example, teachers saw the importance of teaching multiple methods and drawing connections between the two. Again, the connection between factoring and dividing common factors and "canceling out" common factors was more readily seen when solution methods were presented side-by-side.

On the pedagogy side of the discussion, the presenting teacher's reference to the solution by cancelling common factors as "cross-cancelling" became a source of interest. What exactly does the word "cross" refer to in this method? Is it the misuse of a word? A conflation of meanings? Does "cross" refer to the "crossing-out" procedure used once the numerator and denominator are factored? One person remarked that in general, "cancelling out kind of 'sounds like' you should get zero." Here, teachers were trying to rectify the language commonly used with the mathematical steps involved in the problem solving method. Such pedagogical talks sparked specifically by comparison discussions were common.

*Topic 7: Finding the least common multiple*. The final topic was chosen as an illustration of the potential disconnect between a common method for finding the least common multiple [LCM] of integers and the method later commonly taught for variable expressions. In particular, many students first construct lists of the multiples of the two integers and then find the first number that appears on both lists as the LCM. However, this method has its limitations when applied to two variable expressions such as $4x^2$, $x^2 + x$. Consequently, a method involving prime factorization and then finding common factors must be introduced as students progress to such expressions.

The participants saw that when they tried to apply the "listing" method to the variable expressions, it was not clear how they would actually do so. For instance, participants started by writing a list such as $\{4x^2, 8x^2, 12x^2, \ldots\}$, and quickly realized that they would need to factor in more factors of *x* as well, as in $\{4x^2, 4x^3, 4x^4, \ldots\}$. Eventually, participants saw that it would be nearly impossible to write an exhaustive list of all the varying multiples of $4x^2$, and furthermore they were uncertain how to organize their lists to obtain the LCM of $4x^2(x+1)$. This example illustrated the need to understand the connection between the procedure of listing factors, used primarily with numbers, and the prime factorization method that is taught for finding the LCM of variable expressions. Once more, when teachers had a chance to explore the content they teach and the methods they know in the context of comparison, they were able to challenge their own understanding of concepts and procedures in new ways.

*Teachers' Comments on the Post-Activity Survey*

Recall that participants were asked to complete a written survey following the professional development activity. Teachers' responses generally indicated an increasing appreciation of the potential of comparison for improving students' flexibility. One teacher noted, "If students look at several ways of doing the same problem, they can start to generalize what's really going on." Similarly, another teacher noted the potential power of the comparison conversation, noting, "[The discussion] is a great tool to get students to defend their ideas and explain their reasoning." One teacher began to see how comparison could be used to help students review and consolidate material at the end of a unit: "If I were to use this comparison as a review or a recap of the concepts, the students would then be able to engage in fruitful conversation about the various methods." In addition, another teacher noted how comparison could be used for students to check their work: "Comparison would also be a way for students to check their own work, because they should get the same answer for both methods."

However, other teachers noted the challenges of implementing comparison effectively in their own teaching. As one teacher noted, "Comparison is a tool I have already used in various lessons, but I do not question my students successfully. I tend to want to lecture and give them my comparisons instead of asking them what they notice." More generally, teachers seemed especially daunted by the difficulties of facilitating a classroom discussion where students would be allowed to share their own thoughts when comparing multiple strategies. Several teachers noted their own tendency to do most of the talking in their classrooms and their trouble with allowing students to discuss ideas. Another concern noted by teachers was whether students might be confused by seeing multiple methods; as one teacher noted: My worry is that some students will be confused if I introduce more than one way to solve a problem on the same day."

In addition to commenting on the potential impact that comparison can have on students' knowledge, teachers also commented on the ways that the professional development activity influenced their thinking about their own teaching and learning of mathematics. One teacher noted, "I learned that in my own thinking and strategic competence that I already have a mental map of comparison strategies which helps me quickly decide upon a certain strategy to solve a particular problem…[The discussion] allows students to take ownership of their own learning." Similarly, another teacher noted, "In all of the textbooks, there is the sequential set of examples and students often become confused… I know that when I was learning math, I often fell back on one way of solving a problem. I think this did not allow for a better understanding of the topic because I was so focused on one solution method. This one-way method put up a sort of roadblock in my understanding." Similarly, another teacher noted, "I realize that intuitively I choose a method that is best/most efficient/easiest for me when I work on the board, but I have never taken the time to express why or even let the students suggest why."

## Discussion

The results of this small study suggest that using comparison in a professional development institute can provide teachers with an adaptable instructional tool as well as a chance to examine their own flexibility as problem solving. Survey results suggest that introducing comparison techniques to teachers increased awareness of student flexibility. As a result, teachers began to see that flexibility is a valuable instructional goal that can be incorporated into their curriculum. In addition, teacher discussions during the professional development indicated that teachers expanded their own flexibility. Teachers reported that they were challenged to see the connections between different solution methods, and subsequently questioned why they taught a certain solution method over another. The current study suggests that the three recommended instructional practices described above represent a practical way to begin to teach for flexibility in the algebra classroom.

Future studies on the use of comparison with teachers should focus on effects of teachers' use of comparison in their classrooms, including direct observations of teachers' practices to determine when and how teachers are implementing the practices described here. In addition, a more detailed study of the correlation between mathematics teachers' knowledge of multiple strategies and the effectiveness of their use of comparison in the classroom would be educative. Furthermore, the work of Star and colleagues (e.g., Rittle-Johnson & Star, 2007) suggests that simply changing the method of presentation to side-by-side is not enough; students must have the opportunity to engage in comparison conversations and to try to derive their own problem situations for comparison to be effective. Hence, a comparative study on the impact of comparison in the classroom both with and without the discussion component would more strongly inform teacher best practice for developing flexibility using comparison.

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# THE DESIGN OF DIDACTIC OBJECTS FOR USE IN MATHEMATICS TEACHERS' PROFESSIONAL DEVELOPMENT

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*In this paper we provide an analysis of the work of a video-based case development team whose goal was to produce didactic objects to be used the professional development of secondary mathematics teachers. In order to generate artifacts for use in the creation of the cases, the research team conducted a classroom intervention in an Algebra I classroom. The daily videotapes, copies of all the student work, and interviews with the teacher comprised the resources for the case development effort. As design researchers, we engaged in interactions of design and research as we tested and refined our development efforts. An important aspect of the work is its focus on the unifying mathematical concept of covariation.*

## Introduction

In this paper we analyze the work of the Case Design Project [Cadept][1] that is part of the TPCC [Teachers Promoting Change Collaboratively] [2] Project. As background, the larger TPCC project entails multiple stages of research and development employing a design research perspective (cf. Brown, 1992; Cobb, Confrey, diSessa, Lehrer & Schauble, 2003). Following this design orientation, the TPCC research team first addressed the need for a strong mathematical basis for teachers by engaging them in a series of three graduate-level courses called Extended Analysis of Functions [EAF]. The mathematical content of the EAF courses focused on developing a coherent understanding of the secondary mathematics curriculum from the complementary perspectives of (1) functions and quantitative relationships (with covariation being a foundational idea for both) and (2) representational equivalence. In addition, these courses were designed and implemented as a model of the type of interactions and discussions that supported the mathematical thinking being developed during the course.

Next, the work of the grant was extended to the school setting as groups of teachers enrolled in the EAF courses met weekly in the format of Professional Learning Communities [PLC's][3] with the goal being that of reflecting on practice as it related to the big mathematical ideas of the courses. Our intent was that PLC meeting agendas would be tightly linked with issues that emerged in the EAF courses. As a result, the relevance of the issues to the teachers' classroom practices provided the link between the courses and teachers' classrooms (cf. Zhao & Cobb, 2006). Each PLC was assigned a facilitator from the TPCC project with the expectation that within a three-year period each PLC would become self-facilitated. The appointed facilitator initially set the meeting agendas and conducted these meetings. The means of support used to initiate teacher reflections typically included (1) teacher developed student interviews, (2) Japanese-style lesson study or (3) sharing a self-recorded video of a teacher in the PLC teaching a particular lesson in his classroom.

During the preliminary analysis of both the EAF courses and the PLC's, the research team realized that its work was not supporting the teachers' ability to formulate an *image*[4] of the practices that were being promoted in the project. The teachers also had difficulty imagining the kind of interactions that would support students in understanding the big mathematical ideas from the EAF courses. Examples of teachers' difficulty in understanding the ideas promoted in the project emerged particularly during PLC sessions. When discussing student interviews the teachers conducted, their focus was on students' answers and not how the students thought about obtaining their answers. When a teacher would share a video recording of a teaching segment from her classroom, other teachers' were (1) either hesitant to share their opinion so as not to offend the teacher or (2) focused on classroom management issues. In their discussions, little focus was placed on student thinking.

In order for the TPCC research team to better understand these perceived difficulties and then create conversations that would address the difficulties, it decided to generate artifacts for use in the professional development settings. The goal was to create video-based cases in which teaching took the form of a long-term coherent approach to significant mathematical ideas in a classroom setting where students' current ways of reasoning were at the forefront of decision making and planning. As a result, the Case Development Project [Cadept] was developed. The goal of Cadept was twofold. First, the members of the Cadept design team wanted to create potential didactic objects[5] (cf. Thompson, 2002) that could be used with teachers to reflect on teaching in relation to student learning; and second, these objects needed to provide comprehensive understandings of the struggles teachers encounter as they attempt to implement what they understand to be the big mathematical ideas in their classrooms.

In order to generate the artifacts necessary for creating the potential didactical objects, the TPCC research team determined that it needed to conduct a classroom intervention[6] with one teacher in order to produce a record of her attempts to teach a conceptually oriented course. The team selected a ninth-grade Algebra I course for non-honors students —students with whom the team would later work in Geometry and Algebra II. The teacher, whom we call Augusta[8], was a full participant in the classroom intervention. Augusta was chosen as the teacher for the experimental classroom because she was comfortable taking risks and trying out instruction for which the eventual outcome was unclear. She also was willing to collaborate with the TPCC research team in the process of designing the course. In addition, Augusta's principal was eager to have this project in his school. This, therefore, removed some potential institutional constraints. During the year of the intervention, each class session was videotaped for two purposes: for our own understandings of the struggles that teachers face, and for potential use in generating artifacts.

The development of the cases was an iterative process of ongoing analysis, modification and refinement. Much like Simon's (1995) *Mathematics Teaching Cycle* the TPCC research team engaged in both *meta* and *mezzo* levels of design and revision during which it focused on both the design of the professional development courses for the teachers and the design of activities for the classroom.

Against this background, we next document the evolution of the need for the classroom design intervention. We follow by documenting the research and design cycle that was employed in the development of video-based cases from Augusta's classroom. We then give a summary of the current state of our work. We conclude with an analysis that provides implications of our work for other university collaborators and the field at large.

## The Evolution of the Need to Design a Classroom Intervention

As noted earlier, the need to generate classroom artifacts emerged from ongoing analyses of work in both the EAF courses and the PLC's. Further, the teachers' curricular knowledge—their understandings that corresponded to textbook material they felt compelled to teach and their image of problems that students must know how to solve—overwhelmed their ability to imagine teaching a series of lessons that developed ideas relationally, coherently, and longitudinally independent of their text. As a result, the research team decided to conduct a classroom design intervention with one teacher to produce a record of that teacher's attempts to teach a conceptually oriented course.

It was conjectured that the record of Augusta's classroom would provide a data source for us to use in documenting the process of both the teacher and her students' learning conceptually oriented mathematics. In addition, the team conjectured that the struggles emerging as part of this learning would also be documented. As a result, our design was focused on (1) Augusta's reconceptualization of Algebra I, (2) students' mathematical learning, (3) appropriate instruction to teach what Augusta reconceived so that students could learn it, and (4) the means of support for Augusta's transformation.

## The Artifact Collection Process

In addition to the daily-videotaped classes, the TPCC research team also created an electronic record of the lesson designs for the year, videotaped daily debriefing sessions with Augusta after the class period, audio recorded weekly collaborations with Augusta, made copies of all student work, and videotaped student interviews with the research team. This extensive data corpus not only provided the resources for use in understanding the difficulties associated with teaching conceptually oriented mathematics, but it also provided artifacts that could be used in the design of potential didactic objects. As a result, the Cadept team's initial design conjectures for the artifacts was focused on (1) instances of Augusta's coming to conceptualize an instructional sequence to promote students' mathematical learning, and (2) means of support for Augusta's transformation. It is therefore important to note that the intention of the design was not to focus on Augusta per se; but rather on the generation of artifacts which could be used to focus other teachers' attention on Augusta's reconceptualization of her teaching practices. The motivation for focusing the design of the case study on Augusta's reconceptuatlizations and teaching practices were based on the observations made from the PLC and EAF courses.

This was a highly interventionist and time-intensive process. As part of this process, frequent exchanges occurred between Patrick Thompson and Augusta both after class and during their Saturday planning sessions. The goal of these exchanges was to support Augusta's ability to reason logically with the innovative materials while using the student's ways of reasoning as an important aspect of planning. In addition, these meetings assessed the effectiveness of the materials in developing student thinking and Augusta's understanding of these materials. The meta-level goal of these exchanges was to gain insight into Augusta's difficulties as she was teaching with these innovative materials while getting her input into subsequent design.

## The Design of Video-based Cases

As the TPCC research team reflected on the unfolding "story" from Augusta's classroom and on the changes in Augusta during the teaching of the Algebra I course, it saw the video as a potential source of didactic objects for professional development. The team determined that the video would make a compelling case for other teachers. In particular the research team identified appropriate instances from the classroom video to choose as segments that would form the basis of the cases. The Case Design Project [Cadept] was therefore developed to create a series of video-based artifacts to be used in a professional development setting. These video artifacts would be part of a larger package of artifacts that included problem sets for the teachers, curriculum critique and development, and analysis of student work. These materials were being designed to provide opportunities for the teachers to reflect on their practices by examining Augusta's classroom.

The initial exploration of the data yielded six potential cases: (1) covariational reasoning, (2) linear functions, (3) systems of equations, (4) sums of functions, (5) factoring and polynomials, and (6) quadratics. As the design team worked, each case required condensing the classroom video into sequences of short video stories that could be supported with additional resources from Augusta's classroom. The video was edited both to make these stories of practical viewing length, and also to emphasize specific plots. These plots involved the students struggling with mathematical ideas, the teacher struggling with implementing those ideas, the development of discourse in the classroom, and the cognitive development of the students including significant mathematical benchmarks and shifts.

## Pilot Studies as Part of the Design Cycle

As noted, the design team took a *design research* perspective in its development process. As a result, selected video segments were piloted with teachers throughout the development process. For example, the third EFA course served as one pilot study. The goals of this study were to draw teachers' attention to student thinking and the subtleties of covariational reasoning and instruction. Initially, the teachers in the functions course did not focus on content nor student understanding. Their original focus was on Augusta's classroom management. Their assessment of the success of the lesson was directly related to how well the students' behaved. Moreover, the teacher's focus was on Augusta rather than on the students she was teaching to. They did not discuss the students' thinking, nor notice the role of Coordinating Quantities Tool[8] in the lesson. They viewed the tool as a "nice activity." When they broke into groups to watch individual video clips, their discussions indicated that they did not have a theory of learning or a notion of an epistemic student. It was only after discussion and probing by Patrick Thompson (the teacher of the functions course) that the teachers attempted to focus on student thinking. As a result, were able to articulate evidence as they built models of student thinking. As an example, they were able to examine one student's use of the Coordinating Quantities Tool to make conjectures about her ways of reasoning about the coordination of the two quantities. Also, the teachers' image of covariation changed. They shifted from shape thinking[9] to covariational reasoning.

Throughout the study, it was apparent that simply changing the curriculum or improving teacher's content knowledge would not provide a sufficient stimulus for change. These issues must be addressed in the context of exploring classrooms (cf. Zhao, 2007). Zhao makes a strong argument for the necessity of "conceptualizing the relations between teachers' learning in the setting of professional development and their instructional practices in the classroom" (p. 3). She argues that

> [r]egardless of researchers' continuous efforts to design and support teachers' professional development, changes in classroom mathematics instruction do not always occur as intended. Thus, an immediate and pragmatic challenge posed to

> teacher educators necessarily involves how to design professional development activities so that teachers can relate what they learn to their classroom practices and, as a result, become willing to engage in changing their current ways of teaching. (p. 4)

In order to address this conundrum, the design team therefore focused its pilot efforts on understanding the relation between the classroom-based video artifacts and the teachers' reactions with respect to their practice.

As a result, the research team constructed an epistemic model of teachers' images of the classroom. The model included the fact that teachers would not attend to student thinking without readily available evidence and someone pressing them to hold to that evidence. Further, the text emerged as the dominant resource for planning. Also, possible distracters in video emerged. For example, teachers paid more attention to classroom management issues than to the intended focus of the video segments. The epistemic model of teachers' image of Augusta's classroom was used as a factor in choosing the story that was to be told of the important issues to be discussed around the video based cases. As a result, selected video segments were continually piloted throughout the development process. The iterative process was crucial in the success of our final design.

## Results of Analysis, Conclusions and Implications

Numerous scholars in the field of mathematics education have advocated the importance of teachers having strong knowledge of the content they teach (cf. Ball, 1990; Bransford, Brown, & Cocking, 2000; Grossman, 1990; Ma, 1999; National Research Council, 2001; Schifter, 1995; Sowder, et al., 1998). This sentiment is echoed in the *No Child Left Behind* legislation that articulates a demand for highly qualified teachers who display mastery of subject matter. There is, in fact, general agreement in both the political and educational arenas that knowledge of content is a necessary condition for an effective mathematics teacher.

However, we have learned that this knowledge is necessary, but not sufficient. Being able to take newly acquired knowledge and transpose it into a new image of teaching is challenging at best. As we have noted, teachers must also develop *images of good teaching*. These images must be grounded in the teaching of significant mathematics where student thinking guides instructional decision-making. Here we have argued that the investigation of a well-designed video-based case can provide the context in which to make explicit the complexities involved in innovative mathematics classrooms. In doing so, we provide a context in which to examine the use of video-based cases in supporting teachers' professional growth, including their understandings of issues of both mathematical content and pedagogy. It is in this context that opportunities for teachers to reflect on their teaching practices arise.

However, investigations of classrooms provide both potential resources and pitfalls. Teachers view classrooms through the lens of their prior beliefs, thereby negating any efforts for issues of teaching and learning to be made explicit through their observation. For this reason, teachers' discussions of classrooms often take on the characteristics of "storytelling" during which the teachers in professional learning communities share their interpretations of accounts from the classrooms. The judgments they make about what they observe and experience can become traps that prevent professional growth. Overcoming this can be a formidable task. We therefore cannot assume that the issues that are focused upon during collaborations will be made explicit and then acquired naturally through teaching.

However, the effectiveness of video-based, multi-media cases has been documented by

Richardson and Kyle (1999) who state that "the use of multimedia cases significantly impacts teachers' cognitions" (p. 131) They note that the power lies in the cases' ability to present "a visual, moving picture of teaching in a real-life classroom" (p. 136). Video-based cases also allow easy access to numerous facets of the classroom to facilitate in-depth investigation of issues of content, the teacher's decision-making process and students' diverse ways of reasoning. Through their investigation and critique of a case, teachers have the opportunity to develop and refine their skills in critiquing, evaluating and creating learning experiences. Their image of teaching is changed as a result.

The critical aspect of this process is the guiding and framing of the experience by the facilitator. Just as we view the role of the teacher as critical in supporting students' developing understandings of mathematics (or any other content area), we view the role of the facilitator as critical in supporting teachers' understandings of what it means to teach mathematics effectively. We do not believe that the cases are transparent carriers of meaning. Nor do they have agency. They are, in fact, tools to be used in the course of teacher collaborations (cf. Kaput 1994; Miera, 1998; van Oers, 2000). The goal is then to create the settings in which these cases can become genuine didactic objects. For this reason, our next cycle of design and research will focus on the development of facilitators' guides. However, like Carpenter and colleagues (Carpenter, Blanton, et al), we do not believe that forms of professional development can be codified and handed over as a means of scaling up. Therefore, the next steps in our design process will involve cycles of design and revision while working closely with other university collaborators.

Although our process is still ongoing, we claim to have documented evidence to support the following guiding principles:

1. *The thoughtful design of a video-based case is essential in creating effective means of supporting teacher professional growth and development because it provides a bridge between the professional development setting and the classroom.*
2. *Video-based cases must support the larger goals of any collaboration.*
3. *The strength of a video-based case is limited by the quality of instruction and the nature of the student discourse captured in the video.*
4. *Video-based cases can only become didactic objects when thoughtful consideration has been given to their design and use.*

In order for the cases to meet these guiding principles and therefore support teachers' ability to re-conceptualize their practice, they need to provide resources to support the teachers' construction of an image of a conceptually oriented mathematical conversation with students. A conceptual conversation is one that has a diminished emphasis on technique and procedure while having an increased emphasis on images, ideas, reasons, goals, and relationships. People conversing conceptually speak in ways that make their meanings, ideas and ways of thinking clear to others in the conversation. To avoid speaking in ways that could possibly hide their meaning, these individuals are aware of possible interpretations of their words another may have which are different from the meaning that they intended. The design, testing and refinement of our cases and the supporting material can therefore provide this opportunity. This is significant in that it offers a means of supporting teachers' transitions in professional development setting.

## Endnotes

1. The Case Design Project Team [Cadept] is composed of Kay McClain, Scott Adamson, Ted Coe, Carlos Castillo-Garsow, Sharon Lima and Patrick Thompson.

2. The research team is composed of Patrick Thompson (Principal Investigator), Scott Adamson, Ted Coe, Carlos Castillo-Garsow, Sharon Lima, and Kay McClain. Research reported in this paper was supported by National Science Foundation Grant No. EHR-0353470 under the direction of Patrick W. Thompson. Any conclusions or recommendations stated here are those of the authors and do not necessarily reflect official positions of NSF.

3. We use the term Professional Learning Communities to denote the cohorts of teachers within the schools who met on a weekly basis to discuss issues related to the college course. An analysis of the development of the cohorts into communities is beyond the scope of this paper. We therefore realize that we are taking liberties with the term *community* and do not intend to imply that we have conducted analyses to confirm that these cohorts actually transformed into communities (cf. Dean, 2005; Wenger, 1998).

4. By "image" we build from what Maturana (1978) describes as a conceptual system through which we may anticipate another system's behavior. These images are highly related to what Cobb has in mind when he speaks of an envisioned practice as a goal of instructional design.

5. Elsewhere, we have used the phrase didactic object to refer to "a thing to talk about" that is designed with the intention of supporting reflective mathematical discourse (see Thompson, 2002). In doing so we note that objects cannot be didactic in and of themselves. Rather, they are didactic because of the conversations that are enabled by someone having conceptualized them as such. In this sense, a didactic object is a tool, but one designed to produce desirable conversations.

6. We make a distinction between a classroom intervention and a classroom teaching experiment or a classroom design experiment. In the intervention, the goal of the TPCC research team was to elicit certain ways of reasoning and certain struggles from both the teacher and the students.

8. The Coordinating Quantities Tool (or finger tool) makes use of the index finger on each hand by asking students to track the changes in the quantity of the independent variable with a horizontal movement while simultaneously tracking the quantity of the dependent variable in a vertical movement.

9. Thompson makes a distinction between "shaping thinking" and covariational reasoning. In shape thinking, students can imagine the shape of a graph from the scenario such as the distance of a bungee jumper from a bridge as he bounces back and forth. The graph is then a static trace of an event that has occurred. Covariational reasoning requires the student to think about how two quantities vary in relationship to each other or co-vary.

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# DEFINING VISIONS OF HIGH-QUALITY MATHEMATICS INSTRUCTION

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*By synthesizing what has been learned with regard to critical dimensions of mathematics classroom teaching and learning, and investigating the ways teachers and other school personnel characterize high-quality mathematics instruction, this study defines the notion of 'instructional vision' and provides an initial categorization scheme. Motivating this work is the need to reliably document change in participants' instructional visions within an ongoing study of the institutional setting of mathematics teaching, in which we hypothesize that improvement in teachers' instructional practices and student achievement will be greater in schools where teachers and instructional leaders have a shared vision for high-quality mathematics instruction.*

## Background

The Middle School Mathematics and the Institutional Setting of Teaching (MIST) research team is working for four years with four urban school districts serving ethnically and economically diverse populations as 'co-designers' of support structures and strategies for meeting ambitious goals for reforming mathematics instruction at a district level. All four of our participating districts have recently formulated and begun implementing comprehensive initiatives for improving the teaching and learning of middle-school mathematics, including the adoption of mathematics reform curricula and the provision of professional development aimed at developing instructional practices in which teachers place students' reasoning at the center of their instructional decision making.

Our aim is to investigate, test, and refine a set of conjectures and formally test a set of hypotheses about support structures that potentially enhance the impact of professional development on middle school mathematics teachers' instructional practices and student achievement. Our conjectures and hypotheses pertain to seven sets of support structures: 1) teachers' professional networks; 2) shared vision for high quality mathematics instruction across teachers and instructional leaders; 3) quality of instructional leadership; relationships of 4) accountability and 5) assistance between teachers and instructional leaders (including principals, assistant principals, department chairs and mathematics coaches) and among teachers; 6) alignment across district units with respect to high-quality mathematics instruction; and 7) particular supports for providing equitable learning opportunities to all students.

It is the second of these hypotheses that motivates the work represented in this paper: Improvement in teachers' instructional practices and student achievement will be greater in schools where teachers and instructional leaders have a shared vision for high-quality mathematics instruction. Eventually, the goal is to establish a means of tracking shifts toward both increased sophistication and 'sharedness' in individual leaders' and teachers' instructional visions. However, in order to determine the extent to which groups of individuals *share* an instructional vision, we must first be able to reliably assess accounts of personal visions of high-quality mathematics instruction. With this paper I define *visions of high-quality mathematics* and describe the initial categorization scheme resulting from preliminary data analysis.

## Theoretical Perspectives

*Professional Vision*

Charles Goodwin (1994) presented a comparative analysis of practices in two professional settings, an archeological field school excavation and the 1992 California trial of four policemen charged with beating Rodney King. He argued that both the senior archeologist and the defense attorneys (with the help of an "expert witness"—an LAPD sergeant who was not present for the alleged beating) utilized particular complex discursive and representational practices to build and contest professional vision, which he defined as "socially organized ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group" (p. 606).

Sherin (2001) extended the idea of professional vision to her work in documenting the evolution of one mathematics teacher's perspective of classroom events. Her teacher, David Louis, had been teaching for five years at the time that Sherin and her colleague began observing and videotaping his classroom and meeting weekly with him to watch excerpts of those video recordings. During the most recent year-and-a-half, Mr. Louis had attempted to change his instructional approach to one of supporting the development of a community of learners (Brown & Campione, 1996; Rogoff, Matusov, & White, 1996). In addition to their weekly video viewing sessions, Mr. Louis and the researchers also participated monthly in a video club with David's colleagues. Over the course of her 4-year collaboration with Mr. Louis, Sherin documented how his interpretation of classroom events captured on video from his classroom changed from a focus on his own pedagogical actions (i.e., what he should have done differently) to one on student ideas and the nature of mathematical discussions (i.e., accounting for what had actually transpired in classroom events of interest). Adapting Goodwin's notion, Sherin suggested that this marked a shift in his professional vision—a "new interpretation strategy" (p. 90) focused more on the aspects of classroom activity to which researchers rather than teachers typically attend.

However, in his conception of professional vision, Goodwin was much less concerned with how individual actors had come to the point of being able to enact the practices of a professional vision than he was in examining how the enactment of a professional vision is accomplished. Although he recognized that the practices must be learned, his conception of professional vision was much more collectively and historically oriented, arguing that "the ability to see relevant entities is not lodged in the individual mind, but instead within a community of competent practitioners" (p. 626). In her account of Mr. Louis's interpretations of classroom events, Sherin explicitly made the leap from archaeology to mathematics teaching, but in doing so mapped Goodwin's (collective) professional vision within archaeology onto (individual) ways of interpreting events in the mathematics classroom.

Though Sherin's analysis of Mr. Louis's shift in perspective might not have adhered faithfully to Goodwin's conception of professional vision, there is much to be learned from her account. Sherin adapted Goodwin's notion of "professional vision" to describe one individual's way of seeing and interpreting classroom events. Across time, she documented which aspects of the classroom the teacher emphasized as being important with respect to mathematics instruction and learning, and the rationale behind his choices. It is this perspective on the classroom that I refer to as simply a "vision"—specifically, a "vision of high-quality mathematics instruction" (for which I will use "instructional vision" synonymously). Just as Sherin was able document an evolution in Mr. Louis's way of seeing and interpreting classroom events, the motivation for the work reported in this paper arose from a need to reliably document change in the visions of high-quality mathematics instruction among our study participants (including mathematics teachers, principals, mathematics coaches, and district leaders) to provide a means for determining the extent to which groups of participants move toward a shared instructional vision. To achieve this goal, it is therefore necessary to build a framework for considering the ways teachers and other participants characterize high-quality mathematics instruction.

*Dimensions of High-Quality Mathematics Instruction*

A considerable body of literature provides insights into one or more important aspects of mathematics teaching and learning, often investigated and reported as discrete elements of the practice. However, few attempts have been made to glean a coherent set of distinct aspects that adequately delineate crucial dimensions of the practice. In the following paragraphs, I summarize three such attempts, namely those of Franke, Kazemi, and Battey (2007); Carpenter and Lehrer (1999); and Hiebert and colleagues (1997). By examining how the various summaries fit together, my intention is to provide an initial frame for the analyses presented later in the paper.

Franke, Kazemi, and Battey (2007) described three features of mathematics classroom practice they viewed as most central: 1) creating and shaping mathematical *classroom discourse*; 2) establishing classroom *norms* for doing and learning mathematics; and 3) building *relationships* with and among students that support participation in the mathematical work of the classroom. The authors further detailed specific aspects of each feature. For example, with respect to classroom discourse, Franke and colleagues stressed four core ideas and practices, including revoicing student thinking (O'Connor & Michaels, 1993) to highlight particular mathematical ideas, to introduce mathematics vocabulary or to position students in relation to each other and their arguments; employing tasks that provide for multiple strategies and rich discussion; identifying and building on the resources English language learners bring to mathematical discussion; and encouraging students to interrogate meaning (Rosebery, Warren, Ogonowski, & Ballenger, 2005) behind mathematical assumptions and ideas, which contributes to developing classroom norms around questioning and challenging. Regarding classroom norms, the authors stressed the importance of distinguishing between social and sociomathematical norms (Yackel & Cobb, 1996), and attending to the consequences such norms have for student learning and defining what it means to 'do mathematics.' Lastly, the authors described the importance of teachers building relationships with students in terms of understanding children's thinking, and also in ways that lead to opportunities for participation, "which requires getting to know students' identities, histories and cultural and school experiences, all in relation to the mathematical work" (p. 243).

Carpenter and Lehrer (1999) described three dimensions of instruction, the examination of which they viewed as critical for enabling students to engage in mental activities necessary for learning mathematics with understanding: *mathematical tasks, tools* and *normative practices*. First, the authors suggested that through task sequencing that is based on children's thinking rather than mathematical structure, the learning of concepts and skills can be integrated. Important in this vein is that tasks be viewed as problems to be solved, not exercises to be completed, and that they be couched in meaningful contexts. Secondly, the authors suggested that tools, such as paper and pencil, manipulatives, calculators, computers and symbols, be used to represent mathematical ideas and problem situations. They argued that "connections with representational forms that have intuitive meaning for students can greatly help students give meaning to symbolic procedures" (p. 25). By considering the use of such tools, which can be introduced by either students or the teacher, students begin to abstract the mathematical ideas behind their manipulations, so that they gradually no longer need the physical representations. But the authors also argued that it is not the tasks and tools alone that will support learning with understanding. Lastly, Carpenter and Lehrer pointed to the role of classroom normative practices, which influence the use and interpretation of tasks and tools and “govern the nature of the arguments that students and teachers use to justify mathematical conjectures and conclusions” (p. 26). A key norm the authors highlighted is that students be expected to regularly discuss alternative strategies and why they work. This practice, they argued, will not only motivate the kinds of reflective mental activity previously described as students come to participate in what has been established as common classroom practice, it will also provide opportunities to make relationships explicit as the class examines how various methods are alike and different.

Predating the work described above was Hiebert et al.’s (1997) book, *Making Sense: Teaching and Learning Mathematics with Understanding*. Based on research conducted by the eight authors in various mathematics classrooms, Hiebert and colleagues identified and devoted a chapter to each of five “dimensions” of mathematics classroom instruction and activity: 1) the nature of classroom tasks; 2) the role of the teacher; 3) the social culture of the classroom; 4) mathematical tools as learning supports; and 5) equity and accessibility. Within each of these dimensions, the authors discussed essential “core features,” necessary for supporting students’ understanding of mathematics. Like the authors whose work is discussed above, Hiebert and colleagues attempted to describe a set of features of mathematics classroom instruction they viewed as critical for providing opportunities to learn mathematics with understanding—the dimensions that ‘matter.’ Additionally, the authors viewed their framework as potentially meaningful to those engaged in the practice of mathematics instruction, suggesting that it could be “used by teachers to reflect on their own practice, and to think about how their practice might change” (p. 3).

| Hiebert et al. (1997) | Carpenter & Lehrer (1999) | Franke et al. (2007) |
|---|---|---|
| Nature of Classroom Tasks | Tasks or activities students engage in and the problems they solve | |
| Role of the Teacher | | |
| Social Culture of the Classroom | Classroom normative practices | Establishing norms for doing and learning mathematics |
| Mathematical Tools as Learning Supports | Tools that represent mathematical ideas and problem situations | |
| Equity and Accessibility | | Building relationships for doing and learning mathematics |
| | | Supporting discourse for doing and learning mathematics |

*Figure 1.* Summary of central aspects of mathematics instruction identified in three works.

In Figure 1 I list the critical dimensions of mathematics classroom instruction identified in the work summarized above, mapping those identified by Carpenter and Lehrer and those of Franke et al. onto the dimensions described by Hiebert and colleagues. The ‘gaps’ in these lists should not be interpreted as omissions; they are typically a consequence of arrangement and classification choices. Since all of the authors acknowledged a systemic relationship among their dimensions, it is not surprising that in any one of these classification choices, dimensions identified as central by the other authors can be found. Two particular instances worth noting in this regard are Hiebert et al's “role of the teacher” and Franke et al.’s “supporting discourse for doing and learning mathematics.” Much of what Carpenter, Franke and their co-authors wrote about pertained very much to the role they envisioned a teacher playing. For example, both teams described the importance of the teacher’s influence on the establishment of classroom norms. Likewise, Franke et al.’s discourse dimension was represented in multiple places throughout the others’ summaries as they discussed the importance and role of communicating about mathematics in the classroom. Although it is to some extent a matter of reorganizing and renaming, I will argue below that the labels on the dimensions are meaningful in that they can represent points of view, or ways of seeing and valuing aspects of a mathematics classroom.

## Research Questions

As stated above, my immediate goal was to build a coding scheme for assessing participants’ visions of high-quality mathematics instruction. Therefore, my question regarded the ways teachers, principals, and others characterize high-quality mathematics instruction. In particular, which aspects of mathematics classroom instruction do they choose to highlight? To what extent do practitioners’ characterizations of classroom instruction map onto the critical “dimensions” described in the literature?

## Methodology

Participating school districts were purposively sampled to represent districts with ambitious goals for mathematics instruction reform to meet the needs of diverse populations. While differences exist among the strategies the districts are attempting to implement for accomplishing their goals, in general, all four districts are working to support mathematics instruction in every classroom that emphasizes rigorous tasks, problem-solving and sense-making, productive discourse, fair and credible evaluations, and clear, high-level expectations for all students.

In each annual data collection we document aspects of the institutional settings in which our participants work, the instructional practices and mathematics content knowledge for teaching of approximately 30 middle school mathematics teachers per district, and the extent to which structures such as those listed above have been established to support the ongoing improvement of mathematics teaching in 6-10 representative middle schools per district. Annual data sources are varied, but include 45-90 minute interviews with each participating teacher, principal, mathematics coach and district leader on issues related to the institutional settings in which they work, as well as their vision of high-quality mathematics instruction.

The data analyzed for this paper come from the interviews conducted in year one (January 2008) with middle school mathematics teachers, principals, and, in districts that employ them, mathematics coaches. I examined transcripts from every participant (teachers, principals and coaches) at each of eight schools (two from each district). These schools were theoretically sampled (Strauss & Corbin, 1998) to provide wide variation in the ways participants talked about mathematics instruction, as indicated in case summaries written for each school. The interviews were conducted and audio-recorded by members of the project team and later transcribed. As is the case with ‘unstructured interviews,’ (Burgess, 1984), they followed a set of guiding questions (that were customized for each district), but were conducted flexibly, intended to afford opportunities for conversations rather than a rigid sequence of questions.

The interviews probed at a number of issues related to mathematics instruction and the institutional setting in which participants work, including participants' understanding of the district's plans for improving mathematics instruction, their vision of high-quality instructional leadership, their informal professional networks, professional development activities in which they participate, the people to whom they are accountable, the sources of assistance on which they draw, and the curriculum materials they use in the classroom. An additional purpose of the interviews—and the focus of my analyses in this paper—was to document teachers', principals', and coaches' visions of high-quality mathematics instruction. Specifically, we asked participants the following question: "If you were asked to observe another teacher's math classroom, what would you look for to decide whether the mathematics instruction is high quality?" Depending on the participant's response, we asked, "Why do you think it is important to use/do _____ in a math classroom? Is there anything else you would look for? If so, what? Why?"

The purpose of this particular question was twofold. First, it was an attempt to circumvent the say-do problem (Gougen & Linde, 1993), a well-known obstacle in the social sciences in that self-reporting typically does not yield reliable data concerning participants' own practices. Thus, in asking our participants to imagine and talk about the activities in a classroom of some hypothetical other, our thinking was that we could release the participant from some of the pressures of accurately describing his/her own classroom and practices, and tendencies to foreground more socially desirable aspects of classroom instruction to the exclusion of those perceived as less desirable . Secondly, and more importantly, in asking teachers (and principals and coaches) to place themselves in the role of observer, we hoped to ascertain aspects of the lens with which each participant would actually view a mathematics classroom, or the way they interpret classroom events (Sherin, 2001). That is, we could interpret their responses to mean, "this is what matters in a mathematics classroom"—the aspects of the classroom on which they focus to determine the quality of instruction. This in turn would enable us to not only establish the kinds of things they might attend to when observing a mathematics classroom (e.g., what the teacher does, what the students are doing, the nature of the mathematical tasks, the nature of classroom discourse, etc.), but also to attribute some measure of depth or sophistication to their criteria.

Focusing in particular on the portion of the interviews including the question and probes mentioned above, I collected more than 200 statements from 54 of 222 participants (8 principals, 41 teachers and 5 mathematics coaches). Following Strauss & Corbin's (1998) open coding technique, I classified these statements (or concepts) into categories based on shared properties, such as whose behavior the statement pertained to (e.g., teacher, students, or both) or which aspect of the learning environment was emphasized (e.g., nature of classroom tasks, structure of lessons, etc.).

My initial classification was guided by a provisional list of codes (Miles & Huberman, 1994) drawn from Hiebert and colleagues' (1997) identification of essential dimensions of mathematics classroom instruction discussed above. I employed these dimensions and their accompanying core features as an initial framework for categorizing mathematics teachers' and instructional leaders' descriptions of what they would look for in a classroom to determine whether what they observed was high-quality instruction. Hiebert and colleagues' framework represented a reasonable starting point for my purposes, since the authors viewed it as a tool that could be used by those engaged in the practice of mathematics instruction for reflection and change. Thus, in initially adopting these authors perspective, I was not merely imposing a priori a researcher's tool on practitioners' views, but attempting to combine etic and emic accounts to create tools in which both communities find relevance and meaning.

## Results

Approximately half of the lines of talk I collected from interview transcript resembled the kinds of ideas expressed in the dimensions proposed by Hiebert et al. Many participants commented on the types of problems they would hope to see students working on (the nature of classroom tasks), what they thought the teacher should be doing (the role of the teacher), or how students would be interacting with other students and the teacher (the social culture of the classroom). Approximately five participants commented on the need to differentiate instruction based on students' individual needs (equity and accessibility), and two participants' remarks pertained to the importance of technology or means of representation in the classroom (mathematical tools as learning supports). Since my primary goal was to describe participants' visions of practice, I decided to drop the category pertaining to mathematical tools because it accounted for so few of the participants' responses, and retain the other four dimensions proposed by Hiebert *et al*. But a considerable number of responses were left unaccounted for. Therefore, I sorted the remaining statements into groups that appeared to share a focus. One set of concepts pertained to student and teacher talk (i.e., classroom discourse—a dimension identified in the framework of Franke *et al.*), one to lesson structure, another to assessment, and another (the largest) to student engagement.

In summary, in order to establish a means for describing participants' instructional visions, I have relied on both our interview data and previous work in identifying important aspects of mathematics classroom practice to identify categories to which teachers' and leaders' instructional visions might pertain. This analysis resulted in eight categories or, in Hiebert and colleagues' language, "dimensions": 1) the role of the teacher; 2) classroom discourse; 3) the organization/purpose of activity (i.e., student engagement); 4) social culture and norms; 5) the nature of classroom tasks; 6) role of student thinking; 7) lesson structure; and 8) equity and accessibility.

## Discussion

With this paper I have attempted to define a construct important to the ongoing MIST project, that of instructional *visions*. In the spirit of Simon and Tzur's (1999) efforts to generate "accounts of mathematics teachers' practice," my aim was to categorize and understand teachers' visions of high-quality mathematics instruction "in a way that accounts for aspects of practice that are of theoretical importance to the communities of mathematics education researchers and teacher educators" (p. 254). Thus, guiding my analysis was this question: What is the minimum number of dimensions of classroom activity and instructional practice that makes a difference? This question represents a two-pronged endeavor. One the one hand, I wanted my final categories to reflect what previous researchers have identified as important aspects of mathematics classroom instruction. On the other, I needed the final categories to sufficiently and meaningfully account for patterns I perceived in our data.

Along each of the eight dimensions listed above, we have elaborated a conjectured trajectory for participants' instructional visions in terms of depth and sophistication of their descriptions (the presentation and discussion of which is beyond the scope of this paper). Over the 4-year duration of our research team's work in school districts, we will use (subsequent refinements of) these trajectories to document shifts in individuals' visions of high-quality mathematics instruction, and the extent to which members of various district units move toward a shared instructional vision.

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# MATHEMATICS TEACHER DEVELOPERS' VIEWS OF A LABORATORY-CLASS-BASED PROFESSIONAL DEVELOPMENT EXPERIENCE[1]

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*This paper examines the use of a laboratory-class-based professional learning experience for mathematics teacher developers (MTDs). An 8-day institute focused on mathematical knowledge for teaching (MKT) featured a laboratory class a means of providing a common experience for observation and discussion. In a follow-up study 2 years later, the MTDs recalled several features of the institute and reported that it had influenced their current thinking and practice. In particular, the MTDs identified several features of the laboratory-class-based experience as significant: (a) observing the instructional practices of another MTD, (b) experiencing a potentially novel model of professional development, (c) investigating MKT, (d) working with MTDs of different backgrounds, and (e) exploring student understanding.*

Recent reports have cited an urgent need for improving both the quality and size of the mathematics teacher workforce in the United States (see, e.g., Business-Higher Education Forum, 2007; National Science Board, 2007). Part of this improvement entails mathematics teachers shifting their practices away from teacher-centered models of instruction to a focus on students' mathematical understanding and developing students' ability to solve problems, communicate, and work together. Undertaking a task of this scale requires a cadre of competent mathematics teacher developers (MTDs), those who are charged with the initial and ongoing professional education of mathematics teachers.

MTDs are a diverse group of professionals that includes community college and university faculty from both mathematics and mathematics education departments, privately practicing professional developers, and school district leaders who offer workshops for teachers. Although some MTDs do not have degrees in education or consider themselves teacher educators, they do teach courses designed for teachers. Sztajn, Ball, and McMahon (2006) described the literature

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We are grateful to Kathleen Banchoff not only for facilitating one of the focus groups but also for helping us formulate the protocol for those groups and for sharing her extensive experience in using the focus-group research technique. Tom Ricks, Shelly Allen, Patricia Wilson, and Jeremy Kilpatrick contributed to the design of the survey and focus group protocols. Ricks and Allen also each facilitated a focus group session.

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on MTDs' professional learning as essentially nonexistent. Similar to the shifts recommended for teachers, it is becoming clear that those who work with teachers need to also "undergo shifts in their knowledge, beliefs, and habits of practice that are more akin to a transformation than to tinkering around the edges of their practice" (Stein, Smith, & Silver, 1999, p. 262).

But how do MTDs acquire their expertise? Along with the investigation and development of professional development models for mathematics teachers, attention must also be given to models of professional development for MTDs (Sztajn et al., 2006). In this study, we investigated a model of professional development for MTDs that featured a laboratory class.

In summer 2004, 65 MTDs from across the country gathered in Ypsilanti, Michigan, to attend an 8-day institute entitled "Developing Teachers' Mathematical Knowledge for Teaching". The institute was organized by the Center for Proficiency in Teaching Mathematics (CPTM), an NSF-funded project led by researchers from the University of Michigan and the University of Georgia. The central feature of the institute was the observation of six consecutive sessions of a university-credit mathematics content course entitled Mathematical Content and Applications for the Teaching of Elementary School Mathematics. Sixteen preservice elementary teachers enrolled in the course, which was taught by Deborah Ball from the University of Michigan.

To better understand how a laboratory-class-based model may facilitate the professional development of MTDs, we examined the 2004 CPTM Summer Institute and its laboratory class to address the following research questions:

1. When asked to recall their participation in a laboratory-class-based professional development experience, what do mathematics teacher developers report as being significant?
2. What features of a laboratory-class-based professional development experience do mathematics teacher developers report as influential in their current practices?

## Theoretical Perspective

The professional development of MTDs, like that of other teachers, should attend to the work of teaching (Ball & Cohen, 1999; Smith, 2001). This work includes the cycle of teaching: planning for instruction, enacting the plan, and reflecting on the plan (Smith, 2001). Although this cycle oversimplifies the teaching process, it highlights three of its significant aspects.

Current literature on effective professional development also suggests classrooms should be used as a laboratory to explore teaching and learning (Lappan & Rivette, 2004). Hiebert, Morris, and Glass (2003) proposed an experimental model for preservice teacher development, called *lesson experiments*, whose laboratory-like structure parallels Smith's (2001) three-part cycle of teaching. In this model, preservice teachers treat the development and enactment of lessons as an experiment. First, they develop research questions, plan activities, and formulate hypotheses about what students might learn during those activities. Second, they implement those activities and collect data that will help them test their hypotheses. Third, when the lesson has concluded, they examine and interpret their data, developing conclusions and reformulating their hypotheses. By learning how to learn from their practice, participants in lesson experiments will continue to sustain professional growth throughout their careers.

Hiebert et al. (2003) recommended those who teach teachers should also learn about their practice through lesson experiments. By incorporating systematic inquiry within the cycle of teaching, the lesson experiment model fulfills what Cochran-Smith (2003) called the need for teacher educators' practices to have an *inquiry stance*. This stance includes "a way of learning from and about the practice of teacher education by engaging in systematic inquiry on that practice within a community of colleagues over time" (p. 8).

The ultimate goal of professional development for teachers at any level is to improve their students' mathematics learning, yet it is difficult to link the effect of one professional development experience with students' performance over time. Smith (2001) noted:

> If the goal of these efforts is to change knowledge, beliefs, and habits of practice so as to have an impact on students' learning, then changes in what teachers know, how they think about teaching and learning, and what they do in their classrooms might foreshadow future changes in learning outcomes for students. (pp. 51–52)

In this study, we sought to better understand changes in the MTDs' thinking and practice since attending the institute by asking them to reflect on their experiences with different features of the institute.

## The 2004 CPTM Summer Institute

In this section, we first describe several goals that guided the design of the institute activities. Then we provide more detail about the laboratory class and sessions related to it.

### *Institute Goals and Design*

*MKT.* As indicated by its title, the institute's central theme was mathematical knowledge for teaching (MKT). Bass (2005), who led discussions of mathematics at the institute, defined MKT as "the mathematical knowledge, skills, habits of mind, and sensibilities that are entailed by the actual work of teaching" (p. 429). From this theme, the designers posed two questions to focus institute activities:

> 1. What mathematical knowledge and practices play a central role in the everyday work of teaching?
> 2. What are promising approaches for helping teachers learn mathematics for teaching and and learn to use it in their work? (Sztajn et al., 2006, p. 154)

Institute designers used a variety of strategies to support MTDs' discussion and analysis of MKT. These strategies included assigning a particular focus for laboratory class observations, distributing scholarly articles on MKT, and, when appropriate, connecting MKT to group discussions.

*MTD diversity*. The institute designers selected institute participants that would reflect the diversity of backgrounds held by MTDs. In doing so, they hoped to encourage the group to collaboratively develop a professional identity (Sztajn et al., 2006). In addition, the designers sought to better understand how MTDs of different backgrounds "attend to learning opportunities" (p. 154–155). To accomplish this task, the designers used focused observations and participant journals as ways to capture MTDs' observations and reflections during the institute.

*Professional development of MTDs.* The designers also strove to understand how to help MTDs sustain their learning and professional growth. Using their knowledge of the professional development of K-12 teachers, they identified five guiding principles for the design of the institute:

> (a) learn in and from practice; (b) share with each other and from their professional experiences; (c) participate in some aspects of the design of their professional

> experiences; (d) choose professional development opportunities to work on that are most meaningful to them; and (e) be treated as professionals. (Sztajn et al., p. 155)

Efforts to reach these goals were evident throughout the various institute activities.

*The Laboratory Class*

The laboratory class could “be compared to a shared specimen for observation and manipulation” (Sztajn et al., 2006, p. 156). In this model, the MTDs, as the participants, acted as “a research team developing hypotheses and looking for evidence to support or refute claims and assumptions” (p. 156). Although the laboratory class provided a site for inquiry, it also created an environment that encouraged the MTDs to work within a community to generate knowledge and theorize their practice. Specifically, the laboratory class and its supporting sessions paralleled the three-part teaching cycle—planning, teaching, and reflecting—and incorporated features of Hiebert, Morris, and Glass’s (2003) lesson experiments.

*Planning.* Each morning, the MTDs would review and discuss the plan for the upcoming laboratory class session. Although Ball had already created a plan for that lesson, she gave the MTDs an opportunity to make suggestions. Part of the MTDs preparation for their observation involved solving and discussing the mathematics problems in the lesson. This prompted them to hypothesize how students might approach the activities and what difficulties students might encounter. This planning activity was essential for focusing MTDs’ observations of the lesson.

*Teaching.* The MTDs sat in elevated rows on either side of the class to observe the two-hour laboratory class in which the planned lesson was taught. To provide further focus, the institute designers directed the MTDs to specifically consider the teaching, the learning, or the mathematics being taught in the lesson. During the observation, the MTDs looked for confirming or disconfirming evidence to support or refute their predictions.

*Reflecting.* After each observation of the laboratory class, the MTDs reconvened first in small groups and then as a whole to discuss and reflect upon what they observed. Ball met with the MTDs to discuss what had occurred, the rationale for her instructional moves, and her plans for future class sessions. The MTDs revisited their original thoughts about the lesson and discussed how students approached the activities and what seemed difficult for them.

## Method

To understand what influence, if any, the CPTM Institute had on participating MTDs’ thinking and subsequent practice, CPTM researchers designed a two-part follow-up study. First, institute participants were invited in fall 2006 to complete an online survey that elicited information about their current practice, their impressions of the institute, and input for an upcoming reunion for institute participants. Of the original 65 participants, 46 submitted responses to one or more of the online survey questions.

The second part of the follow-up study consisted of four 90-minute focus group interviews at the reunion in January 2007 with a total of 32 participants from the institute. By encouraging participants to explain and discuss their responses, these interviews enabled the collection of rich data not attainable through the survey alone (Kleiber, 2004). The focus group discussions addressed the institute experience (including specific attention to the laboratory class), the participants’ current practice, and MKT.

Although responses in the online survey and focus group were to questions that dealt with a wide gamut of institute issues, we focused specifically on those that concerned the laboratory class. We examined transcripts to identify specific features of the laboratory class recalled by participants and the changes in practice that they reported had occurred as a result. Through several iterations of this process of examination, two dominant themes emerged pertaining to the laboratory class, along with some other themes that were mentioned less frequently or not described as explicitly. Once the themes were identified, we used them to code the data.

## Results

We have organized our results into three sections. The first two address the dominant themes: (1) the opportunity to observe and emulate instructional practices that the participants thought to be effective, and (2) the identification of features of institute design that affected their institute experience and their view of professional development. The third section summarizes some of the other themes, which included opportunities to investigate MKT, explore student understanding, and work with MTDs of different backgrounds. We address both research questions within each section.

*Observing Effective Instruction*

Many participants cited the instruction that took place in the laboratory class as prominent in their view of what they learned at the institute. Although a variety of aspects of this instruction received particular attention from the participants, we detail the three most frequently cited. First, Ball constantly pushed her students to think deeply about the mathematics they studied and the pedagogical decisions she made during class. In a focus group discussion, Sally (all names are pseudonyms) called this technique “the press” and gave examples of the questions Ball asked to focus students on their mathematical thinking: “‘Why are you saying that? What do you mean by that?’” Ball often asked students about her pedagogical decisions, encouraging them to think more deeply about what they were learning. Rachel (focus group) described this behavior as “making [one’s] practice explicit” and reported that she had become “a much better teacher” by adopting this practice. At times, Ball would push the preservice teachers to consider both the mathematics and the accompanying pedagogy by prompting them to write in class journals. Paul (focus group) remarked on this technique:

> I always had students write a lot about things. But I do that a lot more now than I ever did. And getting at them to understand or explain the whys of what’s happening, to get at that greater depth of the thing, I find that that’s very difficult for most students.

By focusing on creating mathematical explanations, asking significant questions, and identifying connections between representations, Ball was able to push the students to consider more deeply the mathematics and pedagogy they would need in their own practices.

Second, Ball used mathematically rich problems whose exploration required large amounts of class time. In his focus group, Andrew (focus group) described his reaction to this practice:

> There was … an enormous amount of time talking about one problem. This elaboration of discussion where people were able to go down all kinds of different paths, some of them down to dead ends and so forth…. It was a really eye-opening experience for me.

The participants reported using some of these problems in their classrooms. More importantly, some noted having redesigned their courses to include fewer problems with more time for exploration allotted to them.

Third, Ball established classroom norms necessary for the students to explore and discuss their mathematical thinking. Molly (focus group) described this practice as providing “a really rich classroom environment where [students] knew it was okay to make a mistake and that everyone was going to learn from that.” As a result of observing the instruction at the institute, Julie (focus group) reported that she spends "a lot more time socializing my students and talking on the first day." She continued to describe how she stresses to them the importance of spending significant amounts of time on problems, finding multiple solutions, and, at times, becoming frustrated.

Although we have identified three general elements of Ball's practice that participants deemed to be significant, this description may oversimplify what she did in the laboratory class. In her focus group, Suzanne described the complexity of Ball's instruction eloquently:

> Her work is often so nuanced. … You can't make a list. That's sort of just a beginning step. But I have thought many times since then: What is it that she did? What were some of the things? And yet I feel like I'm just scratching the surface.

*Institute Features*

In the follow-up study, participants described how the laboratory class, as a feature of the institute, stimulated their thinking about their own professional development as well as the professional development they provide for others. Several participants noted the importance of having a common experience that enabled "deep discussions about the issues we face in preparing teachers of mathematics" (Helen, online survey) by providing something that "we could dig into together" (Jen, focus group).

Of particular significance, participants frequently stated the importance of interacting with Ball before and after laboratory class sessions and her openness to their input. This interaction enabled them to feel like "active participants in her instruction" (Lauren, focus group). Edward (focus group) was drawn in by the interaction with Ball, coupled with the "real time nature" of the laboratory class, commenting:

> There were all of these unpredictable variables that come into play, what inputs [Ball] used and how those influenced her thinking, and how transparent she was about that, how she invited our involvement in that. And so, we were simultaneously going through the same process.

This shared process, which Robin described as an opportunity to "learn in the moment" (Focus Group D) was also cited by other participants as an important feature of the laboratory class.

Several participants reported changes that they had attempted to make based upon their experience with this professional development model. Martha took her class of 25 preservice teachers to a fifth-grade classroom to observe a lesson, focusing on how students learn. Lauren (focus group), citing the importance of having focused observations, no longer allows the teachers with whom she works to interact with students during classroom observations. Several participants reported using video to create the common experience for their own students that was found to be so valuable at the institute. However, Sandy (focus group) noted that, because the use of video was not instantaneous, "you lose something."

*Other Themes*

Although participants discussed the themes detailed above more frequently and explicitly, other themes could be detected within the data. First, some participants cited the importance of exploring MKT at the institute. Considering their own practices, they also noted the teachers with whom they work needed to develop a deeper knowledge of the mathematics they teach. When discussing how to address this need, some recalled their experiences in the laboratory class. In particular, they focused on the importance of having multiple representations for a mathematical concept. Molly (focus group) stated:

> I … remember all the different representations, none of which I would have thought of in a million years, and then how [the students] even struggled to communicate what is was they were doing and why they were doing it. … Giving them a chance to go back and really reflect on that was so important.

Some participants provided examples of how MKT had become a theme in their current work. For example, Andrew (focus group) described how the focus on explanation in the laboratory class carried over to his own practice: “I confess I got kind of hung up on this idea of explanations in these courses that I’ve been teaching.” He went on to describe how he has been trying to elicit explanations from his students that are similar to those that teachers need to produce on their own.

Other participants described their attempts to focus on deeper mathematical knowledge in their work. This shift in focus involved the creation or redesign of their mathematics content courses for teachers and, for Robby (focus group), a move to make his examinations “explanations-based.”

Second, in addition to exploring MKT, participants noted the importance of working with MTDs of different backgrounds while at the institute. By assembling a group incorporating the diversity of MTDs, the institute organizers created an environment in which all could learn from each participant’s unique insights. The participants of our study referred to the diversity of the MTDs and how the laboratory class structure brought out a variety of viewpoints. Jonathan (online survey) described the various perspectives as “critical”, stating that it had “a powerful impact on all of us, that there was not agreement, or even common understanding among those in the ‘same’ workgroup.” Robbie (focus group) cited the presence of diverse viewpoints as a way to bring out “some kind of opposition or polemic” in the group discussions, an element that he claimed facilitated his learning. Erica (focus group) had a similar view, stating that the presence of divergent views forced her to reflect upon and justify her perspectives.

Some participants reported that the interactions with MTDs of different backgrounds affected their subsequent practice. For example, an unidentified participant (notes from focus group) reported having “more confidence to interact with mathematicians about the body of knowledge that I possess & where our worlds intersect.” Julie (notes from focus group) expressed a similar sentiment, stating, “I valued at the institute the input of research mathematicians and work with them seeking deeper understandings of that which I teach.”

Finally, the participants reported that the laboratory class and the student journals that students produced in conjunction with the course provided an opportunity for them to develop hypotheses of student understanding. Considering the utility of understanding student thinking, Kelly (focus group) remarked, “If we can capture student thinking on problems so that we know how they typically respond, then that’s very informative for the next time you teach.”

For some participants, this focus on student understanding carried over into their work. At the institute, Erica had become particularly intrigued by her interpretation of a student-produced representation and how it differed from the student’s interpretation. In one focus group, she had reported that her subsequent thinking and attempts to reconcile that this difference led to her developing a “Building Hypotheses of Student Understanding” theme for her methods courses. Many participants also reported adopting the practice of using student journals after the institute as a way to monitor student understanding.

## Discussion

Comparing the goals for institute held by institute designers with the experiences that participants recalled about it revealed some parallels. The institute designers intended for the participants to investigate MKT (the institute theme), learn from peers with varying professional backgrounds, and learn in and from the practice of MTDs. MKT was a prominent topic in the online survey and focus groups. When asked questions that focused on the laboratory class, however, the participants spoke much more on the practice of the laboratory class instructor rather than on MKT. It was not that MKT was unimportant to the participants, but rather that they mentioned it less frequently in the context of discussion of the laboratory class. This relative lack of attention might be attributed to a struggle to articulate their understanding of MKT. In addition, many aspects of Ball's instruction recalled by participants, such as the focus on explanation or the importance of a deep understanding of mathematics, could be considered an embodiment of MKT in action.

One feature of the institute that participants reported as influential in their thinking and practices was the laboratory class. The laboratory class, through its combination of participant observation of and reflection on a common experience, allowed the MTDs to progress in how they thought about the work of MTDs. Making the laboratory class the central feature of an intensive institute experience appears to have helped the MTDs change their thinking and practice. The responses of MTDs suggest that variations of this laboratory class experience could be incorporated into professional development for mathematics teachers.

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# MEDIATING INFLUENCES OF TEACHERS JOINTLY PLANNING A LESSON

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*This study examines what mediated teachers thinking processes when jointly planning a lesson. The findings reveal that how teachers thought about standardized testing and state standards influenced how they took into account the cognitive, language and social development of students. In addition, the mediation of the facilitator, that involved the tension between creating a product and engaging in discussion, influenced the level of conversation that took place.*

## Background

Teacher lesson planning that supports learning is a complex process (Ball, 2000). Teachers need to consider the students they will be teaching, the mathematical content to be taught, and the methods through which the information will be delivered. Understanding the thought processes that teacher engage in for planning lessons can give professional developers insight on how to support teachers to improve their teaching practices. Lesson study has been advocated by researchers (Lewis & Tisuchida, 1997; Stiegler & Hiebert, 1999) as a viable means for teachers to examine and improve their teaching practices. In lesson study, a group of teachers jointly plan and teach lessons. We adapted the idea of teachers jointly planning a lesson in our work with teachers. Therefore, we present an analysis of what mediated teacher's thinking processes when jointly planning a lesson. Lesson study involves a group of teachers identifying a goal within a content area and jointly planning activities to teach a lesson (Fernandez, Cannon, & Choksh , 2002). Lewis, Perry and Murata (2009) point out that the innovation mechanism of lesson study must be understood as opposed to focusing on the surface features of lesson study (c, Fulla, 2001; McLaughlin & Mitra, 2001).

## Theoretical Perspectives

Lesson planning provides teachers with an organizing structure to conceptualize outcomes, and identify means of delivering instruction in order to influence student learning (Bage, Grosvenor, Williams, 1999; Panasuk & Sullvan, 1998; Yinger, 1979). Therefore, understanding what teachers consider when planning lesson is necessary for supporting teachers to improve their practice. Simon (1995) pointsout that effective lesson planning should involve thinking about the mathematical goals, learning trajectory that students might follow and designing tasks to support learning. However, the plans must be flexible so that the teacher can adapt tasks based on assessment of student thinking. This process involves taking into consideration children's cognitive development in order to support their mathematical thinking. Furthermore, this also means that teachers must consider the context in which they teach. Specifically, they must consider how to support mathematical thinking and language development (Bransford, Darling-Hammond, & LePage, 2005) within the social setting of the classroom. How teachers think about students' cognitive, language, and social development influences how they plan lessons and also the eventual outcome of the lesson.

Planning a lesson that takes student learning into account involves thinking about multiple variables. Joint lesson planning involves using problem solving skills. Particularly, it involves

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working with non-routine problems. This means, the teachers have to define what they want their students to learn and figure out a topic and tasks. The potential for teachers' growth in learning comes from the messy process of solving the non-routine problems. Polya (1957) identified the process of problem solving as involving several phases such as understanding the problem, developing a plan, carrying out the plan, and looking back. Carlson & Bloom (2005) identify *resources, control, methods, heuristics and affect* as dimensions of problem solving process.

## Research Question

What mediated teachers thinking processes when jointly planning a lesson?

## Methodology

This paper examines data gathered as part of a larger three year Math Science Partnership Project involving collaboration among 5 school districts and 22 schools in a western state. There were three different professional development sites. The data analyzed in this paper is from one site. Five teachers participated in a professional development session where they engaged in developing a lesson, teaching it, and debriefing afterword. These teachers had participated in two and a half years of professional development in math education. Therefore, teachers had experience thinking about current research and teaching methods.

Four female teachers and one male teacher in grades 3-7 participated in the study. Additionally, a facilitator who was a district leader and part of the professional development team oversaw the process and incorporated guiding questions with the focus of completing the lesson plan. The first author serving as the project director, sat in during the lesson planning process. The teachers in the study applied to be part of this three year program. Teachers were randomly selected based on their interest, and commitments to serve as a teacher-leader.

The teachers involved in this study took part in a lesson study session, led by a facilitator. The facilitator focused on the completion of a Lesson Study Planning Sheet on a computer. Teachers were asked to complete the following elements: *Learning Activities and Questions*, *Expected Student Reactions*, *Teacher Response to Student Reactions? Things to Remember*, and *Evaluation*. Under the heading of Learning Activities and Questions the following elements existed: Process Standards, Standard, Big Mathematical Ideas, Objectives, Activities, and Possible Homework.

The data presented in this paper was the first 1 and ½ hour segment of the four-hour video tape recording of a lesson planning session that was part of a lesson study group. The data was transcribed and coded using Strauss and Corbin (1990) constant comparative methods. We first transcribed the data and made initial notes of observations. Then we coded the data based on emerging themes. Once the data was coded we looked at patterns in the data to determine what teachers considered when planning the lesson.

## Results

How teachers thought about the state testing and standards influenced the topic they chose to plan a lesson on. Furthermore, the level of conversation was mediated by the facilitator. The facilitator's focus on helping teachers to help students do better on standardized testing, and her focus on keeping teachers on task to develop a product (lesson plan) within the time frame allocated, mediated the level of conversation that took place. A brief excerpt of transcript from the beginning stages of the joint lesson plan is presented here. The teachers were examining the geometry state standards and standardized testing item specification sheet in order to decide on a topic for planning the joint lesson. This transcript illustrates the conversation that took place.

T4 Let's say we are all developing a lesson. And, maybe we have a lesson on linear measurement, and maybe that one lesson would be dealing with inches, feet, yards and miles. You know I mean. It would just be on one thing. And then in a separate lesson you can say we also use….Then they can put it all together. I just think if you teach it all at once with third graders they are going to get very confused. Don't you think? If they can use those units?

T1 It says standard and non-standard. I know I have looked at these before, but I can't remember.

Facilitator Okay, select and use appropriate units, measure to a required degree of accuracy and record results. Estimate and use measures and devices to measure. (Reads standards)

T3 I think the item specs the hint. It says inches and cm. so I think that is all we need to worry about.

Facilitator Another thing to look at and I hate to tell people this, but seriously, is the matrix to see how many questions they are going to be asked about cm and m and if there is only one question….you have to think about how much time

Teachers discussed sequencing of the lesson and what they might be capable of understanding. However, the conversation shifts to focusing on state test.

T3 You know, we did that. The first we got serious about cramming for the (state test). We went through and looked for what they had asked the year before and there were concepts we did not cover. And when we got it back it was really fun because the kids didn't do really bad on the ones we did not cover.

T1 Where is the matrix?

Facilitator: It is on here. Look at this, it tells you. It defines the terminology and it tells you how much they should be… and that is in the vocabulary that is underneath the old standards and we moved it to a glossary because this wasn't satisfactory.

Facilitator: This like unit, inch, meter, pound, it at least defines it a little bit.

Facilitator identifies vocabulary that will be on the state test. T3 teachers comment reveals that teacher cover content only that will be tested.

….

Facilitator: Okay, now what topic do you guys want?

T1 Okay, I posed a topic, now, somebody else.

T3 I agree with you, measurement is frustration in fifth grade, so it is no better in sixth grade because they are still banging their heads against the wall in fifth grade so as soon as they get in middle school they probably have problems. And when I do things around the house with my kids, measurement is still a problem in high school.

Teacher discusses the difficulty that students experience when learning measurement across grade levels.

Facilitator: And it is, to me, am I wrong, or do early elementary not measure as much as we used to. Get your ruler out and measure this….can you jump one inch, or one mile?

T3 Well I think, it has changed a lot.

T1 There is crazy stuff, this one question in here has a sticker star placed in the middle of a ruler and it is an inch and they are counting up from the beginning and it is clearly an inch. I mean it is crazy, weird stuff. (Examine curricular materials)

F2 How about we do a measurement lesson at grade three in your classroom?

T1 All right.

Facilitator suggests that they do a lesson on measurement. As teachers start sharing stories from classroom the facilitator makes a comment to get teachers back on the task of creating a lesson plan.

T5 They read the ruler backwards.

T1Let me show you. Here it is. (shows paper) This is three inches according to my students.

T5 We talked about this in the teachers meeting the other day that you need to step and this was one of the most missed questions in our data. So, what do you need to do to teach them to zero in, or compare the beginning to the end? Ours had a truck with wheels and some kids measured the distance between the wheels instead of the end of the truck, you know? And so we discussed what kinds of things do you have to do in a lesson to remedy your kids missing those lessons? Well, one is you have got to give them practice measuring things not with the beginning of a ruler.

T4 And two, when you are working on in the overhead you are showing them about starting at zero and you do it intentionally and you are modeling and then you just kind of place it down starting from and they will say that to you. OH Mrs. W, Mrs. W, you are not starting at zero! And then I could say, well, does it matter if I am starting at zero and here is when you could say this is at

T1What is a third grade standard we are looking at.

T2 You wanted the vocabulary right?

F2 No, I am going to be ornery and keep you guys or we won't get done.

T5 What she was talking about measurement.

F2 I know, I know. Do you actually want linear measurement or do you want to do something else?

(reading standards)

…

T1 Here is 3.3.1. compare, order and describe objects by compare attributes for area and volume/capacity. Or 3.3.2. select and use appropriate units of measurement required to a degree of accuracy.

T3 We can do both, can't we?

F2 Mm, there is no reason you can't cover both, I mean depending on the standards.

T3 I mean not covering all of both of them, but touching on both of them.

T1 We could very well be touching on both of them.

T3 Usually when you are doing that you are doing multiple anyway.

T5 You know what this is a question I guess I should have asked back when we were looking at our standards, but volume is not mentioned , I don't think, in sixth grade standards and they put it in seventh grade and it used to be in sixth grade standards. Now, I am hearing volume in third grade standards?

T3 Not any more. I don't think.

T5 Didn't you just read volume?

T1 Area and volume/capacity and like I said, I am confused about how much differentiation to make between volume and capacity. Do I strictly say, volume is the amount of space and capacity is the amount it will hold? And volume is measured in square units and capacity is measured in liters, gallons, bla, bla, bla?

T4 I really didn't even differentiate in fifth grade, I don't think.
F2 Remember too, we are talking about one lesson and how much you are going to be able to teach in one lesson. Not everything you are going to be teaching because you may not actually be teaching both of those in one lesson.
T1 Right.
F2 For our purposes and narrowing it down.
T4 Let's look at some lessons and see what we can find that comes to fit us close and then start modifying it.

This vignette illustrates that teachers not only looked at the state standards to make decisions about the focus for the lesson content. They also examined the specific number of items on the test specification sheet that was available on the internet to determine what topic to teach. The data illustrates that preparing students to be successful in standardized testing mediated how teachers interpreted the task and also made decisions. In addition, teachers also considered students ability levels and sequencing of lessons. For example, T4 pointed out that if you teach too many concepts together the students might get confused. Teachers relied on their classroom teaching experiences to make sense of what students might do such as reading the ruler backwards and T5 shared that measurement was a most missed question in their school data.

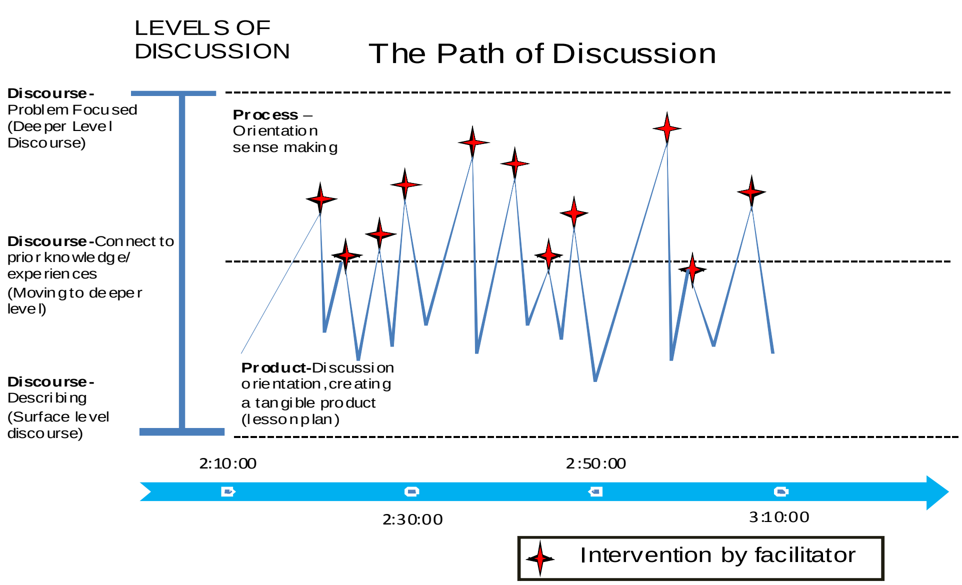
Teachers shared stories from their classroom experiences of teaching measurement and stated that measurement is a frustration in fifth, sixth grades and even middle school. During this process, the teachers were seeking to find a topic that would be beneficial for all teachers at different grade levels present in the lesson planning session. Sharing stories of difficulty in teaching measurement involved the process of beginning to define a problem that students have with measurement. The conversation had shifted from originally reading the standard and test matrix and item specification to talking about students. The facilitator interjected at this point to get the teachers back on task to develop the lesson plan. She even asked the teachers if they wanted to stay with the topic or do something else. This comment shifted the focus away from linear measurement to talking about volume and capacity. Again, the conversation shifted to reading out loud what the standards stated about volume and capacity in the state standards. As a result, the conversation shifted to a lower level of description. Once, teachers started to discuss their own experiences in the classroom and confusions about the topic the conversation shifted to a deeper level of thinking. For example, a teacher commented that teaching volume was more difficult than linear measurement. During this time, the facilitator reminded teachers not to get off track but to focus on one lesson.

The facilitator's actions were mediated by the need to have a finished product in form of a lesson plan within the allocated time for the planning session. The activity of the group shifted to looking at curricula to plan the lesson. The facilitator's interjections resulted in teachers changing the activity of discussion to focusing on developing a product. As teachers started looking at lesson activities, the facilitator was reading ideas from the state standards of other topics they could focus on. The facilitator told the teachers they could choose another topic other than geometry because the topic was taught earlier by the teacher and the geometry standards were not "meaty enough". Therefore, the conversation shifted again to another topic. One teacher was doing her master's project on problem posing and division and suggested that they work on problem posing. Eventually, teachers decided upon a lesson that involved problem posing involving division. They developed a lesson that involved students writing a division problem so that others could solve them. The teachers decided to create bags for group of students so that they could create their own problem. Their goal became to help students develop problem solving skills by posing problems.

**Discussion**

The first one and a half hours of the 4 hour lesson planning session involved finding a topic to plan a lesson. This process involved defining the problem in order to plan a lesson. The first phase of problem solving involves understanding the problem (Polya,957). Carlson et. al (2005) point out that that defining a problem is an important part of the problem solving process. Therefore, this process should be messy because it involves working with a problem that is not defined.

Lesson planning can be compared to the problem solving process. What we found interesting in the data was the tension between the facilitators role in keeping teachers on task of developing a product, verses allowing teachers to engage in discussion. When teachers began to relate a problem about their students and share stories from the classroom, the facilitator refocused teachers on the task of creating the lesson plan. The intervening comments changed the direction of the conversation that was taking place. Many times, discussion about what took place in their classrooms was viewed by the facilitator as an "off task" behavior. The conversation shifted to a descriptive level of reading standards or looking at activities. Whereas, when a teacher shared stories such as the difficulty they have teaching volume and capacity as opposed to measurement, they were posing problems to the group for discussion that required a deeper level of thinking. Carlson, Bowling, Moore & Ortiz (2007) point out that a facilitator needs to "*decenter*" in order for meaningful conversation to take place. Figure 1 illustrates the trajectory of conversation that took place.




Lesson study is a widely used form of professional development. Understanding what happens and how problems get defined can provide professional developers insight on how to support teachers during lesson study. Lewis et. al (2009) had pointed out that the innovation must be understood as opposed to the surface features being implemented. In this 1 1./2 hour discussion, a deeper discussion of defining the problem did not occur. Allowing teachers to share stories from their classroom, discussing what the standards mean in relation to their students can be a process of identifying a significant problem in order to find a solution. Eventually, a lesson on problem posing involving division was planned. The lesson did take into account student reasoning. Further research is needed on how teacher define problem during lesson study.

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# REFLECTION-ON-ACTION OF MIDDLE SCHOOL MATHEMATICS TEACHERS

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*In this study, we examined the reflection-on-action of four middle school mathematics teachers from the perspective of Cohen and Ball's instructional triangle (1999). We addressed questions of how teachers reflect on their students' understanding differently. The four teachers were asked about their students' thinking while watching video clips taken from their lessons. Findings indicate that the teachers showed differences assessing and interpreting students' thinking, differences by teaching experiences, and changes over time in ways they talked about the interaction between materials, students, and teachers as they related to students' understanding.*

## Introduction

In his discussion of the characteristics of practical knowledge, Schön (1983) noted, "There are actions, recognitions, and judgments which we know how to carry out spontaneously; we do not have to think about them prior to or during their performance (p. 54)." However, we sometimes think about what we are doing, and that is reflection. Two important forms of reflection are: *reflection-in-action*, which involves thinking about actions while engage in them and *reflection-on-action*, which involves thinking about processes after actions (Schön 1983). We are interested in teachers' reflection, in particular, reflection-on-action, because a growing body of literature suggests that reflection can help teachers understand the relationship between their cognition and teaching practice (Artzt & Armour-Thomas, 2002).

While there are a number of definitions for reflection in the literature, for this study, we consider reflection to involve teachers' analysis of teaching and students' learning. This is consistent with the *Mathematics Teaching Today* (Martin, 2007), which emphasized the analysis of teaching and learning as one of the professional standards. The Mathematics Teaching today, which is newer version of the NCTM teaching standards, asserts that teachers need to engage in this kind of reflection because it helps them develop deeper understanding of students' learning and development and impacts the ways in which they plan their lessons. Because of the potential for shaping teachers' practices, we assert that reflection is a part of the professional knowledge of teachers.

Research on inservice teachers' reflections has examined how reflection affects teachers' practices as well as their learning. In one line of reflection research, Sherin and her colleagues (Sherin, 1998; Sherin & Han, 2004; and van Es & Sherin, 2008) suggested the concept of *teacher noticing* as an important aspect of teacher learning that develops as part of the reflection process. Teacher noticing includes "(a) identifying what is important in a teaching situation; (b) using what one knows about the context to reason about a situation; and (c) making connections between specific events and broader principles of teaching and learning" (van Es & Sherin, 2008, p.245). Sherin and her colleagues found that teachers commented more on students' understanding and were able to recall more detailed information of classroom events over time as they worked as a group in video clubs. These are considered desirable outcomes of engaging in this kind of reflection activity.

In contrast to Sherin and her colleague's work, this study focuses on individual teacher's reflection. While we relied on video, we did not set up a learning community among the teachers and, in fact, did not engage the teachers in professional development as an intentional activity. Rather, we report the effects of engaging teachers in reflecting on classroom activities as part of a research effort. In further contrast to Sherin and her colleague's work, the researcher in these interviews directed the teacher's attention to particular events in the classroom rather than allowing the teachers to determine which incidents they wanted to discuss.

In this study, we examine four inservice teachers' individual reflection-on-action in the context of being interviewed for a larger research study. Our goal was to understand the development of the teachers' reflections over time in terms of their discussions about student thinking and to consider how the teachers differ in their reflection on student thinking. To this end, we examine teachers' reflections on their students.

## Theoretical Framework

We used Cohen and Ball's instructional triangle (1999) as a metaphor for thinking about teachers' reflections (see Figure 1). The triangle indicates that the learning environment is comprised of the interactions among the teacher, the students, and content as embodied in the instructional materials.

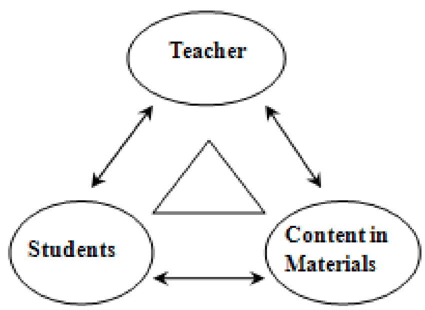


*Figure 1*. Cohen and Ball's instructional triangle.

The triangle forms the basis of our analysis because when teachers reflect on their lessons, they think back to the situations and interactions on the lessons so that the triangle of interactions can be a metaphor for thinking about teachers' reflection. That is, we take the perspective that reflection on these interactions is a critical element for improving the interactions and in helping the teacher begin to think about how the three elements of the triangle interact. Our underlying assertion is that a teacher with better reflective thinking will be more likely to connect a reflection on student understanding to the content and to her/his own actions as the teacher in the learning environment.

## Methodology

In our previous work, we examined one teacher's reflection and its change over time (Authors, 2007). In that study, we found that the teacher moved toward incorporating more discussion of the interaction between teachers and materials in her reflections on student understanding. In the present study, we replicated our analysis on a larger sample of teachers who participated in the NSF-funded Coordinating Students' and Teachers' Algebraic Reasoning (CoSTAR) project.

*The CoSTAR Project*

The CoSTAR research project focused on broad questions about mathematics teaching and learning by considering students' and teachers' knowledge, interactions, and sense-making of shared events. The CoSTAR data included daily videotaped observations of each teacher working with a single class of students using *Connected Mathematics Project* (CMP; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002) materials for an entire unit of instruction at a time (typically 6-8 weeks). The CoSTAR team interviewed teachers using the classroom video clips and student interview video clips.

*Participants*

The four participants were mathematics teachers at Pierce Middle School[1], which had recently replaced traditional instructional materials with the standards-based CMP materials. Three of the teachers were experienced mathematics teachers each having over 10 years of teaching experience. The fourth was a new 6th grade teacher who had some experience teaching 7th grade CMP while serving as a long-term substitute in the year prior to her case study. The data analyzed for this study came from our first case study with each of the teachers, though Ms. Moseley[2] had participated in a two-week pilot study before these data were collected. Ms. Moseley and Ms. Reese were teaching 6th grade and in their second year of implementation of the materials during this study while the 7th grade teacher, Ms. Bishop, was in her third year of implementation at the time of data collection.

*Data Collection*

Each teacher was interviewed weekly for the duration of her case study. In each interview, the interviewer used video clips from class sessions and student interviews. The video clips were selected by the interviewer and the investigator who was interviewing the students in the same classroom. In each case, the focus of the effort was on developing a deeper understanding of how the students and teachers understood the shared events of the classroom.

The interviews were videotaped using two cameras—one focused on the teacher and the other focused on either the computer on which the student interview and classroom clips were shown or on any written work or gestures the teachers made during the interview. These videos were then transcribed verbatim and these transcripts formed the basis of our analysis.

For each participant, we selected three interview transcripts—first, middle, and the final—from each of the cases to investigate these teachers' reflection patterns. The transcripts were the primary data analyzed for this study; however, the interview videos and classroom videos were used to clarify any points of confusion from the transcripts.

*Data Analysis*

Data analysis occurred in two steps. In the first step, the first author separated the transcripts into three categories based on the questions the teachers were asked: questions about teaching, questions about students, and questions about curriculum and materials. Questions about logistics of the research itself were separated out as well. For this study, as in our earlier research, we chose to focus only on interactions that arose from questions about students.

In the second phase of analysis, we used a modified version of the categories developed by Wallach and Even (2005): (1) *Assess*, (2) *Describe*, (3) *Interpret*, (4) *Justify*, (5) *Extend*. *Assess* instances were those in which the teachers evaluated students' ability. *Describe* instances were verbatim descriptions of actions without any analysis. *Interpret* instances were the teachers' analyses of what their students were doing. *Justify* instances provided a rationale of the teachers' beliefs about what was happening. *Extend* instances included interactions in which the teachers reflected on their teaching, the content, or the curriculum materials.

Each researcher coded all of the instances in the subset of data concerned with student questions within the three selected transcripts for each teacher. A single paragraph could have one or more instances within it and each instance was assigned to one of the five categories. Inter-rater agreement was achieved on both the instances and the categories for each instance. We discussed each instance about which there was disagreement to achieve a 100% agreement on the coding. Finally, we used *inductive analysis* (Patton, 2002) to compare the four teachers' reflection on students' understanding.

## Results

Initially, we only considered the relative percentage of instances of each category for each teacher (see Table 1). However, the relative frequency of each kind of instance was insufficient for understanding the teachers' reflection-on-action. Hence, we examined each instance in various ways to further understand the teachers' reflections.

*Table 1*

Percentage of Category for each Teacher

| | **Ms. Moseley** | | | **Ms. Reese** | | |
|---|---|---|---|---|---|---|
| **Date** | **3/13/03** | **4/08/03** | **5/22/03** | **3/17/03** | **4/08/03** | **5/13/03** |
| **Assess** | 6.90 | 15.56 | 14.29 | 16.13 | 6.56 | 4.17 |
| **Describe** | 16.09 | 8.89 | 5.36 | 12.90 | 9.84 | 8.33 |
| **Interpret** | 42.53 | 46.67 | 41.07 | 35.48 | 49.18 | 43.75 |
| **Justify** | 9.20 | 15.56 | 16.07 | 8.06 | 13.11 | 8.33 |
| **Extend** | 25.29 | 13.33 | 23.21 | 27.42 | 21.31 | 35.42 |
| **Total** | 100 | 100 | 100 | 100 | 100 | 100 |
| | **Ms. Archer** | | | **Ms. Bishop** | | |
| **Date** | **2/10/04** | **3/01/04** | **5/25/04** | **3/17/03** | **4/08/03** | **5/13/03** |
| **Assess** | 13.04 | 10.45 | 10.13 | 3.85 | 9.52 | 10.64 |
| **Describe** | 26.09 | 5.97 | 10.13 | 11.54 | 11.90 | 8.51 |
| **Interpret** | 13.04 | 50.75 | 44.30 | 55.77 | 30.95 | 40.43 |
| **Justify** | 13.04 | 11.94 | 11.39 | 3.85 | 10.71 | 10.64 |
| **Extend** | 34.78 | 20.90 | 24.05 | 25 | 36.90 | 29.79 |
| **Total** | 100 | 100 | 100 | 100 | 100 | 100 |

### *Positive/Negative Comments*

As shown in Table 1, Ms. Moseley and Ms. Bishop generally increased in their frequency of Assess instances, but Ms. Reese and Ms. Archer decreased overall. Hence, we cannot say that there is a pattern in general. Nonetheless, we found an interesting result in those teachers' Assess instances. During interviews, Ms. Moseley provided 17 positive comments (80%) of the total 21 assess instances and Ms. Bishop offered 12 positive comments (80%) of the 15 assess instances. Most of these positive Assess instances focused on students' understanding or things that the students were able to do. For example, "She got the right percent though, so that is good" (Ms.

Moseley, 4/08/03 interview) and “I think she understood the concept” (Ms. Bishop, 9/24/03 interview). In contrast, in their Assess instances, Ms. Reese and Ms. Archer talked about their students’ misunderstanding or things that the students were not able to do. For example, “She’s misunderstanding that they’re different sizes” (Ms. Reese, 3/27/03 interview) and “Yeah… he would’ve found some stuff out. It wouldn’t have come out right” (Ms. Archer, 5/25/04 interview). Of Ms. Reese’s 16 Assess instances, 11 were negative (69%). Ms. Archer’s Assess instances included 12 negative comments (67%) out of the 16 Assess comments analyzed.

The positive or negative comments showed up in different patterns of the frequency of *Assess* instances. Ms. Moseley and Ms. Bishop who focused on students’ understanding positively showed a kind of increasing pattern in the percentage of instances of Assess. On the other hand, Ms. Reese and Ms. Archer who commented on negative students’ understanding showed a decreasing pattern in the instances. We claim that the decreasing pattern of the instances of *Assess* of Ms. Reese and Ms. Archer aligned with an increase in their discussion of their students’ understanding in contrast to focusing on students’ mistakes. Further, we assume that the teachers who evaluate their students positively make more efforts to understand their students’ mathematical thinking unlike the teachers who assess their students negatively focus on students’ misunderstanding or their mistake.

*Novice/Expert Teacher*

We had one teacher, Ms. Archer, who was in her first year full-time teaching at the time of data collection in contrast to the other three teachers who all had more than 10 years of experience. The case of Ms. Archer was curious in that she used many descriptions and a few interpretations to explain her thoughts about her students’ understanding at the first interview. However, by her later interviews, she changed to have fewer descriptions and more interpretations, making her similar to the other teachers. This may be because she was nervous about being questioned about her students and her practice or it may be related to her limited pedagogical content knowledge (Shulman, 1987), which necessarily limited her ability to interpret classroom instances. Ms. Bishop’s shift from Description to Interpretation suggest that providing this novice teacher with opportunities to watch and analyze her students’ work with the mathematics in their classroom supported her in moving quickly to being able to analyze and interpret her students’ work. If this is common in other settings, it could suggest that simply reflecting on student understanding can help develop a kind of professional knowledge for teaching much faster than it might develop without this reflection.

*Change in the Instances of Extend*

In this study, Extend instances included a wide variety of comments including reporting, evaluating, reasoning, reconstructing the teacher moves as they related to their students’ understanding, and assessing aspects of the curriculum materials. In the first interview, Ms. Archer provided the highest percentage of instances of Extend. However, the ratio of comments on her teaching decreased in later interviews. In our analysis, we found that Ms. Archer shifted over time to include less reasoning more reconstructing in her comments. Despite the decrease in Extend comments, we found that the Extend instances changed to reconstruct her thinking about teaching as it related to her students rather than just reasoning about her teaching practice itself. Building from our theoretical framework, we assert that this is a more sophisticated focus as the teacher began to attend to the interactions between the elements of the interaction triangle rather than simply focusing on each element independently. Given that the instructional environment is shaped by these interactions, having the teachers attend to them should impact teacher practice.

All four teachers showed changes in their instances of Extend from reporting or reasoning their teaching to considering various aspects including evaluation of their teaching or curriculum materials. For example, in their first interviews, the teachers commented specifically much on a few aspects such as reporting and reasoning; however, in later interviews, the teachers generally reflected on various aspects of teaching while they were talking about student thinking (see Table 2). This implies that these teachers' reflections improved in the sense that the teachers noticed more aspects of teaching and situations in their teaching as they gained experience in analyzing the video clips. This experience allowed them to not simply focus on their students' understanding and report their teaching actions, but also to evaluate and reason about their actions and the curricular materials as they related to student understanding. Again, building from our theoretical framework, this showed a movement to looking at the interactions between what they did, what the curricular materials included, and how the students understood the mathematics of interest. In other words, it built the teachers' professional knowledge in ways that allowed them to coordinate their understanding of how the three elements of the classroom work together and reconstruct their teaching with their students in the context.

*Table 2*
Instances of Extend for each Teacher

| **Extend instances** | **Ms. Moseley** | | | **Ms. Reese** | | |
|---|---|---|---|---|---|---|
| **Date** | **3/13/03** | **4/08/03** | **5/22/03** | **3/17/03** | **4/08/03** | **5/13/03** |
| **Report** | 4 | | | 8 | 7 | 2 |
| **Evaluation** | 4 | 1 | 4 | 1 | | 2 |
| **Reasoning** | 3 | 2 | 4 | 4 | 5 | 5 |
| **Reconstruct** | 8 | 2 | 3 | 2 | 1 | 4 |
| **Others** | 2 | 1 | 2 | 2 | | 4 |
| **Total** | 21 | 6 | 13 | 17 | 13 | 17 |
| **Extend instances** | **Ms. Archer** | | | **Ms. Bishop** | | |
| **Date** | **2/10/04** | **3/01/04** | **5/25/04** | **3/17/03** | **4/08/03** | **5/13/03** |
| **Report** | 1 | 2 | 3 | 6 | 9 | 2 |
| **Evaluation** | 2 | | 2 | | 5 | 2 |
| **Reasoning** | 3 | 7 | 5 | 6 | 7 | 2 |
| **Reconstruct** | | 3 | 4 | | 6 | 4 |
| **Others** | 2 | 2 | 5 | 1 | 4 | 4 |
| **Total** | 8 | 14 | 19 | 13 | 31 | 14 |

*Increased Incidents of "no idea"*

We anticipated that, over time, the teachers would become clearer and less hesitant in explaining their interpretations about students' understanding. However, in our analysis, we found the opposite to be true for these teachers. This shift in the instances of *Interpret* occurred for all teachers, indicating that experience was not a factor in this aspect of our analysis. In the initial interview, the teachers provided interpretations of their students' mathematical understanding, but in later interviews, they often said "I don't know how he/she gets this" or "I have no idea." Perhaps, as they became more aware of individual student thinking through viewing these videos, the teachers were beginning to question their preconceived ideas about student understanding. In contrast, it could be that the teachers were becoming more comfortable with the interviewers and felt safe admitting when they did not understand a student's thinking. Regardless of the underlying cause in the increase of "no idea" comments, it was clear that the teachers did not always know how to interpret the students' thinking. We expect that reflection built into the regular practices of teachers might impact the teachers' abilities to interpret student thinking; however, the duration of this study was too short to test this assumption.

## Discussion

The data presented here suggest that the teachers who participated in this project began to change in the ways they reflected regardless of their years of teaching experience. While not large, these changes indicated that these teachers gained more insights about students' understanding as they gained experience in analyzing student understanding. The role of the researchers in this study is important because the researchers persisted in asking questions about students' understanding each week and selected video clips to support the teachers in discussing student understanding. This forced the teachers to reflect on their students' understandings week after week for the six to eight weeks of the unit. Without the researchers, the teachers likely would not have reflected on the students' understanding. We assert that the consistent interviews supported ongoing development of teacher knowledge that would not have occurred without the intervention. However, it should be noted that this study was not intended as a professional development study, rather the changes in reflection appeared to be the products of the teachers' changing in their reflection abilities simply through regular engagement in reflection.

Consistent with Sherin and Han's study (2004), we found that our teachers also shifted to focus more on student conceptions than pedagogies over time. This is especially interesting given that our context was quite different from theirs in that our study focused on one-on-one interactions between the teacher and the researcher rather than group interactions among teachers. Also, in our study, the researchers selected the classroom events on which the interviews focused whereas in Sherin and Han, the teachers attempted to interpret their students' mathematical thinking and, in the process, they struggled to explain their interpretations. These results suggest that reflection may be the critical aspect in promoting attention to student understanding as a product of the classroom interactions. Further, the video clips used in this study played a role of useful sources to engage teachers in reflecting on. This suggests that teachers can quickly develop relatively the ability to analyze student mathematical understanding if they were provided appropriate sources for reflection.

Clearly, further research is needed to understand the impact of reflection on teaching and learning. Reflection, as a form of professional knowledge, involves teachers' analyzing their own practice and using the conclusions they draw from the analysis to drive their future teaching. In this study, we examined teachers' reflection-on-action as prompted in one-on-one interviews. Our findings suggest that professional development may consider ways of using reflection and ways of engaging teachers to reflect on in order to raise teacher awareness of the interactions among the three elements of the interaction triangle.

## Endnotes

1. All names are pseudonyms.

2. The pseudonyms for teachers used in this study are consistent with those used in other publications of the CoSTAR project.

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# SUPPORTING TEACHERS TO INCREASE RETENTION

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*U.S. national reports have identified the need to increase the pool of highly qualified mathematics teachers as a way to improve mathematics education. However, providing high quality mathematics education for all students goes beyond the recruitment of knowledgeable teachers. This paper offers an examination of the role that professional development plays in the work and retention of new teachers and/or teachers in hard-to-staff settings. Based in California and including 10 sites and more than 250 teachers, first and second year results from data collected through large group surveys, online logs and site level focus groups helps to explain why the attrition rate over the first year of the study dropped from 20% to 11%.*

## Introduction

The preparation, support, and retention of mathematics teachers in grades 7–12 merits careful examination. Several U.S. national reports have pointed to the need to increase the pool of highly qualified mathematics teachers as a way to improve mathematics education (National Academy of Sciences, 2007; Glenn Commission, 2000). However, providing high quality mathematics education for all students goes beyond the recruitment of mathematically knowledgeable teachers to encompass issues of teacher preparation, support, professional development, and retention. Analyses of teacher employment patterns reveal that new recruits leave their school and teaching shortly after they enter. Ingersoll, using data from the School and Staffing Survey concluded that in 1999-2000, 27% of first year teachers left their schools. Of those, 11 percent left teaching and 16 percent transferred to new schools (Smith & Ingersoll, 2003). Earlier research revealed that teachers with the highest qualifications tend to leave first (Schlecty & Vance 1981). This "revolving door" is even higher in large urban districts; for example, 25% of the teachers new to Philadelphia in 1999-2000 left after their first year and more than half left within four years (Neild and other 2003). In Chicago, an analysis of turnover rates in 64 high-poverty, high-minority schools revealed that 23.3 percent of new teachers left in 2001–2002. In California, 10% of the teachers working in high-poverty school transferred out in 2006–2007 (Posnick-Goodwin, 2008).

Reasons for the attrition of new teachers and teachers in high-poverty schools are often related to "working conditions" and lack of support (Ingersoll, 2001; Smith & Ingersoll, 2004; Johnson et al., 2004), though pay also plays a role (Hanushek, Kain, & Rivkin, 2001). This support includes both professional and collegial support such as working collaboratively with colleagues, coherent, job-embedded assistance, professional development, having input on key issues, progressively expanding influence and increasing opportunities (Johnson, 2006). Preparation, support, and working conditions are essential to teachers' effectiveness and their ability to realize the intrinsic rewards that attract many to teaching and keep them in the profession despite the relatively low pay (Johnson & Birkeland, 2003; Liu, Johnson, & Peske, 2004; Lortie, 1975).

Most of the research on the "support gap" has dealt with elementary teachers and national data. To focus attention on the retention and the impact of support and professional development

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on mathematics teacher retention, both those who move and those who leave teaching altogether, a study was developed in the Fall of 2006 by the California Mathematics Project. This study is entitled California Mathematics Project Supporting Teachers to Increase Retention (CMP STIR) and was funded by the California Postsecondary Education Commission under the Improving Teacher Quality Grants to address both dimensions of teacher retention across California. The project focuses on teachers from schools and/or districts eligible under the No Child Left Behind guidelines who are in their first five years of teaching or teachers in hard-to-staff schools.

A major component of the project is research that would extend and deepen the knowledge base on mathematics teacher retention. Since the base for this study of support and sustainability of mathematics teachers involves 10 CMP regional sites, its design is complex, diverse, and builds on a 25 year-old professional development network, thus embracing a diversity of perspectives on the retention of mathematics teachers through support and professional development.

This paper will include an overview of the project, an outline of the research design and the results of the first two years. Results will include project as well as site specific analyses, and reflect upon opportunities and challenges.

## Research Design and Methodology

The project is multifaceted in both the range of professional development models and the research design. In general, CMP STIR is a 5 year intervention project with the first three years focused on systematic and sustained support, year 4 supports leadership development, career advancement and school and district support, and year 5 emphasizes collaboration, communication, and dissemination. Although specific dimensions of the professional development vary from site to site, the general model for the first three years is (1) intensive professional development and (2) systematic and sustained academic year support. The intensive PD consists of Institutes and follow-up, content, and communities of practice, while the support may include coaching, lesson study groups, school site networking, data driven reflection, access to resources, district and/or school support.

To study the major question of teacher retention, the project design consists of both quantitative and qualitative data. Overall, the research design encompasses project level longitudinal data, site level data, and case studies in three of the sites. The project level component includes baseline data, annual surveys and exit surveys. The site level data involves baseline data providing a history of attrition for each site across a five-year period from 2002–2007, teacher content knowledge, student achievement data, site yearly reports, teachers' monthly logs, focus group interviews, administrator interviews, and exit interviews. The case studies include classroom teacher interviews and observations for both project teachers and control teachers.

Project level, site level, and case study observations and interviews are linked through the inclusion of similar questions, such as a question about confidence and one about competence are found in the annual survey, the focus group questions, the log prompts and the case study interviews. The question of how much longer the participant expects to teach and why is also found in each level. Since most of the data collected is self-reported, triangulation helps provide validity and reliability. Yearly data allows us to trace trends within and across years. Tracking and understanding teacher retention patterns as related to professional development relies on site level models and input. Site level perspective provides the basis for on-going discussions and a serious element of reality. In this paper we choose to focus project findings by describing the unique features of one particular site's model, California State University Bakersfield (CSUB) in Kern County, and relate this site's results regarding opportunities and challenges translated at the site level. At CSUB, teachers are supported in taking one course per quarter in the Master's of Arts in Teaching Mathematics (MATM). The Department of Mathematics being very collegial, this model not only impacts subject matter content knowledge, but also allows for effective collaborations among participants while working together on course work, or sharing teaching ideas and other concerns with one another and the faculty in an atmosphere of nurtured community. The heavy technology focus of most courses such as Dynamic Geometry, Discrete Mathematical Models, or Numerical Approach to Calculus, engages teachers in first-hand experiences with educational technologies they are then able to transfer to their own classroom. This choice of support model draws both from the program coordinators' knowledge of the District's needs, and from the nature of the MATM. Kern County recruits heavily from the upper Midwest. These teachers generally obtain their teaching credential along with their baccalaureate degree, whereas in California, the credential generally requires an additional year of post-baccalaureate courses. Thus, many teachers obtaining their credential outside of California fall behind in graduate credits and end up at the bottom of the pay scale. Historically, these teachers would earn a graduate degree in Counseling or Administration and leave the math classroom. Furthermore the collegial, technology intensive and constructivist nature of the MATM meets some of the immediate needs of these individuals by breaking their classroom isolation while providing the tools to implement new teaching strategies with their students. Early in the data collection, one teacher shared with us "I already know what the unsuccessful lesson feels like. I hope to learn the successful strategies that will make my lessons work!"

In addition to this intensive course work, participants at CSUB have the option to attend yearly summer institutes aimed at addressing specific pedagogical content knowledge needs. These needs are self-identified during the school year to promote a model of professional development that is not only Content Knowledge and Community centered, but also focused on Teacher, and Assessment and in-line with the recommendation from the "How people learn" framework (Bransford & Brown, 2003). In summer 2007, 20 participants attended a "Meaningful Algebra" one-week intensive workshop that addressed the key California algebra standards, developing activities adaptable to any algebra book, modeling instructional strategies that teachers can use in their classrooms, and providing research-based information about learning. In summer 2008, 6 participants attended a Proportional Reasoning workshop during which Susan Lamon's books on teaching ratios and fractions for understanding (Lamon, 2006) as well as integrated-type curriculum supported the teachers' inquiries and discussions. Other support and engagement activities for participants include attending additional technology-based workshops and conferences, engaging in leadership positions at their school, self-reflecting on teaching practices through action research projects, and presenting at regional conferences.

## Results

Project sites work with teachers from schools and/or districts across California. Locations of the sites range from Northern California to the border of Mexico and include urban and rural communities. Each site is expected to work with at least 27 teachers. Data from the initial survey conducted in the Fall of 2007 included 266 teachers with site participation ranging from 17 in a small costal community to 60 in an urban setting. Each site also identified teachers for a project level control group that consists of 83 teachers across the state. The CSUB site currently serves 25 teachers in 17 high-need schools. Even though all participants are in their first five years of teaching mathematics, demographics vary across the group from newly graduates, to second career, with some individuals more seasoned in serving the district.

Baseline retention data for the state-wide project was collected in the site proposals which included a five year study of retention. That is, for mathematics teachers who were teaching in June, how many returned to teach in September. Each site compiled this data for the period of 2002–2007. Across this time period the attrition rate was consistently 20% both across years and sites. In addition to the attrition, the need across the five years, as reflected in the number of mathematics teaching positions reported each September, showed an increase of more than 60%. That is, the 600 math teaching positions reported in 2002 grew to 982 teachers in 2006. One site was able to trace individual teachers across the five years. The attrition rate for this site was 40% when the new teachers were tracked across the five-year period. In no case did the baseline data reflect how many left teaching and how many moved to another teaching position or administrative position.

Initial surveys included a baseline for retention expectation which provided a sharp contrast with the pre-project baseline data. In 2007, the participating mathematics teachers across the 10 sites reported that 94% expected to continue teaching for at least 3 years and 81% expected to teach for at least another 6 years. Additionally, the individuals in the 6% who expected to teach for only one or two year more were those who anticipated retiring or moving into administrative roles. Expected retention was traced through multiple instruments. Similar retention expectations were reported in the focus groups administered during the intensive professional development of 2007. A log prompt completed in March 2008, however, signaled an increase in expected attrition ranging from 10% to 20%. Actual first year attrition results varied from 4% to 29% with an average across the sites of 11%. What happened in these teachers lives that changed their expectations? Data collected across the multiple instruments and levels are designed to help understand these changes but the exit interviews conducted at each site are especially important. For example, since the CSUB project started in 2007, 5 participants have withdrawn due to conflicts of interest but are still teaching in the district, and 2 participants discontinued their participation to a change career: one went back to his prior engineering career after being unable to follow through with his alternative license, while another followed her parents' advice to pursue a profession in Law. The increase in expected attrition identified in the spring log coincided with a time when budget cuts where announced throughout the state and teachers were receiving Reduction in Force notices, especially those new to teaching.

Across all sites, beginning in September of each year, six electronic logs are collected throughout the academic year. Due to serious follow-up by each site, the response rate was very high varying from 53% to 100% with an average of 78% across sites and logs. The first two logs addressed teaching goals and support for classroom teaching. Results from these responses reflected the impact of the professional development with 84% indicating that the professional development affected their goals and/or supported their classroom teaching. Logs 3 and 6 looked at their perceived success as a teacher and the success of their students. In log 3, the teachers looked at opportunities and challenges and whether their students were as successful as they would like. Only 13% of the participating teachers were satisfied with their students' achievement. But, when they talked about the opportunities and challenges, student achievement became both a challenge and an opportunity. Instead of deciding to leave teaching or looking only at the challenges, the majority discussed the opportunities for learning how to teach. The final log asked the participants to describe a lesson that they especially liked and the support they received in developing this lesson. Seventy-nine percent attributed contributions to the lesson from the professional development and/or community developed through the professional development.

As we conduct research to gauge the effects of our intervention on teacher retention and support, each site gains valuable feedback on the model of professional development offered, which in turn helps refine the support provided. At CSUB, self-reported needs seem to be met by STIR, whereas a greater support from the school site is requested regarding assistance in instructional approach and curriculum. Some participants, even though they acclaim the community found at the university, still suffer from isolation at their site and express frustrations regarding workload, curricular pace, and lack of parental, administrative or collegial support. At times these frustrations seem enhanced by a new awareness of what teaching for understanding could potentially look like, and the evident discrepancy with their school environment or the reality of their classrooms, as one of our teachers points out, “There is a disparity between what I know how to do in my classroom and what I actually do. Much of the difference is caused by a lack of time to prepare, and lack of chance to reflect on what was done. This disparity is the cause of building frustration.”

As we enter the second year of intervention, online feedback also reflects enthusiasm for the program. The Summer Institutes have helped some implement successful lessons in their classrooms. Some have turned what they learn in graduate school into enrichment activities for their students, especially through history and technology. Teachers welcome the opportunities provided to use technology in their classrooms as they have become very familiar with it as users in their university courses. The MATM is technology-focused; mathematical concepts are discovered and discussed through inquiry-based activities encouraging participants to try new teaching strategies with their students as one teacher sums it up: “I have used Geometer’s Sketch Pad with my LCD projector onto my classroom whiteboard. I learned how to use the program in my class thru the CMP STIR program. Likewise, I have used my knowledge developed from the program about cooperative learning exercises with my classrooms. Overall, my students have been very pleased about how the classroom lectures are conducted and how much more they are involved compared to previous math teachers.” Some mathematics faculty teaching in the MATM have become role models for these beginning teachers who express a desire for their students to access deeper conceptual understanding of the material. Teachers also want guest speakers to enhance classroom motivation, feel empowered by their participation in STIR, and supported by a learning community that exists beyond the MATM courses; confidence levels have increased. Recently, teachers started asking for “more”: more classroom games enhancing students’ motivation; more enrichment ideas; more on-site support; more insights on how to teach lower-level mathematics; more involvement from administrators and parents.

Focus Groups are group interviews conducted at each site during the summer intensive professional development. The purpose of the Focus Group interview is to talk with a randomly selected group of participants during the intensive PD about their experiences to date, and to obtain data that might not surface by other methods. All of the Focus Group interviews are conducted by the same person and last between 45 to 70 minutes. Responses are coded and the data is compiled for all sites. Due to group dynamics, not every participant answers every question. In some cases, several teachers would nod in agreement to a response, visual affirmations that are not recorded while most of the verbal responses are transcribed during the interview.

The 2007 Focus groups included 59 participants consisting of 28 middle school teachers (including 3 special education teachers), 20 high school teachers, two pre-service students not assigned to a school, a middle school/high school coach, and a special education teacher of grades 4–6 (data for seven teachers are missing). When responding to “why are you participating?” and “what did you hope to gain?,” participants often responded with what they had already gained through networking with each other and with institute leaders: ideas for working with students, as well as for teaching and understanding mathematics conceptually. Ideas that linked to the classroom and time allocated for planning with colleagues were identified as successful aspects of the institute. Three key challenges emerged: how to transfer what was learned to the classroom, how to balance home, institute, coursework, and summer school, and how to address mathematics content that was challenging. As a result of the institute, many were realizing the importance of teaching conceptually, with concerns that they won’t be able to implement it with fidelity. Several teachers shared that they felt comfortable with the lecture method and that changing would be a challenge for them. Accordingly, they want the site to support them in the classroom through regular observations and coaching.

About 85% of the respondents said that teaching would be a lifelong career. In response to why they went into teaching, most said they liked the idea of teaching and helping others, liked helping students specifically, and influencing them with positive experiences in math. Other reasons included making an impact, coming from a family of teachers, and liking mathematics. About 10% mentioned that teaching was a second career. Most indicated that they would be interested in a leadership positions if they did not have to leave the classroom, such as become department chair or give presentations. A few talked about going into administration or becoming a college instructor.

For the 2008 Focus Group interviews, another group was randomly selected to participate. Future plans still showed high retention expectations with 88% expecting to be lifelong teachers and 5% expecting to leave teaching in 5 years. One-third of the participants mentioned an improvement in their classroom teaching. Regarding changes in the delivery of instruction, 35% identified higher level cognitive demands of students in questioning and problem solving. In terms of content, teachers reported deeper understandings of math, and greater accessibility to new mathematical ideas. Table 1 below shows changes in confidence that the participants can make a difference in student learning of mathematics, and competence in their knowledge of mathematics.

*Table 1*
Self-Reported Confidence and Competence in Teaching Mathematics (4-Point Scale)

| | Confidence | | | | Competence | | | |
|---|---|---|---|---|---|---|---|---|
| Year/Level | 4 | 3 | 2 | 1 | 4 | 3 | 2 | 1 |
| 2007 | 15 | 32 | 12 | 0 | 14 | 34 | 12 | 0 |
| 2008 | 31 | 26 | 1 | 0 | 23 | 29 | 6 | 0 |

Support, networking and community are reflected in the responses to what has been gained to date as well as from prompts that looked at support provided, most useful support, what was gained in summer of 2008, and unexpected benefits. The following is a quote from the last log of 2008 that resonates with many of the Focus Group responses:

> It was great to work together and talk about what worked and what didn't. It was great to hear about the "aha" moments when they caught on that the number was always approximately 3. We discussed many ways of teaching this concept and it was very useful for us to bounce ideas back and forth to see what has worked or not worked in the past.

## Conclusion

This project is only in its second year but critical questions are being addressed. The key issue of the role of professional development on teacher retention shows reductions in mathematics teacher attrition for teachers in their first five years of teaching or teachers in hard-to-staff schools. Through the multiple levels of data collection and the triangulation of the data, dimensions related to this connection are surfacing. They include increased teacher perceived content and pedagogical knowledge and the development of professional mathematics teacher communities. Another major theme that has emerged across the sites and will need to be examined in future years is the issue of teachers' confidence in and competence with learning and teaching essential mathematical understandings.

As we look ahead, a greater presence at the school site will be necessary for the CSUB model. This need may be addressed through seeking administrators' involvement in supporting the program, focusing the teachers' thesis on action research projects, and moving into the leadership phase to help broaden the community. On-site leadership positions are already assumed by some of the participants. Additionally, 6 potential leaders were identified and/or expressed interest in leadership and participated in the San Joaquin Valley Mathematics Project Winter Leadership Retreat this winter. They shared very positive impressions with us: "The retreat was one of the most positive experiences of professional development I have ever participated in". Further leadership opportunities will include an involvement as workshop leaders in the induction program for beginning teachers, and presentations at regional conferences.

To conclude, success of a retention initiative takes root in a variety of needs: the need to know your District and its teachers—a necessity that often relies on established, long-term relationships between the university and district leaders; the need to offer sustained support as opposed to punctual interventions in order to break the isolation of beginning teachers and create a sustainable community; the need to establish relevance in the professional development activities proposed by engaging participants in deep introspection of their own knowledge gaps; the need to involve all actors of the community to prevent miscommunication from annihilating attempts made towards change; the need to nurture the community created by moving its members forward into roles and responsibilities they are ready to take on; and last but not least, the need to refine even successful models to keep the momentum.

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# PROFESSIONAL DEVELOPMENT FOR TEACHING IN CONNECTED CLASSROOMS

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*A U.S. federally funded nationwide field trial in 118 school classrooms of algebra 1 implemented Texas Instruments TI-Navigator™. The year 1 control group implemented the treatment in the second year. The purpose of this paper is to describe the professional development program and evaluate the summer institutes for both cohorts of teachers.*

## Purpose and Context

Teaching reforms and innovative interventions require carefully planned professional development programs to support implementation (Clarke, 1994). Large-scale research efforts in education often utilize teacher training as part of the intervention in order to effect change in student outcomes. *Classroom Connectivity in Promoting Mathematics and Science Achievement* (CCMS)[1] is a national research study examining the impact of classroom communication technology on student achievement, dispositions toward mathematics and science, and self-regulated learning. The intervention for this project consists of professional development to support algebra 1 and physical science teachers' implementation of a modern classroom connectivity technology, the TI-Navigator™ that allows classroom teachers to wirelessly communicate with their students' handheld devices. The research design is a randomized cross-over field trial in 118 Algebra I classrooms. Cohort 1 teachers attended a summer institute and implemented the intervention in the first year. Cohort 2 was a control group in the first year using graphing calculators and implemented the TI-Navigator™ in the second year of the study. This paper reports the practices and findings related to the professional development (PD) efforts for the mathematics teachers in the CCMS project.

## The Connected Classroom

Connected classrooms in this study were equipped with the TI-Navigator™, a system connecting each student's handheld graphing calculator with the teacher's computer. Four handhelds are wired to a hub. The students and teacher communicate wirelessly via the hubs through an access point connected to the teacher's computer. Using the *Quick Poll* feature, the teacher can pose an individual question, and *LearningCheck*™ is a feature by which several questions can be sent to the calculator. Student responses may include multiple choice, true/false, and open-ended responses. The results can be displayed as bar graphs using a projector. A third feature is *Screen Capture* by which the teacher can take a “snapshot” of all calculator screens. The teacher can hypothesize about and diagnose errors or display a selection of screens to the class to foster class discussion of students’ work. The fourth feature of the system is *Activity Center* in which a coordinate system can be displayed. A typical Activity Center lesson might include the teacher displaying a line with the assignment, “match my line.” Similarly, the teacher can ask students to send the equation of a line through a given point parallel to or perpendicular to a line entered by the teacher. Finally, the teacher can aggregate data collected by students, display it on the screen, and send the lists to students to analyze. Activity Center is typically used to develop conceptual knowledge. Quick Poll and Learning Check are typically used as tools for formative assessment. Screen Capture may be used for either of these purposes. The teacher has immediate information that may lead to adjustment of instruction. Using any feature, the teacher can engage the class in discussion diagnosing incorrect responses that can be displayed anonymously. As aggregate class results are shown to the class, students receive immediate feedback in a private non-threatening way that can encourage them to reflect and discuss their understanding or methods of solution in small groups and with the class as a whole (Roschelle, Penuel, & Abrahamson, 2004).

## Perspectives on Professional Development

Various models of professional development are described in the literature (Guskey & Huberman, 1995; Loucks-Horsley, Love, Stiles, Mundry & Hewson, 2003; Rodriquez & Knuth, 2000; Smith, 2001). Well designed PD programs are teacher-driven (Borko & Putnam, 1995; Little & McLaughlin, 1993) and focused on documented learning needs for students (Fullan, 1993; Howey & Collinson, 1995). Loucks-Horsley et al. (2003) offer a framework for professional development in science and mathematics that includes a commitment to a vision and standards, analysis of student learning, goal setting, planning, execution, and evaluation in the context of classrooms, with consideration of teacher knowledge and beliefs.

Clarke (1994) describes ten important research-based principles of successful professional development: 1. Determine initial teacher interest in the topic of professional development and provide an element of individual choice; 2. Develop collegial working groups within schools with broad community and administrative support; 3. Identify and manage possible classroom, school, and district level barriers; 4. Model actual classroom approaches to help teachers develop better understanding of the change; 5.Request teacher commitment to active and sustained participation in their individual classroom context; 6. Acknowledge the importance of classroom practice on teacher beliefs by encouraging teachers to validate the practices in their own classrooms; 7. Provide time for reflection and opportunities for group discussion and feedback; 8. Allow teachers to develop ownership by encouraging the development of their professional judgment regarding implementation; 9. Acknowledge the slow and incremental process of change and celebrate small successes; 10. Encourage goal setting for continued professional development. The CCMS Summer Institutes aimed to enable expansion of teachers’ professional knowledge base including their pedagogical content knowledge, their subject matter knowledge and their beliefs, experiences and habits (Borko & Putnam, 1995; Eraut, 1995).

### CCMS Professional Development

Given the complexity of teaching with the TI-Navigator, a significant initial professional development program was required. We designed a weeklong summer institute for the summer before a teacher's implementation of the TI-Navigator. Given what we know about the need for regular continuing professional development (Loucks-Horsley et al., 2003), the teacher participants meet one day before subsequent Teachers Teaching with Technology ($T^3$) International Conferences for targeted CCMS professional development and attended the three day conference. Web-based follow-up training tutorials on graphing calculators and TI-Navigator were available on the $T^3$ website, and a project listserv was made available to the participants as an opportunity for questions and discussion.

In technology-laden training, an unwarranted focus on the technical aspect of making the technology work is appealing. Yet an equal emphasis on the pedagogy of using the technology is extremely important. Important components of the summer institutes included: a) focus on student learning in algebra 1; b) extensive hands-on practice in both 'student' and 'teacher' roles; c) curriculum-specific applications; d) the pedagogy of the connected classroom, especially self-regulated learning and formative assessment; e) experienced teacher-instructors who use the technology in their own secondary mathematics classrooms; and f) differentiation and practice based on the teacher's technology ability level. Discussions about pedagogy for using the technology were built into the workshops in a practical, hands-on manner. The goal was to teach theory through real-life examples of classroom activities. Often these were introduced spontaneously as the participants played the part of students for the workshop instructor's "expert teacher," as s/he illustrated how to use features of the connected classroom system. In this way, the context became sensitive to specific classroom needs of the participating educators (Pink & Hyde, 1992).

The model used for the Summer Institute was based on lessons learned in the $T^3$ PD program founded at The Ohio State University in 1988 (with NSF support) that closely parallels the Clarke (1994) principles stated above. The $T^3$ trained instructors were secondary school teachers who had used the technology in their mathematics classrooms and modeled the methods from their classroom experience. In addition to the three primary $T^3$ instructors who assumed responsibility for the PD planning and delivery, three other $T^3$ instructors who were also classroom teachers provided support during practice sessions. This enabled the differentiation of practice sessions based on technology ability levels. Following the $T^3$ model, we held daily debriefing sessions so that adjustments could be made for the next day. Mathematics content for the institute was from typical Algebra 1 curricula. Activities included in the institute were technology-focused, hands-on instruction and practice in "student" and "teacher" roles. Participant products for the week were lesson projects and presentations.

Finally, faculty lectures infused theoretical and pedagogical focus on productive classroom discourse, formative assessment (FA), student self-regulated learning strategies and the teacher's role in supporting self-regulation, and learning environments from *How People Learn* (HPL) (National Research Council, (NRC), 1999). Self-regulated learning (SRL) is consistent with mathematics education reform. For example, in *Adding It Up*, Kilpatrick, Swafford and Findell (2001) call for strategic competence, adaptive reasoning, and conceptual understanding as well as procedural fluency as components of mathematical competence. Among contexts that support SRL are multiple representations and rich mathematical tasks, classroom discourse, environmental (classroom) scaffolding of strategic behavior, an evaluation system that emphasizes feedback, and self and peer evaluation (Pape, Bell & Yetkin, 2003). Among the components of contexts that support SRL are teacher's higher order questions, teacher press for student involvement, teacher press for elaboration, explanations and justifications, soliciting multiple answers or solution methods, mastery orientation, and scaffolding and social support (high expectations, respect, and inclusion of all students in the learning process) for student achievement (De Corte, Verschaffel, & Eynde, 2000; Pape, 2005).

Another component of faculty lectures was HPL "centerednesses" for the design of learning environments. *Learner-centered* environments "pay careful attention to the knowledge, skills, attitudes and beliefs that learners bring to the educational setting" (NRC, 1999, p. 133). These environments are termed "culturally relevant," or "culturally responsive," and include "diagnostic teaching." "*Knowledge-centered* environments take seriously the need to help students to become knowledgeable" (p. 136). They "focus on the kinds of information and activities that help students develop an understanding" (p. 136) of the algebra content. *Assessment-centered* environments "provide opportunities for feedback and revision and what is assessed must be congruent with one's learning goals" (pp. 139-140). Included are formative assessments and appropriate summative assessments. *Community-centered* environments include "the classroom as a community, the school as a community, the degree to which students, teachers and administrators feel connected to the larger community" (pp. 144-145). Using a TI-Navigator with pedagogically sound techniques has the potential to support a learning environment with these centeredness characteristics.

The last component of faculty lectures was formative assessment. Cowie and Bell (1999) define formative assessment as "the process teachers and students use to recognize and respond to students' learning and enhance it *before it is complete.*" In addition to the timeliness of the formative assessment process, Black and Wiliam (1998) direct attention to the "*teacher use* of assessment information to modify and improve their teaching effectiveness." In an analysis of more than 40 studies, high quality formative assessment was linked to significant learning gains (Black, Harrison, Lee, Marshall & Wiliam, 2003). Black et al. identified enhanced feedback loops, active student participation, teacher modification of instruction based on knowledge of student learning, and increased motivation and engagement as important characteristics of successful formative assessment studies.

## Participants and Data Collection

The Institute was designed to last one week. For Cohort 1 in the first summer, 30 participants attended the first week, and 30 participants attended the second week. Out of 60 total teachers, 42 were female (70%) and 18 were male (30%). For Cohort 2 in the second summer, 27 mathematics teachers participated in week 1 and 20 attended week 2. Of the 47 teachers in the second summer, 35 were females (74%) and 12 were males (26%). Each day of the Summer Institute was videotaped to document the progress of the PD. In addition, participants completed an Institute Evaluation Survey in the afternoon of the last day of the Institute.

## Evidence of PD Instruction

The videotapes from the Institute provide evidence of teacher-instructors incorporating pedagogical issues into their instruction. In the segment below, one of the instructors described insisting on student explanations and justifications. He then described how to use Quick Poll for teacher error analysis during or after class.

Instructor 1 (I 1 discussing the Screen Capture feature): Before I show it to the class, I could group all the 'good' wrong answers to the top. I could ask, "How did someone get such and such. I could have kids defend. Why is 3 the correct answer? Have them tell why do you think it is. Why is it not? Having kids starting to talk mathematics . . . I can take a picture of the entire record. When I have a question and I want to know who is really with me on this topic, . . . I can capture this entire [class] snapshot to look at later during my conference period. Also, in addition to [surveillance], saving a [Screen Capture] picture is crucial for us deciding which kids need remediation, which kids need intervention, who do I need to focus on, all this educational stuff we have to be concerned about. We can save it to look back at later. In the heat of the battle of the classroom we might not be able to address every single misconception. [July 31, 2006; year 2, day 1]

On another day the teacher-instructor reinforced pedagogical strategies of speaking mathematically in class discussion, classroom discourse, and engagement.

Participant 1 (P 1): We have to reset our window.

I 1: Ahh – This is a great discussion because I would hope the kids would say our window [on the calculator] is not right now. What do we need to change our window to?

P 1: Something above 84.

I 1: So we have students in the calculator change the window to something above 84. Now we have a discussion. I want to pause for a second. Why do you think it might be more powerful to have every student look at this graph versus the teacher standing up here just graphing "watch."

P 2: Ask them what they thought.

P 3: And you could do a Quick Poll of what they thought.

P 4: That way you would have everybody engaged.

I 1: You said the magic word, engaged! They are taking active participation in the activity. . .

I 1: (summarizing a few seconds later) Obviously we are talking about discourse, and this is great discussion. In the Navigator world this is what should be going on in your classroom. It is different from the days when the teacher lectures and the kids sit there . . . and write stuff down. . . . So we get engagement. [August 2, 2006; Year 2; Day 3; PM]

In the brief episode below, one instructor emphasized the importance of planning good pedagogy, speaking mathematically in classroom discourse, and student error analysis and revising incorrect answers. In discussion a participant turns the topic by asking about indicating the correct answer for a Quick Poll. Another instructor discussed the potential for increased argumentation if the correct answer is not indicated.

I 2: When it comes to your presentations on Friday, keep everything in mind. You should use all of this technology in a pedagogically sound way. [I 2 asks a question followed by a brief participant response; I 2 continued:] That's the discourse; when we get to the point where we can talk about mathematics and talk about error analysis and get kids to revise their incorrect thinking and get their incorrect thinking to be correct. I use a little trick: I say give me a good case for why we choose 'false;' or make a good case for choice 'c.'

P 5: If they saw the correct answer [bar] in green, there would be no discussion. If you set the system so the correct answer is not indicated we can go back later to see the correct answer.

I 3: Certainly there is tremendous discussion without knowing the correct answer, because the kids argue more vehemently. Sometimes, I seed my questions so the obvious [apparent] answer is the wrong one. [August 1, 2006; Year 2; Day 2; PM]

**Institute Evaluation Survey Results**

The Institute Evaluation Survey was administered at the end of each summer institute (see Table 1). Overall, participants rated the Summer Institute very highly on a Likert-type scale with means on appropriateness of content of 4.5 and 4.8. Participant's perceived usefulness of the pedagogy and their self-confidence in using the TI-Navigator exhibited means of 4.1.

Table 1. *Institute Evaluation Survey Results*

| Construct | Cohort 1 (N=60) | | Cohort 2 (N=47) | |
|---|---|---|---|---|
| | *Mean** | *SD* | *Mean* | *SD* |
| Appropriateness of institute content (5 items; see Table 2) | 4.54 | *.70* | 4.76 | *.54* |
| Perceived pedagogical usefulness (8 items; see Table 3) | 4.10 | *.85* | 4.08 | *.81* |
| Perceived self-confidence in components (15 items) | 4.09 | *1.18* | 4.10 | *1.02* |

*Note: 1 – strongly disagree; 5 – strongly agree

Table 2 presents participant responses to specific items about the appropriateness of institute contents. Appropriateness included materials and discussions about pedagogy. These resulted in the indicated levels of comfort and confidence that they can teach with the TI-Navigator. Across the two cohorts, the percents of participants who agreed or strongly agreed ranged from 87% to and impressive 98%.

Table 2. *Participant Responses about Appropriateness of Institute Contents*

| Statement: single item | Cohort 1 % somewhat agree or strongly agree | Cohort 2 % somewhat agree or strongly agree. |
|---|---|---|
| The content of this institute was at the appropriate level | 90 | 93 |
| The institute materials were useful and helped me learn how to teach more effectively with the TI-Navigator | 98 | 98 |
| The institute helped make me feel more comfortable about using this TI-Navigator technology in my classroom | 95 | 98 |
| The discussions about how and why to teach with the TI-Navigator (i.e. pedagogy) were valuable to my everyday teaching | 87 | 98 |
| As a result of this institute. I will be able to use TI-Navigator technology in my teaching | 98 | 98 |

Table 3 presents participants rating of their learning of various specific pedagogical issues. The statement in the item stem reads: “As a result of this institute, I have a better understanding of the following pedagogies.” In Cohort 1, 77% to 90% agreed that the institute had helped their understanding. In Cohort 2, 74% to 89% agreed that the institute had improved their understanding of various pedagogical issues.

Participants were asked to respond to the components with the item stem: As a result of this institute, “I have a better understanding of the following TI-Navigator components.” Regarding the primary component uses: Create a class, Quick Poll, LearningCheck, Screen Capture, student inquiry/data aggregation, Activity Center, Class Analysis, send lists applications and programs. Participants agreement (combining somewhat agree and agree strongly) over the two summers ranged from 87% to 100%. On other facilities: network manager, sending internet data to the class, percent of agreement ranged from 22% to 48%, and collecting files was 83% and 89% agreement for cohorts 1 and 2, respectively.

Table 3. *Participant Ratings of their Improved Understanding of Specific Pedagogies*

| As a results of this institute, I have a better understanding of the following pedagogies: | Cohort 1 % somewhat agree or strongly agree | Cohort 2 % somewhat agree or strongly agree |
|---|---|---|
| Classroom discourse to reveal student strategies | 83 | 89 |
| Formative assessment | 74 | 68 |
| Strategies to support developing self-regulated learning | 83 | 77 |
| Questioning strategies | 77 | 83 |
| Critical junctures when I might use the TI Navigator | 82 | 96 |
| Classroom norms that foster understanding and development of strategic behavior | 77 | 80 |
| Mathematical explanations and justifications | 85 | 74 |
| Uptake of correct and incorrect student responses | 90 | 89 |

## Conclusion

A professional development program patterned after the $T^3$ model was designed for the national field trial research study of implementation of the TI-Navigator in Algebra 1 classrooms. An important component of the professional development was teacher-instructors modeling their use of TI-Navigator based on their classroom teaching experience. The teacher-instructor roles were supplemented with faculty lectures on applying theory to classrooms. Additional professional development activities were designed to implement the principle of ongoing PD. Participants met for CCMS specific learning one day per year and attended the $T^3$ International Conference. Additional online training and a listserve for participant communication were made available.

The summer institutes were evaluated highly. Participating teachers generally agreed that they had learned facility with the TI-Navigator, pedagogical issues and that the institute activities were appropriate. Future work will undertake an evaluation of annual daylong PD activities and use of the project listserv.

## End Note

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# REFRAMING FAILURE: HIGH SCHOOL MATHEMATICS TEACHERS' LEARNING ABOUT STRUGGLING STUDENTS THROUGH PARTICIPATION IN WORKPLACE COMMUNITIES

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*This paper examines teachers' developing understanding of student failure in freshman mathematics classes. The question organizing my inquiry was: How do high school mathematics teachers who are engaged in equity-oriented reforms learn about struggling students in their workplace communities? I found that teachers' conversations shifted away from personal reflections and moved toward understanding the classroom systems that contribute to student failure. Teachers' learning was signaled by changes in their framing of student failure. By joining concepts of frame analysis and learning in a community of practice, this study contributes conceptual tools for understanding teachers' learning at the level of mechanism.*

Teachers' ideas about student ability directly shapes the culture of learning in their classrooms. If teachers believe that student ability is fixed, then there is little they can do about students' learning difficulties. If, on the other hand, teachers have a more developmental view, believing that ability is dynamic, they are more likely to adapt their practice and send students messages that encourage persistence in the face of difficulties (Weinstein, 2004). Yet teachers vary in the extent to which they adapt their practice to the diverse learners they encounter (Stodolsky & Grossman, 2000). When faced with underprepared students, teachers often feel a tension between upholding subject matter standards and meeting learners where they are (Horn, 2007). Teachers' conceptions of struggling students are thus important sites for their reconceptualization of student ability. Research suggests that teachers' participation in a strong teacher community has the greatest potential for yielding the kinds of teacher learning that produces equitable student outcomes, though what that learning is or how it might be taking place is largely unaccounted for in the literature (Gutiérrez, 1996; Horn, 2005; Little, 2003; McLaughlin & Talbert, 2001). As such, the overarching research question organizing my inquiry was *how do high school mathematics teachers who are engaged in equity-oriented reforms learn about struggling students in their workplace communities?*

## Rationale

My study targets high school mathematics because mathematics consistently plays a gatekeeper role for students (Moses, 2001; NRC, 1989; Schoenfeld, 2002): "More than any other subject, mathematics filters students out of programs leading to scientific and professional careers […] Mathematics is the worst curricular villain in driving students to failure in school" (NRC, 1989, p. 7). What is most disturbing is the fact that a disproportionate number of poor and minority students compose this group, a disparity that is quantified in the "achievement gap" that plagues our nation. By targeting high school mathematics, I situate my study in the context of a critical gateway/gatekeeping subject.

I focus my study on teachers engaged in equity-oriented reforms for two reasons. First, in keeping with the prior reasoning, a specific portrait of teacher learning about equity-oriented reforms concerning struggling students can directly speak to single-system attempts to change disparities in student achievement by educators. To achieve this goal, I selected a group of teachers who not only chose to engage in equity-oriented reforms but who also had some success with their efforts to improve equitable outcomes. This particular group is made more exceptional as a case of teacher community because it was designed for optimizing teachers' learning (e.g., attending to issues of equity through conversations about curriculum and pedagogy became a part of teachers' daily work) and had considerable external support by our research team. Second, there is presumably a greater impetus for teachers engaged in equity-oriented reforms to question their assumptions and practices, rendering their learning about students, teaching, and subject matter more visible. I make this assumption because a major goal of equity-geared reforms is providing all students with rich opportunities for making sense of essential mathematics ideas. It follows that the conditions surrounding teachers' enactments of reform, such as instruction and classroom culture, must also align with this goal in order to yield equitable outcomes. As such, my focus on teachers who are collectively engaged in equity-oriented reforms is a strategic choice for increasing observable instances of teachers' sensemaking on struggling students.

## Theoretical Framework

*Teachers' Learning in a Community of Practice*

Teachers' participation in their professional communities is a social endeavor. This activity catalyzes a dual process of participation and reification, which is the fundamental process through which learning happens (Wenger, 1998). This learning-as-a-social-phenomenon stance influences my more general conception of teacher community, meaning that the communities do not necessarily have a certain level of functioning, improvement-oriented stance, or meet some other criterion. In other words, these teacher communities are "neither intrinsically beneficial nor intrinsically harmful. Rather, they constitute the *places* in which organizational and individual learning unfolds" (Coburn & Stein, 2006, p. 28, emphasis in original). As such, by adopting a *community of practice* perspective – which Wenger (1998) characterizes as communities where members are mutually engaged in an activity, held together by a joint enterprise, and have a shared repertoire of customs for praxis – I identify *learning* as change in participation within that community. This definition of learning recognizes the co-construction and distribution of knowledge across teachers and takes the wider social context into consideration (Kelly, 2006). Individuals' learning can be conceptualized as movement from legitimate peripheral participation by newcomers to fuller forms of participation by old-timers: "As they progress they acquire the skills, the identity, and the ways of acting and interacting valued by the community" (Coburn & Stein, p. 44; see also Kelly, 2006; Lave & Wenger, 1991; Wenger, 1998).

This theoretical framing is critical for this paper because it places my study of an equity-oriented teacher community in a broader community of practice landscape, which means that the community in my study is not privileged for its equity orientation. In other words, when looking *across* teacher communities, an equity-oriented community is not more or less of a community *as an entity* than another teacher community; when looking *within* the teacher community itself, the community's equity-orientation does not lock it into one state of being (i.e., "strong learning" or "mature"). More importantly, such a framing allows for an equity-oriented description with the understanding that such a description is not necessarily unitary or consistent. Rather, a community of practice framing helps me see such communities for what they are: key sites for negotiation of the meaning of equity-oriented reifications about students, (Coburn & Stein, 2006), which makes observable instances of teachers' learning around these issues more likely (Kelly, 2006). To be clear, this means that it is possible for the community in my study to look like a developmentally mature, strong teacher learning community in one instance, and then perhaps an evolving weak community in the next instance (Grossman, Wineburg, & Woolworth, 2001; McLaughlin & Talbert, 2001); I will need to look well beyond the labels – including equity-oriented – to make sense of teachers' learning through interactions in their workplace communities. This framing provides a value-neutral basis for my analysis of teachers' learning because it allows for the nonlinear, dynamic, zigzag nature of teachers' learning about students, teaching, and subject matter through interactions in their workplace communities.

*Frame Analysis as a Means for Capturing Learning*

My study aims to understand teachers' learning about struggling students in a community of practice context, and so I need conceptual tools that will help capture learning as changes in participation within teachers' interactions – changes that may be subtle, ambiguous, and most certainly complex. I turn to theoretical and empirical work on frame analysis for ways of making sense of this interactive learning process as it unfolds (Benford & Snow, 1986; Goffman, 1974; Snow, Rochford Jr., Worden, & Benford, 1986). Frame analysts look at the interactive process by which frameworks are created in social interactions, and focus on "how people use interpretive frames strategically to shape others' meaning-making processes in an effort to mobilize them to take action" (Coburn, 2006, p. 347). I take framing interactions to be evidence of learning because these processes – processes such as framing (e.g., prognostic, diagnostic, and motivational), reframing, offering counterframes, aligning to frames, and frame amplification (Benford & Snow, 1986; Snow et al., 1986) – mark and describe changes in participation in a community of practice. As such, examining the ways teachers engage in framing interactions in relation to struggling students stands to result in more manageable units of teacher interactions for the analysis of their learning (Evans, 2002; Russell & Munby, 1991).

## Methods

This research takes place in the context of a larger project, *Adaptive Professional Development for High School Mathematics Teachers* (Ilana Horn, Principal Investigator), a design-experiment project situated in part at Septima Clark High School (all names are pseudonyms), a diverse, large, urban comprehensive high school in a large northwestern school district in the US. Our research team worked with the Clark mathematics teachers using a mutual appropriation approach – that is, we collaborated with the teachers to create activities that fit theoretical principles about equitable mathematics teaching while serving the teachers' goals (Cole, 2006). Our precepts included pedagogical principles about equitable mathematics teaching, such as the use of pedagogical strategies to engage learners in important mathematical ideas (Boaler, 2002, 2006; Horn, 2006; Moses, 2001). In addition, we used learning principles for teachers, such as prioritizing providing teachers with collaborative time in the school day to make sense of new practices in their classrooms (Horn, 2005, 2007; Horn & Little, under review; Little & Horn, 2007). For this paper, I examine teachers' learning about struggling students in context of their interactions during this collaborative time.

During the 2004-2005 school year, I followed the interactions of the mathematics department at Clark in my role as a research assistant on the project. I observed classrooms, attended department meetings, and provided classroom support to teachers. One teacher in particular, Susan, struggled with issues related to students, teaching, and mathematics. She asked for my help, and so I provided her with additional classroom-based support several days per week, such as co-planning instruction, doing mathematics, modeling teaching, making sense of student work, and interpreting student interactions. However, even with my classroom-based support Susan still faced a crisis: over 75% of her freshmen students were failing her first-year mathematics course. This crisis caused the other teachers of first-year mathematics to examine their pass rates, and the results were stunning: more than 50% of students taking the first year ($9^{th}$ grade) mathematics course at Clark were failing. The Clark teachers were in a panic over this crisis and asked our research team to help them make changes to their existing curriculum and pedagogy with the aim of improving all the success of all students.

Realizing the ambitious nature of the Clark teachers' plans for implementing starkly different pedagogical and curricular equity-oriented reforms, our team designed an intervention for the 2005-2006 school year to support their reforms. We created the "Freshman Team" intervention by providing the four teachers of first year mathematics with an extra planning period (in addition to their personal planning period) so that they would have dedicated time during the school day to collaborate around issues of teaching and curriculum. We also helped Clark find a new teacher trained in equity-geared teaching practices who could take on the "missing" four first year classes, in addition to being a part of the collaborative team and having her own personal planning period. We aimed our intervention at freshman mathematics because it is – and was at Clark – in this course where students are historically most likely drop out of high school mathematics, a group disproportionately represented by poor and minority students. Clark's principal supported this intervention by crafting the master schedule so that all five team members had a common overlapping planning period. The Freshman Team met during every sixth period, and was composed of five teachers: Susan, Zack, Rose, Julie, and Linda.

Through active participant observation, I collected a variety of qualitative data about the teachers' work, including audio records and fieldnotes of the Freshman Team's weekly meetings, artifacts of classroom practice, records of professional development activities, and teacher interviews. In addition, I have the "headnotes" I collected through my experiences working at Clark that allow me to build connections between events and have a deeper knowledge of the place and participants (Emerson, Fretz, & Shaw, 1995). I crafted a case study around the Freshman Team at Clark because this method will help me concentrate my investigation and analysis on the complexities and particulars of teachers' learning around about struggling students in context of equity-oriented reforms (Merriam, 1998). By conducting an in-depth analysis of the Freshman Team teachers' learning around issues of struggling students I am able to use the case of Clark to theorize about teacher learning inside of a community of practice more generally, which responds to a need for case studies of this nature (NAE, 2008).

*Data Analysis Procedures*

I began my analysis by content logging all of the recorded Freshman Team meetings in chronological order. Then, I systematically examined the content log and timeline resources with my unit of analysis: the *episodes of pedagogical reasoning* (EPRs) (Horn, 2005) that are related to struggling students. Horn (2005) defines the EPRs to be

> units of teacher-to-teacher talk where teachers exhibit their reasoning about an issue in their practice. Specifically, EPRs are moments in teachers' interaction where they describe issues in or raise questions about teaching practice that are accompanied by some elaboration of reasons, explanations, or justifications. (p. 215)

My decision-rule for locating EPRs is based on topical shifts that are related to struggling students. Identifying these episodes allowed me to systematically reduce the larger data corpus into smaller portions of Freshman Team meetings that are the most proximal and relevant to teachers' learning about struggling students.

Next, using my theoretical framework, I coded the transcripts from the most proximal and relevant episodes of pedagogical reasoning based on teachers' sensemaking around struggling students, and in particular, noted instances where teachers use interpretive frames strategically to shape others' meaning-making processes and in what way these frames are being used (e.g, prognostic framing versus diagnostic framing, reframing versus counterframing, etc.). Once the data were coded, I looked for themes that helped make a case for teachers' learning (or not) around issues of struggling students. I then generated findings based on my analysis of a strategically reduced data set.

## Results

I examined teachers' learning about struggling students through a close analysis of one conversation that emerged as significant from the larger data analysis process. This conversation took place at the end of the first term (January 2006) after the research team initiated an activity to support an investigation into the reasons for students' failure. On this day, teachers used their two-hour meeting time to go to the counselor's office as a group to review the student history files of their students they identified as struggling students. They did this activity in conjunction with another planned "make-up packet" intervention for their struggling students.

The teachers used the first 90 minutes of this meeting for review of student histories. During this time, Zack privately commented to a member of our research team that he did not understand the value and utility of this review. At the end of this review, Zack, the Team-designated facilitator of this meeting, prompted the group to debrief this activity by asking the following question: "Did you find anything interesting out, and what are you gonna use it for?" Rose responded to Zack's question first, and launched the debriefing with a finding about student transitions:

> I found a lot of transitions for ELL and special ed kids, and that kids are having trouble with these transitions. […] They don't get any support when they transition out, which doesn't make any sense to me. I don't know if there's anything we can do about that, because that seems like the time when the kids need the most support.

Rose's report suggests a call to action around supporting struggling students through important transitions (motivational framing). Julie went next, and reported that she thought that one of her struggling students was just a lazy kid: "my perception is that he's lazy, not getting it, troublemaker." But then she found evidence contradicting her perception in his records:

> You know, he had pretty good grades up until, you know, this year. So I know he can do the math. His test scores show that he can do the math. Or at least he tests well, you know. So, you know, kind of gives me new glasses to look at him through.

Julie demonstrates a change in participation in the community of practice (learning) by her demonstration of a change in her framing of a problem of practice about one of her struggling students. Her view shifted from one that linked a struggling student with laziness (diagnostic framing) to a more multidimensional view that this student is capable of doing mathematics, and by extension, indicates her agency in helping this student reclaim this capability (prognostic framing). What is more, even though it was Linda's turn to reflect when Julie finished speaking this thought, Rose interrupted the sharing out process to say, "Maybe something recent happened with that kid. Maybe that's a good thing to try to find out!" Rose's comment opened the conversation back up on Julie's student (motivational framing and frame amplification), which generated discussion around supporting him and other students like him (prognostic framing).

The debriefing continued in this manner, where each teacher reported and reflected on what she or he found. Zack was the last teacher to report out. I interpreted his reflection as an account that confirmed and amplified what he already knew: struggling students are students who have poor work habits and perform poorly, and always have. Based on his report, I infer that though Zack believes teachers amplify the struggling student problem with practices like social promotion, the main problem is mostly attributed to the ways in which students work in class. As an analyst, I interpreted Zack's report as one that frames the problem of struggling students in a way that primarily locates the problem of student failure with the behaviors of the students (diagnostic framing). Despite Zack's negative outlook on this exercise and his framing of the struggling student problem, Rose responded to Zack's comment by wondering about kids who have been failing for ever and ever and ever. What happened? What do you do? How did they get so behind? They must not understand anything that's going on. This statement oriented and opened up the conversation to an alternative framing of the struggling student problem (reframing using prognostic/motivational framing). Linda aligned with and extended Rose's potential reframing by saying that students "must get just so used to it," which implied that it must be really difficult for struggling students to overcome a history of failure, or to believe that success is possible (frame alignment using motivational framing).

Lisa – the external professional development specialist funded by our project – then made a conversational move that connected this conversation to status ("the first thing that popped into my head is status") and asked, "How much does he believe that he's not capable if he's been being told the same thing over and over and over again?" I interpret Lisa's statement as a move to align with and extend Rose's and Linda's reframing with status (students fail because they get multiple messages that they are not capable), which also served as a contrast Zack's framing of the problem (students perpetually fail because of their work habits). Lisa's comment "finished" the "community" reframing by linking the struggling student problem to status (frame bridging), which further served to engage teachers' agency with this problem.

I claim that Freshman Team aligned with a community-based "status reframing" that was built by Rose, Linda, and Lisa. This reframing locates the problem inside teachers' classrooms, giving the Freshman Team agency over the problem. The data show how teachers' participation in this conversation on sensemaking around struggling students shifted away from Zack's framing that locates the problem inside the student, and moved towards alignment with the community-based framing started by Rose. What is more, teachers' participation in this conversation shifted away from personal reflections, and moved toward understanding the within classroom systems that contribute to student failure. Taken together, I argue that these shifts show changes in teachers' participation in a community of practice (Wenger, 1998). This indicates that the Freshman Team learned through this conversation about the conditions that contribute to student failure and how to better support them. In other words, these teachers' learned about struggling students through participation in their workplace community in a way that led to their reconceptualization of student ability, an important process for adapting their practices and changing the culture of learning in their classrooms (Weinstein, 2004).

## Discussion and Implications

In this paper, I examined teachers' conversations in their workplace communities for the purpose of understanding the nature of teachers' learning about struggling students. I found that teachers' participation in a workplace community conversation brought about alignment toward or away from a developmental framing of student ability. I interpreted these shifts in framing as a marker of teachers' learning about their struggling students. Theoretically, this analysis contributes to our understanding of teachers' learning about struggling students through participation in their workplace communities. By connecting the conceptual tools of frame analysis with analysis of teachers' learning in a community of practice, this study contributes to the development and use of conceptual tools for understanding teachers' learning at the level of mechanism, which responds directly to the need for more literature clearly explaining what teacher learning is within a workplace group or how it might be taking place (Grossman, Wineburg, & Woolworth, 2001; NAE, 2008).

This study also serves to highlight emergent issues that warrant further study, such as the effects of teachers' roles within the workplace community and looking at individual teachers' learning alongside group learning. For example, Rose consistently made conversational moves that pressed the group for deeper understanding and opened up the conversation for alternative framing, such as when she commented about Julie's student ("Maybe something recent happened with that kid"). This comment engaged Julie's agency with that student, which changed the group's sensemaking around Julie's student, thereby helping Julie and the rest of the Freshman Team learn that there is something more that can be done to support that student. I theorize that Rose's participation in the conversation catalyzed a collaborative reframing process that resulted in teachers' learning.

Furthermore, I hypothesize that Zack's ambivalence towards this exercise – an exercise connected to the equity-orientation of the group – and his consistent orientation to diagnostic framing for problems of practice closed off this learning opportunity for him. For example, at the end of the debriefing, Zack remains uncertain about the utility of their shared activity, and makes a comment about struggling students who get "passed along" in spite of their failure. His participation appears relatively constant, which indicates little visible learning around issues of students in this conversation. My hypothesis brings up an emerging analytic issue concerning Zack's learning opportunities alongside an analysis of his learning. In addition, analysis of group dynamics alongside individual learning has not yet been done and will help us better understand the relationships between individual and group learning. Though these emergent issues raise questions that need to be addressed in my future work, this analysis highlights the possible levers for bringing more teachers into a developmental view of student ability.

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# EXAMINING THE INFLUENCE OF LEARNER-CENTERED PROFESSIONAL DEVELOPMENT ON ELEMENTARY MATHEMATICS' TEACHERS ENACTED AND ESPOUSED BELIEFS

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## Background and Literature Review

American students continue to perform poorly on tests of mathematics achievement (National Center for Educational Statistics [NCES], 2000; 2004). Analyses of student scores on large-scale tests have gone beyond identifying student performance shortcomings, and have identified specific factors that influence student achievement. Studies have shown that students' mathematical learning can be positively influenced by allowing students to explore hands-on tasks that focus on students' higher-order thinking skills (Wenglinsky, 1998). Further, students' learning has been linked to specific pedagogies, such as posing questions about students' mathematical thinking (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). While these practices echo the recommendations for mathematics education reform (National Council for Teachers of Mathematics [NCTM], 1989, 1991, 2000; RAND, 2003; Schoenfeld, 1992), the enactments of these pedagogies are still rare in today's classrooms.

How do we support teachers' enactment of these pedagogies? A recent synthesis of research about teachers' enactments of mathematics curricula suggests that numerous teacher factors, such as content knowledge, pedagogical content knowledge, beliefs and their interpretation of the curriculum influences how learner-centered activities are enacted in classrooms (Remillard, 2005). Teachers must be given opportunities to develop an understanding about these pedagogies while also participating in experiences that develop each of the teacher factors mentioned above.

*Professional Development's Role in Improving Student Learning*

In the past decade, professional development leaders have presented theoretical perspectives about how teachers learn (Cohen & Ball, 1999; Putnam & Borko, 2000; Richardson, 1996) and recommended principles for effective professional development programs (e.g. Guskey, 2003). These recommendations include:

- focusing on issues related to student learning (Hawley & Valli, 1999);
- allowing teachers to take ownership of their learning (Hawley & Valli, 1999; Loucks-Horsley, Love, Stiles, Mundry, & Hewson 2003);
- addressing specific content and pedagogies (Desimone, Porter, Garet, Yoon, & Birman, 2002);
- providing opportunities for teachers to reflect and learn from their own practice (National Partnership for Educational Accountability in Teaching [NPEAT], 2000a, 2000b; Putnam & Borko, 2000);
- allowing teachers to collaborate with each other and with project staff (Sparks & Hirsch, 2000); and
- providing ongoing and comprehensive activities (Loucks-Horsley et al., 2003; Richardson, 1996).

In essence, these documents call for learner-centered approaches to professional development (NPEAT, 2000a, 2000b).

In mathematics education, promising approaches to learner-centered professional development (LCPD) have been advanced. These programs allowed teachers to focus on student learning by having them watch videos of their own classroom instruction (Sherin & van Es, 2005) examine student work samples (Carpenter, Fennema, & Franke, 1996; Fennema et al., 1996), collaborate with university faculty to develop and implement reform-based curricula into their classroom (Silver, Smith, & Nelson, 1995; Silver & Stein, 1996), and make instructional decisions based on their analysis of student work (Fennema et al., 1996; Schifter & Simon, 1992).

While learner-centered principles have been widely embraced, empirical research is needed to examine how LCPD programs influence teachers' classroom practices and their students' learning. Typically, professional development research includes only teachers' self-report about their perceptions, experiences and intentions to apply their new knowledge and skills in their classroom (Guskey, 2000). While this information is useful, teachers often overstate how they intend to use what they have learned from professional development in their classroom (Buck Institute for Education, 2002). LCPD research must study participants' enactments of pedagogies emphasized during workshops.

## Methodology

Based on the need to examine teachers' enactments of pedagogies emphasized in a professional development project, I conducted a naturalistic study (Patton, 2002). Two research questions guided this research:

1. To what extent (and how) do teachers enact the practices emphasized in a learner-centered professional development during their mathematics teaching?
2. How do teachers' enactments of the practices emphasized during learner-centered professional development compare with their espoused and intended practices?

*Context*

Two teachers participated in this naturalistic, qualitative study (Patton, 2002). Both teachers taught in an urban elementary school located near the downtown area of a major city in the southeastern United States. Seventy-nine percent of students at the school qualified for free or reduced lunch. The participants, along with colleagues from other elementary schools in the district, took part in a professional development program designed to prepare them to integrate learner-centered mathematical tasks and associated pedagogies into their classrooms. During the program, teachers completed mathematical tasks while the project staff modeled learner-centered pedagogies, worked with related technologies, examined cases from the Developing Mathematical Ideas curriculum (Education Development Center, 2006) and discussed how to address the state mathematics standards by having students complete mathematical tasks.

*Participants*

*Shantel.* Shantel, an African-American female, has been teaching the 5$^{th}$ grade for 13 years. During the study, Shantel taught three departmentalized mathematics classes daily: one with students in the Early Intervention Program (EIP) and two with students at grade level (AGL-1 and AGL-2). During her baseline interview, she indicated her intention to use professional development-related practices in order to change her teaching in what she referred to as a "good way" to help her students learn.

*Keisha.* Keisha, an African-American teacher, has completed six years of teaching, including four years as a 4th grade teacher. Keisha finished her specialist degree in Educational Leadership in August, 2005, and described herself during her baseline interview as “a lifelong learner.” In her first year, Keisha did not teach mathematics, so this year was Keisha’s third year of teaching 4th grade mathematics. Keisha frequently characterized herself as a “different” teacher because she used manipulatives, games, songs, videos and other instructional strategies to teach mathematics to her students.

*Data Collection*

Data were collected related to intended (i.e., what they planned to do), enacted (i.e., what they were observed doing), and espoused practices (i.e., what they believed they did). Teachers were observed when they indicated their intent to implement practices consistent with the professional development goals and were interviewed to identify their intended and espoused practices. During each implementation a video camera and a wireless microphone were used to record the classroom activity. Further, I recorded field notes about the students’ work and the teachers’ interactions with the students. I interviewed each teacher after the observations about their intended and espoused practices.

*Analysis*

The Video Analysis Tool (VAT; http://vat.uga.edu) was used to code instances of the six instructional practices emphasized during the professional development (i.e., tasks, questions, algorithms, technology, student communication, and mathematical representations) using a lens that codified the extent to which they implemented the pedagogies. The lens (Figure 1) was constructed based upon scales that were developed during prior research studies (Fennema et al., 1996; Hufferd-Ackles et al., 2004) and was refined after initial pilot testing. Interview data were analyzed using inductive analysis. The instructional practices in the scale were used as primary codes during the analysis of the interviews.

| **Practice** | **The teacher…** |
|---|---|
| Mathematical Tasks | 0- does not provide opportunities for students to work on mathematical tasks<br>1- provides opportunities for students to work on tasks that do not use resources (e.g. manipulatives or technology) and involve completing a procedure given by the teacher<br>2- provides opportunities for students to work on tasks in which students use appropriate resources and follow a procedure given by the teacher<br>3- provides opportunities for students to work on tasks in which students use appropriate materials, choose their own approach and provide a solution |

*Figure 1:* Sample scale.

## Findings and Discussion

Several patterns from the data analysis warrant further discussion: These are discussed in this section.

*Little evidence was found to indicate that participants' enacted practices aligned with the professional development intended practices.* Consistent with prior research studies (e.g. Cognition and Technology Group at Vanderbilt [CTGV], 1997; Doyle, 1988; Henningsen, Stein, & Grover, 1996), a majority of the enacted tasks did not align with the professional development goals. Both teacher-participants implemented didactic tasks that did not include resources or used them for rote procedures rather than to complete the tasks. One explanation for teachers' enactments of low-level tasks might be their desire for their students to have success in mathematics. Previous studies about the enactment of mathematical tasks (Doyle, 1988; Henningsen, Stein, & Grover, 1996; Kim & Stein, 2006; Tarr, Chavez, Reys, & Reys, 2006) found that teachers often provided rote procedures, skills-based practice problems and explicitly told students how to complete the tasks in order to ensure students' success.

*Subsequent implementations were more likely to feature learner-centered tasks and high-level questions.* Professional development researchers examining teacher questioning of students' mathematical thinking reported that teachers needed time to make substantive changes to their teaching practices (Richardson, 1994; Orrill, 2001) and to recognize instances where questioning would be appropriate (Sherin & van Es, 2005). In the present study, both participants asked more high-level questions during their latter enactments. The increase in high-level questions as the study progressed may be evidence of the cumulative impact of ongoing professional development activities. During the workshops, teachers observed high-level questioning strategies modeled by the professional developers, reading and watching teachers' implementation episodes and discussing questioning approaches. It seems likely that initial attempts to apply target strategies were influenced by limited familiarity and few opportunities to practice. Thus, with ongoing workshop and planning support, paired with prior opportunities to apply the methods with their students and emerging familiarity and comfort, teachers were more likely to demonstrate learner-centered practices in their classrooms.

*Participants' espoused practices did not align with the professional development goals.* During this study, teachers' interpretation of the professional development goals rarely matched the actual goals. While teacher-participants' reported that each of their implementations would align with the professional development goals, few were consistent with the project goals. Prior studies reported similar results: researchers observed teachers as they employed didactic instruction, but teachers' indicated they were implementing reform-based mathematics instruction (Peterson, 1990; Wilson, 1990).

*Although scaffolding influenced classroom enactments, didactic components were evident even during highly scaffolded tasks.* Tharp and Gallimore's (1988) application of Vygotsky's Zone of Proximal Development to teacher learning contended that teachers require extensive support and guidance when first learning new pedagogies. This support can be scaffolded and gradually removed when teachers are able to independently enact these new pedagogies. Studies of enacted curriculum (Remillard, 2005; Kim & Stein, 2006) found that teachers were more likely to implement learner-centered curriculum when instructional materials adequately supported instruction. The present study confirmed teachers' need for support; classroom implementations were most closely aligned with the professional development goals on tasks that were scaffolded by the professional developers (i.e., tasks the professional developer modeled or co-planned with the participants).

## Implications for Future Research

*Scaffolding Implementation*

While the scaffolding tended to increase the likelihood of learner-centered task implementation, the teachers did not receive the type of progressive guidance recommended by Tharp and Gallimore (1988). The workshops transitioned from directly adopted, to co-planned to independently planned tasks, but participants varied in the order in which they implemented those tasks in their classrooms. Participants may have been more likely to adopt the professional development practices if their first implementation was directly adopted from workshops and subsequently followed by co-planned lessons and independently planned lessons. Perhaps initial enactments might be more effective if focused on directly adopted tasks modeled during the initial workshops and scaffolded via on-site support. Research is needed to examine the benefits and tradeoffs involved in explicitly imposing and scaffolding tasks developmentally.

*Clarifying Links between the Enactments and Student Learning*

Future studies should examine how evidence of student understanding and measures of student learning, are influenced by the enactment of learner-centered tasks. The progressively scaffolded approach suggested previously may complement this line of research. Implementation of adopted tasks might promote consistent student learning outcomes (e.g., similar types of student-generated mathematical representations, communication about students' mathematical thinking, and representations of mathematical work). As teachers assume increased ownership of the implementations by co-planning and independently planning tasks, and begin personalizing their approaches consistent with learner-centered tenets, student learning outcomes might then demonstrate greater variation. Research that attempts to link the implementation of learner-centered tasks to student learning outcomes must start by examining measures of student learning that are embedded within the tasks themselves.

## Conclusion

This study provides evidence that scaffolding teacher's implementations increases the likelihood of the enactment of learner-centered tasks—especially after teachers gain greater familiarity through professional development workshops and have opportunities to practice the methods with their students. However, even highly scaffolded tasks were sometimes implemented didactically. Due to the inconsistency between teachers' self-report and their observed behaviors, in situ observations are needed to sufficiently examine participants' implementation of professional development practices. Further, professional development researchers must continue to examine the links between teacher learning, teachers' implementations of their new knowledge and skills, and student learning outcomes.

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# EXAMINING THE IMPORTANT FEATURES OF LESSON STUDY

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*This study examined similarities and differences in the learning of two teachers in the context of Japanese Lesson Study. The goal of this project was twofold: 1. To assess teacher change as a result of participating in Lesson Study: 2 To uncover mechanisms in Lesson Study that support teacher change. Both teachers began the lesson study with comparable mathematical knowledge; both came from the same school, used the same textbooks and faced similar challenges in terms of student learning profiles and attitudes, but only Brenda demonstrated significant changes in her practice. An analysis of the transcribed video data revealed that Brenda spent more time than Francis examining student work from the practice lessons and also appeared to focus more on her students' developing understandings than Francis did. The implications arising from this limited but detailed study are explored.*

## Background

Lesson study is becoming an increasingly popular professional development method for mathematics educators despite a lack of evidence of its effectiveness in a North American context and little research into the processes by which lesson study might lead to improved teacher practices and student learning (Lewis, Perry, & Murata, 2006). In lesson study - a professional development method credited with improved student learning in Japan over the last 50 years - teachers cyclically plan a lesson together, observe the lesson implemented in a real class, scrutinize student learning, and then re-teach an improved lesson (Stigler & Stevenson, 2001). Lewis, Perry & Murata (2006) argue that there is a need for formative evaluation to summarize the essential processes, and to determine the mechanisms by which lesson study can lead to teacher professional development and ultimately to improved student learning.

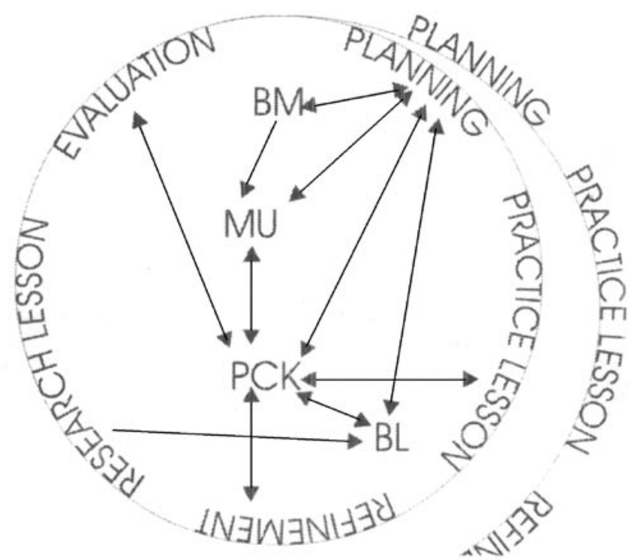


***Figure 1.*** **Theoretical model guiding study.**

This study introduces a methodology for examining lesson study mechanisms and teacher learning outcomes associated with improved student learning in mathematics: teacher beliefs, classroom practices and mathematical knowledge. A literature review suggested a tentative model (see Figure 1) of how these closely-interconnected concepts might interact during lesson study to affect teacher implementation of reform mathematics.

The circle captures the cyclical stages of lesson study, beginning with planning. The spiral arm conveys the potential generative nature of lesson study, building and learning from previous cycles, resulting in gradual improvements in mathematics teaching. Improved mathematics instruction in this research implies changes in teacher beliefs, knowledge, and practices towards those associated with reform mathematics teaching. These changes include teacher's use of authentic learning tasks that promote multiple solutions or strategies, instruction that builds connections between mathematical ideas through classroom discourse, and teaching that adapts to the student understandings (Ross et al., 2002; Sherin, 2002).

In order to make these desired changes in mathematics teaching, research has documented the increased knowledge demands on teachers, both of mathematics and mathematics teaching (Ball et al., 2004). In this research, mathematical understanding (MU) has been defined as the ability to see connections between mathematical concepts and flexibly apply this knowledge to new situations –a synthesis of Woodruff's (2007) flexible understanding and Hill, Schilling, and Ball's (2004) subject knowledge. In contrast, pedagogical content knowledge (PCK) refers to the teacher's ability to choose the most appropriate instructional strategies and representations and ability to anticipate and interpret student's understanding and misunderstandings (Hill et al., 2004). The proposed model suggests planning activities—discussing problematic mathematics topics and considering possible teaching strategies—will affect both mathematical understanding and pedagogical content knowledge.

The model proposes that planning activities affect beliefs about mathematics (BM) and learning (BL); as teachers explore rich mathematical tasks and reflect on teaching strategies to overcome traditional student misconceptions, they need to explicitly examine their beliefs. Changing teacher's beliefs about mathematics and beliefs about learning are considered central to reform teaching. Traditional mathematics instruction implies a belief that mathematics is a static field comprised of set procedures leading to set answers. In contrast, reform mathematics envisions mathematics as a dynamic field emphasizing problem solving continually enhanced through conjecture, exploration, analysis, and proof (Smith, 1996).

The proposed representation also expects that the practice lesson —one of the more unique features of Lesson Study— will build teachers' pedagogical content knowledge as teachers observe student conceptions (and misconceptions) of the activity. Similarly, the model predicts that, as teachers collegially elaborate their comprehension of student understandings and adjust the lesson during the refinement stage, teachers will further develop pedagogical content knowledge. It is anticipated that the focus on student learning during the research lesson will build teacher knowledge of student understandings and reinforce teacher pedagogical content knowledge. As this knowledge increases, it will affect mathematical understandings, beliefs about learning, and all stages of subsequent lesson study cycles. The spiral in the model can also represent the gradual, yet continual enrichment of these conversations.

## Context and Methodology

To evaluate this model, teachers' mathematical understanding (MU) (specifically of fractions), pedagogical content knowledge (PCK) and beliefs about mathematics (BM) and learning (BL) were probed prior to, during and following the lesson study. Before and after the lesson study, the teachers individually solved problems using a talk-aloud protocol and were individually interviewed to assess their knowledge of fractions, mathematics teaching, student understandings, as well as their beliefs about mathematics and learning. Additional data were collected by recording all lesson study sessions and by having teachers complete short reflections on the lesson study process and their own learning.

This present study comes from a larger study of four teachers conducted in a small-town elementary school in the rural school district where the first author works. The school, with approximately 450 students, is a dual-track school in an economically-depressed area, resulting in classes in the regular stream having a high proportion of special needs students. The first author worked with the teachers as a participant researcher, as the group explored lesson study for first time. The group met formally ten times within a five-week period, with many more informal meetings in the hall, the staffroom, and during a shared half-hour commute each day. Two formal meetings occurred during school hours, with coverage provided by administration. The remaining lesson study sessions occurred after school and at lunch. The teachers, eager to demonstrate the lesson study process to their colleagues, arranged a public lesson, attended by two staff from their school, six teachers from other schools, and two special assignment teachers from the board office.

In the larger study, more than 25 hours of interview and session transcripts were read closely in their entirety to identify themes and categories using peer and participant **reviews** to develop validity and manage subjectivity (Bogden & Biklen, 2003; Macmillan & Schuster, 2001). Matrix queries within NVivo 7 (Network Solutions E-Commerce, 2007) investigated relationships among the different stages of lesson study and codes identifying changes in teacher beliefs and knowledge. This paper contrasts the learning of two teachers; Brenda who made significant changes in her practice, and Francis who became more rigid in some of her teaching. Both worked in the same school, teaching grade 5/6 and 6 respectively. Both used the same textbooks and faced similar challenges in terms of student learning profiles and attitudes. Interview and session transcripts were re-examined to identify differences during the lesson study that would explain the variation in teacher change. As tentative hypotheses developed, the video and audio sessions were rechecked in search of discrepant evidence.

## Results

*Pre-Lesson Study Mathematical Understanding (MU)*

Both Francis and Brenda were confident completing their pre-lesson fraction knowledge assessment. Francis answered every question correctly except one, and using a mixture of procedures and intuitive strategies, she moved flexibly between graphical, verbal and formal representations to solve the problems. She used sophisticated counting strategies such as counting by arrays, and quickly caught any calculation or reading mistakes by checking the reasonableness of her answer.

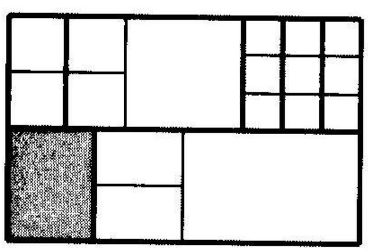

***Figure 2.*** **What fraction is shaded?**

Brenda also answered almost every question, struggling only on the more unusual conceptual-based questions such as determining the shaded fraction of a complex shape such as Figure 2 (where she estimated rather than find the determining the specific fraction). She used arrays to determine fractions of the cookie set and easily converted percents to fractions, hesitating only at 0.33. She immediately recognized it as an important fraction, but needed to use 3/10 as a benchmark to determine it was 1/3. She ordered fractions instinctively, without once using a procedure for finding common denominators.

*Pre- lesson Study Pedagogical Content Knowledge*

To assess the teacher's PCK , the teachers were asked to anticipate how students would answer the fraction items that appeared on the teacher interview. While both teachers were confident in their own ways of solving these questions, neither Francis nor Brenda was able to identify mathematical misconceptions that students might have. Francis could only identify areas where students might have difficulties with the presentation of the material, noting for example her students might ask, “How come this number has two decimals?” where one dot was the period at the end of the sentence. When asked how her students would think about specific questions, Francis initially answered in absolutes—students would or would not be able to answer: “I think that half of my students would know this.” Brenda’s conception of student understanding was limited to whether, in her view, they understood each step of the procedure required to solve the problem.

*Pre-Lesson Study Beliefs about Mathematics and Teaching*

The first author observed one of each teacher’s mathematics classes before the pre-interviews and the lesson study. In Francis’s class, students were observed answering teacher questions in a number sense activity followed by students making up questions for the last answer. Of the 40 minutes observed, 35 minutes focused on the teacher asking closed questions with only one right answer. In her initial interview, Francis reported that three days a week her class worked on new skills. On Wednesdays her class worked on consolidating number facts, and on Fridays the class would work in homogenous groups to solve problems. Francis also noted that her students benefited from, and enjoyed drill and practice activities:

> It’s probably old school but I do believe that practice can make perfect and I think it’s lacking but I don’t think it does them any harm to practice.

Brenda reported that, for the most part, she followed the textbook, supplemented by a skills duotang. During the observed class, Brenda provided a teacher-centered lesson on creating nets, asking closed questions to review terminology, and provided detailed hints for students to create nets. In her interview she noted that she liked traditional math and had difficulty with students explaining their work:

> I’m the type of person that I just liked to do math. I don’t want to explain, I don’t want to talk, so I’m awful;… Every time I’m telling them [to explain their solution], I feel like a hypocrite, because I know I hate it. That’s why I don’t really like teaching math, because I like doing, I like thinking about it.

*Post Lesson Study: Changes in Teachers*

Despite their similarities before the lesson study, Brenda’s and Francis’ teaching practices and thoughts on teaching appeared to diverge significantly by the end of the Lesson Study experience. In post interviews, Brenda reported that that since the Lesson Study she had begun to ask her students questions to probe their understanding, such as “How do you know?” Do you have a different way of explaining that?’ She also reported that she began to value class discussions about student thinking—in sharp contrast to her earlier comments about hating such discussions. In addition, when asked to describe student strategies, Brenda revealed a much more analytical approach than she described prior to participating in the lesson study. For example, in describing a specific lesson involving ordering of fractions, Brenda referred to an example where students had to consider which of two fractions was the greater, 20/15 and 10/4. Brenda noted how some students were able to reason by either drawing pictures or converting to mixed fractions. She also noted that several students reasoned erroneously asserting that 20/15 was bigger because both the nominator and denominator were bigger.

In another example, where students were asked to find equivalent fractions, Brenda described students’ grouping of cookies. When asked whether she thought more students would use the grouping strategy or divide the numerator and denominator by the same term, she initially responded that they would most likely divide—the strategy that she would have used. However, she then reconsidered from the students' perspective:

> I think the dividing strategy. Or would they? They had a hard time going from a lot to less. They had more success paper folding and drawing lines—thinking I am going to group this. I think you have to present it both ways the abstract and the physical.

Brenda summarized changes in her teaching from as moving from search for the perfect resources to a focus on student understanding:

> I am going to think about what they did, or I am going to look at their work. I am just going to read this rather than think of more practice for them, because that's not reaching them. It's okay to sit back and think about, and read about data management or stem and leaf graphs. It's fine if you don't bring something new everyday. It's fine if you just go over where they are coming from.… Not just more practice. Mulling over. What's going on in their brain?… We put away the textbook and just discuss. I've done a lot of that, and I get a lot more out of them.

In contrast, Francis reported much less change in her teaching: during post interviews she seemed at times to be more insistent on traditional mathematics instruction than she had appeared to be prior to the Lesson Study. She insisted on the dominance of rules and the need to practice set procedures. Even though, as part of the Lesson Study, she had been involved in teaching three lessons with the goal of fostering students' conceptual understanding of equivalent fractions, at post interview she asserted that finding common denominators was the only way of adding fractions—this traditional method is no longer even in the grade six curricula. She said that students did not need to explore why rules worked:

> I think a lot of my students had the rule... I don't know that we are so concerned with [the understanding why]. I thought that the lesson was very good for a lot of students to show that they had an understanding of the material. They had an understanding of the rule, of the abstract.

### Accounting for Differences in Teacher Changes: A Focus on Student Work

Given that Francis and Brenda began the lesson study with similar knowledge and beliefs and taught similar students in similar classrooms, what differences in their lesson study participation might explain their different outcomes? An analysis of the transcribed video data revealed that Brenda spent more time than Francis examining student work from the practice lessons. She also appeared to focus more on her students developing understandings than did Francis.

After each lesson study session, the student work was distributed to teachers to examine. The video record showed the other teachers browsing the student samples for five minutes and then proceeding to discuss their notes on the lesson. In contrast, Brenda is visible studying the work even after the discussion begins. For example after the first practice lesson, Brenda examined the student artifacts for 20 minutes, at times appearing to sort the work into categories. When Brenda participated in the discussions she did so by referring to the student work she was trying to analyze. For example, in one point she directed the group's attention to several students' strategies including that of Paul and Adele:

> Paul: He was adding one to the top and four to the bottom [to determine equivalent fractions to ¼]. I was thinking, "Where did that come from?" I would have never thought like that.
>
> Adele: Its not a fraction until you cut it up.

The video of the debrief session for the second lesson shows a similar pattern in regards to Brenda's interest and focus on student work.

In contrast, video analyses show that Francis examined the student work for less than three minutes after each practice lesson but then was the most active participant in the discussion with more than ten turns in each discussion. During the debrief of the first practice lesson, Francis listed the students who "just didn't get it at all" and several times during the lesson study meetings and post interview she reported that it was attention rather than misunderstanding that led to students poor achievement. She described students who did not attend to the class review of a worksheet and noted " It's not that they are totally confused with things but they dismiss so much."

An examination of the lesson study video indicated that Francis differed from Brenda in how she used the materials in her own classrooms, outside of the formal observation lessons. Francis tried some of the problems in her class, and reported to the group the students who could or couldn't solve the question, and the frustration students experienced when the teacher refused to immediately tell them "the" "right" answer.

When probed "What were your students thinking" by the first researcher, Francis detailed the students who wanted detailed procedures, while Brenda responded she had no idea what her students were actually thinking. Francis moved on to other topics, but Brenda persevered. At the next group meeting, Brenda reported that she had covered equivalent fractions in her class for three periods days before she could see what the students did not understand, and before the students could actually do the work. She indicated that "It's so hard when you are there by yourself and trying to observe," but her comments suggest that she had set up a feedback loop within her own class. In these lessons, she analyzed student responses to questions to determine their thinking and how instruction should proceed.

## Conclusion

Lesson study consists of a complex web of activities that have the potential to support professional development. Although limited, this study comparing the changes in two teachers over three months suggests that concentrated examination of student knowledge and their developing understandings is an important mechanism of lesson study. While longitudinal studies of larger groups of teacher are necessary to determine if this focus on student thinking leads to long-term teacher change, this study suggests that teacher knowledge and focus on student conceptions through Lesson Study is a powerful supporter of teacher change.

This central finding of this study has implications for North American teachers undertaking lesson study. While a focus on student understanding is an integral component of the Lesson Study process, North American teachers may not always be amenable to a careful focus on student understanding (Fernandez, Cannon and Chokshi, 2003). In this study only Brenda spontaneously examined student work in detail and this attention was followed by significant changes in her teaching beliefs and practices. The results of this admittedly small study appear to be aligned with the findings of Franke, Carpenter, Levi, & Fennema (2001) where the focus on student understandings is an important element of teacher change.

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# DEVELOPING AS MIDDLE GRADES MATHEMATICS TEACHERS: CAREER CHANGERS

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*This qualitative study examined the career paths of 12 career changers who completed a field-based Master of Arts in Teaching program to become middle level mathematics teachers. Researchers examined the sources of mathematical knowledge for teaching mathematics as well as the affect on their mathematics teaching of the context in which these teachers learned and used mathematics in their previous careers.*

## Background

The findings presented here are the result of a pilot study for a larger, multi-state longitudinal study that examines the development of career changer mathematics teachers are the middle and high school level. This was a qualitative study of 12 career changers who became middle level mathematics teachers. Teachers were graduates of a Master of Arts in Teaching (MAT) program at a research university in the southeastern portion of the United States. The program had a heavy emphasis on field experience. The purpose of this study two fold. The first goal of the study was to investigate the sources of professional knowledge that career changers use in their teaching of middle level mathematics. In this study researchers sought to describe sources of professional knowledge such as *knowledge for practice, knowledge in practice and knowledge of practice* (Cochran-Smith & Lytle, 1999; Sowder, 2007) necessary for career changers to teach middle level mathematics. The second goal of the study was to determine how these career changers develop their vision of what they need to know to teach middle level mathematics, their mathematical content knowledge for teaching, their understanding about children's thinking about and learning of mathematics, their pedagogical content knowledge, and their sense of self as a mathematics teacher (Sowder, 2007). Data sources included surveys, interviews, classroom observations, and licensure examinations. In this paper, the researchers report the results on the development of the aforementioned types of knowledge and sources of development of such knowledge.

## Theoretical Framework

*Problem Statement*

The teacher shortage in the United States is a two-fold problem. As student enrollment is increasing and teacher attrition rates are also increasing (Ingersoll, 2000, 2007). In its recent report, *Teacher Attrition and Mobility: Results from the 2004-05 Teacher Follow Up Survey* (2007) the National Center for Educational Statistics (NCES) reported that in 2004-05 269,600 teachers left the profession. This number represents 8.4% of the teaching force in that year. In California, it is estimated that approximately 33 1/3% of the teaching force is approaching retirement age and that 25% of teachers leave the profession in their first five years (CFTL, 2001). Ingersoll (2000, 2007) reports that the attrition rate for mathematics and science teachers is not significantly different from that of the all teachers combined. Because of current attrition rates and retirement rates of mathematics teachers at all levels, there have been several attempts to increase the number of teachers entering the profession. These attempts include programs for

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mid-career changers such as Troops to Teachers, programs for recent college graduates with degrees in content areas such as Teach for America, and programs that provide an alternative route to certification, for example the South Carolina Program for Alternative Certification for Education (PACE) program and the American Board for Certification of Teacher Excellence (ABCTE). The National Science Foundation (NSF) funds programs such as the Noyce Scholarship and Fellowship program to encourage science, technology, engineering, and mathematics (STEM) majors to pursue careers in mathematics and science education. Many universities offer Master of Arts in Teaching (MAT) programs for career changers who are seeking second careers in education. A major goal of these projects is to encourage career changers who have experienced successful careers in occupations in which mathematics was used to seek second careers as mathematics teachers at the middle and high school levels. Researchers report that the inservice and preservice professional development of mathematics teachers is an important aspect in the improvement of mathematics education in public school systems. (Ball & Cohen, 1999; Elmore & Burney, 1999; Nelson & Hammerman, 1996; Sykes, 1999; Thompson & Zeuli, 1999) In this study, we seek to understand how novice teachers develop their mathematical knowledge for teaching in the context of a field-based MAT program designed for career changers who seek certification in middle level mathematics.

*Prior Research*

There has been significant research in the area of teachers' professional mathematical knowledge and its development (Ball & Cohen, 1999; Borasi & Fonzi, 1999; Hill, Ball & Schilling, 2008; Ma, 1999; Simon, 1997; Spillane, 2000; Sowder, 2007). Researchers have defined professional knowledge for teaching mathematics in various ways. For example Ball and Hill (2008) refer to mathematical knowledge for teaching as "the mathematical knowledge that teachers use in the classroom to produce instruction and student growth" (p.374). Spillane (2000) discusses procedural knowledge as well as principled knowledge for teaching mathematics that focuses on the conceptual knowledge that provide a basis for procedural knowledge (p.144). Simon (1997) describes the intersection of eight areas of knowledge for teaching mathematics. The aspects of knowledge for teaching mathematics described by Simon (1997) include knowledge of and about mathematics; knowledge of a meaningful model of mathematics learning; knowledge of the way students develop relevant mathematical concepts; knowledge of a meaningful model of mathematics teaching; knowledge of students' interaction with mathematics; knowledge of goal setting for students; knowledge of what student learning might occur; and knowledge lesson planning that is consistent with one's model for teaching mathematics. Ma (1999) discusses at length the fundamental knowledge of mathematics that is required for teaching elementary mathematics. In the *Mathematical Education and Development of Teachers*, Sowder (2007) synthesizes the research on the professional development of mathematics teachers. In particular, Sowder uses the framework proposed by Cochran-Smith and Lytle (1999) which identifies *knowledge-for-practice, knowledge-in-practice,* and *knowledge-of-practice* to make sense of the body of research on the development of professional knowledge for teaching mathematics. Knowledge – practice is defined as the shared knowledge already know by others such as those that provide teacher education and professional development experiences. Is knowledge is acquired as a result of formal professional development activities (p. 250). Knowledge – in – practice is defined as the particular knowledge of teaching known "as embedded in practice and in teachers' reflections on practice" (p.268). This type of knowledge is acquired when teachers reflect on their own practice. Knowledge – of – practice is the knowledge that teachers develop while practicing their craft in their own classrooms and school sites while investigating the interaction of learning, knowledge and theory.

Several researchers have studied mathematics teachers who have had successful careers in other fields in which mathematical concepts were applied (Adler & Davis, 2006; Ensor, 2001). In particular, Adler and Davis have researched diverse populations of mathematics teachers, career changers in the UK and teachers with insufficient training in South Africa and how their knowledge for teaching mathematics develops. Adler and Davis studied the works of Hill, Ball, & Schilling, (2008). with respect to unpacking mathematical knowledge for teaching mathematics with respect to mathematics courses required for inservice teacher professional knowledge. It was determined that mathematics courses required for inservice teachers in South Africa did not require teachers to unpack knowledge for teaching mathematics, even though this unpacking of mathematical knowledge is required to teach reform mathematics. Of particular interest to this study are the works of Bernstein (1996) and Ensor (2001) have studied the conceptualization of mathematics for teaching. The Bernstein and Ensor studies lead to the questions, what effect does the context in which one learns and applies mathematical concepts have on ones ability to teach middle level mathematics and how do career changer mathematics teachers learn from their professional communities about teaching mathematics

## Research Questions

The participants in this study became mathematics teachers after successful careers in engineering, accounting, retail sales, applied sciences, and the military. Career changers who become mathematics teachers face many challenges. Some of these challenges are similar to those who completed the traditional route to teacher certification others are very different. The researchers are interested in tracking the development of the professional knowledge for teaching mathematics for this unique population of career changers. The research questions for this study are:

1. What are the sources of professional knowledge such as *knowledge for practice, knowledge in practice and knowledge of practice* (Cochran-Smith, M. & Lytle, S., 1999; Sowder, J., 2007) necessary for career changers to teach middle level mathematics?
2. How do career changer mathematics teachers learn what they need to know for teaching mathematics?
3. In what ways do career changers develop:
    a. their vision of what they need to know to teach middle level mathematics:
    b. their mathematical content knowledge for teaching:
    c. their understanding about children's thinking about and learning of mathematics;
    d. their pedagogical content knowledge; and
    e. their sense of self as a mathematics teacher (Sowder, 2007).
4. How do career changer mathematics teachers learn from their professional communities about teaching mathematics?

## Methodology

*Participants*

There were 12 participants in the study. Each of these participants has experienced a successful career in which mathematics concepts were applied on a regular basis. Participant undergraduate degrees included civil engineering, electrical engineering, architectural design, marketing, economics, meteorology, statistics, business management and marketing. As candidates in the MAT program, these teachers were required to complete a minimum of 250 hours in field-based internships in addition to the traditional student teaching experience. The extent of the field experiences required of this program are consistent with Ensor's (2001) recommendation that extensive field experiences are important in the development of best practices in the mathematics classroom.

Participants in the study ranged in age from 22 – 50. Three of the participants in the study were males, the remaining nine participants were female. There was one African American participant in the study. After successful completion of the MAT program, participants had completed a minimum of one year of teaching middle level mathematics.

Participation in the study was voluntary. Participants were recruited from recent graduates of an MAT program. Respondents received no incentives for participation in this study.

*Data Collection*

The researchers collected data through surveys completed by candidates in the MAT program and recent graduates of the MAT program. Video-tapes of the candidates conducting instruction in a simulated classroom, a requirement of the first semester Middle Grades Mathematics Methods course is the second source of data for this study. Another video tape of their teaching was collected during the regular academic year after completion of their teacher-training program. Data was also collected from interviews with participants after they have completed a minimum of one semester of teaching in a middle grades mathematics classroom. Participants Praxis mathematics exam scores were also provided.

*Data Analysis*

The researchers considered participants Praxis scores as indicators of their mathematical content knowledge. To analyze the video lesson plans collected in this study researchers used Instructional Quality Assessment Classroom Observation Tool Rubrics created by Lindsay Clare Matsumura, Helen Garnier, Sharon Cadman Slater, and Melissa Boston. The interviews were transcribed, coded, and analyzed. The researchers noted common themes that immerged from the interview data and survey results. These themes included but were not limited to the following themes: a) teachers vision of what they need to know to teach middle level mathematics; b) their mathematical content knowledge for teaching; c) their understanding about children's thinking about and learning of mathematics; d) their pedagogical content knowledge; e) their sense of self as a mathematics teacher; and f) identified sources of professional knowledge to teaching. With every new theme, the researchers re-evaluated prior interview data and survey results.

## Preliminary Results and Discussion

Preliminary research finding of this pilot study include the following:

1. Based on survey data and interviews, the researchers found that all participants relied heavily on their prior professional training in the field outside of education to bring rich contextual examples into their classrooms.
2. Ninety percent of participants also reported that mathematics methods classes as well as classroom management classes (that they took as part of the teacher training) were most important for their transition into a classroom. They felt that pure "mathematics classes were very helpful". However, teachers wished they "had an opportunity to take more mathematics methods classes." Their sense of self as a "mathematics teacher" versus "mathematics user" continued to develop during these formal classes and as they spend more time in the field.

3. Based on the interview data, several participants who worked in schools with established ‘communities of practice’ were influenced by these communities. These teachers/career changers had significant growth of knowledge-in-practice, and knowledge-of-practice. However, two participants reported that they “stayed away from majority of their teacher fellows in order to avoid constant complaining about students, school, administration, etc….”
4. During the interviews, all teachers reported that they developed foundation of their mathematical content knowledge during formal mathematics classes at the university. They continued working on this content knowledge after graduating from each of the academic programs
5. Based on the preliminary video analysis, career changers vision of what they need to know to teach middle level mathematics changed with time. 90% of participant started their teaching practice with a behaviorist approach to teaching mathematics. They focused on students’ correct answers and attempted to create a “perfect sequence of procedures or steps” for students to remember. Over time teachers focused on their own understanding of children’s thinking about mathematics and their own pedagogical content knowledge. The classroom focus also shifted towards learning environment with opportunities for problem solving.

*Limitations*

One of the main limitations of this study is the fact that the sample for this study was a convenience sample. In addition, one of the researchers was the instructor for the mathematics methods class in which the participants participated during their formal class work for the MAT. Participants in the study were overwhelmingly female Caucasians. The results of the study cannot be generalized because of the small sample size and lack of diversity of the sample. This was a pilot study. The researchers intend to collect additional data through classroom observations after career changers first year of teaching. This classroom observation data analysis was not included in the current report.

*Final Thoughts*

Additional research is needed to better understand the transition STEM majors, career changers experience when becoming middle school mathematics teachers. In this pilot study, we observed and described some of the patterns in such transition. However, we are interested in learning more on how these middle grades teachers contextualize their prior knowledge in their daily teaching practice. We are also interested in knowing in what ways will the analysis of classroom observations (that researchers collected) contribute to the project findings.

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# LANGUAGE DEMANDS OF MATHEMATICAL THINKING: AN INVESTIGATION OF THE MATH ACCESS PROJECT

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*This study investigated the design and impact of a professional development project, Math ACCESS (Academic Content and Communication Equals Student Success) with a unique focus – working with teachers to understand the language demands of student participation in higher order thinking and justification in mathematics classes, particularly related to supporting linguistically diverse students in urban schools. The results demonstrate increases in teachers' content knowledge, perceptions of knowledge and confidence related to supporting students' development of academic language, and awareness of challenges and strategies related to vocabulary and language development for ELLs as well as other students.*

## Objectives and Purposes

In this paper, we document the design and impact of a professional development (PD) project, Math ACCESS (Academic Content and Communication Equals Student Success). This on-going project has a unique focus – working with teachers to understand the language demands of student participation in higher order thinking and justification in mathematics classes. There is evidence that promoting the development of academic language and engaging students in the practices of justification and argumentation support students' learning of mathematics (e.g., Brenner, 1998; National Research Council [NRC], 2001). These practices may be particularly important for supporting lower attaining students (Boaler & Staples, 2008) and the growing population of English language learners (ELLs) (Brenner, 1998; Moschkovich, 2002).

The development of students' academic language is a central function of schooling (Schleppegrell, 2007). A key aspect of this work is to help students move from everyday, informal language toward academic language and use of the mathematics register (Halliday, 1978; Pimm, 1987). This goal is pertinent for all students, but particularly so for students whose first language is not English (Cummins, 2000; Schleppegrell). Too often, attention to language in mathematics classrooms focuses on vocabulary. Language-related instruction should move beyond simple vocabulary; it should include attention to how language is used to express mathematical ideas (functional linguistics) and the development of the mathematics register (Moschkovich, 2002; Pimm; Schleppegrell). Most classrooms, however, do not support such practices. Many teachers remain unaware of the language demands involved in learning mathematics, especially with attention to justification and higher order thinking.

Developing command of academic language is not only a valued end in and of itself, but it also supports students' learning (Halliday, 1978; Pimm, 1987; Schleppegrell, 2007). Language and thinking are intertwined; we use words to think and reflect; when we name something, we can come to understand it in a new way (Vygotsky, 2002). Building on students' everyday language and *bridging* to academic language is a key strategy to develop mathematical proficiency (Echeverria, Vogt, & Short, 2002). Math talk is dense. Phrases like "Given that the sides of the triangle are…" and "For all x…" are not structures used in everyday language. We must help students understand these meanings (Pimm, 1987).

The need for teachers to be knowledgeable of the linguistic demands of learning and doing

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mathematics is perhaps heightened in urban areas where students are more likely to be linguistically diverse. In particular, intermediate level ELL students who demonstrate *Basic Interpersonal Communication Skills* (BICS), but who have not yet achieved *Cognitive Academic Language Proficiency* (CALP) (Cummins, 2000), are at risk of not being able to access necessary academic language to support justification and higher-order thinking. These students may be socially fluent, yet may need strategic linguistic support for cognitively challenging mathematical tasks (Janzen, 2008). Attention to these areas can help to ensure that students who vary in prior mathematical background, language ability and other characteristics will have more equitable access to math learning opportunities (Cohen & Lotan, 1997).

As noted, teachers often remain unaware of the language demands of doing mathematics beyond a consideration of vocabulary. In a recent review of the literature related to teaching ELLs in content areas, Janzen (2008) emphasized the need for further research on PD that can help teachers to develop "understanding of the relationships among language, content, teaching, and context, and how they can implement that knowledge in their disciplinary fields" (p. 1031).

To provide a sense of the complexities involved, consider an open-ended prompt that asks grade 4 students to find a way to purchase *at least* 40 buns that costs *the least* amount of money, given particular package sizes and prices. The task includes potential for cognitive challenges, but also contextual and linguistic challenges beyond vocabulary. These may include unfamiliarity with "everyday" words and phrases (e.g., package and purchase), as well as other words that are germane to the mathematical work they are expected to engage. For example, phrases such as "*at* least" and "*the* least" may *seem* familiar, but may be misinterpreted or not understood, and lead to very different mathematical work. Students must make sense of what is required of them mathematically and must be able to represent their mathematical thinking and processes in written form—ideally, including justification in the response. The intersection of language, content, and context are complex for students and challenging for teachers to teach.

Given these complexities, it is important to consider how the mathematics education community might expand teachers' pedagogical expertise in these areas. There are examples of PD that have attended to the development of language, though they typically target the teaching of ELLs in *any* content area (e.g., SIOP, Echevarria, Vogt, & Short, 2004). Conversely, there are examples of PD that foreground mathematical justification or higher order thinking (e.g., QUASAR, Silver & Stein, 1996), but that do not include explicit attention to language issues. There are limited examples of projects that attend to both language and cognitively challenging math (e.g, *Project Challenge,* Mitchell, 2007 and the *Center for the Mathematics Education of Latinos/as,* CEMELA, 2009). Given the importance of these areas for student learning, there is a clear need to develop and provide research-based PD for teachers that addresses these issues.

## Context

In response to these needs and issues, we created a conceptual model to guide the development of a professional development program. The model comprises three "pillars": *Academic Language and the Mathematics Register, Student Justification and Collective Building of Arguments,* and *Access by all Students.* Each pillar addresses a core component of instruction that has a strong research base documenting its value for student engagement and learning, and promoting more equitable outcomes. In this paper, we focus on the first pillar, *Academic Language*, and document and discuss relevant activities, outcomes, and issues from our work with a group of teachers during the ACCESS Summer Institute.

The Math ACCESS PD program was supported by the *Teacher Quality Partnership Grant* program from the Connecticut State Department of Higher Education. We partnered with two K-8 public schools, one public high school, and one private high school in an urban district. In this district, 45% of the students speak a language other than English at home, 94% of students qualify for free/reduced lunch, and 96% of the students are categorized as “minority students” (Connecticut Strategic School Profiles, 2008).

Given our goals and the requirements of the grant program, we organized the PD into two main components. The “instruction” comprised a one-week intensive summer institute (40 hours) during July and a half-day (5 hours) session in September. The “follow up” comprised a modified form of lesson study where teachers, organized in grade-band teams, collaborated to develop, implement, and debrief lessons that used pedagogical strategies related to each pillar. Space precludes an in-depth discussion of each pillar and the corresponding activities. We provide an overview and discuss in more depth some of the instructional activities that targeted promoting academic language during the summer institute.

The first pillar, *Academic Language and the Mathematics Register*, focuses on the role of language in mathematics, specifically, developing students’ academic language and their ability to respond to open-ended prompts. Our learning objectives for the teachers focused on helping teachers recognize language-related aspects of math prompts that may be challenging for students and generate and/or purposefully select strategies that support students in managing the language demands. Examples include: expanding students’ math vocabulary, having command of certain phrases (e.g., at least, the least, for each person/for every person), and helping students develop proficiency with explanations and justification. Towards these ends, we developed a range of activities, including: providing background in the Sheltered Instruction Observation Protocol (SIOP) model (Echevarria, Vogt, & Short, 2004); unpacking language demands within curriculum materials, state testing materials, and student work samples; writing language objectives for math lesson plans that focus not only on vocabulary, but also functional language (Schleppegrell, 2007) (e.g., *Students will continue to build an idea of what makes a good explanation by using a language frame: “___ is correct/incorrect because ___.”)*; and modeling explicit attention to language during algebraic and proportional reasoning content instruction.

An example activity that demonstrates the intersection of language, content, and sense-making was designed around the prompt: *Before, tree A was 8’ tall and tree B was 10’ tall. Now, tree A is 14’ tall and tree B is 16’ tall. Which tree grew more?* (Lamon, 2006). Lamon notes that proportional reasoning problems are filled with language challenges because “the same words that we use to discuss whole number relationships, take on different meanings in different situations” (p. 33). In this case, the phrase “grew more” had more than one interpretation, allowing teachers to wrestle with language related to absolute and proportional reasoning.

Activities related to the other two pillars were also critical, as these pillars are mutually supportive. Throughout the Institute we focused on scaffolding, norm setting, appropriately challenging each student, and particular teaching-learning strategies. The teachers analyzed cognitive demands of tasks (Stein, Smith, Henningsen, & Silver, 2000), modified existing tasks to infuse higher order thinking (HOT), discussed and analyzed qualities of good justifications, analyzed student work, and worked in teams to plan and implement HOT lessons.

The following research questions are addressed in this research project:

3. What are the outcomes and effects of the ACCESS professional development activities for participating teachers?
4. What issues arise as teachers work at the intersection of academic language and justification?

This paper focuses primarily on the first research question, with particular attention to the first pillar, *Academic Language*. We are currently engaged in the academic year “follow up” so data collection and analysis are ongoing. Please see Staples and Truxaw (under review) for focus on the second pillar, *Justification,* related to this project.

## Methods of Inquiry

Twenty-four grades 4-9 teachers participated in the ACCESS Summer Institute; 20 teachers continued the program during the academic year. The 20 continuing teachers included 11 teachers who taught the single subject of mathematics and 9 teachers who taught multiple subjects. Teaching experience ranged from 0 years to over 21 years. The first two authors did the majority of the instruction; all authors are supporting school-based follow-up activities.

This study employed a mixed-methods design in order to investigate outcomes and effects of the ACCESS PD. Participants completed two assessments at the beginning and completion of the Summer Institute in order to assess the following: a) the growth of teachers’ content knowledge and b) change in their ability to identify and generate strategies to support the development of students’ academic language, especially among ELLs. Assessments included both content questions and teaching scenarios where situations are analyzed and action is proposed.

Items for the *content knowledge assessment* were drawn from previously validated sources (e.g., CT State Dept. of Education, 2008; Healy & Hoyles, 2000; *Learning Mathematics for Teaching Project*, 2008) and were selected by a team of mathematics educators to fit the content themes of the institute, algebraic and proportional reasoning. The items were field tested for appropriateness and timing by administering them to non-participating elementary and secondary mathematics teachers. Based on these results, the final items were selected. The final instrument included 7 multiple-choice questions and 2 open-ended questions.

The *language assessment* was developed specifically for this investigation in order to uncover participants’ growth related to supporting the development of students’ mathematics academic language, especially with attention to ELLs and higher order thinking. We identified the key constructs of the survey based on the themes of the institute. Following recommendations of Gable and Wolf (2001), content-validity was sought through the use of research literature and experts’ content validation (i.e., mathematics educators, linguistic experts, second language learner specialists, and methodological experts) that noted the adequacy of the items as representative of the specified constructs. The final instrument included 6 open-ended questions related to language use, challenges, and strategies, and 7 Likert-type questions asking for self-reported knowledge of language issues addressed during the PD. The content validity questionnaires and the final instruments are available from authors upon request.

Analysis of the multiple choice and scaled items were performed using standard statistical techniques (Green, Salkind, & Akey, 2000). The open-ended responses were analyzed using standard qualitative methods (Strauss & Corbin, 1990). For example, for questions that asked teachers to identify the linguistic demands of a problem, researchers employed open coding of emergent themes, discussed the themes, and then employed axial coding to make connections among the categories and to refine the coding schemes (Strauss & Corbin). Coding categories and definitions for each question were developed and tested by at least two researchers. Disagreements were resolved in discussion with a third research as needed. Applying these codes allowed us to look for change on both the group level and the individual level.

## Results and Discussion

Results from the pre-post assessments indicate that the Math ACCESS Project did have an impact on teachers' knowledge of content and language-related issues and strateigies. We will describe content results briefly and then focus on the language-related results.

*Evidence of Reaching our Objectives Related to Content*

Twenty-four participants completed the pre- and post- assessments related to content—algebraic and proportional reasoning. The results demonstrate increases in content knowledge overall. Scores of 15 participants (63%) increased; scores of 7 participants (29%) decreased, scores of 2 participants (8%) remained the same. Overall content knowledge showed a statistically significant difference in mean scores ($p < .05$), as shown in Table 1.

Table 1. *Assessment of Content Knowledge of PD Participants, n = 24*

| Administration | Mean | SD | *t* | df | *p* |
|---|---|---|---|---|---|
| Pre | 9.33 | 3.67 | | | |
| Post | 10.63 | 3.63 | 2.24 | 23 | .035 |

*Evidence of Reaching Our Objectives Related to Language*

The analysis of our data indicates that there were (Institute-compatible) observed changes in teachers' proficiencies identifying language-related issues and planning in ways that would promote the development of academic language. A summary of related results follows.

*Teachers' self-assessment of knowledge.* We asked teachers to self-assess their knowledge of various issues by rating themselves on seven 7-point Likert item questions (1= not at all knowledgeable; 7=expert knowledge). For example, they rated their knowledge of *Generating strategies to support students in managing language demands of CAPT and CMT-like open-ended prompts.* Nineteen participants completed all items on the pre- and post-assessments. The overall pre-assessment mean was 4.09; the post-assessment mean was 4.77 (4=moderately knowledgeable; 5=very knowledgeable). A one-sample *t*-test was performed, showing a statistically significant increase in total mean scores for all participants ($p < .05$). Similarly, mean scores for each of the 7 individual items increased from pre- to post-assessment. These data demonstrate a positive and relatively strong impact of the PD with respect to language.

*Open-ended items related to language challenges and strategies.* In addition to the Likert items, the Language Assessment included 3 open-ended questions that provided evidence of change with respect to the teacher's ability to identify challenges related to language and/or strategies to address challenges and support the development of academic language.

*Developing academic vocabulary.* Question 2 asked participants to identify 3-5 general strategies they might use to help students build understanding of mathematical vocabulary. Taken as a group, participants increased the number of strategies they generated to address vocabulary in the classroom and specifically increased the number of *ACCESS related* strategies (i.e., strategies associated with the 3-pillars; e.g., "Think-pair-share activities to build vocabulary," was considered *ACCESS related*; "flashcards," was *not*). Table 2 shows the mean number of strategies per teacher and the percent of those strategies identified as *ACCESS related*. We also include results for single subject teachers as compared with multiple subject teachers as we found some interesting differences among these two groups.

Table 2. *Language Survey Question 3 – Strategies to Address Vocabulary*

| | All n=20 | | Single subject n=9 | | Multiple subject n= 11 | |
|---|---|---|---|---|---|---|
| | Pre | Post | Pre | Post | Pre | Post |
| Mean # strategies per teacher | 3.2 | 3.3 | 2.9 | 3.5 | 3.6 | 3.1 |
| % ACCESS related strategies | 33% | 50% | 16% | 40% | 50% | 72% |

The mean number of strategies per teacher increased for all participants and for the single-subject subgroup; the mean number of strategies per teacher decreased for multiple-subject teachers. However, perhaps more important than the number of strategies (especially since the prompt suggested a specific range of 3 to 5 strategies), was the inclusion of *ACCESS related* strategies. These percentages increased overall, as well as for the two subgroups. Sixty-five percent of the teachers added at least one *ACCESS related* strategy.

*Identifying challenges and strategies in open-ended math prompts.* In question 3, participants were given an open-ended prompt designed for grade 7 or 8 students. Participants were asked to identify words and/or wording that might be challenging for students and then describe why the words would be challenging. Responses were coded for increased detail or specificity if they included additional words or phrases and/or included increased detail in their description of the challenges (e.g., adding the word "explain" [not included in pre-assessment] accompanied by, "Students may need some language support to be able to explain the process they followed to come up with the answer.") The results suggest that, overall, the participants increased detail or specificity related to challenges of language for this specific prompt: 12 participants (60%) showed an increase; 2 participants (10%) showed a decrease; 5 participants (25%) had similar detail and/or specificity, and 1 (5%) had missing data. There were slight differences in results for single-subject versus multiple subject teachers—55% of single subject participants showed increases; 67% of multiple subject participants showed increases.

Question 4 included an open-ended prompt designed for grade 4 students. Participants were asked to identify a) features of the problem that would be challenging to ELLs and b) strategies the teacher could use to help students understand and answer this type of problem. Data were coded for increased specificity and detail, as well as by thematic category. For Question 4a (challenges of the prompt), results were mildly positive: 10 participants (50%) showed an increase, 4 participants (20%) showed a decrease, 5 participants (25%) showed similar level of detail, and 1 participant (5%) had missing data. Most notably, teachers seemed to show increase in responses that included specific words, short phrases, and symbols *and* were accompanied by reasons why they may be challenging (e.g., use of prepositional phrases or relational words). This demonstrated some movement beyond simply identifying vocabulary words.

For question 4b (strategies), results related to increase/decrease of strategies were mixed: 7 participants (35%) showed an increase; 5 participants (25%) showed a decrease; 7 participants (35%) showed equal numbers, and 1 participant (5%) had missing data. While the number of strategies did not show marked increases, the thematic coding revealed some interesting trends. Forty percent of the teachers' increased the explicit mention of language in the strategies they described. Interestingly, the increase in strategies related to language was much higher for single subject teachers (55%) than for multiple subject teachers (22%). To further make sense of the strategies, those coded as specifically mentioning language were subcoded thematically. This coding revealed a decrease in responses involving simple definitions and an increase in responses where the teacher described building meaning about language. This showed a shift toward *ACCESS related* language awareness.

**Implications and Conclusions**

This research reported on PD designed to help teachers understand the language demands of student participation in higher order thinking and justification in mathematics classes—with particular attention to linguistically diverse students in urban schools. Recognizing that it is not a simple matter for teachers to change their practices (Darling-Hammond & Bransford, 2005), combining multiple themes seems ambitious. However, it is this very recognition that PD is too frequently unconnected to complex issues and practices of real schools (Kazemi & Hubbard, 2008) that suggests to us its timeliness.

Although the Math ACCESS project is ongoing, the preliminary results suggest that this model of PD has promise for increasing teachers' knowledge of the language demands of student participation in higher order thinking and justification in mathematics classes. The results demonstrate measurable increases in teachers' content knowledge, perceptions of knowledge and confidence related to supporting students' development of academic language, and awareness of challenges and strategies related to vocabulary and language development for ELLs as well as other students. Thematic analysis demonstrated subtle shifts in perceptions about language use in mathematics classrooms. For example, some of the teachers moved from suggesting rote strategies for learning definitions of math words to providing evidence that they were beginning to grapple with contexts and functions of language (Schleppegrell, 2007); this suggests a shift toward thinking that language is not separate from mathematics, but rather, can be integrally involved with the process of *doing* mathematics.

An additional interesting finding concerned differences in results between single subject teachers (predominantly secondary math) and multiple subject teachers (predominantly elementary). While there were clear group differences in certain results, the differences were inconsistent—that is, for some questions, single subject teachers showed more improvement and for other questions, multiple subject teachers showed more improvement. We conjectured that some of the differences in results may relate to teachers' background experience, expertise, and comfort with content or language. For example, multiple subject teachers were likely to have more extensive backgrounds in language-use, though they may not have considered applications of language to mathematics. Secondary teachers, on the other hand, were likely to have deeper content area expertise, but less experience with language. Questions that have arisen for us include: How do interactions of single subject and multiple subject teachers over the course of the PD impact their perceptions of and competencies with content or language? How do prior knowledge and background influence teachers' openness to ideas? Considering impact of these differences is worthy of further investigation.

As we consider the implications of this research, it is useful to note that these results relate only to the Summer Institute. The project is ongoing, with regular follow-up work in the schools. With this in mind, we look forward to further uncovering important distinctions between "knowledge" and "knowing" (Cook & Brown, 1999; Kazemi & Hubbard, 2008). The results reported here relate predominantly to the teachers' *knowledge* from the PD; as we continue to work with these teachers and document their practice, we will be better able to say how the knowledge translates to *knowing in teaching practice*. This translation from knowledge to knowing in practice has the potential to transform teaching and learning.

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# MODELING THE COLLECTIVE INQUIRY PROCESS IN MATHEMATICS TEACHER EDUCATION

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*Starting with the notion that teachers learn as teacher educators model effective mathematics pedagogy, we examined what teacher educators model from the perspective of researcher, teacher educator, and teachers. Using the NCTM Professional Teaching Standards as a starting point of our analysis, we highlight the idea that teachers were not only engaged as learners of mathematics in a classroom community but also as professionals in a community of mathematics teachers. Here we redefine and elaborate on the NCTM Professional Teaching Standards as they apply to mathematics teacher education with respect to teachers' engagement in a community of mathematics teachers.*

## Introduction

The NCTM Professional Standards for Teaching Mathematics (1991) express the vision of teachers who are well prepared to teach mathematics using student-centered instruction. This vision highlights the importance of teachers' ability to select tasks that engage students' intellect and deepen students' understanding, orchestrate mathematical discourse, use technology and tools to pursue mathematical investigations, make connections to previous or developing knowledge, and guide individual, small group and whole class work. As recognized in the mathematics teacher education literature, such a shift in the classroom environment requires changes in the core dimension of mathematics instruction that are not easy to accomplish, may take several years, and require appropriate professional development (Clarke, 1994; Friel & Bright, 1997; Fennema & Nelson, 1997; Loucks-Horsley, Hewson, Love, & Stiles, 1998).

It has been argued that teachers' instructional practices are shaped by their own learning experiences long before entering teaching in what Lortie (1975) calls an "apprenticeship of observation". Thus, a major difficulty for teachers working to transform their teaching practices in accordance with the NCTM Professional Standards for Teaching Mathematics is that many teachers' experiences as learners of mathematics stand in stark contrast. One way to help teachers make this transition is to engage them as learners in inquiry-oriented mathematics communities where student-centered mathematics teaching is modeled.

The results reported here are part of a larger study in which we investigate modeling by teacher educators in teacher education courses for practicing teachers. An analysis of the data revealed that the teachers are not only engaged as learners of mathematics in a classroom community, but also as professionals in a community of mathematics teachers. In this report, we redefine and elaborate on the NCTM Professional Teaching Standards specifically with respect to modeling professional practice by mathematics teacher educators. We illustrate this elaboration with an example.

## Theoretical Perspective

We view learning as situated within practice. We presuppose that novices develop while embedded in a community alongside experts. Such engagement provides learners with multiple opportunities to build a conceptual model of desired practice. With respect to mathematics

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teacher education this suggests that learning the teaching profession stems, at least in part, from the teaching teachers see and experience as learners and the activity they engage in as professionals. Thus, in mathematics teacher education modeling instructional practices is essential to learning about the practice of teaching since from this perspective the learning of and development of any practice emphasizes the influence of participation, observation and listening in as practice is modeled and mediated by culture and communication.

The constructs of perceptual lived experience, intent participation, apprenticeship and cognitive apprenticeship share a situated perspective on learning from the milieu. They all suggest that knowledge is developed and deployed in activity and is not separable from or ancillary to learning and cognition (Brown, Collins, & Duguid, 1989). In mathematics teacher education, teachers have the opportunity not only to learn mathematics, but also the practice of teaching mathematics from the mathematics instruction they experience as learners. When student centered instruction is modeled, teachers have the opportunity to understand their mature roles as professionals and develop a conceptual model of effective inquiry-oriented teaching.

## Literature

The NCTM Professional Standards for Teaching Mathematics (1991) suggests learning to teach is a process of integration of theory and practice, and teachers should be afforded opportunities to comment and reflect on their own learning and teaching. The current reform movement in mathematics education has a strong underlying theme of the professionalism of teaching. Reform recommendations suggest that teachers ought to collaboratively plan instruction, reflect on practice, create and reflect on new practices, and support one another's professional growth, (NCTM, 1991). This collaborative work of teachers allows teachers to share what they have learned from their experiences as practitioners and then act on what they learn through discussion to enhance their effectiveness as professionals so that students benefit (Astuto, Clark, Read, McGree, & Fernandez, 1993, Hord, 1997). Communication and collaboration such as this among school faculty and staff are important aspects of what researchers call a "professional learning community" or PLC (Dufour & Eaker, 1998; Astuto, Clark, Read, McGree, & Fernandez, 1993, Hord, 1997).

A professional learning community is characterized by a supportive and shared leadership, collective creativity, shared values and vision, supportive conditions and shared personal practice. A major component of the PLC is the collective inquiry process. Dufour and Eaker summarize Ross, Smith, and Roberts' (1994) description of the collective inquiry process:

1. Public reflection–members of the team talk about their assumptions and beliefs and challenge each other gently but relentlessly.
2. Shared meaning–the team arrives at common ground, shared insights.
3. Joint planning–the team designs action steps, an initiative to test their shared insights.
4. Coordinated action–the team carries out the action plan. This action need not be joint action but can be carried out independently by the members of the team. At this point, the team analyzes the results of its actions and repeats the four-step cycle.

For mathematics teachers this includes discussion and reflection about teaching, student learning and the evaluation of both, sharing insights about mathematics teaching and student thinking gained through practice, collaboratively lesson planning and so on. However, engaging in this collaborative process is not automatic. In reform-centered mathematics teacher education the hope is that the teachers are enculturated into a professional learning community of mathematics teachers. In what ways can mathematics teacher educators foster the enculturation of mathematics teachers into the collective inquiry process of a professional learning community of mathematics teachers?

In what follows we report on research conducted in mathematics teacher education courses for practicing teachers. The courses in this study provide teachers the opportunity to deepen their understanding of the mathematics they teach, and engage in activities that are important to the enculturation into collective inquiry process of a professional learning community of mathematics teachers as it is modeled by mathematics teacher educators. In this report we describe our analytical framework and discuss an emergent framework for how mathematics teacher educators model the collective inquiry process and how it parallels the NCTM standards. We further discuss implications for instruction.

**Method**

The mathematics courses for practicing teachers discussed in this report are a part of a university-based professional development group. As with many mathematics professional development programs, the goal is to move teachers forward in their thinking about content and student learning so teachers can work to help increase student achievement in mathematics (Nickerson, 2000; Sowder, 2007). These professional development programs are designed to provide extra preparation for teaching mathematics, not only by communicating pedagogical knowledge, but also by providing opportunities for teachers to deepen their content knowledge by collaboratively reflecting on their teaching and student learning. However, as the focus of this study is on modeling the collective inquiry process the results will focus on the latter.

In this study we observed the mathematics professional development of three cohorts: a primary elementary cohort (grades k-3), an upper elementary cohort (grades 4-6), and a middle school cohort. The classroom data was collected in two consecutive classes for each of the three cohorts. All class sessions were videotaped and a researcher was present at all sessions and took field notes. The videos of the classroom sessions were reviewed to create a descriptive timeline of classroom events to aid in analysis. The teacher educators were interviewed pre and post observation and several participants were interviewed to enable the coding and subsequent creation of an integrated data set of complementary perspectives. Starting with the NCTM teaching standards the classroom sessions coupled with the timeline analyzed in a cyclical process of coding and search for confirming and disconfirming evidence (Strauss & Corbin, 1990) to delineate the categories of modeled instructional acts. Once we developed what we thought to be an exhaustive group of codes, we coded a few episodes separately and compared codes for inter-rater reliability. The coders were in agreement 78% of the time and discussion resolved discrepancies. The primary cause of discrepancies was related to sub-codes of the categories.

## Analytical Framework

The NCTM Standards (1991) advocate a shift in the mathematics classroom environment from an emphasis on mathematics as an individual pursuit that privileges the memorization of algorithms and procedures to an emphasis on mathematics as a collaborative endeavor among members of the classroom community where logical reasoning and argumentation are used to solve problems. In Table 1 following, we provide a brief summary of the NCTM Professional Teaching Standards as they are described by under four headings: tasks, discourse, environment, and analysis.

Table 1. *Brief Summary of NCTM Professional Teaching Standards (1991)*

| | |
|---|---|
| 1) Pose worthwhile mathematical tasks | **Tasks** are the projects, questions, problems, constructions, applications, and exercises in which students engage. They provide the intellectual contexts for students' mathematical development. |
| 2) Orchestrate class discourse<br><br>3) Promote student discourse<br><br>4) Encourage the use of tools to enhance discourse | **Discourse** refers to the ways of representing, thinking, talking, and agreeing and disagreeing that teachers and students use to engage in those tasks. The discourse embeds fundamental values about knowledge and authority. Its nature is reflected in what makes an answer right and what counts as legitimate mathematical activity, argument, and thinking. Teachers, through the ways in which they orchestrate discourse, convey messages about whose knowledge and ways of thinking and knowing are valued, who is considered able to contribute, and who has status in the group. |
| 5) Create a leaning environment that fosters the development of each student's mathematical power | **Environment** represents the setting for learning. It is the unique interplay of intellectual, social, and physical characteristics that shape the ways of knowing and working that are encouraged and expected in the classroom. It is the context in which the tasks and discourse are embedded; it also refers to the use of materials and space. |
| 6) Engage in ongoing analysis of teaching and learning | **Analysis** is the systematic reflection in which teachers engage. It entails the ongoing monitoring of classroom life-how well the tasks, discourse, and environment foster the development of every student's mathematical literacy and power. Through this process, teachers examine relationships between what they and their students are doing and what students are learning |

The NCTM Professional Teaching Standards served as a lens for examining how the mathematics teacher educators' support the teachers' improved participation in a mathematics classroom community of learners. From this emerged parallel categories to describe the mathematics teacher educators' interactions with the teachers modeling the collective inquiry process of mathematics teachers during the class sessions.

**Results**

Starting with the NCTM Professional Teaching Standards as a basis for our analysis, the classroom observations and video were used to redefine the categories of interaction of the mathematics teacher educators with the teachers to shed light on how the teacher educators model the larger practice of the teaching profession. One of the results that emerged was that in the mathematics courses for practicing teachers in this study, the teachers were engaged on two levels, as learners of mathematics in a classroom a community and as professionals in a community of mathematics teachers. In this section we redefine and elaborate on the NCTM Professional Teaching Standards in terms of mathematics teacher education as they are modeled in mathematics courses for practicing teachers with respect to the collective inquiry process of a community of mathematics teachers.

The mathematics teacher educators model the collective inquiry process of a professional learning community of mathematics teachers by engaging teachers in activities that are a part of the practice of teaching mathematics. The activities were often related to evaluating and reflecting on student learning and teaching, lesson planning, and thinking about student thinking, understanding and learning and so on. These activities were mediated by the mathematics teacher educator and provided opportunities for teachers to increase their participation in the collective inquiry process in a community of mathematics teachers. Table 2 characterizes the mathematics teacher educators' instructional acts that have the capacity to foster the enculturation of mathematics teachers into the collective inquiry process.

In this section of the results we will describe a classroom episode from the middle school cohort and discuss the mathematics teacher educators' interactions with teachers as they engaged in an activity that has the capacity to promote teachers' enculturation in the collective inquiry process. Here the teacher educator is Karla, the teachers are the participants in the mathematics teacher education and the term student is reserved for the children that the teachers teach.

The mathematics teacher educator, Karla, began the class by asking the teachers to discuss in their groups the student work that they brought from their own students and choose a few examples that they thought would be interesting to share with the class. The student work the teachers brought was drawn from a predetermined task that all of the teachers in the class tried with their own students, called "try-ons" in this context. This particular try-on was a banquet hall problem that stated as follows:

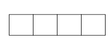
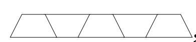
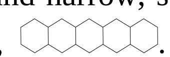
A banquet hall has a huge supply of various shaped tables (square, trapezoidal and hexagonal). Only one person can sit on each side of a table, except the longest side of the trapezoid table, which can seat two people. The same shape tables must be used for each banquet. The banquet rooms are long and narrow, so the tables can only be put together as shown  , . For a given table shape, develop a rule or formula for the number of people that can be seated at 1 table, 2 tables, 5 tables, 100 tables, or n tables. (Adapted from Burns 1992)

*Figure 1.* Try-on task.

As the groups discussed their students' work, Karla walked about the room, listened in on the groups' conversations and briefly joined the discussions of each group in turn. Much of the discussion in groups focused on trying to understand what the students were thinking, the reasonableness of the students' approach, common approaches the students took, etc. After a few minutes Karla asked the teachers if they need more time to discuss. A show of hands suggested that the teachers needed more time. Karla decided to let them continue their discussions for a few more minutes and continued to listen in and discuss with groups. After a few more minutes Karla brought the class back together so the teachers could discuss the task as a class. Ms. K shared a student's work related to using tables to determine linear relationships. Karla placed the student's work on the document camera and asked Ms. K what she could tell the class about this the work. (See Figure 2.)

Square

| Tables | 1 | 2 | 5 | 100 | n tables |
|---|---|---|---|---|---|
| People | 4 | 6 | 12 | 202 | Add 2 each time |

Trapeziod

| Tables | 1 | 2 | 5 | 100 | n tables |
|---|---|---|---|---|---|
| People | 5 | 8 | 17 | 302 | Add 3 each time |

Hexagon

| Tables | 1 | 2 | 5 | 100 | n tables |
|---|---|---|---|---|---|
| People | 6 | 10 | 22 | 402 | Add 4 each time |

*Figure 2.* Student work shared in the professional development class.

Ms. K explained that the student could determine the number of seats at the banquet by her rule of adding a certain number for each table added, but could not generalize that statement with a formula. Ms. N, a teacher in Ms. K's group, further elaborated on the student's work and explained that the student could reason about the situation additively but had not yet transitioned her thinking about repeated addition as multiplication or make use of variables to further generalize the situation. Karla pointed out that the student is making use of a recursive relationship to solve the task but not the functional relationship.

Ms. N expressed that they chose to share that example of student work because many of their students thought about the task in a similar fashion. They discussed another student's work where the student did use multiplication to do the task but did not generalize the linear relationship with a function and was unsure how to think about n tables. This particular student chose n to be 200 and found the number of seats available if there were 200 tables. Karla stated if we had to order the students in their level of understanding it seems this student seems to exhibit a little less understanding than the one in their last example. Karla noted that the students were from 6th grade and that they might not expect that all students at this level would be able to could come up with a function to express the relationship. Karla then asked the teachers to think about what would be next for these students to push them further in their thinking if they were going to be their teacher next year. Ms. N suggested that they could give the kids practice translating words into variable expressions; and once they have practice with that they could go back to the banquet hall task and ask them how they could use their experience translating words into variable expressions to determine the number of seats when there are n tables are put together. Karla reiterated what Ms. N said and Ms. K added that the practice could start with simple translations that yield expressions like 2x and 3x and so on. Karla suggested another possibility could be to ask the students express the relationship in words and later connect the words to the algebra. Karla then opened the floor for additional question or comments before moving on.

Using the NCTM Professional teaching standards to describe classroom episodes like the one above was problematic with respect to tasks, discourse, environment, and analysis because the while some mathematics learning may have taken place the teachers were primarily engaged as teachers of mathematics and not learners. In order to characterize the interaction of the mathematics teacher educators and the teachers we elaborate on the professional teaching standards as it applies to modeling the collective inquiry process.

*Tasks*

In the episode described above the primary focus of the task was not to provide intellectual contexts for the teachers' mathematical development as learners, but to motivate the development of the teachers' understanding of the students' mathematics through discussion, exploration and experimentation. With the try-on task, the mathematics teacher educator modeled the coordinated action and analysis of the collective inquiry process.

*Discourse*

While like in the NCTM professional teaching standards the discourse during this episode reflected the ways of representing, thinking, talking, and agreeing and disagreeing that mathematics teacher educators and teachers use to engage in tasks, the discourse in this episode is embedded in the teachers' classroom experiences as teachers within a community of teachers. The mathematics teacher educator orchestrated discourse that modeled the public reflection aspect of the collective inquiry process.

*Environment*

The environment represents the setting or context in which the tasks and discourse are embedded for learning. When the teachers are engaged as learners of mathematics, the setting is a mathematics classroom community. However, when the teachers are engaged as professionals, the setting shifts to a community of mathematics teachers. As the teachers engage in this professional learning community the mathematics teacher educator models the interplay of intellectual and social characteristics of the members of the community that shape the evolving knowledge base of the community.

*Analysis*

Analysis in the NCTM professional teaching standards refers to the ongoing monitoring of classroom life-how well the tasks, discourse, and environment foster the development of every student's mathematical literacy and power. In terms of the collective inquiry process the mathematics teacher educator works to foster development of every teachers' enculturation in a professional learning community of mathematics teachers. One way a mathematics teacher educator fosters this enculturation is exemplified in the above episode as she listened in on the conversations of the groups. Through this process, mathematics teacher educator examines the group members' participation in the collective work of teachers. The teacher educator modeled the analysis of student thinking as a means of thinking about where to go next. She also modeled, we argue, by listening in on groups and selectively sharing what she was thinking about their learning from the perspective of analysis.

Table 2 characterizes the mathematics teacher educators' instructional acts as they model the collective inquiry process.

Table 2. *The NCTM Professional Teaching Standards and the Collective Inquiry Process*

| | |
|---|---|
| 1) Worthwhile Tasks or Activities | **Tasks/Activities** motivate the development of the teachers' understanding of the students' mathematics through discussion, exploration and experimentation. |
| 2) Orchestrate class discourse<br><br>3) Promote teacher discourse | **Discourse** refers to the ways of representing, thinking, talking, and agreeing and disagreeing that teacher educators and teachers use to engage in those tasks. The discourse embeds fundamental values about knowledge and authority. Mathematics teacher educators, through the ways in which |

| | |
|---|---|
| 4) Encourage the use of tools to enhance discourse | they orchestrate discourse, convey messages about the collaborative work of teachers. |
| 5) Create a learning environment that fosters the enculturation of teacher collective inquiry process | **Environment** represents the setting for engaging in the collaborative work of teachers. It is the unique interplay of intellectual, social, and physical characteristics that shape the ways of knowing and working that are encouraged and expected in the community of teachers. It is the context in which the tasks and discourse are embedded; it also refers to the use of materials and space. |
| 6) Engage in ongoing analysis of teaching and learning | **Analysis** is the systematic reflection in which mathematics teacher educators engage. It entails the ongoing monitoring of classroom life-how well the tasks, discourse, and environment to foster teaches participation in the Collective Inquiry Process of a professional learning community of mathematics teachers. |

### Concluding Response

Collins, Brown and Newman (1989) and Lave and Wenger (1991) hypothesize that through observation, learners develop a conceptual model that provides them with an advanced organizer and interpretive structure for reflecting on a given practice. It can be argued that understanding how to participate in the collective inquiry process of a professional learning community of mathematics teachers is important because it provides insights into the nature of the professionalism of teaching advocated by the NCTM. In mathematics teacher education, teachers have the opportunity not only to deepen their mathematics content and pedagogy skill, but also the opportunity to engage in the collective inquiry process of mathematics teachers. The hope is that this research informs the body of knowledge about teaching the practice of teaching mathematics.

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# BEGINNING TEACHERS AND NON ROUTINE PROBLEMS: MATHEMATICS LESSON STUDY GROUP IN AN URBAN CONTEXT

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*This paper reports on a professional development and research initiative that engages a group of beginning middle school teachers in studying non-routine mathematics problems and investigates the effects of this intervention. Participants in this group engaged in the guided study of: a) non-routine mathematics problems, b) samples of students' written work on NRP, c) sample mathematics assessments that include NRPs, d) case studies of NRP-centered mathematics instruction, and e) 'vertical' analysis of various curricula. The study group improved teachers' lesson planning but varied in the authenticity of using the NRPs among individual participants. An instructional design framework was developed linking the NRPs, the lesson and the unit of instruction.*

## Background

This study draws on data from a larger professional development and research initiative that engages beginning middle school mathematics teachers in a lesson study group and studies the effect of their participation in this in-service initiative on the quality of their mathematics lessons. Lesson study participants are teachers in their first years of service who work in 'hard-to-staff", high poverty, urban schools. Our long term research goal is to explore the impact of participating in this study group on teachers' practice and their retention in the high need schools were they work.

In this report we describe how beginning middle school teachers interacted with lesson study activities centered on the solving, teaching, and learning of non-routine problems (NRP). Participants in this group engaged in the guided study of: a) non-routine mathematics problems, b) samples of students' written work on NRP, c) sample mathematics assessments that include NRPs, d) case studies of NRP-centered mathematics instruction, and e) 'vertical' analysis of various curricula.

## Theoretical Perspectives

Influenced by the seminal work of Polya (1945), Schoenfeld (1985) and the NCTM focus on problem solving (1989, 2000), mathematics education researchers highlight the need for instruction to engage students in solving rich, challenging, high level, and open-ended tasks. Researchers might vary in how they refer to these problems based on their wide range of frameworks ranging from cognitive science to social constructivism and activity theory. The literature refers to these tasks as high level tasks (Stein et al., 1996), model-eliciting tasks (Lesh & Harel, 2003), realistic modeling problems (Verschaffel & de Corte, 1997), spiral tasks (Fried & Amit, 2005), and multiple-solution connecting tasks (Leikin & Levav-Waynberg, 2007). What is common among all the above types of problems, which we refer to as non-routine problems (NRP), is their valuable role in eliciting thinking and reasoning, communication, critical attitude, interpretation, reflection, creativity, and generalization, all of which are central to the activity of mathematizing (Freudenthal, 1991).

Yet there is increasing concern that in many classrooms, especially in those attended by minority and low SES students, instruction focuses almost exclusively on mechanical ways of applying algorithms and formulas to the solution of stereotypical word or ‘story’ problems (Oakes, 2005; Boaler, 2002). The poor quality of mathematics instruction in schools attended by low SES and minority students is seen as a critical contributor to social inequality (Moses & Cobb, 2001) in that these groups of students are denied access to high level mathematical thinking as well as important pathways to economic and other enfranchisement (National Action Committee for Minorities in Engineering 1997; National Science Foundation 2000).

It is well documented that teachers rarely make non-routine problem solving an integral part of their instruction (Henningsen & Stein, 1997; Silver, 2005; Leikin & Levav-Waynberg, 2007). Therefore, it is hardly surprising that students have difficulties with these kinds of problems (Verschaffel & de Corte, 1997; Cooper & Harris, 2002). A modality of mathematics instruction that focuses only on routine problems is seen as very unlikely to prepare students to successfully tackle and solve novel problems in and out of school settings. While this poverty of mathematics instruction could be blamed to a great extent to mandated curricula and standardized testing (Haydar, 2009), it is also a result of limitations in teachers’ appreciation of the educative value of those kinds of problems, their own level of comfort in solving such problems, and their ability to handle the pedagogical demands that this type of problem solving activity entails, in particular in orchestrating whole-class discussions about multiple strategies for solving a given problem (Silver et al., 2005; Shreyar et al., 2009).

Mathematics teacher educators advocate the use of lesson study as a model of teacher-initiated and mentor-facilitated professional development (Stigler & Hiebert, 1999). Issues of adaptation of the lesson study model to the US context have been the subject of many recent studies (Fernandez & Yoshida, 2004; Lewis, 2002). However, we know little about how lesson study activities centered on non-routine problems affect beginning teachers’ planning and assessment skills, especially when in urban school contexts. This study is an attempt to fill some of these gaps.

## Research Questions

In this research we aimed to explore to what extent and how does NRP-centered lesson study group increase participants’ ability to effectively incorporate non-routine problem solving into their classroom practice. In addition, our goal was also to use this Lesson Study group as a laboratory and draw insights on the design, try out and documentation, revision and dissemination of a sequence of NRP-based activities and materials for the professional development of middle school mathematics teachers.

## Methodology

The participants in the Lesson Study Group (n=10) were either recent graduates or in their final year in a middle school mathematics master’s program. They were within their first 5 years of teaching in ‘hard-to-staff”, high poverty, urban schools. They teach seventh to ninth grade mathematics in urban school settings attended by a predominantly African-American, Latino, Asian, and/or recent immigrant school population. All of them expressed in writing, as a response to the invitation letter from the researchers, their interest in, understanding of, and commitment to the proposed project.

In line with what is typical to the Lesson Study modality of professional development in mathematics (Stigler & Hiebert, 1999; Fernandez & Yoshida, 2004), a central focus of the sessions was the design, try out, documentation, revision, and write up of mathematics middle school lessons. More specifically participants were engaged in the following activities:

- Solving and studying NRP
- Selecting and sequencing from a list of NRP
- Designing NRP-based lessons
- Trying out, documenting, discussing NRP-based lessons
- Analyzing curricula in search for NRP
- Analyzing assessments in search of NRP
- Inventing, finding, adapting NRP
- Transforming a routine problem into a NRP
- Analyzing student work samples on NRP

Parallel to the three face-to-face sessions, the pilot lesson study group project engaged participants in three a-synchronous on-line discussion boards via Blackboard.

All Lesson Study sessions were audio and video-taped and a record was kept of all the asynchronous online discussion board contributions. Portions of these sessions were analyzed in search for evidence of an increase in participant teachers' ability to: appreciate the value of NRP; solve and discuss alternative solution strategies for NRP; recognize NRPs in textbooks and assessments (as well as the lack thereof); design NRP-centered lessons, and organize NRP into a unit; and incorporate non-routine problem solving into their classroom practice.

In order to be able to assess the effect of participating in the LSG for each individual teacher, we collected pre- and post data from each participant regarding: a) solving and explaining in detailed write ups their solution to NRP, b) selecting NRP from a given list of 'scrambled' problems and sequencing those problems into a unit, c) transforming a routine problem into a non-routine problem, and d) designing a NRP-centered lesson for students in one of their classes for a given unit/topic.

We also conducted a follow up, open-ended survey four months after the last Lesson Study session. Survey questions focused on their recollection of the most memorable moments of the lesson study, their narratives of what NRP-related elements they incorporated or planning to incorporate in their mathematics lessons and their interest in future engagement in similar professional development activities.

In analyzing the above data, we looked for indicators of improved skills in studying, solving, and describing the solutions to NRPs as well as evidence of an enhanced ability on the teacher participants' part to search for, design, adapt, and sequence NRPs. We developed a coding scheme based on the PISA cross-disciplinary problem-solving framework and Competency Clusters (OECD, 2003).

To analyze the manner in which participant teachers incorporate non-routine problem solving into lesson planning we developed a lesson template based on the Japanese lesson study and its various adaptations (Fernandez & Yoshida, 2004).

## Snapshots from Lesson Study Sessions

To illustrate how teacher participants typically engaged in NRPs during the lesson study group activities we present below snapshots from one of the sessions. These snapshots are paradigmatic of the kinds of conversations that occurred during the sessions. Participants worked on the cross-to-square problem (fig.1). We selected this problem as a rich context that includes all the following features which we view as highly relevant to the classroom practice of middle school teachers:

- Conveyed through a diagram (invites diagrammatic thinking)
- A geometric dissection
- A constant area problem
- As a puzzle, it generates puzzling, puzzlement
- Linked to rotated or 'tilted' squares (transformational geometry: from a square to a tilted square)
- Connection to Pythagoras theorem
- Lends itself to other related problems (other square dissection puzzles)

<u>Cross-to-Square Puzzle</u>

Divide the shape below into four parts that can fit back together to form a square

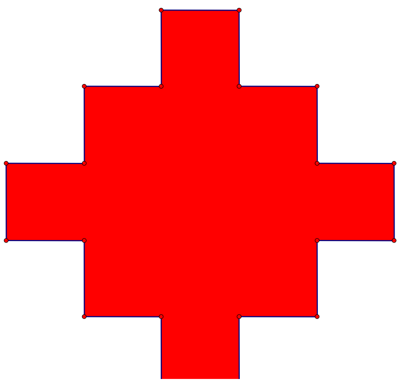

Figure 1.

The problem was introduced during a brief whole-group exchange following a "thinking aloud together" modality (Zolkower & Shreyar, 2007).

*Framing the NRP*

BZ: Can we dissect this rectangle in three pieces so the pieces may be re-arranged into a square? (*Draws a 2 by 8 rectangle on chart paper)*

*A few minutes later*

Ms. E: I got it! It will give you a tilted square with an area of 8.

*BZ draws the tilted square and shows with arrows how it results from dissecting the rectangle.*

BZ: Easy, right? Now the real problem consists of dissecting this cross-like figure… (*Draws diagram).*… in four pieces so that the pieces may be re-arranged to make a square.

Ms. S: How can this work?!

BZ: What do you mean?

Ms. S: I mean, if there are 13 squares. 13 is not...

Mr. H: Can we split the squares?
BZ: Split them how?
Mr. H: Cut them diagonally in some way
*BZ Draws a small square and splits it in half diagonally*.
BZ: What do you all think?
Mr. Z: It won't work unless we cut the squares.
*Shifting the Context: From the LSG to the Classroom*
After the solution was found.
Ms R: I have a question… How do we help our students solve these kinds of problems?
*BZ writes R's question on the board, crosses out 'help,' and substitutes it with 'teach.'*
BZ: I'd like to rephrase R's question: How do we TEACH students to solve these kinds of problems? What kinds of problems would they need to work on before tackling this one?
Ms. G: Unless they're familiar with the idea that a square can be tilted.... *Makes a hand gesture to indicate 'tilted'*
Ms. S: Also… as soon as they see 13, they may think, like I did, it's impossible to do it!
BZ: Those two issues seem quite related, right? So it may beneficial to first engage students in activities that involve sketching and finding the area of tilted squares… on graph paper, of course.
*From NRP to Pre-Requisite, Sub-Problem*

The above shift in perspective, from solving the math NRP problem to raising the question of how a problem such as this one could be introduced in a middle school classroom, led us to introduce a likely candidate for a pre-requisite (or sub-) problem (fig. 2).

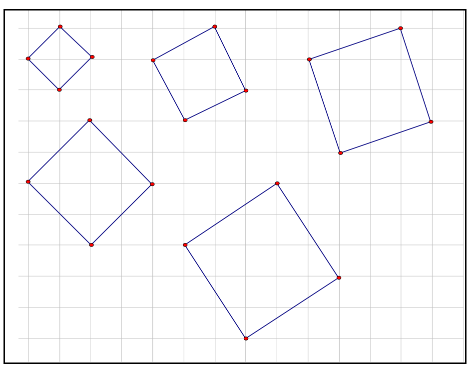

Figure 2.

*From Problems to Units*

And, in turn this was followed by an activity whereby participants were given a set of nine 'scrambled' problems and asked to solve each of them, identify the mathematics in each of the problems then select three to five problems to make an instructional unit with them. They were also asked to justify their choice of problems for the unit as well as its sequencing.

**Results**

The lesson study activities centered on NRPs proved to be a rich context for teachers' engagement with mathematics, learning, and teaching. Our analysis of the data revealed the following

- Mirroring the classroom: By asking teachers to try out and report on both their own problem solving and their incorporating of NRPs in their classrooms, teachers developed a better understanding of what their students go through when they solve mathematical problems and led them to realize the importance of providing a challenging and, at the same time, supportive classroom environment. As one participant put it: *"The engagement in solving non routine problems so resembled what happens in my classroom. Students demonstrate different approaches as did we as teachers. The environment needs to be safe so that all feel comfortable to share. I also rediscovered my own strengths and weaknesses as a problem solver."*
- The laboratory context: the participation in the lesson study encouraged teachers to experiment with incorporating non-routine problems in their classrooms and being able to experience gradual successes *"I have started incorporating NRPs occasionally in my class. I noticed that all my students are becoming more actively engaged and are asking if we are going to continue to do problems of that nature."*
- Lesson planning improvement: As noted before, teachers showed improvement in planning sequences of related lessons: *"I am planning to continue the use of proper sequencing of lessons and use more NRP's.*"
- Analytical pedagogical tools: Working on NRP and analyzing curricular and assessment materials with respect to non-routine problem solving gave teacher participants tools for examining their classroom practices. For example one teacher noticed: *"Comparing and contrasting the different assessment forms and questions from around the world was most memorable to me. I found it very interesting and insightful to critique the NYS math curriculum in comparison to other countries"*
- Community of practice: Teacher participants found in the group a learning community and enjoyed "*the experience of being able to work with teachers who have a different method or strategy of presenting a lesson different from what [they] will usually do in [their] classroom."* One teacher noticed how the group provided a structured time for collaboration and how other teachers became a resource: "*Not only did I have structured time to create a higher-level multi-topic activity to complete with my class, but now I can take the other activities created by the other teachers back to my school and use them with my students in the future.*
- School Leadership: One teachers played a leading role in replicating some of the lesson study activities at their schools. This participant wrote in a journal entry*: "Our professional development sessions at school are currently planned around solving non-routine problems as we wish to enhance teachers own skills (at their request)."*

The analysis of the lessons planned by individual participants both at the beginning of the study and at the end of the sessions showed that participants: (1) moved from the brief isolated lesson formats to more complex ones that place the lesson in the context of broader unit of instruction; (2) made more effort after the sessions to include mathematical problems in their lessons; (3) The success in incorporating non-routine problems into classroom practice on a routine basis varied by individual participants. Also only two teachers introduced the non-routine problems systematically as authentic contexts for mathematizing while others often contrived some of the mandated curriculum standards to the problems.

## Discussion

Given that NRPs are challenging for students and given their rich mathematical content create the need for teachers to act as curriculum and instructional designers and locate a given NRP (Pn) with another that should precede (Pn-1) and/or follow after (Pn+1) , this vertical analysis constitutes the backbone for designing a unit of instruction based on NRPs. Also NRPs are analyzed and compared horizontally to problems from other mathematical strands. NRPs are by their very nature amenable to a variety of approaches. The chart in Fig.3 describes our framework on the relationship between NRP, other NRPs, Problem-based lesson and problem-based unit. NRP-based Lesson study should guide teachers in relating and moving back and forth between these different instructional design dimensions.

Other than the curricular side, NRP-based teacher education activities were also found to help teachers improve their ability to conduct classroom interaction.

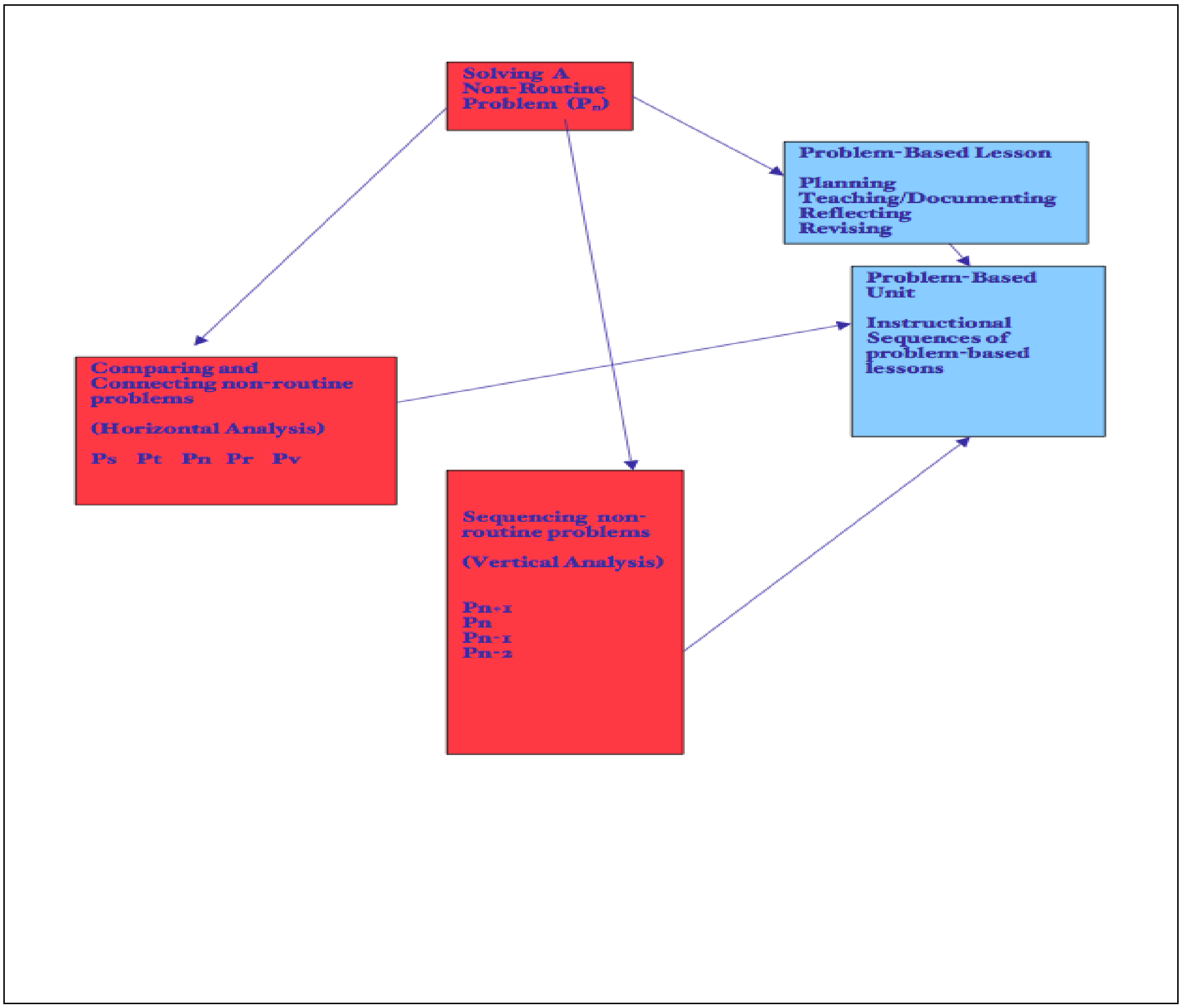


Figure 3.

Non Routine Problems require from teachers a connected understanding of mathematics where the isomorphic connections between the different mathematical strands and structures are developed and used to solve the problem. The uneven results found in this study in how individual participants incorporated NRPS in their lessons are due in part to the level of comfort with NRP mathematical content and linked to participants' disposition to try these in their classroom. Engaging teachers in NRP-based LSG proved to be a safe professional development modality strengthening teachers' own mathematical and problem solving content knowledge and guiding that shift from thinking about incorporating non-routine problem as potential tasks that beginning teacher will only do after they become expert teachers to that belief that this is essential part of being a better reform mathematics teacher.

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# PROMPTING MATHEMATICS COACH DEVELOPMENT OF MATHEMATICAL KNOWLEDGE FOR TEACHING

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*This teacher development experiment examined the development of mathematical knowledge for teaching (MKT) of three coaches of mathematics teachers (Hill, Rowan, & Ball, 2005). Coaches were graduate students in the field of mathematics who had an interest in teaching. Coaches developed knowledge of content and teaching as they reflected and collaborated with classroom teachers to implement inquiry-oriented lessons. Coach knowledge of content and students developed through observing and interacting with students. Finally, coaches developed specialized content knowledge as they discussed perturbations from lessons with teachers.*

## Background

The field of professional development coaching currently enjoys steady growth in mathematics education as schools search for effective ways to support the learning of in-service teachers. Although coaching is gaining popularity as a means of professional development, its forms and effectiveness of implementation vary from context to context (Olson & Barrett, 2004). For example, due to the difficulty in recruiting highly qualified coaches from the field of mathematics education, some school districts have looked to hiring their best mathematics teachers as coaches. As a result, districts rob students of qualified teachers and position the newly hired coaches to find their own way in supporting teachers. Thus, finding qualified, cost-effective coaches remains a difficulty for schools.

One alternative model of coaching matches graduate students in the fields of science, technology, engineering, and mathematics (STEM) as content specialist coaches with K-12 classroom teachers. The National Science Foundation (NSF) has established the Graduate Fellows in K-12 Education (GK-12) program that provided the context for this study. The GK-12 program places Graduate Fellows into K-12 classrooms as *collaborative coaches* (Olson & Barrett, 2004). The partnership is designed as a mutual professional development opportunity for both the coaches and the classroom teachers. Teachers develop content knowledge related to the subject matter they teach. (See Knapp, Barrett, & Kaufmann (2007) for teachers' development of mathematical knowledge for teaching through this model.) The teachers voluntarily participate in the partnership. The mathematics graduate coaches may or may not have prior teaching experience, but they have had summer and bi-weekly training on topics such as the National Council of Teachers of Mathematics (NCTM) Standards (2000) and social constructivism. Graduate coaches collaborate with practicing teachers and other graduate coaches on planning and delivering standards-based lessons. Coaches generally model lessons for teachers with the teachers' own students. On some occasions, however, the teacher leads the lesson with the coach's assistance. Most often, teachers and graduate coaches work in pairs, although at times teachers request the assistance of multiple coaches. NSF expects that graduate coaches will develop useful teaching abilities through the program that will prepare them for faculty positions that involve teaching. In order to examine the viability of the partnership, this study examines the impact of the coaching relationship on the teaching abilities of the *coaches*. More specifically, I investigate coaches' development of mathematical knowledge for teaching, which is the mathematical knowledge and habits of mind needed for *teaching* mathematics (Hill et al., 2005). In addition, I seek to ascertain aspects of the coach-teacher relationship that might lead to coach development. Thus I ask, "In what ways do graduate coaches develop mathematical knowledge for teaching as they engage in collaborative coaching with classroom teachers?" For the remainder of this paper, I refer to graduate coaches as *Coaches*, K-12 classroom teachers as *Teachers*, and K-12 students as *students*.

## Theoretical Framework

The construct of *mathematical knowledge for teaching* (MKT) has been linked to student achievement, and thus provided a framework for analysing the Coaches' development in this study (Hill et al., 2005). MKT includes these six elements: common content knowledge; specialized content knowledge (SCK) needed specifically for the mathematics classroom; knowledge of content and students (KCS) which is knowledge of how students learn mathematics; knowledge of content and teaching (KCT) which includes knowing the best representations for teaching mathematics; knowledge of curriculum; and knowledge at the mathematical horizon. This study focused on KCT, KCS, and SCK (See Table 1) (Hill et al.).

Table 1. *Components of Mathematical Knowledge for Teaching*

| Subject Matter Knowledge | Pedagogical Content Knowledge |
|---|---|
| Common Content Knowledge (CCK) | Knowledge of Content and Students (KCS) |
| Specialized Content Knowledge (SCK) | Knowledge of Content and Teaching (KCT) |
| Knowledge at the Mathematical Horizon | Knowledge of Curriculum |

For a theoretical framework, the emergent perspective appeared suited to this study because mathematical knowledge for teaching (MKT) is related to the construct of classroom social norms as outlined in the emergent perspective (Ball, 2003; Cobb & Yackel, 2004). The emergent perspective takes the social aspects of learning and the individual psychological aspects to be reflexively related. In this study, I investigated the individual and social construction of MKT of Coaches as they collaborated with teachers.

## Methodology

I chose to employ qualitative, multi-tiered teacher development experiment (TDE) methodology because the goal of a TDE is to generate models for teachers' mathematical and pedagogical development, closely matching our research aims for the collaborative coaches (Lesh & Kelly, 2000; Presmeg & Barrett, 2003).The methods for this qualitative teacher development experiment involved year-long case studies of three mathematics coaches: Melvin, Dave, and Marsha. I also conducted case studies on four Teachers, but I do not report on the Teacher development in this paper. Melvin had four years of teaching experience at the secondary level before pursuing his graduate degree in mathematics. Dave had two years of teaching experience, and Marsha had no former teaching experience or preparation.

Data sources for Melvin's development of MKT included 87 written pre/post lesson reflections, transcripts of two audiotaped interviews, 4 video-taped lessons, and transcripts of 9 audiotaped pre and post planning sessions done in collaboration with a 6-8$^{th}$ grade mathematics teacher, Mrs. Gerber. Data sources for Dave included 74 written pre/post reflections, transcripts of two interviews, 9 video-taped lessons, and transcripts of 7 audiotaped planning sessions with Teachers. Data sources for Marsha included 87 written pre/post reflections, transcripts of two interviews, 12 video-taped lessons, and transcripts of 11 audiotaped planning sessions with Teachers. Pre lesson reflections asked Coaches to describe the lessons that they would teach and to explain how students would be expected to invent knowledge and think through the content. Coaches were also asked to predict areas that would be difficult for students to understand. Post lesson reflections required Coaches to describe how the lesson went and whether students understood the content. These data sources were analysed for development in mathematical knowledge for teaching with regard to SCK, KCS, and KCT.

In order to analyse the data, SCK, KCS, and KCT were broken down into 17 codes relating to different aspects of each construct. A question accompanied each code in order to highlight ways that MKT development might occur. Questions came from the elements of the work of teaching elaborated by Hill et al. (2005) and from salient aspects of the pilot study. For examples of the codes and accompanying questions, see Table 2. Three transcripts were analysed by both the researcher and Melvin, Dave, and Marsha respectively until an interrater reliability of 80% was reached. After this, the researcher coded the rest of the transcripts. Each time a portion of transcript was coded as SCK, KCS, or KCT, the accompanying question was answered based on the data. Ways in which these elements developed were then categorized and tabulated.

Table 2. *Analysis Codes and Questions*

| Category | Analysis Question (Code) |
|---|---|
| KCT | How did the lesson study environment affect the Teachers'/Coaches' instructional choices and use of curriculum? (KCT1) |
| KCT | How did the Teacher/Coach encourage student construction of knowledge? (KCT4) |
| KCT | How did the Teacher/Coach provide explanations, examples, or counterexamples? (KCT7) |
| KCT | Did the Teacher/Coach ask students to justify their reasoning? (KCT6) |
| KCS | How does the Teacher/Coach notice students' knowledge/reasoning/thinking as they engaged in lesson study? (KCS12) |
| KCS | How does lesson study help Teachers/Coaches question their students [not as an instructional tool but to learn about students' thinking]? (KCS13) |
| KCS | How does lesson study help Teachers/Coaches see/hear student misconceptions? (KCS14) |
| SCK | How did mathematical discourse between Teachers and Coaches foster reasoning? (SCK17) |

## Results and Discussion

I coded Coach reflections for knowledge of content and students (KCS), knowledge of content and teaching (KCT), and specialized content knowledge (SCK). This meant that I coded quotes which I felt indicated that the Coach developed or had an opportunity to develop these elements of teaching. For example, if a Coach noted changes he would make to a lesson after teaching it, I coded an opportunity to develop KCT. I avoided counting quotes which showed teaching knowledge that the Coach possessed prior to the lesson. In addition to reporting results from reflections in this section, I also report codes from transcripts of audio taped reflections with teachers, videotapes of lessons, and interviews. The final tallies revealed 91 expressions of developing KCT, 41 expressions of developing KCS, and 14 instances of developing SCK. In addition, Coaches reported learning general pedagogy.

After coding the data, I looked back at each code, and identified *how* that type of knowledge developed. To focus on the primary ways in which Coaches developed, I eliminated all ways that were expressed less than ten times. Some ways I collapsed together into a single category. The compression phase of data analysis revealed five primary ways that mathematical knowledge for teaching developed for the Coaches. I list the five primary ways in Table 3, and I provide evidence for the ways in the sections following the table.

Table 3. *Primary Ways in which Coaches Developed Mathematical Knowledge for Teaching*

| Frequency | Way |
|---|---|
| KCT (46) | |
| 17 | **Reflecting** about changes and things to keep from a lesson taught |
| 11 | Having students construct their own knowledge and make discoveries and conclusions; **Teaching for conceptual understanding** |
| 18 | Reflecting on lesson with teacher; Considering perturbations or misconceptions prompting revision; **Collaborating with Teacher** in planning, with both sets of expertise working together; Repeating lesson or teaching strategy |
| KCS (32) | |
| 32 | **Observing**, listening, and interacting with students during lesson; seeing what's hard for them and misconceptions; seeing level of material students could handle, construct, or conjecture (raised expectations) |
| SCK (10) | |
| 10 | Finding an application to model the content; **Discussing perturbations** with Teachers or other Coaches; Testing a student conjecture |

*Coach Development of Knowledge of Content and Teaching*

The primary ways in which Coach mathematical knowledge for teaching developed focused on knowledge of content and teaching (KCT) and knowledge of content and students (KCS). The first primary way Coaches developed KCT was in reflecting about lessons they had taught and deciding to either retain or change elements of the lessons. For example, Marsha stated the following in a reflection:

> The kids really had a lot of fun with this game, and it really was simple in terms of materials and set up. They practiced a lot of multiplication problems and were really checking each others work since they wanted to win the cards that round. It also helped them to think about strategies of what is going to give you a big number when you multiply, which I think helps them to develop their estimating and telling if an answer is reasonable.

She learned a classroom activity to support students' estimating strategies (KCT). This finding substantiates Mumba et al. (2003) who found Coaches to have Technocratic-oriented reflections. In other words, they sought solutions to improve their teaching. Furthermore, like Mumba et al., the nature of the Coach reflections were descriptive, dialogical, and critical. Dialogical referred to reflecting with teachers during and after lessons as well as reflecting with other Coaches before and after lessons. Critical referred to expressing dissatisfaction with lessons and suggesting alternatives. The second way Coaches developed KCT followed from the first, in that Coaches often chose to change lessons towards inquiry. In other words, they valued to a greater degree students making their own discoveries and conclusions, individual construction of knowledge, and teaching for conceptual understanding. The third way KCT developed was similar to the first and second, but allowed for collaboration with Teachers. Coaches collaborated with Teachers on planning lessons, in which both sets of expertise went into the lesson design. The teacher contributed knowledge of her individual class and pedagogy. The Coach contributed content knowledge and knowledge about inquiry-based instruction. After lessons, Coaches and Teachers as collaborative partners discussed perturbations relating to student misconceptions, pedagogy, or technology. The Coaches and Teachers would then revise and sometimes repeat lessons based on their discussion of the issues. Finally, Coaches would at times repeat the cycle of revision, perhaps with another Teacher or another Coach.

*Coach Development of Knowledge of Content and Students*

Knowledge of content and students (KCS) primarily developed during lessons as Coaches observed, listened to, and interacted with students. Coaches learned what concepts were difficult for students to understand as well as misconceptions that students possessed. For example, all three Coaches independently encountered trapezoid as a challenging concept in middle grades classrooms. Dave and Marsha, independently in different lessons and different schools, found that students' conceptions of 'trapezoid' are often of an isosceles trapezoid where both legs are the same length. The Coaches then challenged this misconception by presenting them with right trapezoids, and discussing with them the properties of a trapezoid. Marsha used geoboards as a teaching tool and Dave used Geometer's Sketchpad to address the misconceptions. Melvin addressed the misconception during a class discussion. In a post lesson reflection he wrote,

> The best discussion I felt like came from discussing the trapezoids, as the students felt there was only one possible trapezoid, the picture we frequently see in books of an isosceles trapezoid. After determining the definition of a trapezoid to be one set of parallel sides, the students debated me as to whether or not a right trapezoid was really a trapezoid. This discussion really brought out the misconceptions created by always using the same figure to describe a family of figures and was a good chance to discuss what properties are necessary to classify a quadrilateral.

Thus, the Coach learned that students hold limited concept images of figures, developing knowledge of content and students (Vinner & Hershkowitz, 1980).

In addition to learning about student misconceptions, Coaches learned about the level of material the students could handle. Moreover, Coaches learned how students created conjectures and constructed knowledge. At times, Coaches' expectations were raised by observing students' productions in an open-ended environment. For example, in an open-ended lesson in which students were to conjecture about quadrilateral properties using Geometer's Sketchpad as a tool, the Coach, Melvin, expected students to come up with perhaps four or five conjectures. He and the teacher were amazed when the students produced twenty conjectures, some of which can be seen in the following portion of transcript:

Student: 2 sets of parallel lines.

Melvin: 2 sets of parallel lines. What do we think?

Students: Yeah, No [chorus of no's]

Melvin: Who says no? OK right there, you say no? Alright, so far she's the winner. She says that's not the case, but let's go back and let's look at this, OK. On your screen you might have parallel sides, but remember what a property is. A property is something that's true for every quadrilateral. So let's look over here. Do these have parallel sides? [motioning to a student screen and dragging] Are they parallel now?

Melvin: Look at mine up here [referring to the projected Sketchpad screen]. What I'm going to do is I'm going to start dragging this thing around, right? I can drag this around. Are those parallel? [chorus of no's] So it doesn't. Even though I can take these and I can make them parallel, and according to that observation right there, they are parallel. So I would agree with the young lady. However, we need to be true for all of the quadrilaterals, and so that isn't going to be one for all of them.

Student: All have four letters.

Student: Four segment lines.

Melvin pointed out that "segment lines" was contradictory and wrote, "Four segments." The transcript continues.

Student: They're all enclosed areas, I mean, inside it's all enclosed.

Melvin: OK, I'm going to tie all that up with one word – polygon, OK. Enclosed.

In the audiotaped post reflection between the Coach and the teacher, the Coach said,

> Yeah, I felt, I was very happy with their creativity, and their coming up with the properties. They all were involved. One thing we talked about might happen is that they'll just stare at me, and look at me. That didn't happen at ALL. So, so by that I was taken a little bit aback and was kind of enthused by it. Um, was maybe a little bit overwhelmed..."

Melvin learned that students are curious and eager to conjecture, and he also learned that student conjectures could appear very different from properties listed in a textbook. Thus, at times, the results of inquiry-based lessons took Coaches by surprise; knowledge of content and teaching had spawned knowledge of content and students.

*Coach Development of Specialized Content Knowledge*

The last area of Coach development related to specialized content knowledge. Although this type of development occurred less frequently than KCT and KCS, it surfaced through rich discussion as can be seen in the following example. Melvin, like Dave and Marsha, encountered trapezoids when he and a teacher developed a quadrilateral taxonomy in which a trapezoid was defined as having exactly one pair of parallel sides (Battista, 1998). During a debriefing session, the following discussion ensued between Melvin and the 7th grade Teacher, Mrs. Gerber.

Melvin: But see now the tricky thing, actually, I'm learning a lot, Mrs. Gerber, because having done this kite, now I see the relationship a kite… rhombus,

Mrs. Gerber: OK

Melvin: Every rhombus is a kite, by my, by our definition, and see this is the thing. Um, I'm finding out that that definitions vary from book to book. (That's true.) Do you guys have, do you guys talk about kite in your book?

Mrs. Gerber: No, we don't talk about it. No, we don't.

Melvin: I'm finding a lot of differences in the definition of trapezoid, a lot of differences in the definition of kite.

Mrs. Gerber: What are you finding in the differences in trapezoid?

Melvin: Some books just say that it's got um at least two [one] pairs of opposite sides [parallel], so that would mean a parallelogram is a trapezoid as well.

Mrs. Gerber: OK

Melvin: But it doesn't have to have all, it doesn't have to have all, it doesn't have to have two pairs of opposite sides parallel. It just has to have at least two opposite sides that are parallel, so that gives it a little bit of more flavor, so every parallelogram is a trapezoid, and every trapezoid is a quadrilateral, it would fit in that flow.

Mrs. Gerber: Oh, OK, now I've never seen that either.

Melvin: Yeah, actually, I like that better, if we're going to make a note on how things are going to be done next year, I would suggest doing that.

Mrs. Gerber: Well, you know, that that make the trapezoid not such an odd guy out.

In his final interview, Melvin stated,

> I will be honest with you, I was growing in my understanding as well, right with her [Mrs. Gerber]. I mean, because I, we had presented trapezoids as only one pair of opposite parallel sides as opposed to at least [one pair of parallel sides]. And that change makes quite a difference…

Thus, Melvin developed specialized content knowledge (SCK) about definitions of a trapezoid as he delved into the topic of quadrilateral relationships with the teacher. In another planning session, Melvin and the Teacher wrestled with adding arrows to their taxonomy to denote generality (Battista, 1998). For example, they showed that a square is always a rectangle with a down arrow, but that a rectangle is sometimes a square with an up arrow (See Knapp et al., 2007). They decided to give the same task to students. The collaboration between Melvin and Mrs. Gerber inspired a didactic problem situation, and thereby provided for the development of knowledge of content and teaching (KCT).

*Coaches' Path to Mathematical Knowledge for Teaching*

In reflecting about the ways in which Coaches developed mathematical knowledge for teaching, I summarize the development with the following cycle. In the *Coaches' Path to Mathematical Knowledge for Teaching*, Coaches develop specialized content knowledge (SCK) as they research the topic for a lesson. As they search for curriculum and collaboratively plan for instruction with Teachers and other Coaches, they develop specialized content knowledge (SCK) and knowledge of content and teaching (KCT). During the lesson, Coaches develop knowledge of content and students (KCS) through observation of and discourse with students. Finally, in the reflection/debriefing phase, Coaches develop KCT, KCS, and SCK as they consider perturbations from the lesson with the Teacher or another Coach and reflect on the lesson (Hart, Najee-Ullah, & Schultz, 2004). Finally, the lesson may be repeated at another school with another teacher, and the cycle continues. The debriefing phase is perhaps the most valuable aspect of the cycle because both mathematical and pedagogical issues get discussed and intertwined. For example, the definition of a trapezoid discussion between Melvin and Mrs. Gerber occurred during a debriefing meeting. Also during the debriefing sessions, the lessons are fine-tuned based on the knowledge of content and students gained from the teaching phase. It is important to note that not all Coaches follow this path for all lessons. Rather, it is when and to what degree Coaches follow this path or elements of it that they appear to develop mathematical knowledge for teaching.

In conclusion, Graduate Coaches developed MKT through collaborative coaching with teachers. In particular, development occurred during collaborative planning and subsequent debriefing with Teachers. This aspect of the Coaching relationship is critical and should not be shortchanged. This research implies that the coaching relationship can serve as a form of professional development for both Teachers and Coaches (Knapp, et al., 2007). Thus, I recommend that universities involved in the preparation and professional development of STEM professors or K-12 teachers consider offering coursework, student teaching assignments, or assistantships which involve coaching of K-12 teachers.

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# ANY RIGHT TO GET IT WRONG? BEGINNING URBAN TEACHERS AND STUDENT MATHEMATICAL ERRORS

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*This paper reports on how beginning mathematics teachers who are participating in the New York City Teaching Fellows program view and respond to their students' mathematical errors. It describes the error analysis-coding model used when identifying and analyzing the error-handling situations. The study focuses on (a) elicitation of the identified errors, (b) type of errors (c) teachers' immediate reactions, (d) teachers' follow-ups and (e) the correction and post-correction processes. The other focus of the study is teachers' views of error attribution, and their own role in responding to students' errors in urban context as expressed in end-of-year interviews. Results show the importance of considering influences of school contexts on individual teachers' error-handling role.*

## Theoretical Perspectives

Most mathematics education professional organizations and reform voices call for mathematics classroom environments where students participate actively in trying to understand what they are asked to learn. Effective learners in such environments recognize the importance of reflecting on their thinking and learning from their errors. As Lannin et al. (2006) emphasized "creating an environment where students can learn from their errors is paramount for supporting their mathematical learning" (p.186).

The NCTM standards (2000) elaborate that, in such environments, mathematical errors are seen not as "dead ends but rather as potential avenues for learning." This view considers the fact that "students feel comfortable making and correcting mistakes" as a basic and main reform characteristic. Teachers have a crucial role in building such environments, where students understand that it is acceptable to struggle with ideas, to make mistakes, and to be unsure (NCTM, 2000). However, there are fewer studies on error-handling activities of teachers in mathematics lessons (Heinze, 2005).

The research design and methods for this study are informed by a framework that articulates theoretical standpoints deriving from teacher education, learning theories and comparative education. First, we adopt the view of errors as resources for promoting learning rather than simply as diagnostic tools or stumbling points, we look developmentally at the immersion of beginning teachers in the teaching profession and realize the importance of the first years in shaping their teaching practices. We believe in the importance of supporting beginning teachers and considering their professional needs within the growing alternative certification context, we finally derive from the cross-cultural lens of comparative educational studies their emphasis on the importance of contextual nature of the teaching practices.

Researchers in mathematics education have long realized and studied the role of errors in learning (Baruk, 1985). Most of the earlier research held a diagnostic approach and tried to study the nature of students' errors and suggest remediation strategies (e.g., Radatz, 1979). Calls for teachers to capitalize on students' errors are also abundant in the literature (Borasi, 1994; Ashlock, 2005; Lannin et al., 2007; Smith et al. 1993). For example, Borasi (1994) proposed "using errors as springboards for inquiry" as a strategy to use students' errors in stimulating and supporting mathematics inquiry instruction while Lannin (2007) stated that " when students seek to understand the general nature of the errors, they engage in a critical boundary-defining process that can lead toward a normative view of the application of a particular concept [and] when students engage in considering the applicability of a particular concept, they define the boundary for when to and when not to apply a particular idea." (p.57). Martinez (1998) looked at errors as unavoidable part of the process of problem solving. He called for both teachers and learners to be more tolerant of them, arguing that "if no mistakes are made, then almost certainly no problem solving is taking place." In this view errors are not seen as signs of failure or weakness as other learning theories might suggest. Expanding on how different learning theories view errors, Santagata (2005) noted that different learning theories assign to mathematical errors rather fundamental roles: either as obstacles as in the case of behaviourists or as tools for learning with constructivists); however, "the extent to which they inform teachers' practices is a question yet to be investigated" (Santagata, 2005, p.492).

Comparative education studies showed differences in how teachers in different cultures and countries vary in their approaches to students' mathematical errors: for example Osborn and Planel (1999) reported on how teachers in England made more effort to protect students' self-esteem and avoid negative feedback while teachers in France were observed responding directly to students' wrong answers and sometimes yelling at students. Stevenson and Stigler (1992) reported that Japanese teachers viewed mathematical errors as having a positive function. Students in Japanese mathematics lessons are called to the front of the classroom to share their own problem solutions with their classmates even when they are wrong. These wrong solutions are regarded as sources of useful discussions. Japanese teachers also plan their lessons taking into account common mistakes made by their students. To the contrary, they found that US teachers avoided discussions of students' mathematical errors and showed more concern about students' self-esteem.

## Research Questions

This study aims to investigate whether and how beginning teachers, like the NYC Teaching Fellows, are aware, develop or practice their error-handling role. In particular, it aims to answer the following questions: (1) How beginning mathematics teachers respond to their students' mathematical errors in their classroom settings and (2) How do they view students' mathematical errors and look at their own role in responding to these errors.

## Methodology

This study draws on data from a larger research project facilitated by MetroMath (The Center for Mathematics in America's Cities at the Graduate Center- CUNY in NYC) that explores the nature of teaching and learning in "high-needs," urban schools having a large proportion of alternative-routes teachers. More specifically, the MetroMath study examines the impact that the NYCTF program, an alternative teacher certification program, is having on mathematics education in the NY City classrooms.

The NYCTF program is examined both from a macro study – using large scale surveys of the 2006 and 2007 cohorts, as well as a micro study. In the micro study, we draw on data from case studies of eight teachers whose mathematical classrooms we observe about once a month using field notes, video taping, written reflections and post-observation interviews to understand teachers' experiences in the classroom.

For this particular paper we focus on the video- -tapes and field notes that show occurrences of students' mathematical errors and analyse how the eight case study teachers responded to and handled these situations. A second set of data we used was end of year interviews that we conducted with the eight NYC fellows in order to better understand their views about errors and their perception of their own practices when they recognize mathematical errors.

## The Error Profile and Coding Scheme

In this study, a mathematical error was defined as a student's wrong answer to a teacher's question or mathematical task. To analyse teachers' responses to the error we developed an error profile form adopted from Haydar (2002) and Santagata (2005). For every identified error, the error profile included teacher name, date and time, instructional task, narrative of the teacher-student interaction followed by a table that organized the analysis of each error according to: elicitation activity; nature of talk; type of error; immediate reaction; teacher interaction; error correction and post-correction behaviour.

A coding scheme was developed to look at each of the aspects listed above. After we identified that the teacher had recognized a student error, we would fill out the identifying information for the first part of the error profile. We then analysed the elicitation activity (in what part of the lesson did the error occur? ) then the nature of talk: was it public (whole class)? private? (one-to-one)? or semi-private? (small groups). We then coded the type of error (conceptual, procedural, drawing, computational, distraction, notation). As for the teacher's immediate reaction we looked whether the teacher gave a negative verbal reaction or asked a question related or not to the content. We then analysed the intervention that followed (Did the teacher give another task?; Interviewed the student?; Repeated the same question?; Used picture or other teaching aids?; Explained?). The focus that followed was to locate who gave the correct answer at the end (Was it the teacher him/herself?; Another student? ; The same student?; Or was no correct answer given at all?). Finally, we looked at what happened right after the correction was done (Did the teacher use the error as a learning moment?; Did he/she use it as a warning for students?; Or did the teacher delay the response?; Or just moved to another task?). Following are two examples that illustrate the error profile, the identifying information were taken off for space and anonymity considerations, clarifications were added in italic.

Example 1: Private error profile.
Task or question: Simplify the following algebraic expression: 3) $3x^3 + x^3 - x + 2x^2 + x$

Narrative:
A.N. (seventh grade student) has written the following answer:
$3x^3 + x^2 + 36$

T (teacher) pointing to the $x^2$
"you forgot the 2 here" she adds the two then asks him:
"why did you put 36 here?"

Coding:

| | |
|---|---|
| EA (elicitation activity) | 4 (student work time) |

| | |
|---|---|
| EN (elicitation: Nature of talk) | 3 (private) |
| TE (type of error) | 8(not enough information) |
| IR (immediate reaction) | 4 (other: points to 2) |
| TI (teacher intervention) | 6 (other) |
| CR (correction) | 1 (teacher corrects) |
| PC (post-correction) | 4 (moves to another error) |

*Figure 1*: Example of a partial private error profile.

Example 2: Public error profile.
Task or question: Simplify the expression: (n + 2) + (n + 4)

Narrative
T (teacher) moves to the front of the room and asked S1 (8th grade student) to answer #3: (n + 2) + (n + 4)
S1 immediately says **$n^2 + 6n + 8$.**
T moves to the back and asks if the answer was correct.
S2 (Another student) says no, and gives the answer 2n + 6.
T looks at S1 and explains that it is very important to see the difference between # 3 and # 4 in these two questions. He says that one has the plus sign in between parenthesis and the other has the imply multiplication and that these problems must be solved by different methods.
T looking at the whole class says: "We need to be careful not to use the FOIL method because we have the plus sign" and assures S1 that he did not ask this to trick him, but to show that we need to be alert.

Coding:

| | |
|---|---|
| EA (Elicitation Activity) | 1 (Do Now) |
| EN (Elicitation: Nature of Talk) | 1 (Public) |
| TE (Type of Error) | 2 (Procedural) |
| IR (Immediate Reaction) | 2 (Content Question) |
| TI (Teacher Intervention) | 2 (Teacher interview students, i.e Q& A) |
| CR (Correction) | 3 (Other student corrects) |
| PC (Post-correction) | 1 (use of error for explanation) |

*Figure 2*. Example of a partial public error profile.

## The Interview

In order to better understand how teachers looked at their students' mathematical errors and how did they view their roles in responding to students' errors, we included four related questions as part of an end-of-year interview conducted within the larger research methodology mentioned at the beginning of the methods section. The questions were: (i) When a student makes mistakes or errors in math, does it mean he/she is a weak student? Explain. (ii) When a student in your class makes math mistakes or errors, how do you react? That is, describe the different ways you typically respond to student errors. (iii) Some educators think that allowing students to present and explain wrong answers to their classmates, puts students at risk of acquiring wrong information or skills. What do you think? and (iv) How much can you do to improve the understanding of a student who always makes mathematical errors? Do you have such students? Explain.

When analysing teachers' answers, we focused on teachers' error attribution (why do they think student make errors?); their views of their own interventions; their views of the role of errors in learning; and their own sense of efficacy in dealing with students errors.

## Results and Discussion

Our findings show beginning teachers moving over time toward more private talk about errors situations with students in their classrooms. This fact may be a reflection of the workshop model being taught to the teachers in NYC in middle and high school math instruction. The workshop model was originally designed to promote interactive pedagogy and creative student learning and has been adopted in NYC public schools since 2003 (Traub, 2003).

Our analysis of the data is also showing variations in how individual teachers develop the way in which they respond to students' mathematical errors. This is due in part to the contextual differences in the culture of the schools where they each teach. The main influence themes that we detected:

Influence of the emphasis on successful test preparation strategy at the school. The focus on the State test preparation led two of our case study teachers to minimize any interactions around mathematical errors. This resonates with other studies that show how high stake tests are narrowing the mathematics curriculum and how teachers filter the policy messages according to their priorities and experiences (Haydar, 2009).

Influence of the workshop model at the school resulted in a shift from teacher ownership to including more students in the correction process. The participatory nature and structure of the workshop model helped some of the teachers especially in the schools where the model was systemically emphasized to delegate some of the correction role to students.

Influence of limitations in content knowledge. Teachers who showed limitations in their content knowledge tended to have more negative immediate reactions, less interviewing during the intervention and more of teacher correction.

Influence of discipline issues. In classrooms where teachers were still struggling with discipline issues, we found a narrowing in the correcting agent to the teacher or the student who made the error especially within public talk. This is in line with what other researchers described as beginning teachers facing challenges in "survival skills" (Kirby et al., 2006). Kagan (1992) affirmed that until beginning teachers have established standard routines and resolved their images of self as teachers, they will continue to be obsessed with discipline and class control.

The analysis of the teachers' interviews permits to draw the following observations:

Error attribution: six of the eight teachers interviewed distinguished between making errors and being weak in mathematics but for different reasons. For some, making mistakes is at least a sign of on-task work : “if they’re making mistakes, they’re doing something, which is half the battle.” For other teachers errors may result from simple confusion or wrong assumption: “I would say that a student who makes a mistake is either confusing something or is making an assumption that he or she shouldn’t make”.
Teachers self-scripted role: half of the teachers explained that they would seek the help of students’ peers: “ I would just tell the student who made the mistake to check with so and so and if you got a different answer try and figure out who’s right”. Others mentioned that, especially in public situation they would start by highlighting any positive aspects in the method that led to the wrong answer and then move to some questioning regarding the error itself: “I always tell them what they did correctly first.”
Errors and learning: six teachers thought that displaying individual students’ errors in front of the whole class can be beneficial. Some saw in that an opportunity to learn from common errors: “They’re going to make mistakes in math all the time anyways, so we might as well learn from the most common mistakes that the students are making.” Others thought that other students in the class are always critical to each other and hence playing the correcting agents “I think it’s very uncommon that a kid leaves the classroom thinking that that was the right answer because of just what the other kids do. They don’t let them get away with that”. Two teachers out of the eight saw some risks in the error public displays, one of them thought “it needs to be made explicit by the teacher and the student that what students presenting at this time is wrong answer”. The other teacher emphasized students’ short attention span as source for a risk of getting stuck on the error without paying attention to the correction: “ so all of the sudden now they think what the person presented was correct because they didn’t pay attention to the rest of the presentation.”
Self-efficacy: teachers sense of efficacy in going about helping students who make continuous mathematical errors varied between admitting a role and ability to help by “ keep working with them”;“ working one-on-one” ; “ try to identify their way of thinking”; “giving them individualized homework”. However, teachers were concerned with time constraints “You can’t sit with them every day either because you’ve got, you know; 29 other kids to worry about.” and students’ lack of cooperation “Some students just don’t even try”. These two factors were the most reported by teachers as hindering their role in helping students who always make mathematical errors.

This paper presented an inside view of how beginning teachers in the NYTF program think and behave when facing their students’ mathematical errors. The influences at the school and policy levels shown in this study along with teachers views of their error handling role and their content and pedagogical knowledge need to be considered seriously in any teacher education effort whether pre- or in-service aiming to help novice teachers reach a comfortable zone whereas they can create challenging yet safe environment where students feel comfortable making and learning from their errors.

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# APPLICATION OF THE NECESSITY PRINCIPLE IN INQUIRY BASED MATHEMATICS LESSON DESIGN

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Recently there has been increasing emphasis on constructivist-oriented teaching practices within professional development mathematics communities (Loucks-Horsley et al. 1996). Aimed at promoting inquiry based learning, this poster illustrates an example of curricular innovation inspired by Piaget's notion of *cognitive disequilibrium*, which maintains that "knowledge develops as a solution to a problem" (Piaget, 1997). Harel (2000) reformulated Piaget's ideas as a fundamental principle for teaching and learning in the *Necessity Principle*, which states that:

> For students to learn, they must see an (intellectual, as opposed to social or economic) need for what they are intended to be taught.

A widely pervasive mathematical relationship related to the development of perhaps all biological systems finds expression in the surface area to volume ratio. This poster describes lesson concepts, procedures, sample discussions, and activities coordinated using the Necessity Principle to allow student opportunities for genuine cognitive disequilibrium based on their familiar knowledge of dogs, combined with intellectual and mathematical justifications for topics and methods used. This poster illustrates how subjects were seen to:

1. Create cube models of small dogs and dimensionally doubled large dogs, while mathematically studying comparative surface area to volume ratios in an applied context.

2. Discover mathematical justifications and necessities for using concepts such as proportion, surface area, volume, and unifix cube manipulatives in addressing the 'cooling problem.'

This example of application of the Necessity Principle to lesson design represents preliminary pilot research in a study investigating the effectiveness of intellectual necessity for the transfer of familiar problem solving knowledge to less familiar, non-isomorphic problem settings.

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# RESPONDING TO JOURNAL WRITING IN THE MIDDLE GRADES MATHEMATICS CLASSROOM

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In *Principles and Standards for School Mathematics* (2000), the National Council of Teachers of Mathematics (NCTM) asserts that written and oral communications are vital processes for the learning of mathematics. In fact, there exists a plethora of literature asserting the importance of writing in the mathematics classroom. However, the literature lacks analysis of the difficulties mathematics teachers may have in implementing writing in the classroom, beginning with the creation of writing prompts and proceeding through to responding to students' writing. Therefore, efforts to document the process of using writing in mathematics classrooms could prove invaluable to mathematics teachers and mathematics teacher educator.

The purpose of this poster is to report results from a research study where the aim was to develop a picture of how middle school teachers respond to student journal writing in mathematics. This poster focuses on how the teachers responded to student journal writing and the implications that has for the use of written communication in their mathematics classes.

Looking at the very nature of the teachers' responses allowed me to understand how teachers "listen" to their students thinking, and in turn make statements about what the teachers valued in the student writing. A lens for looking at this work is Davis's (1996) framework for listening. Davis defines three types of listening, evaluative, interpretive, and hermeneutic.

Teacher responses were disappointing. Teachers' responses much like the student responses were short and unelaborated. Most teacher comments on students' papers focused on assessing the quality of student responses, listening evaluatively, something that has been reported in the literature to lack benefit. In fact, almost all teachers' responses were evaluative although the literature reports that to achieve positive results from journal writing teachers should focus on nonevaluative comments (Borasi & Rose, 1989). Evaluative responses denote teachers' listening to students to determine correctness. Consequently, because of this practice teachers were unable to elicit better responses from their students, due to comments, which focused on correcting student thinking rather than trying to interpret student thinking (interpretive listening).

Another factor that caused difficulty in this study was the teachers' inability to respond to student writing in a timely manner. Often times, teachers would become bogged down in responding to student writing and this caused them to have to respond to a lot of writing at one time, in turn affecting the quality of the teacher response. Borrowing ideas from Language education, there exist alternative ways to respond to student writing that can help teachers manage time factors and the amount of writing assigned. Instead of commenting on each individual student response, teachers could engage students in peer sharing and responding, individual conferences, or respond to each student once per week.

As outlined by NCTM (2000), being able to communicate mathematically is a vital process to learning mathematics. A teacher's ability to encourage written communication is important. Therefore, responding to student journal writing is a crucial step. Teachers need to understand that with journal writing the point is not to necessarily critique the student work but is to push students' thinking, not imposing necessarily their own views on the student but to have the student think more deeply about mathematics (Borasi & Rose, 1989).

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# THE IMPACT OF A MATHEMATICS IMMERSION EXPERIENCE ON MATHEMATICS EDUCATION FACULTY

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This poster will report on the Education Development Center's (EDC) NSF-funded one-week summer institute for mathematics education faculty. The institute was guided by the principle that mathematics education professionals need first-hand experience with coming to understand mathematics and was inspired by a course for graduate students developed at the University of Maryland. It was designed to immerse the twenty-four participants (17 faculty members, 6 graduate students, and 1 retired faculty member) in mathematics, exploring problems posed by the organizers and investigating personally relevant questions posed by individuals or small groups. The group considered how such an experience could guide the development of mathematics courses for graduate students in mathematics education at their own institutions. The third author was the Principal Investigator for this project, and the other two authors conducted the project evaluation. The poster provides a general overview of the program and will focus on two questions that guided the evaluation component of the project: 1) How well does the institute serve the participants' mathematical needs?, and 2) How do participants' courses for pre-service teachers and graduate students change as a result of participating in the institute?

To address the first question we will describe the participants' responses to the mathematical experience. Data sources include participant journals (time was allotted twice daily during the institute for journaling), a follow-up survey, and observation and interaction with the participants. A thorough review of this data highlights several general themes among the participants' reactions to the mathematical immersion experience. For instance, several felt invigorated by engaging in and exploring mathematics in ways that were new, unfamiliar, or uncommon in their professional lives. Many participants related their own experiences to those of their students, and the social and community aspects of the work were impactful. There was some rejection of a perceived insinuation that mathematics educators in general lacked such experiences and would benefit from collaboration with mathematicians, but others anticipated a need for such collaborations in their own contexts. Overall, a general excitement about open-ended, low-stakes mathematical explorations for their own sake developed in the group, and many participants projected their own fulfilling experiences as valuable for graduate students.

To address the second question we will describe the various implementation strategies and professional reflections that occurred following institute participation. At least two participants developed collaborations with faculty members in other departments and taught graduate courses modeled after the institute at their institution. One participant reported moving from a Department of Education to the Department of Mathematics and Statistics as a result of an "awakened desire . . . to teach and 'do' mathematics".

# ANALYZING TEACHERS' MATHEMATICAL ACTIVITY IN AN ONLINE ENVIRONMENT

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We will discuss our efforts to create an online environment that supports teachers engaging with the practices of "doing mathematics." We feel it is important to differentiate this from U.S. students' (and therefore U.S. teachers) common experience of mathematics as "doing procedure after procedure." In our work, we seek to engage teachers with practices common to professional mathematicians: asking mathematical questions, making conjectures, testing conjectures, proof, and generalization. Hiebert (1999) argued that teachers need opportunities to experience mathematics in the way they are expected to teach; teachers have few opportunities to do so. We believe this could be one reason for the inconsistent adoption of instructional processes recommended by the NCTM process standards (NCTM, 2000), as well as the methods of implementation of reform curricula by teachers (Senk & Thompson, 2003).

The model we use for our interactions is Online Asynchronous Collaboration (OAC) in Mathematics Education (Clay & Silverman, 2009), which was developed primarily to scaffold participants' engagement in legitimate mathematical practices online. OAC involves cycles of individual, small group, and whole class interaction and collaboration. In this poster, we will discuss our recent implementation of OAC in a professional development setting. We will include analysis of teachers' "regularities of practice" – the way they "do mathematics" personally and with their students – and the ways in which the structure and scaffolding provided by (a) the OAC environment, (b) the professional development (PD) facilitator(s), and (c) their colleagues supported the emergence of these mathematical practices. While we feel that professional development of this sort is invaluable for teachers, we also note that it is significant that we are having measurable success doing PD online, which opens up significant opportunities for scaling and for serving traditionally underserved communities.

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# DISTRICT-WIDE IMPLEMENTATION OF STANDARDS-BASED MATHEMATICS INSTRUCTION

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Standards-based curriculum embodies an approach to teaching mathematics that differs substantially from traditional, didactic approaches (Smith & Smith, 2006). Standards-based instruction employs rich mathematical tasks that allow students to explore mathematical concepts, determine their approach to finding a solution, and connect their answer to mathematical concepts by discussing mathematics and their strategies with their colleagues. Research on the use of standards-based curriculum indicates three things: 1) when implemented with a high level of fidelity students using standards-based curriculum significantly outperform their peers who use traditional curriculum on measures of problem solving (Smith & Smith, 2006; Stein, Remillard, & Smith, 2007); 2) teachers require sufficient amounts of support in the form of workshops and in-class assistance while beginning to implement standards-based curriculum (Cohen, 2006, Henningsen & Stein, 1997; Tarr, Reys, Reys, Chavez, Shih, & Osterlind, 2008) and 3) professional development that is content specific and develops teachers' content knowledge in conjunction with teachers' skills related to teaching with standards-based curriculum can positively influence teachers' instruction (Cohen, 2005; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Makros, 2003).

This study employed survey data from over 200 teachers in 18 schools. Each teacher was involved in the piloting of a standards-based mathematics curriculum in a large urban district. Preliminary analysis of descriptive statistics revealed that a great deal of variance exists across schools regarding implementation. Further, teachers employed varying practices to assess students' understanding and supplement the material with other curricular resources. The poster will provide more elaborate details of the findings.

# STRIVING TO MEET THE CHALLENGE OF READING MATH

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Reading math is not like reading a basal reader or novel. Illustrations accompanying situational problems require students to begin at different points to correctly retrieve information. During mathematical testing today's students are expected to read lengthy real life situations and determine logical answers to questions. They are no longer expected to only calculate or manipulate the numbers Reading mathematic texts has put additional processing demands on the math readers that contradict with how readers process narrative and expository texts. The math reader may need to read 1) right to left as well as left to right. i.e. reading a number line, 2) from top to bottom or bottom to top. i.e. reading mathematical tables or 3) diagonally. i.e. reading mathematical graphs.

In addition, the math reader may have very little prior knowledge about lengthy mathematical word problems and a paucity of text material to activate prior knowledge (Patton, 2007). Brennan and Dunlap (1985), Culyer (1988), Thomas (1994) and Thomas (1988) stated in their research that mathematics texts contain more concepts per word, per sentence and per paragraph than any other type of textual material. Young math readers have difficulties visualizing mathematical concepts as the concepts have been addressed previously by memorization or in a very abstract format.

This study's purpose was to determine if teacher candidates were knowledgeable of the various approaches needed to interrupt (read) illustrations accompanying math problems. Subjects were 35 females enrolled in required EC-4 math/science methodology. The seven question instrument was researcher -designed and modeled very closely to the state released 4th grade test. A panel of experts was utilized to equate the proper procedure/s to solve each problem

Since the problems were modeled closely to a state achievement test released items, the success rate of participants (teacher candidates) was expected to be near 100% correct. Participants were successful on most problems. However, only 26% stated that they approached one of the problems in the best manner according to the panel of experts. It is enlightening, and alarming, that eighty-one percent of the participants stated they approached the problem in an inappropriate manner (Patton). More than one approach, therefore it is possible to have the sum of the answers selected greater than 100%.

To effectively teach elementary mathematics, teacher candidates must abandon misconceptions about mathematics teaching. Misconceptions may arise from the teaching practices of the past decades when the students were expected to memorize the facts with little emphasis placed on higher level thinking skills Teacher candidates' views and perceptions of mathematics must be broadened to encompass how teacher candidates more effectively facilitate and interpret the nature of children's thinking. It is time for teachers of elementary mathematics to stop memorizing facts and start developing the metacognitive awareness they need to select appropriate mathematics strategies for learner success (Patton, Klages & Fry 2008). ) Metacognition allows learners to make adjustments to different problem-solving tasks (Montague, 1998). Reading situational math problems with corresponding illustrations are considered to be different problem-solving tasks. Results conclude teacher candidates need more math instruction if they are to be truly successful with their students in the quest of learning.

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# AFFORDANCES OF VISUAL REPRESENTATIONS: THE CASE OF FRACTION MULTIPLICATION

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*This study focuses on two instructors who taught a mathematics course designed for prospective elementary teachers and explores which interpretations of fractions they addressed and how they used visual representations when discussing fraction multiplication. Our findings indicate that the distinct interpretations of rational numbers can turn out to be quite intertwined during actual practice. As a result, it might be challenging to extract meaning from the visual representations, especially when the problems are not situated in a context, unless instructors explicitly attend to the interpretations underlying those representations.*

## Introduction

It is challenging to blend research on fractions with classroom teaching for several reasons. First, the development of fractions in the classroom is complex and non-linear. Second, "teachers are not prepared to teach content other than part-whole fractions" (Lamon, 2007, p. 632) and thus we may not see in reality the ideas and constructs that research suggests in theory. Third, teachers may not explain details supporting their choices of problems and representations, making them invisible to research. In this study, we explore some of the complexities of multiplication of fractions in the context of preservice teacher education.

There is evidence that part-whole has been the most dominant realization of fractions for students as well as preservice and inservice teachers (Domoney, 2002; Sowder, Philipp, Armstrong, & Schappelle, 1998; Tirosh, Fischbein, Graeber, & Wilson, 1999). Besides part-whole, Kieren (1980) proposed four other subconstructs or interpretations for fractions: measure, operator, quotient and ratio. Each of these interpretations may be illustrated with multiple representations, including numbers, and various discrete, linear, and area models and more. In her longitudinal study of six classes from grades 3 to 6, Lamon (2007) noted that the students who were exposed to these five subconstructs developed deeper understanding of rational numbers and proportional reasoning compared to students in the control group who received traditional instruction that did not explicitly attempt to use multiple interpretations of rational numbers. (Note that, in this paper we use "fraction" and "rational number" interchangeably, acknowledging that there are important and contested mathematical differences between the two.) As a result, she considers being able to "move flexibly between interpretations and representations" as one of the key elements of understanding fractions (Lamon, 2007, p. 636). Therefore, understanding of fractions entails experience with multiple interpretations (Kieren, 1980) as well as experience with multiple representations of fractions.

According to Izsák (2008), research on teachers' knowledge of fraction multiplication is not as extensive as the research on fraction division and decimal multiplication. There is a body of research indicating that teachers find it difficult to construct appropriate representations for fraction multiplication (Armstrong & Bezuk, 1995; Sowder et al., 1998; Tirosh et al., 1999). In this paper, we look at two instructors who teach mathematics content for undergraduate prospective elementary school teachers. The study addresses the following question: Which interpretations of fractions do instructors teaching elementary mathematics content to undergraduate preservice teachers concentrate on as they teach fraction multiplication, and how do they represent these ideas to students?

## Theoretical Framework

The *part-whole* realization of rational numbers entails "the partitioning of a continuous quantity or a set of discrete objects into equal-sized parts…" (Sowder et al., 1998, p. 8). Therefore, this subconstruct requires understanding of the whole and the ways in which it may be partitioned. The realization of a rational number as a *measure* "occurs when we want to measure something but the unit of measure does not fit some whole number of times in the quantity to be measured" and so "demands that the rational number be understood as a number, as a quantity, as how *much* of something" (Sowder et al., 1998, pp. 9-10, italics in original). A rational number acts as an *operator* when it is interpreted as:

> a function that can operate on a continuous region as a stretcher or shrinker or on a set as a multiplier or divider, in either case serving as a function machine that operates on one value to form an output of another value. (Sowder et al., 1998, p. 11)

A rational number can also be realized as a *quotient*. "A fraction $a/b$ can also represent the quotient $a \div b$ ; that is, $a$ and $b$ are integers satisfying the equation $a=bx$" (Sowder et al., 1998, p. 11). Finally, when we realize a rational number $a/b$ by means of the comparative relationship between $a$ and $b$, we are thinking of the rational number as a *ratio*.

Among the five subconstructs of rational numbers, the operator subconstruct seems to be the most effective for fraction multiplication (Behr, Harel, Post, & Lesh, 1993; Izsák, 2008; Sowder et al., 1998). Lamon's (2007) findings also suggest using the measure subconstruct for fraction multiplication since it might enable the extension of the operator subconstruct. She exemplifies the measure and the operator subconstructs and their meaning for the fraction $3/4$ as follows:

| **Interpretations of 3/4** | **Meaning** |
|---|---|
| Measure "$3(\frac{1}{4}$-units)" | $\frac{3}{4}$ means a distance of $3(\frac{1}{4}$-units) from 0 on the number line or $3(\frac{1}{4}$-units) of a given area. |
| Operator "$\frac{3}{4}$ of something" | $\frac{3}{4}$ is a rule that tells how to operate on a unit (or on the result of a previous operation): multiply by 3 and divide the result by 4 or divide by 4 and multiply the result by 3. This results in multiple meanings for $\frac{3}{4}$: $3(\frac{1}{4}$-units), $1(\frac{3}{4}$-unit), and $\frac{1}{4}$ (3-unit). |

Figure 10. *Portion of the table Lamon (2007, p. 654) uses when she addresses alternative instruction strategies to the part-whole interpretation of fractions.*

In this paper, we investigate how instructors of preservice elementary teachers utilize these interpretations as they represent fraction multiplication.

## Methods

Data for this study comes from a larger project that explores the mathematics content taught to undergraduate prospective elementary teachers. This paper uses data from two instructors, collected through observations of their classes when they taught fractions. Particular attention was given to the visual representations instructors used when addressing fraction multiplication. Fraction lessons were videotaped and portions of the tapes where the instructors discussed fraction multiplication were transcribed. Field notes taken during instruction supplemented the data.

We report on two instructors for whom we will use the pseudonyms Eliot and Sam. These instructors form contrasting cases with respect to the number of visual representations they used when discussing fraction multiplication. Eliot primarily relied on a single visual representation across the problems she solved whereas Sam used multiple representations for each of the problems she worked on. Given this, we investigated the possible impact of this difference on the interpretations of fractions the instructors facilitated in their classrooms. We used snapshots from the classroom videotapes for Eliot's representations to keep the authenticity of her representations on the whiteboard in her classroom. Sam's video snapshots were not clearly visible since she used a blackboard so we used field notes to reproduce her drawings. The researchers checked the fidelity of the field notes with Sam's actual representations in the video clips. We initially analyzed the data with respect to the subconstructs underlying the visual representations the instructors used for each problem individually and then compared our results until we reached consensus. In this respect, we used a form of competitive argumentation (VanLehn & Brown, 1982) during our data analysis.

## Results

### *Eliot's Representations of Fraction Multiplication*

Eliot based her initial discussion of fraction multiplication on whole number multiplication. Although her initial introduction encouraged students to think about fraction multiplication in terms of repeated addition, Eliot also mentioned that multiplication does not necessarily lead to a larger number in the case of fractions. After these, she explicitly pointed out the word *of* means *to multiply* and started modeling fraction multiplication problems using diagrams. Throughout her discussion of multiplication of fractions, Eliot consistently used a visual diagram in which two hexagons were considered as one whole. At the very beginning of her fraction lessons, Eliot introduced these hexagons and their subunits consisting of triangles, rhombi (Eliot used the word rhombuses instead of rhombi so we will stick with her word use), and trapezoids using pattern blocks. Afterwards, she kept on using the same idea by drawing the hexagons on the board. Below is the relationship among the units and the subunits:

2 hexagons (the whole) = 12 triangles = 6 rhombuses = 4 trapezoids
1 rhombus = 2 triangles
1 trapezoid = 3 triangles

Eliot assumed her students knew how to compute fraction multiplication and briefly mentioned the rule. She focused only on modeling during her discussion of multiplication of fractions. For all the problems she worked on, except for one, she used the diagram in which two hexagons referred to one whole. For example, she modeled $\frac{2}{3} \times \frac{1}{2}$ as follows:

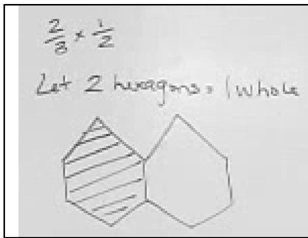
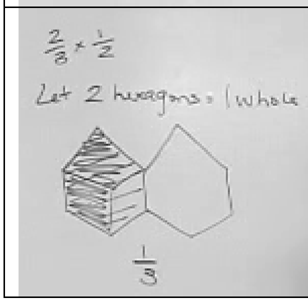
| | |
|---|---|
|  | Eliot first modeled $\frac{2}{3} \times \frac{1}{2}$ by considering it as $\frac{2}{3}$ of $\frac{1}{2}$. She shaded one hexagon noting that it was the half of her whole, which was two hexagons. |
|  | Eliot: "What is two thirds of my half? I need to split my half into three pieces and shade two again" (The class was familiar with representing thirds by rhombuses). Therefore, Eliot split the hexagon into three rhombuses and shaded two of them one by one. She then asked what the double shaded region was in terms of the whole. Two rhombuses made up a third of the whole, so Eliot wrote $\frac{1}{3}$ as the answer. |

Figure 11. *Eliot's representation of $\frac{2}{3}$ of $\frac{1}{2}$*

Eliot's wording while shading $\frac{2}{3}$ of one hexagon might be considered as recursive partitioning when she split the hexagon into three equal pieces and shaded two of them. Given this, she might be using the idea of finding part of a part by applying the notion of part-whole recursively. On the other hand, it is also possible that Eliot operated on the hexagon that represented $\frac{1}{2}$ as she split it into three parts (divide by 3) and then shaded two of them (multiply by 2). In this respect, given the description in Figure 1, she might have also used the operator subconstruct. Therefore, for this problem, it is not explicit which subconstruct she is particularly attending to. Eliot then modeled $\frac{2}{3} \times \frac{1}{2}$ again, this time considering it as $\frac{1}{2}$ of $\frac{2}{3}$:

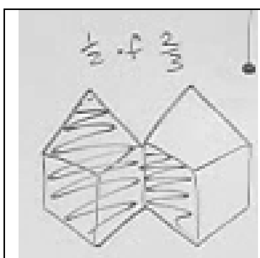
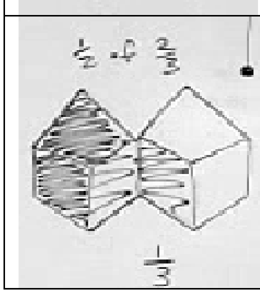
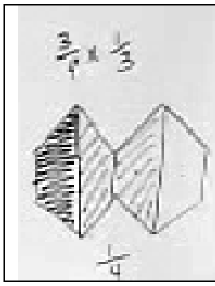
| | |
|---|---|
|  | Eliot noted that in order to take the half of $\frac{2}{3}$, we first needed to know what two thirds of the whole was. She asked, "two-thirds is how many rhombuses? Four" (Eliot used the equivalent fraction $\frac{4}{6}$ for $\frac{2}{3}$ to be able to represent it with rhombuses. The students were familiar with this use). She then divided the hexagons into six rhombuses and shaded four of them. |
|  | Eliot: "What is half of my four rhombuses? Two rhombuses. How much of my whole is shaded twice?" (Note that she used the names of the geometric shapes rather than saying 'what is a half of two thirds'?)<br>She then shaded two rhombuses out of the four rhombuses she already shaded and wrote $\frac{1}{3}$ as the answer. |

Figure 12. *Eliot's representation of $\frac{1}{2}$ of $\frac{2}{3}$*

When finding the half of four rhombuses, Eliot considered four rhombuses as another whole. It is likely that she used part-whole interpretation with recursive partitioning here since she made the number $\frac{2}{3}$ concrete by naming it "four rhombuses". Then half of four rhombuses would be equal to two rhombuses. On the other hand, she often emphasized in her previous classes on fractions that students needed to think about this model in terms of area. For example, when modeling addition of fractions with these hexagons, she said, “we are merging the areas together to find out how much of our same whole the new area takes up”. Similarly, when she explained why $\frac{3}{4}$ of two hexagons would be equal to three trapezoids, she mentioned, “because we can cover three fourths of the area of our whole using three trapezoids”. Given this, she might also be attending to the measure subconstruct (See Figure 1) as she modeled $\frac{1}{2}$ of $\frac{2}{3}$.

Eliot modeled $\frac{3}{4} \times \frac{1}{3}$ by considering it as $\frac{1}{3}$ of $\frac{3}{4}$:




Eliot mentioned that three fourths of the whole would be three trapezoids and shaded them in her drawing. She then asked, “what is a third of three trapezoids? One trapezoid. *Because when you have three of anything, a third of it is one of them*”. She then shaded one trapezoid out of the already shaded three trapezoids and asked how much of the whole was shaded twice. Students answered by saying a fourth. Eliot wrote $\frac{1}{4}$ as the answer.

Figure 4. *Eliot’s representation of* $\frac{1}{3}$ *of* $\frac{3}{4}$

Note that Eliot talked about one third of $\frac{3}{4}$ as one third of three things (trapezoids), which would be the recursive application of the part-whole subconstruct. Here, it seems relatively clear that Eliot used recursive partitioning for finding the part of a part rather than attending to the operator subconstruct. However, when the numerator of the operator is equal to 1, it might also be difficult to identify from the visual representation whether the operator subconstruct or the part-whole interpretation is used.

Eliot might have used the operator subconstruct when she modeled $1\frac{1}{4} \times \frac{2}{3}$ considering it as $\frac{2}{3}$ of $1\frac{1}{4}$. The following is the picture she drew:

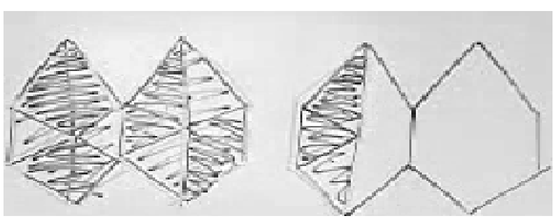


Figure 5. *Elliot’s representation of* $\frac{2}{3}$ *of* $1\frac{1}{4}$.

Eliot noted that $1\frac{1}{4}$ would be equal to a whole plus an additional fourth. Since a pair of hexagons corresponded to one whole, she drew two pairs of hexagons. She shaded all of the first pair and a trapezoid in the second pair that corresponded to the additional fourth. She then said, “You want to ask yourself what is two thirds of the shaded portion? I need to think of a way to cut that into three equal portions…I guess I am going to cut everything into triangles”. After this, she split the shaded portion, which is $1\frac{1}{4}$, into triangular portions. She then noted she was going to shade “two out of every three”. Given Eliot’s previous examples and arguments, one can again assume that she used part-whole interpretation with recursive partitioning. On the other hand, it is relatively clear in this example that she operated on the fourths in $1\frac{1}{4}$ since she split every fourth in $1 + \frac{1}{4}$ into three equal portions (divide by 3) and then took “two out of every three” *fourths* (multiply by 2), which would suggest she was attending to the operator subconstruct. Eliot’s consideration of $1\frac{1}{4}$

as $1+\frac{1}{4}$ when splitting each fourth into its thirds and taking two out of three fourths also signaled an implicit use of the distributive property.

*Representations of Fraction Multiplication in Sam's Class*

Sam also started her discussion of fraction multiplication with whole number multiplication. She structured her class around the three cases: a whole number multiplied by a fraction, a fraction multiplied by a whole number, and a fraction multiplied by another fraction. Similar to Eliot, Sam also mentioned the relationship between the word *of* and multiplication. Sam did not discuss mixed number or improper fraction multiplication. Unlike Eliot who used a single visual representation for each problem, Sam used a variety of visual representations (some of which were initiated by her students) for the problems she worked on. Her representations consisted of pie diagrams, rectangular area models and also the number line. For example, while modeling $\frac{1}{2}\times\frac{1}{3}$, Sam considered it as $\frac{1}{2}$ of $\frac{1}{3}$ and used the following visuals:

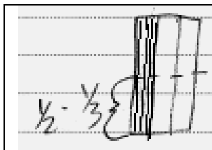
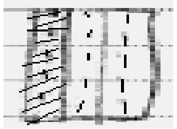
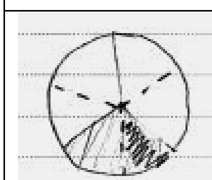
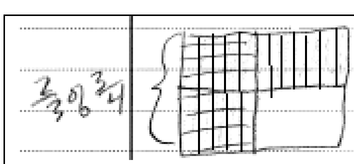
| | |
|---|---|
|  | Sam drew this model mentioning, "we draw a whole and then have one third. We then think about what one half of that third is". She split the third she shaded into two parts horizontally following a student's suggestion. She then labeled the portion that corresponded to $\frac{1}{2}$ of $\frac{1}{3}$. |
|  | Sam then asked students if she could split the shaded third into two parts vertically by saying "I will split this to halves." She drew dotted lines and shaded one strip as shown . The students agreed that she could do this vertically. |
|  | Sam: "How about trying other models? How about a pie?" After saying this, she drew a pie diagram and divided it as shown with solid lines. She then added the dotted lines and said "because of that, we have to cut the other parts, so this one half of one third will be what?" A student answered "one sixth of the whole". Sam confirmed this was the correct answer. |

Figure 6. *Sam's representations of* $\frac{1}{2}$ *of* $\frac{1}{3}$

Sam seemed to be using the part-whole subconstruct with recursive partitioning for these representations. However, it is also possible that she might be using the operator subconstruct when she the split the third into two parts and shaded one part. On the other hand, Sam went on and split the other thirds into halves after this step for each of the visuals. In this respect, it is more plausible that she used the part-whole subconstruct with recursive partitioning rather than the operator. In general, it is difficult to distinguish the operator subconstruct from the part-whole subconstruct in situations where the numerator (of the operator) is 1 or when the denominator is equal to the numerator of the operand. The latter is shown when Sam elicited the following representations for $\frac{2}{3}\times\frac{3}{4}$:

| | |
|---|---|
|  | A student drew this representation first by drawing the rectangular region as the whole, then cross-partitioning it and shading three fourths. The student then said, "I will take two thirds of this three fourths" and shaded the two parts as shown on the right. Sam did not add any other explanation for this picture and asked what other models the students could use. |

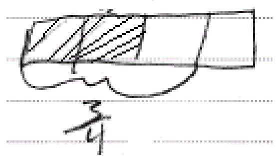
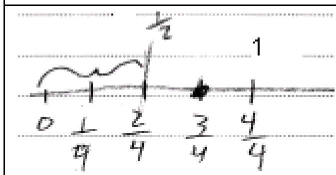
| | |
|---|---|
|  | Another student drew this picture. In Sam's class, this model is referred to as the bar diagram and is mostly used for situations involving measurement. The student first split the bar into four parts and then labeled three fourths of the whole. She then shaded two parts one part at a time. Again, Sam did not have any additional comments about the picture and asked the class if they could model the same problem using the number line. |
|  | Sam drew this model herself. She first put the numbers 0 and 1 on the line and then divided the interval into four parts. She marked three fourths. She then labeled other points in terms of fourths and asked, "what will be the two thirds of three fourths of this line |
| | (pointing to the region between 0 and 1)? Two fourths (pointing to the region between 0 and $\frac{2}{4}$)". She then labeled the portion of the number line from 0 to $\frac{2}{4}$ as $\frac{1}{2}$ and concluded, "so we can use different models to show the idea of multiplication of fractions". |

Figure 7. *Representations of $\frac{2}{3}$ of $\frac{3}{4}$ in Sam's class*

Sam's primary goal seemed to be providing a variety of representations for this. The student might have drawn the first model using part-whole (recursive partitioning) or operator (split the region into three s and shade two) interpretation. Yet, because the student did not explain her thinking process fully and Sam did not follow up , it is hard to identify which interpretation was in use. Similarly, for the second drawing, although Sam often used a bar diagram for measurement situations, the student might have used the part-whole interpretation with recursive partitioning if she thought about the problem as part of a part. That the part already consisted of thirds blurs whether the student attended to the operator subconstruct when finding two thirds of three fourths. The last drawing seems to clearly use the measurement subconstruct given Lamon's (2007) definition of the notion (See Figure 1). However, Sam did not refer to the numbers in terms of their distances or measures from 0 since she asked what two-thirds of three-fourths would be pointing to the line segment between 0 and 1. If she considered this portion of the line segment as the whole that was partitioned, she might be attending to the part-whole interpretation with recursive partitioning.

In summary, both Eliot and Sam used visual representations to illustrate solutions to fraction multiplication problems. While doing so, Eliot relied on a single representation across problems whereas Sam used multiple representations for each problem. It remains unclear whether their use of visual representations also facilitated understanding of the different mathematical interpretations underlying fraction multiplication.

## Discussion

Identifying the relationships between visual representations and mathematical interpretations was challenging in our study possibly because: (a) different interpretations of fraction multiplication could result in the same representation, and (b) instructors did not explicitly address which interpretations they were attending to as they represented fraction multiplication. For example, in using a subdivided area as both instructors did, whether they interpret fractions as part-whole or operator depends on the language they use to explain the representation and, in some cases, the order in which they subdivide the object. Making the steps clear could tie the fraction more closely to the interpretation or subconstruct. Another possibility is that using real contexts for fraction problems could lend meaning to the fractions that is absent in the abstract representations both of these instructors used.

One difficulty we encountered in analyzing these cases is that *representing a fraction* and *representing an operation with fractions* create different requirements for the teacher. Representing a single fraction using one of the subconstructs is relatively straightforward. Representing an operation, though, is not so easy. A subdivided area, as in Sam's pie diagram, can represent a part-whole fraction. But dividing each piece in half can be seen as creating smaller pieces (part-whole) or as operating on a single piece (operator). The language surrounding the representation as well as the choice of numbers in the multiplication problem is important for what idea the picture evokes for the student.

Does it matter? About this we have little evidence in this study, but previous work by Lamon (2007) suggests that it does matter. If K-8 students end up with a better understanding of and greater fluency with fractions by specifically learning about different interpretations of fractions, then it makes sense that teachers should themselves recognize these interpretations. We see in this case study, however, that the subconstructs of fractions can be intertwined during actual classroom practice. Our findings also indicate that it might be difficult to extract meaning from visual representations unless instructors clearly attend to the interpretations underlying those representations.

This study suggests several important areas for further research. In our view, it is especially interesting and important to understand more fully how explicit instructors of future teachers need to be about fraction interpretations and representations to equip their students – the future teachers of K-8 children – to teach fractions effectively.

## Acknowledgements

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# DEVELOPING PROSPECTIVE TEACHERS' KNOWLEDGE OF ELEMENTARY MATHEMATICS: A CASE OF FRACTION DIVISION

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*Prospective elementary teachers must understand fraction division deeply to be able to teach this topic to their future students. This paper explores how two university instructors help prospective elementary school teachers develop such understanding. In particular, we examine how instructors teach the meaning of division, the concepts of unit, and the connections between multiplication and division.*

## Purpose of the Study

The National Mathematics Advisory Panel has identified "proficiency with fractions" as a major goal for k-8 mathematics education because "such proficiency is foundational for algebra and, at the present time, seems to be severely underdeveloped" (p. xvii). However, as acknowledged by the authors of *The Mathematical Education of Teachers* (Conference Board of the Mathematical Sciences (CBMS), 2001) and supported by prior research studies, many prospective and practicing teachers possess shallow understanding of fractions (e.g., Ball, 1990; Ma, 1999; Simon, 1993; Tirosh & Graeber, 1989), and some are convinced that "mathematics is a succession of disparate facts, definitions, and computational procedures to be memorized piecemeal" (p. 17, CBMS, 2001). This characterization stands in stark contrast to the depiction of mathematical knowledge needed for teaching that has arisen from research on the mathematical knowledge that teachers draw upon in the context of teaching. This research (cf., Ball, Thames & Phelps, 2008) suggests that prospective teachers need mathematical knowledge and skills beyond basic competency with the topics they intend to teach. They need, for example, to be able to give or evaluate mathematical explanations, and to connect representations to underlying mathematical ideas and other representations.

How can college mathematics courses help prospective elementary teachers develop the deep understanding of mathematics that they will need for their future teaching? In this paper, we analyze two sets of fraction division lessons for prospective elementary teachers to highlight both the content and nature of two different approaches to achieving this goal. We chose to focus on fraction division because of the well-known and well-documented struggles of U.S. prospective and practicing teachers with fraction division. For example, In Ma's study (1999), 20 of the 21 U.S. teachers were unable to come up with correct story problems for the given fraction division sentence 1 ¾ ÷ ½. The findings of this study provide paradigm cases to highlight the challenges of designing mathematics courses for prospective elementary teachers.

## Theoretical Framework and Prior Study

Based on interviews with U.S. and Chinese elementary teachers, Ma (1999) proposed a 'knowledge package for understanding the meaning of division by fractions" that teachers should have as illustrated in the diagram below (from Ma, p. 77). In this paper, we focus on three specific aspects of this knowledge package, as suggested by the bolded objects in the diagram: the meaning of division for both whole numbers and fractions, the concepts of unit, and the properties and relationships among four basic operations.

Fishbein et al. (1985) identified two primitive models for division: a partitive model and a measurement model. For measurement division, one tries to determine how many times a given quantity is contained in a larger quantity. For partitive division, an object (or collection of objects) is divided into a given number of equal parts (or sub-collections), and the goal is to determine the quantity in (or size of) each part (or sub-collection). The "primitive" version of partitive division restricts the number of equal parts to a whole number, thus precluding division by a fraction, and reinforcing the idea that division makes smaller. Tirosh and Graeber (1989) found that partitive division was the dominating model held by U.S. prospective teachers, which led many of them to believe that in a division problem, the quotient must be less than the dividend, even though they could apply procedures to solve problems with divisor less than one correctly. This finding has prompted increasing attention to fraction division in the measurement context as well as calls for a modified interpretation of partitive division by capturing the idea of division as an inverse operation of multiplication. For example, Parker and Baldridge (2003) suggested thinking about 12 ÷ 2/3 as "12 is 2/3 of what?"

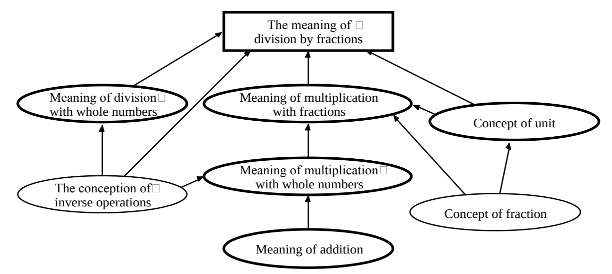

Fig. 3.2. A knowledge package for understanding the meaning of division by fractions
From Ma, 1999, p. 77.

The concept of unit in definitions and in operations is a key part of developing a deeper understanding for fraction division. For example, solving a measurement division problem such as "How many 2/3's are in 2?" requires the students to conceptualize "2/3" as a reference unit and interpret the "2" in terms of chunks of that particular unit: a process called "unitizing" by Lamon (1996). In the context of partitive division, such as when Parker and Baldridge (2003) suggest that students think of 12 ÷ 2/3 as "12 is 2/3 of what?" the unitizing process is more complex. In partitive division an object –the initial unit – is divided into a given number of equal parts, in this case 2/3 of a part. The goal is to determine the size of each part, a new unit, in this case 18. To solve a partitive division problem, a student must conceptualize the "unknown" quantity as both a unit itself and a fraction of a different unit.

Finally, the properties and relationships among operations (addition, subtraction, multiplication, and division) are needed when developing alternative algorithms (e.g. solving division word problems through repeated subtraction) or when explaining why the "flip and multiply" algorithm works.

## Methods

This paper reports findings from the case study component of a large-scale research project that investigates mathematics content courses taken by prospective elementary teachers during their undergraduate education. We focus here on two of seven case studies in the larger study, the cases of Pat and Eliot. During the units on fractions, we videotaped, wrote observation notes, and collected artifacts from students and from the instructor. We interviewed the instructors to probe their ideas about teaching the course, and both instructors completed an extensive written survey about their teaching. As part of the larger project, students in theses courses took pre- and post-tests assessing their mathematical knowledge. Results of these tests suggest that both of these instructors were successful at teaching their students mathematics, producing among the highest gain scores of all 42 instructors in the larger study. (For additional information about the larger project and the pre-post-test results, see McCrory, 2009.)

Both Pat and Eliot taught at universities that prepare large numbers of future teachers in their respective states. They provided the greatest theoretical contrasts in their professional backgrounds and instructional approaches to fraction division among the seven participating instructors. Eliot was a new instructor who had received her Ph.D. in mathematics the previous year. This was the second time she taught this course and her instructional approach was a combination of lecture and individual seatwork. Pat was an experienced math instructor with a Ph.D. in mathematics education and several years' experience teaching high school math. He had taught this mathematics course for future teachers over 20 times. The majority of his class time was spent on a combination of small group work and students explaining and justifying their solutions in front of the class, interspersed with his comments, questions, clarifications, or explanations. He occasionally gave a prepared short (15 minute) lecture. Finally, the course taught by Eliot was a 3-credit math content course that met for 50 minutes three times a week, while the course taught by Pat was a 4-credit integrated content and methods course taught for 80 minutes twice a week.

Data from multiple sources for each instructor was compiled. Tabular materials chronicled the major goals and instructional events of each lesson as well as narratives containing initial memos about the research questions were generated to form the case study database for each participating instructor. The research team went through the videotaped lessons to identify the opportunities prospective elementary teachers had to develop deeper understanding of fraction division. Episodes that illustrated the development of a particular aspect of the knowledge package of fraction division were selected and transcribed for further comparative analysis.

## Results

Our analysis on the fraction division lessons taught by the seven participating instructors uncovered a wide variety of approaches and emphases. The discussion of Eliot's and Pat's lessons helps illustrate such diversity. In the following we will first provide a summary of the main activities for each instructor's instruction of fraction division. Then we will discuss the main differences between these two different instructors using specific episodes from their lessons.

Table 5

*Summary of Eliot's and Pat's Lessons on Fraction Division*

| | |
|---|---|
| Eliot | 2/29/08 (50 min.)<br>• Model fraction division with pattern blocks using two hexagons as the whole.<br>• Explain why the invert and multiply algorithm works.<br>3/03/08 (8 min.)<br>• Discuss the patterns of fraction division when the divisor is smaller, equal or larger than one.<br>• Discuss the patterns of fraction division when the divisor is smaller, equal or larger than the dividend.<br>3/05/08 (24 min.)<br>• Use reasoning and logic to estimate the result of fraction division.<br>• Review of fraction division with pattern blocks.<br>3/07/08 (8 min.)<br>• Review of fraction division with pattern blocks. |

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

| | |
|---|---|
| Pat | 4/10/08 (40 min.)<br>• Model the solution of a (single) measurement fraction division word problem.<br>• What number sentence can be used for solving this word problem?<br>• Why is it a division problem (vs. a multiplication problem)?<br>• Why is it hard for elementary students to connect their solution for a word problem to a number sentence?<br>4/15/08 (70 min.)<br>• Model the solution of a (single) partitive fraction division word problem.<br>• Compare and contrast the type of mathematical knowledge needed for solving this word problem with a number sentence vs. with a pictorial model.<br>• Why can the same number sentence be used to represent both partitive and measurement division word problems?<br>• Connect both fraction division word problems with whole number division problems.<br>• Discussed the invert and multiply algorithm and ask students to think about why it works for both types of fraction division as homework. |

As noted in the introduction the three key features of a deeper understanding of fraction division include: the meanings of division for both whole numbers and fractions, the concepts of unit, and the properties and relationships among four basic operations. In terms of the key fraction division concepts, Eliot's lessons were based exclusively on the measurement interpretation of division. Pat's students had opportunities to make sense of both measurement division and partitive divisions, and also spent considerable time to contrasting the two. In the context of measurement division, both instructors emphasized the process of unitizing, conceptualizing the divisor as the "unit" to represent the given quantity (dividend). ). Eliot explained why the division algorithm worked by utilizing various properties of operations, while Pat facilitated his students' own discovery of the logic and reason behind this algorithm. Next we describe in detail episodes from each of the instructors.

*Eliot' Lessons*

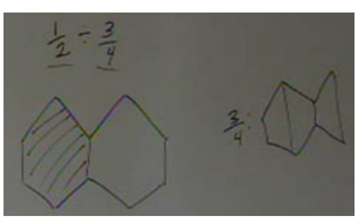


*Figure 1*. Eliot's board drawing for 1/2 ÷ 3/4.

In Eliot's lessons, students were familiar with the "2-hexagon as the whole" model when using it to model operations with fractions. She expected that her students could move flexibly among representations and interpretations such as "1/2 ÷3/4", "How many ¾ in ½?" and "How many 3-trapezoids in one hexagon?" and the drawings associated with them. This was an approach that required developing an understanding of the model itself, as well as an understanding of the operation for division applied to fractions. Eliot modeled such processes for her students as shown in the following episode (see Figure 1 for Eliot's board drawing).

Eliot: Let's illustrate a half which we decided is one hexagon [Draws on board], and just to refresh our memory, three-fourths, we decided was three trapezoids. [Draws on board]. Now do I have an entire set of three trapezoids in my half? No. How much

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of three trapezoids do I have in my hexagon? [Some students responded two, others responded two-thirds.]

Eliot: Two-thirds. That's exactly it. I have two-thirds of three-fourths in one-half. So I have two-thirds of three trapezoids in two trapezoids. So I have two out of the three I was looking for in my shaded region. Two-thirds. (Transcript, 2/29/08)

In terms of the explanation of why the invert and multiply algorithm works, Eliot used 2/3 ÷ 5/7 as an example. As she wrote on the whiteboard, (Figure 2) she explained each step:

> What you are really doing when you flip and multiply is multiplying by one…. If I were to multiply by something over itself, I would be multiplying by the number one... So I want to multiply by seven-fifths over seven-fifths Have I changed a thing? No. …What is something multiplied by its reciprocal? One. So now all I have is two-thirds times seven-fifths divided by one. Well one is also the division identity. So guess what I have here? Two-thirds times seven-fifths. So what have I done effectively? Flipped it and multiplied. …This is why you can do that, because all you are really doing is multiply by the multiplicative identity. (Transcript, 2/29/08)

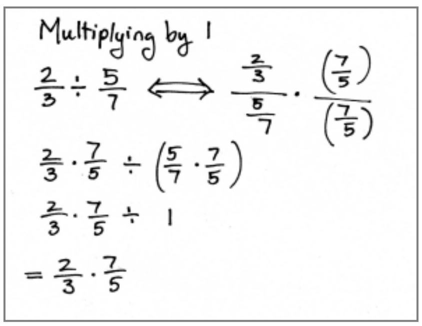


*Figure 2.* Eliot's explanation of "flip and multiply".

During all of her lessons on fraction division, Eliot designed her lessons around modeling with pattern blocks. She provided her students with clear, step-by-step explanations of the process and ample opportunities for them to practice on similar problems both in class where they could get additional support from her and as homework. She provided them with actual pattern blocks during class so that they could physically manipulate them. She acknowledged the struggle some of her students were having and re-visited this topic two more times, once after the quiz and once before the final exam, to address some of the common mistakes her students made. Eliot also modeled for her students how to use reasoning and logic to determine the reasonableness of their answers. She wanted her students know why the division algorithm works.

*Pat's Lessons*

Pat used the following word problem to discuss fraction division in the measurement context. *A batch of waffles requires ¾ of a cup of milk. You have two cups of milk. Exactly how many batches of waffles could you make?*He gave students time to work on the problems in small groups (a mode of work that they were used to), and after about 30 minutes, asked the class what answers they got. Individual students gave answers – 2, 2 ¼ (later changed to 2 ½ after discovering a computation error), 2 2/3, 2 3/8. After some discussion the class agreed that there was enough milk to make 2 batches, and the computation 8/4-6/4=2/4=1/2 was carried out to get the answer 2 ½. One student who thought the answer was 2 2/3 batches was asked to explain his reasoning. He first wrote down 2/4 and ¾ and explained that 2/4 is 2/3 of ¾. He then drew the following diagram (Figure 3) while explaining his reasoning:

Student 1: This is two cups of milk [draws the two rectangles and then sub-divides each into four equal parts]. This is going to be batch number one right here [shades three parts of the rectangles, angle to the right]. And then batch two would be this [shades another three parts, angle to the left]. You got two boxes left. So there would be two boxes left. You need three boxes for a batch, so we have two boxes, we need three for a batch so we are only going to fill in two of these boxes [draws the three circles and shades two of them], 'cause this is one batch right

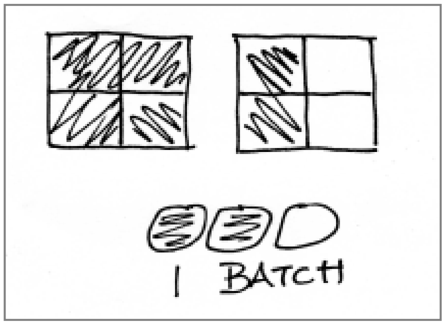


*Figure 3.* Student 1 waffle problem drawing.

here [points to the three circles and writes "1 BATCH" underneath them]. It's every three boxes. We have two left, which means there is two thirds of the last batch. So we have one batch right here [points to first three rectangle parts shaded], two batches [points to the next three], and two thirds of the last batch. ...the number two is two-thirds of three. ... I changed the wholes or whatever…. two-quarters is two-thirds of three-quarters. (Transcript 4/10/08)

Pat re-iterated the entire explanation and emphasized the point of "2 is 2/3 of 3, so you have 2/3 of a batch" which he wrote on the board. He also wrote "2/3 of a batch" and "½ of a cup" on the board for the shaded two of the three circles. He then asked the class what the student meant by changing the whole.

Student 2: Every time you make a batch of waffles, your whole or what is left over changes.

Student 3: You are changing from the whole as being the cup to the whole as being one batch of waffles.

Pat: So a cup is a whole and a batch is a whole… rather than writing 2 ½, you have 2 batches and ½ a cup of milk left over…. These two boxes [pointing to the bottom of the students' drawing] have double meaning. (Transcript, 4/10/08)

Instead of providing his students with a representation like the 2-hexgon, Pat asked them to generate their own drawings. He continued to push his students on being explicit about their explanation. In the process, he provided ample opportunities for his students to make connections among multiple representations: the story context, the drawings, the words, and the number sentences. His students were comfortable with being pushed and they also started to push each other for clear explanations or other their own elaboration without being prompted.

Pat introduced the question of why "flip and multiply" works after discussion of the waffle problem. Many students settled on the number sentence "2 x 4/3" for the waffle problem, but Pat pointed out that 4/3 was not a number in the problem (a requirement for an acceptable number sentence). One student offered an explanation using algebra, $\frac{3}{4} \cdot x = 2$ so $x = 2 \cdot 4/3$. Pat asked for another justification that would work in their teaching, pointing out that an algebraic equation was beyond the comprehension of elementary students. Another student proposed thinking of 4/3 as the number of batches that could be made with one cup of milk. Pat delayed the rest of the discussion until the next class, during which the class worked on an additional contextualized division-by-fractions problem, this one a partitive problem:

*You have 2 cups of flour to make some cookies. This is ¾ of what you need for one full recipe. How many cups of flour are needed for a full recipe?*

Pat again asked students to work on the problem in groups, then to share and explain their solutions at the board. At the end of the second class, he asked students to use the pictorial representations of the two problems to figure out why the invert and multiply algorithm makes sense as a homework problem.

Following the principles of Cognitive Guided Instruction (Carpenter, et. al, 1999), Pat's lessons on fraction division were built upon story problems embedded in daily contexts. His students were encouraged to develop their own solution methods and models to explain their reasoning. Not apparent in the short episode discussed earlier were attempts both Pat and his students made to compare and contrast different models based on different solution methods of the same given problem.

## Discussion

In this paper, we described the opportunities to develop deeper understanding of fraction division offered by two mathematics instructors in their courses. In terms of content, Eliot's lessons addressed topics that were not discussed in Pat's class, while Pat's lessons went deeper in connecting the meanings of whole numbers, fractions, multiplication and divisions through contextualized problems. Furthermore, Pat insisted on developing language and representations accessible to elementary students, while Eliot used concepts and terminology that would not be familiar to elementary school students.

Some of these differences are surely the result of the difference in course purposes: Eliot's a mathematics content course; Pat's an integrated content & methods course. They may also be a result of the difference in instructor backgrounds: Eliot a mathematician and Pat a mathematics educator and former high school math teacher. The effectiveness of these different curriculum models, math first then methods vs. integrated math and methods, is beyond the scope of the current study, as is the effectiveness of their very different approaches to teaching these ideas. We can observe, however, some differences in the mathematics of the lessons and provide conjectures about what these differences might mean for future teachers.

One big difference is the representations and how they were used. Eliot relied on pattern blocks and modeled reasoning with pattern blocks for her students. Pat encouraged his students to generate their own diagrams and expected them to use the context of the story problems to support their explanations. Both approaches get at the meaning of fractions and require moving flexibly across representations, which Lamon (2007) noted as a key part of fraction division understanding.

We also noticed a difference in the level of abstraction different representations demanded. For example, in Eliot's case, students need to be able to relate the actual physical blocks with the fraction quantities each block represents. Although the manipulatives are "real", the association of the block with the fraction is abstract and requires learning to connect the two. In Pat's case, the students need to create diagrams that connect with fraction quantities and the corresponding unit. In this case, the connections have meaning outside of the realm of mathematics, and may not be experienced as abstract. They also needed to attach each number and picture to something in the context of the word problem. The uses of both manipulatives (e.g. pattern blocks, fraction bars) and student-generated diagrams have been the primary focus of prior research investigations, and the findings have highlighted the complexity of making such contexts meaningful in elementary classrooms (e.g. Olive, 2000). The question, "How might these different uses of representations affect the development of deeper understanding of fraction division among prospective elementary teachers?" is worth pursuing.

The goals of mathematics courses specifically designed for prospective teachers should go beyond K-12 mathematics in order to distinguish themselves from mathematics courses for non-teachers. Both instructors did this. Eliot takes the students to mathematical explanations of the underlying mathematics that would not be appropriate for elementary students (e.g., the "flip and multiply" explanation) but, if successful, serves to provide the future teachers with deeper understanding of why the algorithm works. Pat's use of story problems requires students to understand why the problem is division and write number sentences for the problems they are learning. They give public explanations for their reasoning, thus teaching each other.

In this paper, we analyzed two sets of fraction division lessons for prospective elementary teachers and highlighted how two different representational contexts were used to achieve this goal. Even though mathematics courses for prospective elementary teachers are just a small component of the professional development continuum, these courses provide a common context to reach a large number of prospective elementary teachers. Thus it is important that we continue to explore how such classes are taught and how instructors choose and use representations to help future teachers learn mathematics.

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# LEARNING TO INTERPRET STUDENTS' MATHEMATICAL WORK: STUDYING (AND MAPPING) ELEMENTARY PRESERVICE TEACHERS' PRACTICES


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*Research has painted a dreary picture of preservice elementary teachers' preparation to teach school mathematics. A response to this problem has been to focus on the mathematical knowledge that is needed for teaching. However, another response remains largely uninvestigated—the learning to enact mathematics teaching. In this paper we report on our ongoing investigation of elementary preservice teachers' learning of mathematics teaching practices. Here we focus more narrowly on one of our focal practices and report our progress defining and unpacking the ways prospective elementary teachers at different stages in their teacher preparation program perform interpretations of students' mathematical work. We report on the conceptual and empirical work we have done to define the practice of interpreting with greater precision and with the goal of broadening what might be considered competent performance, especially for those just beginning their studies of mathematics teaching.*


## Introduction

The practice of listening to and making sense of students' mathematical ideas comprises much of the work of mathematics teaching. Either on the fly or when grading papers, teachers are constantly reading and interpreting what students say, write, and do. Such interpretations inform teachers' instructional decisions and actions—which in turn affect students' access to, and opportunities to learn, mathematics. It is therefore no coincidence that there is so much emphasis in reform documents and rhetoric on the need for teachers to develop new and better ways of seeing, interpreting, and handling students' mathematical ideas.

Noteworthy here is that although we can all recognize that the practice of interpreting students' mathematical work is essential to teaching, this practice is not named anywhere as an explicit object of study in teacher preparation. The study of teaching is focused on the knowledge, skills, and dispositions that support teaching practice, and not on the practices themselves. Yet increasingly we find preservice and inservice teachers working on instructional activities that prompt them to interpret students' work and to sometimes construct a teacher response. While there is an increasing collection of such materials for teacher learning, much less work has been done to understand the practices of mathematics teaching that those instructional activities aim to develop, especially at the early stages of teacher preparation.

In this study we are concerned with unpacking the practice of *interpreting* students' mathematical work by characterizing the performances of elementary preservice teachers who are at different stages in their teacher preparation. By examining closely the differences and similarities among these prospective teachers' interpretations we seek to generate more detailed descriptions of how the practice of interpreting students' mathematical work is performed by those who are just beginning their formal studies of mathematics teaching.

## Theoretical Framing

We draw on the research literature in mathematics education to broadly define the practice of interpreting as the work teachers do to figure out students' mathematical ideas expressed in oral or written form. In theory, this is a practice that teachers get better at as they gain more experience with students' ways of thinking and doing mathematics. It is also reasonable to assume that as teacher preparation students move through their program, they accumulate exposure to the work of mathematics teaching, and hence to the practice of interpreting students' mathematical work. As we considered how one might differentiate between an uninitiated and an experienced student of mathematics teaching, in terms of how they might interpret students' mathematical work, we found that such distinctions have been theorized more so than empirically established and mainly with a focus on practicing teachers.

Davis (1996) theorized three orientations to listening in mathematics classrooms (evaluative, interpretive, and hermeneutic) and suggested that a hermeneutic orientation (one in which the teacher and students co-construct and negotiate meaning) is rare and mainly performed by accomplished teachers of mathematics. An evaluative orientation, on the other hand, is quite prevalent and indicative of mathematics teaching of poor quality. In fact, Crespo (2000) found that elementary preservice teachers, as Davis suggested, had difficulties interpreting students' work and an overall tendency to promptly evaluate without carefully analyzing it. Additionally, Sherin's (1997) notion of teachers' professional vision—which she defines as the ability to notice and interpret significant features of classroom interactions—is also an attempt to describe teaching practice. She argues that teachers, as all professionals, develop particular ways for looking and making sense of what happens in classrooms. They develop selective attention to classroom events (focusing and narrowing the landscape of what needs to be made sense of) and then use various strategies to make sense of what is noticed.

Drawing on Davis' (1996) descriptions of teachers' *orientations to listening* and Sherin's (1997) construct of *professional vision* we conceptualized the practice of interpreting students' mathematical work as embodying aspects of 'noticing' (what is selected as the focus of analysis) and also strategies and orientations to the analysis of students' mathematical work (how the analysis is performed). In order to interpret students' mathematical work, teachers then need to purposefully select what is important and use some analysis strategies to unpack that work.

We used the distinctions and descriptions provided by Davis (1996) and Sherin (1997) as starting points but it became evident that we needed more conceptual tools to help us make distinctions among preservice teachers of elementary school mathematics that were at the very beginning of their professional preparation. As Hiebert and colleagues (2007) suggest, we needed to define a more reasonable and a wider set of performances for those beginning their formal study of mathematics teaching than the expert teaching practices that are more readily available and discussed in the research literature. We have therefore adopted three conceptual tools to help us make distinctions across a range of performances and allow for a broader sense of what we might consider competent practice for a beginner. We are conceptualizing mathematics teaching as a set of practices for which teachers build a *repertoire*, as practices that can be bound to, or flexibly used across, *teaching genres*, and as practices that have both *imagined* and *implemented* performances.

In using the concept of *repertoire* we are borrowing from music education, where it typically means a collection of well-practiced pieces (routines) that are ready to be performed publicly. Our hypothesis is that experienced teachers would have a broader collection than inexperienced teachers, and that they would be able to de-compose and re-compose their collection (set of routines) much easier than novices.

By using the concept of *teaching genre* we are hoping to be able to look at a wide range of teaching practices rather than draw a line between teaching that is usually seen as reform or traditional– and think about the ways that those learning to teach execute the practices of posing, interpreting, and responding within and across these genres. By drawing on the notion of genre, we could think about differences between beginning and experienced teachers in the way they did something like assign and correct practice problems as well as differences in more complex genres of teaching like facilitating a mathematical discussion. We expect that some genres might be easier to learn than others.

Finally, because teacher preparation students do not have a fully developed practice, we have adopted the terms *implemented and imagined* enactments of practice to describe a distinction and relation we see between these two forms of teaching practice. To us these two forms are equally important in the construction and development of a repertoire of mathematics teaching. For example, it would be difficult to consistently and thoughtfully enact teaching moves that one has not imagined as possibilities and rehearsed in some imaginary, vicarious, or simulated situation.

## Methods

The practice of interpreting students' work is not performed in a vacuum but rather happens in conjunction with other instructional practices. Hence, we study the practice of interpreting in relation to two other practices not discussed here (that of posing mathematical tasks to students, and of responding to their mathematical ideas). We conceptualize these three practices as interrelated. Even within the simplest of math lesson structures—introduce task, monitor students' work on that task; and then close the lesson—teachers find themselves enacting the practices of posing, interpreting, and responding. Previously (at this conference) we presented about this project's goals, design, and initial insights from pilot data (see Crespo et. al, 2007). In this paper we report on the cross-sectional phase of the study with a focus on the practice of interpreting to look more specifically at how preservice teachers who are at the beginning, middle, and end of their program conduct interpretations of a given students' mathematical work.

### *Studying Teaching Practice with Written Tasks*

As in many other research projects, we use paper-pencil teaching scenario tasks to collect data about the participants' enactments of mathematics teaching practice. While it might seem odd to study teaching practice anywhere but inside classrooms, researchers have used proxies to real classrooms for a long time. In our case and because teacher education students construct performances of mathematics teaching long before they have the opportunity to try them out in an actual classroom, we began our explorations of preservice teachers' performances of teaching practice by using written scenario tasks.

The conceptual tools of repertoire, genre, and imagined/implemented practice played key roles in the design of the paper-pencil instruments used in this project. First, we consider all responses to our paper instruments to be *imagined* performances of teaching practice. Only practice that we observe in an actual classroom is considered as *implemented*. Furthermore, in designing the project's written tasks we paid close attention to the implied *genre* of mathematics teaching that particular scenarios suggested. We have explicitly designed teaching scenario tasks that are not clearly identifiable as belonging to a particular genre of mathematics teaching. We have also included tasks that can be identified as portraying a particular genre, in particular those explicitly promoted in the teacher preparation program where this study is located.

Following on our assumption that the teaching repertoire of experienced and beginning teachers are quite different we also designed tasks that would allow us to peer into the range and depth of the participants' repertoire of posing, interpreting, and responding practices. While it is impossible to get a complete picture of anyone's entire teaching repertoire, we designed tasks to prompt participants to share multiple ways they could imagine performing the target practice. For instance, one particular task prompts participants to "share at least three different ways you can imagine setting up this task in a classroom."

Because we are studying three practices (posing, interpreting, and responding) that are intricately connected and that for the most part happen altogether within a single lesson, we designed our instrument so that each task places the respondent at the start of a math lesson and prompts them to perform the practices of posing, interpreting, and responding within a lesson. All of our tasks' math content was focused on the strand of Number and Operations. We designed and used 6 such tasks but in this paper we only focus on only one (shown below). Task 1(a) shown in Figure 1 focuses on the practice of posing a mathematical task, and 1(b) focuses on the practice of interpreting students' mathematical work.

1. (a) Imagine you are going to ask your class to solve the following addition. What can you imagine saying and doing to get them ready to work on this task?

$$\begin{array}{r} \mathbf{258} \\ \underline{\mathbf{+389}} \end{array}$$

**Teaching Scenario Continued**

1. (b) Imagine that a student shows on the board the following strategy for adding.
   i.) What would you want to make sure your class notices in this student's work?
   ii.) What are three different questions you can imagine asking to the class about this student's strategy, and say a bit about how these three questions are different?

$$\begin{array}{r} \mathbf{258} \\ \underline{\mathbf{+389}} \\ \mathbf{17} \\ \mathbf{130} \\ \underline{\mathbf{500}} \\ \mathbf{647} \end{array}$$

*Figure 1:* Sample P-I-R project task*.

* This task was adapted from a mathematics-for-teaching task used by another project at the authors' institution. Original task was developed by the IMAP (Integrating Mathematics and Pedagogy) project.

*Data Collection and Participants*

The project's instrument was administered in the Spring 2008 to elementary preservice teachers who were at three different stages of teacher preparation (Junior, Senior, and Internship). Roughly speaking the junior year of this program focuses on children as learners. During the Senior year, they study teaching methods in each of four subject areas (mathematics, literacy, science and social studies). During the Internship, they spend a year working with a mentor teacher (and supported by a field instructor) as they take increasing responsibility for teaching in each of the subject areas while also taking four graduate level courses of teaching methods throughout the year.

All preservice teachers enrolled in the program were invited to participate in the study. The tasks were administered during class (when instructors were able to fit us into their class schedule, typically at the end of class so that students could choose to participate or leave the room) and also outside of class (during lunch hour). Participants were given 30-40 minutes to complete either form A, B, or C; with each form including 2 of the teaching scenario tasks. Participants were invited to complete more than one form. A total of 126 Juniors, 152 Seniors, 69 Interns participated in the cross-sectional study. Task 1 was included in Form A and was completed by 46 Juniors, 42 Seniors, and 56 Interns.

## Some Results

Our work in this project has reminded us that even though there seems to be an over abundance of descriptions of mathematics teaching practice, the field lacks clear language to characterize with more precision those practices. This becomes very apparent when attempting to distinguish among a set of questions, which could have been generated by experienced teachers. Figure 2 offers a few sample responses to task 1b (ii) that were constructed by pilot participants who were and were not teachers. Figuring out (and explaining reasons for) which of these questions one might think were made by which group of participants is not that simple.

- Can someone explain why Susie has 130 in the second line and not 13?
- Does someone have a different way to solve this problem?
- Why did this student put zeros in their work?
- Why didn’t this student carry the numbers?
- Will this work for any 3 digit + problems? For any + problem? For any problem?
- Can you see how this student was able to get to the answer?
- Why did they add 17, 130, and 500?
- How do you know that this answer is correct?

*Figure 2*. Sample responses to task 1b(ii) “three different questions you can imagine asking …”

Would we be surprised if we learn that all of these questions have been generated by non-teachers; by experienced teachers? What would we consider ‘typical’ and ‘not typical’? Which of these would we agree should be in all teachers’ repertoire of mathematics teaching (and why)? In order to be able to address these sorts of questions, this project aims to generate more precise language and descriptions of three focal practices of mathematics teaching. We are also proposing that these practices need to be explicit objects of study. But most importantly, as Hiebert and colleagues (2006) suggest, we aim to not just map expert performances of these practices, but to construct descriptions of beginners’ enactments of these practices.

### *Developing Analytical Rubrics*

While designing analytical rubrics to code the participants’ responses to the project’s tasks we have had to continually revise and rework our definitions. Initially we broadly defined interpreting as the practice of figuring out students’ mathematical ideas expressed in oral or written form. Later we had to make our definition more precise and applicable to the project tasks. For the practice of interpreting our current working definition is that it involves the process of selecting (and discarding) and then analyzing and drawing conclusions from data (in this case students' work). It involves narrowing the data (what Sherin calls 'selective attention') and making low and high inference observations—such as describing, explaining, comparing, evaluating—about that data.

Because task 1b includes a student's work that features non-traditional arithmetic algorithms our rubric is also consistent with criteria articulated by Campbell, Rowan, and Suarez (1998)—validity, generalizability, and efficiency—that teachers should consider to support student-invented strategies in teaching situations. Ball, Bass, and Hill (2004) also suggest these criteria when stating that there is some relevant work teachers must do in order to be able to respond to students' strategies they haven't seen before—what, if it exists, is the method and will it work for all cases? What these (as well as many other) mathematics educators suggest is that teachers, upon encountering a non-standard piece of mathematics (especially if proposed by students), would carry out some analysis about the validity of the method, not only whether it produces the correct answer, but how and why it works. They would explore if the method works for other examples and perhaps contrast or connect with other methods. They might also explore if and when it makes sense to use (or not) this method.

Our rubric for Task 1(b), therefore, attempts to capture these aspects of the practice of interpreting students' work. It includes codes for parts (i) *what is important to notice in the student's work* and (ii) *questions one might ask about the student's work*. For both parts (i) and (ii) the response is first coded in terms of where the focus of attention is (A. the answer, B. the method, C. Big ideas) allowing for the possibility that attention can be spread across all three foci (to be consistent with our notion of repertoire). In fact, mathematics educators have often alluded to teachers needing to develop multiple lenses and peripheral vision in the classroom (i.e., Lampert, 2001). Then each category is coded in more detail to characterize more precisely the analysis work that is done on the answer, the method, and/or big idea.

A. Focus on Answer – (A1: Answer is correct; A2: incorrect; A3: other evaluation of answer)
B. Focus on Method – (B1: description; B2: explanation; B3: comparison; B4 evaluation)
C. Focus on Big Ideas – (C1: Mathematical; C2: Pedagogical)
E. Explanations are or not provided for generated questions in (ii) – (E0 and E1)

<u>*Sample statements for task 1b(i):*</u>

(A1): "to notice that the student was able to come up with the correct answer."
(A2): "this is wrong for adding but perfect for multiplication."
(A3): "this is set up as a multiplication problem not an addition problem"
(B1): "that they added each column and put the whole answer down."
(B2): "adding the 2 and 3 in the 100's column does not give you 5, it gives you 500."
(B3): "instead of carrying the 1 the student placed it beneath and worked downward instead of from right to left."
(B4): "this is a different but not incorrect method of finding the sum."
(C1): "that this strategy relies on place value to work."
(C2): "I would also want to see what other students were thinking about this problem."

<u>*Two sample responses to task 1(b)*</u>

(i) I want to make sure that the class notices that there is more than one way to solve math problems. They are free to use whatever strategy they choose to arrive at the correct answer. I want the class to pay close attention to place value (ii.) Can you tell me how you started this problem? Will this strategy work for all addition problems? Can you think of a time when this strategy would not be the most effective? (Student A)

*(i) I would want my class to notice how the student used the different numerical places in order to help him/her solve the problem. (ii) Is this a sufficient strategy to use when adding? Why or why not? Why did the student start from the ones place? Would it have made a difference if he/she began from the hundredths place? Is this the correct answer? How do you know? * These three questions are different because the first one deals with the efficiency of the problem (is it the fastest way to get to the answer), while the second questions deals with the place values. The third question focus on the answer, and whether it is correct and why. (Student B)*

Student A's response was coded as follows. Part (i) was judged as exhibiting attention to the answer and a mathematical big idea—place value. More specifically the response was characterized as C1, A1. The three questions generated in response to (ii) of the prompt were judged as exhibiting attention to the answer and the computational method. It was assigned the following codes: B1, A3, B4. In turn, student B's response was coded as follows. Part (i) was judged as exhibiting attention to a mathematical big idea – "numerical places." More specifically the response was characterized as C1. The three questions generated in part (ii) were judged as exhibiting attention to the method and the answer and the response was characterized with the codes B4, B2, A1. We chose these two students as examples not because they were the 'best' or because they stood out from the rest, but rather because these statements were common across all the cohorts of participants. What is noteworthy also about these and other similar comments is the breadth of what these two students were able to notice about the given student's work. If task1b had only used prompt (i) or (ii), we would have missed the range of what these students are considering (answer, method, big idea) when asked to interpret a student's mathematical work.

*Contrasting Cross-Sectional Interpretations*

Our initial observations from the pilot data were that as a group the non-teachers had a narrower repertoire (in both what they noticed and in the types of questions they generated about the students' work) than the experienced-teachers group. We also noticed that the experienced teachers made noticing statements and generated questions that were not at all made by the non-teachers group. However, we also found that there were also unexpected overlaps between the two groups. We were therefore curious about what we might learn by examining the performances of those who were at the very beginning, right in the middle, and at the very end of their formal study of elementary mathematics teaching.

Simple frequencies (see Table 1) of the coded performances in tasks 1b (i) and (ii) using the developed rubric showed that when interpreting the given student's mathematical work the main focus of attention for all the cohorts (including students who were enrolled in a pre-requisite course but not yet applied to the teacher preparation program – Pre-TP) was mainly on the computational method (B). Additionally, the most frequent type of analysis was that of description (B1), regardless of how far along the participants are in the program.

*Table 1*
Distribution (in percents) of Participant Statements across Rubric's Criteria

| TE Stage | No. Participants | Task | A1 | A2 | A3 | B1 | B2 | B3 | B4 | C1 | C2 | E0** | E1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Pre-TP | 32 | 1b(i) | 16 | 22 | 0 | 50 | 22 | 16 | 16 | 34 | 9 | N/A | N/A |
| Juniors | 46 | 1b(i) | 4 | 2 | 22 | 65 | 35 | 22 | 17 | 33 | 15 | N/A | N/A |
| Seniors | 42 | 1b(i) | 12 | 5 | 7 | 57 | 26 | 26 | 14 | 88 | 12 | N/A | N/A |
| Interns | 56 | 1b(i) | 9 | 2 | 20 | 70 | 52 | 14 | 16 | 70 | 16 | N/A | N/A |
| Pre-TP | 32 (91 Q*) | 1b(ii) | 25 | 22 | 6 | 59 | 16 | 19 | 38 | 25 | 25 | 53 | 28 |
| Juniors | 46 (123 Q) | 1b(ii) | 20 | 0 | 37 | 48 | 26 | 24 | 37 | 30 | 2 | 46 | 54 |
| Seniors | 42 (113 Q) | 1b(ii) | 14 | 2 | 12 | 69 | 26 | 26 | 36 | 79 | 5 | 55 | 45 |
| Interns | 56 (172 Q) | 1b(ii) | 29 | 0 | 46 | 63 | 48 | 29 | 38 | 34 | 4 | 11 | 89 |

*Note.* * (# Q) means total number of questions generated by the participants
** E0 means – no explanation; and E1 means explanations were written

In terms of differences, the interpretations made by those who were farther along in the teacher preparation program (Interns) had, in addition to the focus on the description of the method (B1), a prominent focus on the explanation of the method (B2) more than the other three groups. They also provided the highest percent of explanations for the questions they generated about the student's mathematical work. The Interns and those in the middle of the program (Seniors) also had a strong focus on the mathematical big idea(s) that were involved in the solution method of the student. It is hard to tell whether this could be a product of being in the classroom more or having completed more coursework. While we have not yet delved into making finer categories about the quality of their noticing and questions, one thing that is remarkable, however, is that although the practice of interpreting is not explicitly taught at this institution's teacher preparation program, the preservice teachers who are farther along in the program do pick up and learn to enact some aspects of this practice on their own. We wonder what more they might learn should this practice become an explicit and systematic focus of instruction in teacher preparation.

## What Next?

Coding and analyzing the cross-sectional responses to task 1(b) showed us some important differences and similarities in the ways prospective elementary teachers at different stages in their teacher preparation program perform interpretations of students' mathematical work. Our rubric has helped us to identify salient responses for each cohort of participants and to also identify aspects in the practice of interpreting students' work that seem productive to build on and aspects that appear to be challenging for those at the very early stages of teacher preparation.

Because we are looking at the collective rather than individual performance of preservice teachers and because we are using the conceptual lens of repertoire, we have been able to elicit and see a wider range of performances from preservice teachers than we expected. Our analysis so far suggests that preservice teachers may have a much broader collection of interpretive moves than the research literature has reported in the past. There is however much work still to be done to further unpack these participants' interpretations of students' mathematical work. In terms of what comes next, we are currently in the process of exploring clusters and combinations of interpreting moves performed by participants. We are also continuing the work of developing more detailed descriptions and finer distinctions among the cohorts as to what might be considered reasonable performances at different stages of teacher preparation.

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# USING SEMIOTICS TO TEACH RATIONAL NUMBERS TO PROSPECTIVE ELEMENTARY SCHOOL TEACHERS

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*Prospective elementary mathematics teachers should be able to understand how their future students understand number concepts. A difficult concept is that of rational numbers. Rational numbers often have complicated means of representation, signifiers and signifieds, making them difficult for students to understand and teach. In this paper, we describe how one teacher integrates a theory of semiotics when instructing prospective elementary school teachers about rational numbers. We propose that by teaching prospective teachers about semiotics, connections between signs and units are made explicit and prospective teachers will be more equipped to approach the instruction of rational numbers to future students*.

## Background on Obstacles to Understanding Rational Numbers

Lamon (2007) suggests that fractions are a subset of rational numbers, in that fractions are notational and are "non-negative rational numbers" (p. 635). We are interested in how to instruct preservice elementary school teachers to move beyond simply the use of symbols (notational systems in which there are two integers written with a bar between them) to a conceptual and transferable understanding of what those symbols represent.

Preservice elementary teachers often have trouble understanding and ultimately teaching future students about rational numbers (Graeber, Tirosh, & Glover, 1989; Harel et al., 1994; Simon & Blume, 1994). One key issue that prospective teachers face when they are trying to understand and work with rational numbers is whether a fraction is related to division, is a type of multiplication, or is a ratio of some sort (Ni, 2001). For example, Behr and his colleagues (Behr et al., 1993, 1994) demonstrated that transformation involved in solving problems with rational numbers use compositions and recompositions of units. The part–whole construct for 2/3 suggests two interpretations: two-third as parts of a whole are two one-third units, i.e., 2(1/3-unit), or two-thirds as a composite part of a whole is one two thirds unit, i.e., 1(2/3-unit). In the number sentence $2 \div 3 = 2/3$, fractions are related to the idea of division. However, in another sense, in order to get 2/3 you must first define 1/3 and then multiply it by the number of 1/3rds that you have. Further, if the numerator and denominator are meant to express a ratio—like there are two dogs for every three cats, then 2/3 cannot be thought of in either of the above ways. You cannot have 2/3 of a dog.

## Purpose of this Study

In this paper we consider how semiotics instruction can be used to help preservice elementary school teachers learn about rational numbers. While there may be many reasons why rational numbers are hard for students to understand, from a lack of prerequisite knowledge to a lack of working memory to hold multiple numbers in mind at one time, we are going to focus on how different meanings denoted by a single representation of a fraction in different relevant contexts may lead to an inability to fully understand rational numbers. We describe how one professor used a dyadic, or two part, semiotic framework developed by Ferdinand de Saussure (1957) to help instruct preservice elementary school teachers about rational numbers. Furthermore, by using this framework, the teacher was able both to decompose the problems themselves and to give the preservice teachers a way to understand how their future students will view the problems.

## Theoretical Framework: Why the Study of Rational Numbers is a Semiotic issue

Semiotics, broadly conceived, "is concerned with everything that can be taken as a sign" (Eco, 1976, p. 7), including "images, gestures, musical sounds, objects…these constitute, if not *languages*, at least systems of signification" (Barthes, 1967, p. 9). Ferdinand de Saussure (1957) described a linguistic sign as a two-sided entity made up of a signifier and signified. A signifier is the material aspect of the sign whereas the signified is the mental concept associated with the material symbol. For example, the English word "tree" is made up of the material sounds /t/, /r/, and /e/ (the signifier) as well as the mental concept we each hold of what it means to be a "tree" (the signified).

In addition to the simple model of a signifier and signified, one must also bear in mind the community in which this relationship takes place. Saussure (1957) reminds us that regardless of the signifier a linguistic system uses "to designate the concept 'tree,' it is clear that only the associations sanctioned by that language appear to us to conform to reality" (p. 66-67). Thus, there are many different signifiers that can represent the same signified. In the field of semiotics, "langue" refers to the collection of signs, the overall system of signification, that permit individual speech utterances. Put another way, "langue" is "language minus speech," the structure that permits individual utterances (Barthes, 1967, p. 14). Depending on how the words are combined in a phrase or what words are chosen in a particular phrase, the concept of the signified may change. In other words, the meaning of a sign is not contained within the signifier or signified alone, but develops within a phrase and within a community as well. To make sense of the sign, not only do the signifier and signified need to be taken into account, but the context that contains that sign must also be considered. One type of sign may be signified by a written symbol. Therefore, it follows that one way to study and begin to understand rational numbers is by using semiotics.

Taking the idea of rational numbers as signs, we can see how it can be difficult for students to make sense of them. A single signifier of a rational number, like two whole numbers with a bar in between them (ex – ½), can take on multiple meanings. Consider the following uses of rational numbers taken from a popular book used to teacher prospective elementary teachers mathematics (Billstein, Libeskind, Lott, 2007, p. 299) (Table 1):

*Table 1*
Uses of Rational Numbers

| Use | Example |
|---|---|
| Division problem or solution to a multiplication problem | The solution to 2x = 3 is 3/2. |
| Partition, or part, of a whole | Joe received ½ of Mary's salary each month for alimony. |
| Ratio | The ratio of Republicans to Democrats in the Senate is three to five. |
| Probability | When you toss a fair coin, the probability of getting heads is ½. |

In the example, a rational number could be used as division, to partition something, as a ratio, or even as a probability. Similarly, Kieren's semantic analysis of rational number identifies five "subconstructs" of rational numbers. They are part–whole, ratio, quotient, operator, and measure (Behr et al., 1992; Kieren, 1976). Depending on the context, the meaning of the rational number—even if it is represented by the same numbers—changes.

Consider the following question out of context. What does 2/3 mean? This signifier, 2/3, could mean any of the following (the list is certainly not exhaustive):

- 2 candy bars shared equally by 3 people, each person gets 2/3
- 2 dogs for every 3 people
- 2 ÷ 3 = 2/3
- 1 ÷ 3 = 1/3 (partitioning), there are 2 1/3rds = 2/3 (iterating)
- 2 parts, with 3 equal parts to make a whole
- 2/3 = 4/6 = 6/9, etc.
- It could be 2/3 of a number greater than one, like 6, which is equal to 4
- It could be 2/3 of a number less than one, like 1/6, which is equal to 1/9

We posit that one difficulty students have when trying to first understand and then later teach rational numbers is that they must navigate across multiple meanings, or signifieds, for the same symbol, or signifier, a relatively common task in spoken and written language, but not necessarily common in the understanding of numbers. We use the phrase 'semiotic dissonance' to describe the difficulty or inability for a person to meaningfully construct a sign from a signified (meaning) and its associated signifier (symbol). Further, when a teacher develops the ability to navigate across meanings, he or she must be able to step back and understand how future students must then navigate across meanings. This understanding is the basic semiotic framework that we will refer to for the rest of this paper.

## Mode of Inquiry: A Case of One Teacher Using a Semiotic Framework to Teach Rational Numbers to Prospective Elementary Teachers

As part of a larger study that explores the mathematics content taught to undergraduate prospective elementary teachers, this paper focuses on one of seven instructors who were video-taped while they taught fraction lessons. The video data was supplemented by the field notes taken during instruction and of an interview of the instructor. For more information about the project, including student test data and teacher surveys, see McCrory (2008). Since in this paper we are concerned with overcoming the difficulties students face when trying to work with rational numbers, we looked at one college professor in more detail. For this project, over 15 hours of video data was gathered for this instructor. The analysis of the video tapes used an Iterative Refinement Cycle (Lesh & Lehrer, 2000) model in which multiple interpretive cycles were used on the data. The first interpretive cycle was used to identify those issues that pertained to general pedagogy and classroom culture. The second cycle was used to identify those issues that reflected the semiotic framework that framed this research investigation. The third cycle was used to establish explicit connections between the instructor's pedagogical decisions in order to make clear to the students the semiotic issues in their problem solving process. Finally, throughout the entire iterative viewing and interpretive process, the analysis of the other data source, including field notes and the exit interview with the instructor, was used to help inform the context and nature of the classroom discourse.

Below we provide excerpts of the interpretive narrative and discuss how Pat (a pseudonym), a professor of prospective elementary school teachers, used semiotics to instruct his students about rational numbers. The following episodes from Pat's class show how an understanding of semiotics can help preservice elementary teachers understand rational numbers and ultimately how to understand how to navigate these teachers' future students' misconceptions of rational numbers.

## Results

Before delving into the difficult concept of rational numbers, Pat did a short presentation introducing the students to semiotics. In the presentation, Pat first defined a linguistic sign as a "…two sided entity, a dyad, between the signifier (symbol) and the signified (meaning)." He then gave the following visual example of linguistic sign using the concept of 'dog':

**sign → <u>Spoken word "Dog"</u>**
**Concept of Dog**

Pat would refer back to this simplified semiotic framework when semiotic dissonance arose in problems that had the same numbers, but different units, to explicitly show students how the semiotic framework provided insight into their confusion. Pat specifically chose the waffle and cookie problems below to include two meanings of division: partitive/sharing (how many/much per group?) and quotative/measurement (how many groups?).

- The Waffle Problem: *A batch of waffles requires ¾ of a cup of milk. You have two cups of milk. Exactly how many batches of waffles could you make?*
- The Cookie Problem: "*You have 2 cups of flour to makes some cookies. This is ¾ of what you need for one full recipe. How many cups of flour are needed for a full recipe?* (Class handout, 4/10/08)

In his class, Pat began by having students work in six groups of four students to solve the above problems. When Pat asked for solutions to the first problem, four of the six small groups each gave a different answer (2, 2 ¼, 2 2/3, 2 3/8), only one of which was actually correct. Clearly this question posed a series of problems for this class. Pat proceeded to have the group that had 2 ¼ go to the board and write how they solved the problem for the class. One student said, "I knew that 1 cup of milk was four fourths," and then wrote out on the white board 4/4 + 4/4 = 8/8 to represent the 2 cups of milk. He continued saying, "I know ¾ a cup is batch so I took away 6/8 for two batches," while writing 4/4 + 4/4 = 8/8 – 6/8 = 2/8. The student concluded by saying that he had 2 batches so far, represented by the 6/8, and a ¼ leftover, simplified from the 2/8, giving a final answer of 2 ¼ batches. Another student quickly pointed out that 4/4 plus 4/4 was actually 8/4, not 8/8. Using that fact, Pat reworked the problem on the white board writing 4/4 + 4/4 = 8/4, and 8/4 – 6/4 (for the two batches of waffles) = 2/4 = ½. The class then began to discuss the meaning of the ½, whether it meant ½ a batch or ½ a cup. After some class discussion on how to interpret the ½, a third student pointed out that the ½ left over was not ½ a BATCH, but rather was ½ a CUP and that ½ cup was the same as 2/3 of a batch. To illustrate his point, the student drew a pictorial representation of the problem (figure 1) trying to show how ½ cups of milk was equivalent to 2/3 batch of waffles.

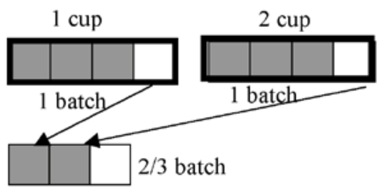


*Figure 1*. Pictorial solution to the waffle problem.

After the students explained why they got their answers, Pat brought the class back and explained how both a cup and a batch could be a whole. Pat stressed that the students needed to attend to the context, and said, "technically the ½ is not wrong until you put a name to it. You say ½ a batch. That's not true, because it's half a cup" (from video on 4/10/08 and 4/15/08).

This is clearly a semiotic problem, where the signifier, or visual mark "1/2," had taken on two different meanings, depending on the signified, or concept, with which the students associated the signifier to construct their sign. One student incorrectly associated the ½ to 'batches' while another student correctly associated the ½ to 'cups,' which was equivalent to 2/3 of a batch. In this case, the single signifier, ½, could only be meaningfully associated with one meaning, cup. We describe this phenomenon with the phrase 'semiotic mismatch,' in which a signifier is incorrectly associated with a particular signified as determined by the context. As illustrated in this excerpt, by explicitly showing students how signs can mean different things depending on the context, the students gained insight in how to determine what the accurate signified, or intended meaning, was in the problem.

When Pat moved to a discussion of the cookie problem, he encouraged the students to use a semiotic framework not only when approaching their own understanding, but also in future instruction of rational numbers to elementary school students as well. As with the waffle problem, Pat first had the class work in small groups to come up with a way to model their solution. Pat had two groups put up two solutions. The first solution, which was an algebraic solution, looked like this:

2 cups = 8/4 cups
2 divided by ¾
8/4 divided by ¾
8/4 x 4/3 = 8/3
2 2/3 cups

The second solution (figure 2), copied in the notes as replicating the board drawing, was a pictorial strategy.

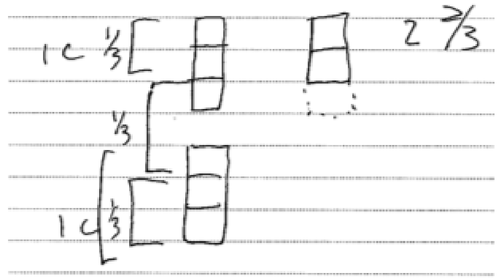


*Figure 2*. Pictorial solution to the cookie problem.

At this point, Pat gave the students some time to think about how they would explain each of the above answers to their future students. Pat discussed how this was a content pedagogy course, so he wanted them to understand the content, but then, as future teachers to be able to explain the mathematics with clarity. While no one seemed eager to attempt to explain the first solution strategy, one student did label and explain the second strategy. First, she labeled the model (figure 3) and noted that she thought it was important to label cups on one side and ¼ batch on the other side so students would not get confused. Here the student had moved from having semiotic dissonance to realizing the importance of being consistent with the use of symbols when there are multiple signifieds, quantities of cups and recipes, as a result of the context.

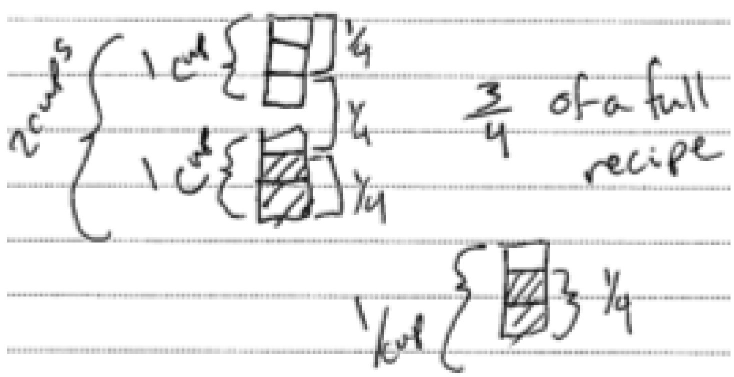


Figure 3. Pictorial solution to the cookie problem with student explanation.

Pat waited for the students to comment as the model was explained. He then pointed out that the two shaded boxes in the drawing, 1/3 of 2 cups and ¼ of a recipe were also both equal to 2/3 cups. In this case, the signifier represented by the two shaded boxes, could be associated with three different meanings, the signified, depending on context.

After students shared solutions of the cookie problem, Pat asked the groups to think about what mathematics was necessary for kids to solve the problem using the different strategies presented. The class decided that to solve the problem using strategy 1, the algebraic solution, the child would need to know improper fractions, the division of fractions, fraction algorithms, and whole number operation facts. On the other hand, for strategies 2 and 3, the pictorial strategy, the child would need to know how to partition pictures, split quantities up, and recognize the concept of a changing whole. One of the students specifically pointed out that in strategy 1, number facts were necessary while in strategies 2 and 3 a conceptual understanding of division was needed. Furthermore, looking back over the three representations, one could see that while the first strategy was mathematically correct, a student would not necessarily need a conceptual understanding to solve the problem. There were no units labeled, and thus semiotic confusion could arise in children. In the second solution, the diagram was not labeled adequately, which could also lead to semiotic confusion. The third solution had taken into account much of the semiotic framework that had been taught, including labeled pieces so that a child could more easily figure out what is signified by the images and numbers.

By analyzing the three representations of the cookie problem and determining what langue, or semiotic context, was needed to solve the problem using the two strategies, Pat enabled his students to begin to think about how they could instruct future students about rational numbers. He emphasized the importance of labeling the units and how the changing whole could be tricky for their future students to understanding and learn. He presented them with a semiotic framework to be able to create meaningful signs by connecting the appropriate signifiers with the appropriate signifieds, depending on the knowledge available to his students as well as his students' future school aged students.

After discussing the waffle and cookie problems individually, Pat asked the groups to determine the difference between the two problems. While both problems used the same numbers (2 and ¾), the same operation (division), and had the same answers (2 2/3), the contextual difference led to different solution strategies and conceptions of the fractional quantities. Again, Pat wanted the students to understand that the langue of the problem affects the ultimate meaning of the signifieds (amount) and of the signifiers (visual or auditory number itself). By now, the students had decomposed both problems and quickly answered that: 1) In the problems they are asking two different questions; 2) In the first you have everything you need. In the second problem you need to figure out what extra represents. You don't have everything you need. 3) In the first problem, ¾ makes a whole, in the second it is ¾ of what is a whole; and 4) In the first problem, the measurement is division, where you know the amount of groups and want to know what the whole is and the second is partitive division, in which you know how many groups/parts are in the whole. By analyzing the problems and giving students a semiotic framework to allow them to determine the meanings of the symbols being used, the students were able not only to understand the rational numbers themselves, but were also able to conceptualize what kinds of mathematical knowledge their future students would need to understand such problems.

## Discussion

As noted in the introduction, preservice elementary teachers often have trouble understanding how to conceptualize and use rational numbers. Without adequate conceptual understanding, this lack of content knowledge will be perpetuated when these students go to teach their future elementary school students about rational numbers. This paper has shown how one teacher has used a framework of semiotics to help students understand, use, and ultimately instruct each other about rational numbers. Whether this knowledge transfers to the elementary classroom is an area that needs further study. Nonetheless, the preservice teachers in this case study clearly showed growth in their understanding of rational numbers through the use of a semiotic framework. The study of semiotics helps students to interrogate "how" things mean, not just what they mean, that a secondary system underpins the superficial representations of concepts with which they have become so familiar. Further, breaking down the problems using semiotics makes explicit what we often think we are doing implicitly. Finally, this explicit decomposing of problems allows students to begin to see where their future students may encounter difficulties when trying to understand rational numbers.

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# PRE-SERVICE ELEMENTARY TEACHERS' KNOWLEDGE OF GEOMETRY AND MEASUREMENT

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*This paper reports on a study of the development of knowledge of geometry and measurement of more than 450 pre-service elementary teachers (in two cohorts) that were taking a required mathematics course focused on these topics. They completed pre- and post- tests consisting of multiple choice and open response items. Pre-service teachers' mean percent correct doubled between the pre- and the post- test. Such large increase was also evident in some individual items; however, this improvement in mean scores was not at all uniform. This suggests that after a semester of study on these topics preservice teachers made significant progress but there were still concepts they continued to find difficult. Implications of these and other results are discussed.*

## Introduction

Mathematicians and teacher educators have been studying what mathematics is needed for teaching at elementary grades for decades. These studies have been summarized in the recommendations of the Conference Board of the Mathematical Sciences (2001) and include individual studies conducted by scholars such as Ball (1990), Borko et al. (1992), and Ma (1999), and the reviews by Begle (1979), Ball, Lubienski, and Mewborn (2001), and Hill, Sleep, Lewis and Ball (2007). The book by Even and Ball (2009), a study of the International Commission on Mathematical Instruction (ICMI) about the education of mathematics teachers, indicates that the concern about mathematical knowledge for teaching is worldwide. Most of the studies of mathematical knowledge of prospective or practicing elementary teachers have tended to focus on knowledge of number and operations. In particular—except for a few studies about pre-service teachers' van Hiele levels, e.g., Mayberry (1983)—little is known about prospective teachers' knowledge of geometry, measurement, or spatial reasoning. Yet, after number and operations, the topics of geometry and measurement are the most frequently taught content strands in elementary grades (Rowan, Harrison & Hayes, 2004).

In this paper we report on a study that addresses this gap in the literature on mathematical knowledge for teaching. A mathematics research group originally set up by the Teachers for a New Era Project [TNE] at Michigan State University has been conducting self-studies of what prospective teachers in the elementary teacher certification program learn about knowledge for teaching mathematics from the required courses in mathematics and mathematics education. In this paper we report on one part of our work: an investigation of the knowledge of geometry and measurement acquired during a required undergraduate mathematics course about geometry and measurement designed for students in the elementary teacher education program. In particular, the study reported here addresses the following research questions:

1. What knowledge of geometry and measurement do pre-service teachers have at the beginning of a mathematics course addressing those topics?
2. To what extent and in what ways does the pre-service teachers' knowledge of geometry and measurement change by the end of the course?

## Participants

The population of concern is the set of students enrolled in the 5-year elementary teacher preparation program at the authors' University. Students in this program are required to take two mathematics courses designed especially for future teachers of Grades K–8 (The State's elementary certification is for Grades K–8 and secondary is for Grades 7–12). In the first course, called MATH 1 here, students study whole numbers, rational numbers, number theory, ratio and proportion, and elementary ideas in algebra. In the second course, called MATH 2, students study topics in geometry and measurement, including spatial visualization, and properties of 2-D and 3-D figures, such as congruence, similarity, transformations, perimeter, area and volume. This choice of content was influenced by the recommendations and research mentioned earlier, as well as by national and state standards for students in Grades K–8. To develop what Ma (1999) calls "profound understanding of fundamental mathematics," both MATH 1 and MATH 2 emphasize developing multiple representations for mathematics concepts, writing and solving word/story problems, and explaining why mathematical statements are true or why procedures "work."

MATH 1 is required for admission to the elementary teacher preparation program; so most students taking it are freshmen or sophomores. MATH 2 may be taken at any time prior to the 5th year internship; so students are a mix of freshmen, sophomores, juniors and seniors. Both MATH 1 and MATH 2 are taught in sections of 25–35 students. Typically, one section each semester is taught by a professor, and other sections are taught by teaching assistants, most of whom are doctoral students in mathematics or mathematics education. In order to investigate the stability of results, this study was conducted in two consecutive years in MATH 2. Students enrolled in MATH 2 in Spring 2007 used the first edition of Beckmann's text (2004), and in Spring 2008 they used the second edition of the same book (2007). A different faculty member was course supervisor each semester. Students in Spring 2008 spent more time on volume and less time on properties of quadrilaterals than the students in Spring 2007. Other aspects of curriculum and instruction were the same.

## Methods

Each semester students were given a pre-test on the second day of class. As part of the consent process students were informed that their pre-test score would not count towards the course grade. They were also informed that these questions were about material that would be studied and examined later in the course and that the results would help instructors evaluate the effectiveness of their instruction. Most questions on the pre-test were also embedded in the final exam (post-test) for the course.

*Instruments*

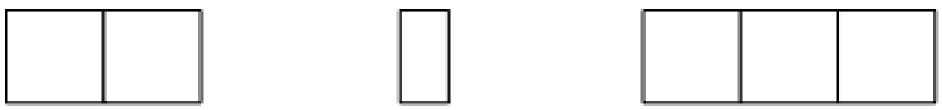
Topics on the pre-test included properties of polygons and polyhedra, the meaning of measurement units, perimeter, area and volume. Items addressed various strands of mathematical proficiency, including conceptual understanding, adaptive reasoning, and strategic competence (National Research Council, 2001). Several items were taken from previous research involving school children (e.g., Battista, 2007), or teachers (e.g., Ma, 1999). In this paper we report on a total of eight items—three multiple choice and five open-ended. Several of the free response items have multiple parts, and some items can be solved in more than one way.

Figures 1 and 2 show two items dealing with concepts of area. Figure 1 shows a multiple choice item assessing conceptual understanding of the effect of changing the unit of measure on the area of a figure. Understanding such transformations is essential for most real world applications of geometry. In addition, a successful response to this item would indicate the ability to integrate what Battista (2007) calls measurement and non-measurement reasoning.

Figure 2 shows a free response item asking students to identify and explain the relation between the areas of two triangles. As Beckmann (2002) notes:

> *prospective teachers should learn to explain mathematics not only because they will explain mathematics to their future students, but also because explaining mathematics enhances their own understanding of mathematics and their own mathematical reasoning abilities.* (p. 2)

Juan, Kim, and Angelo each measured the area of the same shape using the area units shown below.



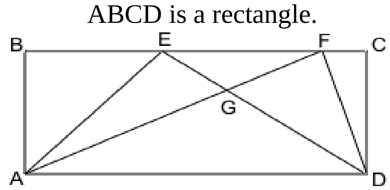
Juan's unit    Kim's unit    Angelo's unit

Which of the following could be a correct set of area measurements for the shape?

(a) Juan: 60 units Kim: 15 units Angelo: 90 units
(b) Juan: 60 units Kim: 240 units Angelo: 40 units
(c) Juan: 120 units Kim: 30 units Angelo: 80 units
(d) Juan: 120 units Kim: 480 units Angelo: 180 units

*Figure 1.* Item G4 – a multiple choice question about units of area.

ABCD is a rectangle.




a. How do the areas of triangles AED and AFD compare? (Mark one.)
   (a) The area of triangle AED is equal to the area of triangle AFD.
   (b) The area of triangle AED is greater than the area of triangle AFD.
   (c) The area of triangle AED is less than the area of triangle AFD.
   (d) There is not enough information to determine how the areas of triangles AED and AFD are related.

b. Justify your answer.

*Figure 2.* Item G6 – assessing reasoning about areas of triangles.

*Scoring Items*

Each correct multiple choice item was given one score point. For each free response item the research team developed a scoring rubric, with 4 points maximum allotted for each free response item. This results in a maximum total score of 29 points.

Influenced by the work of Malone et al. (1980) and Thompson and Senk (1998), rubrics typically assigned 4 points to a model solution, 3 points to a solution that is conceptually correct but has a minor computational error, 2 points to an item that indicates a chain of reasoning but contains either a conceptual error or gets only about halfway to a solution, 1 point to a solution that does not contain a valid chain of reasoning, but has at least one correct relevant statement, and 0 points to a completely incorrect solution.

Rubrics were developed iteratively, generally over a period of several weeks for any given items. Typically, a team of two or three researchers would select sample solutions that they thought illustrated the general scoring guidelines described above, and present them for discussion and potential revision to the full 7-person research team. Once agreement on the language of the rubric was reached, anchor papers were identified, and the full team practiced scoring a small set of other papers. Once acceptable levels of reliability were reached, the team that proposed the rubric scored all remaining papers. Figure 3 shows the rubric developed for Item G6 (b). Once rubrics had been developed for all items, other researchers who had not been involved in the original scoring, rescored a random 15% of the samples. For Item G6 (b), the rescoring resulted in 98% agreement.

| | |
|---|---|
| 4 points | (a) Complete, correct argument based on Cavalieri's principle (must include a statement that side BC is parallel to side AD because figure ABCD is a rectangle, or other equivalent statement). (b) Responses that use the formula Area=(base)*(height)/2 and justify why the bases and heights of the two triangles are the same [must include the statement that the distance between the opposite sides of a rectangle are equidistant because they are parallel]. (c) Responses that correctly show some combination of categories (a) and (b) or other method. |
| 3 points | (a) Same criteria as 4(a) but minor errors/omissions (e.g., student fails to mention that rectangles have parallel sides). (b) Same criteria as 4(b) but minor errors/omissions. (e.g., failing to mention opposite sides of a rectangles are parallel, parallel lines are equidistant) (c) Same criteria as 4(c) but minor errors/omissions. |
| 2 points | (a) Explanation that mentions Cavalieri's Principle and/or shearing process (or other related term). (b) Explanation that uses the formula Area=(base)*(height)/2 but does not show how the bases and heights of the two triangles are equal. (c) Combination of (a) and (b) or other methods that shows a chain of reasoning but either has major conceptual errors, or half way done. |
| 1 point | There is at least one correct and relevant fact in the response. |
| 0 point | Incorrect, irrelevant, or blank statements |

*Figure 3.* Rubric for Item G6 (b).

## Results

The mean and standard deviation of the overall score earned on the pre- and post-tests for the two samples is given in Figure 4. Also given are the Test Difficulty, defined as the ratio of the mean score to the maximum possible score (29), and the distributions of scores. Performance on the pre-tests was quite stable across semesters, with a mean score about 9 of 29 points possible in each semester. Mean scores rose to almost 20 points on the post-test in 2007 and 18 points in Spring 2008. In each semester the growth from pre- to post-test was more than two standard deviations. Table 1 reports descriptive statistics for the items shown in Figures 1 and 2, including the mean, standard deviation, and item difficulty (defined to be the ratio of the mean score to the total number of points possible on the item).

| *Administration of Test* | *N* | *M* | *SD* | *Test difficulty* |
|---|---|---|---|---|
| Spring 2007 Pre | 176 | 8.74 | 4.24 | 0.30 |
| Spring 2007 Post | 219 | 19.93 | 4.56 | 0.69 |
| Spring 2008 Pre | 232 | 9.03 | 4.39 | 0.31 |
| Spring 2008 Post | 232 | 17.79 | 4.27 | 0.61 |

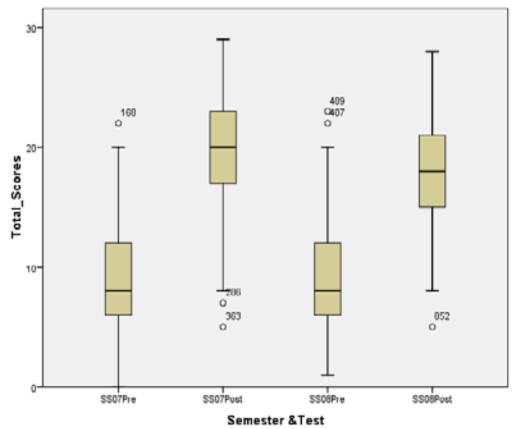



*Figure 4*. Descriptive statistics for students' overall scores on TNE geometry items and boxplots of distributions of total scores.

*Table 1*
Descriptive Statistics for Items G4 and G6, Shown in Figures 1 and 2 (respectively)

| *Item (Max. Points)* | *Test* | *N* | *M* | *SD* | *Item difficulty* |
|---|---|---|---|---|---|
| Item G4 (1 point) | | | | | |
| | Spring 2007 Pre | 235 | 0.23 | 0.42 | 0.23 |
| | Spring 2007 Post | 235 | 0.55 | 0.50 | 0.55 |
| | Spring 2008 Pre | 232 | 0.18 | 0.39 | 0.18 |
| | Spring 2008 Post | 232 | 0.43 | 0.50 | 0.43 |
| Item G6 (a)(1 point) | | | | | |
| | Spring 2007 Pre | 235 | 0.28 | 0.45 | 0.28 |
| | Spring 2007 Post | 235 | 0.80 | 0.40 | 0.80 |
| | Spring 2008 Pre | 232 | 0.36 | 0.48 | 0.36 |
| | Spring 2008 Post | 232 | 0.73 | 0.45 | 0.73 |
| Item G6, (b) (4 points) | | | | | |
| | Spring 2007 Pre | 235 | 0.22 | 0.58 | 0.06 |
| | Spring 2007 Post | 235 | 2.12 | 1.12 | 0.53 |
| | Spring 2008 Pre | 232 | 0.25 | 0.65 | 0.06 |
| | Spring 2008 Post | 232 | 1.54 | 1.08 | 0.39 |

Both Items G4 and G6 were difficult for the students in this study. The correct answer to Item G4 is choice (b) and on the post-test this was the most commonly chosen answer, but only 55% in 2007 and 43% in 2008 chose the correct answer. Each semester, the most commonly chosen wrong answer on the post-test was distractor (a) with about 30% choosing it in 2007 and about 55% in 2008.

For Item G6, although only about a third of the students recognized on the pre-test that the areas of the two triangles are equal; by the end of the semester about 80% in 2007 and about 73% in 2008 recognized that the two areas are equal. However, justifying this conclusion was more difficult for students even at the end of the course. In 2007 and 2008 combined only 1% of students on the pretest and 24% on the post-test scored 3 or 4 on part (b) of Item G6. That is, by the end of the semester less than one student in four was successful at justifying why the areas are equal. Figure 5 shows two valid arguments illustrating the two most commonly used solutions for this item.

| Student A | Student B |
|---|---|
| We know through Cavaleri's principle that we can shear an object along a parallel line and preserve it's area. We know through def of a rectangle (w/ 4 right angles) that opposite sides must be parallel. If we sheared $\Delta AED$ on the parallel line $\overline{BC}$ to point F, we would have $\Delta AFD$. Since we can shear this, we know the area's of both triangles are the same. | $\square ABCD$ is a rectangle, so it's also a parallelogram and $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$. Therefore, the distance btwn. these lines is constant and the heights of $\Delta AED$ and $\Delta AFD$ are equal. They share base AD, so their areas $= \frac{1}{2}bh$ are also equal. |

*Figure 5.* Sample solutions to Item G6(b) shown in Figure 2.

Student A used Cavalieri's principle to compare the area of triangles AED and AFD. Whereas there were no students who used this particular principle on the pre-tests in 2007 or 2008, on the post-tests there were 45 students altogether who used this principle to justify successfully their response that the area of the two triangles must be equal. This type of response is interesting because Cavalieri's principle is not typically taught in high school geometry, and it is clear that these students did not know this principle at the start of MATH 2; but by the end of the course about 10% of the samples were able to use a newly-learned abstract principle in a problem context and to construct a well-argued explanation for their responses.

Student B used the formula A=b*h/2 for the area of a triangle to argue that if the bases and heights of two triangles are equal, then the two areas must be equal. Whereas on the pre-tests there were only 5 students who invoked this particular argument with some success (all scored 3 points), on the post-tests there were 67 students altogether who used this argument to justify successfully (scored 3 or 4) their response. Students typically enter MATH 2 familiar with the area formula for triangles. Responses like Student B's are interesting because they show that what students seem to have learned in the course is not new content but rather how to apply what they already knew to a new problem situation involving the construction of an explanation or justification. In both cases, these students seemed to understand that areas can be compared without calculations or measurement—one can use properties of shapes and general principles to determine areas. More generally, solutions such as those of Students A and B show how pre-service elementary teachers can use formulas and general principles to understand relationships, contrast shapes and even make arguments about relationships among mathematical objects.

## Discussion

This study investigated the knowledge of geometry and measurement that pre-service teachers displayed at the beginning of a mathematics course addressing those topics, and to what extent and in what ways did pre-service teachers' knowledge of geometry and measurement change by the end of the course. The overall scores on the TNE Geometry Test show that consistent with results reported by Mayberry (1983) the pre-service teachers in this study began their course with weak knowledge of geometry, with mean scores of about 9 of 29 points on a pretest about properties of polygons and polyhedra, the meaning of measurement units, perimeter, area and volume. These results were stable across two semesters. The post-test scores show that they learned quite a bit about geometry and measurement in MATH 2 with average scores on the same items on the post-test increasing to about 18 - 20 points. However, when examining specific items we see that some topics explicitly taught in the course were learned successfully, but that there were other concepts that many students continued to find difficult.

Item G4 about the size of the unit and the resulting measure of area addresses a fundamental concept of area that shows up in many elementary curricula. This item was quite difficult even at the end of the semester with slightly more than half the sample getting it correct in 2007 and slightly less than half successful in 2008. It is not clear how much the complexity of the item, involving three different measurement units, contributed to its difficulty. But this item taps a concept that is fundamental to understanding area. It is therefore important for researchers and teacher educators to figure out why this concept is so difficult and how to help pre-service teachers understand it.

Item G6 assesses reasoning about areas of triangles that is typical of mathematics included in some state standards for Grades 7 or 8. Results show that during this study, the percent of students who were successful on this item increased between the pre-test to post-test from about 30% to 75% on part (a) and from about 1% to 24% on part (b). Thus, after completing a whole semester of a geometry course that places considerable emphasis on explaining your thinking, the majority of preservice teachers made excellent progress on recognizing that two areas were equal, but continued to have difficulties explaining and justifying their responses. Most had no trouble recalling the area formula for triangles, but applying the formula and properties of quadrilaterals to construct an argument about the equivalence of the area of the two triangles was difficult for many students. However, it is heartening to see that other students, about 15% of the sample, not only were able to learn a new mathematical principle, Cavalieri's Principle, but they were also able to apply it to write an argument about areas of triangles. Thus, Beckmann's (2002) proposition that explaining mathematics will help future teachers to enhance their own understanding of mathematics and their own reasoning abilities is somewhat supported by our research. However, for the majority of students in our sample, the development of strong reasoning abilities is still an elusive goal.

This study was undertaken with the joint goals of contributing to the research on knowledge for teaching mathematics and of investigating what prospective teachers in the elementary teacher certification program at our university learn about mathematics for teaching from the required courses in mathematics and mathematics education. To the former, we have contributed items about geometry and measurement that are lacking in the literature, and rubrics for scoring those items that can be used reliably by other scholars. Starting with the current semester, Spring 2009, we have begun to share the results of this work, with faculty and teaching assistants who are currently teaching MATH 2, with the belief that knowing about pre-service teachers' performance in previous years will engage instructors in conversations about ways to improve teaching and learning.

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# FROM "THESE" CHILDREN TO "MY" CHILDREN: SHIFTS IN DISCOURSE ABOUT THE NEEDS OF CHILDREN FROM POOR COMMUNITIES

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*In this work we examined the impact of a 10 weeks long experience of one-on-one work with children from disadvantaged communities at a learning center on a cohort of thirty middle school mathematics teacher candidates' views about children's learning and teaching matters. Data collected through reflective journals indicated that personal and sustained experience of working with children allowed the teacher candidates to shift their focus from identifying what they perceived to be children's academic and social "deficiencies" to importance of making mathematics meaningful to children. The teachers however, continued to overestimate their ability to influence children's academic and social growth.*

## Introduction

The importance of designing educational experiences that prepare teachers to work productively with children from economically disadvantaged communities has been voiced and recognized by teacher educators in various areas for quite some time (Mason, 1997). In mathematics education, the NCTM's call for making mathematics accessible to *all* children, including those in poor communities, has generated concerns about how teacher candidates might be assisted to develop necessary skills and dispositions to so. There is certainly legitimate ground for such concern since research findings continue to suggest that teachers maintain negative attitudes towards low income, culturally under-represented student groups, assuming them incapable of academic success (Garcia, 2004). Though not widely explored in mathematics education, research on teacher candidates' beliefs confirm that they have lower expectations for economically disadvantaged students' academic performance and undermine the role of ability in explaining their academic success when they occur (Tiezzi & Corss, 1997). Challenging such views is a must if the goal of empowering all children to achieve mathematics is to be accomplished.

The overarching goal of the study we report here was to investigate the impact of a 10 weeks long field experience which required teacher candidates to work, in an after-school program, with individual and small groups of children from urban schools on teachers' conceptions about the academic needs and intellectual capacity of children from disadvantaged economical backgrounds. Of particular interest to us was documenting what teacher candidates viewed as important for children to know, ways in which they examined children's mathematical thinking skills, as well as orientations they adopted in their interactions with children. More specifically, we aimed to document and analyze how the experience of sustained one-on-one work with children from poor communities impacted their conceptualization of the children's needs and their mathematical knowledge.

## Literature Review and Conceptual Framework

In recent years one of the foci of attention of research in mathematics education has been on understanding the nature of professional knowledge base and beliefs of future teachers. As some researchers have studied the substance of knowledge of mathematics teacher candidates as it relates to specific pieces of content, others have explored their beliefs and perceptions of teaching and the subject matter (Phillip, 2008). The body of research suggests that teacher candidates enter teacher education programs with well-established teacher role identities (Bush, 1986) and strong convictions about teaching and intentions on how to teach (Phillip, 2008). Their beliefs about effective teaching and learning can be strongly held and appear to be resistant to change (Pajares, 1992). These beliefs strongly influence the extent of teacher candidates' willingness and ability to learn while in teacher education programs. In studying the source of teacher candidates' beliefs about mathematics teaching and learning, researchers have noted that those beliefs, for the most part, are formed in contexts dating back to teachers' own schooling years (Phillips, 2008). Families and social communities provide additional contexts in which teachers form their beliefs and perceptions about not only teaching but also about the impact of race, culture and socio-economic status on academic success and ability (Ladson-Billings, 1997). Bush (1986) labeled the combined impact of these forces as "enculturation." Several researchers have documented that pre-service teachers about to begin student teaching expected teaching tasks to be less problematic for themselves than for others (Weinstein, 1988). These scholars have suggested that pre-service teachers may have an unrealistic optimism about their future teaching performance, and that this optimism may be associated with a lack of motivation to become seriously engaged in critical examination of their own knowledge base for and beliefs about teaching. Assuming beliefs to be context-bound *raises concerns about preparing teachers to teach in contexts drastically different from their own culture, race, ethnic background and social class background* (Tiezzi & Cross 1997, p.113-114). Teaching force in the US consists primarily of white, middle class females whose goal is to pursue teaching appointments in settings similar to where they were raised (Gutierrez, 1999). Combined with knowledge that teachers' stereotypes about children are sufficient to impact student performance (Oaks, 1990) and that many teachers have negative attitudes about individuals from cultures different from their own (Irvine & York, 1993 cited in Gutierrez) heightens the need to design experiences that raise teacher candidates' awareness of the intellectual strengths and capabilities of children from poor communities and to challenge their perceptions about what this student population is capable to accomplish. In mathematics education this need is paramount since national and international data on student achievement in mathematics continue to indicate that children from low socio-economic backgrounds perform at significantly lower levels compared to their white, middle class peers (NCTM, 2008). While it is naïve to ignore the non-instructional factors that both implicitly and explicitly impact student performance, the significant role of the teachers on advancing student learning even in the presence of non-instructional elements cannot be contested (Ladson-Billings, 1997) and one which is of concern in teacher preparation. Our study aimed to explore potential for creating change in teacher thinking about mathematical ability of children from disadvantaged communities. We conjectured that an intensely supervised experience in which teachers worked with individual and small groups of children would allow them to focus on instructional factors that could assist in advancing children's mathematical thinking.

Our design of the experience was informed by three perspectives guiding current thinking on teacher preparation. These include: The use of concrete contexts for learning about the profession (Brown et al, 1989), the value of sustained reflective practice on development of professional knowledge (Schon, 1987), focus on children's cognition as a means to advance teacher thinking (Fuson et al, 1999).

The notion of "contextual learning" accompanied with reflection proposes that teachers' engagement in intentional efforts to think about their experiences potentially create cognitive change in both their beliefs and practices (Schon, 1987; Tobin, 1990). Additionally, as Thompson and Thompson (1996) proposed we espouse the teachers gain knowledge about teaching and mathematics through sustained and reflective work with students and mathematical ideas, analyzing students' work, their own teaching, and reflecting on what they intended as well as what they achieved.

## Setting

The goal of our research was to collect data on mathematics teacher candidates' perceptions about teaching children from poor communities as they worked with individual and small groups of children in an after-school program for a period of 10 weeks. During this experience teacher candidates worked with approximately 40 children ranging in age from 6 to 16 years. All children were of African American heritage and from low socioeconomic background. The after-school program was housed in the learning center on the university campus and provided free tutorial and assistance to community members. The children were transported directly from their schools to the learning center. Each child was assisted approximately three hours a week on two different days (1.5 hours each day). Each child worked with the same teacher candidate for the entire 10 weeks. Hence, the number of hours of contact with each child by each teacher candidate was approximately 30 hours. With the exception of three, each teacher candidate was assigned two specific children, based on schedule and grade level.

*Participants*

The participants were 30 teacher candidates pursuing an undergraduate degree in middle level mathematics teaching. Nine of the participants were male and 21 were females. All participants were Caucasian and had completed at least one field experience in the previous quarters. They had also completed a minimum of 25 credit hours of mathematics coursework. Lastly, they all had completed 12 credit hours of coursework in general education. One of the education courses completed by students focused on global multicultural issues.

At the time of data collection, the participants were enrolled in a course titled, "Field Experiences in Mathematics Teaching." Traditionally, this course required the teacher candidates to observe teachers of their own choice once a week for 8 weeks. An examination of past data indicated that a majority of the teacher candidates enrolled in the course had opted to complete the required field experience in their own hometown, frequently observing a former teacher of their own. The quarter during which the study took place was the first attempt at modifying the field experience requirement so to assure teacher candidates gained a different learning experience, moving away from their own comfort zone, focusing their attention on children's cognition.

In addition to the 3 hours of work at the learning center, the participants attended course sessions on campus. During these sessions, the instructor of the course granted teachers the time to talk about their experiences, ask questions about teaching techniques or resources they could use with children, and discuss course readings which linked closely to teaching specific content pieces at different grade levels. The articles were selected from professional journals published by the National Council of Teachers of Mathematics. Additionally, during the campus sessions the course instructor used samples of children's work on specific problems the teacher candidates had used in the after school program to comment on the mathematical ideas that children seemed to be grasping, problem solving strategies they had used and reasons that could have contributed to their choice of representations and/or solutions to tasks. These discussions were framed to capitalize on approaches that could help or hinder children's progress.

*Data Collection and Analysis*

The teachers' weekly reflective journals served as primarily data sources for the study. While these journals were augmented with field notes on on-campus course session discussions, and observational data completed on site (learning center), in discussing the impact of the experience we are relaying solely on the written documents submitted by teacher candidates. Data was collected with the intent to trace a trajectory of the impact of the experience on participants' to cognitive needs of children and ways in which their socio-economic background influenced these views. Since no previous reports on such an experience in mathematics teacher preparation were published our research was of exploratory nature. Indeed, in coding the data the following categories guided our analysis: (a) experiences influential in candidates' thinking when assessing children's work and ability; (b) challenges and tensions that the candidates were experiencing in their work with children; and (c) frames that teacher candidates used when analyzing and interpreting children's work.

## Results

Throughout the 10 weeks long field experience each of the teacher candidates submitted 8 reflective journals on their work at the center (n=240), 3 progress reports on each of the children with whom they worked and one final report on their experience working with children and what they extracted about teaching and learning through their work. In their reflective journals teacher candidates documented their impressions of the children's needs, procedures they had used in detecting these needs, issues they struggled with in their work with children, as well as areas in which they felt children and they themselves were making progress. Additionally, they were asked to comment on teaching approaches and problem types they were using and whether they found children responsive to these tasks. Children's progress reports had a specific structure. In each report the teacher candidates identified specific mathematical areas they had worked with children each week, level of progress in those areas as well as plans for their subsequent work in the follow up sessions. Lastly, the final report was structured so to provide the teacher candidates with an opportunity to assess not only the experience of working with children but also their overall assessment of their own growth and development as teachers. In analyzing the data, we divided the reflective journals into two groups reflecting teachers' ideas during the first three weeks of experience (Phase I) and the last three weeks of experience (phase II). Constructing such a phase analysis was essential to study the impact of two particular interventions: (1) Modeled teaching by the field experience supervisor, (2) implementation of specific problem solving tasks to be used as a means to assess children's mathematical thinking.

*Assessment of Learners and Teaching*

Tables I and II summarize a typology of teacher candidates' comments as well as the frequency of each comment type that they made about teaching and children as reflected in their first three journals (Phase I) and last three reflective journals (Phase II). During the first phase, despite the instance of the course instructor that the candidates must rely on multiple sources to determine learners' facility with different mathematical concepts they relied primarily on on-line diagnostic tests which measured (in a narrow way) children's mastery of basic skills. These instruments failed to capture the abilities of children in a meaningful way and in problem solving contexts. Hence, the candidates' perception of children's needs was shaped by these results. In places where children offered correct solutions to questions using non-traditional strategies the merit of these approaches were dismissed. The children's authentic approaches were labeled as "inefficient" or "inadequate" to help them launch answers on tests "quickly." This assessment was universal among the candidates even in places where children were enrolled in advanced or honors courses (middle level and high school children).

In addressing these perceived procedural gaps, the candidates eagerly relied on using various games and puzzles to engage children in learning however, they failed to connect the content of these games and puzzles to mathematical competencies they had hoped to address. Therefore, due to the disconnect between what they were asked to do during their time with candidates, children became less and less engaged in tasks, less responsive to teacher candidates' expectations and "demands," and at time refused to do what they were assigned. These results directly confronted teacher candidates' initial optimism towards their ability to teach. However, rather than a retrospective analysis of children's behaviors, they "blamed" their home cultures and backgrounds to their lack of interest. Reflective journals submitted during this phase were indicative of the candidates' resistance to examine their own actions. Rather than exploring the impact of instructional choices they had made on children's behaviors they assumed children's own lack of motivation to learn as primary issue they struggled with. Indeed, on several occasions when teacher candidates felt they had made progress with a child at the end of a session (the child had successfully imitated the process the teacher had modeled through the use of several examples) they were disappointed that during the subsequent session the child had exhibited difficulty performing the procedures again. In explaining this phenomenon the teacher candidates referenced lack of support and/or reinforcement at as the primary reason for children's "failure." The following statements are typical of the type of comments teacher candidates made in the reflective journals during this phase.

> *When she left last week I knew she understood borrowing, she showed me she could subtract two digit numbers from three digit numbers. She came and we were back on the same step as we were last week. I think that had she practiced more at home this would not have happened. I think as teachers we need to be able to rely on families to get kids to their work. I know these kids are not getting that at home. (S1)*
>
> *I feel so sad for these kids, they come here and we help them but then they go home and I am guessing there is no one there to help them. (S2)*

These results, while disheartening, were not unexpected. Previous studies had documented similar findings during the initial exposure to children of similar backgrounds among teacher candidates (Haberman, 1996).

Table 1. *Teacher candidates' comments about children (Numbers reflect the total number of times statements had occurred on the first three journal entries. Numbers larger than 90 indicate that in the same journal a remake may have been repeated more than once by the teacher candidate)*

| Comments about Children | | | | | | | |
|---|---|---|---|---|---|---|---|
| | It is sad what these kids can't do | Not sure if they get support at home | It is sad how little they know | They are capable but problems are boring | They are not getting the support they need at home | They don't want to do their homework, they probably are not getting reinforcement at home | They don't sit still long enough to do the worksheets |
| Phase I (First four journals) | 120 | 56 | 84 | 12 | 75 | 65 | 62 |
| | | | | | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Phase II (Last for journals) | 34 | 42 | 32 | 48 | 29 | 28 | 10 |

During the first phase the candidates primarily commented on what learners were unable to do, referencing them as "these children," using a passive voice in identifying the impact that their own actions or choices could have had on children's learning. A prominent portion of the reflective journals concerned their perceived notions of children's home life, assuming their families to be less supportive of academic pursuit. Indeed, in nearly 28 of the 30 cases children's lack of mastery of basic skills was attributed to lack of support at home and their families low expectations for success. These assumptions were not supported with data and mirrored the teachers' pre-conceived notions of what children experienced at home and value attached to education by their guardians and parents. Additionally, the candidates were unable to understand the children's lack of interest in completing drill and practice exercises that were either assigned at school or asked to do at the center by them. Almost each of the teacher candidates made comments reflecting greater concerns for children's social behaviors and habits of children than their mathematical thinking. They seemed unwilling or unable to go beyond what they perceived as "inappropriate behavior" (walking, not sitting down while working on problems, or not completing work as they were told by the teacher candidates) and to analyze differences due to cultural practices (Ladson-Billings, 1997).

Table 2. *Teacher candidates' comments about teaching and their expectations ((Numbers reflect the total number of times statements had occurred on the first three journal entries. Numbers larger than 90 indicate that in the same journal a remake may have been repeated more than once by the teacher candidate)*

| Comments about Teaching and expectations for children | | | | | | | |
|---|---|---|---|---|---|---|---|
| | My job is to make sure they feel good while they are with me | I don't want to push them too much | Teaching **these** kids is hard | Teaching is harder than I thought | I need to learn more about how to teach (CONCEPT) | If I could find a way to connect their estimation skills to algorithms | I am surprised at their problem solving skills |
| Phase I (First four journals) | 76 | 65 | 45 | 3 | 7 | 0 | 12 |
| Phase II (Last for journals) | 23 | 12 | 12 | 44 | 46 | 43 | 47 |

*From "**These**" Children to "**My**" Children*

During the second phase however, frequency of comments that indicated some awareness of the need to learn alternate ways to effectively work with children increased significantly. Having developed a personal knowledge of each child due to extended weekly interactions with them, allowed for the development a personal bond that motivated greater investment in children's success on their part. This personal bond appeared to have also made them conscious of the impact of their own choices on what children did or gained from their time at the center. Although the teacher candidates continued to make unsupported claims about what they perceived to be the conditions of children' home lives, the frequency of their comments decreased significantly. Additionally, while they seemed to overestimate their own influence on children. This is consistent with the findings of Tiezzi and Cross (1997) as they reported that in their reflections over the experience of observing teaching in an Urban setting the teacher candidates offered idealized notion of teaching children in such settings, believing they could "help." Unlike their findings though, it seemed that the desire to be helpful (on an affective way) facilitated the development a disposition to focus on children's thinking and cognition as a means to achieve this goal. The following comments were typical of statements teacher candidates made in their journals during this phase.

> *I know my kids are doing better, I know they can do just as well as others. I feel I have made a lot of progress with them. They are happier when they are with me here and I know I am more at east working with them. (S1)*

Indeed, an examination of progress reports written for children also indicated a shift in quality of teacher candidates' assessment of children's work over time. Whereas in the first 3 progress reports a majority of teachers' comments concerned what children were unable to do; highlighting their lack of proficiency in some basic procedural domains; the last two reports manifested a deeper appreciation for children's problem solving skills and their resourcefulness in solving problems in authentic ways on their parts.

*An Analysis of Major Influences*

While the depth of analysis of teacher candidates remained relatively native, the shifts in their foci are particularly important and promising. These shifts were due to two major interventions implemented starting the fourth week of experience: (1) on site modeling by the supervisor, (2) assignment of specific problem solving tasks on which teacher candidates were asked to implement at the center.

The field experience supervisor attended the learning center three times a week for approximately 3 hours each for 5 weeks. During the time at the center, she held small group problem solving sessions with children while teacher candidates observed her interact with kids and her teaching. In organizing the sessions, she deliberately chose to work with those children whose tutors had diagnosed as having learning problems due to their lack of responsiveness during the sessions. During these sessions the supervisor presented both the children and the teacher candidates with the same mathematical problem as they sat at the same table using identical resources and concrete materials. As children worked on the task, the supervisor asked them to explain their thinking and repeatedly inquired whether the teacher candidates understood what concepts they were using. On several occasions, teacher candidates were puzzled by the fact that the children could solve problems in ways that they themselves were unable to grasp. Indeed to assure continuity in challenging the teacher candidates' views about children's ability, the supervisor assigned specific problems to be administered to children. Many of these tasks involved algebraic reasoning, guessing and estimation skills. Classroom discussions then focused on analysis of methods children had used and ways in which these methods could be connected to different topics in school curriculum.

## Epilogue

Teacher education programs have traditionally utilized an urban field experience prior to student teaching in teacher preparation Mason (1997) assuming that such an enculturation allows for development of positive professional insight and heighted awareness about children in urban schools. Results of research on the impact of this approach on teacher candidates' beliefs however have been inconclusive. In fact, several scholars have questioned the effectiveness of such experiences suggesting that these exposures can reinforce, rather than challenge, negative attitudes towards students from low Socio-economic backgrounds (Haberman, 1996). After an extensive review of literature, Mason (1997) concluded that the *overall research fails to provide evidence supporting the value of field experiences on teacher candidates' conceptions about teaching and learning in poor communities of the abilities of children in achieving success in such settings* (p. 120). He punctuated the need for identifying elements of supervised experiences that are necessary to provide positive learning opportunities for teacher candidates.

In the current study, rather than placing teacher candidates in urban schools as observers of teaching, we organized a quarter long supervised tutorial experience during which they worked one on one with children from disadvantaged communities. Our goal was to determine whether such an experience would assist the future teachers in learning about the potential of children for learning and their capacity for problem solving. This experience, supported with extensive supervision and teacher modeling proved to be a useful vehicle for engaging teacher candidates in learning to focus attention to children's strengths as opposed to their shortcomings in procedural domains. Such a content focused supervision impacted teachers candidates' professes assessment of their own professional needs as reflected in their journals.

While it would be naïve to make claims to long term impact of this experience on teacher candidates' beliefs and views about teaching and learning in poor communities, results of our exploratory research provide some evidence that such an experience can be valuable when carefully supervised by a content specific specialist who can help them make sense of mathematical merit of children work and introduce them to techniques and tasks they could immediately test and verify. This approach is consistent with current recommendations for using children's cognition as a springboard for advancing teacher development. Providing teachers opportunities to establish personal relationships with children of different culture and socio-economic background might serve as a good starting place for advancing their professional growth.

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# RELATIONSHIPS BETWEEN CONTENT KNOWLEDGE, AUTHORITY, TEACHING PRACTICE, AND REFLECTION

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*This paper presents a description of the ways that elementary teachers' mathematical content knowledge and locus of authority are related to their instructional practices and their reflections on their teaching. The cases of four teachers are used to illustrate the four possible combinations of content knowledge (high/low) and authority (internal/external). Although four cases are presented as exemplars, teachers' practices were not consistent across time and setting, and the changes were not necessarily reflective of "growth" toward more reform-oriented teaching. Data were drawn from a 5-year study in which two cohorts of elementary school teachers were followed from their junior year in college through their first two years of teaching.*

## Background

In my previous research, I identified connections between the locus of authority (internal or external) from the viewpoint of preservice teachers and their propensity to think reflectively about pedagogical dilemmas (Mewborn, 1999). I proposed that when preservice teachers see themselves as having the authority to raise and solve pedagogical dilemmas, they are more likely to think reflectively. In contrast, when preservice teaches see authority as external to them (residing in a teacher educator, an experienced classroom teacher, or in a textbook), they are inclined not to think reflectively about pedagogical dilemmas. In the study reported here, I have extended this connection to include the way that authority and mathematical content knowledge are interconnected in shaping teaching practice and reflection on that practice. In particular, I describe four possible interactions between authority (internal/external) and mathematical content knowledge (high/low) and associate each with a particular approach to teaching mathematics and reflecting on that teaching.

## Theoretical Perspective

This project is situated within the interpretive paradigm for teacher socialization (Zeichner & Gore, 1990). This interpretive approach involves an attempt to understand the nature of a social setting at the level of subjective experience. The purpose of this approach is to gain an understanding of the situation from the perspective of the participants and within their levels of consciousness and subjectivity. The goal is to "capture and share the understanding that participants in an educational encounter have of what they are teaching and learning" (Kilpatrick, 1988, p. 98). Eisenhart (1988) noted that the purpose of the research questions posed by researchers using the interpretive paradigm is to first describe what is "going on" and second to uncover the "intersubjective meanings" (p. 103) that undergird what is going on in order to make them reasonable.

## Methods

The data reported in this manuscript were collected as part of a 5-year research project entitled Learning to Teach Elementary Mathematics[8] in which two cohorts of preservice teachers were studied for two years of their preservice program and their first two years of teaching. The overall goal of the project was to develop conceptual frameworks for understanding teaching and learning in elementary mathematics teacher education by studying how novice teachers craft their teaching practices across time as a result of their personal theories, teaching experiences, and teacher-education programs.

The participants for the project were selected from two cohort groups who began the four-semester teacher education program in the fall of 2000 (Group A) and 2001 (Group B) and remained together as a cohort for the four semesters. Some data were collected on all students from each cohort, but the majority of data collection focused on two target subsets—6 students from Group A and 9 students from Group B. The target students were selected by purposeful sampling (Bogdan & Biklen, 1992) to represent a range of personal theories about mathematics. And to the extent possible, the target students were reflective of the diversity of students enrolled in each cohort. The 15 target students consisted of 13 White women, 1 White man, and 1 African American man. Although the racial and gender composition of the targeted students was fairly homogeneous, there was considerable diversity in their experiences with mathematics and their personal theories about the teaching and learning of mathematics.

The data set includes a mathematics beliefs survey (Ambrose, Phillip, Chauvot, & Clement, 2003), a content knowledge assessment (Hill & Ball, 2004), and all written work produced by the students during their two mathematics methods courses. In addition, the target students were observed four times teaching a mathematics lesson and individually interviewed on four occasions during their preservice years. The observations occurred during field experiences in the second and third semesters of the program (one observation each) and during their student teaching experience (two observations). The four interviews were semi-structured and took place at the end of every semester of the teacher education program. The first interview was conducted near the end of the first mathematics methods course and focused on eliciting autobiographical data from the participants regarding their views of mathematics and their prior experiences as a mathematics learner. They were also asked to reflect on their experiences working one-on-one with a student in mathematics during the methods class. The second and third interviews occurred at the end of the second and third semesters of the teacher education program and asked participants to reflect on their practicum field experiences and how these experiences differed (or not) from their one-on-one teaching experience and from what they had learned in their mathematics methods course. The fourth interview, conducted at the conclusion of student teaching, focused on their reflections about their various teaching experiences during their teacher education program and the changes (if any) in their conceptions about mathematics and mathematics teaching and learning from the beginning to the end of their teacher education program. Most of the target students (those teaching within reasonable driving distance) were observed teaching mathematics monthly and interviewed twice per year during their first 2 years of teaching.

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All data were transcribed and organized for coding purposes, and the research team defined an initial set of 25 codes. Line-by-line coding of data took place chronologically for all 15 target participants with different researchers coding data for different participants and then writing summaries, called "data stories." These chronological and parallel data stories facilitated comparison and contrast among the participants. Finally, the data stories and original data were recoded across participants.

## Results

Figure 1 depicts the hypothesized relationship between content knowledge and view of authority and the resulting nature of both lessons and post-lesson reflections. Following Figure 1 I describe a teacher who typifies each cell and give an example of a lesson she taught that depicts the type of teaching and reflecting that is suggested by the category.

| | | Content Knowledge | |
|---|---|---|---|
| | | High | Low |
| Authority | Internal | Lessons are student-centered, mathematically rich, developmentally appropriate, and teacher is reflective about self, students, and content after the lesson. | Lessons focus on fun activities with some mathematical substance, but they sometimes fail due to the preservice teachers' lack of mathematical foresight or inability to "think on the fly." Teacher is reflective after the lesson, but reflections center mostly on self and teaching actions. |
| | External | Lessons consist of attempts to provide clear and concise procedural explanations. Preservice teachers assume that this is how children learn mathematics best and that improving their teaching is merely a matter of giving better/clearer explanations. Little reflection after the lesson. | Preservice teachers try to provide lessons that make math fun and easy in order to spare students the agony of learning. The content of the lesson is often unclear and sometimes mathematically inaccurate or unimportant. There is little reflection on the lesson beyond whether or not the students enjoyed it. |

*Figure 1.* Relationship between content knowledge, authority, teaching practice, and reflection.

*Jayne*

Jayne provides an example of a teacher with high mathematics content knowledge and an internal locus of authority. She described herself as favoring language arts and mathematics, with a particular affinity for algebra. She also described herself academically as "creative," "original," "independent," "wanting to stand out from the crowd," and "not afraid to ask for help." Her scores on the Learning to Teach Mathematics instrument placed her in the top 10% of all students participating in the study. Jayne had an internal locus of authority and wanted to help her students develop a similar sense of independence. Even as a preservice teacher, her view of textbooks provided a good example of her internal locus of authority: "I feel that [the textbook] gives a lot of good suggestions and activities for teachers to pick and choose from. Some activities just need a little adjusting." When asked how she planned her lessons, she noted, "I went through them [the curriculum materials] and saw what I thought. I went through and picked what I like and what I didn't like and some stuff I thought was a good idea and just kind of

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modified it to what I thought my classes need." As a classroom teacher Jayne challenged her principal's decision to ability-group students across classes for a portion of mathematics instruction every day to help students with weak procedural knowledge prepare for criterion-referenced testing. Jayne participated as required but continued to state her case, and, after test results came back and Jayne's students had the highest scores in the building, her principal relented and allowed her to manage mathematics instruction for her students for the full period.

A lesson that typifies Jayne's teaching style and that of teachers fitting in the first cell of Figure 1 involved introducing first-graders to addition sentences. Jayne made the lesson engaging by using a song about frogs sitting on a log being joined by others. She allowed the students to pick numbers and to act out the problems, and then she introduced the corresponding notation. At one point a student proposed to act out a story that corresponded to 10 – 0. Jayne asked him to state his story as a number sentence, which he did correctly, and then asked him if it was an addition sentence. The child acknowledged that it was not. Jayne then asked him to recast his story so that it would correspond to an addition sentence, which the child did successfully. Throughout the lesson, Jayne asked questions such as "What are we going to do with the numbers?" "How does the addition work?" "What can you tell me about adding two numbers?" "What is the number sentence? Why?" "Does it make sense?"

In a post-observation conference following this lesson, Jayne was able to reflect on her teaching, students' learning, and the content of the lesson. She articulated the purpose of her lesson this way:

I wanted to see if the students could understand the concept of numbers sentences in a context, almost like a story problem. Because I have notice that my kids have problem with story problems, you know, trying to deicide if they need to add or subtract—knowing what to do with the story—where to start. I wanted them to have some basic comprehension of mathematical sentences and understand to go back and check their work, and to determine if their number sentence made sense—all the basics.

She also reflected on the seatwork portion of the lesson, noting that the individualized nature of the task allowed children to self-differentiate by selecting numbers with which they were comfortable. She also noted that she challenged particular children by asking them to choose larger numbers.

*Cynthia*

Cynthia also possessed strong content knowledge in mathematics (and in other subjects, graduating from college with perfect 4.0 grade point average). Cynthia considered herself to be "fairly good" at mathematics and attributed her success to teachers' clear explanations rather than her ability. She was often one of the first people to catch on to new material in her high school math classes and her college math class for elementary teachers, and her peers often came to her for help. If she did not understand something right away, she made a point of going to the teacher for help immediately.

However, Cynthia was a "teacher pleaser" who had an external locus of authority both as a student and as a teacher. As a college student, Cynthia would often read ahead in the syllabus and ask detailed questions about expectations for future assignments. She frequently turned in drafts of assignments early and asked if she had done it "right." During interviews for this research study, Cynthia would ask, "Am I telling you what you want to know?" "Am I giving you what you need?" Cynthia's lessons, both as a preservice and inservice teacher, were characterized by following the textbook and presenting clear, logical, well-organized explanations to students. Students did engage in activities, including hands-on activities, but these were always tightly structured and almost always procedural in nature. Cynthia's interactions with students were generally quite directive. As a preservice teacher, Cynthia offered the following response to a written case: "To break this habit of writing the problems incorrectly, I would show them many times how to write the problem and have them do many practice problems themselves because practice makes perfect!" In providing suggestions for a peer who was having difficulty teaching a child the standard addition algorithm, Cynthia wrote, "First, the child needs to learn to start every addition problem in the right-hand column, or she will never remember to add the one. Model for her the correct way to do two-digit addition problems while you are teaching her how to actually do them." These suggestions typified her teaching during her field experiences and her first two years of teaching.

During one observation Cynthia was circulating around the classroom as students worked independently. One student was confused, and she looked at his paper and said, "Write 3. Just write the number 3." Two more students had questions, and Cynthia told them what to write on the worksheet. Similarly, in a lesson on graphing in a second grade classroom, Cynthia was working with a small group of children who had just finished collecting data from their peers. She asked the group what color most people chose as their favorite. One student noted that he had a tie–the same number of people selected two colors as their favorite. Rather than asking the student what he meant by a "tie" or engaging the children in a discussion about what they should put on the worksheet if two colors were tied, she simply told the student to write the names of both colors in the blank. Later in the lesson, the worksheet contained the question, "How many people liked horses or dogs the best?" The children were not sure how to interpret the "or" part of the question. Cynthia quickly told them to count how many people picked dogs and how many people picked horses and write the total in the blank.

Cynthia's reflections on her lessons were generally confined to assessments of student behavior and her organization in preparing for and implementing the lesson. For example, after one lesson she noted that students were engaged because "I made my word problems contain aspects of the Halloween season, such as trick-or-treating, candy, and toys, because those things were what [they were] interested in at the time."

*Shelly*

Shelly provides an example of a teacher with low content knowledge and an internal locus of authority. Shelly struggled with mathematics, taking the Praxis I basic skills test several times before passing the mathematics portion. She admitted to feeling a great deal of anxiety and to having a lack of self-confidence regarding mathematics teaching and learning.

Shelly demonstrated that she had an internal locus of authority by making modifications to textbook lessons even as a preservice teacher. During a field experience where she was in a school that had adopted the Saxon curriculum, she felt comfortable going outside the scripted lesson to add her own touches. In particular, she often incorporated children's literature and hands-on activities in her lessons. She also demonstrated her internal locus of authority in her reflections on her teaching. At the conclusion of an 8-week one-on-one teaching experience she noted, "I have made some improvement through trail and error….I stopped worrying so much about planning things that were on her grade level but rather planning things based on her success during a lesson."

Although Shelly planned lessons that engaged students, her lack of mathematical knowledge often caused her lessons to collapse. She frequently started lessons with children's literature books that were only tangentially related to the topic at hand. For example, one day during her second year of teaching she had planned a lesson on graphing in late October. The lesson involved children making glyphs based on their favorite things about Halloween. Shelly began the lesson by reading a book about Halloween, but it had no connection to the mathematics of the lesson, so 15 minutes of the lesson were spent discussing ghosts, witches, and other Halloween creatures.

A more illustrative example of how Shelly's weak mathematical knowledge interfered with her lesson planning came from a lesson on even and odd numbers that she taught in her first-grade classroom during her second year of teaching. After an introduction, Shelly handed each child a piece of paper and told them to write their first name on it. She then placed a pile of cubes at each table and told the children to count out enough cubes to equal the number of letters in their first name and snap them together. Shelly then told the children to "see if you can divide your cubes into 2 groups evenly." She chose a student's cube train and said "This is what I'd do." She modeled snapping off one cube at a time and alternating placing them in one pile or another. She then asked if the result was fair, if the piles were the same (her informal definition of "even"). After giving the children time to work independently, she called the class back together and asked one child at a time to say how many letters were in their name, to show their cubes and tell whether they could put them into 2 equal groups, to declare whether their name (not the *number of letters* in their name) was odd or even, and to come to the front of the room to attach their name to a poster under "odd" or "even." She had one child snap his cubes off one at a time and place them in her hands, alternating between left and right. Then she had him count 6 in one hand and 5 in the other and asked him whether it was fair or even. The child said "yes," so, she ended up telling him it was odd when he said the amounts were not equal. After the lesson I examined a child's textbook and discovered that the book taught odd/even by having children use cubes to represent a quantity and then snap the cubes off in 2s (pairs) rather than putting them in two groups (measurement division rather than partitive). When I asked Shelly if she had already taught even and odd numbers this way she seemed completely unaware of this approach and said that she would try it because it might be easier. This is an illustration of Shelly feeling comfortable to go beyond the textbook to design her own lesson but not having the mathematical foresight to realize that she was creating problems with students' understanding by introducing a method that was contrary to the method the students had to use to do their homework. In our post-observation conference Shelly indicated that she planned to delay the test on this chapter because she did not think the students had a firm grasp on even and odd numbers.

*Tracey*

Tracey was not confident in her mathematical ability, claiming that "math has always been my least favorite subject throughout school, and I've always called it my worst subject." She said she was not good at quick computations and memorizing algorithms, and this had worked to her disadvantage in school. She hoped not to have to take any math in college but said she enjoyed her math for elementary teachers course "once I figured out what I was supposed to be doing."

I claim that Tracey had an external locus of authority for herself as a future teacher and that she saw herself as that external authority for her students. As a preservice teacher she noted that time would be a constraining factor in the classroom and that it would not be possible to hear ideas from many children. She stated that moving through the curriculum at a predetermined pace and adhering to school system requirements was a higher priority than listening to children. She also noted that sometimes what a child says is "off the wall and would only confuse mattes more or lead the topic off on a tangent."

In Tracey's first field experience, I saw evidence that she viewed herself as an authority for her students. Her lesson was very directive, and she was spoon-feeding the children to enhance their short-term success. In helping second-graders complete a two-digit subtraction problem, she asked questions such as, “Where do I start?” “What problem do I do first?” “What is 8 minus 5?” “Where do I put it?” “Am I done with this part?” “Where do I go next?” Her lesson was a string of such bite-sized questions with no “why” questions at all. For much of the lesson she was helping students create a bar graph about circus animals in order to answer word problems. She directed students to “put your finger on the yellow box. Find the line that says ‘elephants.’ Point to the number of elephants. Who can tell me how many elephants there are?” Then she told a student to come to the board and put the prescribed number of elements on the graph she had created.

Tracey’s overriding concern as a teacher seemed to be to “protect” children from the pain of doing mathematics by either breaking it down into tiny steps and asking structured question that they could answer without much thought or by doing “fun” activities with them. An example of a fun activity occurred during a lesson when she was teaching kindergarten. The goal of the lesson was for children to tell time to the hour. She began the lesson by reading a children’s literature book (*The Grouchy Ladybug* by Eric Carle) and then had the children make their own clocks in the shape of ladybugs using paper plates, construction paper, brads, and pipe cleaners. The bulk of the one-hour lesson was spent on the craft activity of the children constructing the ladybugs. After they had spent 40 or more minutes on their creations, Tracey directed the children to fill in the numbers on the clock face. She did not anticipate that this would be difficult for kindergarteners (cognitively and in terms of fine motor skills), but most of the students ended up with inaccurate representations of clock faces (starting with 1 where the 12 should be, numbers unevenly spaced, numbers going past 12, etc.). Her plan had been to have the children bring their clocks back to the reading rug while she read the book again and have them show each hour on their own clocks. As the time allotted to the lesson began to wind down, she realized that the clocks were not functional because the clock faces were not accurate and because the crafts contained too much wet glue to withstand handling, she omitted the final part of the lesson. In reflecting on the lesson later in the day, Tracey was disappointed in the lesson, but she mainly focused on the effort she had put into cutting out the ladybug wings and spots and bending construction paper to make ladybug legs and her frustration that the clocks did not turn out to be functional because of the glue issue. She did not mention the fact that the clock faces were not accurate or that the lesson involved virtually no mathematics. A similar but slightly less dramatic example comes from Tracey’s second year of teaching second grade in a lesson on “grater than” and “less than.” In an effort to make a lesson “fun,” she gave each student three colored index cards–a pink one with a less than sign on it, a green one with a greater than sign on it, and a blue one with an equal to sign on it. She put two numbers on the white board and asked the students to hold up the correct card (with the sign facing her). In planning this lesson she failed to take into account that the greater than and less than signs are really only one sign and that when the students held up the cards to face her, they would be facing the opposite of the way the children saw them at their desks. After about 10 minutes, she called a halt to the lesson because she could not tell whether the children understood greater than and less than. In reflecting on the lesson, she noted that she had tried to make the lesson more fun and interactive than a worksheet but that a worksheet was probably a better way to assess this content.

## Discussion

Although I have presented examples of four teachers who seem to fit neatly into the cells of Figure 1, the reality is that teachers fit in one cell for some lessons and in another cell for other lessons. Teachers' practices are not consistent across time and setting, and the changes are not necessarily reflective of "growth" toward more reform-oriented teaching or the upper left cell of Figure 1. For example, some teachers were very comfortable asking open-ended questions and building instruction from children's mathematical thinking in their first mathematics methods class field experience in which they worked with one child. In subsequent whole-class field experiences, however, they became more tied to the textbook and focused on correct answers. In most cases, the teachers were aware of the contrast in their instructional style across settings. The teachers attribute these differences to a variety of factors: curriculum constraints, the difficulty of orchestrating discourse in a large group, the challenge of managing children on so many different levels, and the pressure to "cover" the curriculum prior to testing. These findings support Tabachnick and Zeichner's (1984) portrayal of the socialization of teachers as a negotiated and interactive process rather than as one that is predetermined. Our challenge is to refine this claim by explicating the factors that influence this negotiation and in what ways. If we can better understand the ways in which individual teachers integrate the messages they receive from various sources to shape their teaching practice, we can develop ideas about how teacher education programs, induction year support programs, and professional development programs can best assist teachers in developing a practice that leads to rich mathematical activity in the classroom.

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# FACTORS IN THE ACHIEVEMENT OF PRESERVICE ELEMENTARY TEACHERS IN MATHEMATICS CLASSES

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*While it has been found that teacher knowledge affects mathematics student achievement, to date, little research has explored how professors affect preservice elementary teacher's mathematical knowledge. This study explores future teachers' learning in undergraduate mathematics classes. Our data includes pre and post tests from over 1000 students in classes of 41 instructors at 17 institutions in four states as well as teacher survey data from those 41 instructors. In our multilevel models, we identify three key variables that influence gain: students' prior knowledge, use of specifically designed textbooks, and the methods the professor uses to teach the course.*

## Introduction

Improving the mathematical knowledge of elementary teachers is key to improving children's mathematical knowledge (e.g., Conference Board of the Mathematical Sciences, 2001; Hill, Rowan, & Ball, 2005; Monk, 1994; Rowan, Correnti & Miller, 2002). Mathematics classes required for certification provide a unique opportunity to influence teachers' mathematical knowledge, yet there has been little research about either what is offered or what students learn in these undergraduate mathematics classes. In this study, we begin to address these questions. On average, undergraduate programs require teachers to take 2.7 mathematics courses for elementary certification and in states where there is a separate endorsement or certificate, 5.6 classes for middle school certification, up from 2.4 and 3 respectively in 2000 (Lutzer, Rodi, Kirkman, & Maxwell, 2007). Although the number of mathematics classes required of these future teachers has been increasing over the last decade, we know almost nothing about the content of these courses, who teaches them, or what their impact is on future teachers' mathematical knowledge. The goal of our research is to understand what these future teachers learn and what accounts for differences in learning across instructors at various institutions. Specifically, we ask what characteristics of students, courses, instructors, and institutions explain variation in achievement across mathematics classes for future teachers that are focused on number and operation?

This work is important because these undergraduate classes are a unique sustained opportunity to influence teachers' mathematical knowledge. If we could learn more about how to make these classes better – to have a greater impact on teachers' knowledge – through changing the content, the textbook, the way classes are taught, or other variables, our findings could have an enormous impact on the preparation of elementary teachers for teaching mathematics

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## Background

To investigate this question, we designed a multilevel study at undergraduate institutions in 4 states. We collected data from future teachers (students in undergraduate mathematics classes), instructors, and mathematics department chairs. We developed hypotheses about what variables might predict student learning and tested these hypotheses using multilevel analyses. The complete study is described in detail in other documents including McCrory (2009).

*Literature Review*

The National Math Panel (NMP) report, (see p. 5-1) which summarizes research on teacher knowledge, suggests that the effect of teacher quality on student achievement is large. In one study, 12-14% of the variation in student achievement was attributed to differences in teachers. In another study, the difference in outcomes for students of the worst and best teachers was 10 percentile points on a mathematics assessment. Yet, the NMP report points out that understanding the individual differences that constitute teacher quality remains elusive. One likely candidate is teacher knowledge of mathematics, but there, the research is inconclusive.

Studies of teacher knowledge have typically relied on proxy data (number of math classes, certification status, years of experience, test scores such as SAT or ACT) to investigate the question of what mathematics teachers bring to their teaching. One exception is the work of Ball and colleagues at the University of Michigan (e.g., Hill, Rowan, & Ball, 2005; Hill Schilling & Ball, 2004). In their project, Learning Mathematics for Teaching (LMT), they developed measures of elementary teachers' mathematical knowledge and went on to show that the knowledge measured by their instrument was a significant predictor of K-8 student achievement in mathematics. That is, teachers who scored higher on the LMT measures had students who scored higher on standardized mathematics achievement tests. Their work shows that there is specific mathematical knowledge that contributes to or is an indicator of teacher quality.

What we do not know is whether or how prospective teachers might learn this content. It seems unlikely that they learn it from their high school mathematics classes or from conventional college mathematics courses. If that were the case, the problem of elementary teachers' mathematical knowledge would be nonexistent or at least easy to address. Although some have argued that more required mathematics classes would improve teacher quality, research suggests otherwise: taking more mathematics courses does not necessarily result in teachers who teach mathematics more effectively. Wilson, Floden and Ferrini-Mundy (2002) point out in their report on teacher preparation that studies about subject matter preparation "undermine the certainty often expressed about the strong link between college study of a subject matter and teacher quality" (p. 191). What is missing is a better understanding of the content that matters and how best to offer it to future teachers.

*Purpose of this Study*

This study begins to address what affects prospective teachers' mathematical content knowledge in their mathematics courses. In particular we investigated a number of factors that we hypothesized might influence their mathematical achievement, measured with the LMT items that we know correlate with teacher quality. Factors include the textbook used, the instructor's attitude toward the class and experience teaching the class; and how the textbook was used; and methods of teaching. Other factors, related to the students themselves, are prior knowledge, socio-economic status, and attitude toward mathematics.

## Method

*Population*

Data for this study were collected at institutions in 4 states, chosen to reflect variation in state policy and K-8 mathematics outcomes. The analysis here includes data from 41 instructors and 1706 students at 17 institutions. The data reported in this paper were collected between September 2006 and December 2008.

*Instruments and Data Collection*

Students completed pre- and post-tests using items from the LMT project. The pre and post-tests were different, equated through Item Response Theory (IRT) methods to make results comparable. We used two forms and 6 additional common items. Each student took one of the forms plus the 6 common items for the pretest and the other form for the posttest, making the pretest somewhat longer than the posttest. Thus, every student completed every item, but had completely different pre- and post-tests. The student tests also included attitudes and beliefs items and demographic questions as explained below. The pretest was administered in the first two weeks of class; the posttest in the last two weeks. LMT items are not generally available for public release, but a set of released items is available on the LMT Web site, http://sitemaker.umich.edu/lmt/measures.

Instructors completed an extensive survey developed for this project. It includes questions about content coverage, teaching methods, contextual issues, personal demographics, and more. The complete instrument is available at our Web site, http://meet.edu.msu.edu. The instructor survey was administered at the end of the semester during which student pre/post tests were administered.

*Data Analysis and Results*

The first analyses were on the test scores themselves. We used IRT parameters from the University of Michigan project to calculate pre- and posttest scores for all students. IRT scoring takes into account the difficulty of items and thus makes it possible to compare pre- and posttest scores on the same scale. The scores, however, do not correspond to percent correct and are more like z-scores. For reporting purposes, we set the average pretest IRT score to 50 with a standard deviation of 10. The range of scores is theoretically infinite, but practically, scores fall within 3 standard deviations of the average. Posttest scores are calculated using the same parameters and are placed on the same scale as the pretest. In these data, the pretest mean is 50.00, with a posttest mean of 59.16.

Table 1. *Student (Future Teacher) Descriptive Data*

| Variables | Coding and Range | Mean | SD |
|---|---|---|---|
| Pretest Score | 17 – 79 | 50.00 | 10.00 |
| Prior Knowledge (CACT = SAT or ACT on a common scale) | 12 – 36 | 23.17 | 4.38 |
| I like Math | 0 = Strongly disagree, disagree, undecided<br>1= Strongly agree or agree (Used in model)<br>On 5 point scale (used in correlations) | 0.46<br>3.03 | 0.50<br>1.26 |
| College Math Coursework | 0 = none<br>1 = 1 | 2.47 | 1.12 |

| | | | |
|---|---|---|---|
| | 2 = 2<br>3 = 3<br>4 = 4 or more | | |
| SES (Mother's Education) | 0 = no higher ed, 1 = some higher ed | 0.46 | 0.50 |

Descriptive data for students and instructors are shown in Tables 1, 2 & 3. We developed a measure of prior knowledge for students using self-reported ACT or SAT scores. We put these on a common scale (1-36) using conversions published by ETS and named that variable CACT. We used a single question from the student survey to measure their attitude toward mathematics. They ranked from 1 (Strongly disagree) to 5(Strongly agree) the statement "I like mathematics". We converted their responses to a two-point scale: 0 for a response of 1, 2 or 3; 1 for a response of 4 or 5.

Table 2. *Correlations of Student (Future Teacher) Descriptive Data*

| | 1 | 2 | 3 | 4 | 5 | Variable Explanation |
|---|---|---|---|---|---|---|
| 1 | 1.00 | **0.49 | **0.48 | -0.02 | **0.20 | Pretest Score |
| 2 | | 1.00 | **0.41 | -0.33 | **0.21 | Post test Score |
| 3 | | | 1.00 | -0.01 | **0.31 | Prior Knowledge (C_ACT) |
| 4 | | | | 1.00 | **0.13 | College Math Coursework |
| 5 | | | | | 1.00 | "I like math" (5 point scale) |

*Correlation significant at the .01 level, two- tailed.

For instructors, we asked them what textbook they used for the course. Based on their response and on the list of 13 textbooks in print written specifically for such a course (list available on the Web site http://meet.educ.msu.edu), we created a variable that had the value 1 if they use one of the textbooks on the list as their primary textbook, 0 if they use some other textbook or do not use a textbook at all. We also asked them about their interest in teaching the course before the current semester, and their interest in teaching the course again. Results of these questions are shown in Table 3, along with information about class size, and experience.

Table 3. *Instructor Descriptive Data*

| Variables | Coding and Range | Mean | SD |
|---|---|---|---|
| Primary Textbook from choice of 13 | 1 = a primary textbook on our list<br>0 = not a textbook on our list | 0.65 | |
| Class Size | 4 – 102 | 26 | 13 |
| CACT (mean SAT or ACT on a common scale) | 20 – 27 | 23 | 1.6 |
| Years College Teaching Experience | 0 – 41 | 16 | 11 |

| | | | |
|---|---|---|---|
| Interest in teaching this course | 0=no interest at all<br>1=limited interest<br>2=some interest<br>3=a great deal of interest | 1.26 | 0.59 |
| Interest in teaching this course again | 0=no interest at all<br>1=limited interest<br>2=some interest<br>3=a great deal of interest | 1.22 | 0.55 |
| Teaching Methods (Mean of 11 items) | 1.45 – 4.00 | 2.74 | 0.58 |

Another variable we developed was a measure of teaching methods. For this variable we asked instructors: “In your mathematics course, how often do your students engage in each of the following activities? Please check the box that best describes what happens in your course.” The scale was from 1 to 4: 1. Never or almost never; 2. Some lessons; 3. Most lessons; 4. Every lesson. The 11 items instructors ranked included (complete list available in McCrory, 2009):

- Explain the reasoning behind an idea
- Work on problems for which there is no immediate method of solution
- Listen to you explain terms, definitions, or mathematical ideas (Reversed)
- Listen to you explain computational procedures or methods (Reversed)

Since these are questions about what the instructor expects *students* to do, the last two on the list were reverse coded to create a scale that indicates student’s personal engagement with the mathematics as compared to listening to the instructor. At one extreme (4), students would be doing mathematics at all times. At the other extreme (1), students would be listening to the instructor at all times. As table 3 shows, the mean score on this variable was 1.81 suggesting that these instructors use a mixture of methods, leaning slightly toward student engagement. Although the dataset includes many more variables than described here, we include only those used in the models developed thus far.

To put the student and instructor data together and investigate our hypotheses, we developed a growth model using HLM (Raudenbush & Bryk, 2002). Although growth models are most often used with more than two data points to measure growth, we chose this model because it allowed us to interpret the data more completely than a two level model with either gain or posttest score as outcome, and it allowed us to use level 1 data from students who took only the pre or post test. Because we are using scores from an Item Response Theory (IRT) model, it is possible to estimate the growth model with only two data points. In this model, we define three levels. Level 1 is the growth level with time set to 0 for the pretest, 1 for the posttest. Level 2 is the student level. Level 3 is the instructor level. We do not have adequate data for an institutional level, and have not developed a state-level model (which would include only 2 states because of the sparse data in the other 2 states).

*Figure 1.* Unconditional model for allocating variance.

Level 1- Growth:

$$Y = P0 + P1*(TIME) + E$$

(TIME is 0 or 1)

Level 2 - Student:

$$P0 = B00 + R0$$

$$P1 = B10 + R1$$

Level 3 - Instructor:

$$B00 = G000 + U00$$

$$B10 = G100 + U10$$

(E, R, and U are random error)

The unconditional model is used to allocate

*Figure 2.* Model 2 with preditors at student and instructor levels.

Level 1: Growth Model
    Y = P0 + P1*(TIME) + E
    (Time is 0 or 1)
Level-2: Student Level with CACT
    P0 = B00 + R0
    P1 = B10 + B11*(CACT) + R1
Level-3: Instructor Level with Textbook and Methods
    B00 = G000 + U00
    B10 = G100 + G101(TEXTBOOK) + G102(METHODS) + U10
    B11 = G110

variance. There are no predictors in the model and what we learn from it is how much of the variance is within instructor and how much between instructors. Although we are interested generally in explaining achievement, the primary purpose of this analysis is to explain variation between instructors. This model is shown in Figure 1.

From this model we get the following data:

Mean Pretest score (B00): 50.56
Average gain (B10): 7.73
Gain Score student level variance: 3.72
Gain Score Instructor level variance: 7.80

Note that the mean score of 50.56 differs from the set mean of 50 because even in this simple model, Bayesian estimation techniques are used and error is in play. For our models, we assume that there is measurement error, and we use expected a posteriori (EAP) procedure for calculation of parameters. Variance is a pure number without units, and this tells us that if we assume there is measurement error, we have a lot of variation between instructors relative to the variance within classes.

We built a number of models to test hypotheses and found no significance for student SES, instructor experience, instructor attitude toward the class, or class size. Significant predictors include student CACT, student attitude toward mathematics, textbook used, and method of instruction. We found that CACT and student attitude were correlated, so that if both are in the model, one loses significance. We chose to use CACT rather than student attitude in our models because of the measure's greater reliability. Although we have not completely tested all hypotheses, the model that best predicts student outcomes so far is shown in Figure 2.

Results of Model 2, shown in Table 4, are:

Mean Pretest: 51.48
Average Gain, no primary textbook, average methods: 4.79
Average Gain, Add primary textbook 4.58
Average Gain, Change in methods by one point: 2.92
Average Gain, Change in CACT by one point: 0.52

Again, the mean pretest is different from 50 and also different from the unconditional model because the estimation method is recursively using the data to come up with the most likely "true" value of the mean pretest. In this model, we predict that an instructor with average methods score of 2.73 (see Table 3) who did not use one of the 13 textbooks on our list would have an average posttest score in his/her class of 51.48 + 4.79 = 56.27. If the instructor used one of the textbooks, the predicted posttest score would increase by 4.58 points; and if the instructor scored a point higher on the methods measure, the predicted score would increase by 2.92.

There are numerous issues with these models and with the results, not least of which are the set of decisions about the role of error; which parameters are fixed and which are allowed to vary by instructor (e.g., in Model 2, we did not let CACT vary by instructor, indicating an assumption that instructors are equitable with respect to prior knowledge); and which IRT parameters to use. We have tried the models with different assumptions, and although the absolute numbers are different, we consistently see that the use of one of the textbooks and the use of different teaching methods are significant – statistically and practically – at the instructor level; and the CACT score (a measure of prior knowledge) is significant at the student level. The more prepared students are for this class (as measured by SAT or ACT), the better they do on the tests. That is no surprise, but it may be a surprise that the CACT and attitude toward mathematics are equally good predictors (raising a chicken and egg question that cannot be answered with our data). We also checked the assumption that instructors are equitable with respect to CACT and found it to be correct.

Table 4. *Model 2 Results*

| | Coefficient | Error | T-Ratio | d.f. | P-Value |
|---|---|---|---|---|---|
| For INTRCPT1, P0 (pretest score) | | | | | |
| For INTRCPT2, B00 | | | | | |
| INTRCPT3, G00 | 51.48 | 0.60 | 85.82 | 37 | 0.000 |
| For POST slope, P1 (Gain score) | | | | | |
| For INTRCPT2, B10 | | | | | |
| INTRCPT3, G100 | 4.79 | 1.08 | 4.42 | 35 | 0.000 |
| TEXTBOOK, G101 | 4.58 | 1.35 | 3.40 | 35 | 0.002 |
| METHODS, G102 | 2.92 | 1.10 | 2.70 | 35 | 0.011 |
| For CACT, B11 | | | | | |
| INTRCPT3, G110 | 0.53 | 0.07 | 7.73 | 952 | 0.000 |

## Discussion

Several results stand out in this analysis. First, students' attitudes toward mathematics are very important. In particular, whether they like math or see themselves as capable of doing mathematics are both significant predictors of their achievement. This is no surprise given the history of research on attitudes toward mathematics, but it signals a possible leverage point for improving teacher knowledge. If we could make inroads into improving future teachers' confidence in their mathematical ability or their attitude about mathematics, we might be able to teach them more effectively. An alternative hypothesis is that they dislike mathematics because they are bad at it and that their self-assessment is accurate. This depressing view is not borne out by research that suggests that students' mathematical abilities are not fixed, but depend on their effort and engagement.

Second, students' prior knowledge is important. If students enter the class with a weak background, these classes generally do not make up for these deficits, although some instructors are more successful here than others. On the encouraging side, even students with weak prior knowledge are, on average, learning from these classes.

Third, students are learning a lot from these classes. A gain of 9 points is almost a full standard deviation. This is a big gain by any standards, and since we have reason (from LMT) to believe that what they are learning matters in their future teaching, this is an important finding. Fourth, students in classes of instructors who use one of the textbooks designed for a mathematics class for future elementary teachers are more successful than those in classes that do not use such a book. This may be because the books tend to treat the mathematics more coherently or appropriately than self-developed materials or other mathematics textbooks not intended for such classes. Alternatively, it may be that the textbook signifies something else about the class that matters. For example, it could be that instructors who use one of these textbooks teach in a more consistent manner across the semester, or that the department syllabus based on the textbook is better developed. There are many other possibilities that could explain this result. Without more data from different textbooks, there is little we can say to explain this further.

Finally, and perhaps most surprisingly, the teaching method matters. Teachers who report more student engagement with mathematics and less lecturing tend to have greater student achievement. This may be related to the first point in this discussion: students do better when they have a more positive attitude toward mathematics. A class in which they have an opportunity to work directly on mathematics rather than only listen to mathematics may be instrumental in developing a more positive attitude. One could argue that agreeing to participate in the ME.ET project signals a different set of beliefs or commitments to mathematics education, making the method result an artifact of the particular instructors involved in this study. But we did have considerable variation on this measure, suggesting that even if these instructors are different than nonresponders, they are also different from one another.

We are still analyzing these data, and will have a more complete report on model development. We will also develop structural equation models to try to tease out more complex relationships among variables (e.g., that the teaching methods may result in more positive attitudes that then impacts learning). In the meantime, our results suggest that there are leverage points in the courses that could be used to improve the mathematical knowledge of future elementary teachers.

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# INSIGHTS FROM MATHEMATICS TEACHERS IN URBAN/SUBURBAN CLASSROOMS: FIRST YEAR AFTER TEACHER PREPARATION

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*This study examines alternatively prepared secondary school mathematics (SSM) teachers' thinking and what they actually do to facilitate teaching and learning in their classrooms. After teacher preparation, these SSM teachers complete their first year in an urban or suburban school. A phenomenological approach captures and unfolds the SSM teachers' perspectives. Deep caring and other authentic strategies were their pro-activities that led to success among their students and provided further insights and new directions to the preparation of future alternatively prepared secondary school mathematics teachers.*

## Objectives and Purposes of the Study

The impact of teachers on the education of students in urban schools has continued to be of concern at both local and national levels (Junor Clarke, 2008; Ingersoll, 2001). In particular, a high percentage of these teachers who were going into the urban classrooms were alternatively prepared (Wilson, Floden & Ferrini-Mundy, 2001). As a faculty member in an alternative teacher preparation program at an urban institution in the southeastern United States, I continue to inquire about the experiences of our teachers as they commit themselves to teach in urban or suburban classrooms. Four out of the five schools to which a cohort of our graduates returned would be considered moderate- to high-diversity schools because they have greater than 10% ethnic minority student population (Freeman, Brookhart, & Loadman, 1999). In such environments, understanding the way these teachers think and facilitate teaching and learning becomes necessary. The objective of this inquiry is to gain insights and to think further or rethink how these teachers are being prepared for the urban classrooms.

In this study, I examine five alternatively prepared secondary school mathematics (SSM) teachers who have completed their first year in an urban or suburban school after teacher preparation at an urban research university in the southeastern United States. To understand the teachers' thinking and what they actually do to facilitate teaching and learning in their classrooms, I ask the following research question: *What are alternatively prepared secondary school mathematics teachers' perspectives of the reality in their urban/suburban classrooms?* For the purpose of this proposal, I report on two alternatively prepared SSM teachers.

## Conceptual Framework and Context of the Study

A phenomenological approach provided the framework to capture and unfold the teachers' perspectives as they continued to teach students in urban and suburban schools. Phenomenology research design allowed me to study the deep human experiences of the secondary mathematics teachers in their own classrooms and provided the flexibility to take every word that accurately depicts the rich descriptions of the teachers' experiences (Blodgett-McDeavitt, 1997; Husserl, 1970; Junor Clarke, & Thomas, 2009; Moustakas, 1994; Schwandt, 2001). This level of engagement enabled me to see data from new and naïve perspectives such that fuller, richer, and authentic descriptions can be rendered (Junor Clarke, & Thomas, 2009; Blodgett-McDeavitt, 1997) and that I make meaning based on the SSM teachers' human experiences (Patton, 2002).

*Context for the Study*

The participants of the study are secondary school mathematics teachers, who were prepared at an alternative teacher preparation program in an urban institution in the southeastern United States. They were teaching with a provisional licensure and pursuing certification and a master's degree in the program. The duration of the program was for four semesters. Upon graduation, the SSM teachers continued to teach in their urban/suburban schools.

## Methodology

The participants of this report are two women out of the study's five SSM teachers—three female and two male. The two women were assigned pseudonyms, Anna and Laura. Anna was chosen because she was the only one at a suburban school, and Laura was randomly selected from the remaining four participants who were in urban schools. While in the preparation program, they taught during the day and came to classes in the evenings. These teachers were already teaching on a provisional licensure and were in the program to gain their full certification and a master's degree. After graduation, the participants remained teaching in their schools. Class descriptions (Table 1) and the demographics (Table 2) of Anna and Laura's schools are provided below:

Table 1. *The Secondary School Mathematics (SSM) Teachers' Teaching Experience, Ethnicity, Content Taught, Grades Taught, and Class Size*

| Teacher | Experience | Ethnicity | Content Taught | Grade(s) | Average Class Size |
|---|---|---|---|---|---|
| Anna | 3 yrs | Latina | Algebra, Calculus | 10-12 | 10 |
| Laura | 4 yrs | Hispanic | Algebra | 9 | 18 |

Table 2. *The Demographics of the Secondary School Mathematics (SSM) Teachers' Schools*

| | | Demographics of Students | | | | | |
|---|---|---|---|---|---|---|---|
| Teacher | School Type | Black | White | Hispanic | Black & Latino | Latino | Asian and/or White |
| Anna | Suburban | n/a | 95% | n/a | 5% | n/a | n/a |
| Laura | Urban | 40% | n/a | 20% | n/a | n/a | 40% |

*Procedures*

A graduate research assistant (GRA) conducted the 1.5 hour-interviews using a storytelling approach to capture the SSM teachers' stories that depicted their experiences (Creswell, 1998, p. 54). Stories are defined as socially constructed accounts of past events that are important to members of an organization (Junor Clarke, & Thomas, 2009; Hansen & Kahnweiler, 1993). These accounts are seldom factual; however, they reflect what people believe should be true. They differ from gossip because they have a moral. Stories permit researchers to examine perceptions that are often filtered, denied, or not in participants' consciousness during traditional interviews. "Stories happen naturally as a way of telling one's perceptions of past events, problems, or people . . . They are easy to follow, generally entertaining and are more likely to be remembered than other forms of written or oral communications" (Hansen & Kahnweiler, 1993).

This approach allowed the SSM teachers to convey their thinking and understanding of the reality in their urban/suburban classrooms (Eisner, 1998; Mertens, 1998). The follow-up questions in the interviews explored the meaning of each teacher's lived experience (Creswell, 1998). Through the interviews, data for this study was collected. Use of storytelling allows a unique aspect of analysis with respect to story components. Stories can have main characters, motivating difficulties, heroes, villains, turning points, and morals. After the teachers told their stories, the following questions were asked: (1) Who is the main character in your story? (2) What is the motivating difficulty? (3) Who or what is the hero(s) in your story? (4) Who or what is the villain in your story? (5) What is the turning point in your story? (6) What is the story's moral? (7) Is there anything else you would like to add? In the interview, the following additional questions were asked before bringing the interview to a close: (1) What effect did your preparation have on your practice? (2) What were your experiences in the teacher preparation program that prepare you for your profession?

*Data Collection and Analyses*

The GRA informed the SSM teachers about the goals of this study and asked them to tell a story about mathematics teaching and learning. They were encouraged to include in their stories their perspectives on the impact made on their students. Their stories could be about any event that occurred within the past 6 months, or they could simply relate an incident that was of particular interest. The story should have a hero, a villain, a turning point, a moral, and anything else the SSM teachers may want to add.

The GRA transcribed the teachers' stories and I coded the data. Using the codes, I was able to dissect the stories in searching for essential structures. Data were extracted from each SSM teacher's story about the reality in their urban/suburban classrooms. The additional questions in the interview provided supplemental data for the analysis. In the data analysis, I exercised the research process, known as epoche (Holstein & Gubrium as cited in Denzin & Lincoln, 1994) in which I set aside my taken-for-granted orientation about the phenomenon. The SSM teachers' experiences that highlighted the content and pedagogy were the themes used to demonstrate the reality in their urban/suburban classrooms.

## Findings and Analyses

The SSM teachers' stories reflected on common themes that were evident in their classrooms and that had drawn their interest as being important to them at the time, such as (1) classroom culture, (2) management of students, (3) teaching content, (4) cognitive aspects of student learning, and (5) affective aspects of student learning.

*Anna*

Anna's story was based on her lived experience in a calculus class, where she changed her students' work ethics and development during one year. Her story describes how one student became upset with her after receiving the results of his test, recognizing that problems on the test were not the same like the ones Anna gave them in class. Anna was persistent with her approach, maintained the standards set, remained in charge, and helped her students. Subsequently, she was able to build a learning community where her students realized success and developed great attitudes that other teachers recognized. She established a positive reputation from her efforts not only in her classrooms but also in the school.

*Culture of the classroom.* The main character of her story was the student who helped her to make all the other students understand what she was expecting from them. The students were expecting a different teacher, not one who was tough and demanded much from them. They found that lots of work and "hard" tests became the norm in the class. Anna had high expectations for her students to do well at any cost and she was willing to pay the price. One student said to her after receiving his test, “This is not exactly what we have been doing, this is something new.” Anna responded, “Yes, it is something new! But there is some supplemental material you are supposed to get for yourself after what we have studied in class; this is part of your expectations.” She further expressed to her students that their expectations of her giving them examples of questions on the test would not happen and her expectation is that they must be able to work on problems they have not seen before. Anna began a new classroom culture where students do not have to like her but trust her. She admits that trust is important.

*Classroom management.* Anna’s class was always pre-occupied with work so there was no opportunity for negative classroom management, except for the time when the one student challenged her. There was a back and forth response on which she did not give in to the student but tried to let him understand her expectations. This confrontation was not helping her because she was trying to establish positive relationships with the students. However, Anna maintained her position on the expectations and worked on building relationships with the students all year. She sets the tone for the class through maintaining a class that is well managed and organized and has procedures and high expectations for her students.

*Content focus.* The student who challenged Anna was actually very bright, but he was apprehensive based on his previous experience of getting what he wanted, that is, samples of problems on the test. Because Anna did not accept his unwarranted needs, he responded: “Well, I think you need to start giving us examples of the questions you are going to put on the tests.” Anna made it clear to the student that, “Well, if that is your expectation then that is not going to happen. I expect you to work on problems you have not seen before on a test.” She continued to give her students problems where they could be successful but at the same time allowing them to explore and solve problems they did not see before. She insisted that students had to overcome their preconceptions about mathematics and how they learn mathematics. She maintained her support and fostered commitment of her students to mathematics all year.

*Cognitive aspect of learning.* The one student who had complained earlier in the class did a wonderful job. Other students of her calculus class would attend after school sessions to work with students from different classes. However, one day her students’ previous teacher came to Anna and said, “I want just [for] you to know that I am so amazed of what you have done with this group of students this year because they were all in the after school [session] working and even though you were not even there, they had a great attitude and even though it was a tough problem and they did not know what to do, they were discussing the math[ematics], they were arguing.”

*Affective aspect of learning.* Though the challenges from her students and her persistence for them to be accountable in the learning process, Anna recognized that over the year she and her students were able to build a learning community at the school, which she thought was a very positive and encouraging experience for their learning. She believed that she was responsible for establishing the relationship and helping students to achieve success.

*Connection to her teacher preparation.* Anna’s confidence in teaching was credited to her preparation in the teacher preparation program. One of the NCTM standards is to maintain high expectations and Anna surely made her point and followed it through to the end with her students in the suburban classrooms. She also felt that she was successful in establishing a true learning community that she experienced with her colleagues during her teacher preparation program. Her definition of a high quality teacher depicts what she did with her students.

*Laura*

Laura's story is about the development of a notebook in one of her algebra classes. This was similar to an idea from one of her colleagues in a teacher preparation class where they worked in groups. While her colleague used the idea of a toolbox, she used the idea of the notebook as a "BRAIN." When Laura tried to implement this idea at first it was "really a disaster" because she claimed that the students were overwhelmed and the idea just went for naught. "Trying to implement new ideas in the classroom and how to be successful" was the motivating difficulty in Laura's story. However, she decided to be a little more organized and more committed on her next trial in the next semester. She gave the students time during the semester like on Fridays every two weeks she gave the students 30-45 minutes for the activity. As a teacher she invested a lot of time and resources to ensure students had their work organized. Laura struggled with the students' lack of skills and the time constraints to implement the idea on a regular basis.

*Culture of the classroom.* The main character and hero of Laura's story was Ashley, whose work had impressed her. The activity had created a new culture in the classroom where Laura felt "students were talking about math," and they were sharing and using the mathematical language in asking each other: "Do you have this one about solving equations [correct]? – No, I don't have that one." Laura realized that her students were really beginning to use that math language related to mathematical concepts that they never use. She also recognized that they had to be attentive to the vocabulary and be more detailed oriented for them to create notes for the notebooks. This was different from what they would do, as Laura said, "If you send them home and tell them okay this is due at the end of the month they could care less, but if you put time in the classroom, then they see more value"

*Classroom management.* However, on her two efforts to implement the notebook "BRAIN" idea, she realized that she needed to give the students class time, for example, every two weeks on a Friday she provided 30-45 minutes to the students during the semester to work on this activity. Intermittingly, she found that she fell behind in the curriculum because of the students' lack of skills and the time constraints. The villain of Laura's story was her struggle to manage the lack of skills and the time constraints to do so effectively.

*Content focus.* Laura's students were extremely weak in the mathematics skills. The activity that Laura gave to her students was to reinforce the mathematics concepts they have learned and learning in class. The final product of the activity at the end of the semester was anticipated to be an organized notebook called the "BRAIN" with the important concepts and skills learned in the class. Her intention was that students would have a product "that they feel proud of and that was related to the material they were learning."

*Cognitive aspect of learning.* Laura had a plan for her students. She wanted them to see the value in the work of the notebooks. But she realized that she had to communicate what she wanted to the students and to give them time in class to process that way of thinking. Laura stated, "that is what made them [the notebooks] valuable to them [the students]." Though her first attempt was a "disaster," she realized that in her next effort she would do it much better because "I knew it was something of academic importance for my students."

*Affective aspect of learning.* The hero of Laura's story was one of the exemplar in the class. As Laura expressed,

> The second semester was much, much better. Most of my students participated during the semester. What I really liked about it was a girl who put her heart [the drawing of a heart] in the notebook. That showed me the talent and the potential,

> this girl had. She really put all her heart into it [the assignment]. It [the notebook] had everything I had asked for. She was very proud and I was also very proud of her because she really dedicated her time to it. There were other stories a little bit similar, students who showed their personality, it was a good experience because I got to know the students a little bit better.

*Connection to her teacher preparation.* Laura aluded in her story to her process in teacher preparation ending positively saying:

> It was positive because students like Ashley turned in really good work. It was like going through a process, sometimes when you go through college, the kind of work that you do in college, you have to make sure that everything is in there. Sometimes it is hard, but you have to go through the process and finish it and I think I gave that opportunity to the students. It was good because at the end of the semester most of the students made sure it was finished and they were talking about their content, in what order and I think it was a good experience.

Laura's persistence in retrying her notebook activity came directly from her experience in the teacher preparation program with one of her colleagues who tried a similar activity several times and convinced Laura that it will work. Therefore, Laura was confident that with a few trials, she would be successful, too. She revealed, "So although the first time it was not successful, I knew I could try again and do it much better. I knew it was something of academic importance for my students."

## Discussion and Insights

The stories were actually small testimonies of their teaching that provided multiple meanings and interpretations (Newkirk, 1992). The two SSM teachers' data indicated that they were proactively reflecting and using the experiences they had in the teacher preparation program. Through their lens, deep caring and the authentic strategies they explored with their students brought them successes that were much needed in their urban/suburban classrooms (Blodgett-McDeavitt, 1997; Junor Clarke, & Thomas, 2009; Moustakas, 1994; Schwandt, 2001). The meanings that I perceived from their voices (Patton, 2002) are described in the following sections:

### *Prior Experiences from Teacher Preparation*

In the methods courses, the SSM teachers and I developed a learning community built on sharing, trusting, and caring. In this nurturing environment, they were expected to pull from the experiences of each other. Therefore, it was not surprising but fulfilling to realize that the graduates were actually drawing upon the expertise of each other. Specifically, Laura enacted an activity that Anna had shared in our learning community. I believe the graduates had a sense of community which began in the preparation program.

### *Reflecting on Practices*

The theme of our teacher preparation program was *"Teacher as Reflective Practitioner."* The SSM teachers were consistently reflecting as part of their preparation. I realized that this practice became routine in the program and even in their K-12 classrooms. Teachers who reflect on their practices are expected to improve, but when they do so in learning communities, the results are powerful (Wenger, 1999).

*Insights and New Directions*

Unlike other mid-career changers, the SSM teachers in this study have at least 3 years' experience in the mathematics classrooms. These teachers have the ability and commitment to remain in their schools and community. Their impetus to work with children was paramount. Their stories were actual experiences in their classrooms and interestingly they provided the challenges they face in implementing some of the ideas they learned within our program. Although this may be limited evidence on which to make judgments about the shared views of their classrooms and the discourse, some inferences could be made. The SSM teachers provided authentic experiences, which were growth statements depicting the changes brought to their classroom to assist students to achieve. However, for the episodes they focus more on classroom management, specifically the attitudes of the students and developing their confidence in engaging the students during implementation of the ideas. I felt a sense of silence in the collective stories (Newkirk, 1992)—no one particularly talked about their own mathematics knowledge for teaching students and how that knowledge was perceived by their students. This is the second cohort of students that I worked with in collecting similar data. From this experience, I believe that the nature of the methods was limiting in providing more holistic perspectives of what happened in their classrooms. To gain more insights would definitely demand different methods of data collection.

I am encouraged to enhance the emphases on mathematics knowledge for teaching and the benefits/challenges of technology integration in their mathematics classrooms in such a way that it also becomes a continuing conversation for our graduates.

## Endnotes

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# A COMPARATIVE LONGITUDINAL STUDY OF MATHEMATICS BELIEFS AND KNOWLEDGE IN A CHANGING ELEMENTARY EDUCATION PROGRAM

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*This study compared mathematics beliefs and content knowledge for teaching of two groups of preservice teachers in a changing elementary teacher preparation program: (a) those who completed a program with three mathematics content and two mathematics methods courses, and (b) those who completed a program with four mathematics content and one mathematics methods courses. Results show both of these programs increased personal teaching efficacy, teaching outcome expectancy, and cognitively-oriented pedagogical beliefs. Exchanging a methods course for a content course had no significant effect on content knowledge for teaching; further, greater content knowledge for teaching correlated with more cognitively-oriented pedagogical beliefs.*

## Objective

A teacher preparation program at a large, urban university in the southeastern United States recently responded to a mandate from its university system by increasing the number of mathematics courses for elementary preservice teachers. This change resulted in the exchanging of a second mathematics methods course for a fourth mathematics content course. That is, the Old Program included three courses in mathematics content for elementary teachers and two courses in mathematics teaching methods; the New Program now includes four mathematics content courses and a single mathematics teaching methods course. To examine the longitudinal impact of these programmatic changes, we created a research effort we call the Mathematics Education Research Project (MERP). This specific study compares longitudinal changes during the Old and New Programs.

## Theoretical Perspectives

Relationships between teachers' beliefs and teaching are well-established. Beliefs influence teacher behavior and decision-making (Thompson, 1992), and change in beliefs is a crucial precursor to real change in teaching. Beliefs develop over time (Richardson, 1996), are well-established by the time a student enters college (Pajares, 1992), and develop during what Lortie (1975) terms the *apprenticeship of observation* while a student. Teacher preparation programs have a limited amount of time to effect any changes.

Many studies on changing mathematics pedagogical beliefs have focused on aligning these beliefs with a reform perspective. These studies often looked at change during only one course or semester; although some reported achieving desired effects, others did not. Similarly, most studies of preservice mathematics teaching efficacy beliefs have examined a single methods course. However, these studies more uniformly reported significant increases in mathematics teaching efficacy. While these studies contribute to our understanding of these beliefs, it is also

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

important to examine programmatic effects on these beliefs over a longer period. Longitudinal effects of the Old Program on these beliefs are documented in Swars, Smith, Smith, and Hart (2007; 2009/2008 online) and Swars, Hart, Smith, Smith, and Tolar (2007).

## Research Question

What effects has this programmatic change (one more mathematics content course and one fewer methods course) had on longitudinal changes in preservice teachers' efficacy beliefs, pedagogical beliefs, and content knowledge for teaching?

## Methods

In the elementary teacher preparation program studied, groups of teacher candidates completed the 4-semester professional program together in cohorts. The first three semesters each included on-campus courses and 2-day-a-week field placements, followed by a semester of full-time student teaching. Field placements prior to student teaching (and coursework, when applicable) followed a developmental model, with the grade level focus starting in pre-kindergarten and finishing in fifth grade.

The Old Program included three courses in mathematics content for elementary teachers (in addition to university general education mathematics requirements). This sequence of courses, taught by faculty and adjunct instructors in the Mathematics Department, was started before admission to the professional program to ensure the sequence was completed before student teaching. The Old Program also included two mathematics methods courses, occurring during the second and third semesters of the teacher preparation program. The first methods course focused on grades PreK-2, and the second methods course focused on grades 3-5. The methods courses were taught by faculty in the elementary education department, who shared a common philosophical orientation toward the teaching and learning of mathematics. Important goals of the courses included: (a) developing beliefs consistent with the perspective of the *Principles and Standards* (NCTM, 2000), (b) understanding children's thinking about important mathematics concepts, (c) creating problem-solving learning environments to facilitate discourse and understanding, and (d) building confidence as a lifelong learner and doer of mathematics.

The New Program exchanged the second methods course for a fourth mathematics content course. Although the focus of the remaining methods course was changed to P-5, the field experiences during the second semester continued to focus on grades 1-3.

This study includes data from two 5-point Likert-scale surveys, the Mathematics Beliefs Instrument (MBI) and the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) (Enochs, Smith, & Huinker, 2000). The data also includes scores from a multiple-choice content knowledge for teaching assessment, the Learning Mathematics for Teaching Instrument (LMT) (Hill, Schilling, & Ball, 2004).

The MBI assesses beliefs about the teaching and learning of mathematics and the degree to which these beliefs are cognitively-aligned (Peterson, Fennema, Carpenter, & Loef, 1989, as modified by the Cognitively Guided Instruction Project). The three MBI subscales include: (a) relationship between skills and understanding (Curriculum), (b) role of the learner (Learner), and (c) role of the teacher (Teacher).

The MTEBI consists of the Personal Mathematics Teaching Efficacy (PMTE) subscale and the Mathematics Teaching Outcome Expectancy (MTOE) subscale. The PMTE subscale addresses teachers' personal beliefs in their capabilities to teach mathematics effectively; the

MTOE subscale addresses teachers' beliefs that effective teaching of mathematics can bring about student learning regardless of external factors.

The LMT is designed to identify specific knowledge and reasoning that are important for teaching mathematics from a reform perspective, including such understandings as how to generate representations, interpret student work, and analyze student mistakes. Items are identified in one of three content strands: Number and Operations; Geometry; and Patterns, Functions, and Algebra. However, only aggregate LMT scores were used for this study.

Data from six groups of students are used in this study (three Old Program groups and three New Program groups). MBI and MTEBI data were collected at five points in the program, including the beginning of the 2-year program and the end of each of the four program semesters. The LMT was administered once at the end of the program (following student teaching), using Form E04-A. Sample sizes in the data are 1236 (630 Old, 722 New) for the PMTE and MTOE, 1215 (611 Old, 714 New) for the MBI Curriculum subscale, and 1218 (614 Old, 714 New) for the MBI Learner and MBI Teacher subscales.

## Analysis and Results

The data in this study involve a 3-level nested structure that warrants the use of hierarchical linear growth modeling (HLM) for analysis. Individual measurements over time (L1) are nested within persons (L2) that are in turn nested in groups experiencing Old and New Programs (L3). In addition, there are some variances in the number of measurements per group as well as per person. Both issues are compensated for when using HLM (Singer & Willett, 2003).

Prior research suggests that content knowledge for teaching is correlated with more cognitively-oriented pedagogical beliefs and higher teaching efficacy beliefs (Swars, Smith, Smith, & Hart, 2009). Thus, the results of the LMT were included at the teacher candidate level (L2) of the conditional model. Teacher candidates in both programs completed one mathematics methods course (M1) during the second semester of the program. Only the Old Program teacher candidates completed a second mathematics methods course (M2) during the third semester of the program. The full model equation reduces to the following predictor equation (final model):

$$\hat{Y}_{ij} = \gamma_{000} + \gamma_{010}(\mathrm{LMT}) + \gamma_{100}(\mathrm{TIME}) + \gamma_{110}(\mathrm{LMT})(\mathrm{TIME}) + \gamma_{200}(\mathrm{M1}) + \gamma_{301}(\mathrm{Program})(\mathrm{M2})$$

This model was used to analyze five outcome variables: personal teaching efficacy, teaching outcome expectancy, and pedagogical beliefs about curriculum, the learner, and the teacher. Coefficients from this analysis are given in terms of values on the 5-point Likert scale used in the instruments. Coefficients of influencing factors indicate the changes in the linear model attributable to each significant factor. For each of the five variables analyzed, significant variances remain unexplained.

The difference in mean LMT score (as a percentage correct based on raw score) for teacher candidates from the Old and New Programs was not statistically significant (p = .065), indicating that exchanging a second mathematics methods course for an additional mathematics content course did not have a significant impact (either favorably or unfavorably) on content knowledge for teaching as measured by the LMT. However, the LMT score was found to be a statistically significant factor in accounting for change in most of the other outcome variables, indicating again that teacher candidates with greater content knowledge for teaching also have more cognitively-oriented beliefs and greater personal teaching efficacy.

Table 1 shows the results for personal teaching efficacy beliefs (PMTE), which were positively impacted by LMT score, time, and the second mathematics methods course.

Table 1. *PMTE Significant Results**

| Fixed Effects (final model) | Coefficient | S.E. | t-value | p-value |
|---|---|---|---|---|
| Mean PMTE Score at t = 0 | 3.048 | | | |
| Influence of LMT | 0.685 | 0.250 | 2.734 | .041 |
| Change per semester (TIME) | 0.179 | 0.066 | 2.713 | .042 |
| Influence of 2nd Methods Course (M2) | 0.207 | 0.054 | 3.867 | .017 |

* In this and all other results tables, only statistically significant results are reported.

Table 2 shows the results for teaching outcome expectancy beliefs (MTOE), which were positively impacted only by the second mathematics methods course. This variable is not correlated with the LMT score.

Table 2. *MTOE Significant Results*

| Fixed Effects (final model) | Coefficient | S.E. | t-value | p-value |
|---|---|---|---|---|
| Mean MTOE Score at t = 0 | 3.54 | | | |
| Influence of 2nd Methods Course (M2) | 0.191 | 0.005 | 3.853 | .017 |

Table 3 shows the results for pedagogical beliefs about the curriculum (MBI Curriculum), which were positively impacted by the LMT score and the first mathematics methods course.

Table 3. *MBI Curriculum Significant Results*

| Fixed Effects (final model) | Coefficient | S.E. | t-value | p-value |
|---|---|---|---|---|
| Mean MBI Curriculum Score at t = 0 | 2.719 | | | |
| Influence of LMT | 0.693 | 0.196 | 3.532 | .022 |
| Influence of 1st Methods Course (M1) | 0.460 | 0.062 | 7.394 | <.001 |

Table 4 shows the results for pedagogical beliefs about the learner (MBI Learner), which were positively impacted by the interaction of time and LMT score and by the second mathematics methods course.

Table 4. *MBI Learner Significant Results*

| Fixed Effects (final model) | Coefficient | S.E. | t-value | p-value |
|---|---|---|---|---|
| Mean MBI Learner at t = 0 | 2.850 | | | |
| Influence of (TIME)(LMT) | 0.179 | 0.050 | 3.560 | .022 |
| Influence of 2nd Methods Course (M2) | 0.145 | 0.039 | 3.756 | .019 |

Table 5 shows the results for pedagogical beliefs about the teacher (MBI Teacher), which were positively impacted only by the LMT score, indicating that teacher candidates with greater content knowledge for teaching also held more cognitively-oriented beliefs about the role of the teacher.

Table 5. *MBI Teacher Significant Results*

| Fixed Effects (final model) | Coefficient | S.E. | t-value | p-value |
|---|---|---|---|---|
| Mean MBI Teacher Score at t = 0 | 3.036 | | | |
| Influence of LMT | 0.495 | 0.146 | 3.386 | .025 |

## Discussion and Conclusions

Previous studies have shown that preservice teachers enter teacher preparation programs with relatively well-entrenched beliefs about mathematics teaching and learning (Pajares, 1992). However, our results indicate that this cohort-based, developmental teacher preparation program had significant impacts on many of those beliefs.

The findings reveal that during both the Old and New Programs teacher candidates' pedagogical beliefs became more cognitively oriented, thus more consistent with a reform perspective. In general, those with greater content knowledge for teaching had more cognitively-oriented pedagogical beliefs. In the case of cognitively-oriented beliefs about learners, greater content knowledge for teaching correlated with larger increases in these beliefs as indicated by a greater slope in the linear model for this variable for those with higher LMT scores (i.e., the TIME-LMT interaction).

Teacher candidates' beliefs about the elementary curriculum were significantly impacted by the first methods course in both the Old and New Programs. This course emphasizes problem solving and other mathematical processes fundamental to a standards-based curriculum, as well as student construction of both conceptual understanding and procedural knowledge.

The second methods course in the Old Program significantly impacted teacher candidates' personal teaching efficacy, teaching outcome expectancy, and pedagogical beliefs toward learners. The New Program did not significantly influence teacher candidates' teaching outcome expectancy. Pedagogical beliefs related to the teacher were not effected significantly by either the first or second mathematics methods course.

Both the Old and New Programs were effective at increasing teacher candidates' personal teaching efficacy and pedagogical beliefs during the three semesters of coursework and field experiences prior to student teaching. During student teaching, as expected, personal teaching efficacy continued to increase, while teaching outcome expectancy and pedagogical beliefs remained relatively stable.

Interestingly, the exchange of a second methods course for a fourth mathematics content course affected these program outcome variables in both expected and unexpected ways. Previously reported increases in personal teaching efficacy and teaching outcome expectancy attributed to the second methods course are, as expected, not present in the data for the New Program. Changes in MBI Curriculum beliefs during the first methods course continue to be significant in the New Program. Changes in MBI Learner beliefs are still influenced significantly by time in the program, even though the significant effect of the second methods course is no longer available. While greater content knowledge for teaching mathematics (as represented by LMT scores) continues to be correlated with higher scores on most of these belief variables, it is interesting that the additional content course did not result in a significant increase in LMT scores. Also, MBI Teacher beliefs continue to show an influence from LMT scores and continue to be more cognitively oriented at the end of the program. Unexpectedly, this increase is not attributable to time in the program or a specific methods course and remains unexplained by this analysis.

Limitations of this study include significant unexplained variances that warrant further exploration. We expect some of these variances may be explained by post-hoc analyses of differences due to methods course instructor or field experience location. In considering other research directions, we also hope to acquire data for these same teacher candidates from the ECE mathematics content subscale of the Georgia Assessments for Certification of Educators (GACE) as an additional indicator of mathematics content knowledge.

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# PRE-SERVICE ELEMENTARY TEACHERS' UNDERSTANDING OF AN EQUIPARTITIONING LEARNING TRAJECTORY

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*In this design study, we investigated 56 pre-service elementary teachers' understanding of equipartitioning. Teachers' understanding was assessed before and after an eight-week long teaching experiment that used a learning trajectories approach. Findings from this study indicate that after using a learning trajectories approach in instruction, teachers' knowledge of equipartitioning and their knowledge for teaching equipartitioning increased significantly.*

## Introduction

In the last couple of decades, research on learning has focused on understanding how students reason and how this thinking changes and evolves over time. Some researchers have verified consistent findings relating to these constructs, which they have articulated in the form of learning trajectories. While this has contributed greatly to the knowledge base of how students learn, the field has just begun to explore the extent to which learning trajectories can be integrated into the practice of teaching or in the preparation of pre-service teachers (PSTs).

## Theoretical Perspective

Different terminology and definitions have been used to describe learning trajectories in the literature. Simon (1995) indicates that a *hypothetical learning trajectory*, a teacher's anticipation of the progression of the learning path, provides a rationale for designing instruction, taking into account the learning goal that defines the direction, learning activities, and the teacher's prediction of the potential reasoning and learning of students. Clements, Wilson, and Samara (2004) indicate a learning trajectory is comprised of a mathematical goal, domain-specific developmental progressions that children advance through, and activities corresponding with distinct levels of progression. Catley, Lehrer, and Reiser (2005) suggest learning should be viewed as the process of developing key conceptual structures (Case & Griffin, 1990), or *big ideas*, which coordinate and integrate isolated conceptual components, indicating instruction can be viewed as an orientation towards core ideas that direct teaching and assessment around foundational concepts. They suggest teaching should trace a prospective *developmental corridor* (Brown & Campione, 1996), or *conceptual corridor* (Confrey, 2006), spanning grades and ages, with central concepts introduced early in the school experience and are progressively refined, elaborated, and extended (Catley et al., 2005).

A common theme among the various terminologies is that knowledge progresses from less sophisticated to more sophisticated levels of understanding in a relatively predictable way. Building on the work of others, Confrey, Maloney, Nguyen, Wilson, and Mojica (2008) define a *learning trajectory*, as:

> a researcher-conjectured, empirically-supported description of the ordered network of experiences a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time.

We view a learning trajectory as a tool that can be utilized by PSTs to inform key instructional activities, such as planning, teaching, and assessing. While student understanding cannot be observed directly, learning trajectories seek to identify and describe key items, constructs, and behaviors, which can be observed.

Confrey (2008) defines equipartitioning as the "cognitive behaviors that have the goal of producing equal sized groups (from collections) or pieces (from continuous wholes) as 'fair shares' for each of a set of individuals." Confrey makes a distinction between creating unequal sized parts, which she refers to as "breaking," "fracturing," "fragmenting," or "segmenting," and creating equal parts of a group or whole (i.e., equipartitioning). Confrey et al. (2008) conducted a synthesis of the literature on equipartitioning and other areas of rational number reasoning, where they articulate a learning trajectory for rational number reasoning concepts and organize children's reasoning of equipartitioning into four cases: (A) sharing a discrete collection, (B) sharing a continuous object, and sharing multiple continuous objects between (C) more people than objects and (D) more objects than people. Confrey et al. (2008) has built a progress variable for equipartitioning (see Table 1), describing the behaviors and verbalizations of different levels of understanding of equipartitioning. Within each level (i.e., 1.1, 1.2, etc.), Confrey at al. (2008) describes another level of the progression of knowledge: methods, multiple methods, justification, naming, reversibility, and properties.

Table 1. *Equipartitioning Progress Variable in Relation to Cases A, B, C, and D*

| Case | Equipartitioning Progress Variable |
|---|---|
| D | 1.8 $m$ objects shared among $p$ people, $m > p$ |
| C | 1.7 $m$ objects shared among $p$ people, $p > m$ |
| B | 1.6 Splitting a continuous whole object into odd # of parts ($n > 3$) |
| B | 1.5 Splitting a continuous whole object among $2n$ people, $n > 2$, & $2n \neq 2^i$ |
| B | 1.4 Splitting continuous whole objects into three parts |
| B | 1.3 Splitting continuous whole objects into $2^n$ shares, with $n > 1$ |
| A | 1.2 Dealing discrete items among $p = 3 - 5$ people, with no remainder; $mn$ objects, $n = 3, 4, 5$ |
| A, B | 1.1 Partitioning using 2-split (continuous and discrete quantities) |

## Methods

This paper reports findings from a larger on-going design study that took place during eight weeks of a semester long mathematics methods course within the elementary education department of a large southeastern university in the U.S. The study occurred during part of the course that focused on the teaching and learning of equipartitioning. Participants included 56 PSTs who were enrolled in two sections of the course, which met for 75 minutes twice a week. The first author was the instructor of both sections of the course.

One goal of a design study is to create instructional activities or tasks for classroom use (Cobb, 2000) and to systematically investigate "those forms of learning with the context defined by the means of supporting them" (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, p.9). Thus, a series of instructional activities, or interventions, were designed and implemented to investigate PSTs knowledge of an equipartitioning learning trajectory. These interventions included the following:

- engagement with equipartitioning tasks to develop PST content knowledge;
- an introduction to the articulation of an equipartitioning learning trajectory;
- instruction in the conduct of clinical interviews with different types equipartitioning tasks; and,
- instruction in the analysis of video and student work with respect to the equipartitioning learning trajectory.

During the course, PSTs were first introduced to the construct of a learning trajectory. Next, they were introduced to the equipartitioning learning trajectory for rational number reasoning (Confrey et al., 2008) that situates equipartitioning within this realm of rational number reasoning, in order to help PSTs recognize the foundations of equipartitioning in developing a more robust understanding of a rational number than is currently enacted in U.S. classrooms (Confrey et al., 2008). Over the eight-week period, PSTs were exposed to parts of the equipartitioning learning trajectory and progress variable one case at a time.

As each case was introduced, interventions initially focused on equipartitioning tasks to assess and support the development of PSTs' content knowledge of equipartitioning. The equipartitioning/splitting construct (Confrey, 2008) was new to all PSTs. After PSTs engaged in equipartitioning tasks and discussed their own solutions, as well as the underlying mathematical structures of the tasks, they were introduced to the components of the learning trajectory and progress variable. When these components were introduced, video exemplars of K-5 students, engaged in working with equipartitioning tasks, were presented. Next, instructional activities focused on analyzing other video exemplars of K-5 students. The video exemplars illustrated a range of students' verbalizations and activity as they participated in clinical interviews on the same equipartitioning tasks with which PSTs had previously engaged. Class discussion focused on analyzing student thinking with respect to the equipartitioning learning trajectory. Lastly, PSTs implemented equipartitioning tasks with students in K-2 classrooms.

Data for this study includes pre- and post-tests developed by the research team to assess PSTs content and pedagogical content knowledge of equipartitioning. Many items came from or were modified from our synthesis work on equipartitioning or rational number reasoning (Empson & Turnner, 2006; Lamon, 1996; Pothier, 1981; Pothier & Sawada, 1989). Rubrics were created to score the assessments so that responses could be categorized into four distinct levels per item. Instruments were piloted with teachers who were either teaching elementary school at the time or who had previously taught elementary school. The pre-test was administered prior to the interventions and the post-test was administered on the last day of the study. Both assessments were administered during regular 75-minute class meetings under the same conditions. For the pre-test, half of the PSTs were randomly assigned to form A and the other half were assigned to form B; alternate forms of the assessment were assigned for the post-test.

## Results

Pre-test scores ranged from 16 to 41 on a 54-point scale. Post-test scores ranged from 22 to 48 on a 54-point scale. Pre- and post-test data were paired by PSTs and analyzed for differences using the Wilcoxon signed rank test. The median of the gain scores was 6 points, which was significantly greater than zero ($S = 70.5$, $p < 0.0001$). Four items have been selected to illustrate the ways in which PSTs did and did not show changes in their knowledge. Tasks were selected to provide an example of items used to assess pedagogical content knowledge (i.e., Tasks 1 and 2) and content knowledge of equipartitioning (i.e., Tasks 3 and 4). Each item and scoring rubric will be described, followed by a report on the frequencies of responses and illustrations.

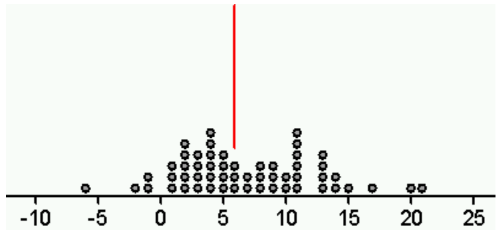


*Figure 1*. Distribution of gain scores.

Equipartitioning a continuous object (i.e., Case B) involves coordinating three components: i) using the entire whole; ii) creating equal sized parts of the whole; and, iii) creating the appropriate number of equal sized parts of the whole (Confrey, 2008). Tasks 1 and 2 required PSTs to generate imagined student responses for students who pay attention to only one of these three components (Task 1: component iii; Task 2: component ii), requiring them to consider how the other two components could be varied to produce an incorrect or incomplete result. Task 1 (form A) and Task 2 (form B) are stated below, and the scoring rubrics can be found in Table 2:

- Draw a picture of how a student might respond to the following task given the indicated understanding. Provide an explanation: Sharing a round cookie among 3 people if the *focus is on the number of pieces*.
- Draw a picture of how a student might respond to the following task given the indicated understanding. Provide an explanation: Sharing a (rectangular) pan of brownies among 5 people if the *focus is on the size of the pieces*.

Table 2. *Rubric for Scoring Tasks 1 and 2*

| Level | Response Characteristics for Task 1 | Response Characteristics for Task 2 |
|---|---|---|
| 3 | Correct number of pieces (with either unequal sized pieces, or not filling the whole) with full explanation | Equal sized pieces (with either the incorrect number of pieces, or not filling the whole) with full explanation |
| 2 | Same as above with incomplete/poor explanation or no explanation | Same as above with incomplete/poor explanation or no explanation |
| 1 | Same as Level 3 with incorrect explanation, or correct number of equal sized pieces that uses whole with full explanation | Same as Level 3 with incorrect explanation, or correct number of equal sized pieces that uses whole with full explanation |
| 0 | Incorrect response or no attempt | Incorrect response or no attempt |

As the results presented in this paper are part of a larger on-going study, we are considering the appropriateness of coding correct predicted responses (i.e., correct number of equal sized pieces that use the whole) on Tasks 1 and 2 of the pretest as Level 1. While these responses meet the criteria for the tasks, they over define it. While coding these responses as a Level 1 on the pre-test may be too strict, it is appropriate for the post-test, since the language used in the items was made clear in the context of instruction. These items will be modified in the future to clarify that PSTs should *only focus on the number of pieces* or *size of the pieces*.

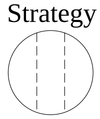
The percent of PST responses in each of the four levels is located in Table 3. On the post-test, more PSTs provided Level 3 and 2 responses, 43% in comparison to 13% on the pre-test. Fewer PSTs provided Level 0 responses on the post-test (25%) than on the pre-test (46%).

Table 3. *PSTs' Pre- and Post-test Responses to Tasks 1 and 2*

| Level | Pre-test | Post-test |
|---|---|---|
| 3 | 9% | 34% |
| 2 | 4% | 9% |
| 1 | 41% | 32% |
| 0 | 46% | 25% |

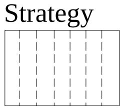
Examples of PSTs' representative responses to Tasks 1 and 2 can be found in Tables 3 and 4, respectively. The responses are PSTs' prediction of student strategies for that task, not their own strategy. The most common Level 3 response can be characterized in the following way: PSTs indicated that students might create 2 parallel cuts that result in 3 unequal sized pieces of the cookie. These PSTs explained that students using this strategy would create the appropriate number of pieces, but the pieces would not be the same size. Another Level 3 response by PSTs suggested students might split the cookie in half, and then half one of the halves, using the same explanation as above. PSTs providing a Level 2 response indicated that a student would use repeated halving to create four equal sized groups, and then throw away one of the pieces; however, the explanation in Table 3 does not explicitly state the child intended to create the appropriate number of pieces. The most common Level 1 response suggested a student might use the entire cookie by creating 3 radial cuts, coordinating equal sized pieces, appropriate number of pieces, and using the entire cookie. Even though Task 1 asks PSTs to provide a strategy for students who *focus on the number of appropriate pieces*, these PSTs presumably assumed the student would focus on creating equal sized pieces, as well. The Level 0 response provided in Table 3 does not show a student strategy that would result in creating the appropriate number of pieces; the PST broke the cookie into 32 pieces, which is not divisible by 3 without a remainder.

Table 3. *PST Responses to Task 1 (Form A)*

| Level | Strategy | Explanation |
|---|---|---|
| 3 |  | |

With respect to Level 3 on Task 2, PSTs suggested that students might create both the appropriate number of pieces, as well as equal sized pieces of brownies, by creating a composition, explaining that extra pieces would not be used or would be saved as leftovers. This was the most common Level 3 strategy and explanation provided by the PSTs. Another Level 3 response was that students might create *n* cuts, instead of *n* –1 cuts, explaining that students who focused on the size of the pieces would create 6 equal sized pieces, not 5. PSTs indicated Level 2 responses included the strategies from Level 3, but these responses by PSTs did not provide explanations. Almost a third of the PSTs (32%) responded with a student strategy that coordinated size, appropriate number of pieces, and use of the entire cake by creating *n* – 1 cuts. Even though the task asks PSTs to provide a strategy for students who *focus on the size of the pieces*, these PSTs presumably assume that students might focus on the number of appropriate pieces, as well. The Level 0 response provided in Table 4 shows a student strategy that focuses on the appropriate number of pieces, but these pieces are not the same size.

Table 4. *PST Responses to Task 2 (Form B)*

| Level | Strategy | Explanation |
|---|---|---|
| 3 |  | |

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

methods, whereas Level 1 responses listed one or two methods for folding the paper. Level 0 responses included one of the following: a) the method of creating $n$ - 1 (i.e., 11 or 17) parallel folds, b) an incorrect method, or c) no attempt. Distinct methods for solving Task 3 include some permutation of folding the paper in the following ways: i) half, half, third; ii) fourth, third; iii) half, sixth; and, iv) 11 parallel folds or a permutation involving a 12$^{th}$. Methods for solving Task 4 include some permutation of folding the paper in the following ways: i) half, third, third; ii) sixth, third; iii) half, ninth; and, iv) 17 parallel folds or a permutation involving an 18$^{th}$.

Table 6. *PSTs' Pre- and Post-test Responses to Tasks 3 and 4*

| Level | Pre-test | Post-test |
|---|---|---|
| 3 | 2% | 4% |
| 2 | 2% | 9% |
| 1 | 20% | 59% |
| 0 | 76% | 29% |

On the pre- and post-tests, the majority of PSTs were unable to describe more than one method for folding the paper to obtain the correct number of equal sized regions. The most typical response for Task 3 was some permutation of folding into a half, a half, and a third; the most common response for Task 4 was some permutation of folding into a half, a third, and a third. On both the pre- and post-tests, PSTs who did not provide the above strategies, often responded with an unproductive strategy that involved folding in half or repeated folding in halves. Others did not attempt to respond. Very few PSTs utilized the strategy of creating $n$ - 1 (i.e., 11 or 17) parallel folds; however, more PSTs used this strategy on the post-test than the pre-test, sometimes in conjunction with other methods. Although very few PSTs were able to list three or more distinct strategies for folding a piece of paper to create a given number of equal sized partitions, PSTs showed a better understanding on the post-test. More than half of the PSTs (59%) were able to describe one or two strategies on the post-test, while more than three-fourths (76%) of the PSTs could not elucidate one strategy for folding a piece of paper to create a given number of equal sized partitions on the pre-test. Of all the items from the pre- and post-tests, PSTs performed the poorest on Tasks 3 and 4. Responses to these items were scored significantly lower than any other item.

## Discussion

PSTs knowledge of partitioning and knowledge for teaching equipartitioning increased significantly after their participation in instruction that used a learning trajectories approach. There is also evidence to suggest that PSTs used components of the equipartitioning learning trajectory (Confrey et al., 2008) in their responses relating to students' understanding. For example, consider Task 3. Current work by the DELTA research team and earlier work by Confrey (2008), have identified distinct behaviors of students who cannot coordinate creating both equal sized parts and the appropriate number of parts when splitting a circle into three parts. In this specific case, students often create 2 parallel cuts or might split the circle in half, and then half one of the halves. Almost half of the PSTs (43%) showed evidence of anticipating and adequately interpreting such a student response on the post-test. To be prepared for the complexity and range of diversity in student knowledge that they will encounter as they enter the profession, drawing such distinctions is key.

Although PSTs' gain scores significantly increased after the intervention, most PSTs scored very low on Tasks 3 and 4 in comparison to the other items. In a study examining children's multiplicative reasoning, Empson and Turner (2006) investigated the role of repeated halving in relation to first, third, and fifth graders' multiplicative thinking as they engaged in paper folding tasks. They found that children initially connected the action of folding and the outcome in *non-recursive* ways. Other children used an *emergent recursive strategy*, such as recursive doubling, understanding that creating a half fold could double the number of partitions. Few children used a *recursive* strategy to connect the fold and subsequent number of partitions. On the pre-test, 76% of PSTs used non-recursive strategies that were nonproductive. On the post-test, only 29% of PSTs utilized non-recursive strategies in the paper folding tasks. Like the children in the Empson and Turner study, very few PSTs used recursive strategies. Even on the post-test, the majority of the PSTs needed a concrete material for the paper folding task and only provided one or two methods. The first author observed the majority of the PSTs folding paper as they worked on both assessments. This is further evidence that few PSTs were using recursive strategies. Further on-going item analysis needs to be completed before definitive conclusions can be made about PSTs multiplicative reasoning.

Further analysis is underway to gauge the impact of a learning trajectories approach on PSTs. A complete analysis will include examining how their participation in the course tasks generated changes in their knowledge of both student thinking and equipartitioning. In addition, a detailed analysis of their subsequent interactions with children will begin to permit us to evaluate the degree to which a learning trajectories approach can be a productive strategy in pre-service teacher education.

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# THE NATURE OF MATHEMATICS IN MATHEMATICS TEXTS FOR PRESERVICE ELEMENTARY TEACHERS: A CRITICAL LINGUISTICS ANALYSIS

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*In this study, I use the tools of critical linguistics analysis to explore how preservice elementary school teachers' beliefs about the nature of mathematics may be influenced within a primary source of mathematical activity in classrooms: textbooks. The language within these textbooks implicitly promotes particular views about mathematics, which we analyze here. I discuss how the textbook authors' choices related to Halliday's (1973) ideational function of language—which includes the actors within the text as well as the processes present—can promote three particular views about the nature of mathematics:* the Platonist view, the instrumentalist view, *and* the problem-solving view.

## Introduction

Though it has been well-established that textbooks play a substantial role in teachers' decisions about what is taught in K–12 classrooms (Stein, Remillard, & Smith, 2007), less is known about the nature of preservice teacher mathematical beliefs and textbook interaction before entering those classrooms, particularly within their teacher preparation programs. Just as textbooks influence what is taught and learned in elementary through high school mathematics classrooms, they "exert a major influence on the content and approach of courses for prospective elementary teachers" (McCrory, Siedel, & Stylianides, 2007, p. 5), reaching over 80,000 students in teacher preparation programs each year. Textbooks work from an underlying philosophy about the nature of mathematics as a discipline, as well as the assumed knowledge that students bring with them to the classroom. These ideas "combine to create a distinct view of mathematics and mathematics learning that permeates each textbook" (McCrory et al., 2007, p. 13), one that has the potential to influence the development of the beliefs of their readers. While this study does not provide a complete picture of that development, it aims to provide a good starting point by using the tools of systemic functional grammar (Halliday, 1975) to address my research question: what views about the nature of mathematics are being promoted in textbooks for preservice elementary school teachers?

## Conceptual Framework

The critical discursive framework developed for this analysis is shown below (see Figure 1). Its foundation is in the work of Morgan (1996), whose ideas about critical discourse analysis closely reflect those used by linguist Norman Fairclough (1993) and are grounded in his assumption that every text somehow contributes to an individual's identity within their culture. Both claim to have their ideas rooted in the multifunctional linguistic theories represented in Halliday's (1985) functional systemic linguistics.

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| | | | View about the Nature of Mathematics | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | ***Platonist*** | | | | ***Instrumentalist*** | | | | ***Problem-Solving*** | | | |
| Components of Linguistic Analysis | | **Actors** | We | You | 3rd person | Non-human | We | You | 3rd person | Non-human | We | You | 3rd person | Non-human |
| | **Processes** | ***Material*** *(outside the self)* | | | | ☆ | | ☆ | ☆ | ☆ | ☆ | ☆ | | |
| | | ***Mental*** *(inside the self)* | ☆ | ☆ | ☆ | | | | | | | | | |
| | | ***Relational*** *(denote relationships)* | | | | | | | | | ☆ | ☆ | | ☆ |

*Figure 1. Conceptual Framework.*

The framework has two dimensions, the first comprised of the components of linguistic analysis and the second being the different views about the nature of mathematics. I adopt the three distinct views Ernest (1988) detailed in his review of empirical studies focusing on teachers, namely the *problem-solving view*, the *Platonist view,* and the *instrumentalist view*. While others have suggested different characterizations about the nature of mathematics, my choice to use these three is supported by Thompson's (1984) findings that these views parallel those most frequently observed in mathematics teaching.

This framework proposes ways in which the linguistic components of *actors* and *processes* can be connected to these different views about the nature of mathematics by indicating the probable actors present as well as the types of processes in which they would likely participate within each view, with these intersections indicated by the stars. For example, according to the star placement in the framework, the Platonist view supports the intersection of non-human actors participating in material processes. Next, I elaborate further on the different dimensions of the framework and how they intersect.

*The Platonist View*

So named due to its roots in ideas of Plato, the Platonist view portrays mathematics as a static body of knowledge, "bound together by filaments of logic and meaning" (Ernest, 1988, p.10), waiting to be discovered as opposed to being created. A textbook working from the Platonist view would indicate the presence of the reader as an actor in the discourse by using the personal pronoun *you*, with phrases such as *you see* controlling the ways in which the mathematics is seen and understood. Mathematics is not created but is depicted as an entity that exists independent of the student. In addition to human actors, this view would support the frequent use of *nominalization* within the text, where mathematical objects themselves are positioned as actors in control of the understandings and the human actors positioned as passive recipients of the knowledge. Given that both human and nonhuman actors are present, use of both the active and passive voice is possible, obscuring the agency of the reader as well as any other human participant. Morgan (1996) suggests that having a large portion of "mental processes (e.g., seeing, thinking) may suggest that mathematics is a pre-existing entity that is discovered" (p. 4), in agreement with the followers of Plato. These processes involve the inner experience of the participant, representing their perceptions, desires, and emotions such as *think*, *observe*, and *recall*.

*The Instrumentalist View*

Textbooks working from an instrumentalist view see the discipline as a collection of "unrelated but utilitarian" (Ernest, 1988, p.10) facts and procedures used by those trained with the tools to accomplish a particular end. Emphasis on procedures here necessitates the presence of human actors to carry out those procedures, yet with a very restricted role in the creation of the knowledge. Though the use of an active voice would be expected, the reader's actions would stress the importance of systematic, rote exercises in order to promote precision and mastery of tools. Of all the views, it is the instrumentalist view that is most likely to use the pronoun *you* with expressions of the form *you* + (a verb), such as in *you calculate*, often accompanied by temporality sequences that consequently restrict the reader's activity. Such a focus on procedures indicates a large proportion of material processes, which "may be interpreted as suggesting a mathematics that is constructed by doing" (Morgan, 1996, p.3), procedures taking precedence over conjectures. These processes deal with the actors participating in some kind of activity outside of themselves in the physical world involving some other actor within the situation, using verbs such as *write*, *draw*, and *calculate*.

*The Problem-Solving View*

Textbooks grounded in the dynamic problem-solving view suggest that new mathematics is constantly being invented through a process of inquiry that is open to revision. Given the emphasis placed on the student's role as an active participant in the creation of knowledge, a textbook promoting this view would not only suggest the presence of human actors within the activities, but that the reader is actively included as a decision maker within those activities. Pronouns such as *I* and *we* may point to the acknowledged presence of the reader, and the high level of interaction on the part of the student required by this conception suggests the use of the active voice promoting the actors present performing the action. Since "a high proportion of material processes may be interpreted as suggesting a mathematics that is constructed by doing" (Morgan, 1996, p. 4), that is the mathematics promoted within the problem-solving view. Relational processes are those that identify and associate one experience with other experiences, using verbs such as *is*, *connects*, and *relates*.

## Methodology

*Textbook Selection*

The textbooks included in this analysis were chosen based on the work done by McCrory et al. (2007), which explored and compared the content of 14 textbooks used in mathematics courses for prospective elementary school teachers. One of many dimensions discussed there, the *mathematical stance* of a textbook was described as "addressing the conception of mathematics that the book presents: What is important? What is the nature of mathematics? How does mathematics work as a discipline?" (p. 13). Given that my goal was to analyze the nature of mathematics being presented in these textbooks, I concentrated my exploration on three of the four textbooks described as possessing *explicit* mathematical stance in the McCrory et al. (2007). These textbooks were found to be consistently *explicit* in describing the general function and importance of mathematical ideas. However, this earlier analysis was not able to illustrate what the authors' proposed as *being* the general function and importance of mathematical ideas, indicators of the view about the nature of mathematics being promoted. Therefore, these explicit textbooks offered the greatest opportunity to capture a text's promoted view using my analytic tools. Written by Darken (2003), Parker & Baldridge (2004), and Wu (in preparation), the texts will be referred to by their authors' names from this point forward. I could not obtain a copy of the fourth book classified as *explicit* in time to include it in this analysis.

*Focus on Number Theory and Definitions*

Here, we restrict our discussion to only definitions found within the chapter in each book related to number theory topics. The fact that number theory is a customary topic covered in most mathematics courses for preservice elementary school teachers serves as the rationale for its selection. At the author's request, Wu supplied an electronic copy of his number theory chapter for my analysis, as the complete textbook is still being revised for publication.

The choice to focus my analysis on definitions in textbooks was not arbitrary. In addition to playing an important role in mathematical activity, the presentation of definitions in textbooks can capture how the textbook promotes different mathematical views. For example, one text may simply provide the reader with a single definition and present it as being the only definition possible, while another may engage the reader in the creation of a definition that is open to modification. These two textbooks would be promoting vastly different views about the nature of mathematics, the first Platonist and the second problem-solving.

*Coding*

Though one of the three textbooks did not contain a chapter explicitly titled "Number Theory", the chapter chosen had the greatest intersection of topics with the other two textbooks' number theory chapters. The analyses that follow focus only on the following concepts: factors, divisors, divisibility, even and odd numbers, prime numbers, LCM, and GCD. I determined the section to be analyzed to include the introduction, statement, and discussion of each definition. This did not include examples or related theorems or lemmas. Each sentence in these sections was coded according to the actors present and the type of process used. Human actors were coded according to personal pronouns used (*you*, *we*), while third-person participants (e.g. *the student*, *someone*) and non-human actors (e.g. mathematical objects) were simply coded as belonging to one of the two groups. A list of all verbs used in within the sections of interest were generated and classified as representing either a mental, material, or relational process and then shared with two other mathematics education doctoral students. These classifications were discussed and modified until all three parties reached agreement, and each sentence was given the appropriate process code. Since there were several sentences in each section, it was possible for concepts to have multiple codes. In addition to the quantitative code frequency measures, I also share sample qualitative analyses of the findings.

## Results and Discussion

*General Findings*

Table 1 shows the results of my analysis, with the values representing the percentage of codes falling into each intersection of the different actors and processes. By summing the columns or rows, this table also provides the frequencies of each individual actor or process, respectively, for each analyzed text.

Considering the high proportion of both human and non-human actors participating in material processes (62% overall), Parker and Baldridge (2004) appears to strongly support an instrumentalist view of mathematics. The very first sentence of the chapter supports this categorization and sets the tone for all of the activities that follow: "Mathematics is built on precise definitions and proceeds using clear reasoning" (p. 109). As described in the case of prime numbers and to some degree common to all three texts, the majority of the discussions surrounding the various terms defined in the chapter acknowledged the presence of human actors (the student), yet those actors had a very restricted role in the creation of the knowledge being developed. While the use of *we* may imply that the author is establishing solidarity with the reader and placing them at equal positions of power within the text, there are several instances within discussions that indicate this *we* is more *exclusive* than *inclusive*. For example, in the section dealing with divisibility tests, the authors precede the definition of divisible with the following statement: “In the remainder of this chapter, *we* will often use letters A, B, …, k, l, … and a, b, … to represent whole numbers. At any time, *you* may assign them specific values (like A=20, k=5) to aid *your* understanding” (p. 113, italics added). Therefore, while the students are certainly present, the processes in which they are to engage are purely material, constructing a mathematics which is about practical activity that is carried out in a procedural way. I also found a large proportion of the codes (20%) showed non-human actors participating in relational processes, which my framework suggests would be promoting a problem-solving view of mathematics by focusing on connections between mathematical ideas. Non-human actors engaged in material processes points towards a more Platonist view.

*Table 1*
(Rounded) Percentages of Actors and Process Found in the Textbooks

| | | **Frequencies (in percentages)** | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | ***Darken (2003)*** | | | | ***Parker and Baldridge (2004)*** | | | | ***Wu (in preparation)*** | | | |
| | **Actors** | We | You | 3rd person | Non-human | We | You | 3rd person | Non-human | We | You | 3rd person | Non-human |
| **Processes** | ***Material*** | 13 | 21 | 8 | 11 | 14 | 12 | 16 | 20 | 33 | 2 | 4 | 9 |
| | ***Mental*** | 5 | 5 | 2 | 0 | 3 | 6 | 6 | 1 | 15 | 12 | 2 | 1 |
| | ***Relational*** | 0 | 5 | 0 | 30 | 0 | 0 | 2 | 20 | 1 | 1 | 0 | 20 |

The picture of mathematics presented by Wu (in preparation) has some similarities to Parker and Baldridge (2004). Though I found *we* to be the most prevalent actor in the text (accounting for almost half of the codes), which suggests human agency in the discussion, there is strong evidence that the reader is not assumed to be actively participating in the knowledge being introduced. When describing “our enduring interest in the primes” (p. 10), Wu places himself in the authoritative role, considering that prime numbers had only been defined in the previous sentence and the reader has not yet had a chance to fully understand what prime numbers are, let alone develop such a powerful interest in them. The mathematics is introduced in a very matter-of-fact manner, with each new idea logically derived from the former in a linear fashion. Like Parker and Baldridge, the presence of non-human actors involved in material processes is evidence of what linguists call *nominalization,* which “describes language structure that obscures human agency” (Herbel-Eisenmann & Wagner, 2005, p.123), where inanimate objects (in this case, mathematical objects) perform activity usually related to humans. Almost every definition described in the chapter, and more generally the entire book, is followed by a collection of theorems using those definitions. In this area, it is common to find a phrase such as “the theorem tells us”, or “the theorem says”, which “depicts an absolutist image of mathematics, portraying mathematical activity as something that can occur on its own, without humans" (Herbel-Eisenmann & Wagner, 2005, p.123). Indeed, the reader has little to do with the creation of definitions or related mathematical knowledge, asked to serve as a mere spectator of the activities, and to accept that which has been "clearly" outlined. Though material processes such as "check", "remove", and "define" were most commonly found throughout the chapter, at 30% Wu contained the largest proportion of mental processes of the analyzed text, processes that Morgan (1996) claims "may suggest a mathematics that is a pre-existing entity that is discovered by mathematicians" (p. 2). These things considered, I suggest that Wu's text works from and portrays a more absolutist view of mathematics, having instrumentalist tendencies but more Platonist considering the overwhelming restriction of reader activity (the exclusive *we*) and the 12% of codes describing mental processes in relation specifically to the reader (*you*), which was the was the highest frequency any processes.

Though being the least common human actor present (at 18% of the total), Darken's (2003) *we* not only suggests an awareness of human agency, but the ways in which it is used also includes the reader as a participant of mathematical activity. Most of the 21% of *you* actors participating in material activities were in the form of imperatives, commands which were seen as *inclusive*, welcoming into the community as an active participant first given the chance to construct their own knowledge with the formal definition coming later. This is in sharp contrast to the similar activity described in Parker and Baldridge (2004), where the definition was given with little reader contribution, after which the student is lead temporally through a procedure. Whereas Parker and Baldridge employed a narrow to broad pathway (meaning they began with the definitions and then used a variety of problems to discuss the term), Darken's approach started broad, narrowed, and then broadened once again, starting the conversation with a reader-driven activity that better prepared them for a more formal discussion of prime numbers. Despite a clear emphasis on certain procedures, there is evidence to characterize this text as promoting a problem-solving view of mathematics, with some instrumentalist tendencies. At 35%, the book contained the largest proportion of relational processes found in any of the analyzed texts, with most of the 52% material processes related to human actors (excluding the 11% with non-human actors) being classified as inclusive imperatives. More importantly, these processes are combined with a picture of the reader/student that often appears to be invited to share in the mathematical activity of defining, as well as to be engaged in considering the evidence supporting the developed argument.

Now that I have provided an overview of the general findings of my discourse analysis, I next focus on the particular definition of prime numbers. In describing the actors and processes used in the different text in relation to this common concept, I attempt to illustrate the different views about mathematics that may be promoted by each textbook.

*Prime Numbers: An Illustration*

Though each textbook acknowledges that the student has a role within the discussion of prime numbers, the authors' choice of processes in which the students are involved seem to promote different ideas about what that role entails. In addition to using *we* as a means to construct the student as an active member in the mathematics, Darken (2003) encourages the student to initially share in the responsibility of creating their knowledge about primes. With prime numbers in particular, Darken asks the student to initially share in the responsibility of creating new knowledge instead of simply stating the definition for the reader to accept unquestioningly. The chapter opens with an activity meant to fuel the students' investigations into possible emergent patterns that moves them toward the notion of prime numbers.

Instructions are given about how to use the chart, but the authority is quickly given to the reader, where the author makes comments about "*your* work" and "*your* patterns" at the heart of the activity. By participating in the relational processes of investigating and observing patterns, the students are presented with "a picture of mathematics as a system of relationships between objects or between objects and their properties" (Morgan, 1995, p. 3), in this case natural numbers, their factors, and the property of being prime.

In contrast, Parker and Baldridge (2004) and Wu (in preparation) restrict the student's role as an active participant in the mathematics. While Darken (2003) asks the student to explore the mathematical terrain of prime numbers and to share in decision-making activity, a student using Parker and Baldridge is directed to carry out material tasks related to a described procedure to follow, using imperative instructions. While the former suggests a view of mathematics that is dynamic and creative, the focus of material processes in the latter "may be interpreted as suggesting a mathematics that is constructed by doing" (Morgan, 1995, p. 3), procedures taking precedence over conjectures. Though not immediately clear, the use of imperatives in both texts is related to their seemingly more exclusive use of *we*. Instead of interpreting imperative commands like *find*, *list*, and *explain* as mathematical convention, the reader may interpret their role in relation to mathematics as only that of a follower of rules in a procedure-centered discipline. As in earlier discussions, I use Rotman's (1988) dichotomy of imperatives as being either *inclusive* or *exclusive*, with the first constructing a reader whose actions are *included* in a community of people doing mathematics, whereas the other constructs one whose actions can be *excluded* from such a community. I illustrate the difference between these imperatives as used in two of the texts, and discuss the corresponding consequences.

Parker and Baldridge (2004) and Darken's (2003) main activities within the discussion of prime numbers appear at first glance to be superficially similar. While both seem to acknowledge the student as an active participant, their efforts to demonstrate the relationship between the factors of a number and the attribute of being prime, the differences and the processes that are entailed highlight contrasting views about where the authority in that knowledge lies. Before any mention or definition of the term *prime*, Darken opens the chapter with a mostly blank chart that is meant to fuel the students' investigations into possible emergent patterns. Instructions are given about how to use the chart, but the authority is quickly given to the reader, where the author makes comments about "*your* work" and "*your* patterns" at the heart of the activity. The commands are *inclusive*, welcoming the reader into the community as an active participant. Parker and Baldridge on the other hand, acknowledge this relationship between factors and the condition of being prime, but instead of promoting exploration the students are guided through a process driven by *exclusive* imperatives such as "proceed, circle, cross out", not giving them a choice to act otherwise or question the command. Though both are grounded in the same mathematical ideas, the different processes engaged in by the reader (mental vs. material) in discussing prime numbers have the potential to present two very different views about mathematical activity, the first situating them as creators and the second as followers.

## Conclusions and Implications

This analysis explored how linguistic choices made by textbook authors can promote different views about the nature of mathematics. I observed distinct contrasts in the ways the authors of the three textbooks examined have chosen to create, discuss, and use definitions, affecting the ways in which those views are being promoted. Are pronouns such as *you* and the inclusive *we* present with imperative verbs, inviting the reader to play an active role as a problem solver in the discussion? Does the author place actions in the hands of inanimate mathematical objects, Platonistic and existing independent of human activity? Does the student feel present in the mathematical conversations, but only as an individual meant to accept given material and procedures to follow as unquestionable?

In conjunction with the work done by Morgan (2005), the sample analyses of this study can provide textbook authors, curriculum developers, and teacher educators charged with selecting textbooks with linguistic tools that can help them "anticipate the meanings, both substantive and positional" (Morgan, 2005, p. 9) relating to views about mathematics that is being supported by textbooks for elementary school teachers. Also, it may help those authors realize the views about mathematics being promoted in their texts about which they may have not even been conscious. These tools can serve as a guide to develop textbooks that construct the roles and authority, the relationships between the reader, author, and mathematics, and the overall vision about the nature of mathematics more purposefully in alignment with the view of mathematics desired by the author. In attempts to investigate consistencies in linguistic choices, an interesting direction for future research would be to examine other content areas in addition to number theory and compare the results with those found here. While textbooks are certainly not the only potential source that influences preservice elementary teachers' beliefs about the nature of mathematics, they are certainly a prevalent feature in the majority of classrooms in this country, from the elementary to the university level. Therefore, whether intentional or not, the language choices made by textbook authors promote particular views about the nature of mathematics that may influence the reader's view of the nature of mathematics and their position with respect to the discipline, and so those choices should be made carefully and consciously.

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# EXPLORING PRESERVICE TEACHERS' CONCEPTIONS OF NUMBERS VIA THE MAYAN NUMBER SYSTEM

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*Preservice elementary school teachers (PSTs) struggle to understand numbers in our base-10 number system, but uncovering their base-10 conceptions is difficult because the underlying mathematical structure is masked by language and prior experience operating on numbers in base-10. PSTs' conceptions of numbers were explored through work on identifying numerals in the Mayan number system (base twenty) during which PSTs drew on their base-10 conceptions. Of 24 participants only 6 could identify a 3-digit and a 7-digit Mayan numeral correctly. Both those numerals were the digit equivalent to 1 with 2 and 6 zeros, respectively, attached. The PSTs' answers are categorized and explained. Implications for mathematics education are discussed.*

## Background and Theoretical Framework

Place value is an essential part of the elementary school mathematics curriculum. However, PSTs as well as teachers often struggle explaining numbers beyond procedural applications (Ball, 1988; Ma, 1999; Ross, 2001; Thanheiser, 2005). Previous work (Thanheiser, 2005) has shown that only 3 of 15 PSTs were able to draw on the underlying structure of our number system to interpret the values of the digits in a number and relate them to one another. Because of this underlying structure of the base-ten power sequence, the relationship among adjacent digits is a multiplicative relationship with a factor of 10. Each place represents a value 10 times as large as the next lower place. Thus, if one moves a digit one place to the left, its value is 10 times as great as previously. One difficult aspect of understanding the underlying structure on which our numbers are built is that it is implicit, not explicit. Many PSTs use a *tens and ones language;* that is, they use phrases such as "I borrowed a ten." However, that *tens and ones language* is often only procedural (Ball, 1988; Thanheiser, 2005). For many prospective teachers, facility using base-ten language masks lack of understanding (Ball, 1988). Even though PSTs read 367 as "three hundred sixty-seven," they may not think of the individual digits as representing 3 groups of 100, 6 groups of 10, and 7 groups of 1 or that a hundred is 10 tens and a ten is 10 ones— both important aspects of understanding numbers in our base-ten number system. To avoid the use of this tens-and-ones language and other masking aspects of our base-ten language, we explored a different number system (the Mayan base-twenty system[1]) and related it to the base-ten system. We conjectured that considering mathematics through the lens of a different numeral system would help teachers recognize the complexities of the base-ten system.

Our research grows out of a rich cognitive-science paradigm focused upon children's prior knowledge in learning situations, a consideration that is equally important in work with adults (Bransford, Brown, & Cocking, 1999). To help PSTs develop a solid understanding of mathematics, we need to build upon their initial conceptions, which both determine what they understand when looking at a number and serve as a basis for building more sophisticated conceptual structures. Our goal in this study was to further understand how PSTs think about numbers in our number system and how we could use those conceptions to design instruction.

## Method

The data analyzed are drawn from four 160-minute teaching sessions and a preinterview, and a pre- and post-assessment with each with 24 preservice teachers at a large, urban, comprehensive state university. All participants were enrolled in a preservice teacher elementary mathematics methods course and were about to begin student teaching. The teaching sessions were part of the regular methods course.

In the pre- and post-assessments, we asked the PSTs to explain regrouped digits in addition and subtraction problems (e.g., see Figure 1).

```
  1 1
  389
+ 475
-----
  864
```

1. Please explain what the “1” above the 8 represents.
2. Please explain what the “1” above the 3 represents.

*Figure 1.* Sample task from pre- and post-assessment.

During the teaching sessions, the PSTs explored Mayan numerals, a base-twenty place-value system that uses three symbols. A dot represents one unit, a bar represents five units, and a shell represents zero units. These symbols are grouped to create digits from 0 to 19. Five dots are grouped into a bar. Thus, the number 13 would be represented by two bars and three dots. Four bars are regrouped as one dot on the next higher level. Place values increase from bottom to top. That is, symbols on the lowest level represent groups of ones, and thus have values 1–19, whereas symbols on the second level represent groups of 20 and thus have values 20–380, and so forth. Participants began by exploring Mayan numerals in a homework assignment in which they converted between Mayan numerals and standard (base-ten) notation and performed simple addition problems with Mayan numerals. The assignment was taken from *Teaching Mathematics in the Middle School* (Overbay & Brod, 2007). Included in the homework was a reference table that listed the Mayan numerals equivalent to 1–29 in base-ten notation (for some sample references and problems, see Figure 2).

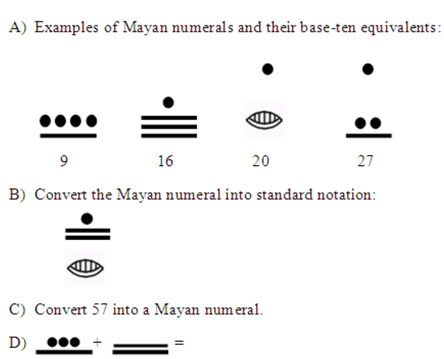


*Figure 2*. Examples of references and questions on homework assignment.

During the class period following the Mayan homework assignment, PSTs worked in groups to answer the questions in Figure 3. The homework assignments and in-class writing assignments were collected and analyzed using open coding (Strauss & Corbin, 1990). The analyses of the answers to the questions in Figure 1 are reported in this paper.

Please answer the following questions

1. How would you represent 20 in Mayan Numerals, why?
    - How would you know that it represents 20 and not another number like 1?
2. What value does the number below have in our base-ten system?

Mayan Symbol

Please explain why you think your answer is correct:

3. What value does the number below have in our base-ten system?

Mayan Symbol

Please explain why you think your answer is correct:

*Figure 3.* In-class task for preservice teachers.

## Results and Discussion

Only 7 of 25 PSTs in the class correctly explained the regrouped 1s in a three-digit addition problem (see Figure 1) at the beginning of the class. Nine PSTs said that both 1s represented 10, and 8 PSTs said that both 1s represented 1.

Table 1. *PSTs' Explanations for the Regrouped 1s in Three-Digit Addition on Pretest*

| Values attributed to the 1s in<br>1 1<br>389<br>+ 475<br>864 | 1—10 regrouped from the one's place<br>1—100 regrouped from the ten's place | 1—10 (from the 14)<br>1—10 (from the 16) | 1—1 (from the 14)<br>1—1 (from the 16) |
|---|---|---|---|
| Number of PST responses | 7 | 9 | 8 |

Developing an understanding of the values of the digits in a number is nontrivial (Thanheiser, 2008). The Mayan-numbers task was designed to illuminate the PSTs' conceptions of how numbers are built on the underlying structure of the number system.

All PSTs correctly represented 20 as *dot, shell* for Question 1. This result is not surprising inasmuch as PSTs used the Mayan numeral for 20 in their homework assignment. The correct answer for Question 2 is 400 in the base-twenty system in which the value of *dot* in the third place is $20^2$, $20 \times 20$, or 400. Only 11 PSTs correctly answered this question (see Table 1), and

13 PSTs incorrectly answered 200 on the basis of (a) interpreting the *dot, shell* as 20 and the *shell* as a zero that is appended, (b) interpreting the *dot, shell* as 20 and appending another place value, or (c) using the fact that the number had three symbols to assign the highest place value as hundreds. J, for example, explained, “Dot, shell = 20, and shell = 0. [The] shell symbol represents the 0 place holder, so the 0 will be added [meaning *appended*[2]] to the 20 making the number 200.” E stated, “There is a symbol for 20 with one more for the place value.” And T explained, “Because dot, shell represent 20 and another shell underneath represent another place value, hundreds.” These answers reflect the rules of our base-ten system in which a symbol with 1 zero has a value in the tens and a symbol with 2 zeros has a value in the hundreds (e.g., a 3 with 1 zero is 30, and a 3 with 2 zeros is 300). PSTs are able to read such numbers without thinking of the underlying structure of our base-ten system. This structure relates adjacent place values by a multiplicative factor of 10 (i.e., in 33333, the second 3 from the right is 10 times the value of the first 3 on the right, etc.). This multiplicative relationship between number places is not obvious to PSTs (Ross, 2001; Thanheiser, 2005). The rule of appending a zero for each place value when we move to the left combined with the fact that most adults are fluent readers of numbers masks the fact that the value of each digit in a number is 10 times the value of the digit to its right. More than half of the PSTs in this class did not utilize this multiplicative relationship to determine the value of *dot, shell, shell*.

All 13 PSTs who incorrectly interpreted *dot, shell, shell* as 200 incorrectly answered Question 3. Twelve interpreted a dot with 6 shells as 2,000,000, and one interpreted it as 20,000,000. Justifications were consistent with those used for 200 in that most students discussed appending zeros to a number each time they saw a shell. For example, W explained, “As the above problem, we used 20 because it’s a base-twenty system and then added the zeros after.” W explicated her understanding of a base-twenty system as a 20 with zeros added for increasing place values. Instead of viewing the succeeding values of the symbols in a base-twenty system as 20 times as great as the preceding, W saw the values of the symbols as increasing according to appended zeros. Other PSTs’ ideas were similar to W’s. J explained, “Dot, shell represents 20 plus 5 shell so add 5 zeros to 20, which equals 2,000,000.” C explained, “Dot, shell is our 20, and the remaining shells stand as place holders.”

As noted, all 13 PSTs who incorrectly answered Questions 2 and 3 had correctly said that *dot, shell* represents 20. Their Question 1 explanations often lacked the specificity needed to determine how they thought about the number. H, for example, explained, “The dot above stands for *20,* and the shell is zero. So 20 + 0 = 20.” If not probed further, this answer could represent multiplicative thinking; her explanation for *dot, shell, shell* as 200, however, revealed that she was thinking of the shells in terms of place holders to the right of a *2*: “Two shells are place holders.” Similarly C explained *dot, shell* as “top is one group of 20; shell means zero.” Again this response could be indicative of multiplicative thinking, but her explanation for *dot, shell, shell* as 200 revealed that it was not: “Dot zero is 20 so if you add a zero (shell),” it is 200. Thus, because the PSTs knew that *dot, shell* represented 20, some gave explanations that sounded multiplicative but that when probed we saw were not.

Table 2. *The PSTs' Interpretation of Mayan Numerals*

| Task | | Dot shell shell and dot with 6 shells | |
|---|---|---|---|
| Responses | 200 and 2,000,000 / 20,000,000 | 400 and other incorrect second response | 400 and 64,000,000 |
| Nature of response | Both incorrect | First correct second incorrect | Both correct |
| Number of PSTs giving the response | 13 | 5 | 6 |

Of the 11 PSTs who correctly answered Question 2 as 400, 5 gave incorrect answers for Question 3. Four of these 5 PSTs explained the 400 for Question 2 similarly[3]: Because of the 20 in the second place value, the value is 20 × 20 with a zero in the first place value, yielding 400 + 0. Three of these 4 gave 40,000 as their answer to Question 3.[4] K, for example, explained, "Dot shell is in the raised spot and is multiplied by 20 which equals 400, then the shell is equivalent to 0, so 400 + 0 = 400." For Question 3 she explained, "The top = 400 then the next two reps. a place value of 0 in the 1000's + 10,000's spot." D used a similar explanation for *dot, shell, shell:* "It is 20 groups of 20 (shown by the top) and zero on the bottom—20 × 20 = 400." For Question 3 she explained, "Mayans may have written numbers from top to bottom rather than left to right like we do." D and K seemed to see the *dot shell* as one numeral (20) on the second level worth 20 *twenties* rather than *dot* as a 1 on the third level worth 1 *four-hundred*. This is equivalent to seeing 500 in the base-ten system as 50 tens and 0 ones. This is a correct interpretation of the number, but interpreting the *dot, shell* as one symbol masks the nature of the place-value structure as iterations of multiplication by 20. D's interpretation does not show that she understands 20 *twentie*s as equivalent to 1 *four-hundred*. This connection between 20 twenties and 1 four-hundred is an important connection for understanding these numbers. In our base-ten system it translates to being able to see 100, for example, as 1 *hundred* as well as 10 *tens*. This connection has been shown to be nontrivial for PSTs (Thanheiser, 2005). In addition 100 is connected to 100 *ones,* as is inherent in the place-value language in the base-ten number system. This connection becomes important, however, when we move beyond interpreting the symbol for hundreds to interpreting symbols for larger numbers and operating on numbers. G explained the answer 40,000: "Mayans might've written numbers top to bottom & two symbols (top & bottom) make a number set & just write it out in order. Shell, shell = 0 (because 0 × 20 + 0 = 0) and the second set of shell, shell is also zero, so place it next to 400." Thus G seemed to interpret Mayan numerals in double-digit sets. She seemed to think two shells together were equivalent to appending a zero as a placeholder in the base-ten notation of the number. She seemed to interpret the dot followed by six shells in three parts: *dot, shell, shell* on the top is 400, *shell, shell* is a zero placeholder; and the following *shell, shell* is another zero placeholder, which gives 40,000.

The interpretations of *dot shell* as 20 and thus *dot, shell, shell* as 200 or of *dot, shell, shell* as 400 and thus a dot with 6 shells below it as 40,000 may be rooted in the notion that in base ten we merely append zeros to a number. In base ten, this practice of appending zeros is equivalent to multiplying by 10, but in different base appending zeros is equivalent to multiplying by that base (i.e. in base 4 the number 12 multiplied by 4 is 120). One reason to work with PSTs in a different base is to explicate some of these underlying structures of our base-ten system. PSTs often operate on numbers without understanding the reasons underlying those operations (Ball, 1988; Ma, 1999), and because the language facilitates the connections between numbers and values, these underlying structures are often difficult to explicate in base ten.

All 6 PSTs who correctly answered Question 2 (400) and Question 3 (64,000,000) indicated the use of iterative multiplication by 20 as their source for their answers to Question 3. Z, for example, justified his answer that dot followed by six shells was 64,000,000 as follows: “Each level is 20 of the previous level. $1 \times 20 \times 20 \times 20 \times 20 \times 20 \times 20 = 64{,}000{,}000$.” Similarly, A explained the same answer: “There is a pattern where each dot on each level is 20 times more than [*sic*] the dot on the previous level,” indicating that she understood the nature of a base-20 number system in which the value of each place is 20 times the value of the place to the right. However, not all 6 of these PSTs used this reasoning to answer Question 2. Three PSTs argued that one must regroup to the next higher level once there are “too many” in the second place. M, for example, stated, “Because the dot is the 3rd level, and once you are at 4 bars, you go up another level for a dot. Each bar in the second level equals 5 dots, and thus $5 \times 20$ or 100. Thus four bars equal 400 and would need to be regrouped into a dot at the next level.” This is a valid argument, but impractical for explaining large numbers. Two PSTs used the argument that they had a 20 in the second place value; thus $20 \times 20 = 400$. One PST gave no explanation.

## Conclusion

The Mayan-numeral activities described were designed to assess PSTs’ understanding of the underlying structure of numbers in our place-value system. Although the PSTs had worked with 2-place Mayan numerals for homework, more than half of the participants were unable to correctly interpret 3-place Mayan numerals. Most PSTs did not attend to the underlying structure of base twenty. They either interpreted the *dot* as a 2 and appended 2 zeros for the two shells or interpreted the *dot, shell* as 20 and appended 1 zero. Appending a zero to a digit is equivalent to multiplication by 10 in our base-ten number system, but doing so in the Mayan (base-twenty) system is equivalent to multiplication by 20. Although the PSTs were able to read a *dot shell* in the Mayan system as 20, most were unable to assign the correct value ($1 \times 20 \times 20$) to *dot, shell, shell*. The underlying structure of numbers in our base-ten system remains implicit when PSTs work with numbers in base ten. PSTs become accustomed to reading base-ten numbers, in which each digit is 10 times the value of that to the right of it. In doing so, they may miss the number meaning in the base-ten number system. Making the structure of the base-ten system explicit while working within base ten is difficult because the labels we use for the places (ones, tens, hundreds) are the same as their quantities; thus, whether PSTs are using labels or quantities is unclear. In coming to understand a different base, PSTs must explicate the underlying base structure and relate it to our base-ten system. A thorough understanding of number and place value is essential for teachers, especially elementary school teachers. With follow-up discussion and practice, the activities used in this study may be effective for helping PSTs recognize the complexities of all number systems, including our base-ten system. By working with a number system with which they are less comfortable, PSTs can explore the meaning of a place-value system. We close with two PSTs’ reflections:

- The most important thing that I learned about my own understanding of the base-ten system I realized through discovering the Mayan number system. … Just as the Mayans multiplied each new level by 20 since it was a base 20 system, our system is really just multiplying each place by ten more then the line [place] below. For example in the number 326, this can be seen as 6 + (2*10) + (3*10*10) since each level up is another times ten.
- We know base-ten means that each place value is multiplied by 10, and that we group and regroup when a certain place value exceeds 10. … What really broke it down for me was when we got into working with the Mayan system. It allowed me to compare and contrast the difference between their system and ours. Theirs is a base-twenty system and follows the same concept as us, except they would regroup after 20 in a place value (level) and each level is multiplied by 20 instead. I first struggled with any number beyond the second level in the Mayan system, but as we discussed more in class and I did some self-discovery at home, I came to see how similar our systems are.

## Endnotes

[1] In our class we considered the Mayan numbers as base-twenty numbers. In the literature the Mayan numeral *dot shell shell* can be found as representing 18 × 20 rather than 20 × 20 as we would expect in a base-twenty system. We disregarded this aspect in this lesson (Bennett & Nelson, 2007).

[2] Students often use the *adding* zeros when they mean *appending*.

[3] The other PST gave a generic explanation: "In order to get *400*, … the top number must be multiplied."

[4] The other PST gave 64,000 as her answer; her explanation, however, was insufficient for us to interpret.

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# INFLUENCES TO WANT TO TEACH MATHEMATICS IN AN URBAN SETTING: A PRE-SERVICE TEACHER'S STORY

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*What influences a pre-service teacher to decide to teach in an urban school? Despite challenges associated with teaching in urban schools, there are per-service teachers making this decision. This study takes a phenomenological look at a secondary mathematics pre-service teacher's, who has made the conscious decision to teach urban learners, journey through student teaching. It details factors that influenced her decision to teach in urban schools and how student teaching further effected her decision.*

## Introduction

There has been a plethora of literature on the shortage of teachers in urban environments and an ample amount of research dedicated to acknowledging the need for increased teachers of mathematics (Bracey, 2002; Cavallo, Ferreira, & Roberts, 2005; Follo, Hoerr, & Vorheis-Sargent, 2002; Howard, 2003; Ingersoll & Smith, 2003; Ng, 2003). According to a study by Villegas and Clewell (1998, as cited in Valli & Rennert-Ariev, 2000), 13% of 3000 teachers state they do not want to teach in an urban setting. The participant for this study, however, has made the conscious decision to teach in an urban school. Thus, this research will provide a glimpse into possible influences that encourage or hinder a teacher candidate from continuing a career as a teacher of urban learners after completing student teaching in an urban school. Understanding possible influences may also provide insight into the retention of new teachers in urban schools.

## Philosophical Framework

The tenets of phenomenology provided the philosophical framework for this study. "Phenomenology is oriented…toward describing the experiences of everyday life as it is internalized in the subjective consciousness of individuals" (Schwandt, 2001, p. 191). Therefore, the crux of phenomenology is to understand the experiences of an individual from his or her own unique perspective. Schutz (1967) combined the tenets of Husserl's descriptive phenomenology with Max Weber's sociology. In his version of phenomenology, as one reflects on an experience, the reflection is not purely based on the individual but is influenced by social interactions with others. Conversations, other experiences, and subconscious and conscious thoughts influence how one reflects on an experience. These reflections provide a foundation for the construction of knowledge (Wagner, 1970). Schutz acknowledged the importance of interpreting phenomenon in context and he also valued the social influences that impede upon the perceptions of the experiencer (Schutz, 1967; Wagner, 1970). Schutz's standpoint on phenomenology encompasses both the internal processes involved in knowledge construction and the social aspects that influence that development. The influences of the interactions between Tanjala (the pre-service teacher of this study) and her cooperating teacher, her students, other teachers, and parents factor into how Tanjala interpreted her experiences and thus influences her decisions regarding maintaining a career in an urban environment.

There are two attitudes that are presented in the philosophy of phenomenology. They are the natural attitude and the phenomenological attitude.

> The *natural attitude* is the focus we have when we are involved in our original, world-directed stance, when we intend things, situations, facts, and any other kind of objects. The natural attitude is, we might say, the default perspective, the one we start off from, the one we are in originally. We do not move in to it from anywhere more basic. The *phenomenological attitude*, on the other hand, is the focus we have when we reflect upon the natural attitude and all the intentionalities that occur within it. (Sokolowski, 2000, p. 42)

The natural attitude and the phenomenological attitude offer viewpoints of how the Tanjala experiences student teaching as she develops her views on teaching in urban contexts. The natural attitude illustrates her naïve thoughts on teaching in urban schools. The phenomenological attitude then validates or contradicts Tanjala's initial thoughts as she reflects upon her experience. Both positions are depicted through her journey.

## The Participant and Research Methodology

A phenomenological approach was taken to gain insight into the phenomenon from the participant's perspective. The participant, Tanjala, is a 40-ish African American woman, who was enrolled in an alternative teacher preparation program. The program in which she attended admits individuals who have a degree in mathematics or a related field. Through an intensive four semester program the students earn a Masters of Arts of Teaching degree along with their initial certification. This program, located at a large metropolitan university in the southeastern region of the United States of America, has obtained funding to financially support students who wish to focus on teaching urban learners. Tanjala was awarded $10,000 to support her as she matriculated through the program.

Data was collected in several phases. Tanjala participated in an interview prior to any field experience. She shared stories of her years as a student of mathematics, how she came to want to be a teacher, more specially a mathematics teacher of urban learners, how she defined urban, and what she expected to gain from her student teaching experience which would be in an urban setting. As Tanjala completed her student teaching experience, she maintained journals detailing her successes and challenges. After student teaching was completed, she shared her reflections of the experience during a phenomenological interview.

## Data Analysis

The three data sources were used to construct a textual description of Tanjala's experience. The textual description is a rich and detailed account of the participant's experiences prior to student teaching, during and after. The textual description provides Tanjala's naïve and phenomenological attitudes (Colaizzi, 1978; Moustakas, 1994; Polkinghorne, 1989).

Tanjala was provided an opportunity to read her textual description and provide approval or suggest revisions. Tanjala's has approved her textual description as it appears as accurate and acceptable (Colaizzi, 1978). From the textual descriptions, the meaning of the phenomenon is drawn, providing structural descriptions. To form the structural description, I began the analysis by reading the data to gain an in-depth awareness. Next, I separated phrases that pertained to the phenomenon from non-revelatory material. Once only relevant material remained, the statements were clustered, themes emerged, and then were phenomenological reduced into eight themes. These eight themes are (a) evidence of mentoring versus lack of mentoring; (b) individual versus collective behaviors; (c) relating to the characteristics of an urban teacher; (d) to stay or not to stay (in urban schools); (e) influencing factors; (f) anxieties; (g) classroom management; and

(h) pedagogical style. A zigzag approach between these themes and the participant's narratives was utilized to establish the structural descriptions of the phenomenon (Colaizzi, 1978).

### Tanjala's Textual Description

I open Tanjala's textual description with a story that she shared with me that affected why she wanted to be a teacher.

> I was born into a family of nine children. I am number seven of nine. My father was an alcoholic and he was really abusive to my mother. We would see him give her black eyes, split her arm, do this, do that, and on and on. When I was 7, my father and mother had an argument. My father went into another room and got a gun. I think at this time they had been married 15 years. She was really tired and she let him know. She said I know it is loaded so go ahead and shoot. She told us to get ready to go over to her sister's house. My brother ran out to call the police. When the doorbell rang, my father said, "If it is the police I'm going to shoot you." In the meantime, they were arguing about something, and she was pressing my hair. I remember being on the floor by the stove while she was pressing my hair. My father went to the door and of course it was the police. So he came back and he shot her. We actually saw her die. The next day we went to live with my auntie. My auntie, God bless her soul, was such a strong lady. She took 8 of us. She lived in a project in a large metropolitan city. This happened in 1971. The projects were bad, but they weren't horrid, like they are now, but there was still a lot of crime. There were a million people that were kind of unmanaged living in these conditions. At my auntie's house I learned to love school. School is where I could found safety, where I could find consistency. So I just kept running to school. When I became a teenage, that place [of safety] became church. Church is where I found love and consistency. So I kept running to there. That whole experience happened over two decades or even longer of my life. This is how I developed a love for education and for helping people. My brothers would say if you want to get out of the projects you needed to go to school. So school was no problem. No question for me. I loved it anyway, so I am going. All of my brothers and some of my sisters had a love for education. So that is basically the foundation of why I am here. I really love school and I want to help women who sometimes feel like they are trapped. I think my mom may have felt trapped. It was the 70s. She didn't work. My father worked. He actually had three businesses: a moving business, a furniture business, and a trucking business. So he was doing really well, but I think she felt trapped in that abusive situation 'cause she didn't work and she had 9 kids. So I want to help women or help people, specifically women, who are in situation no matter why or whatever the cause, to know that they have options and education always brings options. It is not the only option, but it helps to open you up to a lot of different options that weren't available before.

Prior to making the decision to become a teacher, Tanjala worked as a data analyst but found no personal rewards in this work. Therefore she took a sabbatical and worked as an urban missionary. Tanjala found this work extremely fulfilling and thought, "I would make an even better impact as a professional teacher." So at the age of 42, Tanjala enrolled in an alternative preparation program to become a secondary mathematics teacher of urban learners.

Through her childhood experiences, Tanjala had a connection with the urban community, which she defined as "a metropolitan area, inside the city limits...a ghetto…ghetto meaning lower economic status." She believed that her connections would aid her in reaching urban youth. Connecting the content to the students' life would assist her in teaching mathematics and her cultural connections with the students would support positive relationships and interactions. Tanjala also believed that creating a safe environment, where the students can ask questions without being ridiculed, provided further benefits to an effective learning environment.

In preparing to teach urban learners, Tanjala wanted to learn how to put information she learned from books into action. She wanted a mentor who would be open to sharing his or her experiences as well as communicate openly. A perfect mentor for Tanjala was "somebody who is more experienced, walking besides [her] through an experience, to guide [her] or just be there as a sounding board and possibly to be a model."

Throughout Tanjala's student teaching experience, she was constantly seeking guidance for improvement in her performance as a teacher. She used resources, such as books, peers, other teachers, her college supervisor, and her cooperating teacher, Ms. Stanley. These resources provided a source for Tanjala to improve upon her pedagogical skills. Below are some excerpts to substantiate this claim:

> My cooperating teacher has modeled a lot of constructivist activities for me and I feel comfortable using them in my lesson plans. However, I am still lost as to how to make Algebra more appealing. I will talk with my peers and supervising teacher for help.
>
> Started using the foldables a few weeks ago and they were a lifesaver. I found that the students are more engaged when using the foldables and they produce "neater" work, namely graphs. I am glad that my mentor teacher introduced me to this tool.
> I observed another seasoned teacher today. The objectives were clearly written along with the agenda. The teacher was VERY effective in leading a class discussion on finding the roots of polynomial equations. When one student inquired about the definition of a root, she asked the entire class if they knew the definition. When no one was able to explain, she said that I will draw a picture and you tell me what you notice. She drew x-y axis, an up-open parabola and from the x-axis began to draw flowers, trees, daisies, etc. The students began guessing and eventually stated that a root is where the parabola crosses the x-axis. This discourse was amazing. The students were actively engaged the entire session. I was so encouraged by seeing her example.
>
> Spoke with my peer about the review and I was given some good tips on conducting review sessions. One method suggested was to give the review on a day prior to the scheduled quiz/exam/test, and provide sample problems so that the students can practice/study for the exam. On the day of the exam, ask students if there are any questions about the problems. Do not work the problems, instead have the students articulate the concept, process or step that presented their problem and briefly discuss that issue.

The advice Tanjala received helped her to form reflective practices as well as be observant of her students' behaviors that may affect their opportunities to learn. Even though Tanjala was audacious enough to seek guidance, she had anxieties about being effective in an urban classroom:

> I am feeling so overwhelmed. Why am I doing this again???? Oh, I love the city and want to serve in the city. Lord, I could really use a lift today. I could use some event or something that would make it easier for me to choose to be positive.
>
> Today, I assessed the students on the four methods used to solve quadratic equations. The assessment was planned to be in the form of a game. Before the assessment, I conducted a review of the key issues. The review took a very long time. I feel as if I had to re-teach the lesson. The students apparently did not "get" it the first 3 times I taught it. I feel frustrated because I am running out of ideas to actively engage the students in learning.
>
> I am so tired and I am not quite sure why, the work of preparing lessons daily or the emotional work of keeping a good attitude in this atmosphere. I know that I am called/purposed/destined to be in education—so I will press on. I sure can use a break.

In spite of these anxieties, Tanjala commanded respect and took a position of authority in the classroom by not allowing students to interrupt the learning environment or break school or classroom rules:

> I told one pair of students, who I caught kissing when I exited the restroom that they had to clear the halls. The male student was very upset that I stood there until they began to walk towards the exit.
>
> During my discourse, I just fell silent, looked straight ahead and said that I will not be able to continue until there is silence. The silence lasted for almost thirty seconds and then there were distractions from two members of the classroom. I asked to speak with one young man, who by co-incidence was White, outside. During our conference, I asked if everything was all right with him and if I could help him in any way. He said that everything was fine; he just did not like me. I began to laugh on the inside and told him that his "liking me" was not a requirement for the class, but his respecting the class guidelines and me was a requirement. I asked him if he was ready to respect the rules of the school and the classroom. He said "no." I told him that he would not be able to return until he was ready to do so. He did not return until after the bell rang to collect his belongings. This challenge of my authority was intense and I felt a little threatened by his influence on the class' behavior.

After student teaching was completed Tanjala responded that she was glad it was over. Tanjala realized that even though she had dealt with challenging situations during student teaching; she had anxieties about her personal safety and the support she would receive from school administration. Once she completes her scholarship requirements, she stated she would like to teach at the college level.

Tanjala saw herself as a compassionate teacher with a passion for teaching mathematics. She viewed herself as someone who strived to help others accomplish their goals. Her confidence in the classroom was correlated to her pre-lesson preparation. If she rehearsed the lesson several times prior to teaching it, she felt more confident in her abilities. Through her experience she viewed teaching as an activity where "I do it, we do it, you do it." However, she planned to incorporate the hands-on method of "foldables" introduced to her by her cooperating teacher. Tanjala attributed the construction of her identity to her faith in God, diligent search of the literature, and the events of her childhood:

> The compassion I believe really came from my faith. It is attributed to God because I really had to learn how to love people when they come to you and say I just can't stand you. Okay let it bounce off kind of thing. That was just something in my personal life that I had to learn. As far as being an effective teacher in other areas, I just kept searching. I was like I know there is somebody out there who has an answer for me on how to do this successfully and that is why I read a lot. I just kept looking and looking and looking. One day the light came on. (Williams, 2007, pp. 50 - 55)

## Conclusions

It has been reported that most pre-service teachers prefer to teach in schools similar to the schools they attended, thus teaching students who are similar to them (Quality Counts, 2000). According to Zeichner (1996) economically disadvantaged students of color traditionally do not excel with teachers from different cultural backgrounds then their own. Therefore, one would presume that Tanjala, who grew up in a poverty-stricken urban environment, would thrive in such an environment. In fact, she mentioned in her textual description that she felt that personal history would help her connect with her students. However, classroom management challenges led her to have reservations about teaching teenagers:

> I had a student. I think I talked about him the journal. He appeared to be a White student and his mom was . . . he was mixed with something. But anyway, from the first day that I started teaching, his class was a geometry class. He [would] just sit in the back of the room and I could feel him like staring at me that did not bother me. So I would just keep teaching and on and on, but like a week or two later he would start really acting out while I was teaching, like I could be in the middle of saying something about a circle or inscribed triangle or whatever and he would just bust out HA or whatever. He would say something disruptive and the first time I did call him on it and I said is there a problem? Is there something I can help you with? No, Huh and he would be sort of a little disruptive. He had two friends that he was always sitting with and they were never disrespectful to me, the two friends. They could get disruptive in the class, but they were never disrespectful to me, but I always felt like he was attacking me. So one day I asked him . . . one day after he was going through his normal "ha . . . I don't want to do this" and his grades was really low and if I looked at his handwriting. He writes like a 3rd grader. I mean it is probably not even a 3rd grader, but a second grader. It is never straight on the line. None of his letters are even, even if he is writing on lined paper. It is always up and down and up and down and his letters are not always complete. The "A" and it would not be in the middle, but one day I asked him to step out into the hall so we could talk for a minute while everyone else was working and I said what is the problem? Are you doing well today or are you having a good day? Yeah and he would never look at me and he would be like Yeah. I would say okay. Is there a problem with your work or do you need some extra help with it? Can I help you do it or something? "No." So why aren't you doing it? "I don't want to" So I said is there a problem with me? I forgot what else I said, but I jumped to the quick of it...is there a problem with me? "No, I just don't like you". So I look at him. At first I was like ouch and then I said okay and I laughed and said you don't have to like me that is not going to affect…that is not what I am grading you on, but you have to respect the rules of the classroom and while I am talking there can be no other talking unless

> I am asking you a question. There can't be any disruptive talking or something I said and he said HUH and I said well are you ready to respect the classroom rules and he said NO. I said well I don't think you should enter classroom the again until you are ready to respect the rules and he was like well and I was like okay then you can stand right here until you are ready, let me know when you are ready and he did not come back until the end of the class. Then I let him in to get his stuff and went on. He was a little challenged (Williams, 2007, pp. 101-103).

Despite the challenges of classroom management, Tanjala remained encourage to be successful at teaching urban learners. After Tanjala narrated this story, I asked her how she felt now about handling challenging situations. She replied:

> He gave me a lot of experience. Now I am . . . I mean really that was good experience and now I can go back I just need to know I am the teacher. That is the fact. I am the teacher and he does not have to like me, but it is my responsibility and role to make sure that the class climate is safe for other people to learn (Williams, 2007, pp. 104-106).

## Discussions

Various factors may influence pre-service teachers to make the decision to establish careers in urban schools. For Tanjala, is was the connection with the urban life and desire to want to help young women know they have options through education. However, student teaching in an urban environment may increase pre-service teachers' ability to teach urban learners, it may also reduce their desire and commitment to establishing a career within urban schools (Wiggins & Follo, 1999). This reduction may be due to the factors often connected with urban schools including, but not limited to, issues of classroom management, lack of resources, lack of teacher support, and lack of student interest in their own education (Bracey, 2002; Brown, 2002; Gormley, Hammer, McDermott, & Rothenerg, 1993; Ingersoll & Smith, 2003; Matus, 1999). Despite these challenges there are teachers, like Tanjala, who want to make a difference in the lives of urban learners. It is the responsibility of teacher educators and the larger community to nurture these teachers into what Haberman (1995) refers to as "star teachers": teachers with persistence, resilience, resourcefulness, a connection with students, and a board perspective on factors that contribute to student failure.

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# DURABILITY OF PROFESSIONAL AND SOCIOMATHEMATICAL NORMS FOSTERED IN A MATHEMATICS METHODS COURSE

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*This study investigated the extent to which seven professional and sociomathematical norms intentionally fostered in a mathematics methods course through the use of a video case professional development curriculum re-emerged in a later course with a different cohort. All seven norms were evident, to varying extents, in the written analysis and group discussion of 11 prospective teachers who engaged in a video case analysis similar to those they had participated in during their first methods course one to four semesters earlier. This paper discusses the norms, evidence, and counterevidence for their re-emergence, and implications for teacher preparation.*

## Objectives

A key intended outcome of our mathematics teacher education program is for prospective teachers to experience *self-sustaining generative change*, defined by Franke, Carpenter, Fennema, Ansell, and Behrend (1998) to involve "teachers changing in ways that provide a basis for continued growth and problem solving" (p. 67). This paper analyzes the durability of one component of our efforts to achieve this outcome—the development of professional and sociomathematical norms embedded in the Learning and Teaching Linear Functions (LTLF) video case professional development curriculum (Seago, Mumme, & Branca, 2004). Specifically, we address the extent to which professional and sociomathematical norms intentionally fostered in a mathematics methods course re-emerge in a similar context later in the program with a different cohort of prospective teachers.

## Perspectives

Since the identification of sociomathematical norms as critical contributors to school mathematics learning (Yackel & Cobb, 1996), a growing body of research has investigated the subtle power of these norms to support the development of mathematical learners (e.g. Kazemi & Stipek, 2001; McClain & Cobb, 2001). More recently, researchers have turned their attention to the role of such norms in supporting teachers' learning during professional development (e.g. Elliott et al., in preparation; Grant, Lo, & Flowers, 2007).

Recognizing that sociomathematical norms have the potential to support teachers' learning, Seago, Mumme, and Branca (2004) incorporated the development of such norms, as well as a set of professional norms, into the LTLF materials that we adapted for use in our mathematics methods course. These professional and sociomathematical norms are listed in Column 1 of Table 1 and form the basis of our study. We see these norms as important to preparing teachers in three ways: (1) supporting the development of their own mathematical understanding;

(2) learning to view and analyze classroom practice in productive ways; and (3) thinking about what norms should be developed in mathematics classrooms with students.

One of the challenges of investigating norms is the difficulty of determining normative behavior from a snapshot of practice. For example, in one class session, there may not be time for each participant to present a solution to a mathematical task, but if most who do include a mathematical argument, a reasonable inference is that mathematical argumentation is a norm for that class. On the other hand, if few include a mathematical argument, it is safe to conclude that it is not a norm. Another way to infer the presence of a norm is when the norm is not exhibited and this is recognized and corrected by other members of the group.

We approach both the development of the program and our research from a situated perspective (e.g. Borko et al., 2000). That is, we generate learning situations that are similar to those in which we intend prospective teachers to use the learning in their future teaching, and we study the way in which they interact in these situations. We also follow Cobb, Stephan, McClain, and Gravemeijer (2001) in our interest in coordinating the social and psychological perspectives. For the study reported here, this means that we have concerned ourselves with evidence of professional and sociomathematical norms in both group interactions (social perspective) and in the prospective teachers' individual written work (psychological perspective).

## Modes of Inquiry

The participants in the study were 11 prospective mathematics teachers (PTs) enrolled in their final mathematics methods course who had used the LTLF video case curriculum in their first mathematics methods course one to four semesters earlier. During one 80-minute class session, these PTs were engaged in a video case discussion similar to those they had participated in during their first methods course. Prior to the session, the PTs solved the mathematics problem (see Figure 1) individually (Data Source 1) and predicted possible student correct (DS 2) and incorrect (DS 3) thinking. The session began with a group discussion about the mathematics. Next, the PTs watched a video of middle school students discussing their thinking about the same problem and responded in writing to questions about what they noticed (DS 4), student thinking (DS 5 and DS 6), and teacher actions (DS 7) in the video, and then engaged in a group discussion about these ideas. At the end of the session, the PTs reflected in writing about what they learned from the discussion (DS 8). In addition to these written data sources, the group discussions of the mathematics and the video were videotaped and transcribed (T).

Study the sequence of cube buildings below. Assuming the sequence continues in the same way, how many cubes will there be in the 4th building? The 17th building? The nth building?

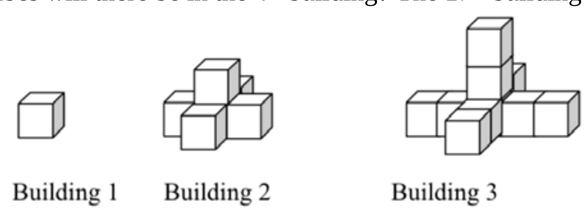


*Figure 1.* Counting Cubes problem solved by the PTs and the students in the video. The problem and the video are from the *Turning to the Evidence* project (see Seago & Goldsmith, 2005).

The first two authors facilitated the session. Both had taught the first methods course, but half the participants had taken it from other instructors. The facilitators made a point of not introducing professional or sociomathematical norms in order to see if any of the norms established in the first methods course would spontaneously re-emerge.

In order to analyze the data for evidence of the seven professional and sociomathematical norms (see Column 1, Table 1), transcripts of the mathematics and video discussion and all written work were coded independently by at least two researchers for examples and counterexamples of each norm. Any differences were resolved through refining the code definitions. The analysis was completed using multiple charts that cross-referenced evidence and counterevidence of each norm by PT. These charts were then collapsed into the summary chart shown in Table 1.

## Results and Discussion

*Overview*

Table 1 summarizes the professional and sociomathematical norms exhibited by each PT, as well as the identifier of the data source in which each norm was exhibited [1-8, T]. “C” indicates that a counterexample of the norm was identified. For example, Abby’s “TC” for *talking with respect yet engaging in critical analysis* indicates that she was critical, but not respectful of the teacher in the video in at least one instance in the class discussion.

The bottom two rows list the number and percent of speaking turns during the discussion. It is worth noting that Hana spoke only during the mathematics portion of the discussion when she presented a possible way that students might think about the problem, Abby spoke at the end of the video discussion after being encouraged to do so by the facilitator, and Ruth did not speak at all, despite a direct invitation.

Table 1.
*Professional and Sociomathematical Norms Exhibited by Prospective Teachers*

| | Norm | Abby | Evan | Hana | Iris | Jim | Ken | Leah | Lily | Lew | Roxy | Ruth |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Professional Norms | Listening to and making sense of or building on others’ ideas | 4, 6 | T, 6, 8 | 7 | 8 | T, 8 | T, 1, 4, 5, 8 | T, 4, 8 | 4 | T, 8 | T, 4 | |
| | Adopting a tentative stance toward practice – wondering vs. certainty | 8 | | | T | 8 | T | T, TC | | T, 8 | T, TC | |
| | Backing up claims with evidence and providing reasoning | T, 7 | T | | T | T, 7 | T | T | 7 | T, 4, 6, 8 | T, 4, 6 | |
| | Talking with respect yet engaging in critical analysis of teachers and students portrayed on the video | TC | T | | T | T, TC, 7, 8 | T | T, 4 | T, 6 | T | T | |
| Sociomathematical Norms | Naming, labeling, distinguishing, and comparing mathematical ideas | 4, 5, 8 | T, 4, 5, 6, 7 | 4, 5, 7 | T, 4, 5 | T, 4, 5, 8 | T | T, 4 | T, 4, 5 | T, 4, 8 | T, 6, 7, 8 | 4, 7 |
| | Using mathematical explanations that consist of a mathematical argument, not simply a procedural description or summary | T, 1, 4, 6 | T, TC, 1 | 1 | T, 1, 5 | T, TC, 1, 6 | T, 1, 6 | T, 1, 6 | T, 1, 4, 6 | T, 1, 5 | T, 1 | 1 |
| | Raising questions that are related to the mathematics and push on understanding of one another’s mathematical reasoning | 7, 8 | 4 | | 7, 8 | T, 7 | T, 4, 7, 8 | T | | 4 | T, 7 | 8 |
| | Total Speaking Turns | 3 | 21 | 9 | 12 | 22 | 18 | 15 | 6 | 17 | 19 | 0 |
| | % Participant Speaking Turns | 2% | 15% | 6% | 8% | 15% | 13% | 11% | 4% | 12% | 13% | 0% |

*Note.* Professional and sociomathematical norms are from Seago, Mumme, and Branca (2004).

As can be seen in Table 1, 3 of the 11 PTs (Iris, Ken, and Lew) exhibited each of the seven norms in the discussion and/or their written work with no counterexamples. An additional 2 PTs (Jim and Leah) exhibited each of the seven norms, but Leah also had a counterexample for the norm of *adopting a tentative stance toward practice*, while Jim had counterexamples of both *talking with respect, yet engaging in critical analysis* and *using a mathematical argument*. Three of the remaining PTs exhibited six norms overall (Abby, Evan, and Roxy), 1 exhibited five norms (Lily), and 2 PTs exhibited three (Hana and Ruth).

In the whole-group discussion, 7 of the 11 PTs (Evan, Iris, Jim, Ken, Leah, Lew, and Roxy) exhibited at least five of the norms. The other 4 PTs exhibited zero (Hana and Ruth), two (Abby), and three (Lily) of the norms, but they all participated in the discussion in a very limited way. As might be expected, there appears to be a general correspondence between the percent of speaking turns during the discussion and the number of norms exhibited, with the largest number of norms exhibited by those PTs who participated the most. Iris is an exception to this, however, as she had only 8% of the speaking turns, yet exhibited five of the seven norms.

To better understand evidence and counterevidence of the norms during the discussion, consider an exchange that occurred at the end of the session and contained Abby's entire contribution to the video component of the discussion:

F2: There's a few people that we haven't heard from yet. Wondering if any of you who've not talked a lot have something that you want to add to [previous comment]. Like Abby.

Abby: I didn't like the teacher. I didn't like that he didn't ask about the picture. I felt like he was feeding them the answers. I hated line 85. So I have nothing good to say about the teacher, which,

F2: Can you say a little bit more about what it was that—

Abby: I feel like he should have tried to make more connections about the picture, because, like we were talking about, group 2 didn't really know what the minus 4 was; it just worked for the numbers, and I think he could have tried to pull more out of them. Same with what Iris was saying about group 3. We know nothing about their method. He just kind of let them put that up there, and then used it to make the, the number connection about the expressions, the algebraic stuff. Like he could have made more connections with the visual, uh, aspect of it. And then I just didn't feel, like he just talked [brief pause] at the end, in line 85, like he could have let them explain that more.

F2: Okay. You were storing up a lot. [laughter] So there's several things to respond to there.

Jim: I felt the same way, because in line, 26, he, you know, "How many cubes would be in a [seventh] building?" And Cassie says "31," and how, you know, he asked how they got that. They've already made it apparent that they know how to substitute, because, before that, Cassie says that they first thought it was $5n + 1$, but they found out that it didn't work for the first one. And so, he just asks a redundant question, and, by asking them what the seventh one was, all they have to do is plug it in. But he never asks *why* it's $5n – 4$. He just, they give him the answer, he asks what the seventh one is, they give the right answer and it's over. But never why, where did the minus 4 come from? Again, never relates it to the picture.

Iris: I guess it kind of depends on what his teacher goal was for the day. [laughter] [T539-564]

Both Abby and Jim were coded as exhibiting a counterexample (TC) for the *critical yet respectful* norm because they did not talk about the teacher's practice in a respectful way during this exchange. Iris, on the other hand, shifted the conversation back toward showing respect towards the teacher by raising the question of whether his actions might have been appropriate for his goals for the day. Although it is difficult to see emotions in a transcript, the points at which the laughter occurred support the facilitators' sense that Abby's and Jim's comments made the group uncomfortable, and thus were not normative. It seems that the laughter served the role of diffusing tension caused by a violation of a group norm and the action taken by Iris to re-establish it. Also interesting to note is that both Abby and Jim used evidence to back up their claims, demonstrated that they had been listening to the prior conversation, and, in fact, raised valid concerns about the teacher's actions. Thus, even though they were not respectful, they were critical in a potentially productive way. In our experience, however, the respectful component is important to developing an atmosphere where teachers feel comfortable talking about their teaching and, in so doing, are able to identify and act on specific opportunities to improve their practice. If the group had not self-corrected, the facilitator would have used this exchange to point out the importance of being critical in a respectful way and to remind the PTs that their analysis should focus on the instance of teaching practice, not the teacher himself.

In the written work, all 11 PTs exhibited at least three norms, with 5 exhibiting five or more of the seven norms. The PTs demonstrated the norms in two ways: by participating in the norms themselves and by making a statement that indicated they recognized that the norm was important in the classroom. For example, Lew exhibited the norm of *using a mathematical argument* both ways. First, he made a mathematical argument to justify a mathematical expression: "A better more visual formula would be $5(n-1) + 1$, since I have five 'arms' that are $(n-1)$ long and one single 'central' block" [DS 1]. Second, he recognized the importance of the norm in the classroom when he reflected that the students in the video "were very thorough and were very aware of what everything stood for in their solution" [DS 5]. Each PT demonstrated between two and five norms in their own analyses and reflections, and made statements indicating they recognized the importance of between one and five norms. Looked at in another way, over half of the PTs made statements alluding to the importance of the norms that could be considered most relevant to their future classrooms: *listening to and making sense of others' ideas; backing up claims with evidence; naming, labeling, distinguishing, and comparing mathematical ideas; using mathematical arguments;* and *raising questions that push on others' understanding.* This is important as these future teachers will not likely work to establish these norms in their own classrooms unless they are explicitly aware of the norms and recognize them as important to developing mathematical understanding.

In sum, all seven norms fostered in the first mathematics methods course through the use of the LTLF materials were evident during a video case written analysis and group discussion that occurred at the end of the program with a different cohort. This is significant in that the PTs had not explicitly discussed the importance of these norms, nor were they reminded of the norms prior to the session. In addition, the PTs had participated in the prior video case discussions with at most three others, and thus had not constituted the norms as a group. Rather, the norms had been developed in four separate classrooms, yet appeared to re-emerge naturally during a similar discussion focused on analyzing teaching and learning.

To give the reader a further sense of what it means to exhibit the professional and sociomathematical norms examined in this study, we now turn our attention to a more detailed analysis of three norms that are particularly relevant to developing students' mathematical understanding and to teachers' continued professional development. We will first discuss results related to the professional norm of *backing up claims with evidence*—a norm that is critical to becoming a reflective practitioner. We then focus on the sociomathematical norms of *naming,* *labeling, distinguishing, and comparing mathematical ideas*, and *using mathematical arguments*, both which are critical to supporting students' understanding of mathematics.

*Providing Evidence*

All seven of the PTs who participated substantially in the video discussion, as well as one who did not, made at least one claim during the discussion that they backed with evidence and/or reasoning, referring either to specific line numbers in the video transcript or directly referencing the transcript in another way (e.g., "on page 1 it says …"). Of the 13 instances where PTs referenced the video transcript, 10 (77%) occurred without any prompting, three were prompted by the facilitator and one by another PT. Notably, all four PTs who were prompted to use evidence also had instances where they used evidence without prompting.

The following exchange illustrates how the practice of providing evidence to back claims was normative in the group:

> Ken: Well, Arden's [expression] works if you have his variable.
>
> Jim: Exactly. So it works for all of them, if you use his variable, which he specified in the very beginning. But apparently the girls weren't listening to what he said.
>
> Ken: Where did he explain that? Because I—
>
> Jim: Line 7. The equation was that 5*n* + 1 equals the volume, and *n* equals the length of one individual arm. So he told people, but everyone was just so caught up on the building number. [T326–332]

Here, one PT, Ken, actually prompts another to refer back to the transcript to back up his claim that a student had clearly defined his variable.

Yet another source of evidence of this norm is the use of line numbers in the PTs' written work, which was completed before the group discussion, and thus before any prompting was provided. In this case, four of the PTs cited specific line numbers in their reflections, while another provided more indirect evidence. It is important to note that using evidence does not come naturally to PTs and, in fact, takes some time to develop in the first methods course.

*Comparing Mathematical Ideas*

All 11 of the PTs *named, labeled, distinguished, and compared mathematical ideas* either in the video discussion or their written reflections, providing strong evidence that doing so was normative in the group. Ten participated in the norm by directly comparing students' thinking, and six, including the one who did not directly compare, made statements that indicated they thought comparing solutions was important.

In the video discussions, all eight of the participants who substantially contributed to the conversation exhibited this norm in at least one of three ways: (1) comparing their mathematical thinking to that of other PTs ("I was just going to say that I came up with the same formula, 5(*n*–1), but I saw it in a different way" [Lily, T65–66]); (2) comparing their thinking to the students' in the videos ("They were adding 1 for that middle cube. They were kind of looking at it the same way I did." [Evan, T338–340]); or (3) comparing the students' thinking in the video to each other's and/or recognizing that it was important that the teacher did so. The following, which immediately preceded the exchange with Abby and Jim cited above, illustrates this recognizing:

> Well, a lot of it seemed to me like [the teacher]'s checking them for their own understanding. … asking them to like compare and contrast is showing like if they understand their own method enough to talk about how it's different from the way someone else did it, and how, how they're the same. [Lew, T533–538]

In the written work, all but two of the PTs responded to at least one reflective prompt by discussing the different ways students in the video were defining their variable, which indicates that they were comparing students' solutions as part of their individual analyses. Evidence of this norm was also shown in a variety of other ways in the written work, including recognizing that the teacher pushed students to compare their solution methods, directly discussing and comparing student solution methods, and discussing whether students really understood each other's solutions. In addition, five PTs noted that comparing and contrasting solutions was important in their end-of-day reflection. For example, Roxy reflected that "It is important to tie everything together and see how or if different solutions are related to each other" [DS 8].

*Mathematical Arguments*

All of the PTs attempted to use mathematical explanations that consisted of a mathematical argument—not simply a procedural description or summary—in their own solutions to the problem [DS 1] and all but two either did so again or noted the importance of doing so when analyzing the student thinking or teacher moves in the video [DS 4–8, T]. In justifying their own solutions the PTs were successful in using a mathematical argument to varying degrees. To illustrate, consider Leah's justification for her expression, $4(n - 1) + n$: "The solution supports the picture since you have four branches that are the size of the previous building ($n - 1$), and one branch that is the size of the current building ($n$)" [DS 1]. Here, Leah justifies each part of her expression in relation to the diagram provided with the problem, though it would have been clearer if she had said "building number" instead of just "building." In contrast, to justify her expression, $5n - 4$, Hana says, "My solution accommodates my visualization of 5 blocks adding every [time] to the original cube: one cube spreading out at its arms" [DS 1]. While Hana adequately justified the coefficient in her expression, she did not make any attempt to justify the – 4, rendering her argument incomplete.

Further evidence that the behavior is normative can be seen in the fact that seven of the PTs made statements in their written reflections on the teacher and students in the video indicating that they recognized the importance of using mathematical argument. In the following reflection, for instance, Abby demonstrated the norm by noticing a lack of argumentation:

> One thing that stood out was that the teacher never asked Cassie to explain where the minus 4 came from. During her explanation, at the end she would just say, "and then you subtract 4." There was no connection to the picture or explanation of where that came from. [DS 4]

In the end-of-day reflection that prompted for insights/connections related to teaching that the discussion generated for them, Iris wrote, "Teachers should ask questions that prompt connections between pictures and expressions/equations" [DS 8].

Similar statements related to this norm were made throughout the video discussion. In total, 24 instances of this norm were coded, involving nine different PTs. In one telling utterance, Roxy discusses how the students in the video were able to justify a part of a mathematical expression that the PTs could not in their own mathematical discussion:

> Yeah, because that's what I had trouble seeing. I couldn't figure out like how to describe where you take away the 4. 'Cause I did it like Leah did it, with the 4—well, I did it in a table, but then I also saw the $4(n - 1) + n$. I was like, oh, well, that's how you get your minus 4. But I like how [the solution looking at the minus 4 as subtracting the overlap when the middle is included in each "leg"] actually shows this is how you take away the 4. [T215–219]

Thus, Roxy recognizes the students' mathematical argument, while at the same time indicating its importance by noting her own inability to fully justify a mathematical expression.

## Conclusions

Professional and sociomathematical norms developed early in a teacher preparation program seem to be durable in the sense that they re-emerged in a similar situation at the end of the program. This is encouraging because these norms support a richness of discussion about the teaching and learning of mathematics that is not prevalent in descriptions of practicing teachers in research on teacher learning. In particular, the PTs in this study focused on student thinking and the implications of the teacher's actions for supporting student thinking in a way that is not commonly seen. The fact that we were able to foster these professional and sociomathematical norms through use of a practice-based video case curriculum in a methods course (Stockero, 2008), combined with the findings from this study regarding the durability of the norms, suggests the value of making the development of such norms a key part of curricula used in university methods courses. Not only will moving PTs further along the teacher development trajectory during their university education give them a solid start, experiencing *self-sustaining generative change* will position them to accelerate their movement along the teacher professional development trajectory as they become more experienced teachers.

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# JUXTAPOSITIONAL PEDAGOGY: DESIGNING CONTRASTS TO ENABLE AGENCY IN METHODS COURSES

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*Powerful forces resist the preparation of teachers, especially the 'apprenticeship of observation' and critical professional colleagues. This paper describes a pedagogy of juxtaposition for mathematics methods courses, namely: (a) critically examining polarized videos of teachers' practice, (b) considering contrasting opinions about mathematics education, (c) participating in modified lesson study, (d) co-teaching the method curriculum to peers, (e) public school teaching during two separate field experience blocks, (f) solving dual part problems, and (g) reflecting on previous personal reflection. Student data is presented that confirms the cognitive dissonance such a juxtapositional approach creates, and positive results for independent student thought.*

## Statement of the Problem

Elementary education preservice teachers hold deeply held beliefs (Ball, 1990), beliefs so entrenched by long years in the "apprenticeship of observation" (Lortie, 1975), that receptiveness to alternative teaching possibilities is unfortunately curtailed. Worse yet, new teachers embracing university methods often encounter conflicting messages from veteran teachers (Skempp, Sparkes, & Templin, 1993) skeptical that university theory is just that—theory unconnected to classroom realities where discipline issues, testing pressures (Webb & Coxford, 1993), and lack of time (Stanley, 1998; Wildman & Niles, 1987) demand a more pragmatic (indeed, straightforward) approach to instruction. And not just new teachers face this hardship; Mr. Murano, an eighth-year middle school teacher selected to participate in a research project because of his dynamic whole-class discussions, admitted feeling considerable pressure, and even animosity, from more traditional colleagues over his unique teaching approaches (Ricks, 2007). Another teacher of 15 years experience, Ms. Auburn, had even considered transferring to another school and hand-picking teachers herself, so as to alleviate adverse colleague pressures. Ms. Auburn was also aware of other pressures new teachers face besides disagreeable peers; she had served as the district's mathematics specialist when the district's adoption of an NSF-approved curriculum resulted in a tidal-wave parent backlash that forced the district to drop the mandatory curriculum implementation (Ricks, 2007).

In the context of these difficulties, I have tried an atypical approach to influence preservice teachers' beliefs about the nature of teaching during the few short weeks they are under my tutelage. Rather than plunging them as rapidly and as deeply as possible in a vat of reform ideas, hoping a pressurized, 16-week saturation will infuse into them enough concentrated reform antidote to overcome the previous poisonings and outlast the upcoming epidemics, I have opted for a milder approach of careful comparison of teaching possibilities through an appeal to *juxtaposition*, which allows preservice teachers the opportunity to develop and defend their own conceptions of effective teaching practices as they experience contrasting situations.

The purpose of this paper is to describe this juxtapositional stance that guides my actions during preservice teacher methods courses at Louisiana State University. This paper draws on ideas being tried during the longitudinal Teaching in Mathematics Education (TIME) study, an ongoing 5-year study at Louisiana State University investigating practical, implementable

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strategies to significantly improve preservice teacher preparation by participation in a reflective curriculum of juxtaposition. As the project is only in its initial stages, this paper is designed to be a descriptive account of the course, a medium known to contribute to furthering avenues of possible research (Bush, 1986).

## Juxtapositional Pedagogy

I consider a *juxtapositional pedagogy* to be a pedagogy respecting the preservice teachers' inherent, inalienable right to choose and pursue for themselves their own pedagogical paths, which in mathematics education involves navigating terrain often overly-stereotyped as an either-or, traditional-reform quagmire. This juxtapositional pedagogy comprises sufficiently-contrasting experiences, coupled with the adequate time and social support, to augment preservice teachers' evaluative cognitive capacities.

In particular, I view teaching as a constant form of judgment-in-action; teachers are relentlessly making decisions, whether privately as they ponder and reflect on past teaching and future plans, or publicly as they think on their feet during dynamic whole class discussions (Yackel, Cobb, & Wood, 1999). Teachers must make instantaneous decisions with limited access to information and time to consider the results of their actions. Good teachers are able to choose among many possible paths of action to further propel learners in productive directions.

The underlying theoretical premise of the course envisions teachers—ultimately—as rational, reasoned, decision-making agents (Bush, 1986; Dewey, 1981/1933). The course attempts to augment teachers' decision-making powers. Believing that social interactions play a pivotal role in forming new understandings (Cobb, Wood, & Yackel, 1990; Vygotsky, 1978), I attempt to present carefully constructed juxtapositional experiences to form the catalysts for intellectual conversations between class members. When presented with polarized viewpoints, preservice teachers can begin to critically reflect on their own beliefs. This reflection becomes especially potent when their own thinking is juxtaposed with others' differing viewpoints in social contexts (Feldt, 1993). The following juxtapositional features are described in this paper: (a) critically examining polarized videos of teachers' practice, (b) exposure to contrasting opinions about mathematics education, (c) the collaborative environment of modified lesson study, (d) co-teaching the method curriculum to peers, (e) two separate field experience blocks in a local public school, (f) dual problem parts, and (g) reflecting on previous personal reflection.

### *Critically Examining Polarized Videos of Teachers' Practice*

Drawing from a variety of video sources such as TIMSS (NCES, 2003), professional development mediums, and a growing, personally-videotaped collection (Ricks, 2007), I show short segments of contrasting teachers' practice. These episodes are strategically chosen to bring a pedagogical issue to the forefront, or to highlight certain principles arising in our own class discussions. By using video mediums, portions of the videos can be re-watched and revisited, aiding class reflection; additionally, encapsuled experience in video form, although not equal to the ideal (observations of student thinking in actual classrooms), do provide a manageable nexus for coherent whole class discussion as the camera is only pointing in one direction, a redundant component helpful in whole-class discussions (Davis & Simmt, 2003).

Watching videos has the added benefit of showing alternative strategies for reform instruction. Using the words of Stigler and Hebert (1999), videos provide "a penetrating and unparalleled look into classrooms" (p. 9). Most of my students have never experienced a mathematics class taught in a reform manner; preservice teachers struggle to conceive of what the NCTM, for example, is describing in their vision of school mathematics (NCTM, 2000), having no concrete mental models. The nebulous terminology of *constructivism* (Kilpatrick, 1986), *community* (Grossman, Wineburg & Woolworth, 2001), and *discovery learning* (Davis, 1994) takes on new meanings with these solidly accessible images of practice. It also is convincing proof that such instruction can be done, and done well; a video of Deborah Ball teaching 3rd grade is one of my personal favorites in assisting the preservice teachers' paradigm shift that students *can* think powerfully about mathematics in unexpected ways even in the early grades, and grapple with issues they themselves as adults have never considered.

*Exposure to Contrasting Opinions about Mathematics Education*

Another fundamental part of the pedagogy of juxtaposition involves preservice teachers reading articles by individuals with sharply contrasting beliefs about mathematics education; the preservice teachers begin to understand the arguments of anti-reformers and the rhetoric of reformers. They recognize that this country is deeply and passionately divided over instructional methods for their children (Jackson, 1997), and that they need not only to better situate themselves on the battleground of the math wars (Ball, Ferrini-Mundy, Kilpatrick, Milgram, Schmid, & Schaar, 2005; Schaar, 2005), but should be able to justify their position as well. Stigler and Hiebert's (1999) *The Teaching Gap* is a particularly good reading that inherently emphasizes juxtaposition within the text itself through international comparisons of Japanese, German, and American classrooms.

*Modified Lesson Study*

I engage my preservice teachers in four-person lesson study groups modeled after the common form of Japanese lesson study discussed in Stigler and Hiebert (1999). Typically, American teachers work in isolation (Hart, Schultz, Najee-ullah, & Nash, 1992; Shulman, 1987; Stepanek, 2000; Valli, 1997); lesson study forces the preservice teachers to work together to plan a lesson (Lewis & Tsuchida, 1998). This can be for them a frustrating form of juxtaposition because how their peers think about mathematics often differs in substantial ways from their own conceptions, providing a rich social environment for discussion, negotiation, and cooperation (Davis & Simmt, 2003; Feldt, 1993). Ideally, the research lesson is eventually taught in a local public school classroom, revised, and taught again to a different elementary class a few weeks later, which forms another opportunity to juxtapose experiences through personal reflection on the two lessons' effectiveness.

*Co-teaching the Method Curriculum to Peers*

I also have pairs of preservice teachers teach portions of the class curriculum to their peers. Having videotaped their teaching, I then provide them with a reflection assignment—they report this experience is one of the most eye-opening experiences in their entire teaching program (especially as for many, this is their first time watching a videotaping of their own teaching). In a way, watching oneself teach is the ultimate form of teacher juxtaposition, because the imagined self usually appears very different from how reality (or at least a video rendering of reality) really is. Co-teaching becomes another opportunity besides lesson study to juxtapose one's ideas against that of another teacher.

*Public-school Teaching: Two Separate Field-experience Blocks*

A fifth form of juxtaposition is teaching two blocks in a local elementary school. This achieves juxtaposition in a variety of ways. First, the exposure to their cooperating in-service teachers' mathematics instruction is eye-opening, especially in light of their developing opinions on good teaching practice formed in the weeks before—the preservice teachers are beginning to see with new eyes and return two weeks later after this first block transformed. Then, for three weeks we engage in energized discussion as the preservice teachers are now recognizing the national generalizations Stigler and Hiebert discussed (1999) about American teachers all teaching the same. The preservice teachers are dismayed, discouraged, and yet determined. Dismayed by what they saw—the lack of high-quality math instruction; discouraged because they themselves tried to teach in a more engaging way, by posing challenging mathematical tasks, but found it exceedingly difficult; and determined because they know it can be done—they have seen videos of Japanese and selected American teachers succeed in engaging students in meaningful, exciting mathematics lessons. As Goethe wrote: "Every new object, clearly seen, opens up a new organ of perception in us" (von Goethe, 1988, p. 39).

Additionally, the first lesson study lesson is taught by each group during the first block. The three-week interim also gives time for lesson study groups to analyze the effectiveness of their first lesson attempt, and re-plan for a better second attempt in a different class later in the second block.

It is during this three-week hiatus between field experience blocks that I detect a palpable shift in the preservice teachers' desires regarding the course—something I can best describe as a new hunger. They now recognize that they lack something needed to teach mathematics at a level they know is possible. I change the thrust of the course during this crucial time. Whereas the first part of the semester focused on recognizing, modifying, or developing challenging mathematical tasks, I shift the course's focus to emphasize the implementation (Stein, Smith, Henningsen, Silver, 2000) of these well-designed tasks. The first half focused on *what* teachers might do effectively, and the second half now focuses on *how* to actually do it successfully. The second round of field-experiences allows the preservice teachers to better practice what they tried in the first go-around.

*Strategically-designed, Dual-part Problem-posing*

These juxtapositional activities, which permeate the course, are designed to illustrate a particular learning issue by dividing an instructional task into two parts to allow the preservice teachers the opportunity to compare and contrast their own experiences in the task solution. For example, to illustrate the difference between doing mathematical computations through rote memorized procedures (which many unfortunately believe is 'mathematical activity') and understanding the underlying conceptual principles, I choose a simple fraction division problem, such as: 1 ¾ ÷ ½ (Ma, 1999). Students can easily solve this problem through an appeal to the invert-and-multiply maxim, but struggle writing a real-life story that would be applicable to this computation.

A second dual-part problem I find effective is 198 divided by 12. Again, the preservice teachers are able to implement the long division algorithm and arrive at the correct answer, but struggle explaining the second part of the problem which asks to detail what mathematical reasoning is encapsulated by each step of the algorithm. This is not at all trivial. Even though many preservice teachers recognize the repeated-subtraction conceptual structure of the algorithm, the most difficult step is understanding the place value placement of the quotient's numerals. Their natural inclination is to think "How many times does 12 go into 1? No times. How many times does 12 go into 19? One time." But 12 is neither being divided into 1 or 19 in the algorithm. Further class discussion reveals asking "How many times does 12 go into 100 (or, similarly, 190)?" The mathematical concepts embodied in this algorithm are deep, fascinating, and accessible to even young children, but the preservice teachers have never thought about the mathematics in this way. It is left to the reader to ponder why in this particular algorithm (knowing there a multitude of different long division algorithms, each with their own conceptual base) students do not write an 8 over the dividend's 1 place, as 12 goes into 100 eight times (hint: asking "How many times does 12 go into…?" is an incorrect understanding of what the algorithm is doing mathematically, although it is often erroneously taught that way, even by mathematics educators). These dual-part problems help preservice teachers ponder the differences between, for example, procedural and conceptual understandings, and how certain tasks may reveal little about one or the other type of learning.

*Reflecting on Previous Personal Reflections*

A final form of juxtaposition utilized in this course is the course's purposeful repetitive nature, designed to allow preservice teachers the chance to reconsider previous ideas from a new perspective, resume work on difficult, unresolved issues, or to allow them to consider how their views are changing during the course. For example, students re-respond to certain notebook prompts later in the semester, such as: *What is mathematics?*, *How is it best taught or learned?*, and *What is the purpose of a methods course?*. These can be contrasted to their previous answers kept in their notebooks. Many of the above mentioned videos are re-watched and re-discussed for a second time. Because of the intervening experiences, the re-watching and resulting discussion is always different than before, revealing substantial changes in perception about teaching mathematics. As the Greek philosopher Heraclitus claimed, one can never step into the same river twice.

We also re-watch a promotional video at the end of the semester which touts the virtues of a traditional drill-and-kill approach, and it is remarkable the change in student disposition by the end of the course toward this regimented approach. In particular, preservice teachers discern the pithy, superficial phraseology saturating the video, as well as the lack of mathematical thinking expressed by the elementary students in the video.

## Discussion and Conclusions

I attempt to design components of my class to be matching compliments of one another to stimulate my preservice teachers' judgmental awarenesses using contradicting and/or paradoxical circumstances for personal introspection. In particular, this juxtapositional approach tries to respond to factors which are known to erode preservice teacher education, namely: (a) the apprenticeships of observation (Lortie, 1975) and (b) corrosive collegial atmospheres undermining university theory supposing it is unconnected to the realities of classroom practice (Schempp, Sparkes, & Templin, 1993).

Although still in its early stages, the TIME project at Louisiana State University is demonstrating through circumstantial evidences and initial indicators the positive effect a curriculum of juxtaposition has had on preservice teachers' fundamental perspectives of mathematical pedagogy. The following excerpts are anonymous student comments from end-of-semester university evaluations that reflect this positive aspect of juxtapositional instruction. A student wrote: "The beginning of the semester was a shock and I thought this class was going to be terrible, but I ended [up] learning so much and [I] became open to a new style of teaching." One student said:

> This was quite an interesting course. At the beginning I was unsure as to where this class would go. However, [the approach] challenged us, our minds, in ways that had never been challenged before. The end result was quite phenomenal.… I believe that I will go into the classroom as a more powerful teacher.

Another student wrote:

> It was a very different form of instruction but it was very interesting to learn. It opened my eyes to a different type of teaching and student learning. Even though this

was a tough course it allowed me to learn lots of new ideas.

Admittedly, the beginning of the course was unusually stressful; the preservice teachers were purposefully placed in overwhelmingly uncomfortable positions of ambiguity (Stigler & Hiebert, 1999). Another student bluntly confirmed this in the end-of-course evaluation: “The course started out rough.” Such tenor is echoed by this student’s comment: “The beginning of the semester was rocky because [the approach was] unconventional… but…. I learned so much about education and myself.” Where they expected a typical methods course’s rigidly determined calendar (i.e., *this* reading to be done by *this* date), concise grading rubrics (i.e., so many points for this aspect of the assignment—checklist-like), and traditional memorize-regurgitate exams, I offered a litany of tasks and activities that had little immediate closure and unusually difficult cognitive demand, forcing them into the position of decider—or common judge—for themselves: “I will admit that there were times I was extremely frustrated but I know that a lot is learned through frustration. I think [the approach to be] very innovative…. [The course] turned out to be really good!”, said another student.

Hopefully, these quotes capture the flavor of the initial shock that gave way to the ennobling freedoms offered by the juxtapositional approach. I believe such action provides for significant learning experiences—issues remain open for weeks, problems unsolved, opinions shared but with no teacher judgment passed (e.g. Carpenter, et al., 1989). They are thinking about this class *outside of class*, chewing things over, letting their subconscious go to work. This approach also mimics typical inservice-practice: no authority to turn to for quick, immediate answers or approval in one’s own classroom. One student expressed in their evaluation:

> Class started somewhat slow and with a disoriented feel…but by fall break things picked up…. By the second half of the semester things were going really well and I enjoyed the class. I feel that I have really learned a lot about how to teach. Real stuff, not busy work, but real applicable knowledge and practice.

I consider that these comments hold some degree of credibility as they were comments turned in anonymously to the university as part of the university’s evaluation procedures—also, not all comments were optimistic. One student grumbled: “He never answered questions.” But such is the way of juxtapositional pedagogies—the answers (or ‘learning’) students seek is not doled out by the professor, but is generated through cognitive consideration of the contrasting curriculum components.

My ultimate strategy with using a juxtaposition strategy in my courses is for the preservice teachers to consider knowledge as problematic, uncertain, fallible, socially determined, and revisable (Boaler, 1999; Ernest, 1990; Richards, 1991; Romberg, 1994; Schwartz & Hershkowitz, 1999; Thom, 1973; von Glasersfeld, 1985; Woods, 1992; Yackel, 2000). They then formulate (construct?) their own understandings and positions on issues. In other words, they must make decisions themselves about what is considered ‘good’ teaching practices, grounded through reflective analysis (Dewey, 1981/1933)—I will not make the decisions for them. Hopefully, self-formed preservice teacher reasonings will better withstand school enculturation buffetings that are known to minimize university teaching.

And I think they may well do just that. Many of the students’ comments hint at a deep, underlying paradigm shift about the way teaching and learning is construed:

> I wish I had been exposed to [these] methods much earlier in my college career. I believe [it] could have a great impact on [the university’s] college of education if allowed to continue.

I ended up loving the class. In the beginning I wanted to drop, but ended up learning

> so much from this class. There should be more [approaches like this] in the college of education.
>
> [The approach] did an excellent job in opening my mind about mathematics. I learned a lot about math concepts through discussion. A lot about math concepts were de-mystified. I feel like I am ready to go into the schools and effectively teach math.

Such is the potential of a juxtapositional pedagogy. In particular, as these comments express, preservice teachers regularly demonstrated the expected cognitive dissonance needed to effectuate an inward-motivated, long-lasting paradigm shift from traditional pedagogical thrusts to more meaningful forms of mathematics education (Carpenter, Fennema, Peterson, Chiang, & Loef, 989; Stigler & Hiebert, 1999). And their lessons reflected such a shift by the end of the semester. The fact that the science educator (who teaches science methods in the same semester as the elementary math methods block) noticed that their *science* lessons were incorporating the core structural features of their math methods lessons adds credence to the belief that the preservice teachers' apparent belief changes were not a mere superficial surface adjustment to my course expectations, but were a deeper paradigm shift about what teachers *do*, a shift that was now spilling over into other aspects of their teaching. I believe the juxtapositional approach helps the preservice teachers to become reasoned, purposeful practioners because it instills in them a reflective attitude toward their actions:

> [Reflective thinking] emancipates us from merely impulsive and merely routine activity…. it enables us to act in deliberate and intentional fashion to attain future objects or to come into command of what is now distant and lacking. By putting the consequences of different ways and lines of action before the mind, it enables us to *know what we are about* when we act. *It converts action that is merely appetitive, blind, and impulsive into intelligent action*. (Dewey, 1981/1933, p. 125, emphasis in original)

A curriculum of juxtaposition may provide a suitable environment for developing reflective habits of mind because the nature of contrasting points of view causes the open-minded to reconsider in the face of seemingly-contradictory yet equally-plausible data their own reasons and justifications for supporting their own actions and beliefs. By placing opposing ideas, each which appears to have legitimate support from other intelligent individuals, next to their own cherished beliefs, the way is open for resisting the urge to dismiss the opposition (Rodgers, 2001), and to begin forming habits of accepting or rejecting positions based on appeals to data (Dewey, 1981/1933). The purpose of juxtaposition strengthens future teachers' decision-making and respects their agency, even if they are not accustomed to this mode of university instruction at the outset of the course. In particular, juxtaposition allows preservice teachers to simultaneously consider various viewpoints and to make judgments themselves about which attributes from the differing positions they really value, enhancing their teaching potentials by grounding their actions in "*wide-awake, careful, thorough habits of thinking*" (Dewey, 1981/1933, p. 177, emphasis in original).

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# WHO TEACHES MATH FOR TEACHERS?

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*Math for Teachers courses are specialised mathematics content courses whose nominal aim is to build conceptual mathematics understanding in prospective elementary school teachers. But who teaches these courses? This paper offers some insights into this question, exploring the views of two particular instructors whose interview responses reveal very different perspectives on their goals for this course with respect to knowledge-for-teaching, beliefs about mathematics, and the attitudes/emotions of their students.*

## Background

Prospective elementary school teachers are often expected to exhibit proficiency in mathematics by completing a university level mathematics content course before entering their accreditation programs. In an effort to provide this group of students with a course that is potentially more appropriate to their needs than a calculus or statistics course, mathematics departments at many colleges and universities have developed specialised "Math for Teachers" courses. Though they differ from institution to institution, these courses typically cover elementary school arithmetic and geometry topics, with an aim to help prospective teachers develop a strong conceptual understanding of the mathematics they will one day teach.

Relatively little research has been done to explore the role that these specialised content courses play in the development of teachers. As part of this larger project this qualitative study focuses on the instructors of these courses. Despite the surface similarities between Math for Teachers courses offered within and between institutions, given the autonomy that most post-secondary educators enjoy, the course-as-delivered has the potential to be significantly influenced by the individuals who teach the course. This study seeks to shed light on who the instructors of the Math for Teachers course are and how they attempt to contribute to the development of future teachers.

## Theoretical Perspectives

Our inquiry is both informed by and informs a number of theoretical perspectives including classic and contemporary views on knowledge-for-teaching and sociomathematical norms.

Initial review of curriculum descriptions for Math for Teachers courses suggested that Shulman's (1986) framework for classifying knowledge for teaching might be helpful in understanding the types of knowledge that instructors of these courses hope to transmit to their students. Shulman identifies three major categories of knowledge: subject content, pedagogical content and curricular content. Given that the Math for Teachers course is intended to be a content course, we anticipated that in interviews instructors would address subject content knowledge, but not curricular content in their descriptions of their goals for this course. However, we were interested to see the extent to which pedagogical content issues would be addressed. Under this type of teacher knowledge Shulman includes: *the most useful forms of representation of those [mathematical] ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others. (Shulman, 1986, p. 9)*He goes on to add: *an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons.* (Shulman, 1986, p. 9)

The work of Ball and Bass (2003) also contributes to our perspective on the scope of pedagogical content knowledge. They identify an ability to unpack (or break-down) mathematical ideas, to understand the connectedness of mathematics concepts both at a particular level and across levels, and how students conceptions of mathematical concepts will evolve over time, as examples of mathematical knowledge required for teaching. Furthermore, they include knowledge of conventional mathematical practices, such as the role of definitions, and what constitutes an adequate explanation.

Though theories of knowledge-for-teaching influenced our initial interview questions, recurrent themes led us to broader and deeper considerations. Early interviews revealed that students' beliefs and attitudes are also of particular concern for instructors of this course. They can affect the students' ability to acquire knowledge content (Ball, 1990), and contribute to the development of the attitudes, beliefs and values that they will carry forward with them into their teaching careers. It also became apparent that any analysis of these affective issues would need to consider not only the aspects of their students' attitudes and beliefs which the instructors hope to influence, but also the instructors' beliefs and attitudes about mathematics and the teaching of mathematics that ultimately influence their own practice (Ernest, 1989).

Cobb and Yackel's (1996) notion of sociomathematical norms offers a lens through which to view this complex interplay between the beliefs and attitudes of the instructors and their students. Sociomathematical norms are normative understandings, negotiated through the interaction of teacher and students, which relate specifically to mathematical activity. Research supports the view that the development of these norms is closely integrated with the mathematics beliefs and attitudes of the teacher and students:

*With regard to sociomathematical norms, what becomes mathematically normative in a classroom is constrained by the current goals, beliefs, suppositions, and assumptions of the classroom participants. At the same time these goals and largely implicit understandings are themselves influenced by what is legitimized as acceptable mathematical activity.* (Cobb & Yackel, 1996, p. 460)

Though the negotiation of norms involves participation of the students as well as the teacher, the teachers' various roles (as initiator, facilitator and validator) underscore the relevance of examining the beliefs and attitudes that the instructors bring to this negotiation. Integral to Cobb and Yackel's (1996) emergent perspective is the view that these social constructs are reflexively related to the individual's mathematical beliefs and values. From this perspective, references to beliefs about the nature or activity of mathematics can suggest the sociomathematical norms the instructors hope to establish in Math for Teachers courses, and vice versa.

## Methodology

Interviews were conducted with instructors of Math for Teachers courses at post-secondary institutions in a metropolitan region. The Math for Teachers courses offered at these institutions are locally developed but are sufficiently similar that students can transfer credit for this course from one school to another.

The interviews, which lasted approximately one hour, began with a standard set of questions about the backgrounds of the instructors, their education, number of years of teaching, and number of years of teaching Math for Teachers. These questions were followed with questions about their initial orientation (preparation) for teaching the course, about what they do differently with this group of students compared to their other mathematics students, about their goals in teaching the course, and the outcomes they believe they achieve. As each interview was completed it was transcribed and coded for emergent themes through a process of constant comparative analysis (Cresswell, 2008; Corbin & Strauss, 2008). As part of this reflexive process the interviews were only semi-structured. This allowed the interviewer to incorporate additional and/or deeper questions to respond to new themes as they arose. To compensate for any bias that our own personal experience as occasional instructors of this course may introduce, the interview coding was corroborated by a more neutral colleague. Furthermore, narrative descriptions of the interview subjects were read and verified by the subjects themselves.

Through the analysis of the transcripts nine themes emerged: two related to teacher knowledge (subject content and pedagogical content), five related to sociomathematical norms (beliefs about mathematics, emotions/attitudes, aesthetics, communication, and community) and two others (teaching methods and tensions). Due to space limitations, in this report we will restrict our discussion to the two themes related to teacher knowledge, as well as two of the themes related to sociomathematical norms: beliefs about mathematics and emotions/attitudes. To facilitate readability, we will further limit ourselves to quotations from only two of our interview subjects, Harriet and Bob. Elaboration on these cases will permit us to illustrate two of the very different approaches that were revealed through our study.

We will begin our discussion of results with brief narrative descriptions of Harriet and Bob, and follow this with analysis of the selected themes.

## Narrative Descriptions

*Harriet*

Harriet is an experienced Mathematics instructor who has been teaching for 22 years. She is relatively new to teaching Math for Teachers, but has taught the course six times over the last three years. She has neither taken any Mathematics Education courses, nor does she have a formal teaching designation. She has a Masters Degree in Mathematics, and has a special interest in the history of mathematics. Harriet was initiated into the teaching of this course by a colleague who has a Masters Degree in Mathematics Education, has taught Math for Teachers for many years, and has a particular passion for the course. This colleague provided information about course materials and the nature of the students and their difficulties. She also provided teaching resources, including suggestions for activities.

Harriet feels strongly about the need for good teachers of mathematics in the elementary schools, and has put a great deal of thought into what can be done in a Math for Teachers course. Her priority when teaching the course is to change students' attitudes towards mathematics and their own mathematical abilities. She hopes students will come to see mathematics as enjoyable, even when it is challenging, and will develop confidence, based on a solid conceptual understanding of elementary mathematics.

*Bob*

Bob has been teaching mathematics for 13 years and has taught the Math for Teachers course nine times over the last nine years. He has a Masters Degree in Mathematics, has not taken any Mathematics Education courses, and has not had any formal teacher training. Bob's first forays into teaching the course were guided by the established curriculum, the textbook that had been selected by colleagues who had taught the course before, and through informal discussions with those colleagues.

Bob is passionate about mathematics. He enjoys its logic, its structure, and the challenges presented by a good problem. He cares about producing students who will be successful elementary teachers in the future, and to that end he hopes to equip them with a solid understanding of fundamental mathematics concepts, good communications skills, and a capacity to enjoy mathematics.

## Analysis of Interviews

*Knowledge for Teaching: Subject Content and Pedagogical Content*

Harriet and Bob's interviews are illustrative of two quite different conceptions of the role of the Math for Teachers course in developing prospective teachers' knowledge-for-teaching.

Harriet's description of her goals and strategies for teaching the Math for Teachers course are permeated with comments coded under "pedagogical content knowledge". When describing the content of her course she mentions varieties of algorithms for arithmetic operations, along with models for their representation. Although these topics are part of the prescribed curriculum, her comments indicate that she goes beyond merely delivering this as subject content. She explicitly considers its relevance for teaching mathematics, noting: "*We spend some time on the basic algorithms and different approaches to them, and how those can lead into different understandings of what you're doing when you're multiplying, or adding…*"

This notion of developing multiple understandings arises again when she is asked if there is anything that she teaches her Math for Teachers students about fractions that she wouldn't teach someone who just wanted to learn how to use fractions. She states: "*The fact that there are different models, there are different ways of picturing what's going on, and that they are appropriate for...what may work well for some situation, or for some student, may not work for some other one*". For Harriet, access to a variety of representations and approaches is mathematics content that is particularly relevant for her students as prospective teachers.

Connections between mathematical ideas also play a central role in Harriet's conception of the knowledge content of the course. She explains: *"…what you can do with a grade three student, and what you can do with a grade six student are quite different and I want them to see that it's all interconnected…*" Her evident appreciation for these connections echoes Ball and Bass's (2003) description of knowledge-for-teaching, addressing both the connections within and across grade levels. In her words:

> *I emphasize it [connections between topics] all the way through. I don't try to plan the course to start from the beginning and go through to the end with an obvious thread, because mathematics is way too big for that. […] But at all times I connect it, as far as I can, to what goes on at different levels. What you might do with a grade 1 class, how that connects to what they're going to see in, you know grade 4 or 5 or something like that, how that connects to what they might do in high school and how that connects to what I'm doing in Calculus. Because they've got to see how it's connected, and how we build bigger and bigger, you know, understandings of sets of numbers, or calculations, or whatever.*

Harriet does not just pay lip-service to these ideas. She describes assignments and activities for her classes that provide them with opportunities to exercise their pedagogical content knowledge: her students engage in analyses of pupil errors, as well as activities that allow them to compare alternative methods for solving math problems. While Harriet is concerned to ensure that her students build proficiency in mathematics subject content, her interview stands out from the others in that she constantly returns to comments related to how the content would be used by her students in their future roles as teachers.

In contrast, Bob's interview stands out for its lack of statements that can be coded as pedagogical content knowledge. His emphasis is instead on the notions of developing both a strong understanding of fundamental mathematics and communication skills in his students. When comparing the Math for Teachers course with his other mathematics courses he notes *"...this one focuses on their ability to communicate and convey the ideas that they should, hopefully, be already familiar with and capable of doing."* Bob describes teaching various algorithms and models as part of the course content, but does not specifically address any comments to consideration of how this information can be used differently at different grade levels.

Bob needed to be pressed by the interviewer to consider what aspects of the course content might be particularly relevant to prospective teachers as opposed to general learners of mathematics. Initially his comments revolve around his methodology, the use of group work and manipulatives (both coded under "teaching methods"), but he makes no reference to any special mathematics knowledge for teaching. Eventually an association between the mathematics that he teaches and the students' future role as teachers appears when he describes challenging his students to think about the kinds of questions that they will encounter as teachers:

> *…what kinds of questions will you encounter? And why is it important that you to be able to communicate your ideas effectively, […], why should you understand this material to the most, [...], fundamental and basic level, and understand all of the structure?*

Even then, this response seems to be a justification for developing strong subject matter content knowledge and communication skills. There appears to be a strong connection for Bob between knowing the mathematics subject content and being able to teach it. He goes on to note:

> *...when you get some of these obtuse questions, that are seemingly [...] obtuse, you have to be able to appreciate it and be able to differentiate whether that's something that can lead you into a teachable moment...*

His focus on subject content knowledge is echoed in his description of what his students leave the Math for Teachers course with:

> *…I think that they […] leave having had some sense of the structure of mathematics, because there's a sufficient amount of that in the course, and I think that they also leave the course feeling that they can solve problems, on their own. […] probably it's the technical skills that they have [...] solidified the most.*

Bob's views on the readiness for teaching of those who complete his course shed further light on this. Though he admits that they are not ready, he describes his students as still lacking maturity, confidence and communication skills. But in regard to their mathematics skills he relates:

> *...[for] my A's and high B students I wouldn't have any problem giving a recommendation in terms of how they're outfitted to [...] go into a classroom, and [...] I think other aspects of their [...] education career, will help fill out all of their [...] professional skills, in terms of knowing what's in the better interests of kids...*

It appears that for Bob, mastery of the subject content along with general pedagogical skills are sufficient for the teaching of mathematics—a traditional and not unfamiliar point of view (Hill et al, 2007).

*Beliefs about Mathematics*

Interview subjects commonly commented on the beliefs about mathematics the prospective teachers brought with them into the course, and about their efforts to influence those beliefs, though both the specific beliefs they addressed and their approaches differed.

Bob describes his students as believing that mathematics is arbitrary and incomprehensible: "*So many things seem magical to them.*" He affirms that "*it's not your standard sort of math group, it's one that has encountered some challenges along the way, and it hasn't always left them with a positive impression of mathematics.*" More than once in his interview he describes the Math for Teachers course as a second start for these students, an opportunity to reshape their beliefs about mathematics and their own mathematical ability.

For Bob, this reshaping is attempted by providing his students with opportunities to see the logical structure of mathematics, which in turn will improve their ability and their conceptions of their ability to do and understand mathematics. In his words, the course "*focuses on a very sound fundamental ability to appreciate it [mathematics], in a theoretical way, why things work, as opposed to technical aspects of how do you do mathematics*". He tries to "*give them a sense that "yes, they can" but also ...force[s] them to dig a little bit deeper, so that they also know that they can understand this as well*". He also challenges their presumptions about the root of authority in mathematics, attempting "*to make sure that they know I'm not just saying this stuff because "It's the way it is" and Math isn't just something handed down by the gods, it's understandable*".

Harriet focuses on quite different aspects of her students' beliefs. Her comments indicate that her students believe that mathematics is rigid (allowing for only one right answer), that mathematics is arithmetic, and that at least some of it is irrelevant. She observes: "*It's never occurred to them that there's more than one way to do something.*" In response to this, she takes the time to reinforce the existence of multiple representations and methods of solution. This is clearly a belief about mathematics that she values highly, as later in the interview she reiterates "*if I get across to them that they have to be aware that there are different ways to think about things, and they are all correct, even if they don't remember the details, then they've learned something*". Harriet also makes an effort, through course content, activities and assignments, to expand her students' conception of what mathematics is, noting that by the end of the course "*they're more open to the idea that geometry is a big part of mathematics than they might have been before.*" She also tries to "*explain why it is that fractions are important*", challenging beliefs that a portion of the elementary school mathematics curriculum is dispensable.

Harriet's comments about beliefs seem directed not only towards students' beliefs about the nature of mathematics, but also to beliefs about the teaching of mathematics. A nice example of this is the following comment which she makes in her discussion of teaching about division of fractions:

> *The students know the algorithm so well, they don't remember learning it. They don't remember a thing about how they learned to do these things. They just remember the rules. And their idea of helping somebody find their mistakes is to say "Oh! You're supposed to cross that out, and carry the one." Which isn't going to help anyone understand anything.*

This quote brings out the students' frequent belief that mathematics is a set of static rules that need to be memorised (Szydlik, Szydlik and Benson, 2003), as well as Harriet's contrary beliefs that not only can mathematics be understood, but it is the responsibility of the teacher to foster that understanding. Harriet provides her students with activities that allow them to analyse and discuss pupil errors. Through their discussion of these errors, she seeks to counter the belief that teaching mathematics merely involves showing someone how to execute the correct procedure.

It is worth noting that although the instructors are conscious of their students' beliefs about mathematics, and comment on how the activities and methods hope to contribute to a renegotiation of those beliefs, the development of positive beliefs and attitudes towards mathematics is not normally listed in the official course curricula for Math for Teachers courses.

*Emotions/Attitudes*

We bring our description to a close with a few comments related to the theme of emotions/attitudes that arose in the interviews. Once again this was a common concern, with instructors categorically describing their students as suffering from mathematics anxiety and lack of confidence in their ability to do mathematics. There were however, considerable differences in instructors' approaches to these negative feelings. Bob and Harriet represent two of the views expressed.

Bob describes his students as suffering from confusion and anxiety, which is closely linked to poor performance:

> *in many cases, some of the very elementary arithmetic operations are in fact, confused in their minds and so when they hit upon things, in particular when you hit rational numbers, as an example, that's one place where students have a great deal of anxiety and they would demonstrate poor understanding of ideas.*

For Bob the source of their anxiety is their lack of skill. It is his belief that improved skills will lead directly to increased confidence, however he confesses that the realities of the course conspire against this. At the beginning of the interview he expresses a wish that the students in his course develop a love of math, but when asked about whether this goal is accomplished, he admits: "*...in terms of the other goal, for love of math? Unfortunately, the course is so packed, that in some ways, I think they do get a little bit beaten by the end, and they're just tired.*" This begins to touch on the theme of "tensions" (not discussed here) however it illustrates Bob's realisation that the volume of subject matter content that he needs to cover in the limited amount of time he has with his students is at odds with his objective of instilling a "love of math" through building subject competence.

Harriet is also very concerned about her students' math anxiety: "*they are very anxious around problem solving. They are just terrified, most of them, of a problem they haven't seen before.*" In contrast to Bob, her efforts to address this seem to be centred on changing their ideas of what the enterprise of mathematics is all about. She tries to convince them that "*we're supposed to have fun with this*" and tells her students that "*you may never have seen it; you might not get all the way through it. But what I'm looking for is how far did you get, and how well can you explain what it is that you got*", shifting the focus away from getting the right answer, towards less threatening goals. By the end of the course she hopes her students have grown in confidence and also "*they have more of a sense of play...I think they're more flexible. They think they're more flexible. They're not as scared if... that someone will ask them a question that they can't answer.*" It is not clear if Harriet needs to compromise on time spent on building mathematical proficiency in order to make time for students to approach the course in this way.

## Conclusion

In her 2002 plenary address to the PME-NA, Ball noted that "we have not put in the foreground the "who" of teacher learning as often as we might" (Ball, 2002). The partial results of our study presented here provide a glimpse into the "who" of instructors of the Math for Teachers course, a course that is often the final mathematics content course taken by prospective elementary school mathematics teachers. We have shown that despite the course's nominal focus on subject content knowledge, there is also potential for instructors to address pedagogical content, beliefs about mathematics and the teaching of mathematics, as well as attitudes.

The comments of the interview subjects reveal intentions for the course that go beyond the transmission of mathematical proficiency, providing examples of instructors' efforts to expand students' conceptions of mathematics and their relationship with it. However, we are aware that our methodology permits only a description of the instructors' espoused beliefs and intentions of practice, which may not be consistent with their actual practice (Liljedahl, 2008). Classroom observations, follow-up interviews, and examination of artefacts (such as assignments and tests) will be useful in providing validation for our conclusions. Furthermore it is our hope that the full report of this study will help inform future investigations into the contribution that mathematics content courses can and do make in the development of mathematics teachers.

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# AN INVESTIGATION INTO TWO PRESERVICE TEACHERS' USE OF DIFFERENT REPRESENTATIONS IN SOLVING A PATTERN TASK

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*Use of multiple representations can positively influence students' mathematical learning and problem solving performance. In this study, we investigate how two preservice teachers utilized multiple representations in solving a pattern task. In characterizing their solution strategies and the difficulties they encountered, we make use of concept images and concept definitions. Our results suggest that translating between multiple representations can help students solve a problem that they have difficulties with in the given representation. Furthermore, once they find the solution they can revisit the initial representation they struggled with and make necessary corrections in their solution strategies.*

## Introduction

NCTM standards emphasize the importance of different representations as they enable multiple ways of thinking about mathematical objects (NCTM, 2000). Using different representations facilitates learning algebra meaningfully and effectively (Friedlander & Tabach, 2001). Students can explain their mathematical strategies in multiple ways such as verbally, numerically, algebraically and graphically to gain a deeper understanding of mathematical concepts. Janvier (1987) supports that all meanings of an object cannot be encompassed within a single representation, as a result students need to translate between different representations to gain a richer knowledge. In order to translate between multiple representations effectively, students should have correct and strong links between each representation (Dreyfus, 1991). This study examines how pre-service teachers utilized multiple representations and the difficulties they encountered as they solved an algebra task.

## Theoretical Framework

NCTM defines representation both as a process and product. It is a process because a representation is the *act* of capturing a mathematical concept or relationship in some form, and it is a product because it is the form itself (NCTM, 2000). The ability to use multiple representations is an important skill for solving mathematical problems, as it allows students to view problems from different perspectives. To use multiple representations effectively, students should know how to translate between different representations, and how to apply transformations within a single representation. Strengthening and remediating these abilities can positively influence students' mathematical learning and problem solving performance (Lesh, Post, & Behr, 1987).

A mathematical object can be defined formally which is called the *concept definition* of that object (Tall & Vinner, 1981). What students usually remember, however, is not the concept definition, but the *concept image* which is a set of all mental constructs and processes that are associated with that object (Vinner & Dreyfus, 1989). A concept image may be contained within a single representation (e.g. slope may be associated only with the coefficient of *x*), or it may encompass multiple representations (e.g. in addition to coefficient of *x*, a student may think of the coordinate system or a numerical pair of *x*-*y* values for slope). Additionally, one may have

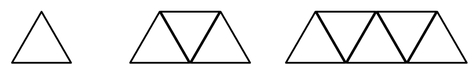
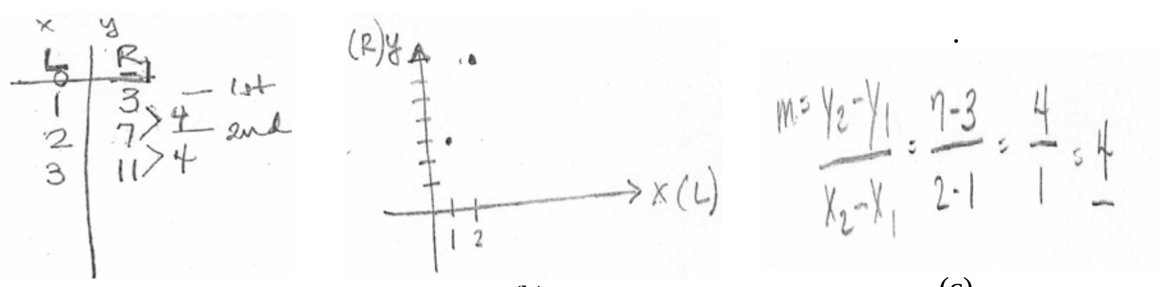



*Figure 1*. The beam task.




(a) (b) (c)

By translating the numerical representation to a graphical one, George was able to find the $x$ and $y$ values that can be used in the algebraic formula. This suggests that his algebraic concept image for slope was more accurate, and therefore he felt more comfortable in this representation. Once George found the slope using the algebraic formula, we asked if he can make sense of what 4 was in his table:

Interviewer: What does this difference of 4 tell you?

George: This 4 here with respect on $R$ [number of rods] is going to be the change. It is going to be the rate of $y$. Well yes of course it is going to be the rate of change because you are doing this [pointing to Figure 2c] to find the slope.

This dialogue shows that George was initially confused about interpreting his table, but once he noticed the correspondence between the algebraic result and his numerical solution, he realized that the difference between the successive entries in his table also gave the slope value. Thus, George was able to make sense of his table by referring to his algebraic solution.

He then continued to solve the problem by trying to find $b$ using the linear equation, $y = mx + b$. Although he described $b$ as the $y$-intercept, he had difficulty to use this representation:

George: For the $y$-intercept we already have the slope. Let's see $y$ equals to zero. No [erase what he wrote]. I can write this in standard form. $y$ equals the same formula $y = mx + b$ where $m$ is 4. I want to find the $y$-intercept [long pause].

This time George had difficulty to use the algebraic representation to find the $y$-intercept. Therefore, he was encouraged to solve the problem by using another representation. He then went back his numerical representation (i.e. table), and noticed that he needed to find the value of $x$ when it is equal to zero to find the $y$-intercept. He found that when $x$ is zero $y$ becomes -1 (see Figure 2a). The following excerpt shows how he used the numerical representation to find the $y$ intercept and the final result (Figure 3):

George: So if $x$ is 0 I have to subtract 4 from 3 which give me -1. So this one now became $y = 4x – 1$ [Then he changes the variables back to $R$ and $L$ to obtain, $R = 4L - 1$]

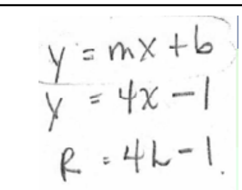



*Figure 3*. George's solution (continued from Figure 2).

George's solution suggest that he had difficulties when using only one representation, but his ability to translate between different representations helped him to find the correct answer. Furthermore, once he found the answer, he was able to revisit the representations that he had difficulties with, and could make better sense of them.

Linda also started to solve the problem by drawing a table. She first wrote the given information in the task into a table as shown in Figure 4. Then she tried the check-and-guess approach to find the general formula of pattern sequence:

Linda: Maybe $2n + 1$. That is not going to work for that one [shows the second value which is 7]. How about $3n + 1$? But this is not going to work for the first one.

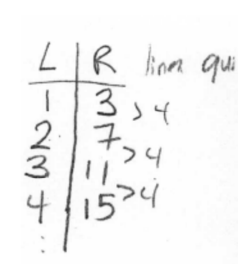



*Figure 4*. Linda's

After she tried several formulas, she found the difference between the numbers of rods ($R$) in successive table entries which was 4. Then she was asked to explain what the number 4 represents in her table. Linda stated that the number shows that it should be plus 4 in the general equation of the pattern task. The following excerpt show how she interpreted number 4 in her numerical representation:

Linda: So it should be plus 4 in the equation, because every time it goes up by 4; from one pattern to the other pattern.

Linda considered 4 as the *y*-intercept instead of the slope. This may suggest a mismatch between her concept image and the concept definition of slope when she is using the numerical representation. Since she had difficulty to find the answer using the numerical representation, she was encouraged to use another approach. Then she drew a graph and plotted the points in the table. After that she connected the points and found the slope geometrically by computing the ratio of vertical change versus horizontal change (Figure 5):

Linda: [Using data points in her table] One is three, two is seven. I can find the slope [Linda finds the slope as rise-over- run by using the graphical representation].

Interviewer: So what is the slope?

Linda: 4

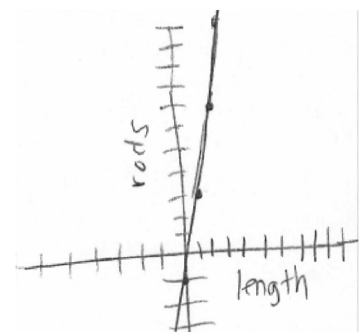


Figure 5. Linda's graph

Once Linda found the slope using the graphical representation, she revisited her table and realized that the difference 4 between the successive table entries corresponded to the slope rather than the *y*-intercept. She then revised her answer as "4*x* plus something" instead of "something plus 4". Thus, graphical representation helped her to translate from numerical representation to algebraic representation. Next Linda also found the *y*-intercept as -1 using the graph (Figure 5).

Linda's solution illustrates the case where the student has multiple concept images of the same object, but these images conflict with each other. As this example shows, in these cases it may be possible for the student to revisit the representation she struggled with and make the necessary corrections.

## Conclusion

In this study, we observed that students used their concept images when solving a mathematical problem. Furthermore, they had multiple concept images, and when their concept images were misaligned with the actual concept definition, they struggled to find the solution. However, students were able to translate between different representations, and as a result they could solve the problem in a representation where their concept images were accurate. This allowed them to revisit the representations they had difficulties with and discover the problems in their understanding.

To conclude our brief analysis, we believe that using multiple representations by having correct concept images of the mathematical objects involved and practicing those representations in different forms of the same task can positively influence mathematical learning and problem solving performance.

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# CARTESIAN PRODUCT–IN-ACTION: VERGNAUD'S THEORY APPLIED TO PRESERVICE SECONDARY MATHEMATICS TEACHERS' INTERPRETATION OF POLYNOMIAL MULTIPLICATION MODELED WITH ALGEBRA TILES

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*This study examines preservice secondary mathematics teachers' understanding and sense-making of polynomial multiplication modeled with algebra tiles. I base this research within a framework of concepts and theorems-in action (Vergnaud, 1983, 1988, 1994, 1997). My data consist of videotaped qualitative interviews. I generated a thematic analysis by undertaking a retrospective analysis, using constant comparison methodology. The main result of this study is that representational Cartesian products-in action at two different levels, indicators of multiplicative thinking, were available to two research participants only.*

## Background

The study of multiplicative structures has been conducted by mathematics education researchers since the 1980s. In his 1983 article, Vergnaud defines the notion *conceptual field* as a "set of problems and situations for the treatment of which concepts, procedures, and representations of different but narrowly interconnected types are necessary." (p. 128). In particular, he views the multiplicative structures, a conceptual field of multiplicative type, as a system of different but interrelated concepts, operations, and problems such as multiplication, division, fractions, ratios, similarity. Although multiplicative structures can to some extent be modeled by additive structures, they have their own characteristics inherent in their nature, which cannot be explained solely by referring to additive aspects. Behr, Harel, Post, and Lesh (1994) developed two representational systems – extremely generalized and abstract – in an attempt to transcribe students' additive and multiplicative structures in which the notion "units of a quantity" plays the main role. According to Steffe, "for a situation to be established as multiplicative, it is always necessary at least to coordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit." (1992, p. 264). Confrey provides splitting, "an action of creating simultaneously multiple versions of an original," (1994, p. 292) as an explanatory model for children's construction of multiplicative structures.

## Theoretical Perspectives

Vergnaud's conceptual field theory asserts that "one needs mathematics to characterize with minimum ambiguity the knowledge contained in ordinary mathematical competences. The fact that this knowledge is intuitive and widely implicit must not hide the fact that we need mathematical concepts and theorems to analyze it." (1994, p. 44). I base this present study within a framework of concepts- and theorems- in action (Vergnaud, 1983, 1988, 1994, 1997). According to Vergnaud, *theorems–in–action* are "mathematical relationships that are taken into account by students when they choose an operation or a sequence of operations to solve a problem" (1988, p. 144). He goes on to state "To study children's mathematical behavior it is necessary to express the theorems–in–action in mathematical terms." (p. 144). Concepts-in-action serve to categorize and select information whereas theorems-in-action serve to infer

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appropriate goals and rules from the available and relevant information (Vergnaud, 1997, p. 229).

Vergnaud (1988) claims that "a single concept does not refer to only one type of situation, and a single situation cannot be analyzed with only one concept" (p. 141). He argues that teachers and researchers should study conceptual fields rather than isolated concepts. He then goes on to define a conceptual field as "a set of situations, the mastering of which requires mastery of several concepts of different natures" (p. 141). Grounded in this theory, I developed a series of terminology in an attempt to reveal my research participants' concepts-in-action arising from their verbal descriptions, actions, statements, hand gestures, and drawings in the context of polynomial multiplication problems modeled with algebra tiles.

## Research Questions

1. How do preservice secondary mathematics teachers make sense of polynomial multiplication problems modeled with algebra tiles?
2. What types of concepts- in action are available to these preservice teachers?

## Methodology

I conducted my study with (2 middle and 3 high school mathematics) preservice teachers enrolled in the Mathematics Education Program in a university in the southeastern United States. I interviewed 5 people individually twice during Spring 2007 semester. Duration of each interview was about 75 minutes and each session was videotaped using one camera.

The focus of this present study is on problems on identities of the form "product = sum" for products of polynomials modeled with algebra tiles. In this model, each little black square tile represents the number 1, purple bar represents the *x*, blue bar represents the *y*, purple square represents the $x^2$, blue square represents the $y^2$, and green rectangle represents the *xy*. The 1, the *x*, and the *y* are called Irreducible Linear (or Areal, depending on the context) Quantities (ILQ or IAQ); whereas the $x^2$, the $y^2$, and the *xy* are called Irreducible Areal Quantities (IAQ). Preservice teachers constructed rectangles with specified dimensions of the form $(ax + by + c)$, where *a*, *b*, and *c* were natural numbers. They were also asked to write their answers for the area of the polynomial rectangle as a product and as a sum.

After the end of data collection, I reviewed each interview data in order to generate possible themes for a more detailed analysis. I transcribed significant events of these interviews. A retrospective analysis, using constant comparison methodology, was then undertaken during which the interviews were revisited many times in order to generate a thematic analysis from which the following results emerged.

## Results

On the first polynomial multiplication task, my instruction was "Use the algebra tiles to multiply the polynomials $x + 1$ and $2x + 3$ on the multiplication mat." Ben first placed the dimension tiles (irreducible linear quantities) on the side and at the top. He then followed a "filling" process during which he tried to fit the areal tiles (irreducible areal quantities) in the polynomial rectangle outlined by the dimension tiles. Rather than a pair-wise multiplication, he relied on a *Filling in the Puzzle Strategy*, a concept-in-action, indicative of his additive thinking; despite the fact that he was asked to "multiply" these polynomials. Figure 1a depicts Ben's polynomial rectangle, which he obtained by the *filling in the puzzle* concept-in-action. Figure 1b depicts what he would have produced if reasoned multiplicatively. Figures 1c-d depict, another preservice teacher, Ron’s *Filling in the Puzzle Strategy*, while commenting “Any chance of fitting this [green tile] there [right next to the purple square]”?

Figure
13a

Figure
1b

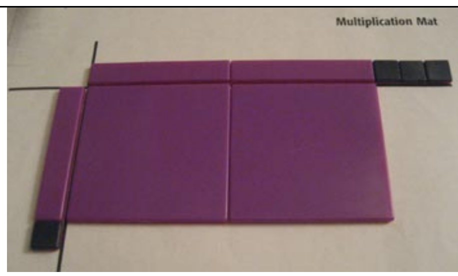


Figure
1c

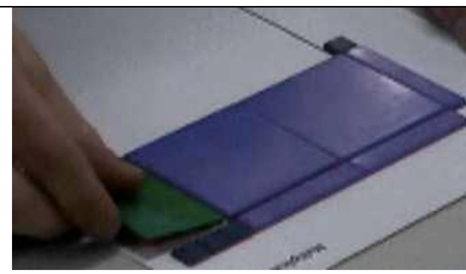

Figure
1d

John, when working on the second task on the $x + 1$ by $2y + 3$ polynomial rectangle, produced a polynomial rectangle with blue squares, blue bars, and black squares only (i.e., the polynomial rectangle was independent of $x$). *Filling in the Puzzle* strategy appeared in that John started with blue squares instead of green rectangles, which indicated that what he was doing was definitely not term wise multiplication (Figure 2a). Below the two blue squares, he placed 3 blue bars (Figure 2b). Right next to the two blue squares, he placed 6 blue bars (Figure 2c). Figure 2c stands as visual evidence that John was not using multiplication. In fact, John said “I am making the rectangle by parts.” Therefore, John's statement validates my previous hypothesis that the “Filling in the Puzzle” strategy seems to be related to an “area as a sum” strategy, namely calling for an additive nature. This contrasts with the “Term Wise Multiplication of Irreducible Linear Quantities” strategy, which naturally defines the irreducible areal quantities (IAQ) as products, namely of multiplicative nature. Finally, he placed 9 black squares right below the previous three blue bars hence completing his *puzzle* (Figure 2d).

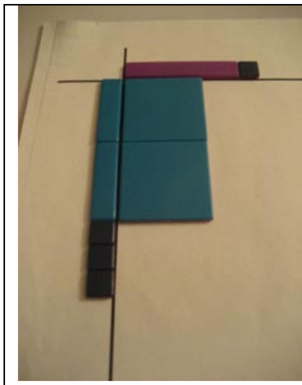
Figure 2a

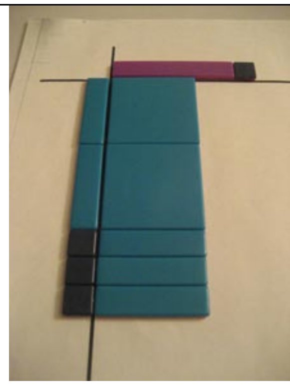
Figure 2b


Figure 2c

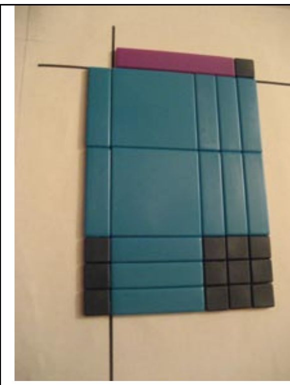
Figure 2d

Though he obtained a totally different polynomial rectangle for this second task, John's written answers and verbal descriptions were consistent in that he was always referring to his *y*–dependent–only polynomial rectangle. Because the initial instruction was to make a polynomial rectangle with length $x + 1$ and width $2y + 3$, at some point he had to write an identity in the last column of the activity sheet. In fact, he wrote the identity "$(x + 1)(2y + 3) = 2y^2 + 3y + 6y + 9$" as his answer. John's written answer warrants a disconnect as well, in that John was unable to write an *area as a product* expression (LHS) based on the *actual dimensions* of his rectangle. If he was able to refer the *actual dimensions* of his rectangle, the correct identity would then be "$(y + 1)(2y + 3) = 2y^2 + 3y + 6y + 9$" instead of "$(x + 1)(2y + 3) = 2y^2 + 3y + 6y + 9$." The following protocol takes this issue into account and reflects how John reconciled the equivalence of *x*– and *y*–dependent LHS with the *y*–dependent–only RHS:

Protocol 1: John establishes the LHS–RHS equivalence.

Interviewer: Are they equal? [about the LHS and the RHS of his identity "$(x + 1)(2y + 3) = 2y^2 + 3y + 6y + 9$"]

John: I mean... they're equal... they have to be equal...

Interviewer: Do you want to verify?

John: Do you want me to multiply that [the LHS] out? [I then ask him to do it on the board. Here is the first step of his verification. Figure 3a]

Interviewer: Is there something wrong?

John: No... It's just that... we don't know what *x* is... so... if you knew what *x* was you'd probably... *x* probably equals... [He looks at his figure] It looks like *x* equals *y* plus 2 [He then substitutes $x = y + 2$ and completes his verification. Figure 3b]

Interviewer: So it works with the condition that...

John: With the condition that *x* equals *y* plus 2.

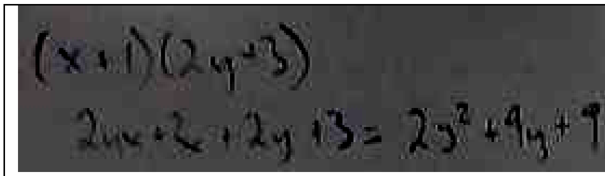

Figure 3a

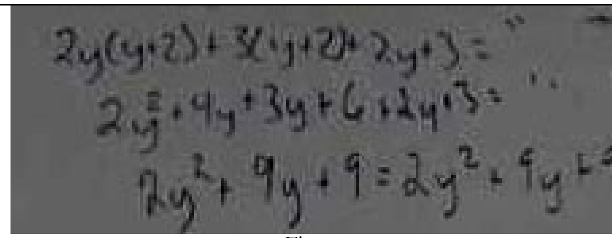

Figure 3b

At the beginning of the conversation, John was so certain about his equality that he did not feel the need to question it. Upon my request to verify his findings, he obtained “$2yx + 3x + 2y + 3 = 2y^2 + 9y + 9$” (Figure 3a). At this point, he realized that the RHS is *y*–dependent–only whereas the LHS has “*x*”s and “*y*”s, and deduced that he somehow had to get rid of the “*x*” on the LHS. He then referred to his figure made of tiles. He actually *measured* the *x* at the top of his figure using the “*y*” and the “1” tiles. In order to get rid of the “*x*” on the LHS, he substituted $x = y + 2$ (Figure 3b), based on his measurements. In other words, *John made sense of the dimension tiles for the first time*. For him, the dimension tiles do not stand as irreducible linear quantities (ILQ) whose term wise multiplication yields the corresponding irreducible areal quantity (IAQ), though. They rather stood as some sort of measurement tools helping John establish the LHS–RHS equivalence of his written identity.

On the third polynomial multiplication task, my instruction was “Use the algebra tiles to multiply the polynomials $2x + y$ and $x + 2y + 1$.” Both Nicole and Sarah, when placing the dimension tiles, followed the “*x* tile followed by the *y* tile followed by the 1 tile” ordering. As was the case with all the polynomial multiplication problems, both Nicole and Sarah actually did each term wise multiplication carefully by pointing to the corresponding irreducible linear quantities, and placed the resulting irreducible areal quantity accordingly (Figure 4).

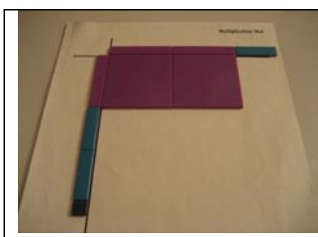

Figure 4a

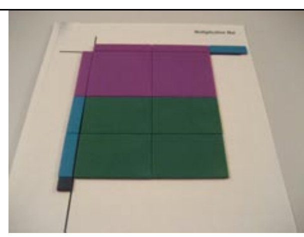

Figure 4b

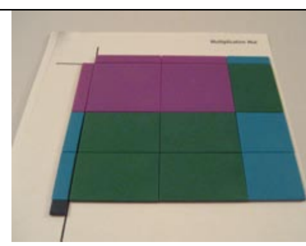

Figure 4c

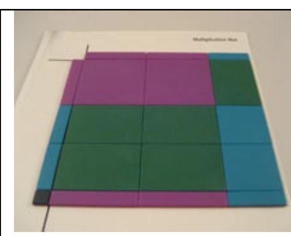

Figure 4d

Both Sarah and Nicole thought aloud and pointed to the irreducible linear tiles at the top and on the side for each multiplication. The “multiplicative nature” of the “irreducible areal quantities” seems to be warranted by Sarah’s statements in the following protocol:

Protocol 2: Sarah’s reference to a Representational Cartesian Product of Type I.

Sarah: This is [pointing to and placing the areal *x* squared tile] *x* [pointing to the linear *x* tile on the side] times *x* [pointing to the linear *x* tile at the top]. This one is also *x* times *x* [in a similar manner]. This one is $x$ times $y$ [pointing to and placing the green tile representing the areal unit $xy$]. And $x$ times $y$ [in a similar manner].

Interviewer: Where is the $x$ times $y$?

Sarah: $y$ [pointing to the linear $y$ tile at the top] and $x$ [pointing to the linear $x$ tile on the side]. And $x$ times $y$ [in a similar manner]. And $x$ times $y$ [in a similar manner]. And this is $x$ times $y$ [in a similar manner]. And then this is $y$ [pointing to the linear $y$ tile at the top] times $y$ [pointing to the linear $y$ tile on the side]. And $y$ times $y$ [in a similar manner]. This is $x$ [pointing to the linear $x$ tile on the side] times 1 [pointing to the linear 1 at the top]. And $x$ times 1 [in a similar manner]. And $y$ [pointing to the linear $y$ tile on the side] times 1 [pointing to the linear 1 at the top].

Sarah did not say "$x$ squared," nor "$y$ squared." She rather said "this is $x$ times $x$," "and then this is $y$ times $y$," i.e., multiplicative in nature. Her language "$y$ and $x$" is also indicative of an ordered pair ($y$, $x$) of linear quantities. In this vein, both Sarah and Nicole can be said to construct a *Representational Cartesian Product* of Type I defined on the *Representational Sets of Irreducible Linear Quantities* (RSILQ). With relational notation, Sarah and Nicole's verbal descriptions accompanied by their hand gestures can be modeled with the following (Representational) Cartesian Product-in-action of Type I:

- $RCP_1 = RSILQ_1 \times RSILQ_2 = \{x, x, y\} \times \{x, y, y, 1\} = \{(x, x), (x, y), (x, y), (x, 1), (x, x), (x, y), (x, y), (x, 1), (y, x), (y, y), (y, y), (y, 1)\}$.

When we discussed the "area of the boxes of the same color as a product" column on the activity sheet for the same polynomial multiplication problem $2x + y$ times $x + 2y + 1$, Nicole's answers, once again, were areas defined as the product of two quantities, i.e., multiplicative in nature. The following protocol illustrates Nicole's multiplicative thinking.

Protocol 3: Nicole's reference to a Representational Cartesian Product of Type II.

Interviewer: You are saying $2x$ times $x$ [About Nicole's expression $(2x) \cdot (x)$, which she wrote on the activity sheet]. And why not 2 times $x^2$?

Nicole: Because I just saw these as a pair together... [pointing to the linear $2x$] same things you can group them together. So it's just $A$ [pointing to the linear $2x$ at the top] times $B$ [pointing to the linear $x$ on the side]. Because when you look at this whole thing, this whole purple area [pointing to the $2x$ by $x$ "same–color–box"] as one area... so you look the length as one number, $2x$, instead of 2 times $x$.

Interviewer: So what is the difference between this and the other way [I am asking her to compare the expressions $(2x) \cdot (x)$ and $(2) \cdot (x^2)$] representationally?

Nicole: I did it [About her expression $(2x) \cdot (x)$, which she wrote on the activity sheet] in terms of the length times the width. Now it [About the expression $(2) \cdot (x^2)$ via which I am trying to challenge her] would be... talking about... how many of these [pointing to the purple squares] you have. In the other case, length times width gives the area of the whole thing [pointing to the $2x$ by $x$ "same–color–box"].

Interviewer: Please do the same for this one... [pointing to her expression $(2x) \cdot (2y)$, which she wrote on the activity sheet] why not 4 times $xy$?

Nicole: Again I did this [pointing to the $2x$ by $2y$ "same–color–box"] as one area. I did it as length times width. Now this [about the expression 4 times $xy$ I am trying to challenge her with] means I have 4 of them [meaning 4 green rectangles] and each one is an $xy$.

Nicole showed a mathematically fruitful performance in creating a *Representational Cartesian Product* of Type II, in her comparison of "$2x$ times $x$" vs. "2 times $x^2$." The "pair" in Nicole's statement “I just saw these as a pair together” refers to the pair of linear “$x$”s in “2$x$” and not to the ordered pair ($x$, 2$x$) of linear quantities. In other words, at the initial stage of defining a *Representational Cartesian Product* of Type II, she first identified the elements of the *Representational Sets of Combined Linear Quantities* $RSCLQ_1 = \{2x, y\}$ and $RSCLQ_2 = \{x, 2y, 1\}$. Her later usage “So it's just $A$ [pointing to the linear 2$x$ at the top] times $B$ [pointing to the linear $x$ on the side]” signaled the onset of a *Representational Cartesian Product* of Type II, another *concept–in–action*, an indicator of Nicole’s multiplicative reasoning. In this way, she established the existence of her concept–in–action: Nicole first picked an element, “$A$” from the set $RSCLQ_1 = \{2x, y\}$ and then picked another element, “$B$” from the other set $RSCLQ_2 = \{x, 2y, 1\}$. She then formed pairs ($A$, $B$) – another concept–in–action – of combined linear quantities (CLQ) by which she generated her RCP. In fact, what she referred to is an abstract definition of a Representational Cartesian Product RCP = $\{(A,B) \mid A \in RSCLQ_1, B \in RSCLQ_2\}$ where $RSCLQ_1 = \{2x, y\}$, $RSCLQ_2 = \{x, 2y, 1\}$ with the set builder notation. $A$ and $B$ can be anything as long as they are coming from the first set $RSCLQ_1$ and the second set $RSCLQ_2$, respectively. Nicole did not describe the expressions $(2)\cdot(x^2)$ and $(4)\cdot(xy)$ as representationally multiplicative, as opposed to Ben and Ron, who embraced the *Filling in The Puzzle Strategy*, which is an indication of their additive thinking. Although $(2)\cdot(x^2)$ and $(4)\cdot(xy)$ are representationally additive, as Nicole explains in Protocol 3 above, for Ben and Ron these expressions are *mis*interpreted as of multiplicative nature. The algebraic symbols $(2)\cdot(x^2)$ and $(4)\cdot(xy)$ can be deduced only within an additive context, according to Nicole, *representationally*.

In the same vein, I asked Sarah to outline the same-color-boxes on her $2x + y$ by $x + 2y + 1$ polynomial rectangle (Figure 4d). The multiplicative nature of the areas of these “same-color-boxes” was prevalent, as reflected in the protocol below:

Protocol 4: Sarah's reference to a Representational Cartesian Product of Type II.

Sarah: This one is 2$x$ [pointing to the linear 2$x$ on the side] times $x$ [pointing to the linear $x$ on the top]. This one is 2$y$ times 2$x$ [pointing to the corresponding linear tiles in a similar manner]. This one is 2$x$ times 1 [pointing to the corresponding linear tiles in a similar manner]. This one is $x$ times $y$ [pointing to the corresponding linear tiles in a similar manner]. This would be $y$ times 2$y$ [pointing to the corresponding linear tiles in a similar manner]. And this would be $y$ times 1 [in a similar manner].

Interviewer: So the product... each time you were doing the same thing... tell me more about that... I just want to make sure that I understand that...

Sarah: I was using the area as a length times width where... this is a length or... and this would be the width... and basing it of like that... otherwise I could have added the insides [pointing to the areal tiles]... the way I did it was length times width.

In other words, Sarah was aware that what she was doing was term wise multiplication of the combined linear quantities, and not addition. Her statement “otherwise I could have added the insides” combined with her gestures indicate that there are only two possibilities: The areas of the “same–color–boxes” could be modeled either via multiplication, or addition, *representationally*. But since she was asked about the areas of these boxes as products, the other option, namely additiveness is irrelevant as she responded “the way I did it was length times width.” This is in contrast to Ben and Ron's written answers and verbal descriptions of these “boxes” indicating an additive nature. Using set notation, Sarah's descriptions can be modeled via a *Representational Cartesian Product*–in-action of Type II defined as follows.

- $RCP_2 = RSCLQ_1 \times RSCLQ_2 = \{2x, y\} \times \{x, 2y, 1\} = \{(2x, x), (2x, 2y), (2x, 1), (y, x), (y, 2y), (y, 1)\}$.

## Preservice Teachers' Concepts-In-Action

In this research study, all five research participants worked on all three tasks. Due to the page limitation, I could not include fifteen different participant-task pieces. In this section, without loss of generality, I summarize the research participants' concepts-in-action pertaining to the 2nd polynomial multiplication task "Multiply $x + 1$ by $2y + 3$ using algebra tiles" for the sake of the constant comparison analysis methodology.

On the first level (Figure 5a); Ben, Ron, and John interpreted the ¨Irreducible Areal Quantities" as meaningless Areal-Singletons, which corroborates their *Filling in the Puzzle* concept-in-action, indicative of their additive thinking. Sarah and Nicole, on the other hand, interpreted these ¨Irreducible Areal Quantities" as Areal-Singletons *resulting from* the multiplication of the corresponding pair of irreducible linear quantities, which corroborates their *Term-Wise Multiplication of Irreducible Linear Quantities* concept-in-action.

On the second level (Figure 5b); Ben, Ron, and John interpreted the "Same-Color-Boxes" additively, whereas, Nicole and Sarah interpreted these areal quantities multiplicatively. Table 1 illustrates my research participants' additive/multiplicative interpretation of the *area of the same-color-boxes as a product* phrase for the $x + 1$ by $2y + 3$ polynomial rectangle.

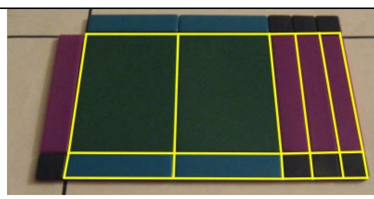

Figure 5a – Irreducible Areal Quantities

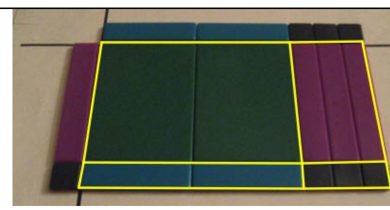

Figure 5b – Same Color Boxes

Table 1

Areas of the Same-Color-Boxes as a Product for the x + 1 by 2y + 3 Polynomial Rectangle

| **Ben, Ron, John** | **Sarah, Nicole** |
|---|---|
| Interpret Same-Color-Boxes **additively**:<br>• The green box as 2 times $xy$.<br>• The purple box as 3 times $x$.<br>• The blue box as 2 times $y$.<br>• The black box as 3 times 1. | Interpret Same-Color-Boxes **multiplicatively**:<br>• The green box as $x$ times $2y$.<br>• The purple box as $x$ times 3.<br>• The blue box as 1 times $2y$.<br>• The black box as 1 times 3. |

Figure 6a shows Sarah and Nicole's construction of $RCP_1$ by applying a Cartesian Product on the Representational Sets of Irreducible Linear Quantities $RSILQ_1$ and $RSILQ_2$. In the same vein, Figure 6b shows Sarah and Nicole's construction of $RCP_2$ by applying a Cartesian Product on the Representational Sets of Combined Linear Quantities $RSCLQ_1$ and $RSCLQ_2$.

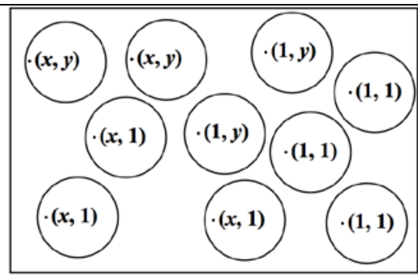


Figure 6a

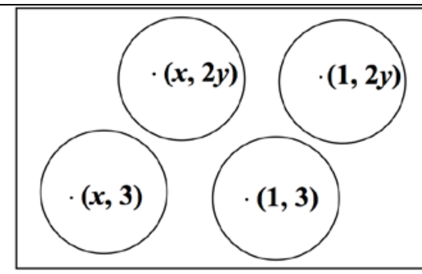


Figure 14b

## Discussion

Concepts– and theorems–in–action framework in this present study helped me produce a set of terminology closely related to mathematical terms (*Representational* Cartesian Products, *Representational* Sets, Irreducible Linear Quantities, Irreducible Areal Quantities, Combined Linear Quantities, Filling in the Puzzle Strategy, Term Wise Multiplication of the Irreducible Linear Quantities Strategy) to describe what my preservice teacher research participants were doing. These notions, as a mathematics teacher educator, helped me make sense of what my preservice teacher research participants were doing and delve further into their understanding of the mathematical situations.

Distinguishing how quantities interact with one another (e.g., additive vs. multiplicative) is an important element of algebraic reasoning. In that regard, *concepts–in–action* and *theorems–in–action* formalisms are powerful instruments to illustrate and explain the continuing progress of students' mathematical proficiency in a certain conceptual field (e.g., multiplicative structures, relational structures, mapping structures, quantitative structures). They also present a way to analyze, compare, and transform students' knowledge intrinsic in their mathematical performance (e.g., hand gestures, actions, operations, verbal descriptions) into the actual known and written algebraic identities and mathematical theorems. In that sense, these tools help teachers and researchers get a better sense of how students make sense of, reconcile, and shift among physical observables at different cognitive levels (e.g., algebraic expressions, their various representations, etc.). Using concepts– and theorems–in–action, teachers and researchers can come up with better strategies to diagnose what students do or fail to understand, to reveal the source of their misconceptions and conceptual flaws, and to help them see the internal and external connections. In this way, students are provided with a set of more interesting, better-prepared activities, and mathematically fruitful situations, which help them strengthen their knowledge, and increase their mathematical proficiency.

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# PRESERVICE TEACHERS' CONCEPTUALIZATION OF FRACTION MULTIPLICATION

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*A classroom teaching experiment, focusing on number and operations, was conducted in a semester-long mathematics content course for preservice elementary teachers. The focus of the study was on preservice teachers' understanding of mathematics related to their development of the mathematical knowledge needed for teaching. This paper presents the results from preservice teachers' development of fraction multiplication. The results indicate that preservice teachers' knowledge develops in three phases.*

## Background

Teachers are entering the profession without a profound understanding of the mathematics they are to teach (Ma, 1999). Mathematics content courses for preservice teachers that provide nothing more than a reiteration of the traditional algorithms do not aid in preservice teachers' development of pedagogical content knowledge any more than what was provided in their K–12 education. Studies show that teachers with a deep understanding of mathematics positively impact student achievement (Kaplan & Owings, 2000); however, little research documents how classroom teachers develop the knowledge base they need to be effective. The knowledge base of effective teachers, which includes using students' knowledge to inform instructional decisions, is beyond the experiences typically received in preservice teacher mathematics education classes (NCTM, 2000). In order for teachers to assess students' knowledge accurately, they themselves need a deep understanding of the content. This is especially important for elementary teachers, as they typically do not have a substantive mathematics background.

The knowledge base that elementary and middle school teachers bring to the classroom is procedurally based and largely misunderstood (Ball, 1990a, 1990b; Ma, 1999; Tirosh, 2000; Tzur & Timmerman, 1997). This is particularly true with teachers' understanding of fraction operations. Several studies have documented the difficulties that preservice and inservice teachers have in conceptualizing fraction operation situations; however, research has lacked in describing the ways in which preservice teachers develop a conceptual understanding of these topics.

Previous studies have documented that teachers need similar experiences to what children need before they can conceptually understand the algorithms used in fraction operation situations (Tzur & Timmerman, 1997). Rather than being presented with an algorithm first and then asked to solve several problems using that algorithm, teachers first need the opportunity to use models and pictures to solve the problems. From there, teachers can develop their own algorithms for solving problems involving fraction operations.

## Methodology

This study was conducted at a large metropolitan university in the southeastern part of the United States during a semester-long undergraduate course focusing on mathematics for teaching elementary school. There were 33 participants in this study. Participants were all female undergraduate students majoring in either elementary or exceptional education. They were all at least in their sophomore year of college. The research team consisted of 2 professors and 6 doctoral students all in mathematics education. One of the mathematics education faculty members was the instructor for the course. The other members of the research team observed every class session.

This study was conducted during a spring semester. The course met twice a week for one hour and fifty minutes each day. Students in the course were situated at tables of at least four and no more than six. The classroom was equipped with a document camera as well.

Data were collected from 10 class sessions, which focused on a rational number unit. The rational number unit was part of a larger study which also included a unit focusing on place value and whole number operations in base 8 (Roy, 2008). The rational number unit constituted the second unit in the course, thus students were already accustomed to being videotaped and observed in every class. The data collected included video and audio recordings, student work, pre and post-test scores, and research team notes.

Three video cameras were used to capture varying aspects of the classroom. Facing the front of the room, one camera was situated at the back right of the classroom and focused on the whole class and individual students. The second camera was placed at the back of the classroom and focused on the work done at the board and the work presented on the document camera by both the instructor and students. The third camera focused on the instructor and individual students from the front left of the classroom.

Audio recordings documented small group interactions during class work activities. Three small groups were chosen to have audio recorders placed at their tables. Each of these groups consisted of at least one person who was interviewed individually, which is how the group was chosen to have the audio recorders placed at their table.

Students were given a pre and post-test before and after the rational number unit. The test used was the Content Knowledge for Teaching Mathematics (CKT-M) instrument developed at the University of Michigan (Hill, Rowan, & Ball, 2005). Scores from the test were used to document the changes in students' rational number understanding.

Five students were interviewed individually before and after the rational number unit. Each interview was videotaped and lasted approximately 40 minutes. During the interview students were asked to solve several rational number problems. Students' work from each interview was collected. These students also participated in a focus group session halfway through the rational number unit were the focus was on students' overall feelings of the classroom structure, their mathematical activities, and thoughts of the rational number unit thus far. Students were selected for interviews based off of their pre-test scores, their normative ways of working mathematically (i.e. some students preferred to solve problems algorithmically whereas others relied on alternative methods), and students whose mathematical beliefs and values were diverse.

Other data collected from students included class work, homework, and exams. The data collected from the research team included field notes and reflective journals from class observations. The research team met after every class session to discuss if the learning goals for the day were met and to plan the next class session. Each of these team meetings were audio taped.

## Theoretical Framework

The emergent perspective was used as the foundation for which the classroom structure was designed so that social and sociomathematical norms were established and sustained throughout the course (Cobb & Yackel, 1996). The social norms established as part of this course included student's explaining and justifying solutions and solution strategies, making sense of others, and questioning others when misunderstandings occurred (Roy, 2008). Sociomathematical norms included understanding what constitutes a different and acceptable solution and solution strategy (Roy, 2008).

The instructional unit used in this study was designed around the theory of Realistic Mathematics Education (Gravemeijer, 2004). The preservice teachers were first presented with problems situated within a context. They were then asked to solve the problem using their own methods. Finally, selected problems were discussed within a whole-class setting. Throughout the course, preservice teachers were encouraged to develop their own methods of solving problems through pictures. Tasks were designed so that the preservice teachers could reconstruct the mathematics for themselves.

## Results

The results indicate that preservice teachers' development of a conceptual understanding progresses through three phases. Within the first phase, preservice teachers revert to a procedure to solve a given problem. The second phase includes developing ways to explain and justify solutions and solution strategies. Within the third phase, new concepts are developed.

For illustration purposes, the results described will pertain specifically to preservice teachers' development of fraction multiplication. Fraction multiplication occurred over a day and half of instruction. When this topic was introduced, preservice teachers were given a set of contextualized situations and asked to solve them without any prompt from the instructor.

*Revert To Procedures*

The following is a contextualized problem that was presented:

Sue ate some pizza. 2/3 of a pizza is left over. Jim ate 3/4 of the left over pizza. How much of a whole pizza did Jim eat?

When presented with this problem, students knew that the problem represented a fraction operations situation, however were not instructed or given any information to indicate that the situation was multiplication. When initially solving this problem, many students reverted to using a known procedure to solve the problem as indicated by the following students' work (Figure 1).

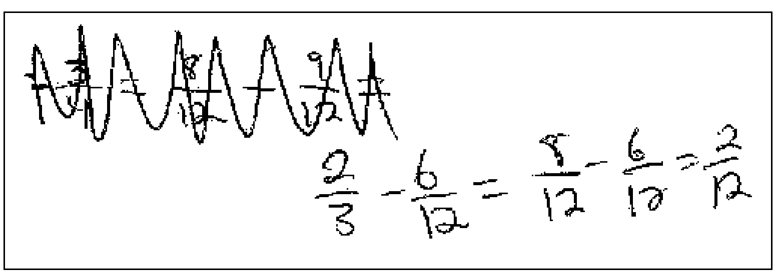



*Figure 1.* Student 15 uses subtraction procedure for multiplication.

$\frac{3}{4} \times \frac{2}{3} = \frac{6}{12} = \frac{1}{2}$

Since there was 2/3 pizza left, & Jim ate 3/4 of that; I multiplied the two numbers to find out how much total pizza

*Figure 2*. Student 2 uses multiplication procedures.

Both students' answers represented a typical solution presented by most students. As seen, both students reverted to using procedures to solve the problem. Though the first student was incorrect in her thinking that the problem represented a subtraction situation, she still used a procedure to solve the problem. The second student was correct in that the problem was multiplication; however, could only solve and explain the problem in terms of the algorithm used to multiply fractions (Figure 2). The class went on to discuss this problem as a whole group in which they established the notation of multiplication representing a 'groups of' meaning and also went on to solve the problem using a model.

*Develop Explaining and Justifying*

The class then moved from contextualized problems to problems situated without a context. To further discuss multiplication, the following problem was presented:

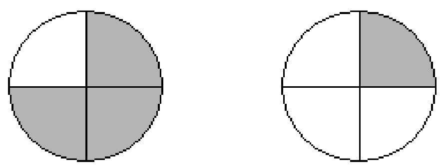
$$1\frac{1}{3} \times \frac{3}{4}$$

The class as a whole was asked to develop a model to solve the problem. When students were having difficulties representing the problem, the instructor further encouraged their understanding development by having them think of the problem in terms of a "groups of" situation.

Instructor: What if we thought about this as groups of objects? 1 1/3 groups of 3/4. How might we approach it that way? 1 1/3 groups of 3/4.
Claire: You draw 3/4, 1 1/3 times.
Instructor: Draw 3/4, 1 1/3 times. Okay.
Caroline: Wait.
Jackie: To figure out 1 1/3 times what do you mean?
Instructor: Well there's one time of 3/4. Are we okay with that?
Class: Yeah.
Instructor: What's a third time of 3/4?
Olympia: Don't you have to break it into pieces? Split the slices.
Instructor: Do I?
Olympia: Yeah because you have to get three wait.
Instructor: What is 1/3 of this?
Caroline: A fourth.
Instructor: A fourth?
Caroline: Yeah out of the 3/4.

Instructor: She didn't split anything up.
Claudia: My answer's 1.
Instructor: I don't really care. That's true but I didn't mean it that way. I don't care about the answer I want to know why.
Claudia: No I know because I don't know I'm thinking of it as if 1/3 is 1/4 so then
Instructor: Is 1/3, 1/4?
Students: No.
Caroline: What?
Claudia: Forget it.
Instructor: No keep going.
Olympia: I understand what you're saying.
Instructor: What is she saying?
Edith: Well because you showed the 3/4 out of 1 and so a third of the 3/4 would be 1/4.
Instructor: Claudia is that what you said?
Claudia: Yeah.

Throughout this discussion, the following picture was drawn on the board.



The first circle represents the one group of 3/4 and the second is the 1/3 group of 3/4. Throughout this discussion, students had to develop ways of explaining so that concepts such as 1/3 being 1/4 could be understood. As seen from this discussion, several students contributed to each others' development of explaining and justifying how to represent this situation. The same problem was then the first problem revisited during the next class session in which the class went on to discuss and develop the answer of one, and how the model represents that solution.

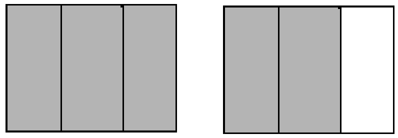
*Develop New Concepts*

After the previous problem was discussed on the second day of fraction multiplication, the class was then presented with the following problem, again out of context:

$$1\frac{1}{5}\times 1\frac{2}{3}$$

When the class was discussing how to solve this situation, one student asked to come to the board to explain 1 1/5 groups of 1 2/3.

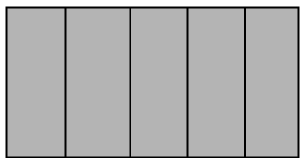
Claudia: So if we start off with two of these and divide them into thirds and then we find 1 2/3. Right? So then it would be this one, this one, this one, that's one. And then 2/3 would be this much.



So let’s like draw because now 1 2/3 is our new whole because we’re trying to find 1 1/5 of it. So if we just draw it altogether and make that our new whole. That’s right, right? Yeah. So then we look at this and we know

Instructor: How did she know that was right? What did she just check? You guys following her so far?

Claudia: So I’m just combining these into here to make that our whole.



Everybody follows?

Class: Yeah.

Claudia: Okay. So now this is our whole and now this is our whole and it’s divided already into five. So then this is one and then one more would be 1/5, right? Because this would be 1/5 since this is our whole and then this is one piece of that.

Suzy: I thought it had thirds, is that not a third?

Claudia: Yeah this is a third of one thing. But then

Suzy: Each of those thirds right?

Claudia: But then of this new whole which is 1 2/3, this is 1/5 of that

Suzy: Alright.

Claudia: because it’s divided into five. Does anybody not follow that?

Alex: I’m not following it. Sorry.

Suzy: Say it again. It took me a second, but that made sense.

Claudia: This is what 1 2/3 are right? Right here what I shaded.

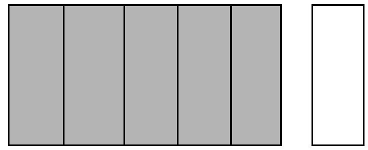
Suzy: Because we’re finding 1 1/5 times

Claudia: And 1 2/3 is going to be our new whole though because we’re trying to find 1 1/5 of that. So I just drew that as a new whole. Everybody okay up to there?

Class: Yeah.

Student: So you have that and then 1/5?

Claudia: Yeah so then one more of these would be 1 1/5 because this is 1/5, right of this 1 2/3? And then one more would be that and it would be two.



From this diagram the classroom conversation then continued with the class taking time to understand where the answer of 2 comes from. Within this discussion, new concepts started being developed. The concept of a new whole is presented within the context of fraction multiplication, which was the first time students started discussing the idea that the whole changes throughout the problem.

## Discussion

Fraction multiplication was used to illustrate the ways in which preservice teachers' develop an understanding of elementary mathematics. The results indicate that preservice teachers' conceptualization of elementary mathematics occurs in three phases. The first phase involves reverting to procedures to solve a given problem. Though these procedures are not always correct, many of the participants in this study initially used known procedures to solve the problem. The second phase involves learning to explain, justify, and conceptualize the mathematics within the problem. As illustrated within the class discussion of the problem 1 1/3 x 3/4, the class as a whole worked together to develop an understanding of taking 1 1/3 groups of 3/4. Finally, the last phase of conceptualization involves the development of new concepts. As seen with the discussion on the second day of multiplication, the idea of a new whole in multiplication was developed out of the idea of taking a group of something. These phases are summarized in Table 1.

Table 1
*Phases of Developing an Understanding of Mathematics*

| Phase | Indicator |
|---|---|
| 1 | Reverts to Procedures to Solve Problems |
| 2 | Developing Ways in Which to Explain, Justify, and Conceptualize Mathematics |
| 3 | Develop New Concepts |

These phases are not meant to be stringent in that if a problem is presented where a procedure is not readily known, students could start at the second phase instead of the first. However, being aware of the progression of preservice teachers' knowledge development could provide a better indicator of the types of instruction they need to receive in college teacher education programs.

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# HOW DO PRESERVICE TEACHERS IN SPAIN AND USA PERCEIVE EFFECTIVE TEACHING OF MATHEMATICS

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*This study investigates how different educational contexts influence student perspectives and how those perspectives are applied to effective teaching of mathematics. Data shows that four criteria—desirable personal characteristics, student-oriented teaching, the use of various instructional strategies, and professionalism—were used to evaluate teacher effectiveness by participants. Preservice teachers in Spain and U.S. tend to identify nonexemplary teaching practice with a teacher's personal characteristics rather than their teaching approach. Results also show differences between two groups. Unlike U.S. students, Spanish students did not distinguish teacher characteristics from their instructional approaches. In conclusion, teaching is complex and multidimensional, and it is important to discuss teaching from multiple perspectives.*

## Introduction

Many mathematics teachers and mathematicians are concerned about students' inability to demonstrate mathematical understanding and convey any interest in the subject. Mathematics educators also have great concern when mathematics teachers are confused or have an unclear understanding regarding effective teaching of mathematics or their own perception of an effective teacher.

## Theoretical Framework

A large body of research suggests that a preservice teacher's entering beliefs greatly influence their development as a teacher (Minor et al., 2002; Tabachnick & Zeichner, 1984; Weinstein, 1989). In addition, "the combinations of characteristics and methods that teachers use to achieve those results may seem endless" (Polk, 2006, p. 23). Other studies also support that the quality of instruction influences student learning and its outcomes (Brown et al., 2006; Darling-Hammond, 2000a & 2000b; Darling-Hammond & Sykes, 2003; Rice, 2003).

Polk (2006) identified ten basic characteristics of effective teachers: good prior academic performance, communication skills, creativity, professionalism, pedagogical knowledge, thorough and appropriate student evaluation and assessment, self-development or lifelong learning, personality, talent or content area knowledge, and the ability to model concepts in their content area. A similar list of effective instruction characteristics is given by Brown et al (2006): instructional techniques, personal qualities, use of the instructional technology, social skills, consistency, and classroom organization. Studies on student beliefs about poor teaching listed the most disliked teacher qualities: inability to communicate and deliver the subject; boring and monotonous; lack of knowledge, uninformed in subject; disorganized; insensitive to students and their needs; aloofness, arrogance; no sense of humor and unenthusiastic; and unprepared (Kardash & Wallace, 2001).

Walls and his colleagues (2002) investigated beliefs about the characteristics of effective and ineffective teachers and characterized teacher behaviors in five categories: emotional environment, teacher skill, teacher motivation, student participation, and rules and grades. Some studies compared perceptions of effective teachers and instruction between prospective teachers and experienced teachers. Prospective teachers are far more concerned with teacher enthusiasm, the student-teacher relationship, and teacher stimulation of interest in the subject or the course, while an experienced educator's emphasis is on teaching technique, the teacher's ability to provide intellectual challenge and to stimulate students' intellectual curiosity.

According to Barns (1992), student beliefs on education are already formed by the time students begin the university. Teacher's beliefs can be challenged or nurtured throughout the teacher preparation program. The need to challenge preservice teachers' existing knowledge, beliefs, and attitudes is strongly addressed by many teacher educators (Doyle, 1997; Driel et al, 2001; Sunal et al, 2001). Thus, "it seems imperative that teacher education programs assess their effectiveness, at least in part, on how well they nurture beliefs that are consistent with the program's philosophy of learning and teaching" (Hart, 2002, p. 4). In order to identify appropriate experiences, discussion on effective teachers and teaching is a common activity in a teacher preparation program. Minor et al. (2002) suggest to have students examine textbook definitions of effective teachers, exemplify characteristics of effective teachers in their experience, think about past teachers, or list characteristics they believe reflect effectiveness.

The purpose of this study is to provide preservice teachers with an opportunity to reflect on their knowledge and beliefs about teaching mathematics. We hope to better understand how preservice teachers apply their understanding of effective teaching in evaluating teachers in practice. Data from Spain and the US were collected in order to see how different educational contexts and programs influence student perspectives and how those perspectives are applied to effective teaching of mathematics.

## Data Collection and Analysis

In Spain, we conducted a survey study with 245 Spanish undergraduate students, enrolled in their first course of an education program. Fifty percent of the participants intended to become elementary school teacher; the other fifty percent were students of physical education. Participant age ranged from 18 to the 40s, but almost all were in their early 20s.

In the United States, thirty-five preservice teachers, enrolled in a 5th year initial teacher preparation program at a Midwest state university, participated in the study: twenty-three early childhood education (ECE) and twelve middle childhood education (MCE) graduate students. Students in the early childhood education program will be licensed to teach pre-K – 3, while the middle childhood preservice teachers desired to teach Grades 4–9. Participant age ranged from the early 20s to the mid 40s.

During the first week of class, participants were asked to describe the characteristics of the best, respectively, worst, mathematics teachers they ever had. Students were also asked to identify their perspectives on effective teaching of mathematics, and what teachers need to know in order to be an effective mathematics teacher.

A multistage qualitative-quantitative analysis was used to analyze the data. The data were analyzed sequentially. The first stage consisted of a phenomenological mode of inquiry (inductive, generative, and constructive) to examine student perceptions of their best, respectively, worst, teachers (as well as the characteristics of effective teaching and an effective teacher). These data were rephrased in simpler terms. We categorized units that appeared similar in content. Each one of the categories represented a distinct theme. Several themes were revealed during this process. Next, we used descriptive statistics to quantify the themes. The data were coded by giving a number to represent each theme, and frequencies of these codes were calculated. From these frequencies, we computed percentages to determine the rates of each theme.

## Results

*An Effective Teacher*

Figure 1 depicts participant perceptions about an effective teacher. Spanish students' views on effective teacher characteristics were more divergent than those of U.S. students. Preservice teachers in the U.S. valued teacher characteristics in the following order: communication skills, caring nature, higher order thinking, motivation skills, flexibility, and classroom management skills. On the other hands, Spanish students value knowledgeable teachers the most, secondly motivation skills and enjoying teaching, flexibility, and caring nature. Interestingly, no Spanish students considered communication skills as a necessary characteristic to be an effective teacher. Spanish students also believe that an effective teacher should enjoy teaching, believe in lifelong learning, interact with students, and see teaching as a mutual learning experience. No American students mentioned these characteristics to define an effective teacher.

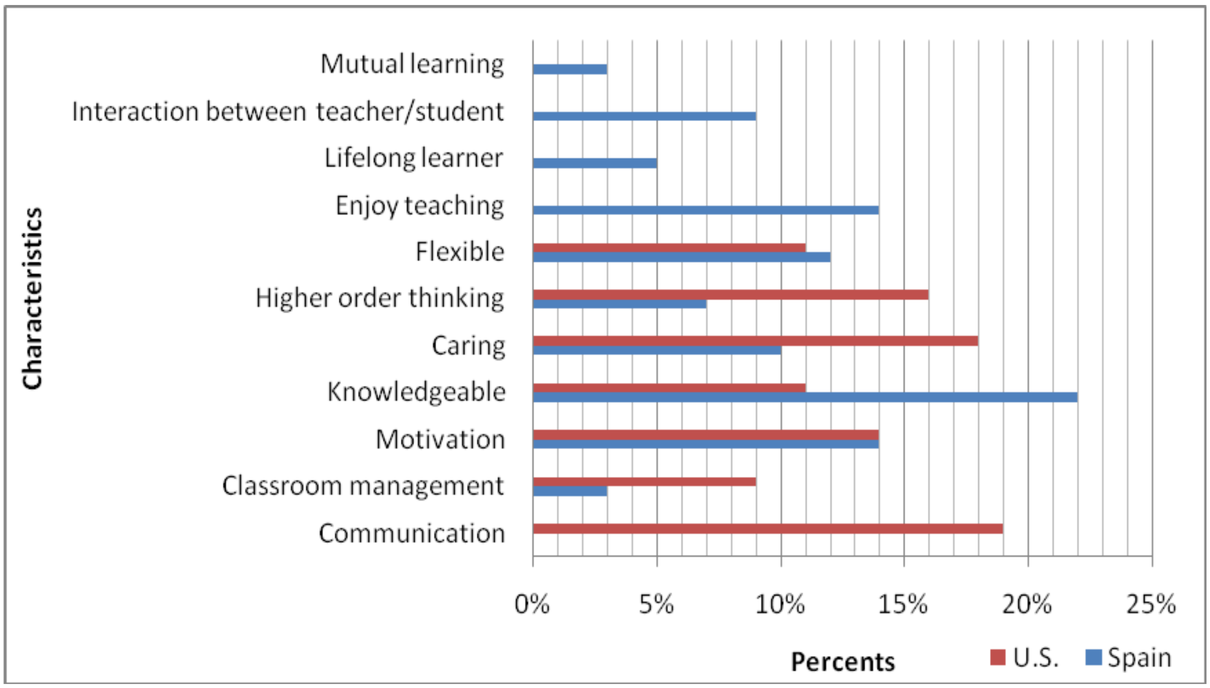


*Figure 1.* Characteristics of an effective teacher.

*The Best Teacher, the Worst Teacher, and Their Teaching Approaches – Spain*

Spanish students used an average of 5 items to describe the best/worst teachers and their teaching styles: a total of 1225 items for best and worst teachers, 1162 items for best teaching styles, and 1175 items for worst teaching styles (see Tables 1 & 2). As can be seen in Table 1, fifteen categories for best teachers and 16 for their teaching styles are identified. Twelve categories are found in both areas, and students emphasized the following areas: Innovative, creative, and uses various instructional strategies (17.6% for best teachers and 21.3% for their teaching styles); Close, pleasant, and amiable relationship with students (16.7 % for best teachers and 15.8% for their teaching styles); and Explains with clarity (9.8% for best teachers and 8.5% for their teaching styles).

Table 1
*Characteristics of the Best Teacher and Their Teaching Style – Spain*

| Characteristics of best teacher | % | Best teacher's teaching style | % |
|---|---|---|---|
| **Caring: understand and aid students** | **16.7** | **Close relationship with students, caring, pleasant, amusing** | **15.8** |
| Patient and tolerant | 7.0 | Patient and tolerant | 5.0 |
| Good communicator | 4.7 | Good communicator | 4.9 |
| Encourage students to be confident and motivate them | 7.9 | Motivate students, concern about student learning | 7.4 |
| **Innovator: creative, active, uses various teaching methods** | **17.6** | **Uses several instructional methods and activities. Active and innovative classes** | **21.3** |
| **Explains well** | **9.8** | **Explains with clarity** | **8.5** |
| Respects students | 7.0 | Respects students and has confidence in them | 7.7 |
| Desire to teach, enjoys teaching | 6.5 | Desire to teach, has a passion for their profession | 7.0 |
| Organized deliberation in teaching content | 3.6 | Orderly: well organized, and well prepared explanations | 6.2 |
| Impose authority, earn respect | 2.8 | Know to prevail when it is necessary, put order and respect | 3.4 |
| Demanding | 2.8 | Demanding: encourages students to challenge | 2.3 |
| Flexible: Allow various solutions | 1.4 | Corrects solutions | 2.5 |
| Uses games and activities in group work | 6.1 | Teaches in a practical way and use real life examples | 2.7 |
| Encourages students to reason and not to memorize | 1.4 | Creates pleasant class atmosphere | 1.2 |
| Holds strong professionalism | 4.7 | Competent in subject matter knowledge | 3.2 |
| | | Uses positive reinforcements, awards student work | 0.9 |

As shown in Table 2, 11 categories for worst teachers and 16 for their teaching styles are identified, which indicate a similarity up to the 11th category in both areas. In the area of worst teachers the order of the categories are: No desire to teach (21.4%), Authoritarian, unpleasant, strict, demanding, edgy (20.0%), and Boring (14.4%). In the order of teaching styles of worst teachers are Unpleasant, disagreeable, poorly educated, authoritarian, severe, strict, and demanding (20%), Not interested in teaching (18.7%), and Lack of respect for students, and favoritism (10%).

Table 2

*Characteristics of the Worst Teachers and Their Teaching Style – Spain*

| Characteristics of worst teacher | % | Worst teacher's teaching style | % |
|---|---|---|---|
| **Authoritarian, edgy, unpleasant, strict, demanding** | **20.0** | **Unpleasant, disagreeable, poorly educated, authoritarian, severe, strict, demanding** | **20.0** |
| **Boring** | **14.4** | Boring | 7.5 |
| Distant, lack of communication with students | 4.7 | Keeps a distance from pupils | 5.4 |
| **Favoritism, discrimination, lack of respect to students** | **12.1** | **Lack of respect to students, favoritism** | **10.0** |
| **No desire to teach** | **21.4** | **Not interested in teaching** | **18.7** |
| Imposes fear and tension in class (threatening, aggressive). | 5.1 | Abuses the norms (threatening, aggressive) | 3.7 |
| Explains poorly | 9.8 | Does not know how to explain/answer questions | 7.5 |
| Disorganized in explaining. | 2.8 | Disorganised in explaining, lack of preparation | 1.3 |
| Intolerant, not flexible | 6.5 | Not comprehensive, intolerant, or impatient | 6.0 |
| Poor communicator | 1.8 | Poor communicator | 2.5 |
| Does not encourage student participation | 1.4 | Inactive classes | 1.5 |
| | | Lack of class direction (do not impose respect or control of the class, disorder) | 1.7 |
| | | Does not correct student work or lacks criteria | 2.4 |
| | | **Poor teaching ability (traditional method: read, copy and follow book)** | **9.8** |
| | | Does not dominate the matter, incompetent | 1.3 |
| | | Does not have "education/teaching style" | 0.7 |

*The Best Teacher, the Worst Teacher, and Their Teaching Approaches – USA*

Tables 3 and 4 include characteristics of the best and worst teachers and their teaching approaches, as identified by U.S. students. A difference from the Spanish data occurs in the number of items used in student descriptions: an average of 4 items (total 128) to describe best teachers, 2 items (total 86) for the worst teachers, and 3 items for instructional strategies used by best and worst teachers (total 94 and 93).

As can be seen in Table 3, ten categories for best teachers and 14 categories for their instructional styles were identified. Only three categories are found in both areas. Students listed teachers who are: Caring (28.6%), Helpful/motivating/encouraging (18.4%), and Make learning fun/interesting (15.1%) as best teachers. With regard to best teachers' instructional styles, students described their best teachers as using: Various teaching methods (14.3%), Scaffolding/guided instruction (13.5%), and Higher order thinking (11.9%).

Table 3
*Characteristics of Best Teacher and Their Teaching Style – USA*

| Characteristics of best teacher | % | Best teacher's teaching style | % |
|---|---|---|---|
| Classroom management | 7.3 | Classroom management | 11.2 |
| Knowledgeable | 6.1 | Knowledgeable | 3.7 |
| Higher order thinking | 5.0 | **Higher order thinking** | **11.9** |
| **Caring** | **28.6** | Review and support | 6.0 |
| **Made learning fun/interesting** | **15.1** | **Variable teaching methods** | **14.3** |
| Good listener | 2.2 | Concise/good communicator | 11.2 |
| Sense of self | 6.7 | Individual projects | 2.2 |
| Likes teaching | 5.6 | Integrates subject areas | 6.7 |
| **Helpful/motivating/encouraging** | **18.4** | **Scaffolding/guided instruction** | **13.5** |
| Has high expectations | 5.0 | Direct instruction | 3.7 |
| | | Group work/discussion | 5.2 |
| | | Hands-on | 2.2 |
| | | Student centered | 6.0 |
| | | Explores real world | 2.2 |

There are 9 categories in the description of worst teachers and 11 categories in their instructional strategies; three categories are common to both. The most frequently listed worst teacher characteristic is Uncaring (51.0%) followed by Closed-minded (11.3%) and Disliked teaching (10.5%). Students said that instructional strategies used by worst teachers are: Poor teaching skills (29.7%), Only reading (16.4%), and Not helpful (14.8%).

Table 4
*Characteristics of Worst Teacher and Their Teaching Style – USA*

| Characteristics of worst teacher | % | Worst teacher's teaching style | % |
|---|---|---|---|
| **Uncaring/mean** | **51.0** | Uncaring/mean | 5.5 |
| Poor communication | 4.5 | Poor communication | 7.0 |
| Poor classroom management | 5.3 | Poor classroom management | 12.5 |
| **Dislikes teaching** | **10.5** | **Poor teaching skills** | **29.7** |
| Student fear or didn't want to be there | 3.8 | **Not helpful** | **14.8** |
| Boring | 3.8 | **Only lecture** | **16.4** |
| Unknowledgeable of subject | 6.0 | Teacher centered | 3.1 |
| Favoritism | 3.8 | Rote memorization | 1.6 |
| **Closed-minded** | **11.3** | Worksheets | 1.6 |
| | | Too much homework | 5.5 |
| | | No hands-on/exploration | 2.3 |

*Cross Comparisons*

Student responses are grouped in broader categories in Table 5, which compares the characteristics and instructional strategies of the best and worst teachers. Student responses may be grouped among four themes: Desirable/Undesirable personal characteristics; Teaching with/without consideration for students; Uses/Does not use various instructional strategies; and Professional or not. Table 5 also compares Spanish and U.S. student perspectives in each category.

*Table 5*
Comparative Analysis

| **Categories** | **Best teacher** | **Instructional strategies of best teacher** | **Country** | **Worst teacher** | **Instruction-al strategies of worst teacher** | **Categories** |
|---|---|---|---|---|---|---|
| **Desirable personal characteristics** | 28.4% | 25.7% | Spain | 33% | 33.9% | **Undesirable personal characteristics** |
| | 37.5% | 11.2% | USA | 62.3% | 20.3% | |
| **Teaching with consideration for students** | 17.7% | 18.3% | Spain | 17.2% | 13.7% | **Teaching without consideration for students** |
| | 33.5% | 6.0% | USA | 12.1% | 10.1% | |
| **Uses various teaching strategies** | 39.9% | 45.6% | Spain | 28.4% | 33.7% | **Does not use various teaching strategies** |
| | 11.1% | 71.6% | USA | 15.1% | 69.6% | |
| **Professional** | 14% | 10.4% | Spain | 21.4% | 18.7% | **Not professional** |
| | 17.9% | 11.2% | USA | 10.5% | 0% | |

Table 5 indicates that Spanish students did not distinguish characteristics of teachers from instructional approaches. On the other hand, students in the U.S. described their best and worst teachers in terns of their personal characteristics, while their teaching styles were discussed in terms of the instructional strategies implemented in class. Students in both countries described their worst teachers in terms of their personal characteristics rather than evaluating their teaching strategies.

## Implication

The results support other studies that effective teaching requires more than technique (Schaeffer et al., 2003; Shannon, 1998; Walls et al., 2002). As can be seen in Figure 1 and Tables 1–5, the characteristics of effective teaching and an effective teacher are complex and multidimensional constructs that are unpredictable, ambiguous, and contextual. Therefore, it is important to view the teaching enterprise from multiple perspectives. Research has explored variables that impact teaching, characteristics of teachers, students, and contexts in which teaching and learning occur (Shannon, 1998). For that reason, it becomes necessary to extend the different studies of educational contexts and countries so that new data may be gathered.

Recognizing and discussing specific effective teaching behaviors is one piece of the puzzle. "The commitment to use what has been found to be effective is one of the avenues that could help raise the teaching profession above the level it currently occupies…. If such qualities are not modeled we are far less likely to adopt the presented idea in classroom practices" (Sheeran & Vermette, 1995, p. 26).

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# UNDERSTANDING BASE-TEN NUMBERS: WHY DON'T YOU JUST TELL THEM IT IS TEN?

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*Six preservice teachers (PSTs) participated in a 1-week teaching experiment designed to assist them in developing more sophisticated conceptions of numbers using tasks based on a framework for PSTs' conception of multidigit numbers developed in previous work. All 6 PSTs developed more sophisticated conceptions; however 3 PSTs did not develop a conception that enabled them to relate adjacent digits multiplicatively. In this study I describe and analyze the first of 5 teaching sessions. I trace the development of the PSTs' conceptions and show that (a) exposure to correct conceptions is insufficient to help PSTs develop more sophisticated conceptions and (b) conceptual change is slow. These facts counter the commonly held notion that if one simply tells PSTs the value of the digit, they will understand base-10.*

Number is the central theme in the elementary mathematics curriculum. "Proficiency with numbers and numerical operations is an important foundation for further education in mathematics and in fields that use mathematics"(Kilpatrick, Swafford, & Findell, 2001, p. 1). However, although most PSTs and teachers can execute algorithms, many struggle when asked to explain them conceptually (Ball, 1988/1989; Ma, 1999; Thanheiser, 2005). Consider, for example, one PST's explanation for the regrouping in 527 – 135 (see Figure 1): "You put a 1 over next to the number and that gives you 10. … I don't get how the 1 can become a 10. One and 10 are two different numbers. How can you subtract 1 from here and then add 10 over here? Where did the other 9 come from?"

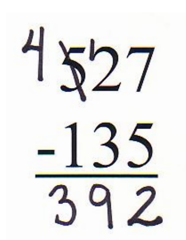

*Figure 1.* Standard subtraction algorithm for 527 – 135.

**For students to come to understand the underlying concepts, teachers (and thus PSTs) must have "solid understanding of mathematics so that they can teach it as a coherent, reasoned activity and communicate its elegance and power" (Conference Board of Mathematical Sciences, 2001, p. xi). The PST described above would be unable to help a child make sense of the regrouping.**

## Theoretical Framework

My research grows out of a rich cognitive-science paradigm focused upon children's prior knowledge in learning situations, a consideration that is equally important in work with adults (Bransford, Brown, & Cocking, 1999). To help PSTs develop a solid understanding of mathematics, we mathematics educators need to build upon their initial conceptions, which both determine what they understand when looking at a number and serve as a basis for building more sophisticated conceptual structures.

In previous work (Thanheiser, 2005), I categorized PSTs' conceptions of multidigit whole numbers into four major groups: thinking in terms of (1) *reference units*, (2) *groups of ones*, (3) *concatenated-digits plus*, and (4) *concatenated-digits only*. The distribution of the conceptions of the 15 participants in that previous study can be seen in Table 1. Because 10 of the 15 PSTs held an incorrect conception of number and only 3 explained all aspects of regrouping in detail, I considered how to assist PSTs in developing more sophisticated conceptions, keeping in mind that adults cannot simply forget the algorithms they know and value to invent new ones.[iv]

Table 1
*Distribution of Conceptions in the Context of the Standard Algorithm for the 15 PSTs in the Study*

| Conception held | # (%) of PSTs |
|---|---|
| *Reference units.* PSTs with this conception reliably conceive of the reference units for each digit and relate reference units to one another, seeing the 3 in 389 as 3 *hundreds* or 30 *tens* or 300 ones, the 8 as 8 *tens* or 80 *ones*, and the 9 as 9 *ones*. They can reconceive of 1 *hundred* as 10 *tens*, and so on. | 3 (20%) |
| *Groups of ones*. PSTs with this conception reliably[a] conceive of all digits in terms of groups of ones (389 as 300 *ones*, 80 *ones*, and 9 *ones*) but not in terms of reference units; they do not relate reference units (e.g., 10 *tens* to 1 *hundred*). | 2 (13.3%) |
| *Concatenated-digits plus.* PSTs with this conception conceive of *at least one* digit as an incorrect unit type at least sometimes. They struggle when relating values of the digits to one another (e.g., in 389, 3 is 300 ones but the 8 is only 8 ones). | 7 (46.6%) |
| *Concatenated-digits only*. PSTs holding this conception conceive of *all* digits in terms of *ones* (e.g., 548 as 5 *ones*, 4 *ones*, and 8 *ones*). | 3 (20%) |

[a]*Reliably* in these definitions means that after the PSTs were first able to draw on a conception in their explanations in a context, they continued to do so in that context.

## Method

The data analyzed are drawn from the first of five 150-minute teaching sessions and pre and post interviews with each of 6 PSTs at a large, urban, comprehensive state university. All participants were volunteer preservice teachers who participated in a 1-week summer teaching session. Two PSTs had completed their methods course and were about to begin student teaching. The other four PSTs had not yet taken their mathematics methods course.

The pre and post interviews, each 60–90 minutes in duration, were conducted immediately before and after the sequence of teaching sessions. Questions for the interviews were designed to elicit the PSTs' conceptions. The tasks for the teaching sessions were designed to build on the

PSTs' initial conceptions and to enable them to develop more sophisticated conceptions. Tasks for Day 1 were designed on the basis of the preinterviews and those for the succeeding days were based on PSTs' changing conceptions. Tasks for the teaching sessions included addition and subtraction of 3-digit numbers using various manipulatives; discussion of artifacts of children's mathematical thinking, such as video clips of children solving problems; discussion of various manipulatives; and exploration of Mayan numbers (a base-twenty system). All teaching sessions and interviews were video taped. The interviews and critical moments of the teaching episodes were transcribed. The analysis focused on the PSTs' developing content knowledge throughout the teaching sessions using the framework introduced above. In this paper, I analyze the first day of the teaching experiment.

## Results and Discussion

Results indicate that carefully chosen mathematics tasks in conjunction with artifacts of children's mathematical thinking addressing the PSTs' preexisting conceptions enable PSTs to re-examine their conceptions and develop more sophisticated conceptions. Five of the 6 participants developed more sophisticated conceptions (the sixth already held the most sophisticated conception; see Table 2), however, only 3 of the 6 PSTs held the most sophisticated conception in the post interview. In this paper I trace 2 PSTs' conceptions to illustrate the difficulties PSTs encounter when asked to make sense of multidigit whole numbers and operations on such numbers.

Table 2
*Conceptions in the Context of the Standard Algorithm for the 6 PSTs in a Teaching Experiment Before and After the Study*

| Pseudonym | Preinterview | Post interview |
|---|---|---|
| Silvia | Reference Units | Reference Units |
| Holly | Groups of Ones | Reference Units |
| Saskia | Concatenated-digits plus | Groups of ones |
| Jason | Concatenated-digits plus | Groups of ones |
| Beatrice | Concatenated-digits only | Groups of ones |
| Isabelle | Concatenated-digits only | Reference Units |

Because most PSTs enter our classrooms (and the teaching experiment) with one of the concatenated-digits conceptions, the tasks for this teaching session were developed to address those conceptions. For example, students were presented with digit cards (see Figure 1a) and asked to build numbers (see Figure 1b).

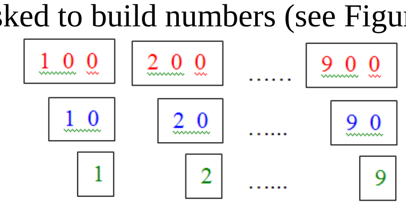


*Figure 1a*. Digit cards representing the place value of each digit 0–9 (increments of 1, in green), 10–90 (increments of 10, in blue), and 100–900(increments of 100, in red).

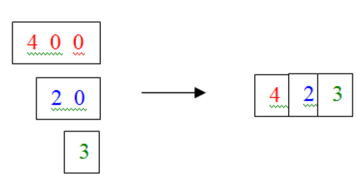


*Figure 1b*. Representing 423 with digit cards.

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

This task was designed to help PSTs connect the symbol to its value. Beatrice and Isabelle both held a concatenated-digits conception at the beginning of the teaching sessions; however their approaches to this task differed. Isabel, on the one hand, used the cards in a concatenated way. She stated, “You can really make so many numbers by putting them together to make them bigger. Putting tens together can give you thousands, and so on” (see Figure 2a). Beatrice, on the other hand, overlaid the digit cards to build a three-digit number: “For me, I actually just stayed with like 3-digit numbers. … I didn’t do the thousands, and, you know, like this was 400 and then I put the 80 and then the 6 to make 486” (see Figure 2b).

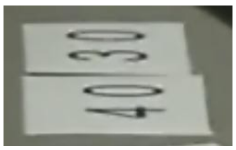

*Figure 2a.* Isabel put 40 next to 30 to make the number 4,030.

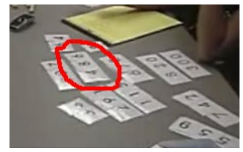

*Figure 2b.* Beatrice overlaid 400, 80, and 6 to build 486.

Thus, whereas the cards enabled Beatrice to connect symbols and values, they did not help Isabel do so. During the class discussion of the usefulness of the cards, Silvia (who held a reference-units conception at the outset) explained, “And the first thing I realized was that this would be a very good way to get students to realize that each number … represents more—like if it … it was 211, that the 2 is not just 2, but 200.” Even though Silvia explicitly stated the relationship between the digit and its value, Isabel did not make that connection for herself. She realized that her way differed from the others’ way but did not evaluate one way as better than the other. When asked how she was thinking about the 40 in her 4,030, she said, “As just a 40.” After they had built numbers with the digit cards, students were asked to solve 389 + 475 in as many ways as they could, using digit cards, base-ten blocks, paper and pencil, and so on. The students were also asked to make notes on their methods in a way that would be understandable to a third grader. Saskia, who held a concatenated-digits-plus conception in the preinterview, developed a horizontal addition strategy using the digit cards (see Figures 3, 3a, and 3b). She explained:

> I split them all up, like into parts, like 300, 80, and 9; 400, and 70, and 5. So I added across: 9 + 5 = 14, so that is a 10 and a 4, so the 10 moves up in the ten’s column and the 4 would stay in the one’s column. Then 80 and 70 equals 150, so that’s 100 and 50 and the 10 from last time. So then the 100 moves up to the next column, and then 50 + 10 would just be 60, and then 700 + 100 is just 800. And then you just combine it all, so it is 864.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Hundreds: | 300 | 400 | ⟶ | 700 | | ⟶ | 700 + 100 = 800 |
| Tens: | 80 | 70 | ⟶ | 100 | 50 | ⟶ | 50 + 10 = 60 |
| Ones: | 9 | 5 | ⟶ | 10 | 4 | ⟶ | 4 = 4 |

*Figure 3*. One PST’s invented addition strategy to add 389 + 475.

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

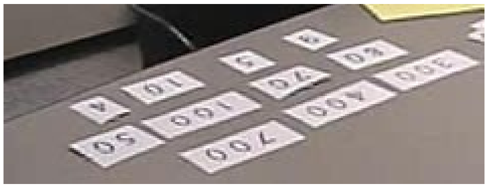

*Figure 3a*. Saskia's strategy to add 389 + 475 before regrouping.

*Figure 3b*. Saskia's strategy to add 389 + 475 after regrouping.

Saskia shared a second strategy for adding the numbers (see Figure 4a) by decomposing the number into their reference units and relating those to one another. She explained, "Fourteen ones is just 1 ten and 4 ones, and16 tens is actually 1 hundred and 6 tens." Thus, through working with cards and blocks to invent strategies to add two 3-digit numbers, Saskia developed more sophisticated ways of conceptualizing numbers. Through inventing these strategies, she was able to draw on more sophisticated conceptions than she had before.

Beatrice, who held a concatenated-digits-only conception in the preinterview, also developed a horizontal addition strategy using the digit cards (see Figure 4b) and translated that into a written strategy (see Figure 4c), seeming to draw on a groups-of-ones conception. However, when working with base-ten blocks, she represented each digit in terms of ones (see Figure 4d) and added those ones, reverting to a concatenated-digits conception. Beatrice showed that providing evidence of holding a correct conception in one instance does not establish that a student holds that conception across multiple contexts.

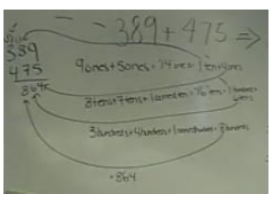



*Figure 4a*. Saskia's second strategy to add 389 + 475 before regrouping.



*Figure 4b*. Beatrice's strategy to add 389 + 475 with cards.

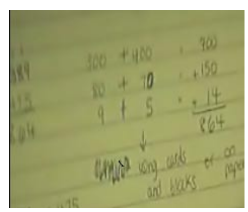

*Figure 4c*. Beatrice's written strategy to add 389 + 475.

*Figure 4d*. Beatrice's strategy to add 389 + 475 with blocks.

Isabel, even after being confronted with various students' conceptions and having access to the digit cards as well as base-ten blocks, still drew on a concatenated-digits conception. She used the cards to explain:

> I did the 389 and then 475, and I was just going by colors. … The pencil and paper way, like that [one's column] would be 14; I took away my 1 [not 10]—and then carry over the 1. ... Like it [using cards] will just help them [children] to see like the 4 goes here; then you add over the 1, and then this lines up.

In her use of the cards (see Figure 5), Isabel clearly showed that she saw the regrouped digit in terms of ones (1 one) rather than in terms of a ten or a group of 10 ones. Thus exposure to more sophisticated, correct conceptions may be insufficient to assist a PST in developing a correct conception herself.

*Figure 5*. Isabel's strategy to add 389 + 475 with the digit cards.

After using various manipulatives to add numbers, the group watched a video clip of a child attempting to add 638 + 476 using the partial-sums algorithm, which is similar to the horizontal addition strategy invented by the participants of this study. In this algorithm, one adds the hundreds first, then the tens, then the ones, and then adds these partial sums (see Figure 6a). The child added the single digits rather than the partial sums, providing evidence of viewing the numbers in a concatenated way.

| | |
|---|---|
| 638<br>476<br>1000<br>100<br>14<br>1114 | 638<br>476<br>10<br>10<br>14<br>34 |
| *Figure 6a*. Partial-sums algorithm for adding 638 + 476. | *Figure 6b.* Child's attempt to use partial-sums algorithm. |

When commenting on the video clip, Beatrice stated, "I think she doesn't see the 600; she is just thinking about single digits. … Instead of doing 600 and 400, she does 6 + 4, and that is why she is getting the wrong answer." Thus, Beatrice again drew on a groups-of-ones conception. In commenting on this video clip, Isabel, for the first time, acknowledged that her conception of numbers (concatenated digits) is insufficient to explain various situations:

> She [the child] is just adding them as single digits, like 6 + 4, and then that's one problem, and then 3 + 7 is another problem, and then taking those numbers and making it a third problem or a fourth problem … instead of looking at it [the numbers] as one whole.

Although this statement does not suffice to establish that Isabel drew on a more sophisticated conception, it does demonstrate that she recognizes the deficiency of her conception.

## Conclusions

The tasks discussed in this paper were developed to address the concatenated-digits conceptions of multidigit numbers and to help students connect the symbols to their values. Although the tasks served to help some (like Beatrice) connect symbols and values fairly quickly, others (like Isabel) maintained their concatenated-digits conceptions throughout most of the activities. Thus, simply exposing students to correct conceptions (telling them the meanings of the digits) does not help them develop such conceptions. Even for students who seem to make those connections, their newly established conceptions may be unstable. Whereas most students were able to use a groups-of-ones conception to explain addition and talk about the video clip by the end of the first teaching session, multiple sessions with additional experiences were needed to solidify the PSTs' understanding. I am often asked whether telling students the meanings of the digits would suffice: "Can't you just tell them it is 10?" This study clearly shows the inadequacy of such an approach. Although all 6 PSTs held a correct conception at the end of the study, developing these conceptions is a complex task, requiring multiple sessions. Mere exposure to correct conceptions is neither enough to enable all students to recognize that their conceptions are incorrect nor to allow them to develop more sophisticated ones. Even after engaging in carefully designed tasks and in-depth discussions, only half the PSTs developed the most sophisticated, reference-units, conception. Thus, the fact that many teachers leave their teacher education programs without a deep understanding of number is unsurprising (Ma, 1999).

## Endnotes

[1]This has, however, been successfully done with children (Ambrose, 1998; Hiebert & Wearne, 1996; Kamii, 1994; Kamii, Lewis, & Livingston, 1993; Sowder & Schappelle, 1994).

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# LEARNING TO TEACH VIA PROBLEM SOLVING AND SUPPORTING PRE-SERVICE TEACHERS IN LEARNING VIA PROBLEM SOLVING

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*This study reports on teacher educators' learning on-the-job. In this study, two novice teacher educators worked with an experienced teacher educator reflecting on the process of supporting elementary pre-service teachers in developing adult-level understandings of foundational mathematical ideas. Using reflective teaching-learning cycles and incorporating ideas from lesson study we reflected on our experiences teaching a mathematics content course for pre-service elementary teachers via problem solving. A dominant theme from our preliminary data analysis indicates that the mathematics knowledge for teaching needed by teacher educators is even more complex when teaching via problem solving than in more traditional instructional approaches.*

Researchers have argued that it is vital that pre-service elementary teachers develop deep and connected understandings of important mathematical ideas (Ball & Bass, 2000, Ma, 1999). The Conference Board of the Mathematics Sciences (CBMS) (2001) has called for engaging pre-service elementary teachers in doing mathematics and supporting them in developing adult-level understandings of foundational mathematical ideas that are taught in grades K-8. It is argued that teachers' mathematical knowledge significantly influences how and what teachers teach and how and what their students learn (Ball & Bass 2000, Borko et al., 1992; Hill & Ball, 2004; Hill, Rowan, & Ball, 2005). Ball and Bass (2000) argue that teachers' mathematical knowledge needs to be strong in order to allow them to deal flexibly with the complexity of teaching mathematics to diverse student populations. They further claim that "not providing teachers with this [mathematical knowledge] undermines and makes hollow efforts to prepare high-quality teachers who can teach all students, teach in multi-cultural settings, and work in environments that make teaching and learning difficult" (Ball & Bass, 2000, p. 94). This need to adequately prepare new teachers has been highlighted in the reform movement in mathematics education and has necessitated not only calls for different approaches to teaching mathematics, but also the reinvention of teacher education (Simon, 2000). An immediate consequence of this is the need to adequately prepare educators of prospective teachers for the job of preparing and supporting the development of teachers.

We propose that one way to foster deep mathematical knowledge development in pre-service elementary school teachers (PSTs) is to engage them in learning mathematics via problem solving (Masingila, Lester & Raymond, 2006; Schroeder & Lester, 1989). In an analogous manner, it is important that teacher educators learn to support PSTs' development through learning via problem solving. We claim that learning to support PSTs' mathematical development via problem solving is also a problem-solving activity for teacher educators. Thus, they are also learning via problem solving, and their learning is about supporting PSTs' mathematical development. Further, in ways that are similar to how in-service teachers grow through interactions with teacher educators and peers in professional development work, we propose that teacher educators learn through working with mentors and peers in reflective ways.

In this study, two novice teacher educators (the first two authors, PK and DO) worked with an experienced teacher educator (the third author, JM) in a supportive professional community, and reflected together on the process of supporting pre-service teachers in developing adult-level understandings of foundational mathematical ideas that are taught in grades K-8. This was achieved through interconnecting theory, participation in a community of practice, and being reflective about our experiences in teaching a mathematics course for pre-service elementary teachers via problem solving. We took a position of *inquiry as stance*—within an inquiry community "to generate local knowledge, envision and theorize" our "practice, and interpret and interrogate the theory and research of others" (Cochran-Smith & Lytle, 1999, p. 289)—throughout this research study.

## Perspectives and Guiding Frameworks

Our theoretical framework is grounded in two bodies of research. The first involves research regarding mathematics teacher educator learning. In recent years, there has been an emergence of literature about how mathematics teacher educators learn (García, Sánchez & Escudero, 2006; Zaslavsky & Leikin, 2004). These researchers draw some parallels between ways in which mathematics teacher educators learn on the job and the ways mathematics teachers learn on the job (Jaworski, 1994). Reflection-on-action and reflection-in-action (Schön, 1983) are the notions underlying the interpretive processes through which these two communities of practice learn. The former refers to ways in which members of a community of practice (in our case, mathematics teacher educators) reflect on past experiences with the intention of refining their work to achieve their instructional goals, while the latter refers to "thinking on your feet" (Schön, 1983, p. 54).

One way in which teachers have operationalized Schön's reflective practice is through lesson study (Lewis & Tsuchida, 1998; Lewis, Perry & Murata, 2006; Stigler & Hiebert, 1999). When lesson study was introduced in the United States, it was initially looked at as a way for American teachers to learn from each other by forming communities of practice. Some researchers have tried to extend this idea to university professors with the view of providing them with professional development opportunities (Cerbin & Kopp, 2006).

Another approach that mathematics teacher educators have used to generate a framework or scheme for thinking about their evolving knowledge about preparing pre-service teachers is the *Teacher Development Experiment* (TDE) (Simon et al., 2000). The TDE provides researchers/teacher-educators opportunities to study the complex interactions that occur in a teaching-learning cycle. Knowledge is developed through "multiple iterations of a reflection-interaction cycle" (Simon, 2000, p. 339) (see Figure 1). A TDE methodology uses both retroactive and ongoing analysis of data to track the development of the students and the researcher/teacher-educator.

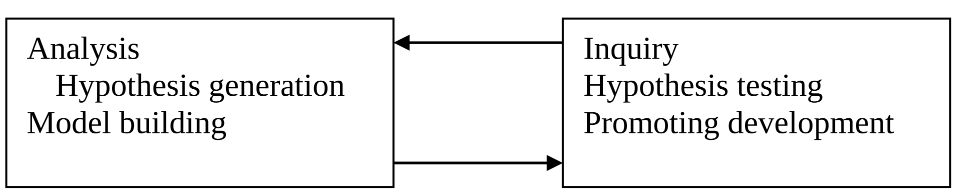


*Figure 1.* Knowledge development (Simon, 2000, p. 339).

The second body of literature involves research regarding the use of problem solving to teach mathematics. This approach to teaching mathematics hinges on the use of a constructivist theory of learning. Researchers argue that students acquire new knowledge by actively participating in the learning process (Grouws, 1992; *Journal for Research in Mathematics Education*, 1994) and not through passive absorption of what their teacher models (Masingila, Lester & Raymond, 2006). In a problem-solving approach, the role of the teacher changes from that of a disseminator of knowledge to that of a facilitator. Masingila and her colleagues (2006) argue that the teacher's responsibility is to "establish a mathematical community in the classroom where everyone's thinking is respected and in which reasoning and discussing mathematical ideas and meanings is the norm" (p. xxi). This means that in the case of learning via problem solving, both the students and the teacher take on roles that are potentially different from what they are used to. In their review of research on problem solving, Stein, Boaler, and Silver (2003) document many advantages of using this approach to teach mathematics — equal or better student performance on standardized tests, more positive and broader student attitudes about mathematics, and more equitable student performance with no achievement differences along social class or gender lines. However, the authors also point out the difficulty of teaching mathematics via problem solving, and state that "much remains to be learned about how to teach mathematics through problem solving in ways that enhance the learning of all students" (p. 254). We hope through this research project to contribute to this knowledge base.

## Research Goal and Methods

Situated in research on how mathematics teacher educators learn, and research on teaching via problem solving, our goal in this study was to reflect critically on the process of learning to teach via problem solving and to support PSTs in learning mathematics via problem solving. In Fall 2007, each of the authors was the instructor for a section of a mathematics content course for pre-service elementary teachers that was taught via problem solving. The content of the course included the concepts of numeration, operations, number theory, probability and statistics, and functions. The course met two days a week for 80 minutes each day; the third author taught the first section of the course, and the first and second authors taught their sections on the same day after they had observed the third author teach the first section. In Spring 2008, each of the authors was the instructor for a section of the other course in a two-course sequence. The content of the second course included the concepts of rational numbers, geometry, and measurement. The emphasis in both of these courses is on learning mathematical concepts through solving problems in a cooperative learning situation. The courses were developed using five principles as a guiding framework: (a) solving many problems is an essential ingredient in becoming a good problem solver, (b) problem solving involves a very complex set of processes, (c) the instructor's role in fostering productive problem-solving performance is vitally important, (d) cooperative, small-group work is encouraged, and (e) assessment practices should be closely connected to instructional emphases.

It is within the context of teaching these mathematics content courses via problem solving that we carried out this research study. We modeled our critical reflection process on the TDE's (Simon, 2000), "reflection-interaction cycles," and also on Schön's (1983) ideas of reflection-in-action and reflection-on-action. While we incorporated ideas from lesson study, since we taught each lesson only once during the semester, it was not practical to model our research directly on lesson study itself; however, the reflection-interaction cycles constituted a type of modified lesson study.

In our version of the reflection-interaction cycle, all three of the authors filled the role of teacher-researcher, and we reflected upon our own teaching while interacting with each other, as well as with our students. We had a number of data sources. The first two authors both observed the third author's lesson each class period. The rationale for this was that she has the most experience teaching in general, and has experience in supporting pre-service teachers in learning mathematics via problem solving. As mentioned previously, following the observations, the first two authors each taught the same lesson in their own sections of the course. As teachers and professionals, we incorporated reflection-in-action (Schön, 1983) while we were teaching. After each class/observation period, each of the authors wrote a memo of the day's lessons. These memos were part of our reflection-on-action. We reflected on the day's teaching and observing, noting things that we wanted to discuss with the group of teacher-researchers. On the day following the second classes of the week, we met for approximately one hour to discuss our observations, how the class was going, particular struggles that we noticed either students having or that we were having. We also used these meetings to plan our future lessons based on our reflections.

We audiotaped these meetings and transcribed portions that we found significant for our results. Thus, our meetings and memos cover the first part of Simon's (2000) model—hypothesis generation, model-building, and analysis—and our teaching and class observations correspond to the second part of the model—inquiry, hypothesis testing, and promoting development. We continued this reflection-interaction cycle throughout the semester. The data sources for this project are our collective memos, the transcribed notes from our meetings, and our experiences teaching in the classroom itself.

Following the methodology of the Teacher Development Experiment, we used both ongoing and retrospective analyses of the data. The ongoing analysis, which occurred during the teacher development experiment, was the basis for continued reflection on our teaching and learning about our teaching, the testing of emerging hypotheses, and the strategies for promoting further development of the pre-service teachers' mathematical understandings. During the retrospective analysis of the data, which is continuing, we are examining the larger corpus of data through a carefully structured review of all relevant data of the teacher development experiment.

We began coding our data at the end of the first course. We started by each looking at the data (memos and meeting transcripts) from the first two weeks of the semester and identifying themes that emerged. After looking at the data individually, we met and compiled a list of the themes that we had identified. Using this list, we then individually looked at five weeks' worth of data to see if these codes matched our data. We met together again, and as a group refined our list. During this meeting, we compiled a list of code definitions and examples, so that we were using the codes consistently.

We continued to code the data individually, focusing on one month's worth of data at a time. Each of the three researchers coded all of the memo data individually. At our weekly meetings we spent time discussing our coding in an effort to triangulate the coding. We discussed any areas where we were uncertain of the coding, and came to agreement. Following our group discussions, we coded our data electronically using a coding software package and ran reports to analyze the data within and across codes.

## Results

While the retrospective analysis continues, a dominant theme that emerged from the data is that the mathematics knowledge for teaching needed by mathematics teacher educators is even more complex when attempting to teach via problem solving than in more traditional instructional approaches. Our goal as instructors was to support these PSTs' growth both as learners and doers of mathematics, and also as problem solvers. Parallel to this, our main goals as researchers were to understand the processes of learning to teach mathematics via problem solving and learning to support PSTs in learning mathematics via problem solving.

In analyzing our data, we found that we supported PSTs' growth through (a) understanding and deciding on the mathematical goals of individual lessons and the two-course sequence as a whole, (b) choosing and facilitating tasks, (c) using questions and modeling to scaffold PSTs' learning, (d) assessing PSTs' understandings, and (e) monitoring PSTs' dispositions. We discuss the first two of these results in this paper. Since we taught these mathematics content courses via problem solving, our efforts to support PSTs' growth in mathematics and in problem solving were intertwined, as we found ourselves consciously and continually working to support PSTs in deepening their mathematical understandings and become more proficient problem solvers.

*Deciding on Mathematical Goals*

We see mathematics as an active venture in which students are encouraged to explore, make and debate conjectures, build connections among concepts, solve problems growing out of their explorations, and construct meaning from all of these experiences. Perhaps the most important goal of these mathematics content courses for PSTs is to support them in developing adult-level perspectives and insight into the nature of foundational mathematics. Another goal is to expose PSTs to key, recurring themes, processes, and tactics in mathematics and support them in making connections among mathematical ideas through these themes, processes, and tactics. We want PSTs to realize that elementary mathematics is foundational mathematics, and is not "elementary" (Ma, 1999).

Teaching mathematics via problem solving requires the instructor to understand the mathematical ideas involved in a problem in a deep and connected way, and further requires the instructor to anticipate the different approaches that students may use in solving the problem. Our data revealed that we first made sense of the mathematical goals of activities, of units, and of the course, then facilitated student engagement of activities so that these mathematical goals were accomplished, and then reflected upon this process.

Making sense of the mathematical goals sometimes involved thinking about how manipulative materials support or fail to support student understanding. For example, when using base ten blocks to model operations with whole numbers, there are limits to using this representation: "You can think about having three flats, four times, and it makes sense [3 + 3 + 3 + 3]. However, as we found, it doesn't make sense to think about three flats times four flats, at least pictorially" (DO, Memo 9/12/07). We found that as instructors it was important for us to think through carefully the relationship between the materials and the mathematical activities in which we want to engage the PSTs.

Making sense of the mathematical goals engaged us as instructors in thinking critically about what concepts and processes we wanted our students to engage with, and how to facilitate this happening. For example, each of us found the chapter on probability and statistics to be problematic in the sequencing of activities and the inclusion or exclusion of particular ideas. We decided that we did not want the probability activities to become an exercise in which the PSTs tried to determine whether to use the command on their calculator for permutations or combinations, but rather we wanted to engage them with methods of counting and have them reason through the solutions.

> I decided to talk about combinations without really calling them combinations. This worked out all right in class when I was thinking about making the slots and then dividing by the possible number of orders because order doesn't matter. However, when trying to answer the homework questions, this proved to be much more challenging. The question that [student] asked was, if you flip eight coins, how many outcomes have exactly three heads. In my head, I knew the answer was just eight choose three. You have eight coins and you want to choose three of them to be heads. This makes total sense to me and I usually teach it that way. However, since I am not doing "choose," I had a lot of trouble explaining it. In fact, I had so much trouble, that I told her I would figure it out and tell them on Wednesday. Grrr. I really wanted this to be a good way. I finally worked it out that we would need three slots for the three heads. We have eight choices for the first head (coins 1-8), seven choices for the second head, and six choices for the third head. Then, since order doesn't matter (coins 1, 3 and 4 being heads is the same as coins 4, 1 and 3), we have to divide by the number of ways of arranging three things. (DO, Memo 11/5/07)

A challenging aspect of teaching mathematics via problem solving is that in the same way that we want to support PSTs in developing deep and connected mathematical understandings, we as instructors need to have even deeper and more connected understandings ourselves.

Deciding on mathematical goals involves focusing on some things instead of others. As with all instruction, teaching via problem solving engages the instructor in balancing depth of exploration versus quantity of concepts.

> I hesitated with trying to decide whether to get into a deeper discussion of what larger than single-digit multiplication is when discussing multiplying two-digit numbers in bases other than ten. In the end, I did go into the meaning of multiplication using partial products. I think this did prompt the students to think more deeply about multiplication and their talk showed this, but is also slowed them down and was more difficult. Trade offs! (JM, Memo 9/12/07)

We found that deciding upon, implementing and reflecting on mathematical goals are critical aspects of the process of teaching mathematics via problem solving and learning to teach via problem solving.

*Choosing and Facilitating Tasks*

Given that the mathematics courses we taught had an activity-based textbook (Masingila, Lester & Raymond, 2006), most of the mathematical tasks were already chosen for us. We found that our focus was not in picking tasks (although for one lesson we designed alternate tasks), but in sequencing tasks, modifying them and deciding how to have the PSTs engaged with the tasks. One of the biggest challenges, we found, was to scaffold and support students' problem-solving efforts without taking the problem solving out of the work, and to facilitate so that powerful mathematical ideas emerged. Our data reflect our process of making sense of the tasks, facilitating their implementation, and reflecting upon the PSTs' engagement with and our facilitation of the tasks.

As discussed previously, we found the chapter on probability and statistics in need of revision and so we did some revising in the midst of teaching the chapter: "I prepared an alternate activity reinforcing the ideas of mean, median and mode, and then engaged students in working with and representing paired data. Again, we used graphing calculators to input data into lists and plot them as scatter plots" (JM, Memo 11/7/07).

While choosing activities and deciding how to use them is important, an equally important part is facilitating their use. Reflecting on the alternate activity, PK discussed how he facilitated the PSTs' engagement with the activity:

> Contrary to other questions that asked the students to find a five-number summary for a set of data, the first question in this alternate activity asked the students to generate a data set that would have a specified five-number summary. I had the six groups put up their data sets on the chalkboard and led a whole class discussion on the data generated. While all the groups had data sets that worked, it was interesting to see that all the groups came up with data sets that utilized only natural numbers. I asked the students if this was necessary. They decided that other values could be used also and we as a group came up with one such set. (PK, Memo 11/7/07)

A third aspect of the instructor's role related to tasks is reflecting on the tasks used, their facilitation, and the PSTs' interaction with them. One set of activities engaged PSTs in exploring and understanding algorithms, including ones that are different from ones that are traditionally used in school. In our memos, we reflected on these algorithms and their relevance to our PSTs: "I think that the Cashier's Algorithm may be a bit outdated. Nowadays, cash registers just tell you what change to give back, and so many people use credit cards, that the students are not really able to relate giving back change with subtraction. There may be some value in just calling it "adding up," and changing the activity" (DO, Memo 9/24/07). We found that choosing, facilitating and reflecting on tasks are critical aspects of the process of teaching mathematics via problem solving and learning to teach via problem solving.

## Discussion

In the same way that "prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach" (CBMS, 2001, p. 7), novice mathematics teacher educators need experiences that will develop a deep understanding of the mathematics that they will teach to pre-service elementary teachers and support these PSTs in understanding deeply. The importance of the mathematical knowledge for teaching for mathematics educators cannot be overemphasized. We found that using a teacher development experiment and modified lesson study framework allowed us to think critically about many aspects of teaching mathematics via problem solving and supporting PSTs in learning mathematics via problem solving.

This framework, along with our position of inquiry as stance, also facilitated us developing a community of practice among the three of us as the course instructors. Within this community of practice, we were able to establish trust where any of us could make observations about the course materials, the questions we asked, the decisions we were pondering, etc. and these observations were received by the other instructors as valid and were considered within the context as we made instructional decisions. The establishment of trust also mediated the balance of power within our community of practice. JM, as the experienced teacher educator, a faculty member and one of the co-authors of the textbook we used, came with more power, by default, into the relationship. However, because we worked diligently and purposefully to form a community of practice with full participation by all three of us, the power relationship was held in check.

In reflecting on the processes of learning to teach via problem solving and supporting PSTs in learning via problem solving, we recognize the importance and are working toward developing a principled way of making pedagogical decisions. These include deciding when to have PSTs continue working on an activity and when to move on, what level of justification is sufficient, when should mathematical ideas be introduced by the instructor and when should they arise from the PSTs' activity.

There is much more analysis to do on this rich set of data. We anticipate coming to better understand how and what we, as mathematics teacher educators, learn about the processes of teaching mathematics via problem solving and supporting pre-service teachers in learning mathematics via problem solving.

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# ELEMENTARY PRESERVICE TEACHERS' ACTIVE-LEARNING EXPERIENCES IN A MATHEMATICS CONTENT COURSE

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*This study examined 96 elementary pre-service teachers' active-learning experiences in the first of a sequence of three mathematics content courses intended to provide them with opportunities to experience as learners Standards-based mathematics teaching. Participants indicated that they had many opportunities to engage in active learning that they had not experienced in their previous mathematics content courses. Participants also described how active learning influenced their level of confidence in their ability to do and discuss mathematics.*

## Purpose of the Study

In recent years teacher preparation programs have been criticized for failing to adequately prepare K-12 teachers to teach mathematics according to the standards set forth by the National Council of Teachers of Mathematics in *Principles and Standards for School Mathematics* (NCTM, 2000). For example, the National Research Council (2001) notes that elementary teachers "possess a limited knowledge of mathematics, including the mathematics they teach. The mathematical education they received, both as K-12 students and in teacher preparation, has not provided them with appropriate or sufficient opportunities to learn mathematics" (p. 372). Similar concerns have been raised by the National Mathematics Advisory Panel (2008), which argues that "the preparation of elementary teachers must be strengthened by providing teachers with ample opportunities to learn mathematics for teaching" (p. 38).

One way of helping elementary pre-service teachers (hereafter referred to as PSTs) develop and strengthen their mathematics knowledge for teaching (Ball, Bass, Delaney, Hill, Phelps, Lewis, Thames, & Zopf, 2005) is to provide them with consistent opportunities to engage in active-learning activities in their mathematics content courses (Conference Board of the Mathematical Sciences, 2001; Mathematics Association of America, 1998). In such activities, PSTs have opportunities to collaborate with peers, make meaningful mathematical connections through discussion, engage in higher-order thinking, and develop new mental structures (Bonwell & Eison, 1991; Meyers & Jones, 1993). Proponents of this paradigm shift towards active-learning argue that new ideas are formed and discoveries are made not by individual competition but through collaboration with others. Furthermore, they claim that the notion of learners constructing their own understandings of mathematics concepts is central to the nature of the subjects (Atwater, 1994; Becker, 1995). Rogers (1992) states:

A pedagogy that emphasizes product deprives [learners] of experiencing the process by which ideas in mathematics come to be and perpetuates a dualistic view of mathematics in which right answers are known by authorities and are the property of experts. Such a pedagogy strips mathematics of the context in which it was created and is based on misconceptions about its very nature. (p. 42)

The shift from studying mathematics as a *product* of human activity to studying it as a *process* of human activity places greater emphasis on the development of mathematical ideas and encourages K-12 teachers to develop their *mathematical power* – that is, a positive disposition toward, curiosity about, and self-confidence in mathematics; the ability to logically reason about

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and analyze mathematics; a deeper and more connected understanding of mathematics across strands of mathematical content (e.g., algebra and geometry) and to disciplines outside of mathematics; and the ability to communicate mathematical ideas (Baroody & Coslick, 1998; NCTM, 1989; Orrill, & French, 2002). The notion of mathematical power aligns closely with NCTM's (2000) five *process standards* (problem solving, reasoning and proof, communication, connections, and representation). Therefore, if teachers are to provide their students with opportunities to engage in these process standards, they too need opportunities to participate in learning experiences that emphasize the process standards and develop their mathematical power.

The work described in this paper is part of a larger project that seeks to understand elementary PSTs' learning in a sequence of three mathematics content courses specifically designed for this population and to identify the sources of their learning. The focus of this paper is on one source of this learning, namely, the active-learning opportunities PSTs had during the first of a sequence of three mathematics content courses that are intended to provide them with opportunities to engage in active-learning as a way of making sense of mathematics. In this paper, we seek to explore the following research questions:

1. To what extent do elementary PSTs engage in active learning in a mathematics content course?
2. In what ways does active learning influence PSTs' learning and mathematical power?

Thus, the current study considers the opportunities PSTs have to learn mathematics and how engaging in those experiences influences their learning and mathematical power.

## Theoretical Framework

This study was guided by Astin's (1996) theory of student involvement which defines involvement as "the amount of physical and psychological energy that the student devotes to the academic experience" (p. 518) in formal and informal contexts both in and outside the undergraduate classroom. This theory is particularly relevant to studies of the relationship between active learning and academic outcomes because active learning inherently requires undergraduates to invest physical and psychological energy in the learning process. Reform efforts that incorporate active-learning opportunities in undergraduate mathematics education seek to change the focus from faculty members' intentions for the undergraduate experience to students' lived experiences. Similarly, the theory of student involvement shifts the emphasis away from faculty content knowledge, university resources, and individualized instructional approaches to focus on what students actually do by examining how students invest their time and energy and understanding the effect that this has on important learning outcomes. Astin asserts that of the three forms of involvement that have the greatest influence on cognitive and affective outcomes – academic involvement, involvement with faculty, and involvement with peers – involvement with peers has the most powerful influence on undergraduate students' academic and personal development (Astin, 1996).

## Method

### *Setting and Participants*

The study was conducted at a four-year public university in the Southeastern United States. The 96 participants (93 female; 3 male) in this study were elementary PSTs who were enrolled during either the spring 2008 semester (72 participants) or fall 2008 semester (24 participants) in the first of a sequence of three mathematics content courses, each of which had a particular mathematical focus (Course 1 focused on whole numbers and whole-number operations, Course 2 focused on rational number, and Course 3 focused on geometry and measurement). The spring 2008 participants were enrolled in one of five sections of Course 1 taught by two different faculty members, and the fall 2008 participants were enrolled in one of two sections of Course 1 taught by one faculty member. These three faculty members used the national and state standards for elementary mathematics as a backdrop for the course and created opportunities for PSTs to be actively engaged in the learning process. For example, PSTs had opportunities to solve mathematical tasks in multiple ways, share and discuss their thinking with their peers, use manipulatives such as base-10 blocks, and analyze children's mathematical thinking in the form of written work and video.

*Data Collection and Analysis*

The study employed a mixed methods approach that involved both quantitative and qualitative data collection and analysis. First, the *Developing Mathematical Power Learning Experiences Survey* (hereafter referred to as the DMP Survey) was administered to PSTs in electronic or paper form to participants at the beginning and end of the course. The DMP Survey used Likert-scale items to assess PSTs' perspectives of their opportunities to engage in active-learning. For each of the 13 items (shown in Table 1) that addressed in-class experiences, PSTs were asked to indicate the frequency in which they engaged in particular active-learning activities during the course by selecting "Not at All," "About Once or Twice a Month," "About Once Every Week," or "Almost Every Lesson." The DMP Survey also assessed PSTs' perspectives on their mathematical power with the four items shown in Table 2. In these items, PSTs indicated their level of agreement relative to their experiences prior to taking the course by selecting "Strongly Disagree," "Somewhat Disagree," "Somewhat Agree," or "Strongly Agree."

A small subset, six PSTs, participated in semi-structured interviews conducted at the end of the course in which they described their learning and their experiences in the course. A final data source was a writing assignment in which PSTs were asked to consider what mathematical power is, and how they think their mathematical power had changed during the course. This assignment was used by one faculty member, for a total of 18 PSTs enrolled in the course during Spring 2008 completing this assignment.

The DMP Survey data were analyzed using descriptive statistics and correlation analyses to explore the context and frequency PSTs' participation in active learning of mathematics during the course and to examine relationships between and among reported levels of active-learning experiences in previous mathematics content courses, during the current mathematics content course, and participants' mathematical power after taking the course. The interview data were analyzed using the constant comparison method (Strauss, 1987). Patterns and themes were identified, merged, discarded, and revised as the researchers compared and contrasted the interview transcripts. The data from the writing assignments were also analyzed in this way. Quotations selected from the interviews and writing assignment on mathematical power are illustrative of the experiences of the majority of the PSTs and are similar in context and meaning to other PSTs' responses and descriptions.

## Results

In this section, we compare the experiences of PSTs who had previous experience with active learning in mathematics to those who did not. Of the 96 PSTs who completed the DMP Survey at the beginning of the course, only 36 (37.5%) indicated that they had regular opportunities to actively engage in the learning of mathematics prior to taking this mathematics course. 77 PSTs completed the DMP Survey at the end of the course, of those, 70 (90.1%) indicated that they had high-levels of active-learning experiences during this mathematics course. This number includes 41 PSTs who indicated low-levels of active-learning experiences in mathematics prior to taking the course. There were 11 (out of 13) active-learning opportunities in particular for which a majority of PSTs indicated that they agreed or strongly agreed to having had engaged in during the course, as shown in Table 1.

Table 1. *End-of-Course Reported High Levels of In-Class Active-Learning Experiences*

| | Freq (n=77) | Percent |
|---|---|---|
| "I listened to and evaluated other students' ideas, solutions, or points of view." | 74 | 96.1 |
| "I was challenged to defend, extend, clarify, or explain how I derived my answers or ideas." | 68 | 88.3 |
| "I was expected to 'investigate' or 'discover' mathematical principles and ideas." | 74 | 96.1 |
| "I worked with other students to explore new ideas/concepts through problem examples." | 69 | 89.6 |
| "I shared strategies with other students for approaching or solving a problem." | 69 | 89.6 |
| "I justified my reasoning in a problem or steps in a proof." | 58 | 75.3 |
| "I discussed connections between mathematical ideas/concepts with other students." | 66 | 85.7 |
| "I worked with other students to evaluate or construct proofs or make conjectures/propositions." | 54 | 70.1 |
| "When students were working together, we were encouraged to admit confusion and ask questions." | 66 | 85.7 |
| "I taught a particular mathematical idea to the class." | 9 | 11.6 |
| "I directed questions to other students about mathematical ideas/concepts." | 43 | 55.8 |
| "I put individual or group work on the board for classmates to examine or comment on." | 28 | 36.4 |
| "I worked in groups with other students on projects to be turned in for a grade or extra credit" | 39 | 50.6 |

These results indicated that PSTs engaged in a variety of active-learning experiences during the course. It is also important to note that although this data was collected from PSTs who took the course from one of three different faculty members, correlation analyses indicated that the active-learning opportunities were not statistically different across the courses. Moreover, many of these experiences emphasized the process involved in the study of mathematics rather than the product of mathematical activity. These findings were supported by the data from interviews and writing assignment as well. During the interviews, PSTs consistently spoke of their professors' expectations to explain their ideas, solve problems collaboratively with their peers, and share their ideas publicly and in writing. Some PSTs in interviews and on the writing assignment noted how these expectations were different from their experiences in previous mathematics classrooms.

PSTs indicated on the DMP Survey administered at the end of the course that some aspects of their mathematical power increased during the current mathematics content course. As shown in Table 2, a majority of teachers agreed or strongly agreed with the four statements used to assess their mathematical power relative to their experiences prior to the course.

Table 2. *End-of-Course Responses to Mathematical Power Items*

| | Freq (n=77) | Percent |
|---|---|---|
| "I feel more comfortable with elementary mathematics content." | 66 | 85.7 |
| "I am more knowledgeable about pedagogical issues related to teaching elementary mathematics content." | 62 | 80.5 |
| "I feel more comfortable discussing mathematics with others." | 66 | 85.7 |
| "I feel more confident in my ability to do mathematics. | 65 | 84.4 |

In addition to finding that PSTs' perceived mathematical power increased during the course, correlation analyses revealed that all of the mathematical power items were highly correlated with statistical significance at the alpha = 0.01 level. This data indicates that PSTs developed their sense of their ability to understand, do, and communicate mathematics in tandem.

Common themes from the writing assignment were consistent with the quantitative analyses and interview data and suggested that PSTs mathematical power may have been influenced by the types of active-learning experiences in which they engaged. For example, Margorie described how considering different strategies influenced her confidence to participate in discussions about mathematics:

> All the way up through high school my worst subject was math. I would always fear having to speak out in class or answer a problem on the board. I would do what I had to do to get by. It was not until college that this changed. I began to have teachers that would challenge me, but at the same time help me to understand different way of thinking and going about solving math problems. I began to feel more confident in my mathematical abilities and I am now able to speak out in class… I love being able to discuss a problem with other students and get their point of views on how to go about solving a problem. I feel that I learn so much more through interaction rather than just lecture and taking notes.

The data collected from the interviews also indicated that PSTs recognized that the opportunities to talk with their peers and listen to others' solutions played an important role in the development of their mathematical power. For example, Teaghan suggested that her experiences even influenced her confidence in other classes:

> ... because at first, I mean nobody wanted to say anything. We were all like, 'Oh my gosh, I hope [the professor] doesn't call on me.' And now, you're expecting it. I mean, you still may not like it. But at least, I know now, that I can say whatever, and, I might say it one way, or might get kind of stuck on a word, and then the girl in the front will chime in, and then they take over and stuff like that, so it's definitely helped my confidence in this, and I mean, other classes.

## Discussion

The results reported in this paper offer some promising support for on-going efforts by mathematics teacher educators in the United States to enrich the mathematics content courses for prospective elementary school teachers. In particular, the results of the current study show that PSTs who are actively engaged in learning tasks in these courses do feel that their mathematical power has increased. Although our results are based on PSTs' self-reporting, we believe it is essential that prospective teachers perceive the usefulness of the active-learning experiences if we want them to incorporate opportunities for such experiences in their own classrooms. If it is indeed the case that teachers teach the way they were taught, then, an important role of mathematics content courses for prospective elementary school teachers is to help them experience the type of teaching we want them to practice in their future classrooms.

This notion of "modeling" the type of instruction that prospective teachers are expected to implement in their classroom practice becomes even more critical when we realize that in many elementary teacher preparation programs, prospective teachers take more mathematics content courses than mathematics methods courses. It is very difficult, if not impossible, for a single mathematics methods course to develop a new vision of mathematics teaching and help prospective teachers learn to implement this new vision. Therefore, mathematics content courses might be promising sites for not only developing PSTs' content knowledge, but also begin to help them confront their own beliefs regarding what it means to teach, by providing them firsthand experience in being students in mathematics courses that likely look very different from their K-12 mathematics experiences.

The study reported in this paper also suggests some specific features of active-learning experiences that may be useful in mathematics content courses. Those experiences include opportunities for PSTs to talk with each other about their mathematical thinking, justify and explain their ideas to their peers and to their instructors, and explore and discover new mathematical relationships through problem solving.

Opportunities for PSTs to talk with each other may be particularly important given that this particular population is predominantly female. Existing literature suggests that female students are often excluded from classroom discourse about mathematics (Fullerton, 1995; Linn & Kessel, 1996; Moreno & Muller, 1999; Seymour & Hewitt, 1997). Drawing upon Belenky, Clinchy, Goldberger, and Tarule’s (1986) work, Becker (1995) argues that traditional lecture teaching methods support separate knowing and devalue connected knowing and create little opportunity for women to develop the participatory competence that would enable them to be self assertive in mathematics. She describes connected knowing and separate knowing as being analogous to inductive and deductive reasoning in mathematics, respectively, and claims that women are more often connected knowers and men are more likely to be separate knowers. Moreover, these practices disempower women and discourage them from developing their own voice (Mau & Leitze, 2001; Rogers, 1992). Mau and Letize (2001) assert that empowering curricula would place women in a position where they not only *could* participate but *would* participate in the articulation of meaningful mathematical understandings without fear of ridicule.

The current study is limited in a number of ways. However, its limitations and some of the findings suggest some potentially important and fruitful future studies. Philipp, Ambrose, Lamb, Sowder, Schappelle, Sowder, Thanheiser, & Chauvot, (2007) reported the usefulness of incorporating opportunities to examine children's thinking in mathematics content courses for prospective teachers. In this study, a number of participants indicated how their experiences with children influenced their perception of the course work. Some of them noted how much better they were able to help children, whether their own or tutees, with mathematics after their course experiences. For example, when asked how the course has influenced her level of confidence in her ability to communicate mathematics, Teaghan commented:

> …last year when I worked at an after school program, I was helping kids do math. And I was telling them the answers cause I just didn't know how to teach it to them…But this year I've noticed that I'm able to help the kids a lot more…Like one of the kids at [the after-school program], it was just a simple worksheet, 20 minus 8 or something like that, but he just couldn't do it. And I just got out little cubes and we did it that way. And he put 20 together and took 8 [away] and I mean, he got it when he could use things. But he couldn't just look at it and say, 'Ok, 20 minus 8.' And I definitely didn't know how to do that last year…So I mean, I just thought back to when I was in class, how did [the instructor] teach it to us, and just kind of remembering all the different things we did… yeah, it's easier to help the kids at [the after-school program] than it was last year.

Others PSTs indicated how viewing brief video clips of children doing mathematics influenced the way they approach these courses. However, what is not clear is how these experiences influenced their learning of mathematics. In some cases, the role these experiences plays is limited to motivational. In other cases, these opportunities may be directly related to PSTs' learning of mathematics content. A further investigation into relationships between examining children's thinking and PSTs mathematics learning is needed.

Another potentially important future study involves the role of manipulatives in these courses. A number of teachers indicated that the "hands-on" nature of the courses influenced their learning. However, several of the interviewees noted that they found it more difficult to solve mathematics problems using manipulatives. One of them even admitted that she solved the problem first and then matched her use of manipulatives to that solution. Yet, all of the interviewees indicated how manipulatives would be useful in their future classrooms. It is not quite clear why they felt the use of manipulatives to be positive when they themselves did not experience the benefits of manipulative use. A further investigation on prospective teachers' experiences with manipulatives and their beliefs of the roles of manipulatives in their own classroom may be necessary if we want teachers to use manipulatives meaningfully.

Finally, probably the most significant limitation of the current study as reported is the lack of objective measures of prospective elementary school teachers' learning of mathematics content knowledge. We did administer the *Mathematical Knowledge for Teaching* (MKT) measures (Learning Mathematics for Teaching Project, 2006) to some participants, and there were general upward trends. However, due to a small sample size, we are unable to report any conclusive finding. It is hoped that we will be able to incorporate the MKT measures in our future studies.

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# TEACHER TRAINING IN VIRTUAL ENVIRONMENTS

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*This study sought to examine the merits of TeachME – an innovative virtual teaching environment for teacher training – in teacher education programs. TeachME, Teaching in Mixed-Reality Environments, provides an environment in which the students are virtual and the teaching is real. In a semester-long methods course, prospective secondary mathematics teachers developed and taught lessons in this virtual environment. In this collaborative training environment, the prospective teachers focused primarily on delivery of a lesson to accommodate the diversity of students and how to manage the classroom. Our results suggest the use of virtual environment can be beneficial to teacher training.*

## Introduction

Traditional teacher training programs focus primarily on developing prospective teachers' content and pedagogical knowledge – what to teach and how to teach it in ideal environments. Environments for teacher training often include field experiences, microteaching experiences, and internships. These are widely used and accepted methods for training prospective teachers for the classroom, yet teachers in their first years of teaching often face difficulties related to classroom management. Without adequate management of student behavior the content knowledge of the teacher becomes irrelevant. This begs the question as to whether there may be an additional method for teacher training which could assist beginning teachers, particularly in classroom management. This study reported here sought to examine the merits of TeachME – an innovative mixed-reality teaching environment for teacher training.

## Development of TeachME

Given high teacher attrition and turnover in public school settings due to difficulty managing classroom behavior (Swan, 2006; Veenman, 1984; Hollingsworth, 1988), a discussion began amongst the education faculty at a large university in central Florida. The question became, "How can we prepare teachers to manage the classroom and student behavior without putting teachers and students at risk?" The answer was a mixed-reality teaching environment for teachers to practice their skills prior to entering the classroom (Hughes, Stapleton, Hughes, & Smith, 2005).

The mixed-reality environment is called TeachMe (Teaching in Mixed-Reality Environments) and is housed at the University of Central Florida (UCF). TeachMe is the result of a unique collaboration rarely seen in education to develop the educational technology for teachers of the future. The outcome of this collaboration between education, computer sciences, and simulation technology provides a path to address the problem of teacher attrition by creating a working, mixed-reality environment to train beginning teachers (Dieker, Hynes, Hughes, & Smith, 2008).

The initial prototype focused on behavior and classroom management, an area of concern for most beginning professionals (Goodell, 2006; Van Zoest, 1995). One goal of the mixed-reality environment is to create an interactive, simulated environment to train beginning teachers in mathematics, science and special education before they enter the teaching force. The methodology for developing the virtual environment was built on strong, scientifically based-research related to the training of people in the military and corporate America, made possible through UCF's long-standing record as a leader in simulation technology (Dieker, Hynes, Hughes, & Smith, 2008). In order to have a successful virtual environment, students in the virtual classroom must be representative of real middle school students. The characters were developed using the American Academy of Child and Adolescent Psychiatry's description of adolescent development, William Long's classification of adolescent behavior, Rudolf Driekurs' theory of understanding adolescents maladaptive behavior, and the work of other early theorists in human development such as Piaget, Freud, Kohlberg, Erikson and Maslow (Dieker, Hynes, Huges, & Smith, 2008). The result is five students who have distinct and specific personalities designed using the conceptual framework for adolescent development of William Long.

One interactor is the human avatar for all five students. The interactor can escalate or de-escalate the level of behavioral responses depending on teacher interaction. Furthermore, the interactor can create behavior issues between students if the specific needs of students are not addressed, creating a simulated classroom with real student-to-student interaction. The teacher faces a large screen that displays all of the students. By stepping towards the screen and leaning towards the child, the teacher can interact individually with each student, or stand in the front of the room to address the group.

In the mixed-reality environment, the teacher feels a sense of realism in trying to get the students to stay on task and complete a lesson, yet also has the ability to go back and try again as the virtual students can be reset, unlike a real classroom environment. This puts a safety net under the novice teacher and protects actual children from any harm. Teaching in a simulated classroom environment allows for teachers to deliver instruction, self-analyze the teaching experience, make changes in the lesson based upon the teaching results, and re-teach the lesson to increase mastery of teaching and learning concepts in a way that does not put children at risk.

## Theoretical Framework

According to Prensky (2001), our education system fails to teach today's students, growing up with new technologies, and the biggest problem facing education is educators, who are Digital Immigrant educators, who speak an outdated language to today's students, who are Digital Natives, native speakers of new technologies. New technologies are integral parts of our students' lives, and we as educators must use emerging technologies in our teacher education programs to prepare prospective teachers for diverse classrooms. In integrating the new technologies into our mathematics teacher preparation program, we sought a theoretical framework that takes into account the dynamic nature of the virtual environment and at the same time addresses the complexity of designing learning experiences for prospective teachers in a virtual environment. We find the instructional design framework of Model Facilitated Learning (Milrad, Spector, & Davidsen, 2003) particularly useful to achieve such goals. As a theoretically grounded framework, Model Facilitated Learning (MFL) draws on well-established learning theories and methods of system dynamics to manage complexity in technology-enhanced learning environments in order to achieve meaningful learning and deep understanding. The MFL framework allows learners to build models and/or experiment with existing ones as part of their effort to understand the structure and the dynamics of a complex phenomenon, in this case teaching mathematics to diverse learners. Following the MFL framework, we designed learning experiences for prospective secondary teachers. The MFL is guided by four principles: 1) Situate the learning experience: The prospective teachers worked in groups of three, and we asked the groups to plan and write a detailed lesson plan for the same algebra problem, an algebraic reasoning problem involving generalizing a non-linear pattern; 2) present problems and challenges of increasing complexity: We presented the prospective teachers with correct, incorrect, and incomplete student work samples to the algebra problem for which they wrote lessons, and asked them to discuss these work samples; 3) involve learners in responding to a set of increasingly complex inquiries about the problem situation: After each teaching session, each group member wrote a reflection based on his/her role in the teaching (i.e., teacher or observer), then watched their own videos and revised their lesson plans in preparation for the next cycle of teaching; 4) challenge learners to develop decision–making rules and guidelines for a variety of unanticipated situations: In an attempt to help the prospective teachers develop appropriate solutions and strategies for challenging student behaviors in a virtual environment, we challenged the prospective teachers by escalating or de–escalating the level of behavioral responses.

## Design of the Study

We sought to examine the merits of TeachME in teacher education programs through a mixed study with fifteen prospective secondary mathematics teachers. Data were collected in a semester-long methods course for prospective mathematics teachers through videos of the teaching episodes and classroom discussions, interviews, classroom observations, students' lesson plans and reflections. Analysis and data collection were an ongoing process throughout the study, and the research team held weekly meetings during the study.

All teaching episodes were video-taped and student reflections and lesson plans were collected for each teaching cycle. Additionally, members of the research team observed the teaching episodes. Student conversations during class time in which the groups were discussing their teaching episodes were audio-taped. Video and audio tapes were transcribed by members of the research team.

The prospective teachers were randomly divided into 5 groups. All groups wrote lesson plans for the same problem, an algebraic reasoning problem involving generalizing a non-linear pattern, which they had previously solved in class. The groups followed a three-stage cycle of teaching in which one member taught the lesson and the other two members observed. The member responsible for teaching changed with each teaching cycle. An interactor behind the scenes acted for the virtual students and provided realistic interactions as might occur in an urban middle school classroom.

In order to deepen the prospective teachers' thinking about various solutions to the problem, eleven correct, incorrect, and incomplete student work samples for the problem were created by the research team and were then assigned to the virtual students in TeachME (See Figure 1). All groups received the same work samples in each teaching cycle, and the prospective teachers were asked to discuss these work samples during their 15-20 minute TeachME sessions. The work samples were changed for each teaching cycle.

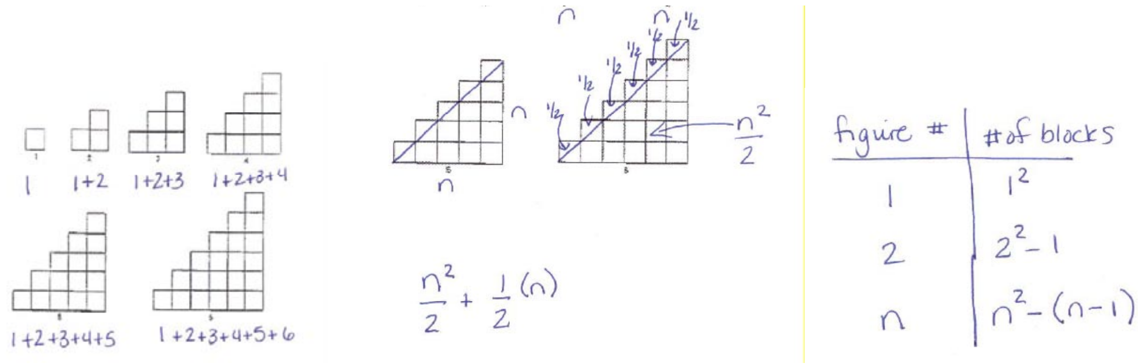



*Figure 1:* Examples of incomplete, correct, and incorrect solutions.

All sessions were videotaped. After each teaching session, each group member wrote a reflection based on their role in the teaching cycle (i.e., teacher or observer). During the next class session, the groups watched their own videos and revised their lesson plans in preparation for the next cycle of teaching.

## Data Analysis

The prospective teachers developed and taught lessons in TeachME environments. From these fifteen teaching episodes, two virtual teaching episodes will be presented which illustrate various challenging student behaviors the prospective teachers encountered in the mixed-reality environment, TeachME. In the first, we examine the teacher, Mr. Jeffrey, interacting with the five virtual students, whom we will call Monica, Michael, Victor, Michelle, and Freddy. Following this, we examine the teacher, Ms. Jennifer, interacting with the same five virtual students.

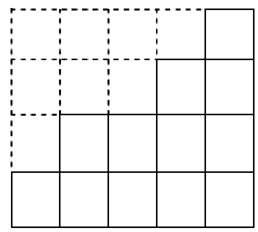
*Episode One*

In this episode, the teacher, Mr. Jeffrey, interacted with the five virtual students during the first teaching cycle. Each student had a different solution and Mr. Jeffrey was asked to teach the classroom discussion portion of the lesson. At one point in the episode, Mr. Jeffrey had to interact with Michael's incorrect solution a shown in Figure 2.

| Figure #1 | # of Blocks |
|---|---|
| 1 | $1^2$ |
| 2 | $2^2 – 1$ |
| N | $N^2 – (N-1)$ |

*Figure 2.* Michael's solution.

Mr. Jeffrey asked Michael what would happen if he used his formula to find the number of blocks in figure 3 of the problem. Despite Mr. Jeffrey's encouragement that Michael had done good work, Michael's response was concern and frustration that his work was wrong. Mr. Jeffrey dealt with Michael's attitude and continued to encourage Michael to find a generalization that would work for all the figures. As the teaching episode continued, Mr. Jeffrey then interacted with Monica related to her solution. Monica had a correct solution as shown in Figure 3.



$$\frac{n(n+1)}{2}$$

*Figure 3.* Monica's Solution.

Monica is a high achieving student, but she does not like to participate in class discussions. As Mr. Jeffrey worked to elicit her thinking, Monica was uncooperative said "I learned this in 6th grade". In an attempt to further explore Monica's thinking with the rest of the class, Mr. Jeffrey asked Monica to explain her solution in a whole group discussion. She was resistant, saying "I just told you." When asked to tell the whole class, she explained her work in an extremely concise way and only provided discussion when directly asked.

As the episode continued, Mr. Jeffrey continued to have to deal with the management of the classroom, particularly in leading a whole class discussion. As he worked with Victor and Michelle, Mr. Jeffrey's goal was to get Victor and Michelle to understand Monica's method for generalizing the formula. Victor and Michelle interpreted the discussion as their solutions being wrong and Monica's being correct, even though their solutions were correct.

Mr. Jeffrey: Michelle, can you tell us about the solution you got?

Michelle: Um, yeah, my solution, I got the same answer I just did a lot more adding and Monica did multiplication and division and adding and I just did straight adding, but I got the same answer.

Mr. Jeffrey: Very good. Did you understand the way Monica did it?

Michelle: Does that mean I am wrong because I didn't do it the way Monica did it?

A similar interchange occurred between Mr. Jeffrey and Victor, with similar thoughts stated by Victor when he said "So I should have done it like Monica?". Both Victor and Michelle knew that Monica usually had a correct answer and interpreted the efforts of Mr. Jeffrey to get the students in the class to make sense of Monica's solution as their solutions were unvalued or incorrect. Mr. Jeffrey tried to manage this situation and to encourage the students that their solutions were as equally valued as Monica's solution.

In leading a whole class discussion, Mr. Jeffrey had to deal with the emotional aspects of teaching children in dealing with Victor's and Michelle's reactions to his asking them to make sense of Monica's work. These interactions provided for a realistic picture of classroom dynamics.

*Episode Two*

In a second teaching episode, Ms. Jennifer was teaching in TeachME during the second teaching cycle. Ms. Jennifer began teaching and was immediately encountered with a confrontation with Michael who had been throwing virtual spit balls at Michelle. Michael admitted to throwing the spit balls and a confrontation ensued as to who was in charge of the classroom.

Ms. Jennifer: Are you throwing spit balls at Michelle?
Michael: Yeah.
Ms. Jennifer: Why?
Michael: Because this is my classroom and that's what I felt like doing.
Ms. Jennifer: Oh, I thought this was my classroom.
Michael: Well you thought wrong, now didn't you.
Ms. Jennifer: Oh, really? Well, we are going to work on our staircase problems and, this is my classroom, and that is what we are going to do.
Michael: Ha. Your classroom! Did you hear that Victor? She said this is her classroom.

This interaction began the lesson which proved to be difficult for Ms. Jennifer. As Ms. Jennifer continued to interact with the other students in the class, the confrontation which began with Michael contaminated the entire classroom, as it often does in reality. Ms. Jennifer's difficulty managing the behavior and responding appropriately to Michael's claim that it was his classroom undermined Ms. Jennifer's authority and affected her interactions with all the other students in the class. The students became hostile and Ms. Jennifer tended to be hostile as well. As Ms. Jennifer interacted with Monica, for example, Monica's response was not only unmotivated as in the episode with Mr. Jeffrey, but was also disrespectful.

Ms. Jennifer: How's it going with the staircase problem? Do you have an answer?
Monica: Yep.
Ms. Jennifer: Can I see it?
Monica: Yep.
Ms. Jennifer: Can you explain to me what you did?
Monica: Used the formula that we learned in sixth grade.
Ms. Jennifer: And how did you get that formula?
Monica: We learned it in sixth grade.
Ms. Jennifer: Okay, so then you can tell me what this means.
Monica: You plug the numbers into the formula and you get the right answers depending on what number you want.
Ms. Jennifer: No. What does the formula mean? What does n squared over 2 plus ½ times n mean?
Monica: You don't know? I thought you were the teacher.

Not only did Ms. Jennifer have to deal with Monica's behavior, she also had to deal with the impact of her inability to manage Michael's behavior and its impact on Monica who challenged Ms. Jennifer's authority in the classroom. The focus of the lesson became managing behavior and the content was left to the wayside. Without the ability to manage the behavior of the students, the content involved in the lesson became irrelevant.

## Conclusions

In the virtual teaching environment, TeachME, the students were real, sometimes disrespectful, unmotivated, and unenthusiastic. The teachers could not rely on anyone but themselves for delivery of content and the added component of behavior management, even on a small scale as in Mr. Jeffrey's case, provided for an enhanced and realistic environment for learning to teach, particularly with aspects of management of behavior.

We conclude that there are potentialities in TeachME for not only deepening content knowledge through discussion of correct, incorrect, and incomplete student work samples, but also for developing behavior management strategies. The virtual teaching environment sharply differs from other teacher training environments (e.g. microteach). Rather than a focus on teaching content, the virtual environment allows for a focus more on managing student behavior in order for delivery of content to occur. This is by no means to say that other training environments, such as microteaching and internships, are not useful in teacher training, but rather that we as teacher educators must see the weaknesses and strengths of these teaching experiences and find ways to enrich the student teaching experience. Moreover, we argue that the realistic aspects of the virtual environment can, in fact, enhance prospective teachers' preparation for classrooms, particularly in urban schools. The incorporation of the virtual in complement to other environments provides for multiple experiences which can focus on both mastery of content and its delivery as well as behavior management strategies which can be effective in schools.

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# PRESERVICE PREPARATION OF MATHEMATICS CANDIDATES IN THE LARGEST ALTERNATIVE CERTIFICATION PROGRAM IN THE U. S.

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*In this empirical study we examine the preservice training the New York City Teaching Fellows (NYCTF) program provides its middle school and secondary mathematics teacher candidates. Drawing on in-depth survey and journal data we present findings about the preservice programs at the four NYC-area universities that partner with NYCTF program to train mathematics fellows. We describe the scope and sequence of the four programs, compare and contrast them, and document the experiences and perspectives of first-year mathematics teaching fellows in these programs.*

## Background

The New York City Teaching Fellows (NYCTF) is the largest alternative certification program in the U. S. and currently places some 300-400 new mathematics teachers in high needs NYC schools every year. Established in 2000, the NYCTF program prepares teachers for in-demand and high turnover disciplines like mathematics, science, bilingual education, and special education. It is a collaborate effort between the New York State Education Department (NYSED), the New York City Department of Education (NYCDoE), The New Teacher Project (TNTP), and a number of colleges/universities. The program, which combines full-time teaching in a New York City public schools with part-time academic coursework, leads to a Master's degree and New York State initial certification in the teaching field (Boyd, Grossman, Lankford, Loeb, Michelli & Wykoff, 2006).

The first cohort of 250 teachers in 2000 was relatively small, but the NYCTF program has grown rapidly over its brief existence. According to the NYCTF website, there were 16,700 applicants for approximately 2500 positions in 2004 including 317 positions for middle and high school mathematics teachers. Since that time, 60% or more of all new mathematics teachers entering NYC schools were participants in NYCTF (Bernstein, 2006). Due to this growth, it is expected that the influence on the entire New York City mathematics teaching force of those teachers prepared through NYCTF has increased substantially.

## Theoretical Framework

We situate this empirical research in studies of alternative certification and alternative route programs (Darling-Hammond & Youngs, 2002; Feistreizter & Chester, 2003; Hawley, 1990). We note that, to our knowledge, none of the extant research literature examines the preparation of alternatively certified mathematics teachers. Additionally, while not focused on mathematics teachers, we find that there are five studies that report results on the NYCTF program and its teachers, including three qualitative studies (Costigan, 2004; Goodnough, 2004; Stein, 2002) and two statistical analyses of student achievement (Boyd, Grossman, Lankford, Loeb, & Wyckoff, 2006; Kane, Rockoff, & Staiger, 2006).

In an early small-scale survey study of the NYCTF program, Stein (2002) found that approximately 90% of the 31 fellows she surveyed were already considering leaving the high needs schools in which they initially were placed, after they fulfilled a two-year commitment.

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

Despite this, Stein concluded that the NYCTF program is an “unqualified success,” citing the fact that it includes a mentoring component and that recruits have strong academic credentials.

Nevertheless, the qualitative and quantitative evidence indicates the fellows may not teach that differently than both uncertified and traditional route (college certified or recommended) teachers in similar NYC schools. Two qualitative studies indicate that novice teaching fellows are similar to other novice teachers in many regards (Costigan, 2004; Goodnough, 2004). In particular, the novice teaching fellows in these studies articulate similar high-minded ideals as traditionally prepared novices (e.g., about “molding students”). However, the fellows switch into “survival mode” once they become teachers of record – a mode characterized by a narrow focus on management concerns and teacher-centered instruction.

It is not clear what teacher preparation programs – alternative or traditional – can do to help novice teachers maintain and realize the student-centered ideals that many aspire to (Flores, 2006). Those who have studied the NYCTF program raise concerns about the program design, particularly, the preservice component that is less than two months in duration. Costigan (2004) writes that the NYCTF “immersion model may not, through a 2 year reduced credit program with full time teaching, adequately prepare self-identified ‘urban pioneers’ … to teach in the increasingly complex and intensified educational situation today, particularly in poor urban neighborhoods” (132). Goodnough (2004) expressed similar concerns, referring to the NYCTF preservice program as a “boot camp” and pointing out that many uncertified teachers had more formal preparation than first-year fellows. To be clear, neither Costigan or Goodnough focused on the NYCTF preservice training of teaching fellows in their research.

The question is what effect, if any, might a shorter “boot camp” style preservice training for mathematics teachers have on student learning? Two quantitative studies of NYC schools address this question, though only one was peer-reviewed. In their peer-reviewed study, Boyd, et al. (2006) find that there are statistically significant, if modest, differences in student mathematics achievement ($4^{th}$ to $8^{th}$ grade tests) between alternative and traditional route teachers; the students of the NYCTF teachers scored about 3.5% of a standard deviation lower respectively on standardized state mathematics exams than that of comparable traditional route (i.e., “college recommended”) teachers. There was some evidence, though not statistically significant, that the fellows catch up to the college recommended teachers if they stay in the classroom for three or more years. This might make sense given that, by their third year, the NYCTF’s have completed a comparable amount of educational coursework to traditional route teachers in NYC (though not necessarily in mathematics). At the same time, Boyd, et al. (2006) include data showing that the fellows have significantly lower rates of retention than traditional route teachers who teach at similar urban schools.

## Research Questions

1. How are alternatively certified middle school and secondary school mathematics teachers prepared to teach during their preservice training?
2. What are alternatively certified teachers’ experiences and perspectives during the preservice component of their training?

## Methods

This large-scale qualitative study of the NYCTF program and its mathematics teachers involves a team of more than a dozen researchers and an extensive amount of data (e.g., videotaped lessons, administrator interviews, in-depth surveys). The study of the NYCTF program for mathematics was designed to compliment aforementioned quantitative research on the NYC teachers from various pathways already under way (Boyd et al., 2006). In this paper we focus on survey and teacher journal data collected about the preservice programs at the four NYCTF partnering universities for mathematics. These preservice programs have three major components: (1) university coursework, (2) fieldwork in NYC summer schools, and (3) “Fellows Advisory.” Fellows Advisory is a highly structured program primarily delivered by former teaching fellows (i.e., those with regular certification and still in the classroom) and is designed to provide pragmatic information about teaching in NYC schools and navigating the NYC public school system.

*Survey Data*

We collected 269 in-depth surveys from first-year mathematics teaching fellows (MTFs) at the four university partners during their last week of their summer preservice programs. This was more than 90% of the MTF population who entered in the summer of 2007. The survey included closed (Likert-scale) items and open-ended items. About 25% of the survey questions concerned aspects of the preservice programs. Another 10% of questions were indirectly relevant and asked questions related to the MTFs personal views of themselves as urban mathematics teachers.

*Journal Data*

In early June of 2007, we met with officials at the four partnering universities and explained our study to them. After getting their consent, we next met with 9 newly recruited MTFs who agreed to collect information on their preservice programs – 3 of the MTFs attended the largest program and the other six were split evenly between the 3 other university partners. The participating MTFs varied in terms of being prepared for middle or high school mathematics teaching. They were each paid $75 a week to: (1) collect handouts (e.g., syllabi, assignments, articles) from university coursework and other NYCTF professional development sessions held off-campus; (2) keep a written daily overview – one or two pages – organized by the lesson, session, or activity, and that provided information about what they and their instructors were doing during that time (i.e., taking notes, math problem solving, role playing); and (3) reflect daily about a particular activity or the day as a whole and, once summer school began, their experiences in fieldwork classrooms and the interplay between their university coursework, NYCTF profession development, and fieldwork experiences. They kept both written accounts and also audio reflections and notes in digital recorders supplied to each of them. The written data and transcriptions of the audio data were then organized to create week-by-week folders for each of the nine fellows.

*Survey Analysis*

Responses to the survey were analyzed using a mixed method approach, including classic statistics, exploratory data analysis, and coding of expository responses, as appropriate. The closed, Likert-scale items were entered into a spreadsheet, electronically counted and compiled into a distribution table. For the open-ended survey items, we used open coding schemes (Emerson, Fretz, & Shaw, 1995). In order to understand how the four partnering university preparation compared, we disaggregated both closed and coded open item responses by the partnering universities.

*Journal Analysis*

The three authors read and reviewed the data folders of the 9 fellowss. We then created one-page overviews that summarized the content that each program covered and activities the individual fellows engaged in on a week-by-week basis. From this we were able to successfully hone the content that each university program generally covered and, to a lesser extent, how this content was organized and presented. Within and across program comparisons of these overviews allowed us to examine both (potential) within program variation and to compare the four programs. These overviews were then analyzed in the context of five survey items that reported on the content and instruction of the four university programs.

**Results**

The MTFs at the four university partners report reviewing similar content in the preservice programs – approximately 1/3 of their time on general education issues (e.g., classroom management, lesson planning), 1/5 on mathematics content, 1/5 on mathematics teaching methods, 1/8 on multicultural education, and 1/10 on educational psychology. Special education received minimal coverage.

In spite of these commonalities, there were also significant variations in the preservice curricula at the four partnering universities, which we refer to as University A, B, C or D in order to organize the results. In particular, fellows at university A reported spending over half their time on general education issues. In their journals, the two MTFs at this university reported viewing a series of videos on classroom management and completing repeated assignments on lesson planning. University A MTFs reported spending less time learning about mathematics-specific teaching methods and mathematics content than MTFs at other university partners. They overwhelmingly evaluated their coursework favorably on the surveys. MTFs at University B spent more time on mathematics content, multicultural education, and mathematics specific teaching methods than the other three partners. Fellows at this program generally evaluated this preservice program negatively on the surveys. To be clear, these negative evaluations have been a result of many factors other than the content of the courses, such as university setting or quality of instructors. As a group MTFs at University B felt less prepared to handle classroom management issues than MTFs at University A – though they may be better prepared to teach mathematics.

In the survey data, the MTFs also reported significant variation in fieldwork experiences. To clarify, there was far more variation in fieldwork within programs than between program. While some MTFs report being in their summer school site for 75 or more hours, another group report completing less than 35 hours. They also appear to have had different opportunities depending on their fieldwork placement. A small number report teaching their classes more than 75% of their time instructing the whole class, while others report spending most of their time working with individual students or observing their mentor. (Still others were not placed in mathematics classrooms at all.)

Initial results from a follow up survey with those math Fellows who remained in teaching after one year provided additional insights about NYCTF summer preservice training. In particular, while most of these experienced MTFs were able to point to strengths of the program – fieldwork was the most oft cited strength – many MTFs reported that the summer program was too short, too accelerated, or too theoretical. That said, the majority of the “experienced survivors” felt prepared – though generally “not well prepared” – to teach their assigned courses. They were about equally split between those who felt prepared and those who felt poorly prepared to handle a range of management/disciplinary concerns and teach using a variety of instructional methods. That said, most did not feel prepared to teach mathematics to ELL students and those with learning difficulties.

## Discussion

The NYCTF program has had a profound effect on high needs NYC schools – replacing uncertified teachers in such areas in mathematics, science, and special education. The NYCTF program has also had a national impact; since its inception in 2000, a number of Teaching Fellows programs, modeled after the NYCTF program, have emerged in a number of high needs districts across the U. S.. While the extant research on alternative certification programs, including the studies of the NYCTF program, suggest that novice alternative route teachers are underprepared in comparison to novice traditional route teachers (who may also be under-prepared in some respects), we know little about the nature of alternative route teacher preparation – both in general and in mathematics in particular. That is, while the research literature points to significant variability in teacher effectiveness and retention, it says little about what type of preparation helps to produce effective teachers and how it might be a factor in keeping them in the classroom. This study begins to address this shortcoming in the existing literature on alternative route programs, in particular, those for urban mathematics teachers.

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# AN EXPLORATION OF PRESERVICE TEACHERS' CREATION AND ANALYSIS OF FRACTION MULTIPLICATION WORD PROBLEMS

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*This article reports on activities within a mathematics methods course designed to deepen PSTs' understanding of the multiplication of fractions. PSTs examined students' work concerning fraction multiplication from the National Assessment of Educational Progress. The PSTs then posed and investigated a set of multiplication problems in groups. These problems and the PST explanations of the solutions of these problems were analyzed. Results revealed features of PSTs' conceptions of fraction multiplication.*

## Background

Often, the arithmetic of whole and rational numbers, which forms the bulk of elementary school curriculum, is taught to PSTs in a maximum of 2 courses. A question we should consider is, *When preservice teachers emerge from an elementary mathematics methods class, what understandings do they possess regarding the multiplication of fractions (including improper fractions) and how will they approach instruction?* This paper describes an investigation of PSTs' responses to questions regarding multiplication of fractions. An instructional approach is described, as are findings concerning PSTs' responses before and after class investigations. These results are explored for implications concerning how best to develop and sequence tasks to help PSTs achieve a more profound understanding of fraction and mixed number multiplication. First, the relevant research literature is examined to provide a conceptual framework.

## Conceptual Framework

Ma (1999) contends that elementary teachers must understand elementary mathematics at a "profound level" (p. X) in order to be able to understand and apply knowledge of appropriate pedagogy. This is consistent with Sowder, Philipp, Armstrong and Schappele's (1998) observations that as teachers' gain in content and pedagogical knowledge, they view students as more capable and are willing to use student-centered teaching approaches. Thus, appropriate experiences in mathematics teacher education have a potential to affect instruction.

Researchers have explored several areas of fraction knowledge development in children and teachers. The development of the fraction concept in children has been explored (Keiren, 1993; Behr, et. al 1993, 1997; Steffe, 2004), as has the development of children's understanding of the multiplication of fractions (Fischbein, Deri, Nello, & Marino,1985), and preservice and inservice teachers' understanding of this concept (Harel & Behr, 1995; Post, Harel, Behr, & Lesh, 1991; Cluff, 2005; Iszak, 2006). Kieren (1976) postulated that fraction concepts were difficult for children to learn because they consisted of many subconcepts that must be constructed first, including partitioning (1993). Behr and colleagues (1993, 1997) expanded these notions in the *Rational Number Project* developing a theoretical model of fraction understanding based on the notions of part-whole, quotient, ratio number, operator, and measure. Recently, Charalambous & Pitta-Pantanzui (2007) used structural equation modeling techniques to study the understanding of fractions among 5$^{th}$ and 6$^{th}$- grade students in Cyprus using the Behr subconstructs as their theoretical frame. The findings confirmed theorized associations between the subconstructs. They also noted that, “a profound understanding of the different interpretations of fractions can uplift student’s performance on tasks related to the operations of fractions and to fraction equivalence” (p. 311).

However, other researchers contend that the Behr model considers fraction understanding from a “top-down” view of an instructional expert, rather than from a learner’s perspective. Hence, in a series of teaching experiments Steffe (2001, 2003, 2004), Steffe and Olive (2001) and Iszak (2006) pursued a separate line of investigation of the development of fraction concepts based on children’s thinking. Steffe’s model is based on the notion of conceptual units and nested levels of units, i.e. the standard part-whole approach to fractions illustrates two levels of units.

In early research concerning children’s understanding of multiplication, Fischbein, Deri, Nello, & Marino (1985) posited that the <u>intuitive</u> model for multiplication is repeated addition. The inherent issues associated with the repeated addition model is that it leads one to assume that the multiplier needs to be a whole number which leads to a further assumption that multiplication always yields a larger quantity. These results were confirmed in several studies that examined elementary school teachers’ knowledge (Graeber, Tirosh, & Glover, 1989; Harel & Behr, 1995; Post, Harel, Behr, & Lesh 1991) and demonstrated that they experienced difficulties in solving word problems where the multiplier was less than one.

More recently, Cluff (2005) defined three case studies of elementary PSTs that described their growth in knowledge concerning fractions, fraction multiplication, and fraction division during a mathematics methods course. Cluff’s study demonstrated that PSTs can exhibit an expansion of their conceptual understanding of fractions and fraction operations based on interactions with multiple representations. Another example of recent work is Iszak’s (2006) study of inservice teachers’ understandings of fraction multiplication which is described below.

Iszak (2006) examined the knowledge applied by two sixth-grade teachers (Ms. Reese and Ms. Archer) in the context of teaching fraction multiplication. The fraction concepts were situated in problems that use lengths and rectangular areas as fraction representations. Iszak’s case studies emphasized that the unit structures produced by these teachers fundamentally shaped the ways in which they used drawn representations and the extent to which they could adapt these representations. This study highlighted disconnections between teachers’ conceptual foundations of fractional multiplication, their inability to use various representations pedagogically, and the expectations of current curriculum in regard to teacher expertise.

## Purpose and Research Design

The purpose of the study is to assess PSTs’ understandings of fraction and mixed number multiplication in the context of an elementary mathematics methods course. The study employed an action research model as defined by del Mas, Garfield, & Chance (1999).

1. *What is the teaching/learning problem?* In addition to the literature cited, the paper describes PST’s responses to two situations intended to provoke discussion of their initial understanding of the concept of fraction multiplication. Fraction multiplication is initially explored by 1) examining a set of problems and student performance data from the National Assessment of Educational Progress (NAEP); and 2) placing PSTs in small groups and asking them to create word problems that can be solved by fraction multiplication and asking them to explain why and how the algorithm works. Examining

NAEP student performance data motivates PSTs and helps them focus their attention on developing sound problems in their small groups. I chose to have them develop fraction multiplication problems in groups based on the strategy employed by Ma (1999) in gathering information about teachers' understanding of fraction division by asking them to create a representative problem.

2. *What techniques can be used to address the learning problem?* Using prior research concerning PSTs' understanding of fraction division as a guide, and Crespo & Nicol's (2006) approach to developing a series of pedagogical tasks for PSTs in the context of division by zero, I designed a series of instructional tasks to assess PSTs' initial understandings of the multiplication of fractions followed by content and pedagogical instruction on the concept.
3. *What type of evidence can be gathered to show whether the implementation is effective?* Following a series of instructional activities, the PSTs answered extensive take-home test questions to assess their understanding of the concept following instruction.
4. *What should be done next, based on what was learned?* Based on results of PSTs' responses to questions about fraction multiplication before and after instructional interventions, I make recommendations for further study.

The research questions for the design and conduct of the study were:

1. How do prospective elementary teachers approach the multiplication of fractions, including mixed numbers, before, during and after instruction regarding the topic? What types of explanations do they use to justify their answers?
2. How do prospective elementary teachers respond to questions about the multiplication of fractions, including mixed numbers following an instructional sequence? Can they use arguments from the instructional sequence to help them solve and explain problems?
3. What do the results for Questions 1 and 2 suggest regarding instructional methods and the amount of instructional time required for PSTs to learn the multiplication of rational numbers (including mixed numbers) to the profound depth required for sound elementary instruction?

In order to assess the readiness of PSTs to provide conceptually-based explanations of algorithms (Ma, 1999) to their students, it is necessary to assess their ability to move beyond rule-based explanations of concepts and construct well-reasoned arguments themselves (Weston, 2000). Well-reasoned arguments explain a concept by: 1) providing examples or counterexamples that confirm or contradict given premises; 2) using deductive logic to test premises; and/or 3) providing an alternative representation that illustrates the concept in a new manner (Crespo & Nicole, 2006). It is also informative to examine the locus of control that appears to underpin PSTs' explanations. Specifically, is this locus of control internal or external?

## Participants and Setting

In the elementary teacher education program at Appalachian State University, PSTs are required to take 9 hours of mathematics and mathematics methods courses beyond general education. The subjects for this study were enrolled in Curriculum & Instruction (CI)-3030 – *Investigating Mathematics and Learning*, a methods course focused on the number and operation strand of the NCTM *Standards*. There were 71 participants in this study enrolled in four sections of CI 3030, two during fall 2006 and two during spring 2007—all four taught by the author. CI 3030 is a 3 hour class, meeting twice a week for 2 hours over 12 weeks, followed by an 8-day practicum in the elementary schools. The demographics of the participants were: 95% below the age of 25; 96% Caucasian; 2% African-American; 2% Hispanic; 94% female; and 6% male.

The paper describes only a portion of the first stage of the study— PST's responses to two situations intended to provoke discussion of their initial understanding of the concept of fraction multiplication. Fraction multiplication is initially explored by 1) examining a set of problems and student performance data from NAEP; and 2) placing PSTs in small groups and asking them to create word problems that can be solved by fraction multiplication, explaining why and how the algorithm works. The results of the after-intervention will appear in a separate paper.

## Data Collection and Analysis

The data included individual subjects' written responses to in-class tasks, individual reflection papers, and tests. The data included sheets of poster paper where groups recorded their responses to in-class tasks, and the instructor's observation field notes kept during two semesters and four sections of the course. The researcher categorized subjects' responses to pre-instruction tasks, in-class group activities, and after-instruction tasks looking for evidence of well-reasoned arguments. Groups were given task sheets on which to record the groups' main points.

The analysis of the PSTs' preliminary notions about fraction multiplication began by classifying their responses to the NAEP problems. Next, I classified the types of fraction multiplication problems they created in their small groups. Their initial responses to the question, *Why does the fraction multiplication algorithm solve your problem?* were grouped and coded by determining their correctness, their locus of authority, and their use of mathematical language and representations. The PSTs' responses to test questions are also described.

## Results: PSTs NAEP Problem and Analysis of Student Performance Results

The classes were given a set of 20 NAEP questions involving fractions to solve individually. (Four of these problems involved fraction multiplication and division and are discussed here. While the paper focuses on fraction multiplication, PSTs were asked to examine and think about fraction division within these problems, in order to help them compare and contrast these concepts.) Then, they were placed in small groups and assigned specific problems to discuss in more depth within the groups. The groups were given a page of questions to guide their discussion and were asked to record their answers. For each NAEP question that the group was asked to discuss, they were asked to respond to the following questions:

> *1) NAEP Problem Number; 2) Complete the question; 3) What math concept is being tested? 4) How is this concept being approached? 5) Does it relate to anything we have talked about in class, thus far? 6) How would you explain how to do this problem to students? 7) What percent of students would you expect to complete this question correctly? 8) What kinds of mistakes would you expect students to make on this problem? 10) How did the actual test results for the problem and the student work you examined compare to your predictions? Were you at all surprised by the results? Why or why not?*

The PSTs solved the problems, predicted erroneous student reasoning, and predicted students' performance results. They then discussed their predictions within small groups. After completing these activities, the PSTs were shown actual NAEP results, including examples of student work for the constructed response problem. PSTs were largely successful in solving these problems.

However, while many could solve the problems computationally, they were unable to provide an explanation of why they chose to use a particular algorithm, and were unable to explain how and why the algorithms work.

## Development of Fraction Multiplication Word Problems by Groups

As the class began the study of multiplication of fractions, I encouraged them to think about what they knew about the topic beyond the algorithms. The PSTs were asked to form groups and to write a word problem that would require students to multiply fractions. They were also asked to show the completed solution for their problem. The PSTs were asked to think about and record any materials, manipulatives, or diagrams that could be used to help explain their problem. They were not given any directions concerning whether they should use proper or improper fractions, or whether whole numbers or mixed numbers could be used. The following shows the list of problems created by the groups of PSTs.

## Fraction Multiplication Problems Created by Preservice Teacher Groups (in Class) During One Method Course

*Problem 1. Ms. Smith's class is going on a field trip. For their lunch* $^1/_2$ *of the students want a peanut butter and jelly sandwich. Of the* $^1/_2$ *who want peanut butter and jelly,* $^3/_4$ *of them want the sandwich with grape jelly? [sic]*
$^1/_2 \times {}^3/_4 = {}^3/_8$
*Models and Representations:* PSTs suggested actually modeling making the sandwiches. PSTs were confused when asked to model the problem in another way.
*Why does the fraction multiplication algorithm solve your problem?* "Well, you know you need to multiply because, 1) you see the word "of", and 2) you know only $^3/_4$ of the group wants jelly."
*Researcher's comments:* This group seemed unaware that their problem did not clearly state an actual question until this fact was pointed out by classmates. When asked if there were other ways of modeling the problem other than actually making sandwiches, the group seemed puzzled, as were several other students in the class. When asked, *What does* $^3/_8$ *represent?* The group explained that you get $^3/_8$ by "multiplying the numerators and putting that answer over where you multiplied the denominators together. $^3/_8$ shows how many students want jelly." Then I asked them, "OK, so if you are going to make the sandwiches with the class, how many sandwiches will you need to make?" There were some audible "Ohs!" as some students grasped the point of my question. During the subsequent discussion, we specified that the problem was really indicating that, regardless of the number of people in the class, and therefore the total number of sandwiches to be made, only $^1/_2$ of the entire class wanted peanut butter and jelly, and only $^3/_4$ of that $^1/_2$ wanted grape jelly on their sandwich. One preservice teacher then made the comment, "OK, so we're taking part of a part!"
*Locus of authority:* External
*Problem 2. There are 6 students in class, each student received and ate* $^1/_2$ *an apple. How many apples did they eat altogether?*
*6 x* $^1/_2$ *= 3*
*Models and Representations:* PSTs recommended modeling the problem with apples and showed a picture of six students, each with ½ of an apple.
*Why does the fraction multiplication algorithm solve your problem?* "You can see that if each of the 6 students gets $^1/_2$ an apple, then you can add $^1/_2$ six times and get 3. That's the same as saying 6 x $^1/_2$ = 3
*Researcher comments:* This problem was interesting because it showed the use of a whole number being multiplied by a fraction. It also provided an example of preservice teachers visualizing fraction

multiplication as repeated addition. The group's solution was also noteworthy in that it was written as 6 x $^1/_2$ rather than $^1/_2$ x 6 or "$^1/_2$ of 6".
*Locus of authority:* Internal. The locus of authority was judged to be internal with this group because they were focused on reasoning within the problem context rather than on the computational process.
*Problem 3. Margaret wants to make her Super-Puppy a cape. Her fabric is $^1/_3$ ft. long by $^1/_4$ ft. wide. How big is the cape?*
*$^1/_3$ x $^1/_4$ = $^1/_{12}$ ft.*
*Models and Representations:* PSTs provided a rectangular diagram to accompany their problem.
*Why does the fraction multiplication algorithm solve your problem?* "It makes sense to use an area representation here. So, we showed it was $^1/_3$ foot on one side and $^1/_4$ foot on the other side, and then we multiplied to get the answer."
*Researcher comments:* As the group read and explained their problem, they encountered some objections from their classmates, notably from one who sewed. "$^1/_{12}$ foot doesn't make sense." I asked the class what $^1/_{12}$ of a foot was and a student replied, "1 inch." Others in the class quickly agreed that $^1/_{12}$ of a foot couldn't be right. I then encouraged the class to think about how they could go about making Super-Puppy's cape. The seamstress pointed out that $^1/_3$ foot would be the same as 4 inches and that $^1/_4$ foot would be 3 inches and so the cape would be 12 *square* inches. Another student then pointed out that this was the same as $^1/_{12}$ of a square foot. At this point, several students stated that they did not think elementary students would understand this problem well because of the measurement complications.
*Locus of authority:* Both internal and external. I ranked this group's work as showing both internal and external authority because, while they attempted to make a diagram to support their computations, the diagram and subsequent discussion revealed that they did not understand completely what the diagram and computation represented.
*Problem 4. Maggie the mechanic had $^5/_6$ of a quart of oil. She only wanted to use $^1/_3$ of that quart. How much oil did she use? $^5/_6$ x $^1/_3$ = $^5/_{18}$*
*Models and Representations:* PSTs made a drawing of a jar with showing $^5/_6$ full and then drew another jar pouring out $^1/_3$. Then a third jar was drawn showing a lesser amount than $^5/_6$ labeled $^5/_{18}$. The PSTs also suggested using beakers and water to model the problem.
*Why does the fraction multiplication algorithm solve your problem?* "Maggie started out with $^5/_6$ of a quart of oil, then she used some oil, $^1/_3$ of a quart. So, we're showing how much is left in the bottle. If you multiply the numerators together and the denominators together, then you get that amount."
*Researcher comments:* The group explained their problem, its diagram, and the solution process to the satisfaction of their classmates. I then asked them why they drew a second jar, labeled with $^1/_3$. "Is this a separate container of oil?" I asked. The group replied, "No, it's really being poured out of the first one, but that's kinda hard to draw." I replied, "OK, can you show me where the 18ths come into play here?" Some of the group members pointed to the algorithm and explained, "It comes from multiplying 6 times 3." To which I replied, "OK, I see, but how can you show the 18ths in your diagram?" At this point, most of the class was puzzled, but a couple of students asked, "Could we use paper-folding here, instead?" (I account for this by some of the students having completed the assigned readings in the van de Walle text prior to class.) I encouraged them to try this approach, and following the creation of their successful model, we ended the discussion of this problem by discussing how the answer represented $^5/_{18}$ or nearly $^1/_2$ of one quart and we also discussed why this physical representation illustrated the solution better than the original drawings.
*Locus of authority:* Both internal and external. I ranked this group's work as showing both internal and external authority because, while they attempted to make a diagram to support their computations, the diagram and subsequent discussion revealed that they did not understand completely what the diagram

and computation represented. The locus of authority for some of the students changed to internal, however, as they decided to use the paper-folding model and could explain its significance.

*Problem 5. Izzy was making cookies. The recipe called for* $^1/_3$ *cup of flour. If Izzy wanted to make* $^1/_2$ *of the recipe, how much flour did she need to use?* $^1/_3 \times {}^1/_2 = {}^1/_6$

*Models and Representations:* PSTs recommended modeling with cups and flour and also made a drawing of a series of cups, showing $^1/_3$ of a cup with half of the contents marked out, and then a second cup showing $^1/_6$ cup of flour.

*Why does the fraction multiplication algorithm solve your problem?* "Well, the original recipe calls for $^1/_3$ cup of flour but since you're only making $^1/_2$ of the recipe, you only need $^1/_2$ that much. So, multiplying by $^1/_2$ is like dividing the recipe in half here."

*Researcher's comments:* The group commented that their problem was, "Sort of like one of the NAEP problems." They explained the problem situation and their application of the multiplication algorithm to solve it. The group was able to explain to their peers that the solution represented $^1/_6$ of one cup. When I asked them if they could show a different diagram, they were initially bothered until one student exclaimed, "We could use paper-folding here, too!"

*Locus of authority:* Both internal and external. I ranked this group's work as showing both internal and external authority because, while they created a diagram and could explain the context of the problem and its solution, they experienced difficulty in applying more than one representation to the problem. It is not clear that if they had not seen paper-folding used in the previous problem, whether they would have thought of using it here.

These problems and their solutions were recorded on large sheets of poster board with markers and posted in the classroom during our subsequent discussions. For the sake of brevity, only one set of word problems developed by the methods classes is explored. However, this class was extremely representative of the problems developed in the other sections. The author asked the PSTs to solve their problems and followed with a class discussion concerning why the fraction multiplication algorithm provided a correct answer to the problems they constructed. The PSTs, were asked to respond to the questions, *So what does this mean? Why does the algorithm solve your problem?* They were asked to respond to these questions within their small groups and record their answers on posters.

## Summary Analysis of the PST Created Problems

Analysis of the problems created by the groups reveals several interesting features. First, most of the groups wrote problems that used only proper fractions. Only one group chose to use a whole number in a problem and none of the groups chose to use a mixed number or an improper fraction. Second, at least one group in each methods class experienced some difficulty in writing a question that made complete sense either in terms of grammar, punctuation, or in phrasing an answerable question. Third, in choosing the numbers for their questions, most groups chose familiar fractions, such as $^1/_2$, $^1/_3$, or $^3/_4$. In our class discussion, I asked each group to explain how they chose the numbers for their problem. They commented that they chose what they thought were "easy numbers" and did not think ahead about whether the numbers they chose would be easy for them (or their students) to model or diagram. Fourth, until the groups developed their own problems, they did not appreciate the intricacies associated with choosing appropriate pedagogical models to illustrate fraction multiplication. For example, at the close of our discussion of Problem 1-The Sandwich Problem, I asked them what models we could use to illustrate the problem in place of making sandwiches. They did not appreciate the differences in using a set model versus an area model until we attempted to illustrate the solution in both ways.

After the problems were posted in the class, and prior to our discussions of each problem, we discussed as a group which problem they ranked as most difficult (Problem 1) and which they thought would be easiest for students (Problem 3). The PSTs thought Problem 1 would be the most difficult for students because of how it was worded and because three numbers appear in the problem. They chose Problem 3 as the easiest problem because they found the group's diagram easy to understand. At the close of the class, I asked them to re-rank the group problems. This time, they ranked Problem 2 as the easiest and Problem 3 as the most difficult. The responses reported in the preceding section are consistent with the types of responses I have encountered in several semesters of using such tasks in the methods class.

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# PRESERVICE TEACHERS' REFLECTIONS ON THEIR INSTRUCTIONAL PRACTICES

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*Purpose of this study was to examine the nature of preservice secondary teachers' instructional strategies and views about teaching and learning mathematics as they planned and taught algebra lessons. The study also looked at how the preservice secondary teachers' views influenced their instructional practices as a result of their participation in reflection activities. The results of this study revealed that reflection activities helped the preservice secondary teachers identify strengths and weaknesses in their lessons and teaching. They also began to think about issues associated with planning, teaching, and assessment as a result of self-reflection and analysis their own thinking.*

## Introduction

Reflection is defined as looking at one's experiences, making connections with his/her thoughts and feelings, then utilizing in actions. Reflection-in-action is described as one's thinking about the teaching/learning process or when he/she is engaged in teaching. Effective reflection-in-action appears when one changes his/her teaching approach and recognize that his/her way of teaching is not working. Reflection leads to building new understandings to describe one's actions for teaching (Schön, 1983).

Reflection is viewed as a key element in an on-going learning process since teachers continue to learn about teaching and about themselves through reflection (Llinares & Krainer, 2006). In addition, learning to teach is a complex process that different factors (e.g., teacher's content knowledge, pedagogical content knowledge, organization of content for teaching) affect the nature of teachers' decisions before, during, and after teaching. In this process, reflection and development of reflection activities may enrich teacher's learning experiences and teaching practices as they become more effective teachers of mathematics (Kaminski, 2003). Davis (2006) suggests that tasks, allowing teachers to integrate different aspects of teaching such as learning, teaching and instruction, be used to promote productive reflections. To answer the question of what makes a reflective teacher, we need to focus on what types of experiences and tasks preservice teachers would need in order to become effective teachers of mathematics.

Curriculum reform for mathematics teacher education has been considered as a crucial aspect in education. With assistance from the World Bank, Development of Ministry of Education Project–Higher Education Council restructured Turkish teacher education programs to increase quality in all primary and secondary education programs in education colleges (Higher Education Council [HEC], 2007) and proposed two teacher education programs. One of these programs, offered under the guidance of education colleges, required high school students to take the nationwide university exam to enroll in this program. The other teacher education program, housed within education colleges, is designed for college graduates with a Bachelor's degree in mathematics. The latter is called Masters of Science degree without thesis (4+1), and in this program, preservice teachers are required to enroll in courses in two major areas: pedagogy courses and mathematics teaching methods courses.

I present data from a semester–long study conducted enrolled in methods courses. In an attempt to investigate how preservice teachers' views about teaching and learning of mathematics influenced their instructional practices, they were encouraged to plan and teach lessons collaboratively and participate in reflection activities. I describe the results of the study and discuss pedagogical implications of these results in a final section.

## Background

Research studies emphasized teachers' experiences as they develop and teach lessons as a focus for reflection (e.g., Jaworski, 1998). Over the past two decades, researchers (e.g., Artzt & Armour–Thomas, 1999; Artzt & Armour–Thomas, 2002; Shulman, 1986) have investigated the relationships between teachers' views and their instructional practices. In an attempt to answer the question of what makes a reflective teacher and explore the relationship between teaching practice and reflection, research studies (e.g., Fendler, 2003; Marcos & Tillema, 2006) have been conducted. Researchers (e.g., Jaworski, 1998) emphasized the importance of enabling teachers to reflect on their practice as a means for the improvement of mathematics teaching. However, these studies have been challenged to explain what has been learned (Darling-Hammond & Youngs, 2002).

On the one hand, Nicole and Crespo (2003, cited in Llinares & Krainer, 2006) suggested that introducing analysis and reflection through teaching practice provide preservice teachers with better opportunities to integrate theory and practice. On the other hand, research studies, conducted with preservice teachers in Turkey, revealed that preservice teachers were limited to discussions and reflections on teaching practices in their teacher education courses (e.g., Çakıroğlu & Çakıroğlu, 2003; Sahin–Taskin, 2006). Moreover, many teacher education programs in Turkey have provided preservice teachers with inadequate opportunities to engage them actively in developing and teaching lessons and to help them connect research and practice as well as develop research–based teaching strategies. I agree with Alp and Şahin–Taşkın that an increased attention should be given on research studies examining preservice teachers' reflections and practices in Turkish teacher education program.

## Theoretical Framework

This study is framed by the research of Artzt and Armour–Thomas (1999). Artzt and her associate posited a model with reflection activities (i.e., pre–lesson, post–lesson reflections, and self assessment) to facilitate preservice teachers' analyses of their views and instructional practices before, during, and after teaching. The model was influential in my descriptions of preservice secondary teachers' reflection activities and was also used as a guide as preservice teachers reflected on their instructional practices. For instance, the preservice teachers were asked to describe how they would teach their lessons by writing reflections on the following components suggested by Artzt and Armour–Thomas: (1) goals for students, (2) knowledge for students, (3) knowledge of content, (4) knowledge of pedagogy, (5) the teacher's role in the lesson, (6) the students' role in the lesson, (7) anticipated difficulties and (8) resources used to get ideas and criteria for selection. These items of the pre–lesson reflections were aimed at helping preservice teachers consider goals of the lessons for their future students. The four components (1–4) were used to encourage preservice teachers to utilize their content knowledge and pedagogical strategies to design their lesson. The three items on the list (5–7) were expected to assist preservice teachers to envision the engagement and interaction between the teacher and their future students, and to be aware and plan for possible difficulties that might occur during their teaching. The last item (8) was designed to encourage preservice teachers to utilize different resources to improve their knowledge and to consider alternatives to design their lesson.

## Methods

This study sought to examine the nature of the preservice secondary teachers' instructional strategies and views about teaching and learning mathematics as they planned and taught algebra lessons. Thirty–eight preservice teachers who were enrolled in secondary teacher education program at a university in the north eastern Turkey (i.e., Masters of Science degree without thesis) participated in the study. Data were collected through the preservice teachers' written reflections, activities, class discussions, and observations in a semester–long study. The preservice teachers were placed in groups of 2 or 3, and each group was asked to choose a topic in algebra. Although members of each group would be teaching the same topic, each preservice teacher was asked to plan and teach individual lessons with two components: 1) Teaching strategies (e.g., student–centered, teacher–centered, problem solving) and 2) Use of various teaching technologies. Since the preservice teachers had a lack of experience in designing a lesson plan, they were asked to use a lesson plan from Ministry of Education as a guideline. Having determined mathematical topics for their lessons and written reflections on the eight components suggested by Artzt and Armour–Thomas (1999), each preservice teacher taught his/her lesson in 25 minutes to their classmates acting as students. In post–lesson reflection, they were asked to reflect on the entire experience considering factors that influence their teaching and ideas to improve their lessons if they were to teach it again. These reflections provided perspectives about what factors influenced preservice teachers' planning and teaching as well as whether or not their views about teaching and learning of mathematics changed as a result of their participation in this study.

## Results

### *Pre–Lesson Reflection*

In the pre–lesson reflection, the preservice teachers discussed the resources such as materials, textbooks, and teaching strategies that they considered utilizing for their lesson planning and teaching. Twenty–seven preservice teachers emphasized that teacher-centered approach would be appropriate for their teaching. Eleven preservice teachers considered student-centered teaching strategy as an effective way to teach their lessons. Seven of these eleven preservice teachers considered using hands-on material for their lesson. Consider the following excerpts from their reflections:

*S1:* I think the most effective way to teach this topic is teacher-centered approach. I would first introduce the key concepts related to this topic. Then, I would solve examples to have students grasp the concept. At the same time, students would be asked to solve problems. Students should listen carefully and participate actively in class activities. I plan to teach my lesson using overhead, I think I would not have any difficult getting students' attention.

*S2:* In my lesson, role of a teacher is to introduce key concepts and then guide students to help them to learn by themselves. With this pre-knowledge, students could be active in the learning process and develop their skills. At the beginning of the lesson, I plan to use real life examples to attract students' attention to the lesson. I will include hands-on material to teach my lesson, but I am going to utilize discovery learning. I am going to ask them questions to have them involve in the lesson. My only concern about the lesson is interaction between me as a teacher and students. That is why I am thinking of using simple language to teach my lesson.

*S3:* I will be guiding the students in my lesson. I am going to help them to reach the goal by asking them questions. My only concern about this lesson is related to the examples. Students might have difficulty in solving examples and comparing their results. Since I will not use teacher-centered approach, it won't be difficult to get students' attention. In addition, student-centered approach would be extraordinary and enjoyable for them. I will give the key concepts and write notes on the board. After that I will start asking questions to students.

In his reflections, S1, a preservice teacher, thought that the most appropriate way to teach his lesson was to use teacher-centered strategy. He also wanted his students to be active in problem solving during his lesson. Therefore, he planned to use an overhead as a teaching technology to attract students' attention during his teaching. S2 considered student-centered teaching strategy with using discovery approach for her lesson. She said the use of simple language (e.g., describing concepts in her own words) in her lesson would help her increase interactions among her students. S3 wanted to utilize a student-centered teaching strategy for her lesson. She said it would be easy to engage her students in the lesson by using a question-and-answer technique in a student-centered approach.

All three preservice teachers were asked to consider the possible difficulties that might arise during their teaching. S1 thought that teaching the lesson with an overhead projector may not motivate his students to participate in the lesson. Both S2 and S3 were concerned about interaction between the teacher and the students. To prevent lack of communications among students and attract their attention for the class activities, the preservice teachers, S2 and S3 said they would create a student-centered learning environment.

Pre-lesson reflections were informative for the preservice teachers. The preservice teachers were given an opportunity to verbalize their ideas and concerns as they envisioned interactions among their students. Descriptions of pre-lesson reflections also provided opportunities for them to understand their envisioned practices and decision making processes.

*Post-Lesson Reflections and Self Assessment*

Most of preservice teachers were concerned about finding ways to make introduction of mathematics concepts interesting or strategies to interact with their students. Although the preservice teachers emphasized that the lessons could be taught using different teaching strategies, most of them claimed that their teaching strategies were more appropriate for their lessons. Twenty–seven preservice teachers described teachers as dispensers of knowledge and chose a teacher–centered approach for their lessons. The remaining eleven preservice teachers preferred a student–centered approach to guide student learning.

Regarding the weakness of their teaching, eighteen preservice teachers found rather difficult to implement the teaching strategies. Thirteen of them thought that they went over the key concepts quickly. Eleven of them were concerned about lack of communication between themselves as a teacher and their students. Seven preservice teachers had difficulty selecting appropriate examples. Following excerpts illustrate the preservice teachers' post-reflections on their teaching:

*S4:* I had a problem with my material. I wanted to put on the board but I had a problem with magnets. Somehow magnets did not hold the material on the board. I ended up spending too much time with material.

*S5:* I knew what I needed to teach, but I could not help myself going over to content quickly.

*S6:* I lectured the students. They were not active during the teaching because I did not ask any question.

*S7:* Number of examples I presented was not enough. I could not relate to real life. I should have solved more examples.

In reflecting on their teaching, the preservice teachers identified appropriate teaching strategies, introduction of the concepts, and communications with students as factors influencing their instruction.

Regarding the strengths of their teaching, one third of the preservice teachers said that that they had a good introduction and selected appropriate examples for their lessons. Some of the preservice teachers, who implemented student–centered teaching, were satisfied with the questions to engage students in the lesson. Few pointed out class management, interactions with students, and the use of technology as the strengths of their teaching. Consider the following excerpts:

*S8:* I did not have any problem introducing key concepts, but it would have been easier for students to understand if I had given examples that started from basic to more complex.

*S9:* I wanted to get the students attention at the beginning of the lesson but I could not say something to attract their attention. Overall, I was able to choose appropriate examples. I selected questions from easy to more complex questions. I believe they were good examples introducing the math content in a logical order.

*S10:* The best part of my teaching was students were active in the learning. That is why I chose student-centered teaching method. If I could teach this topic again, I would also include slides in my teaching.

*S11:* I thought material that I used was simple and fun. I thought I was able to have student involve and participate in their own learning.

The preservice teachers reconsidered several components of their teaching when asked what they would change if they had an opportunity to teach this lesson again. Eleven preservice teachers said that they would use different teaching materials (e.g., worksheets, transparencies) for their lessons. Six preservice teachers wanted to change their examples and pay more attention to interactions with students in their teaching. Due to their difficulties in communicating with students and engaging the class in the lesson, five of them said they would change their teaching strategies (see Table 1).

Table 1. *Preservice Teachers' Suggestions about Their Lesson*

| Changes on teaching | Frequency of responses |
|---|---|
| Pay attention to communication with students | 6 |
| Spend more time on contend | 7 |
| Changing teaching strategies | 5 |
| Selecting appropriate examples | 6 |
| Using different teaching material | 11 |
| Utilizing time efficiently | 3 |

Although the preservice teachers were given an opportunity to teach this lesson once, they expressed various components in their lessons and teaching that they wanted to change. Consider the excerpts from their written reflections:

*S12:* This was my first time for teaching. I could not manage the class well. I don't think I paid enough attention to my students. Overall students were listening. After my teaching I realized that it was very important to plan the lesson ahead of time. I lectured in my teaching but if I could teach it again I would use an activity to teach Cartesian coordinate system.

*S13:* I prepared various math problems, but I was not successful at engaging students in solving the problems. I should have practiced and utilized different questioning techniques.

*S14:* Presenting the concepts with manipulatives was beneficial but I did not spend enough time on introducing the key concepts related to my lesson topic. If I could teach again, I would have introduced the concepts slowly.

In their reflections, it appears that the preservice teachers observed the outcomes of their instructional decisions and actions in their teaching. As a result, they began to think about how mathematical concepts could be represented to teach effectively and consider utilizing teaching techniques, instructional materials, and activities to actively engage students in learning mathematics.

## Discussion

In Turkish teacher education programs, preservice teachers take internship, called School Experience, to be prepared for teaching and exposed to various aspects of the profession under the guidance of both supervisor and cooperating teacher. Preservice teachers are expected to observe mathematics classes and write reports about the entire experience at their internship schools. Preservice teachers do not receive much benefit from classroom observations. Having stated that preservice teachers' internship experiences involving only class observations do not prepare them for teaching and preservice teachers must actively participate in classes by assisting teachers and students, we as educators can create a rich learning environment in which preservice teachers are encouraged to develop lessons collaboratively and teach to their students. In methods courses, preservice teachers should be challenged with issues associated with planning, teaching, and assessment that emphasize self-reflection focusing on the analysis of the teacher's own thinking and dealing with students in a way to plan their education (Ticha & Hospesova, 2006).

In this study, prior to their internship in schools, the preservice teachers gained experience in planning and teaching lessons as they worked with their classmates collaboratively. By reflecting on their teaching and observing different teaching strategies, they identified their weaknesses and strengths in their lessons, and analyzed issues in teaching. It was observed that with the reflection activities, the preservice teachers had opportunities to work together to refine and revise their lessons and improve their teaching.

The model of Artz and Armour–Thomas (2002) provides an insightful framework to describe preservice teachers' analyses of their views and instructional practices before, during, and after teaching as well as assessment of their thoughts and decisions in a structured way. Under the supervision of their professors or supervising teachers, this model could be an excellent tool for professional growth, and preservice teachers utilize this model to assess and reflect on their teaching in a structured way. If preservice teachers get used to thinking about their lesson in such a structured way, it could become a natural part of their reflective practices.

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# PRESERVICE TEACHERS' DEVELOPMENT OF WHOLE NUMBER CONCEPTS

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As learning unfolded in an elementary education mathematics classroom, a classroom teaching experiment was conducted to document preservice teachers'(PST) development of whole number concepts (Cobb, 2000). The development of PSTs' understanding of whole number concepts was emphasized in this study for two reasons. First, whole number concepts are a core component of elementary school mathematics; therefore, the importance of whole number concepts is inherent on future mathematics learning (National Council of Teachers of Mathematics, 2000). Second, PSTs entering the profession need a depth of knowledge necessary to support children's unique understandings of whole number concepts (National Mathematics Panel, 2008, National Research Council, 2001).

As stated by Hopkins and Cady (2007), familiarity with base-10 can prevent adults from fully comprehending various whole number concepts, as such, this study had PST's participate in an instructional sequence where learning tasks were situated in base-8. By having the PSTs reason solely in base-8, it provided them a similar mathematical experience to children learning the same concepts in base-10. In addition, the results of this study found that many of the PSTs reasoned in base-8 in similar ways that children reason in base-10, such as making mathematical connections between place value, number sense, properties of operations, and conservation of number (Huinker, Freckman, & Steinmeyer, 2003; Kamii, Lewis, & Livingston, 1993). Most importantly, PSTs were able to reconceptualize their understanding of whole number concepts, and as a result, strengthen their conceptual understanding of children's unique ways of thinking.

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# PEN-PAL PARTNERSHIPS IN THE PREPARATION OF PRE-SERVICE TEACHERS

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Research on posing mathematical tasks has focused on student-independent task analysis (e.g. Silver, 1996). In contrast, this study used student-dependent analysis by studying responses to mathematical tasks. Continuing the work of Norton and Rutledge (2007) and Crespo (2003), this research examines NCTM process standards (NCTM, 2000) and levels of cognitive demand (Stein et al., 2000), elicited by tasks designed by preservice teachers (PSTs).

PSTs enrolled in a mathematics course were partnered with precalculus high school students and exchanged 6 biweekly letters. In the letters the PSTs posed discrete mathematics tasks seeking to elicit NCTM processes and high levels of cognitive demand, the highest of which is 'doing mathematics' (Stein et al., 2000). The PSTs attempted to predict the NCTM processes and levels of cognitive demand in which their student partners would engage. They later assessed students' actual responses. This arrangement provided PSTs with an authentic experience in designing individualized tasks and analyzing student thinking.

In this 3rd iteration of the study, PSTs engaged in weekly letter-writing workshops in which they shared student responses and worked together to make inferences about the students' mathematical thinking. At the conclusion of the project PSTs completed surveys on their experiences. Our research team is using this data to answer the following questions:

- Did the project help PSTs pose tasks that elicited more processes and higher levels of cognitive demand?
- What factors led to elicitation of processes and high levels of cognitive demand?
- Did the PSTs' predictions of responses improve?

Preliminary results from analysis of PST feedback show a positive response to the project; they found it valuable in learning to pose mathematical tasks and investigate student thinking. Ongoing data analysis will compare processes and levels of cognitive demand elicited from week to week, as well as identifying important factors in PSTs' growth.

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# THE ROLE OF TEACHER IDENTITY IN LEARNING TO ATTEND TO STUDENT THINKING IN MATHEMATICS TEACHER EDUCATION

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Classroom teachers and researchers suggest that understanding student identity, including beliefs, dispositions and self-understandings, may allow classroom teachers to better meet the needs of students as learners. Similarly, teacher educators also need to understand their students—teacher candidates (TCs)—as learners to better prepare them to teach. Research has shown that teacher identity influences a teacher's instructional decisions and shapes whether and how teachers construct and take up professional learning opportunities (Collopy, 2003). Comparing case studies of TCs in two elementary Masters-credential programs, we investigate how their *identity as narrative activity* (Sfard & Prusak, 2005) shapes their participation in practices constituting *attention to student thinking* (Kazemi & Franke, 2004) in math methods courses. Sfard and Prusak (2005) frame the activity of identifying as fundamentally discursive, and define identity as "collections of stories about persons… narratives that are *reifying*, *endorsable* and *significant*" (p. 16, italics in original). This narrative-defined identity can be useful, even as people define their identity differently to different people, because it is "the activity of identifying rather than its end product that is of interest to the researcher" (p. 17).

The two programs in the study attract TCs with different personal and professional backgrounds and structure their field experiences differently. The math methods courses in both programs have the same instructor, goals, activities and assessments. Data consist of video recordings of all course sessions, TCs' course assignments, observations of TCs' teaching, and interviews with selected TCs. Analysis utilized methods of narrative and story analysis to characterize how the TCs see themselves as teachers and learners, grounded theory to develop categories for TCs' participation in the practices of the methods courses, and comparison across case studies to reveal key relationships between aspects of identity and participation.

The poster will highlight how identity as narrative mediates how TCs are able to learn to attend to student thinking and how they integrate experiences with student thinking into their narrative identities. Our analysis reveals three trends: (1) TCs who see themselves as teachers and their work as teaching, rather than someone who is "learning to teach," situate their analyses of mathematical thinking in their broader experience with students; (2) TCs whose vision of mathematics teaching is centered around promoting student engagement tend to approach attention to student thinking as an assessment of the effectiveness of an activity; (3) For TCs who describe their own mathematics learning experiences as personally damaging, engaging in mathematics through student work supported the redefining of their relationship to mathematics.

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# HOW PRE-SERVICE TEACHERS NOTICE PROPERTIES AND WHAT WE CAN LEARN FROM IT

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Early algebra is a topic that is of great interest in the mathematics education community. Pattern finding is an aspect of early algebra that is particularly important in the elementary school. Property noticing is the fourth level of the Pirie and Kieren (1994) theory [PK] of growth of mathematical understanding following image having. When students notice properties it is usually closely associated with the image that they have constructed from the previous levels. They are now about to take that image and begin to identify special phenomena or properties that might be attributable to it. An example would be when a student is able to take an image of ½ and ¼, noticing that ½ > ¼ and that adding these two quantities requires a sum greater than either one. These are properties of fractional addition of unlike denominators that a student may notice. The focus of this case study is to determine what properties a pre-service teacher [PT] notices in a teaching experiment that focused on algebraic reasoning. In this case, algebraic reasoning refers to the ability of the PT to generalize and justify rules that describe patterns. In this analysis the sources of data are Edy's transcripts and written class work from a three-week teaching experiment. Elementary pre-service teacher Edy, a pseudonym, was a 21 year-old Caucasian female in her first semester and had not student taught. Edy was selected for study because her algebraic reasoning was strong even at the beginning of the teaching experiment and she was able to quickly and efficiently generalize and justify rules of the patterns involved. The transcripts were examined and coded for instances of property noticing, with a second coder brought in to assure inter-rater reliability. Then the coded items were examined for common themes. To begin with her property noticing took the form of recognizing properties in the model that she was constructing.

> Edy: Because each time one [block] goes together, you lose two sides. Because here's a side and here's a side and put them together and you lose two sides.

Over time the properties that Edy noticed became increasingly mathematical.

> Edy: Yeah. So, the pentagon's going to be 3n plus two. Because there is a pattern. 1N plus two, 2N plus two, 3N plus two.

Edy noticed properties in a variety of ways that include but are not limited to: recognizing a property of the model, recognizing a pattern in the model, simplifying a pattern in the model, noticing a similarity between two models, or noticing an error in the model.

Property noticing as a level within Pirie and Kieren's growth of mathematical understanding is an important element in educational theory that could inform our practice. If there are certain properties of a task that the PT recognizes, then there could be ways for the PT to encourage learners to find them. This could have bearing upon the way in which we construct our teacher preparation curriculum. Applications can be made to the way that the content is presented and to the way that we question and encourage our pre-service teachers.

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# PRESERVICE TEACHERS' PERSPECTIVES ON A CULTURAL DIVERSITY MATHEMATICS ASSIGNMENT

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This study examined preservice teachers' perspectives on an assignment designed to prepare them to teach in culturally diverse mathematics classrooms. Three cohorts (n=76) of elementary and middle school preservice teachers were asked to research the available literature for an article that specifically addresses the teaching of mathematics to students who are culturally different than themselves. Data were collected from the written reports that specifically asked students to answer the following questions: why did you choose the article, what are the cultural differences between you and the culture(s) discussed in the article, and what is the value of using the strategies discussed in their article. Data was analyzed, coded, and categorized using analytic induction.

*Why this article?* Most preservice teachers chose an article for their *professional preparation* because they felt it would help them become better teachers. Preservice teachers that wanted to learn about a different culture were placed in the *personal gain* category. Lastly, as the title suggest, some preservice teachers chose an article for *both professional preparation and personal gain*. An example of this type of response is, "Latinos are [a] different culture than my own. I do not know very much about Latino education and home life. With the trends in [this state], more Hispanic children will be in schools. This article will give me an idea of how to relate to this different culture." Three students did not answer this question.

*How is the Culture Different*? Eight categories were mentioned as cultural differences between the preservice teacher and the culture they chose to study. Schooling issues such as instructional strategies, curriculum, resources, tracking, class size, and teacher expectations was cited the most (19%). Racial differences (17%), geographic location (16%), language (14%) and Socio-economic status (13%) were next among the most commonly mentioned cultural differences. Differences in family (size, parenting, structure) and customs (birthdays, traditions) were both mentioned by thirteen percent of the preservice teachers. Comments related to differential societal views about cultures (labeling, racism, prejudices) were mentioned by 3% of the preservice teachers.

*What is the Value*? Most preservice teachers found the teaching strategies as having *student value* (i.e., making mathematics more enjoyable and increasing student self-confidence). Thirty-eight percent discussed the *professional value* of the strategies, such as a better understanding the students. About 27% of preservice teachers assigned *conditional value* to the teaching strategies, discussing how the strategies helped certain groups such as English Language Learners. Finally, 5% of preservice teachers discussed the *personal value* of the teaching strategies.

The finding from this study are encouraging and suggest that preservice teachers want to learn ways to reach all of their students. Although their reasons may vary, preservice teachers want opportunities to explore multicultural activities in their mathematics methods courses.

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# SUPPORTING TEACHERS' DEVELOPMENT OF COHERENCE THROUGH A FOCUS ON MATHEMATICAL STRUCTURE

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*We present our emerging model for supporting the development of coherence in teachers' mathematical understandings. Key to this model is the notion that coherence involves coming to see a variety of mathematical ideas as conceptually and structurally related and that while teachers are aware of particular examples that employ a particular mathematical structure, they are unaware of the benefits of connecting the examples. We report on our efforts to support teachers in making these connections in an online content course for teachers.*

## Objectives and Purpose of the Study

The importance of teachers' mathematical knowledge has been well documented in the literature (Ball, 1993; Bransford, Brown, & Cocking, 2000; Ma, 1999; Shulman, 1986) and increasing teachers' mathematical knowledge continues to be a major focus in both education research and policy (Greenberg & Walsh, 2008; National Mathematics Advisory Panel, 2008). Despite the fact that there is widespread agreement as to the importance of teachers' mathematics knowledge, there is little consensus as to the particular content and structure of that knowledge. In this paper, we will discuss a perspective on the question of the mathematical knowledge that is needed for teaching mathematics in which teachers learn mathematics as a coherent and interconnected system of ideas. We focus on a segment of teacher professional development that was designed to support teachers' developing deeper and more sophisticated mathematical understandings that will position the teachers to make connections between a variety of mathematical ideas regularly taught in middle and high school mathematics courses. In particular, the focus of this paper will be how an explicit focus on the mathematical structures of sets, operations, and equivalence relations can serve as a catalyst for conversations that support coherence in teachers' mathematical understandings.

## Theoretical Framework

Ball and her colleagues (Ball, 1993, 2007; Ball, Hill, & Bass, 2005; Ball & McDiarmid, 1990) have focused on understanding the special ways one must know mathematical procedures and representations to interact productively with students in the context of teaching. Their pioneering work has succeeded in identifying a statistical relationship between this *mathematical knowledge for teaching* (MKT) and student achievement (Ball, et al., 2005; Hill, Rowan, & Ball, 2005). We extend this work by focusing not only on particular mathematical understandings but also the conceptual structures within which those particular understandings lie. Our reason for this focus is pragmatic:

> If a teacher's conceptual structures comprise disconnected facts and procedures, their instruction is likely to focus on disconnected facts and procedures. In contrast, if a teacher's conceptual structures comprise a web of mathematical ideas and compatible ways of thinking, it will at least be possible that she attempts to develop these same conceptual structures in her students (Thompson, Carlson, & Silverman, 2007)

Rather than focusing on identifying the mathematical reasoning, insight, understanding and skill needed in teaching mathematics, we focus on the mathematical understandings "that carry through an instructional sequence, that are foundational for learning other ideas, and that play into a network of ideas that does significant work in students' reasoning" (Thompson, 2008). We refer to these understandings as *coherent* understandings: powerful, generative "big ideas" from which an understanding of a body of mathematical ideas and its relation to other bodies can emerge.

It is important to note that coherence is not a characteristic of one's understanding of a particular mathematical idea, for coherence in curricula or students' understandings depends on the way in which they fit together (Thompson, 2008). This notion of coherence problematizes traditional mathematics teacher education efforts that seek to support teachers in "gain[ing] the ability to do the mathematics…and understand[ing] the underlying concepts so they will be able to assist their students, in turn, to gain a deep understanding of mathematics" (Musser, Burger, & Peterson, 2008). When a focus is on coherence, the emphasis is not just on doing and learning "the mathematics," but rather on developing a scheme of understanding within which a variety of mathematical ideas are connected and that can serve as a conceptual anchor for mathematics curricula and instruction.

In this article, we discuss our emerging model for supporting pre- and in-service teachers as they deepen and extend their mathematical understandings and develop schemas within which a variety of mathematical ideas are conceptually connected. First, we begin by exploring mathematical ideas that the teachers teach regularly and ostensibly know well. We then seek to problematize their current mathematical understandings by presenting the abstract mathematical structure that lies behind the math they teach and orchestrating conversations about examples from different mathematical contexts that have similar mathematical structure. We feel the shift to a higher level of abstraction is essential for two reasons. First, it is at this level that the mathematical ideas fit together—on the surface there is little similar about two numbers being equal and two polygons being congruent. It is only when one explicitly focuses on the mathematical structure that the similarities between examples or contexts can "do work" for teachers. It is our goal that teachers come to see the hundreds of concepts that make up the school mathematics curriculum as entailing a small number of big ideas that they can then orient their instruction around. Further, as they learn to engage with these big ideas and mathematical abstractions, they are developing a mode of discourse that they can use as they engage with particular examples of the big idea that can support students coming to see the connections. Second, following Blanton (2002) it is often the case that teachers need to legitimately engage and grapple with mathematical ideas to actually *think mathematically* and that is hard to do in contexts that the teachers (believe they) know well.

## Setting and Participants

In this article, we will discuss the first two weeks of 10-week course titled *Geometry and Geometric Reasoning*. This online graduate course is a content-based course required for the master's degree program in mathematics education at a university in the Northeastern United States. The course was developed and taught by the authors. Tasks and activities were taken from the *Re-Conceptualizing Mathematics* program (Sowder, Sowder, & Nickerson, 2008) and the authors' personal repertoire developed over a combined 25 years of work in mathematics teaching and teacher development. The data for this paper comes from one iteration of our

Online Asynchronous Collaboration model (Clay & Silverman, 2008a, 2008b) and includes private solutions, revisions, and commentary as well as discussions focused both on individual problems and emergent issues (Clay & Silverman, 2009).

The first two weeks of the geometric reasoning course focused on polygons and transformations. We began with the assumption that the teachers were familiar with, and teach or have taught, the ideas of congruence and transformation of polygons. We also assumed that, consistent with typical texts, instruction and state standards, teachers' understandings of congruence was grounded in the idea that congruent polygons have "the same size and the same shape." In addition, almost all teachers were familiar with the rigid transformations of reflection, translation, and rotation.

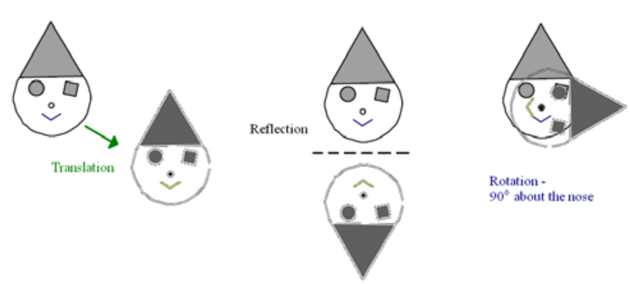
We began the unit by providing opportunities to explore mathematical ideas that the teachers currently teach and were ostensibly familiar with. For example, we initially focused on "standard" problems of the type: "given a polygon and a transformation find the resulting polygon." We then pushed on teachers' current understandings through focusing on the other half of the bi-relational relationship: "given two congruent polygons, you can find the transformation that transforms one to the other." Our goal was then to support teachers in relating their work and developing understandings of transformation to more abstract mathematical ideas like operations on transformations, composition of transformations, and properties of "the set of transformations" such as the set being closed (A set $S$ is closed under the operation $\circ$ if for all $a,b \in S,\ a \circ b \in S$ . Finally, students were asked to re-examine their understanding of congruence by redefining it from "same size, same shape" to "there exists a rigid transformation" from one polygon to another and explore the implications of this new understanding.

## Analysis and Results

*Engaging with the Mathematics*

Teachers were familiar with the notion of transformation and each was able to translate, reflect, and rotate figures in the plane. On a task that asked the participants to translate, rotate, and reflect a given figure, most provided diagrams consisting of the original image and the transformed image and few provided descriptions or explanations of what they did. A paradigmatic example of such a solution is provided below (all names are psuedonuyms).

Quinn: Working with Mr. Conehead here, I've performed the three transformations:
He was translated along the green vector.
He was reflected about the black center line.
And then he was rotated about his nose.



When engaging with the mathematics they teach, participants' posts were mater-of-fact descriptions of what they did. When the class discussed the posts, there was little conversation about the mathematics – conversations were either affective ("I really liked your diagrams") or focused on the technology used to generate the transformations (grid paper, Geometer's Sketchpad, Adobe Photoshop). Examples included:

Carla: I love that you named him! Anywhoo... I was wondering what you used to make your pictures. They are very nice.

Adam: Yours look like perfect replicas....where you able to copy them or did you draw them? If you were able to copy them, how did you do so? The only thing missing for your rotation is the point that its rotated about

Una: I enjoy looking at your work because it is easy to understand. I never thought of using Adobe Photoshop.

Opall: Did Photoshop do everything? If so, would you allow your students to use it or make them follow the directions in the book and produce accurate hand drawn transformations?

It has proven tremendously difficult to orchestrate generative mathematical conversations around mathematical content that teachers' believe they know well. This result is consistent with Blanton (2002), who noted the importance of "legitimate mathematical situations" in supporting teachers' mathematical and pedagogical development. While initial engagement with mathematical tasks that teachers are comfortable with has proven not to be a site propitious for the development of more coherent understandings, it has proven to be a good jumping-off point for discussions that focus explicitly on coherence in mathematical understandings.

*Mathematical Abstraction #1: The Set of Transformations and Operations on that Set*

While participants were familiar with transformations, the idea of transformations as a set that we could define an operation on was unfamiliar territory for the majority of the students. The following three excerpts demonstrate the 'newness' of the idea along with connections the teachers are able to make to something that they do 'know' and teach.

Carla: I also never heard the terminology "composition" of two motions. I remember in high school doing multiple transformations to an object, but the word "composition" never came up. To me in school it would be doing multiple translations, not doing a single translation with multiple steps as this suggests. I found that interesting, and created a shift in thought for me.

Wendy: I have taught reflections, rotations, and translations, but I had never heard of compositions either. I had always treated them as separate motions. But when I think about it, asking the student to use compositions forces them to see the big picture as the sum of the parts. I think both viewpoints are important to get a true understanding of what is happening. The student needs to see the whole as well as its parts.

Therin: Carla, I also don't remember learning about composition of rigid motions before. I am more familiar with composition of functions from teaching Algebra 2. I have a hard time thinking about the composition of rigid motions as a single rigid motions...

Carla admits that she's never heard of "composition" of two motions, although she has literally done the steps of composition. Wendy follows by saying she's taught translations but never heard of compositions. Again, she has done and taught the steps of the composition, just never heard of or used the mathematical terminology. She uses what we believe to be informal, everyday language to describe composition as the "sum of the parts." We believe it to be informal rather than making the connection here that 'summing' is also an operation. Finally Therin makes the connection to composition of functions from teaching Algebra II and is trying to come to terms with this new idea.

In the above excerpts, we see teachers engaging in mathematical discourse and thinking about mathematical abstractions in the context of transformations that they ostensibly already know. We believe that it is significant that these conversations do not take place in the context of school mathematics, but require moving beyond school mathematics and problematizing teachers' mathematical understandings (Cobb & Bauersfeld, 1995). Although we are only claiming at this point that we have problematized the participants understanding, the teachers each claim new understandings of some kind—Carla a shift in thinking, Wendy an impact on her teaching, and Therin connections between algebra and geometry.

*Abstraction #2: Equivalence Relations*

Another example in which students had knowledge but not coherence was in the study of congruence. Teachers are almost universally familiar with the informal notion of congruence of polygons as polygons that have "same size, same shape," and while this notion works for most of school geometry, it is not generative and does not easily generalize beyond the particular case of school geometry. It is possible to define congruence more formally using transformations: two polygons are congruent if there exists a rigid transformation that maps one onto the other. Intuitively this definition of congruence makes sense to the participants, but when asked to explore congruence more abstractly through the equivalence properties—reflexive, symmetric, and transitive—participants are challenged to reconcile their prior understandings with this new notion of congruence. As an example, consider the following posts in response to the task:

Use the meaning of congruence from the transformation point of view to decide:

a) If Shape 1 is congruent to Shape 2, is Shape 2 congruent to Shape 1?
b) If every shape congruent to itself?
c) If Shape 1 is congruent to Shape 2 and Shape 2 is congruent to Shape 3, is Shape 1 congruent to Shape 3?

*Figure 1*. Congruence Task

Adam: The definition of congruent: Two shapes are said to be congruent if one can be transformed into the other through isometry. Since, the definition clearly states two shapes, a single shape cannot be congruent to itself. …Yes, a = a, and 7 =7 and pi=pi, but those are values...not shapes. To say that a shape is equal to itself, seems silly to me. A shape is inherently discrete...from my perspective. So, to say it is equal to itself just seems improper.

Carla: Congruence, by definition, is similar but not the same. When explaining to my students the difference between "equal" and "congruent" I tell them that two congruent objects are so because their properties are the same, same lengths, angles, measurements, etc, but they are not equal because the two objects are still two different objects. They go by different names and in the real world two objects are different because they both occupy their own space. Comparing an object to itself is, what I would think, crossing over from congruence to equivalence.

Pearl: Carla – [C]ongruence and equality…are not the same, but they are related. I know when I have done them with my classes, we use the two kind of interchangeably. Shapes or parts of shapes would be congruent, and the measures are equal. So for example, two angles are congruent if they have the same measure and conversely, if two angles have the same measure, then they are congruent. For my geometry classes, that has been an important part of proving different statements.

Carla: I agree with you completely. I am just not convinced that a shape is congruent to itself. …[The] history that I found also said that the words "equals" and "congruent" throughout the last 4 centuries has been interchangable in translations of Elements. This has basically reinforced that they really are the same. In all this searching I didn't find anything that conclusively stated why we have the separate terms, but I would like to believe that there was at one time a reason. Why, if we specify different terms for equivalence in Geometry verses Algebra, don't we have different terms for addition? Or other operations?

In these posts we see evidence of the teachers' struggling to make sense of the mathematics at hand and reconcile it with their current understandings. We see Adam relying on the definition of congruence "from a transformation point of view" and concluding that this definition implies that a shape cannot be congruent to itself, apparently neglecting the identity transformation (which he was familiar with in other contexts such as the real numbers under addition or multiplication). Interestingly, his post also makes clear the most problematic issue that participants have with this task: "a = a, and 7 =7 and pi=pi, but those are values...not shapes. To say that a shape is equal to itself, seems silly to me."

Looking beyond Carla's informal and unfortunately mathematically incorrect use of the term *similar*, Carla expresses a similar sentiment as Adam, noting that the congruence has to do with the space the objects occupy. She is falling back on her intuitive knowledge rather than the mathematical definition of equivalence. As with many students the intuition comes from her first experience with equivalence, equality. Pearl shifts the conversation to how equality and congruence are really saying similar things, but apply to different kinds of things: "Shapes or parts of shapes would be congruent, and the measures are equal." We believe that Pearl is using the term "measure" to indicate particular values a measureable quantities can take on—in other words, real numbers. She is however not demonstrating clarity as she states, "we use the two interchangeably." We then see Carla still struggling with the different "kinds of equivalence" and posing a potentially significant question: "Why, if we specify different terms for equivalence in Geometry verses Algebra, don't we have different terms for addition? Or other operations?" We do not have data to claim that Carla understands the significance of the question she is asking, but she is asking about what else she might know that is only an example of something bigger and seeking coherence through that bigger idea. Finally, Nora seems to have come to terms with the definition of congruence but is wondering how and when one would decide to use it.

In these excerpts, we see teachers' impoverished understandings of equivalence and the beginnings of a discussion that holds the potential for supporting teachers in moving towards more generative understandings. Their subjective, qualitative notion of equality and congruence as "some kind of sameness" works for particular cases/examples. However, understanding congruence as an equivalence relation supports teachers in helping their students connect aspects of geometry, algebra, functions, etc.

## Discussion

Above, we argued the importance of teachers' coherent mathematical understandings. We have seen teachers willing and able to engage with the broad topics (transformations and congruence) at a limited level: given a particular transformation, they were able to perform that transformation. This activity was not problematic and they were willing and able to relate it to mathematics they already 'know' and teach. It is also clear that as the participants began to engage with transformations and equivalence at a higher, more connected level, they began to become less sure of their understandings. It is important to note that their confusion did not lie in the particular school mathematics ideas; rather it was in the mathematical connections between the school mathematics topics that the confusion became evident: *What do you mean you can compose transformations? Equality and congruence are the same thing?*

At its most basic level, transformations and congruence are mathematical ideas to be taught, but we argue that the significance of these topics for teachers does not lie simply in teaching transformations or congruence. Both are examples of larger mathematical ideas—an abstraction—that it is evident the teachers were not consciously aware of. Transformations *are* functions that map $\mathbb{R}^2 \to \mathbb{R}^2$. Congruence *is* an equivalence relation. While teachers are likely to have experience with transformations and congruence and *may* have some level of familiarity and experience with functions and equivalence relations, they are less likely to be aware of the various different ideas and contexts from which they are abstracted, how they are related, or the value of seeing the relatedness. It is this aspect of understanding—understandings of the ways in which a variety of ideas and contexts fit together—that makes one's understandings coherent (Thompson, 2008).

It is important to acknowledge that we cannot make claims that these experiences have supported our participants increasing their coherence in understandings of transformations, functions, congruence and equivalence—we do not have access to data that would allow us to make that claim (though we are currently involved in a study designed for that purpose). What we can say is that we have created an online system consisting of tasks and an interactional space within which mathematical abstractions of school mathematics ideas problematize teachers' mathematical understandings and support legitimate mathematical engagement. Further, engagement in conversations about the mathematical abstractions of day-to-day school mathematics was able to focus on how the specific examples of the abstractions are related and, thus, holds the potential for participants to develop coherence among previously unconnected mathematical ideas the teachers are currently teaching.

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# MATHEMATICAL KNOWLEDGE FOR TEACHING EXHIBITED BY PRESERVICE TEACHERS RESPONDING TO MATHEMATICAL AND PEDAGOGICAL CONTEXTS

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*Two independent research projects at a large Midwestern university used paper and pencil items to obtain information on different aspects of preservice teachers' Mathematical Knowledge for Teaching. Through happenstance occurrence, several preservice teachers participated in both studies. This paper examines the responses of this subgroup to items from both projects and shows that a richer picture of students' Mathematical Knowledge for Teaching is obtained when looking at students' Mathematical Content Knowledge and Pedagogical Content Knowledge in tandem.*

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One of the persistent issues in mathematics education is the preparation of qualified mathematics educators. A current trend in mathematics education has been one that attempts to describe and quantify types of knowledge required for mathematics teachers. Part of one framework was first proposed by Shulman (Shulman, 1986, 1987) and his colleagues (Wilson, Shulman, & Richert, 1987) which differentiated between content knowledge, pedagogical content knowledge, and curriculum. This framework has since been further defined for mathematics by Ball and others with their Mathematical Knowledge for Teaching framework (Ball, et al., 2005; Hill, Rowan, & Ball, 2005). A recent publication by Ball and others continues to refine and describe in greater detail these categories (Ball, Thames, & Phelps, 2008).

Several research studies have used these frameworks to examine and quantify the mathematical knowledge exhibited by preservice teachers. While most of these projects attempt to describe one aspect of the complete framework for a certain population of preservice teachers, there has yet to be a study that attempts to describe multiple aspects of Mathematical Knowledge for Teaching for a set of preservice teachers within a population. Through a happenstance occurrence, two independent research projects at Michigan State University did just that by collecting responses from preservice teachers who participated in both studies. This paper examines what information can be gained about their Mathematical Knowledge for Teaching when their responses to these two different assessments are examined in tandem.

## Literature Review/Framework

In Shulman's original article (1986), he outlines a shift in state teacher preparation requirements from one that was almost entirely content driven (with very little pedagogy) in the late 19th century, to one that focused heavily on knowledge of pedagogical issues (with little emphasis on knowledge subject matter content). Shulman calls for preparation of teachers that emphasize content knowledge, pedagogical content knowledge, and curricular knowledge. *Content Knowledge* is described in this article as more than just the knowledge of facts or content of a subject, it is also the knowledge of the underlying structure beneath those facts and content. *Pedagogical Content Knowledge* is described as almost a subset of content knowledge, a specialized part of content knowledge. Here, pedagogical content knowledge is made up of not just the content, but knowledge of effective and different ways with which to teach that content. The last of Shulman's categories is *Curricular Knowledge*. This type of knowledge is described as knowledge of the various resources available to teach the content, and knowledge of where the current content fits into the scope of the students' educational program.

Ball and colleagues have described a framework that refines and expands upon these original ideas (and subsequent works) proposed by Shulman and his contemporaries into something more descriptive of the subject of mathematics. Described as a framework that identifies the various aspects of Mathematical Knowledge for Teaching, this "egg" framework (see Figure 1) is split into two main categories of knowledge which are then further refined: subject matter knowledge, and pedagogical content knowledge. The subject matter knowledge category is split into Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), and Horizon Content Knowledge. *Common Content Knowledge* (CCK) is described as "knowledge that is used in the work of

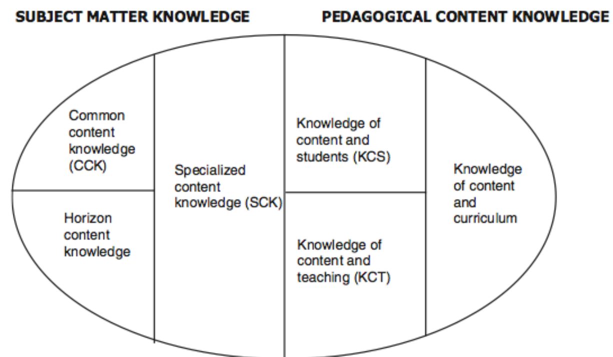


***Figure 16.* Mathematics Knowledge for teaching from Ball, Hill, and Bass (2005).**

teaching in ways in common with how it is used in many other professions or occupations that also use mathematics" (Hill, Ball, & Schilling, 2008, p. 377). In describing *Specialized Content Knowledge* (SCK) they refer to an earlier work by Ball which described SCK as "the mathematical knowledge that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand solution methods to problems (Ball, et al., 2005)" (Hill, et al., 2008, pp. 377-378). Horizon Content Knowledge is described as knowledge of the mathematics as it relates to future classroom subjects. Here it is by knowing the mathematics that is forthcoming on the horizon, teachers can appropriately lay the groundwork with the subjects they are teaching now. It is noted that CCK, SCK and Horizon Content Knowledge require only the knowledge of mathematics, and no knowledge of students or teaching is required to have sufficient knowledge in these categories. The authors elaborate that the categories they have under Subject Matter Knowledge are most closely associated with Schulman's original framework of Content Knowledge (though they argue that CCK is more directly related to Content Knowledge, than SCK). The pedagogical content knowledge category is split into Knowledge of Content and Students, Knowledge of Content and Teaching, and Knowledge of Curriculum. *Knowledge of Content and Students* (KCS) is described as "content knowledge intertwined with knowledge of how students think about, know, think about, or learn this particular content" (Hill, et al., 2008, p. 375). *Knowledge of Content and Teaching* (KCT) is described as knowledge of mathematics as used in the design of instruction. This can include sequencing of tasks for greater effect, or knowledge of what ideas to elaborate on, what to hold for later, and when to pause for clarification. The descriptions used by Ball and colleagues for Knowledge of Content Curriculum and by Shulman for Curricular Knowledge are similar.

The two research projects described below had different goals with the administration of their test items. Although not designed to map directly onto the Shulman, or Ball et al. frameworks, these frameworks do provide a lens with which to look at the tasks and the preservice teacher responses. As will be shown, it could be argued that the TNE test items attempt to capture more information on the subject matter knowledge of preservice teachers, whereas the PIR test items attempt to capture more information on the pedagogical content knowledge of preservice teachers. By examining the responses obtained in two research projects, we attempt to answer what Mathematical Knowledge for Teaching is exhibited by three preservice teachers in both studies.

## Methods

At Michigan State University, two separate research projects are looking at preservice elementary school teachers and their mathematics teaching by using paper and pencil item administrations. These two projects have items that share some of the same question stems, though the focus of the questions is different for the projects. These research projects administered their test items over several years and as such, there are some students who participated in both research projects. This paper looks at some of these students, and the mathematical content that they used in answering these “common” questions.

*Project Overviews*

MSU is one of the universities participating in the multiple-institution project Teachers for a New Era (TNE) funded by the Carnegie Corporation of New York. As part of the work going on at MSU in this project, the mathematics education faculty is performing a self-study of the mathematics in their elementary education program. The purpose of this study is to analyze what mathematical content knowledge preservice teachers learn as part of their content classes, and what is retained through the duration of the program. For this paper, we will use responses from the administration of the TNE items that occurred at the end of the preservice teachers’ math content class that focused on number and operation. At this point, the preservice teachers had not yet taken their math methods course as they were generally at the beginning of their program (freshman and sophomores). This data set was collected in 2005 and 2006.

The PIR project is a NSF funded project looking at the enacted practice of mathematics teaching for preservice and in-service teachers of the MSU elementary education program. The larger research question involves the study of the practices of posing, interpreting, and responding (P-I-R) of current students and alumni of the MSU elementary education program. For this paper, we will use responses from the administration of the PIR items that occurred the semester before students enrolled in their mathematics methods classes (but several years after their math content courses). Preservice teachers in these administrations were generally near the end of their program (juniors and seniors). This data set was collected in 2008.

*Comparing Tasks*

Many items of the TNE tasks and PIR tasks shared the same stem or set-up, but asked students different questions from this common starting point. Here we describe the general nature of the questions used in this paper.

All of the TNE tasks listed ask preservice teachers to assess the validity of students’ alternative methods (is the solution method correct, is the method generalizable, will it work for other numbers, etc.) or to come up with word problems that accurately described a given mathematical situation. Here, the items tend to be asking about the Mathematical Content Knowledge part of MKT. These questions attempt to have the preservice teachers understand student solutions (described as SCK). This paper uses item W1 from the TNE test administration (see Figure 2). An alternate subtraction algorithm is presented to the preservice teacher and two questions are asked. The first question asks if this method is generalizable. The second question asks the preservice teacher how this hypothetical student would use this method on a new problem. Here we can see this question is trying to assess the preservice teachers' SCK regarding subtraction (do they understand the alternative algorithm well enough to use it to solve another problem).

Within the PIR tasks, these questions have a general theme of the posing, interpreting, and responding that preservice teachers imagine themselves doing when presented with these situations. This paper uses items 2(a) and 2(b) from the PIR administration (see Figure 3). The first question asks preservice teachers what they imagine saying and doing with their class when receiving puzzling looks after they pose a task. This question allows for a multitude of different responses from preservice teachers as they are left to determine exactly why students may be puzzled. This confusion could stem from students lack of content knowledge on subtraction, to not knowing where to begin with different ways. The responses here could vary from MCK type responses (CCK or SCK) to more PCK responses to deal with the student confusion (KCS or KCT). The second part of the question states that a student has come up with a new strategy and asks the preservice teacher to describe what they imagine saying and doing. Here again the responses can take on many different forms, from validating or invalidating the method, offering congratulations for unique thinking, or even using the method to connect to the standard algorithm. Here again we can see that these responses could map onto the MKT framework in a variety of different ways.

***W1***

Imagine that one of your students shows you the following strategy for subtracting whole numbers.

$$\begin{array}{r} 37 \\ \underline{-19} \\ -2 \\ \underline{20} \\ 18 \end{array}$$

a. Does this strategy work for any two whole numbers? (Circle your answer.)

Yes

No

I don't know

b. How do you think the student would use this strategy in the problem below?

$$\begin{array}{r} 423 \\ \underline{-167} \end{array}$$

Adapted from: Ball & Hill (2002). *Cultivating knowledge for teaching mathematics: Early fall survey.* Cultivating Teachers' Knowledge for Teaching Mathematics Study. Ann Arbor, University of Michigan.

***Figuee 2.*** **TNE item W1.**

2. (a) Imagine you just asked your class to explore different ways to solve the following subtraction. The students look puzzled. What do you imagine <u>saying</u> and <u>doing</u> next?

$$\begin{array}{r} \mathbf{37} \\ \underline{\mathbf{-19}} \end{array}$$

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2. (b) After giving students some time to work on the task you call on their attention and ask for volunteers to share their strategies. Imagine that one of the students shows the following strategy. What can you imagine <u>saying</u> and <u>doing</u> to the student and to the class? Say a bit about what you would want to accomplish by saying and doing so.

$$\begin{array}{r} \mathbf{37} \\ \underline{\mathbf{-19}} \\ \mathbf{-2} \\ \underline{\mathbf{20}} \\ \mathbf{18} \end{array}$$

***Figure 3.*** **PIR items 2(a) & 2(b).**

The responses of three preservice teachers (Dean, Becky, and Lisa – all pseudonyms) who had completed both TNE and PIR items were chosen for analysis. These preservice teachers were chosen in particular because they all gave identical responses to the TNE item and very different responses to the PIR item. Below is a table of the responses of the preservice teachers (see Table 1).

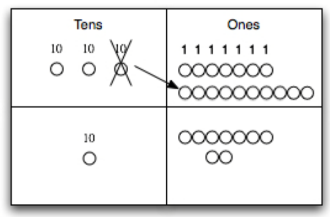
| Preservice Teacher | Response to TNE Item W1 | Response to PIR Item 2(a) | Response to PIR Item 2(b) |
|---|---|---|---|
| Dean | (a) Yes<br><br>(b)<br>423<br>-167<br>-4<br>-40<br>300<br>256 | 1) Ask the students if they can make the problem into two problems<br>37 27<br>-10 -9<br>27 18<br><br>2) Ask the students, what plus 19 equals 37?<br>19 + 18 = 37<br><br>3) Use chips to show the two methods | • I would ask the student to re-write the problem and show each step they took to get to their answer.<br><br>• I would want the students to learn the importance of showing their work and how they can use it to retrace their steps in a problem |
| Becky | (a) Yes<br><br>(b)<br>423<br>-167<br>-4<br>-40<br>300<br>256 | 1: Draw a picture (visual is the best way)<br><br>2: Cannot take nine from seven –<br>3: Take 10 → Pull over to ones<br>And take 9 from 17.<br><br>Or $^{2}\cancel{3}\ \ ^{1}7$<br>$-1\ \ 9$<br>$1\ \ 8$<br>• Slip into two different columns | • I do not like this way – Math for higher on is going to be a lot harder if they learn this now. |
| Lisa | (a) Yes<br><br>(b)<br>423<br>-167<br>-4<br>-40<br>300<br>256 | I would ask the students where we start, in the ones column. We can't take 9 away from 7 so we need to borrow from the tens column. Cross of the 3 and make it a 2 since we borrowed one for the 7. Put a 1 in front of the 7 to make it 17. 17-9=8. Then we do the tens column. 2-1=1 so our answer is 18. | This strategy can work. The student knows that we start in the ones column. 7-9= -2. The tens column is also correct, as 30-10=20. Now what the student did was combine -2 and 20, to get 18. We got the same answer. I would want to let the class know that there is more than one way to solve a problem, and it is important to remember that subtraction of multiple digit numbers involves multiple subtractions, depending on how many places are in the number. |

Table 1. *Student Responses*

## Results

*TNE Item W1*

When responding to this TNE item Dean, Becky, and Lisa all replied with identical answers. Despite the limited amount of response material due to this item not having a written response section, several pieces of information about these preservice teachers can be extrapolated. All three preservice teachers examined the work shown by the hypothetical student, and were able to uncover enough of the student's method to solve an additional problem with the same method. In addition to this, all three preservice teachers circled the response that our hypothetical students' method would work for any two whole numbers. This may have been due to guessing, but based on the preservice teachers correctly applying this method to a new problem, there is reason to believe that Dean, Becky and Lisa were each able to figure out that this method is indeed generalizeable to all whole numbers. Here, there is some evidence that these three preservice teachers have some Specialized Content Knowledge (SCK). It seems reasonable to say that based on the responses to the TNE items, Dean, Lisa and Becky have in their repertoire an ability to analyze this method of subtraction to determine its generalizability, and an ability to take this non-standard subtraction method and apply it to a new problem. Given this, it seems reasonable to assume that these skills could emerge when the preservice teachers were asked to respond to the same subtraction method in a teaching scenario. However, not only do these skills not emerge in their responses to the teaching scenario, three very different responses emerge.

*Dean*

Dean's response to the two PIR items offer some insight into the Pedagogical Content Knowledge he brings to these related items. When asked to describe what he imagined saying and doing after his students looked puzzled when he told them to explore different ways of solving 37-19, Dean suggests three things he might do with his class: breaking the problem into smaller steps, making the subtraction problem into an addition problem, and using manipulatives. All three of these proposed actions display some Knowledge of Content and Students (KCS). Dean appears to be addressing the student confusion by suggesting alternate ways to present the problem that would make sense to students, or be similar to how students would think about a problem. In the second half of this task, the main theme of Dean's response to this question is to emphasize that students need to show their work. What Dean's answer does not contain here is how he might use this student's work productively in a classroom. From Dean's response to the TNE item, we know that at one point in time, Dean was able to circle the response that this method was generalizable and he showed that he was able to understand the method enough to apply it to a new problem. However, in talking about the same solution method to students, Dean only reminds students to show their work. This may be an example of further KCS as Dean believes that students need to show their work in order to be able to retrace their steps, which will lead to more opportunities to learn.

*Becky*

Becky's response to the first question also contains three items. The three things that Becky imagines saying and doing include drawing a visual representation of the tens and ones of the subtraction problem, mentioning that you cannot take nine away from seven, splitting the subtraction problem into two columns, and performing the traditional subtraction algorithm on the problem. What is interesting is that Becky notes that her first item is actually the best as "visual is the best way". This response was given at the end of her math content class and before she was enrolled in any math methods classes for her program so it is not apparent where Becky has come to know this claim. Regardless of where this claim originated, Becky has incorporated it into her KCS as she believes that visual ways of representing content will be the ways that students can best engage with. Additional pedagogical knowledge appears again in Becky's response to the second half of the PIR task. When asked how she would respond to the student and her classroom when someone presented the non-standard method, Becky writes, "I do not like this way – Math for higher on is going to be a lot harder if they learn this now." This claim appears to fall under Knowledge of Content and Curriculum in saying that future mathematics learning will be hampered by learning this method now. Regardless of whether one agrees with the assessments Becky has made, it is clear that Becky is displaying some of her thoughts about Pedagogical Content Knowledge in her responses to these items.

*Lisa*

In responding to the confused looks of her students as described in the first PIR task, Lisa responds by walking the students through the traditional subtraction algorithm. The prompt in the question called for the teacher to ask her students to explore different ways of solving subtraction. It is not clear if Lisa is reforming the question to be about finding one way of solving a subtraction problem, or if her step by step description of this method is supposed to serve as a launching point for her students to then explore other methods. Regardless, it appears that walking the students through the traditional subtraction algorithm is important for Lisa for it will serve either as instruction of the traditional subtraction algorithm or as a review of the algorithm that will serve as a launching point into further explorations and understanding of subtracting. There appears to be some KCS in that Lisa believes this explanation will end the student confusion and they will come to understand subtraction, as no other moves are mentioned. In the next part of the PIR items, Lisa states that she would walk the students through the method used to solve this problem, remind students that there are many ways to solve a problem, and remind students that sometimes a single subtraction problem actually requires several subtractions. Here what we notice is that in both PIR responses, Lisa walks students through the steps of two different methods for her class. This may be due to thinking this is how math instruction is performed, or it may be due to thinking that this is the most effective way that students learn mathematics (possibly evidence of KCS). This is a pedagogical choice that Lisa is making and is yet another way that the MKT of one preservice teacher can be examined through the moves they make in a classroom.

## Discussion

Mathematical Knowledge for Teaching is a way of describing the knowledge that teachers should have in order to be effective teachers of mathematics. This framework is divided into two parts: one part dealing with Mathematical Content Knowledge, and the other part dealing with Pedagogical Content Knowledge. In this study, we looked at a sample of responses from three preservice teachers in an elementary education program. These preservice teachers participated in two research projects that had items that used the same stems. One research project had items that more closely mapped onto the MCK part of MKT while the other research project had items that seemed to map onto PCK. It was by looking at these items in tandem that we were able to obtain a better understanding of the Mathematical Knowledge for Teaching that each of these students possessed. This study looked at only one item that these two research projects shared. Even more information could be obtained about these students if additional items were analyzed.

In going forward, we recognize that this method of using both MCK and PCK items provides a much broader and deeper picture about students MKT than either can do alone. Furthermore, by trying to differentiate, distinguish and study the individual categories of MKT, we loose important information on how these categories interact with each other. This is where much information of MKT is located. While this ad hoc study found itself in a unique position of being able to study not only similar test items, but similar students, it provides a unique insight into what information can be gained if tests and surveys are intentionally designed to capture multiple categories at once. Here we state again, the preservice teacher responses used in this paper were obtained after their math content classes, but before their math methods classes. We expect that a different, yet just as rich, picture would be obtained if preservice teachers were surveyed after their math methods courses. Furthermore, by obtaining this information about preservice teachers both before and after their methods courses, a survey such as the one used in this study could provide institutions with a rich body of information as to how their methods courses are affecting their students' mathematics knowledge for teaching.

There has been much research into the different categories of MKT that Ball and others have described. And while there continues to be opportunities for further explorations into these individual categories, there is also a need to fortify the connections between these categories. The method used in this research is one way that we can start putting the pieces of this MKT egg back together so that we can construct a clearer picture of the knowledge that teachers of mathematics are equipped with and how they can use it in their classroom.

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# MIDDLE GRADE TEACHERS' REORGANIZATION OF MEASUREMENT FRACTION DIVISION CONCEPTS

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*Twelve middle grades mathematics teachers were given division problems in a professional development program designed to help teachers understand rational numbers. This paper analyzes how teachers reorganized their measurement interpretation of fraction division by operating on conceptual units. Past studies have documented that many teachers are not able to reason about fraction division in terms of quantities. The present study extends these past studies by examining teachers' capacities to reason in a sequence of fraction division situations in which the mathematical relationship between the dividend and divisor became increasingly complex.*

## Background

The National Council of Teachers of Mathematics' standards for teaching and learning (NCTM, 2000) require teachers to have a richer understanding of mathematics than traditionally required. One topic that is particularly difficult for teachers is division of fractions. For instance, Ball (1990) asked U.S elementary and secondary prospective teachers to contextualize the problem $1\frac{3}{4} \div \frac{1}{2}$. Her results revealed that teachers could not generate appropriate word problems or situations even though most of them could calculate a solution using the 'invert and multiply' method. She found that teachers were likely to confound division by a half with division by 2 or multiplication by a half when they tried to contextualize the problem. Simon (1993) adapted the problem from Ball's study and asked prospective elementary teachers to write a story problem for $\frac{3}{4} \div \frac{1}{4}$. Twenty-three out of 33 teachers could not create an appropriate problem. Twelve of those 23 teachers used 'invert and multiply' and represented the problem as multiplication thereby creating a problem situation for multiplication rather than maintaining the fraction division. Borko et al. (1992) also examined prospective teachers' knowledge as exhibited during their student teaching and reported the case study of a middle school teacher, Ms. Daniels. When asked by a child to explain why the invert-and-multiply algorithm worked, Ms. Daniels could not clearly explain fraction division and instead showed fraction multiplication, despite having completed a fair number of mathematics courses in her undergraduate program and being able to compute accurate answers with the invert and multiply method. In her study of 23 U.S. teachers and 72 Chinese teachers, Ma (1999) introduced the term *knowledge package*, which refers to pieces of knowledge consisting of numerous subtopics that are related to one another that support more advanced learning. For fraction division, she suggested that teachers' knowledge packages should include whole number multiplication, the concept of division, the concept of division as inverse of multiplication, the meaning of multiplication with fractions, and the concept of unit. Ma's study revealed that pieces of the ideal teacher knowledge package are relatively weak for U.S teachers.

Although previous studies (e.g., Ball, 1990; Borko, 1992; Simon, 1993; Ma, 1999) have stressed errors and constraints on teachers' knowledge of fraction division, they have not considered teachers' capacities to reason in a sequence of fraction division situations. Further, even though studies have revealed teachers' knowledge in detail, they have not examined teachers' reasoning about fractional quantities in terms of conceptual units, although several research programs have investigated children's reasoning with conceptual units in the context of fractions. In this paper, we focus on one approach for supporting teachers' development of mathematical knowledge for teaching fraction division by emphasizing the relationship between the units and operations associated with the measurement conception of fraction division. Conceptual analysis of teachers' knowledge at this grain size allows us to develop a stronger understanding of teachers' capacities to reason about fraction division in detail.

## Theoretical Perspectives

The theoretical framework used for our analysis of twelve teachers' reasoning with drawn quantities in solving fraction division problems draws from the literature on children's development of conceptual understanding of fractions. We draw from these ideas because we noted that the ideas and operations documented with students appeared in our teachers' reasoning. In the literature, the ideas of *iterating* and *partitioning* have been determined to be fundamental for fraction knowledge development (e.g., Piaget, Inhelder, & Szeminska, 1957; Olive, 1999; Olive & Steffe, 2002; Steffe, 2002, 2004; Tzur, 1999, 2000, 2004). This is the notion that a unit whole can be divided into any number of pieces (partitioning) and that any one of these pieces may be iterated to reconstruct the unit. Another critical operation we considered was the *unit-segmenting scheme* for division in a whole number context (Steffe, 1992). This entails the operation of segmenting the dividend by the divisor. In his teaching experiment, Steffe (1992) observed that a child needed to reason with at least two composite units in the unit segmenting scheme: one composite unit to be segmented and the other composite unit to be used in segmenting. For example, consider how many times a child would count if dividing a pile of fifteen books by three. That child would use three as a segmenting unit to divide composite unit fifteen. It is likely that even in cases where the child was using the unit-segmenting scheme in this content, that child may have to reorganize his/her unit-segmenting operations to approach more complex problems.

Olive (2000) defined *common partitioning fractional scheme* to refer to the child's ability to coordinate and compare his two number sequences for two composite units until a common number was found. He also stated that the child who had common partitioning fractional schemes could keep track of how many of each composite he had used to get to the common multiple of both composite units. In other words, it requires units-coordination at three levels that is a coordination of two iterable composite units. To illustrate, when the child was asked to partition a bar that would allow him to pull out both one-third and one-fifth of the same bar, his procedure was to count by 3s and 5s until he found a common number in the two sequences. The common partitioning fractional scheme was a building block for the children's construction of *measurement fractional scheme*. With fractions as *measurement units,* division of fractions becomes meaningful in that children can now answer questions like "How many thirds are in two-ninths?" when children have constructed fractions as measurement units, they could then find a *co-measurement unit* for the two fractions by constructing any fraction from any other.

A co-measurement unit is defined as a measurement unit for commensurable segments, that is segments that can be divided by a common unit without remainder. For example, one child could make one-ninth of a unit stick using one-twelfth of the stick by finding one-thirty sixth as a co-measurement unit for both one-ninth and one-twelfth. In addition to the common partitioning operation children used to find the common partition for two fractions, cross-partitioning using area model also emerged in Olive's teaching experiment. For instance, the children used the *cross-partitioning* operation to solve $\frac{1}{3}\times\frac{1}{5}$ by partitioning a bar vertically into three parts and horizontally into five parts and get 15 of one-fifteenth automatically. In his observations of Nathan, Olive (2000) found the use of cross-partitioning operations in a fraction multiplication context differed from the common partitioning operation in that the former provides for a simultaneous repartitioning of each part of an existing partition without having to insert a partition into each of the individual parts, and it is a fundamental operation to produce *fraction composition scheme* (Steffe, 2004) which is multiplying scheme for fractions. While the Fractions Project's (Olive, 1999; Olive & Steffe, 2002; Steffe, 2002, 2004; Tzur, 1999, 2000, 2004) research on the conceptual analysis of children's construction of fractional schemes made important contributions to the field of knowledge of fractions, the studies were limited in that they did not follow the students beyond the development of fraction multiplication schemes.

In the fraction division, reasoning with conceptual units became increasingly complex as mathematical relationship between the dividend and divisor changes. In this paper, we begin the work of identifying the fraction schemes teachers used to reason about fraction division from a measurement (quotative) perspective. We first considered a simple division problem in which the dividend was evenly divisible by the divisor. In this situation, teachers simply used a *measurement unit* and segmented the dividend into the groups of divisors. In the second case, the dividend did not clearly segment by the divisor, leaving a remainder to be considered. This required teachers to bring forth the conception of the *referent unit* in that it was plausible to quantify the remaining part in terms of one by conflating the referent unit. We say that teachers can attend to the referent unit when they can keep track of the referent whole. The third case involved a divisor larger than the dividend. In this case, teachers reorganized their understandings to move toward conceptualizing the multiplicative relationship between the divisor and the dividend.

## Research Questions

1. What were the primary operations and units that the teachers used when reasoning about the fraction division problems?
2. How did teachers modify or reorganize their measurement fraction conception when faced with increasingly complex problem situations?

## Method

The data reported here were collected as part of the larger NSF-funded *Does it Work?: Building Methods for Understanding Effects of Professional Development* (DiW) project. As part of the DiW research effort, a professional development course called *InterMath – Rational Numbers* (IM) was offered to middle grades teachers in a large, urban district. The course provided teachers with opportunities to develop their content knowledge of multiplication and division of fractions and decimals and to explore direct and inverse proportions by engaging them in solving technology-enhanced, open-ended investigations and by exploring a variety of drawn representations. To support the course goal of raising fluency in the use of visual representations, teachers used a variety of software packages including *Fraction Bars* (Orrill, undated). The course met three-hours per night, once per week for 14 weeks. All meetings were videotaped using two cameras – one to capture the participants' facial expressions, written work and hand gestures, and the other to capture the instructor's teaching. These two sources were then mixed to create a *restored view* of the event (Hall, 2000).

A single class meeting focused on measurement fraction division was considered for this analysis. The first round of analysis involved taking memos of the trends that emerged from the data in the ongoing observation of the lessons as they occurred. Then a retrospective second analysis was conducted after all IM data had been collected. The purpose of the retrospective analysis was to understand teachers' ways of operating mathematically. In qualitative research, notions of reliability, validity, or viability hinge on the quality of thinking and internal consistency of the witness-researcher, since a researcher doing qualitative research is the research instrument (cf. Peshkin, 1988; Richardson, 2000; Weis & Fine, 2000). However von Glasersfeld (1995) emphasized the role others play in developing one's own thoughts and characteristics. He suggested that intersubjective knowledge is the most reliable knowledge in experiential reality. Hence the researchers took advice and shared their thoughts with the teacher-researcher and another witness-researcher of the IM class.

## Results

We considered the teachers' measurement fraction division knowledge across three different types of division problems. The first involves a dividend that can be clearly measured out by the divisor, the second is when there is the remainder as a result of measuring out the dividend by the divisor, and finally when divisor is larger than dividend. By providing teachers with tasks and drawing representations that motivated them to reason with quantities rather than to rely on calculation, we could understand teachers' thinking in detail.

*When the Dividend Clearly Measures Out by the Divisor*

This is the case when there is no remainder in fraction division like $2 \div \frac{1}{4}$. The instructor provided teachers with the task that was comprised of various fraction division expressions such as $2 \div 3$, $2 \div \frac{1}{4}$, $2 \div \frac{3}{4}$, and $\frac{2}{3} \div \frac{3}{4}$. Teachers were asked to find the quotients for each of the expressions without using an algorithm. Moreover the instructor encouraged teachers to use various drawn representations such as area or number line model. When teachers were asked to solve $2 \div \frac{1}{4}$ using drawing representations, most teachers used one-fourth as a measurement unit and segmented the composite unit 2 into 8 parts of one-fourth, and successfully stated that the answer was 8 because 8 groups of one-fourth fit into 2. Even though most teachers derived the answer 8 using measurement division concepts, teachers' strategies were different in that some teachers repeatedly subtracted one-fourth from 2 to get 8 whereas others thought of the problem in terms of how many groups of one-fourth formed 2. Note that the latter method does not imply multiplicative reasoning because we are not clear that the teachers were aware of the multiplicative relationship between the two fractional quantities, a dividend and a divisor, rather than finding the numbers of divisors that fit into the dividend.

*When the Dividend Does Not Clearly Measure Out by the Divisor*

In a problem such as $2 \div \frac{3}{4}$ there is a remainder in the result of measuring out the divided by the divisor. Although our participants did not attend to the referent unit for the first type of fraction division problem, the referent unit concept played a key role here. For instance, one teacher drew two rectangles and divided each rectangle into fourths, then circled three of each of the fourths in each rectangle, and said $2\frac{1}{2}$ was the answer because there were $2\frac{1}{2}$ of the three-fourths fit into 2 wholes (see Figure 1).

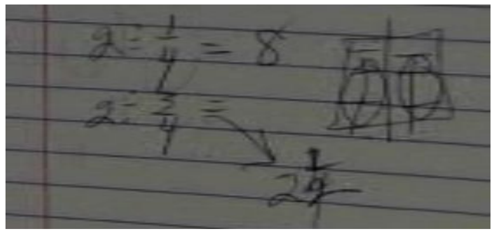

*Figure 1.* A teacher's representation of $2 \div \frac{3}{4}$ using area model and her initial answer $2\frac{1}{2}$.

She made this error at first because she was not attending to the referent unit of the remaining two-fourths that was left after she measured out three-fourths twice. However when the instructor asked her to talk through her reasoning, the teacher reflected upon that reasoning to determine that the remaining two fourths was two-thirds of three-fourths not two-fourths of one whole. So she changed her answer to $2\frac{2}{3}$, which was then correct. In other words, she determined the dividend 2 could be divided into $2\frac{2}{3}$ groups of three-fourths.

Similar issues arose in the whole group discussion as the teachers were considering how a number line could be used to solve this problem. In the whole class discussion, the teachers suggested $2\frac{1}{2}$ but did not recognize that their answer was incorrect until one teacher suggested using the referent unit concept. Teachers' attention to the referent unit was not clear until we observed teachers solving division problems in which the divisor cannot evenly divide the dividend. In the first type of problem, teachers did not have to attend to the referent unit because they could easily figure out the number of divisors that fit into the dividend if they just chose the divisor as the measurement unit. However when teachers were given a situation in which they had to reason with the remainder from fraction division, they needed more than the divisor as their measurement unit. They also needed to attend to the referent unit.

*When the Divisor is Larger than the Dividend*

For people who rely on measurement conceptions of division, problems in which the divisor is larger than the dividend are often problematic because the fractional quantity being measured is smaller than the dividend, leading to an awkward conceptualization. For example, in $\frac{2}{3} \div \frac{3}{4}$, the question being asked from a measurement perspective is how many three-fourths are in two-thirds. In IM, the instructor chose two teachers to explain how they used a number line model and an area model to represent $\frac{2}{3} \div \frac{3}{4}$. The teacher who used the number line model drew his model on a flipchart as he explained his approach. When he tried to use three-fourths as a measurement unit to find the solution, he faced a perturbation because the divisor was larger than the dividend. He was stated, "you can't quite get three-fourths into two-thirds." He continued,

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“for fraction division, we ask ourselves question like how many three-fourths we can put into two-thirds.” We can infer from his statements that his initial conception about measurement division relied on finding the numbers of divisors in the dividend. In order to find how many three-fourths go into two-thirds, he first decided to find the least common denominator (his term) of three and four. He used one-twelfth as a co-measurement unit for two-thirds and three-fourths, and used $\frac{9}{12}$ as a commensurate fraction for $\frac{3}{4}$ and $\frac{8}{12}$ for $\frac{2}{3}$. He drew a number line and divided it into three parts, then subdivided each third into another thirds, using the common partitioning operation. Then, he denoted each $\frac{1}{12}$ segment from 0 to $\frac{9}{12}$ (See Figure 2). He said that $\frac{8}{12}$, which was a commensurate fraction of $\frac{2}{3}$, was $\frac{8}{9}$ of $\frac{9}{12}$, so $\frac{2}{3} \div \frac{3}{4}$ was $\frac{8}{9}$.

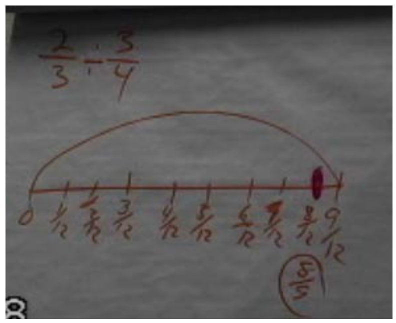


*Figure 2*. The teacher’s representation of $\frac{2}{3} \div \frac{3}{4}$ using a number line.

Even though he used the common partitioning operation and came up with two commensurate fractions, his perturbation was not resolved because he still could not measure out $\frac{8}{12}$ by $\frac{9}{12}$. He needed to use a different approach than the one with which he started. At least in this context, his initial conception of measurement division was modified into a more efficient one as he attended to the multiplicative relationship between the two quantities $\frac{8}{12}$ and $\frac{9}{12}$ instead of looking for the number of divisors that fit into the dividend. In other words, $\frac{8}{12}$ was now for him could be reinterpreted as $\frac{8}{9}$ of $\frac{9}{12}$, and generalizing his measurement division conception in such a way was powerful in that he was not confined by the quantities of dividend and divisor. His attention to the multiplicative relationship between two quantities was novel considering his initial conceptions about fraction division and the comment that he made “you can’t quite get three-fourths into eight-ninths.” The teacher’s clear attention to the referent unit nine-twelfths was another knowledge element that supported his reasoning with this problem.

In contrast to the teacher above who used common partitioning operation with a number line model, other teachers in the class relied on cross partitioning in an area model to reason about this problem. This led them to cut the whole into 12 pieces by partitioning it into thirds one way and fourths the other. Then, the teachers counted the number of rectangles that represented two-thirds of the whole and the number of rectangles that represented three-fourths of the whole to determine that the three-fourths was 8 blocks and two-thirds was 9 blocks (‘block’ was the teachers' to describe the smaller rectangles created by partitioning the area model). The teachers did not discuss the fact that those blocks were $\frac{2}{3}$ or $\frac{3}{4}$ of $\frac{12}{12}$, not $\frac{2}{3}$ or $\frac{3}{4}$ of 12. One teacher explicitly stated that they needed to find "how many 9 blocks fit into 8 blocks?" and "it's eight-ninths". The teachers began to use cross partitioning in the new situation to acquire a co-measurement unit for two-thirds and three-fourths where the denominators of two fractions were relatively prime to each other. We assert this was a reorganization of their conception of measurement fraction division. However, computing the quotient eight-ninths without attending to the referent unit of both the two-thirds and three-fourths is problematic in classroom settings because students may reinterpret the problem to be 8 ÷ 9. We worry that the teachers, in our case, had moved away from their consideration of the referent whole, which was 1 (12/12) at the time. While this was a shortcut for the teachers, such shifting may interfere children's learning.

In summary, we found that some teachers modified or reorganized their initial conception of measurement fraction division conception, which began as finding the number of pieces of the size of the divisor that fit into the dividend. They moved toward using more efficient strategies by incorporating more sophisticated operations and conceptual units. While we do not know whether such modifications or reorganizations were permanent, they were relatively new ways of thinking about measurement fraction division for these teachers as they considered the relationship between the divisor and the dividend in increasingly complex situations.

## Discussion

Although past studies have documented that teachers lack sufficient fraction knowledge, our results suggest that teachers can reorganize and generalize operations and concepts that they have when provided with a professional learning experience in which they can reason in terms of drawn quantities. We observed that teachers' operations and conceptual units necessary for reasoning with the situation where no remainder was left for fraction division did not necessarily transfer to fraction division with a remainder. Teachers needed to incorporate strong referent unit concept in addition to their use of the measurement unit in order to quantify the remainder in terms of the correct referent unit. Moreover, teachers relied on more operations and conceptual units such as common partitioning, cross-partitioning, co-measurement unit, commensurate fraction, and multiplicative reasoning when the divisor was larger than the dividend.

Most studies have not looked at teachers' knowledge any closer than simply saying that teachers' reasoning about fraction measurement division is insufficient. Hence, the idea of examining measurement division more closely using tasks of gradated difficulty as well as understanding where teachers are more and less competent at reasoning about fraction division, and why, in terms of close analysis of operating on conceptual units, is a contribution to the field of mathematics education. This study is only a beginning step toward that understanding.

Note that this study considers only twelve teachers and that we make no claims of the generalizability of the patterns of reasoning observed. We certainly believe that teachers may follow a variety of paths in learning to reason about these concepts. In this study, we considered only those ways of reasoning that emerged from the participants. By building on understanding of teachers' mathematical concepts and operations in the ways illustrated here, we propose that a rich understanding of teachers' understandings could be developed which could be used as the foundation for developing stronger professional learning opportunities for teachers.

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# CONCEPTUALLY BASED TASK DESIGN: MEGAN'S PROGRESS TO THE ANTICIPATORY STAGE OF MULTIPLICATIVE DOUBLE COUNTING (mDC)

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*This study examined the application of a conceptual framework for learning new conceptions to the design and use of tasks/prompts that can lead students to construct multiplicative double counting (mDC) – a scheme underlying the development of multiplicative reasoning. Within the context of a teaching experiment with fourteen 4$^{th}$-5$^{th}$ graders, we analyze the teacher-researcher's work with one student, Megan, as she progressed from having no such conception to the participatory and then anticipatory stage of mDC. Our analysis demonstrates how tasks can (a) draw on available conceptions and (b) be designed to engender the intended learning via orientation of reflective processes.*

## Introduction

How might tasks that promote conceptual understanding of multiplicative operations be designed and implemented based on students' available conceptions? In this study we addressed this critical pedagogical problem, which is consistent with the growing interest of mathematics educators in the role tasks play in students' learning of mathematics (Watson & Mason, 1998, 2007; Watson & Sullivan, 2008; Zaslavsky, 2007). In particular, we examined the application of Tzur and Simon's (2004) stage distinction (see below) to the process of instructional task design. This application contributes to the recent focus on task use, because it is rooted in a framework that explicitly links learning of new (to the learner) conceptions with interventions that can promote such learning. Thus, we applied the stage distinction, reflexively, to both the analysis of students' available conceptions and the tasks used for transforming these conceptions into intended, more advanced mathematical ideas.

We chose the difficult-to-grasp domain of multiplicative reasoning because of the central role it plays in empowering students' mathematics (e.g., algebra preparedness, see Confrey & Harel, 1994). We believe that inadequate conceptualization in this domain is one key cause for the ever-growing gaps among students during the upper elementary, middle, and early high school years. Consequently, this study focused on the commencement of multiplicative double counting (mDC, see details below), a milestone mental operation that constitutes a child's transition from a unitary counting stage to a binary counting stage (Vergnaud, 1994). Our central thesis is that the stage distinction, and the reflection on activity-effect relationship (Simon, Tzur, Heinz, & Kinzel, 2004) framework in which it is rooted, provide useful tools for creating and adjusting tasks/prompts conducive for nurturing mDC at a level necessary for students to independently carry out cross-context problem solving processes proper to a situation at hand.

## Conceptual Framework

In this section, we first briefly describe the general and content-specific constructs that

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guided this study, and then delineate how they were used to design a set of tasks/prompts for teaching mDC. The general constructs constitute the reflection on activity-effect relationship (*Ref*AER*) framework (Simon et al., 2004), itself an elaboration of Piaget's (1985) and von Glasersfeld's (1995) scheme-based theories. *Ref*AER* is the postulated mechanism by which the human mind forms novel conceptions. It commences with the learner's assimilation of a problem situation into her available conceptions, which set her goal and trigger the activities (usually an activity sequence) that the mind and body carry out to accomplish the goal. The learner's goal then regulates, from within the mental system, the progress of her activity sequence and her noticing of effects that this activity brings forth. Through two types of reflection, in the form of brain-based comparisons, the learner first relates the newly noticed effects with the activity and later with the situation in which she should anticipate such an activity-effect relationship (*AER*). *Type-I reflection* consists of comparison between the learner's goal and the actual effect of her activity sequence; *Type-II reflection* consists of comparison across records of experience in which the learner invariantly uses *AER* compounds for solving what then become similar problem situations for the learner. A novel anticipation of *AER* is formed via two stages (Tzur & Simon, 2004). In the first, *participatory* stage, the learner forms a provisional anticipation of *AER* that she cannot access directly from her available schemes. Rather, this anticipation can only be retrieved if the learner is somehow *prompted* for the activity, which generates the effect and hence the *AER* compound. In the second, *anticipatory* stage, the learner forms a robust *AER* that she can independently and spontaneously call up, use, and transfer to new situations. The anticipation encapsulated in the *AER* of both stages is the same; they differ in the learner's access to that anticipation.

The content-specific constructs are rooted in the work of Steffe et al. on children's construction of number schemes, particularly of numerical composite units (CU) through mental activities of iterating the unit of one (1's, see Steffe & Cobb, 1988; Steffe & von Glasersfeld, 1985). Steffe (1994; Steffe & Cobb, 1998) proposed that a child who has constructed CU can operate on such units not only additively (e.g., counting-on to solve a missing-addend problem), but also multiplicatively, via simultaneously applying her counting scheme to CU and to the 1's that constitute the CU (e.g., a child may find 3x4 by counting from 1 to 12 in 'triplets' as in '1-2-3, 4-5-6, 7-8-9, 10-11-12', while keeping track of those triplets on the other hand's fingers, '1, 2, 3, 4'). Most importantly, in using mDC, the child creates a scheme of correspondence, where one CU is distributed across the other. In our example, each CU of 3 is distributed into the composite unit items that make up '4'. Thus, mDC enables a child to quantify, in the absence of objects (i.e., in anticipation), the total number of 1's that are embedded in a given number of same-size CU without having to count each and every singleton. It is important to clarify here that mDC refers to the mental quantification of the units—not to the manner in which it is executed (e.g., using fingers, or making tally marks to monitor each CU, or mentally counting the CU).

To complement the *Ref*AER* with a pedagogical approach, Tzur (2008) elaborated on Simon's (1995) and Simon & Tzur's (2004) teaching approaches by proposing a 7-step cycle. It proceeds from specifying students' available conceptions and the intended mathematical ideas, through identifying an activity sequence they can carry out, designing and implementing tasks that may engage them in such a sequence, to monitoring students' progress and orienting their reflection via intentional introduction of follow-up tasks/prompts. In this study, the tasks were designed to trigger the learner's setting of a global goal of finding the total number of 1's (Unifix cubes) embedded within a given number of CU ('towers' of cubes). Once such situations are recognized, the teacher can hide the cubes to encourage the learner's creation of a fundamental sub-goal – namely, to keep track of how of the number of CU – and introduce the activity of mDC as a means to accomplish that sub-goal. Numbers for tasks were chosen to require more than two hands, hence a transformation in the child's available activity of counting all 1's as a single sequence of numbers (e.g., 1-2-3, 4-5-6, etc.). A child's inability to keep track could be resolved by introducing another set of items on which to keep track of CU accrual, and orient her attention to the stopping point of mDC when she accounted for all of the CU.

## Methodology

This study was part of a teaching experiment (Steffe, Thompson, & von Glasersfeld, 2000) with three 4th graders and eleven 5th graders, designed to develop multiplicative reasoning in elementary school students with (or at risk of) learning disabilities in mathematics[i]. Three teaching episodes with one student, Megan (a student at risk), were conducted over the course of 4 weeks by the sixth author. Megan was selected for this study because, prior to the work presented in this paper, she had constructed the anticipatory stage of using composite units numerically (e.g., for missing addend tasks).

The teaching episodes consisted of students playing a game called "Please Go Bring Me..." (PGBM). It involves one player sending another to a box containing individual Unifix cubes and instructing her to create a tower *m* cubes high. The 'bringer' returns the tower to the table and the process repeats until she brings *N* towers of *m* (henceforth notated $NT_m$). Three principal questions are then asked: (a) How many towers did you bring? (b) How many cubes are in each tower? And (c) How many cubes do you have in all? These questions prompt the child to identify, respectively, the number of composite units (CU), the unit rate (UR), and the total number of cubes (1's). A version of the game that we used frequently utilized "What if?" tasks, which require figuring out her answers in the absence of the cubes (e.g., by asking her to pretend she brought them, or by covering the towers).

Data from the episodes consist of field notes, videotapes, transcripts, and notes from ongoing analysis sessions. The research team initially analyzed episodes soon after conducting them, focusing on significant events and on necessary modifications to the plan for the next teaching session(s). A second round of analysis highlighted critical events in the transcripts of the sessions, where the team inferred Megan's thinking processes at the participatory or anticipatory stages via attending to her language and actions. Final, retrospective analysis involved a team discussion of the highlighted segments, which were integrated into a story line of her growth in multiplicative reasoning. The episodes included in the analysis begin after Megan had become familiar and comfortable with the basic form of PGBM (with cubes).

## Analysis

In session 1, the teacher asked Megan: "Pretend I send you to get towers of 4… and I asked you to bring 7 towers of 4. Can you figure it out, using any other way except bringing those and counting the cubes, how many cubes you would have now?" Megan simply could not answer the question. Even after the teacher offered paper and pencil, and later said she could use her fingers, she threw up her hands and exclaimed, "I can't do it." Her inability to attempt this initial task of $7T_4$ indicated that at this point she had no access to mDC. Below, we present excerpts of critical events from three consecutive teaching sessions and suggest how they promoted Megan's transition to an anticipatory stage of mDC.

### *Promoting Construction of the Participatory Stage*

In order to move Megan to the participatory level of mDC, the teacher suggested that Megan use her fingers to keep track of the number of towers while counting the number of cubes:

*Excerpt 1 (Session 1, Introduction to Double Counting)*

T: Let me suggest the following. I'll give you my fingers for every tower. Okay? Every time we have a tower that's (holds up one finger) one tower of four– how many do we now have? You can use your fingers for [counting] the four [cubes].

M: So 4.

T: What if I brought another tower (raises a 2nd finger)?

M: 8.

T: What if I brought another tower (raises a 3rd finger)?

M: 12? (Doesn't look confident.) No wait.

T: You can use your fingers to figure it out.

M: (Counts under her breath.) 16? No!

T: So we had 4 – now let's use your fingers (shows counting on from 4 on his other hand) 5-6-7-8. And then, 9-10-11-12.

M: Yeah.

T: So now we have 3 [towers], what if we added another one (raises a 4th finger)?

M: 16.

T: Ok another one.

M: (Counts on her fingers under the table) 17-18-19-20.

T: So with 5 we have 20. We still have to go [bring towers] two more times.

M: (Counts on her own fingers) 21-22-23-24; 28.

The exchange in Excerpt 1 enabled Megan to start developing an intentional method for keeping track of the number of towers and to anticipate when to stop counting cubes. The teacher's continual prompts of, "What if I brought another tower?" oriented Megan's Type-I reflections between the accruing effects of the double counting activity and the global goal of finding the total number of cubes. This was possible because she could assimilate indicating a CU by the teacher's finger into her available numerical composite unit scheme. Consequently, Excerpt 1 provides a window to two important facets of the work: Megan's early shift to the participatory stage of mDC and how the teacher used tasks/prompts to promote this shift. Following the prompts, Megan knew to operate on the proper unit (cubes) with her number sequence, including anticipating that 4 cubes comprised a single CU (tower). Thus, the task, which required her to retrieve only one tower at a time and to count after each new acquisition, seemed to promote Megan's *coordination* of the two number sequences as evidenced by her finishing of the last two towers without needing to actually go get them.

Megan's construction of the participatory stage for mDC became evident in the task that followed (Excerpt 2).

*Excerpt 2 (Session 1, Participatory Double Counting)*

T: So you have 6 towers of 3 over there, and you use your fingers or your brain or (jokingly) your hair, or your blinking, or whatever, figure out how many are there all together… Can you put your fingers [above the desk] so I can see what you did?

M: (attempts to skip-count, first by 3's , then by 6's) 3-6-12-19? No, wait. (double counts, nodding her head as though counting, e.g., 4, 5, 6, but only speaking the total after each tower out loud) 3-6-9-12-15-18?

T: So 18? That's very good … I saw you putting 3, almost immediately; then 6 almost immediately. Then you started and recounted… So, could it be you said in your head, 7-8-9, 10-11-12, 13-14-15, 16-17-18 (puts a finger for each triplet)?

M: (nods yes)

Excerpt 2 indicates that the work on the previous task and the availability of counting triplets enabled Megan to solve $6T_3$ while internalizing the differentiation of 1's (cubes) from CUs (towers), distributing her units across each of the re-presented units, and coordinating the addition of the cubes and the number sequence of the CUs (tower). Two interventions were key to the formation of this coordination. The teacher began the task by imposing a constraint on Megan: she was not allowed to use paper and pencil to draw the $6T_3$. This oriented her to move away from simply counting individual (drawn) cubes. The teacher's prompt, "Can you put your fingers so I can see what you did?" provoked a Type-I reflection as Megan revisited the use of mDC *for* finding the total, as evidenced in her immediate self-correction after the first attempt ("3, …, 19") toward using mDC *intentionally* in distributing the unit rate (3 cubes/tower) over the number of CU (towers).

Megan's work on those two tasks indicated that she was in the participatory stage of mDC. We did not expect she could yet spontaneously call up the activity sequence, but we did expect she would use mDC when prompted. To test our hypothesis, we began the next episode by testing if Megan was at the anticipatory stage of mDC by engaging her in a prompt-less situation.

*Excerpt 3 (Session 2, Test Anticipatory/Participatory Double Counting)*

T: Pretend you were going (to retrieve a tower of Unifix cubes), and I sent you to get a tower of 3; another tower of 3, and another (etc.). And you brought, think of $7T_3$. Can you figure out how many cubes are there?

M: (Thinks – uses her fingers to count 1-2-3, 4-5-6, (inaudible speech), gets up to the 6th finger and gets lost.) Ok. I just forgot.

T: Ok. Just take your time. If you need my fingers, you can use them.

M: 3, 6... (Starts using teacher's fingers, but then goes back to her own.) 3, 6... 21. I think.

T: How did you get 21?

M: I added three 7 times.

T: Did you do this? (Demonstrates double counting with one hand monitoring) You raised one finger and said 1-2-3. Then you raised another finger and said 4-5-6 [and so on]. Is that what you did?

M: Yeah.

T: Ok. Let's see if your answer is true.

M: (Builds 7 towers of 3 and counts cubes.) 3, 6, 9, 12; 13-14-15; 16-17-18; 19-20-21.

Excerpt 3 indicates that Megan was yet to construct the anticipatory stage of mDC, as she became lost prior to the teacher's prompt for using *his* fingers. Once prompted, however, she could immediately regenerate an anticipation of the *AER* for mDC. She momentarily used the teacher's fingers, but then internalized the activity as evidenced in her shift to her own fingers for successfully completing mDC to reach her global goal. It confirmed our hypothesis, and led to interventions for promoting transition to the anticipatory stage via tasks with larger numbers.

*Promoting Construction of the Anticipatory Stage*

*Excerpt 4 (Session 4, In transition to anticipatory stage of Double Counting)*

T: Pretend you have a tower of 6, another tower of 6, and another [etc.]. Seven towers of 6. [How many cubes] would you get?

M: Um, 6, 12, 24 (gets lost on her fingers) I don't know. That's hard.

T: You can use my fingers.

M: I can't. That's hard.

T: That's harder, because I gave you larger numbers. [See if you can use] my fingers for the number of towers and use yours to count how many in each. So you said the first one is going to be 6 (puts out one of his fingers), then you said 12. (Puts down another finger) Then you started struggling. Use your fingers to add from 12, 6 more.

M: (Counts-on with her fingers) 12; 13-14-15-16-17-18.

T: Ok that's another tower (puts out another finger). That's 3 towers.

M: (Counting-on with her fingers.) 19-20-21-22-23-24 (Pauses for teacher to put a finger); 25-26-27-28-29-30 (Pauses for teacher); 31-32-33-34-35-36; 37-38-39-40-41-42.

T: Should we stop now, or go on? I said 7 towers.

M: Yeah, that's it.

When finding the number of cubes in $7T_6$, Megan struggled with the size of the numbers because each CU was larger than 5 (her fingers). This brought about her Type-2 reflection, evidenced in her realization ("I can't. That's hard") that using mDC would be difficult for the current situation because, *unlike previous situations*, she would not be able to simultaneously hold the number of towers and count the number of cubes. This Type-2 reflection, however, enabled her to easily assimilate the teacher's suggestion to use his fingers to keep track of CU and she immediately completed the task. It was her spontaneous contribution, evidenced in the intentional pause until the teacher raised his next finger, which led us to conjecture Megan might solve a similar task in the next week's episode at an anticipatory level.

To test our conjecture, we introduced a problem situation at the beginning of the following week's episode that required Megan to use mDC in a different context, asking her to create a PGBM situation (cubes, towers) that would be equivalent to having 7 baskets with 8 chicks in each. Megan immediately built a tower with seven same-color cubes and one different color cube, counting under her breath, "1-2-3-4-5-6-7-8." She then continued building more $T_8$ of the same color pattern until she had $7T_8$, at which point the teacher asked if she could figure out the total number of cubes. Megan spontaneously asked, "Can I use your hands?" and proceeded to count individual cubes on her fingers while counting towers on his, successfully stopping at 56. That is, Megan no longer needed a prompt. Rather, she clearly anticipated and spontaneously carried out the entire mDC activity sequence. The way she built her towers to present chicks and baskets, and her initiative for using the teacher's fingers, indicated that she intentionally (a) distinguished between the CU (towers) and UR (cubes/tower) and (b) used mDC to determine the total. Megan assimilated the chicks and baskets task into her global goal of finding total cubes and independently called up the activity sequence needed for multiplicative coordination of CU: differentiate 1's (cubes) from CUs (towers), distribute her units of 8 across each of the re-presented seven CU, coordinate the addition of the cubes and the number sequence of the CUs (tower), and employ two sets of objects (fingers) to keep track of both counts.

## Discussion

This study demonstrated a fundamental transition to multiplicative thinking. At the beginning of our analysis, we saw that Megan, a student at risk in mathematics, had not constructed mDC, putting her at a disadvantage with her peers (2-3 years behind). Through a *Ref**AER* designed intervention, Megan learned to spontaneously call up mDC for reaching her goal in various multiplicative situations. Megan's intentional translation of and solution to the mDC task in the last episode, including her request to use another set of fingers, indicated the commencement of her anticipatory stage of a units-coordinating scheme to which Steffe (1994) refers as an implicit concept of multiplication. Most importantly, this study demonstrated how Tzur and Simon's (2004) stage distinction for determining a learner's available conceptions could guide selection of tasks and prompts for transforming these conceptions. Such guidance included the introduction of double counting on one's and another person's fingers upon shifting to 'for pretend' tasks as a means to the sub-goal of simultaneously keeping track of clearly differentiated CU and 1's, progressing from small to large numbers, and continually orienting the learner's reflection onto the critical questions of when to stop counting (e.g., when Megan paused for the teacher's next finger).

## Endnotes

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# COMPARING U.S. AND TAIWANESE PRE-SERVICE ELEMENTARY TEACHERS' PROCESSES IN REASONING AND SOLVING FRACTION PROBLEMS

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*This study investigated fraction problem reasoning and solving processes of pre-service elementary teachers from the United States and Taiwan. Eight open-ended problems were given to eighty-three pre-service elementary teachers, forty-one from the U.S. and forty-two from Taiwan. The results show that U.S. pre-service teachers were outperformed by their Taiwanese counterparts and tended to use more intuitive and visual but less formal and symbolic reasoning and solving processes compared to their Taiwanese counterparts. U.S. pre-service teachers had greater difficulty in solving non-pictorial problems embedded in a linear measure or set model than in an area model.*

## Introduction

The purpose of this study is to compare pre-service elementary teachers' processes in reasoning and solving fraction problems across countries. A study attempting a comparison of mathematics education between different educational traditions has been believed helpful to recognize the perpetual challenge to improve the quality of mathematics education in detail, and those based on the West and East Asia appear particularly promising for a cross-national comparison (ICMI, 2000). From the most recent Trends in International Mathematics and Science Study (TIMSS) report, we can find a substantial gap in student mathematics achievement existed between the five East Asian countries including Taiwan, Republic of Korea, Singapore, Japan, and China (Hong Kong SAR) and the United States (Mullis, Martin, & Foy, 2008). Meanwhile, the Programme for International Student Assessment (PISA) conducted in 2006 indicates that Taiwanese students had the highest level of mathematics achievement among participating countries including the U.S. (OECD, 2007). This study chose the U.S. and Taiwan to be the two comparative countries to illuminate the current interests on cross-national research of mathematics education.

It has been revealed that the countries with better quality of mathematics knowledge produced higher mathematics achievement (Akiba, Letendre, & Scribner, 2007; Kulm, 2008). Researchers also indicated that U.S. teachers have a weak grasp of basic mathematics knowledge as compared with their East Asian counterparts (Ma, 1999). With the gap of student mathematics achievement between the U.S. and the East Asian countries, therefore, it would not be a surprise that contrasting performances of mathematics teacher knowledge between the U.S. and East Asian countries can be discovered. Yet, in-depth studies on the relative differences of teacher knowledge are very limited. Without recognizing the relative insights and patterns of their differences in knowledge, the efforts for improving the quality of mathematics teacher knowledge cannot gain a focus and may result in a "mile wide and an inch deep" – the characteristic of current mathematics curriculum in the U.S. (Mirra, 2008).

## Theoretical Perspectives

The theoretical framework for this study derives from Shuman's (1986) discussion of subject matter content knowledge. Mathematics content knowledge refers to knowing a variety of ways in which "the basic concepts and principles of the discipline are organized to incorporate its facts" and "truth or falsehood, validity or invalidity, are established" (Shulman, 1986, p. 9). Mathematics content knowledge is rooted in how the knowledge of mathematics facts is coordinated with deeper understanding and application (Kahan, Cooper, & Bethea, 2003). Our focused topic is fractions – the seeds of advanced mathematics. Difficulty with fractions is a major obstacle for further progression in mathematics, including algebra, as described by the National Mathematics Advisory Panel (2008). Research studies on U.S. pre-service teachers have shown that many possessed limited concepts and operations of non-whole numbers (Azim, 1995; Ball, 1990a, 1990b; Graeber, Tirosh, & Glover, 1989; Tirosh, 2000; Simon, 1993). They concentrated on remembering rules and mastering standard procedures, and they often lacked comprehensive understanding of mathematical ideas and procedures (Ball, 1990a).

As was the case with the international student achievement studies, a substantial gap of mathematics knowledge was found between the U.S. teachers and their East Asian counterparts (Zhou, Peverly, & Xin, 2006). Significant differences were also found on their predication and expectation about students' solutions (Cai, 2004, 2005; Cai & Wang, 2006). With the awareness of a cross-national gap in mathematics teacher quality, teacher education programs in the U.S. have been encouraged to enhance their mathematics courses offered to their pre-service teachers and to provide more professional development opportunities to school teachers (Kulm, 2008). In addition, the approach of having elementary school teachers as mathematics content specialists has been proposed (Li, 2008). Elementary teachers need to know various mathematical topics for teaching, but teacher education programs typically include more training in pedagogy methods than in content knowledge (Hill, Schilling, & Ball, 2004). By focusing the need for expertise on fewer teachers who are specialized in elementary mathematics teaching, there could be a more practical alternative than increasing all elementary teachers' content knowledge (National Mathematics Advisory Panel, 2008). Still, more high-quality research must be undertaken in order to create a sound basis and provide different approaches for the mathematics preparation within pre-service elementary teacher education (National Mathematics Advisory Panel, 2008). Through comparing the problem reasoning and solving processes across countries, this study seeks to find the insights needed for developing a deeper understanding of mathematics called for by Ma (1999) and the Conference Board of the Mathematical Sciences (CBMS, 2001).

## Research Questions

Two research questions guided this study: (a) How well are pre-service elementary teachers from the U.S. and Taiwan capable of coordinating mathematics facts to reason and solve fraction problems, and (b) what are the similarities and differences of problem solving and reasoning processes between two groups of pre-service teachers?

## Methodology

*Participants*

Participants include eight-three pre-service elementary teachers, forty-one from the U.S. and forty-two from Taiwan. The purposive sampling was adopted to determine the participants (Leedy & Ormrod, 2005). On the U.S. side, the participants were selected from a regional teaching institution. On the Taiwanese side, the participants were selected from a traditional teacher education university. The selection unit is a class instead of an individual, but the number of total participants from each country is almost the same. Since most pre-service elementary teachers in Taiwan have their own content concentration, this study excluded the mathematics and science education majors to avoid their content strength contributing to a potential difference.

*Data Collection*

The results were drawn from the participants' solutions for eight open-ended problems designed for supporting another cross-national study. The collected solutions would allow us to know how well pre-service elementary teachers from the U.S. and Taiwan were capable of using mathematics facts to solve problems. The results from their solutions would also provide a comparison of the similarities and differences of reasoning processes between two groups of pre-service teachers. Table 1 below provides a summary of the eight open-ended problems representing combinations of two basic fraction concepts (part-whole relationship and fair sharing) and three common contextual models (area, linear measure, and set). Several problems were also developed into pairs to make an appropriate comparison: Problems #2 and #6 were embedded in the same mathematical concepts and contextual models, but only problem #2 has a pictorial illustration as part of the problem statement. Both problems #1 and #7 involve the concept of part-whole relationship, but they are different in terms of contextual models. Both problems #3 and #5 involve the concepts of fair sharing, but they are rooted in different contextual models. To measure the influence of visual and concrete representations, problems #2 and #8 include a pictorial illustration.

Table 1. *The Problem Concepts and Models*

| Concepts | Models | Problem Numbers |
|---|---|---|
| Part-Whole Relationship | Area | #1 |
| | Linear Measure | #8, #4 |
| | Set | #2, #6, #7 |
| Fair Sharing | Area | #5 |
| | Linear Measure | #3 |

*Data Analysis*

The technique of content analysis was utilized to identify specific characteristics of collected data. Coding the collected data into categories relevant to the research objectives is an essential procedure of a content analysis (M. Gall et. al, 1996). All the mathematical solutions were treated as the data sources and were coded into five success levels: failing (0), poor (.25), weak (.50), fair (.75), and good (1). The success rate for a problem was calculated by averaging the sum of scores from all pre-service teachers in the same group. To gain insights of pre-service teachers' reasoning and solving processes, each data source was coded with respect to its mathematical representations as well as specific characteristics of the solution procedures.

## Results

To compare the problem reasoning and solving processes between U.S. and Taiwanese pre-service elementary teachers in reasoning and solving fraction problems, we list the success rates and calculate their differences for each problem. As shown in Table 2, U.S. pre-service teachers' ability of reasoning and solving fraction problems were much poorer than that of their Taiwanese counterparts. The contextual models appear to be associated with the success rate of U.S. pre-service teachers but not with that of their Taiwanese counterparts. Both problems #1 and #5, the two problems with smaller differences of success rates between groups, are contextualized in an area model. Within the same contextual model and complexity, U.S. pre-service teachers had a higher success rate on a pictorial problem than on a non-pictorial one.

Table 2. *Success Rates and Differences between Groups for Each Problem*

| | #1 | #2 | #3 | #4 | #5 | #6 | #7 | #8 |
|---|---|---|---|---|---|---|---|---|
| U.S. | 56.1% | 70.8% | 29.3% | 33.5% | 76.4% | 47.6% | 25.6% | 28.7% |
| Taiwan | 95.0% | 97.0% | 89.7% | 100% | 88.2% | 97.5% | 89.9% | 95.2% |
| Difference | 38.9% | 26.2% | 60.4% | 66.5% | 11.8% | 49.9% | 64.3% | 66.5% |

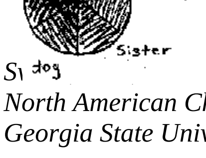
*Problems #1 and #7*

Both problems #1 and #7 are multiple-step problems, but contextualized in different models (see Table 3). It is more challenging for an individual to solve problem #1 than to solve problem #7. Only straightforward reasoning processes are needed to solve problem #1, but unnatural backward reasoning processes would be required to solve problem #7. The difference in the nature of the problem had a greater impact on U.S. pre-service teachers than on their Taiwanese counterparts. For the U.S. group, there was a wide difference between success rates for solving problems #1 and #7. The success rates of Taiwanese pre-service teachers were maintained at a relatively high level compared to their U.S. counterparts. Two groups of pre-service teachers used very different reasoning forms. For problem #1, particularly, most U.S. pre-service teachers used graphic drawing while the majority of their Taiwanese counterparts used symbolic equations to solve this problem. From the Figure 1 and Figure 2, we can see the contrasting reasoning processes between two groups. It is worth noticing that twenty-three Taiwanese pre-service teachers but only one U.S pre-service teacher used the algebraic symbol "*x*" to represent the unknown size of the whole. Figures 3 and Figure 4 are two examples illuminating the difference of problem reasoning and solving processes between two groups for problem #7.

Table 3. *Problem statements, Success Rates, and Reasoning Forms for Problems #1 and #7*

| Problem Statements | U.S. | | Taiwan | |
|---|---|---|---|---|
| #1. Mom baked a cake. Dad ate 1/6 of the cake. Brother ate 1/5 of what was left. Sister ate 1/4 of what was left after that. The dog ate 1/3 of what was left after that. Another kid ate 1/2 of what was left after that. How much of the original cake was left for Mom to eat? | Success | 56.1% | Success | 95.0% |
| | Symbolic | 9.8% | Symbolic | 100.0% |
| | Pictorial | 65.9% | Pictorial | 7.5% |
| #7. At the circus, the clown was busy counting all of the animals that performed. The clown figured that one half of the animals were horses, 1/4 of the remaining were big cats, and the rest were 9 monkeys. How many animals did the clown see altogether? | Success | 25.6% | Success | 89.9% |
| | Symbolic | 26.8% | Symbolic | 95.2% |
| | Pictorial | 36.6% | Pictorial | 21.4% |

*Figure 1.* A U.S. solution sample for problem #1




$\frac{1}{6}$ of the original cake was left for mom to eat after everyone was done. because eac time someone eats, you taking one piece aw from whats left.

*Figure 2.* A Taiwanese solution sample for problem #1

解: $[(1-\frac{1}{6})\times(1-\frac{1}{5})\times(1-\frac{1}{4})\times(1-\frac{1}{3})\times(1-\frac{1}{2})]$

$= [\frac{\cancel{5}}{6}\times\frac{\cancel{4}}{\cancel{5}}\times\frac{\cancel{3}}{\cancel{4}}\times\frac{\cancel{2}}{\cancel{3}}\times\frac{1}{\cancel{2}}]$

$= \frac{1}{6}$

*Problems #3 and #6*

As shown in Table 4, the U.S. group of pre-service teachers had much greater difficulty in solving problem #3, a linear measurement model, than in solving problem #5, an area model. Compared to U.S. pre-service teachers, Taiwanese pre-service teachers appeared to be much more adaptive to problems in different context models, and they were much more successful. Only minor difference existed between their success rates in solving both problems. Same as the finding for solutions for problem #1 and #7, it was found that the nature of reasoning processes demonstrated by the U.S. pre-service teachers is different from those demonstrated by their Taiwanese counterparts. U.S. pre-service teachers preferred to use pictorial processes to solve problems than using symbolic processes. In contrast, their Taiwanese counterparts had an inverse tendency.

Table 4. *Problem Statements, Success Rates, and Reasoning Forms for Problems #3 and #5*

| Problem Statements | U.S. | | Taiwan | |
|---|---|---|---|---|
| #3. A 2 meters long strip of paper was folded into three equal pieces. How long was each piece? | Success | 29.3% | Success | 89.7% |
| | Symbolic | 29.3% | Symbolic | 65.9% |
| | Pictorial | 63.4% | Pictorial | 34.1% |
| #5. Aunt Rachel had 2 cupcakes for the kids to share equally. There were three kids. How much did each kid get? | Success | 76.4% | Success | 88.2% |
| | Symbolic | 23.8% | Symbolic | 61.0% |
| | Pictorial | 73.8% | Pictorial | 39.0% |

Four Taiwanese pre-service teachers used advanced algebraic concepts to reason how to solve problem #5. As shown in Figure 3, a pre-service teacher from Taiwan addressed that each kid would obtain 2/3 of a cake for the situation of equal cakes and 1/3 $(x+y)$g cake for the situation of unequal cakes. None of U.S. pre-service teachers tried to apply the concepts of unknown algebraic symbols to solve this problem. We also found that seven U.S. pre-service teachers used decimals as their answers. A U.S. pre-service teacher used an intuitive reasoning to solve this problem by adding decimals repeatedly to make to make the quantity of 2 meters (see Figure 6). Similarly, seven Taiwanese pre-service teachers also used decimals as their answers. However, instead of using decimals to solve the problem, they directly transferred their fraction results from division computation into decimal answers.

*Figure 3.* A U.S. sample for problem #7

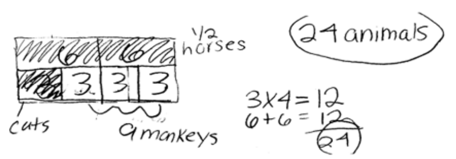
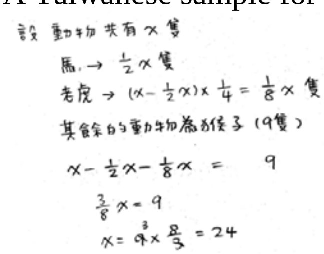



*Figure 4.* A Taiwanese sample for problem #7

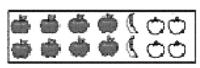



In addition, it was found that fourteen pre-service teachers, seven from each group, conducted a unit conversion. Nine of them tried to convert the given unit “meter” into the smaller scale unit “centimetre,” so the size of the number for representing the length of paper strip has to be expanded to 100 times. Their responses include $200\frac{2}{3}$ centimetres, 66.67 centimetres, and so on. The other five pre-service teachers were from the U.S. group. They tried to transfer the unit “meter” used in the metric system into “foot” or “inch” used in the English system, but did not make a successful unit transformation because of an insufficient knowledge of the metric system.

*Problem #2 and #6*

To examine the influence of the pictorial illustration further, we compared pre-service elementary teachers’ ability in solving the problems #2 and #6 (see Table 5). Notice that both problems involve the application of the same fraction concept and contextual model. From the problem-solving results for #2 and #6, we found that U.S. groups of pre-service teachers had much greater difficulty in solving the problem #6, which does not include a pictorial illustration than in solving the problem #2, which includes a pictorial illustration. It appeared that Taiwanese pre-service teachers were free from the influence of pictorial illustration, and their success rate maintained at relative high level compared to their U.S. counterparts.

Table 5. *Problem Statements, Success Rates, and Reasoning Forms for Problems #2 and #6*

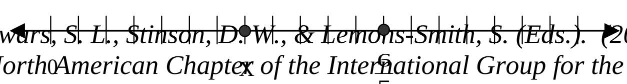
| Problem Statements | U.S. | | Taiwan | |
|---|---|---|---|---|
| #2. A basket contained 8 red apples, 2 bananas, and 4 green apples. What fraction of the apples is green?  | Success | 70.8% | Success | 97.0% |
| | Symbolic | 14.6% | Symbolic | 85.7% |
| | Pictorial | 19.5% | Pictorial | 0.0% |
| #6. Brandon has a box which contains 7 red marbles, 3 purple buttons, and 5 green marbles. What fraction of the marbles is green? | Success | 47.6% | Success | 94.5% |
| | Symbolic | 12.2% | Symbolic | 81.0% |
| | Pictorial | 53.7% | Pictorial | 4.8% |

*Problem #4 and #8*

As shown in Table 6, problem #4 is not a straightforward problem, but all of Taiwanese pre-service teachers provided correct solutions. Consistent with the findings for the other problems, Taiwanese pre-service teachers had a strong preference in adopting symbolic ways to reason and solve problems #4. Although U.S. pre-service teachers used more symbolic process than pictorial processes to reason and solve problem #4, the insights of their symbolic processes are different from those of Taiwanese pre-service teachers’ symbolic processes. U.S. pre-service teachers only engaged in the level of numerical symbols while their Taiwanese counterparts would involve the usage of algebraic symbols.

Table 6. *Problem Statements, Success Rates, and Reasoning Forms for Problems #4 and #8*

| Problem Statements | U.S. | | Taiwan | |
|---|---|---|---|---|
| #4. Jim jogged $1\frac{1}{2}$ miles yesterday. This is $\frac{3}{8}$ of his weekly goal? How many miles does he plan to run each week? Explain. | Success | 33.5% | Success | 100.0% |
| | Symbolic | 36.6% | Symbolic | 97.6% |
| | Pictorial | 22.0% | Pictorial | 14.3% |
| #8. What is the value of x? Explain. | Success | 28.7% | Success | 95.2% |



| | | | |
|---|---|---|---|
| Symbolic | 0.0% | Symbolic | 95.2% |
| Pictorial | 43.9% | Pictorial | 54.2% |

The reasoning processes of U.S. pre-service elementary teachers were more intuitive, informal, and pictorial than those of their Taiwanese counterparts while solving problem #4. Figure 7 and Figure 8 illuminate the difference of their reasoning processes. Problem #8 was one of most challenging problems for U.S. pre-service teachers, but an easy problem for Taiwanese pre-service teachers. Twenty out of forty-one of pre-service teachers from the U.S. omitted this problem. Most U.S. pre-service teachers approached this problem used a trial-and-error way to find the value for x reason while 95.2% of Taiwanese pre-service teachers incorporated symbolic and proportional reasoning processes to solve this problem.

Figure 7. A U.S. solution sample for problem #4

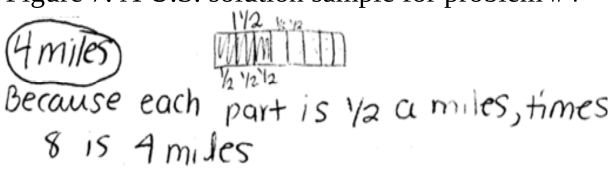
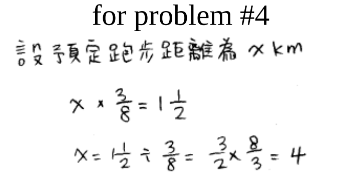



Figure 8. A Taiwanese solution sample for problem #4



## Discussion and Conclusions

This study observed similar reasoning processes across groups. As addressed in the results, several pre-service elementary teachers from both groups converted the fraction for representing the magnitude of length into a decimal and/or converted the measure unit of length for a 2 meter strip into a smaller scale unit. Pre-service teachers might try to do the unit conversion to avoid the existence of fractions on the number line as described by Kerslake (1986). Additionally, they might perceive that the magnitudes of length should not be less than 1. Further investigation such as interviews should be conducted to explore their understanding about the magnitude of a quantity to provide appropriate treatments. For example, if their conceptual and procedural knowledge about fractions with magnitudes less than 1 is not consistent with fractions with magnitudes greater than 1, the background knowledge of fractions with magnitudes in each range needs to be taught directly (National Mathematics Advisory Panel, 2008).

Additionally, this study adds to our understanding of issues in the preparation of teachers' mathematical ability. Consistent with previous cross-national studies, this study did not find sufficient evidence to draw the conclusion that U.S. pre-service teachers have established a satisfactory ability of reasoning and solving fraction problems as their Taiwanese counterparts. U.S. pre-service teachers favored using intuitive, informal, and pictorial strategies more than their Taiwanese counterparts. We suspect that is what these pre-service elementary teachers learned when they were upper elementary or middle school students. Without any intervention, it is very likely that these U.S. teachers will adopt intuitive mathematic methods in their future classrooms without seeking to enter advanced mathematical reasoning and problem solving skills. We conclude then research on what U.S. upper elementary and middle school mathematics teachers should know and what they should expect their students to learn needs to be undertaken.

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# MATHEMATICS-FOR-TEACHING: AN ETHNOPOETIC PERSPECTIVE

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*Ongoing research seeks to discover and describe the knowledge of mathematics-for-teaching teachers employ to facilitate students' understandings. Convinced by Davis and Simmt (2006) that much of this knowledge is implicit, this study seeks to show that teachers' unarticulated mathematics-for-teaching competencies can be studied through ethnopoetics (analysis of the form of teachers' talk). The findings broaden the range of competencies that may be called mathematics-for-teaching and suggest that mathematics-for-teaching is context-specific. Implications for teacher professional development and research are discussed.*

## Purpose of the Study

The mathematics needed for teaching is different than classical mathematical knowledge, but what counts as necessary mathematics-for-teaching is not yet fully known. Ball, Bass, and Hill (2004) have studied what teachers do (job analysis) to theorize mathematics-for-teaching and include practices such as interpreting students' mathematical work and unpacking the mathematical ideas in a problem. Davis and Simmt (2006) expand on those notions by observing what teachers say about teaching in interactions around mathematical content in a teacher study group. They argued that much of teachers' knowledge of mathematics-for-teaching is implicit. They said, "We believe that a key (and perhaps *the* key) competence of mathematics teachers is the ability to move among underlying images and metaphors—that is, to *translate* notions from one symbolic system to another." (p. 303).

These studies have reported on the *content* of teachers' talk about mathematics and mathematics teaching. However, if much of teachers' knowledge of mathematics-for-teaching is implicit, as Davis and Simmt (2006) claim, then looking beyond the content to the *form* of teachers talk is needed. *"How* something is said is a part of *what* is said" (Hymes, 1972, p. 59). How teachers' ways with words perform their understandings of mathematics-for-teaching requires further study.

I analyzed two teachers' narratives of a lesson about fractions on a number line to learn about their mathematics-for-teaching. To examine their tacit knowledge of mathematics-for-teaching, I focused on the *form* of their discourse. I unveiled tacit mathematics-for-teaching competencies in both teachers' stories. Whereas I had begun the analysis with an initial perception that one teacher was more mathematically sophisticated, the results challenged my initial perception.

## Theoretical Framework

### *Mathematics-for-Teaching as Discourse: The Centrality of Metaphor*

Davis and Simmt (2006) argued that the ability to move between metaphors is key to mathematics teaching. Metaphor is a discursive tool. In fact, mathematics may be conceptualized as a discourse. That is, mathematics may be theorized as communication (with oneself or others) about mathematical objects (which are constructed through discourse). Metaphor, or the mapping of language from one domain (for example, physical objects) onto another (such as abstract ideas), is central to creating and understanding mathematical ideas (Sfard, 2008). In the sections that follow, I report on teachers' talk about a lesson on fractions and number lines. Therefore, I will next explain some metaphors that are important to understanding those two concepts.

*Grounding metaphors.* Lakoff and Nuñez (2000) theorized four grounding metaphors that ground arithmetic to physical objects—the *metaphor of object collection*, the *metaphor of object construction*, the *measuring stick metaphor*, and the *metaphor of motion along a line*. Fractions are often understood using the *metaphor of object construction*, because they are constructed by dividing a whole into parts. In that case, a key metaphor is *numbers as objects*. In the *measuring stick metaphor*, arithmetic is conceptualized as the use of a measuring stick. Numbers are physical segments, with the basic physical segment being one. Longer segments are greater and shorter ones are less. Arithmetic is putting segments together and taking them apart.

*Linking metaphors.* Linking metaphors take us beyond the physicality of grounding metaphors to more sophisticated ideas. The metaphor of *numbers as points on a line* is an example. It builds on the measuring stick metaphor and the metaphor of arithmetic as motion along a path but maps these ideas onto the concept of a line (instead of a physical object). This requires understanding several other metaphors: a number $P^1$ as point $P^1$ on a line, zero as a point 0 on the line, one as a point 1 to the right of 0, etc. Sophisticated mathematical understandings and are built by layering metaphor upon metaphor, a process called *conceptual blending*. In contrast, teaching mathematics involves unpacking the layers of metaphor (in effect, unblending) and deftly moving between them to facilitate students' conceptual blending.

*Communicative mathematics-for-teaching competencies.* The teachers in this study had participated in a professional development course that focused (in part) on broadening notions of mathematics and mathematical competence. Teachers discuss competencies of students' that have not traditionally been considered mathematical and ways to help students mobilize them to solve mathematical problems. The goal is for the teachers to ask *how* a student is mathematically competent instead of *whether* they are, recognizing that perceptions of competence are culturally-laden. Likewise, in this paper I investigate the competencies teachers bring and how they mobilize them to teach mathematics. I am assuming that there are competencies required for mathematics teaching that have not yet been considered by the literature.

Viewing mathematics as a discourse, it makes sense to study teachers' mathematics-for-teaching competencies by analyzing their discourse. Survey-based work has increased understanding of mathematics-for-teaching and its importance for student learning (Ball, Bass, & Hill 2004). However, how teachers translate this knowledge into discourse is unknown. Such translation requires knowledge of mathematical content and pedagogy, but also communicative competencies, including effective use of metaphor. Additionally, discursive norms vary between contexts, so such work involves understanding culture. I use the phrase *mathematics-for-teaching competencies* to encompass all these ways of knowing.

*Ethnopoetics and Mathematical Discourse*

If much of teachers' mathematics-for-teaching is tacit, then researchers need new tools to access it. To do so, I employed methods from ethnopoetics, a premise of which is that speakers construct arguments through both the *content* and the *form* of speech (Hymes, 1972). Because mathematics is a discourse, mathematical knowledge and arguments are also constructed through both the *form* and *content* of the communication. Consider the following example from the literature. Staats (2008) highlighted the syntactic parallelism through which a child made a mathematical argument about passing out Cuisinaire rods to a group:

1 Yes

2 Cos every fifth one
3 from William
4 is going to be
5 a white,
6 and every fifth one
7 from the next person
8 is going to be
9 a red,
10 and every fifth one
11 from the next person
12 is going to be
13 a green
14 eh? (Staats, 2008, p. 26)

Staats argued that the parallel structure of this speech is the way this child makes his mathematical argument. That is, through a grammatical pattern, he is performing understanding of the arithmetic pattern and generalizing. She points out the importance of looking across sentences to discern arguments made implicitly by the form of discourse. (Staats, 2008).

However, the discourse of mathematics teachers has been primarily studied in terms of content. Little attention has been paid to the form of teachers' mathematical talk. Therefore, the field is missing some important knowledge about how teachers communicate mathematical ideas in their teaching practice. The research question that guides this study is:

- What can be learned about teachers' mathematics-for-teaching through an investigation of the *form* of teachers' discussions of mathematics and mathematics teaching?

## Method

*Context, Data Collection, and Participants*

This paper describes part of a larger study on experienced elementary teachers' narratives of practice as they implemented new mathematics pedagogies. The teachers had participated in a professional development (PD) course on strategies for teaching mathematics with cooperative groups. They were implementing strategies from the workshop during the eight months of data collection.

I conducted pre- and post-lesson individual interviews with teachers about lessons in which they used cooperative grouping strategies. I asked teachers to tell me the story of the lesson and prompted them to fill in details about students' mathematical ideas, their use of teaching strategies learned in the PD course, and their thoughts and feelings during the lesson.

The teachers and I also met together as a group for a total of six hours over several weeks to discuss their use of the strategies. They came prepared to share classroom stories. One teacher would tell a story about teaching, and the group would discuss it. After the group had finished discussing that teacher's story, the second teacher would tell a story of teaching, so on. I video- and/or audio-taped every interview.

The larger study includes four teachers. All had taken the PD course, chosen to be a part of the study, were interested in reflecting on and improving their mathematics teaching practice, and were attempting to implement the group work strategies learned in the PD. They were experienced teachers (with 10-30 years of experience). They varied in school context, grade-level, race, and gender, and in how they identified as mathematically competent (or not).

I report here on two focal teachers who seemed to speak about mathematics-for-teaching differently. Jonathan, a fifth-grade teacher who identified as "the math guy" in his school, was eager to talk about the mathematical understandings of his students, often spending several minutes talking about his students' mathematical ideas with little or no prompting. In contrast, Glynnis, who taught fourth grade, directly identified as "not a math person". When asked about the mathematics at stake in particular tasks she often responded with only a few words, and when asked about her students' understanding of mathematics, she often had less to say than Jonathan. A first impression might lead one to believe that Jonathan had more sophisticated knowledge of mathematics-for-teaching than Glynnis. However, this paper explores the ways with words Glynnis and Jonathan employed to make mathematical metaphors accessible to students.

Both teachers taught variations of the same mathematics task to their students. Students were given some numbers with decimals and asked to plot them on a number line. Both teachers told me their stories of the lesson afterward. In this paper, I will compare parts of the two stories.

*Data Analysis*

After repeatedly listening and watching the interviews and writing analytic memos about their content, I transcribed analytically salient segments of the data. For this paper, I compare two particular stories about the same mathematics task told by two different teachers. I chose to present in detail a small excerpt of each in which both teachers are trying to accomplish the same thing—helping children to conceptualize the number *one* and *fractions* on a number line. I examined the transcripts and audio for the teachers' use of theatrics (such as dialogue and asides) and poetics (including alliteration, cadence, etc.) After experimenting with several ways of arranging the transcripts, I settled on an arrangement that highlights parallel syntax and ideas, in order to illustrate teachers' use of parallelism for facilitating students' conceptual blending.

## Results

*Connecting Mathematically Important Ideas and Moving Between Metaphors*

*Glynnis.* In the following excerpt, Glynnis is describing an event in which she is talking to a student about placing numbers on a number line. I arranged the following transcript excerpt to highlight parallel ideas. In this case, the subject and action of the sentences are on the left, and the mathematical objects are aligned further left as well as highlighted with bold text. When certain actions were mentioned repetitiously, I tried to align those as well (drew and plotting in lines 38-41, for example). I also divided it into stanzas to highlight shifts between metaphors.

**Stanza 1**

| | | | | |
|---|---|---|---|---|
| 1 | actually | | | |
| 2 | one of the women that was so upset | | | |
| 3 | | Recognizing that from | **here to here** | was the |
| 4 | | | **one whole** | |
| 5 | You know cuz I think | | | |
| 6 | we were doing it yesterday | | | |
| 7 | and they were ordering | | **one and,** | |
| 8 | | | **one** | |
| 9 | | it was all | **one and** | |
| 10 | | | **one and,** | |
| 11 | | and it was still, | | |
| 12 | | hundredths and | | |

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

| | | | | |
|---|---|---|---|---|
| 13 | | stuff like that | | |
| … | | | | |
| **Stanza 4** | | | | |
| 38 | So she | Drew | **the number line.** | |
| 39 | And she began to | Do | **a landmark number.** | |
| 40 | Ok cuz she was | Plotting | **one and three,** | |
| 41 | | | **one and three tenths** | |
| 42 | let's just say | | | |
| 43 | I don't remember what | | **it** | was |
| 44 | So she | Drew | **half** | here. |
| 45 | And so she was beginning to figure out. | | | |
| 46 | And then she goes, oh. | | **It** | |
| 47 | would have to be over | | **here.** | |
| **Stanza 5** | | | | |
| 48 | I says so | | | |
| 49 | you're figuring out that between | | **here and here** | |
| 50 | represents | A | **whole,** | |
| 51 | | and the | **whole** | has to be |
| 52 | | | | divided into what. |

Glynnis brought different metaphors of number into relationship via parallelism as she told this story. In Stanzas 1 and 5, notice the alignment of the words "here to here", "one", and "whole". Glynnis drew on a metaphor these children understood--the metaphor of *number as object* (Lakoff & Nuñez, 2000)--to define the space between zero and one as the *object* on which children should act (divide) to construct fractions. However, in using the words "here to here", she made a *line segment* an object. Therefore, she drew a connection to Lakoff and Nuñez's second grounding metaphor of arithmetic—the measuring stick metaphor—which includes the linking metaphors *numbers as physical segments* and *one as the basic physical segment*. The effect of placing different words for different objects in the same syntactical place is that the audience hears them as synonymous.

In Stanza 4, Glynnis used parallelism to bring "number line", "landmark number", "one-and-three-tenths", "half", and "here" into relationship. This is a different metaphor of number, *numbers as points on a line*. The way Glynnis mapped the metaphors of numbers as objects, physical segments, and points on a line onto each other is in the *form* of her language. She did not explicitly state the different metaphors on which she is drawing. However, she brought the ideas into relationship through the form of her speech—by using the language of different metaphors but using the cohesion strategy of syntactic parallelism (all of these mathematical ideas are used as the objects of her sentences) to help the audience map them onto each other.

In Stanza 5 in lines 47-50, Glynnis returned to language of numbers as line segments. However, in line 50-51, when she said, " the whole has to be divided into what", her language was that of the *arithmetic as construction of objects* metaphor, which Lakoff and Nuñez claim is often the metaphor by which children first understand fractions (because fractions are constructed by cutting an object into equal pieces). Glynnis built on this metaphor, adding the new metaphor of numbers as points on a line, to facilitate a conceptual blend: fractional numbers as points on a number line. By moving back and forth between metaphors and substituting one idea for another across lines of speech, she facilitated students' development of sophisticated metaphorical blends.

*Jonathan*. In the following excerpt, Jonathan is telling the story of the number line lesson:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | I | Had | | | | **meter** sticks |
| 2 | I | held up | | | the | **meter** stick. |
| 3 | I | said to them | | This is | a | **meter** stick. |
| 4 | | | | It 's | **one** | **meter.** |
| 5 | | | So | this is | the | **whole.** |
| … | | | | | | |

As Jonathan continued the story, he began to go through students' work and tell me about various groups of students during the lesson. One group had made a diagram like that in Figure 1.

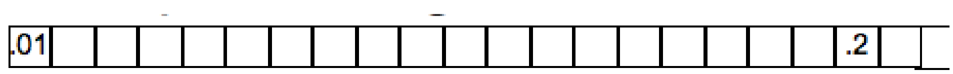


*Figure 1.*

Instead of drawing the kind of number line Jonathan had envisioned before the lesson, the students drew a row of one hundred boxes. They had placed the numbers they were supposed to put on the number line into boxes in the row. Then Jonathan continued with the story:

| | | | | |
|---|---|---|---|---|
| 260 | I guess | | | |
| 261 | I hadn't done enough with | | **number lines.** | |
| 262 | | About what a | **number line** | really means. |
| 263 | **It**'s not | | numbers in order | |
| 264 | **it**'s | | places on a line. | |
| … | | | | |
| 279 | Right so actually | | | |
| 280 | what they did is not said where the | | **spot** | is |
| 281 | but they said | | | |
| 282 | **this square** | Represents | **one (hundredth),** | |
| 283 | so they thought of it more as | | **a fraction** | |
| 284 | Right? | | | |
| 285 | | | **One out of a hundred.** | |

Jonathan also used parallelism to bring ideas together. In lines 1-5 he brought the ideas "meter", "one", and "whole" together. Like Glynnis, he drew on two of Lakoff and Nuñez's metaphors here: numbers as line segments (lines 1-4) and numbers as objects (line 5). Again, these are metaphors that were commonplace in the classroom, and he brought them into relationship by substituting one for another using syntactic parallelism.

In lines 260-264 he added the concepts "number line" and "places on a line" in the same way. However, he explicitly described a number line ("it's") as "places on a line" by placing "it's" in the subject of the sentence (in bold text above). In effect, he used multiple ways with words to build this sophisticated conceptual blend—explicit descriptions of metaphors to clarify concepts, and syntactic parallelism to build connections between them.

In lines 279-286 he used parallelism to bring the terms, "spot", "square", "one hundredth", "fraction", and "one out of a hundred" together with previous ideas. The word "square" (an object) draws on the metaphor of number as object, the words "one hundredth", "fraction", and

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

"one out of a hundred" draw on the metaphor of arithmetic as object construction, and the word "spot" (substituted for point) draws on the metaphor of numbers as points on a line. In lines 279-281 and 283-286 these ideas are grammatically parallel to one another. However, in line 282 Jonathan again placed the mathematical idea (square) in the subject of the sentence to explicate the metaphor his students had demonstrated. Again, he explicitly described a metaphor (resulting in clarity) then used grammatical syntax to facilitate conceptual blending.

*Differences between stories*. Glynnis and Jonathan both used similar poetics to bring concepts into relationship as they moved across metaphors. Why, then, was I initially under the impression that Glynnis's talk was less mathematically sophisticated than Jonathan's? Was it simply because Jonathan made metaphors explicit in his story? In Table 1 I place parts of these transcripts side by to highlight other differences in the form of their talk. Above, I described how Glynnis and Jonathan used parallelism in similar ways to make the moves between metaphors more cohesive. This is indicated in Table 1 as a box with a solid line. However, the rest of their speech is structured very differently. Jonathan's appears to be neat and tidy, only including phrases that line up nicely in parallel (indicated in shaded boxes). In contrast, Glynnis includes an aside (lines 5-13) in which she describes the numbers the students were using in the task. Before and after this aside Glynnis inserts two 3-line stanzas that are grammatically parallel. This brings a sense of cohesion and connection between lines 1-3 and lines 47-49. It seems that a major difference may be in the economy of syntax, as opposed to the level of sophistication of their conceptual blends.

Table 1. *Comparing Glynnis's and Jonathan's Stories*

| Glynnis | | | | Jonathan | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 Actually | | | | 1 | I | had | | **meter** sticks. |
| 2 one of the women | | | | 2 | I | held up | | a **meter** stick. |
| 3 | recognizing that from | **here to here** | was the | 3 | I | said to them | **this** is | a **meter** stick. |
| 4 | | **one whole** | | 4 | | | **It** 's | **one meter** |
| 5 You know cuz I think | | | | 5 | | | So **this** is | the **whole** |
| 6 we were doing it yesterday | | | | | | | | |
| 7 and they were ordering | | **one** and | | | | | | |
| 8 | | **one** | | | | | | |
| 9 | it was all | **one** and | | | | | | |
| 10 | | **one** and | | | | | | |
| 11 | it was still | | | | | | | |
| 12 | hundredths and | | | | | | | |
| 13 | Stuff like that. | | | | | | | |
| … | | | | | | | | |
| 47 I says so. | | | | | | | | |
| 48 You're figuring out that between | | **here and here** | | | | | | |
| 49 | represents a | **whole**. | | | | | | |
| 50 | and the | **whole** | has to | | | | | |
| 51 | | | be divided into what. | | | | | |

## Discussion and Conclusions

Discourse and culture are related--language is spoken and interpreted differently in different contexts. Perhaps both teachers are performing knowledge of their particular students and schools in the form of their talk. Certainly, culture influenced my initial perception their talk.

Both teachers moved among important underlying metaphors, performing a key mathematics-for-teaching skill (Davis & Simmt, 2006). But does Jonathan's story perform a more sophisticated understanding of mathematics-for-teaching than Glynnis's? Maybe. Jonathan did sometimes make ideas explicit in a way that Glynnis did not. However, it is also possible that my initial perception was due to the comparative *elegance* of Jonathan's talk, an aesthetic evaluation of the storytelling that parallels something valued in the culture of mathematics. Einstein said of mathematical proofs, "We are completely satisfied only if we feel of each intermediate concept that it has to do with the proposition to be proved." (Luchins & Luchins, 1990, p. 38). Perhaps values of the mathematics community have been used to judge teachers' mathematics-for-teaching talk and marginalize some mathematics-for-teaching competencies that facilitate student learning. Is it appropriate to apply the same criteria to mathematical pedagogical discourse? An elegant mathematical proof leaves out any indication of the messy work that led to it. If teachers and students are engaging in reasoning and proof, it is likely that the majority of their classroom talk will be ugly by mathematical standards. Additionally, mathematical proofs are written with others in the same discourse community as the intended audience. Teachers' audiences vary widely in regards to culture, age, children's backgrounds, and the like. Acknowledging that teaching mathematics requires more than classical mathematical knowledge means assuming that the discourse of mathematics-for-teaching will be different than the discourse of classical mathematics, in part because the context is different.

My initial perceptions of the teachers were possibly based on how they identified themselves—that is, their own perceptions of their mathematical sophistication. Jonathan identified as "the math guy" and identified as mathematically competent by including mathematics-for-teaching in the content of his talk. Glynnis identified as "not a math person" and identified as less competent by avoiding mathematics-for-teaching in the content of her talk. Glynnis seems to be performing knowledge she is not even aware she has. What would be the influence teachers' practices if they recognized and valued the mathematics-for-teaching competencies they bring to the classroom?

Looking at the form of teachers' discourse for their *mathematics-for-teaching competencies* revealed competencies that the field has formerly overlooked. This study challenges narrow notions of competence. Further attention to the context-specific nature of mathematics for teaching is warranted, as is more work aimed toward broadening our notions of what counts as mathematics-for-teaching, so we can imagine, value, and build on a wider range of competencies teachers bring to the work of mathematics teaching.

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# THE INFLUENCE OF A REFORM-BASED MATHEMATICS METHODS COURSE ON PRESERVICE TEACHERS' CONTENT KNOWLEDGE AND BELIEFS

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## Theoretical Framework

This work is grounded in theories about cognition (i.e., knowledge) and teacher beliefs. Núñez, Edwards, and Matos' (1999) theory of cognition is based on a situated approach that incorporates linguistic, social, and interactional influences. In essence, this theory claims "there is no activity that is not situated" (Lave & Wenger, 1991, p. 33). However, Núñez et al. (1999) argue that thinking and learning are also situated within biological and experiential contexts that shape our understanding of the world. These researchers contend that "knowledge and cognition exist and arise within specific social settings . . . and that the grounding for situatedness comes from the nature of shared human bodily experience and action, realized through basic embodied cognitive processes and conceptual systems" (p. 46). Thus, cognition is a multifaceted social phenomenon that is observed in daily practice and is encompassed by "mind, body, activity and culturally organized settings" (Lave, 1988, p. 1), such as methods courses and everyday mathematics classrooms.

Beliefs are defined as existential presumptions, which are the personal truths everyone holds and are characterized by making judgments and evaluations about phenomena, subject matter, and individuals (Abelson as cited in Parajes, 1992). Individual beliefs endure even when they are contradicted by reason, evidence or experience (Parajes, 1992). Lortie (1975) contends that beliefs about teaching are developed at early ages and are well-established before students enter college. However, changing preservice teachers' beliefs is possible if they have a gestalt shift (Pajares, 1992). Since classroom behavior is the result of beliefs that have been filtered by experience, altering preservice teachers' experiences has the potential to change their beliefs (Parajes, 1992). Teachers' beliefs must be inferred by analyzing their belief statements and behaviors (Goodman as cited in Parajes, 1992). Examination of these belief statements and behaviors are critical to understanding teachers' beliefs about teaching and learning mathematics.

## Purpose of the Study

The purpose of the study was to examine the influence of a reform-based mathematics methods course on elementary preservice teachers' content knowledge and educational beliefs. The research questions that guided these studies were: 1) How did elementary preservice teachers' mathematics content knowledge compare before and after taking a reform-based mathematics methods course? 2) How did elementary preservice teachers' educational beliefs compare before and after taking a reform-based mathematics course?

## Methodology

*Setting and Participants*

Participants were enrolled in one section of mathematics methods during the fall semester of 2004 ($n$ = 25): females (23) and males (2). The study was conducted at a large research university situated in an urban city in the northeastern U.S. The College of Education had an enrollment of approximately 2,100 undergraduate students during the 2004-2005 academic year. The participants constitute a convenience sample since they were arbitrarily enrolled in the reform-based mathematics methods course.

*Data Collection, Data Sources, and Data Analyses*

Mixed methods (quantitative and qualitative) were used to collect data in this study. Quantitative methods were used to compare preservice teachers' content knowledge and efficacy beliefs before and after taking the reform-based mathematics methods course. To measure the development of MCK we used a content knowledge test that measured what elementary teachers were required to teach (Ball et al., 2005). The tests consisted of 50 open-ended items that primarily addressed conceptual and procedural knowledge of fractions, decimals, and percents. We used case studies to collect and analyze qualitative data in this study. Case studies are often focused on a select number of participants for in-depth study (Lincoln & Guba, 1985).
The cases in this study were selected based on the preservice teachers' content scores to obtain a representative sample and to show how beliefs and content knowledge may be related. We used the average scaled score on the pretest ($M$ = 63) and half the standard deviation to determine the cut scores for each category: high (>70); moderate (56-70); low (<56). We used stratified random sampling to select three participants from each category for cross-case analysis to obtain nine cases.

Qualitative data sources included hand-written journals. Preservice teachers' made 8 - 10 journal entries. They were asked to reflect on their previous mathematics background knowledge for the first entry and on what they believed they learned during the semester for the last entry. Using the Constant-Comparison Method (Glaser & Strauss, 1967), we analyzed the preservice teachers' journals to note changes or lack of changes in their belief system. Preservice teachers' belief statements were coded and categorized based on the data that emerged. We used these data to support quantitative data collected in this study and to explain how preservice teachers' belief systems are intertwined with content knowledge.

## Results

*Mathematics Content Knowledge*

Bridget taught one section of elementary mathematics methods in the fall semester of 2004. Participants included 25 preservice teachers who took the mathematics content pretest in early October 2004. However, only 24 preservice teachers took the posttest in December of the same year. As shown in table 1, the results of a paired samples $t$-test (two-tailed) show a statistically significant difference between pretest ($M$ = 63.17, $SD$ = 16.418) and posttest scores ($M$ = 80.04, $SD$ = 12.067), $t(23) = -7.766$, $p$ = .000. The effect size of these results found using Cohen's $d$ was 1.17, which is considered to be a large effect size (Cohen, 1992).

*Preservice Teacher Beliefs*

We selected nine cases (Joy, Mandy, Erin, Edith, Jamie, Yvette, Cathy, Joan, & Anita[9]) for qualitative analysis. Preservice teachers' responded to queries about their mathematics background and beliefs at the beginning of the course and reflections on their beliefs and practices at the end of the course. These data were read and coded by Bridget and another mathematics educator who taught mathematics methods courses in the College for member checks. Analysis of the journals reveal eight factors related to educational beliefs emerged in the case study participants' journals at the beginning of the reform-based methods course: (1) beliefs (i.e. judgment/evaluation); (2) values; (3) educational history; (4) affective states; (5) verbal

[9] Pseudonyms were used for anonymity in Studies 1 & 2.

persuasion; (6) vicarious experiences (7) mastery experiences; and (8) content knowledge. Three additional factors, for a total of 11 factors, related to educational beliefs emerged from case study participants' journals at the end of the reform-based methods course: (9) new knowledge; (10) personal teaching efficacy (i.e. self-efficacy); and (11) teacher efficacy (i.e., outcome expectancy). A descriptive analysis of their belief patterns are shown in Table 3. Preservice teachers' educational belief statements will be compared and contrasted before and after taking the methods course.

*Comparison and Contrast of Educational Beliefs at Beginning and End of Course*

*Belief statements.* Belief statements (i.e. judgments/evaluations) were made about mathematics as a subject domain and/or mathematics teaching and/or learning in all nine cases at the beginning of the methods course. For example:

> *I am a strong believer that children will learn better through inquiry and active learning.* (Mandy)
>
> *Many people dislike mathematics.* (Yvette)

In comparison, preservice teachers made beliefs statements in all nine cases at the end of the course just as they did at the beginning of the course. However, the belief statements were more robust at the end of the course:

> *If math teachers would stop forcing students to memorize facts and lecturing and add hands-on activities, such as the ones we presented in class, math would be a more enjoyable subject. If the children enjoy it, they will perform better.* (Anita)

*Value.* Yvette was the only preservice teacher to make a value statement at the beginning of the course: *I believe that math is very important because it is used in our everyday lives.* However, Yvette's statement, in conjunction with the one highlighted above, reveals some preservice teachers can dislike mathematics and still value it. Likewise, Cathy's case was only one to refer to values at the end of the course: *I still maintain the belief that math is essential and is everywhere.* However, what is not clear from either of the journal entries is how these two preservice teachers might go about helping students to value mathematics.

*Educational history.* In eight cases (all but Joy), preservice teachers included statements about educational history at the beginning of the course. Statements about educational history were prompted by Bridget's request that preservice teachers' describe how their educational background influenced their learning of mathematics in their initial journal entry. For example, Jamie stated:

> *Math was one of those subjects that I used to love. My worst experiences were in 3rd grade and 10th-grade algebra.*

The data in Jamie's case as well as several others were consistent with the findings of Charalambous et al. (2008) and Swars, Hart, Smith, Smith, and Tolar (2007), which suggest that educational histories influence teacher belief systems about mathematics content.

Joan's was the only case to refer to educational history at the end of the course:

> *I was the student that just needed an example from the teacher in order to learn how to use a math concept. I never used interesting materials with math, which may be the reason I never enjoyed math class. Some students need to learn math with a hands-on approach.* (Joan)

However, what is compelling about Joan's statement is she did not allow her educational history to stand in the way of the students' needs. She realized that all students do not learn mathematics in the same way. Yet, it is unclear whether this was a modified, new, or sustained belief.

*Affective states.* Preservice teachers also made statements about their affective states in all nine cases in the beginning of the course. However, the affective statements of preservice teachers who were immigrants in the U.S. provided interesting caveats:

> *I used to attend school in Bahrain. I had to memorize everything. I remember disliking having to memorize the times tables and doing long division.* (Joy)
>
> *I really don't like mathematics. Many people dislike math. I don't like a subject where there can only be one right answer to a solution. I am a foreigner in the U.S. and have been in this country for eight years. My country is now called Ukraine, but my native tongue is Russian. Coming from a place where mathematics was the key to a successful education, I can honestly say that in my former country math was something that you had to know how to do or else.* (Yvette)

These two cases illustrate that emphasis on memorization could have a negative impact on students' belief systems (beliefs, attitudes and values) as they relate to learning mathematics regardless of their country of origin.

In contrast, affective statements were made about mathematics and mathematics teaching in six cases (Joy, Erin, Edith, Jamie, Yvette, & Joan) at the end of the course. In three of these cases, feelings about mathematics changed as a result of the course:

> *This class has opened my eyes to see that math can be fun.* (Joy)
>
> *I am happy to say that I no longer hate math, but I am intrigued at the endless possibilities I have in teaching math.* (Erin)
>
> *Since taking this course, my feelings about math have changed. The practicum had a lot to do with this because I learned how to make math lessons iterative and engaging for students. In this course I saw how important it is to use manipulatives in math lessons. I have never really liked math.* (Joan)

While these statements are encouraging, Parajes (1992) reminds us that newly acquired beliefs are vulnerable. On one hand she states that her beliefs had changed, and on the other hand she admits she "*never really liked math.*" Yet, it was Joan who realized that her dislike for mathematics may have been connected to how it was presented to her and not mathematics itself. Joan's mixed message reveals the importance of field experiences in improving preservice teachers' negative feelings toward mathematics (Hart, 2002).

*Verbal persuasion.* Verbal persuasion was inferred from statements made in three cases (Cathy, Joan, & Anita) at the beginning of the course. One example of verbal persuasion is:

> *My teacher told my parents that I was a "wiz at math."* (Joan)

Interestingly, all statements related to verbal persuasion were made in cases where the preservice teachers had high content knowledge. No statements were made in the journals that could be classified as verbal persuasion at the end of the course.

*Vicarious experiences.* Mandy was the only case study participant to include statements about vicarious experiences at the beginning of the course. She referenced Kay Toliver after watching *Good Morning Ms. Toliver* on the first day of class.

> *I never learned math like that and saw [in Kay Toliver video] how much the kids were learning while they were having fun and connecting it to the real world.*

In contrast, vicarious experiences were mentioned in four cases at the end of the course (Joy, Mandy, Erin, & Edith). In two cases (Mandy & Erin), the influence of Kay Toliver tapes on preservice teachers' belief was evident. These statements support Yoon et al.'s (2006) contention that teachers' beliefs and efficacy can be improved with videos. Interestingly, all four of these preservice teachers also mentioned the positive impact that peer teaching had on their beliefs and efficacy. For Edith, peers had a profound impact on her beliefs:

*Another part of this semester that I truly enjoyed was the learning community we developed. Everyone who was in my practicum was also in science and math methods courses. This was beneficial because we truly got to know and learn from our peers.*

Edith's comment also provides a rather unique finding that suggests the importance of establishing a learning community. One's peers can provide the support needed to sustain newly acquired knowledge and beliefs. Joy's case pinpoints the importance of peer modeling on her beliefs:

*Having my classmates teach lessons also added to my collection of lessons that I could teach to my future students.*

This is also a unique finding that is not prevalent in the literature. Microteaching was a part of the mathematics methods course because Bridget did not believe she could evaluate preservice teachers' pedagogy without seeing actual practice. The influence of peer teaching was a powerful finding that we did not expect, and it reinforced the importance of a positive learning community (Lave & Wenger, 1991; Lowery, 2002).

*Mastery experiences.* In the beginning of the course, Mandy was the only case study participant to refer to mastery experiences:

*Being that math is my favorite subject to teach, I have already experimented with math lessons. For example, during a multiplication review, I had different centers with different games (Multiplication Bingo, dice, and flash cards). I have also visited many math websites and got some good ideas.*

Interestingly, Mandy connects the vicarious experience provided by watching the Toliver tape to her educational history and the mastery experience to her affective state. Moreover, the influence of the Kay Toliver videotapes enhanced the influence of her educational history. The visual images of students learning and having fun in the mathematics classroom was enough to inspire Mandy to try some ideas of her own. These statements support Parajes' (1992) claim that teacher beliefs about best practices can be changed and Bandura's (1997) contention that affective states influence efficacy.

Likewise, mastery experiences were mentioned in one of nine cases at the end of the course:

*I realize from my practicum that my students loved to make music and dance so I came up with a way to use graphs to plot their dance moves. I then reflected on how my students loved treasure hunts and solving mysteries and put that all together to come up with the game: On our way to the treasure spot. Though there was a lot of work that had to go into this lesson, I loved it because it fun and involved learning that was meaningful for my students.* (Jamie)

Jamie's willingness to think outside of the box to develop an authentic task to teach coordinate graphing provided her students with high-quality mathematics teaching. In order for such practices to be sustained, teachers' beliefs must be reinforced with sufficient practice (i.e, time and use) (Parajes, 1992).

*Content knowledge.* Finally, belief statements made in these cases provide information about these preservice teachers' content knowledge before taking the reform-based course. Content knowledge statements, both implicit and explicit, were evident in five cases (Joy, Erin, Cathy, Joan, & Anita). For example:

*Math has always been my poorest subject.* (Joy)

*I was becoming familiar with the multiplication tables. Upon entering school, I found myself ahead of the other children. My interest in math did not wane.* (Cathy)

Joy's statement about content is also a belief statement or judgment about her mathematics ability in relation to other subjects. Cathy's statement suggests a relationship between content knowledge and interest in mathematics. Research studies that explore how content knowledge and interest are linked are needed to shed more insight on these two constructs.

At the end of the course, preservice teachers made references to content in four cases (Joy, Erin, Joan, & Yvette). Joy's belief statement was related to the PRAXIS as well as the mathematics content test:

*This class has been helpful for me because I have been practicing for the Praxis. If I were to take the content test again, I am confident that I would pass.* (Joy)

*Additional Preservice Teachers' Belief Statements at the End of the Methods Course*

Three additional factors related to mathematics teaching and/or learning emerged from the cases at the end of the course: acquisition of new knowledge, personal teaching efficacy and student outcomes. Each of these factors will be discussed in order.

*Acquisition of new knowledge.* Acquisition of new knowledge was evident in five cases (Joy, Erin, Yvette, Joan, & Anita). A few examples of these statements are:

*I have learned so much through this course in the way of teaching theories, strategies, problem solving variations, teaching materials, and numerous math processes* (Erin).

*Honestly, I do not know how I would teach math without this course. Throughout the semester, I have seen that it is not the actual content but the way you present it that really matters. I have also learned that there are many different ways in which children learn best. Some children perform better in a more structured environment, whereas others prefer a less structured environment.* (Yvette)

*I have learned some important things in this class, such as memorization is not the goal of math. As a teacher, I want students to be able to critically think and come up with solutions to everyday problems.* (Anita)

These data imply that Bridget's reform-based mathematics methods course influenced these preservice teachers' learning in a myriad of ways. What is compelling is that Yvette realized that different students had different needs.

*Personal teaching efficacy.* Preservice teachers also made comments about confidence and preparedness to teach mathematics in five cases (Mandy, Erin, Edith, Jamie, & Cathy):

*I feel that the math/science practicum has prepared me to teach, particularly in the areas of math.* (Jamie)

*Through this course and my math/science practicum, I have much more confidence in my teaching abilities.* (Erin)

*Outcome expectancy.* Anita's case was the only one of the nine to address student outcomes. Moreover, her comment specified what she believed to be the relationship between content and student outcomes:

*Knowing how to process information or manipulate facts to come up with sound solutions can increase one's achievement.*

Student outcomes and achievement variables also shape teacher beliefs (Ernest, 1989). Teachers' perceptions of students and their ability to learn mathematics will dictate their classroom behaviors (Walker, 2006). In some cases, negative perceptions of students lead to self-fulfilling prophecies that continue the cycle of low achievement, especially in large urban public schools (Cousins-Cooper, 2000; Walker, 2006).

## Summary

Data show the reform-based mathematics methods course had a huge impact on preservice teachers in these nine cases. Preservice teachers' content knowledge improved considerably. We were particularly pleased that this was true in the cases were preservice teachers' content knowledge fell into the low category. Furthermore, the three cases with the lowest initial content scores (Joy, Mandy, & Erin) had more robust belief statements ($M = 5.00$) as a group than preservice teachers in the moderate ($M = 4.00$) and high groups ($M = 3.67$) at the end of the reform-based methods course. These two findings show the reform-based course had a positive impact on these preservice teachers' content knowledge and educational beliefs. While these data are sparse, they also suggest that content knowledge changed along with educational beliefs. Additional research is needed to validate this assumption.

The study supports previous findings that methods courses and field experiences improve preservice teachers' content knowledge (Ball et al., 2005; Kahan et al., 2003), which is situated in biological and experiential contexts (Lave & Wenger, 1991; Núñez et al., 1999). Content knowledge is also intertwined with educational beliefs, which are not context free (Parajes, 1992). Most of the preservice teachers in the cases understood the importance of teaching mathematics in ways that students learn best: manipulatives, authentic tasks, games, and problem solving. Five cases reveal the importance of personal teaching efficacy (i.e, self-efficacy) on the teaching of mathematics. One case (Anita) reveals the importance of teacher beliefs and student outcomes. In light of these and other findings related to efficacy beliefs, a follow-up study to examine efficacy beliefs and its relationship to MCK is forthcoming.

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# A COMPARISON OF INSTRUMENTS TO EXPLORE DIFFERENT ASPECTS OF MKT FOR ARITHMETIC

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*We have developed a research and educational project that has the aim of improving teachers' mathematical knowledge for teaching (MKT) in elementary and secondary schools in Mexico. Here we describe a part of this project that has the purpose of exploring the arithmetic knowledge for teaching of elementary school teachers. To this end, we designed and applied four different methodological instruments: classroom observations, an open questionnaire, a closed questionnaire and interviews. This diversity obeys our interest of comparing these instruments to determine their advantages and limitations to give a description of the teachers' MKT. This article describes the results of this case study with eight teachers.*

## Introduction

A lot of evidence has been gathered with national and international evaluations, showing no substantial advances in students' achievements, in spite of the curricular changes, new teaching materials and new technologies introduced in Mexico during the past few years at the elementary level. This, points to a very important factor which probably has been overlooked in all these policy changes: teachers' knowledge.

To remediate this situation, we have developed an educational project that has the purpose of improving teachers' math knowledge for teaching at the elementary and secondary level. This started by piloting workshops at different educational levels, designed to help teachers to reflect on their practices and to learn through the interaction with other teachers, guided with an agenda and reading materials provided to them. This part of the project is being reported elsewhere. Parallel to the workshops, we designed and applied several instruments (classroom observations, open and closed questionnaires and interviews) to explore the advantages and weaknesses of each one of them in assessing the specialized knowledge teachers required in their practice within some topics of arithmetic. Our main objective is to gather information about the actual level of teacher's knowledge, to guide us in better defining an effective teachers' development program. This article will report on the findings of an inquiry about the usefulness of these four instruments in revealing and evaluating the specialized mathematical knowledge for teaching held by teachers in some particular topics of arithmetic.

## Theoretical Framework

One fundamental aspect, the mathematical education research community has focused its attention on, is the different kinds of interrelated knowledge a teacher might need in his own practice. The general term Pedagogical Content Knowledge (Shulman, 1987) refers to a complex mixture of knowledge related to many components like content, pedagogy, organization of topics and problems, student conceptions, models, representations, activities, curriculum, etc. Some facets of this teachers' knowledge are more closely related to the mathematical content, like understanding and extending student methods of solution, deconstructing one's own knowledge into its elemental parts to make it more evident (Ball, 2000) or knowing the structure and connections of mathematical concepts and procedures. Ball and Bass (2000) associated this special knowledge with the term: Mathematical Knowledge for Teaching. It is described as an "unbundled" complementary mathematical knowledge teachers need, to manage routine and non-routine problems. These authors identified four core activities related to this knowledge: i) Unpacking math ideas and procedures; ii) Choosing representations to effectively convey math ideas; iii) Figuring out what students understand; iv) Analyzing methods and solutions different from one's own. Teachers' MKT represents a fundamental factor on students' achievement. Evidences of this can be found in Hill, Rowan & Ball (2005).

Based on different frameworks and methods of inquiry, there have been a number of research studies connected to teachers' professional development projects in different countries. Amato (2006), within a mathematics teaching course to student teachers, conducted a study to improve their relational understanding in fractions, playing games (like trading on a decimal board). The author reported improvement in their understanding of mathematical and pedagogical knowledge, especially in multiple modes of representation.

In a study investigating the Pedagogical Content Knowledge (PCK) of elementary school teachers in the topic of decimals, Chick, Baker, Pham and Cheng (2006) proposed a framework with three categories: 1. *Clearly PCK*. Elements included: teaching strategies (math related), student thinking, cognitive demands, representations, resources, curriculum and purposes; 2. *Content Knowledge in a Pedagogical Context*. Elements included: profound understanding of fundamental mathematics, deconstructing content, mathematical structure and connections, procedural knowledge and methods of solution; 3. *Pedagogical Knowledge in a Content Context*. Elements included: goals for learning, student focus and classroom techniques.

Powell and Hanna (2006) explored how teachers develop their MKT in the discursive interaction of practice. They extended the epistemological component of teachers' knowledge, defining it as the teachers' inferential awareness of the students' existing and evolving math knowledge. This knowledge is essential for the teacher to make sense of students' productions, ideas and arguments. Also, like some other previous authors, Seago and Goldsmith (2006) studied the possibility of using classroom artifacts like students' work and classroom videos to assess and promote MKT.

In addition to the quality of the mathematics content and pedagogy knowledge held by teachers, effective learning requires students to construct their own knowledge through exploration and interaction. According to Askew, Brown, Denvir, & Rhodes (2000), this process is productive if the next four components meet some necessary requirements: (a) *Tasks* are challenging, meaningful and interesting; (b) *Talk* facilitates learning and includes all sorts of teacher and students interactions; (c) *Tools* cover a range of modes and types of models; and (d) *Relationships and norms* help towards a social construction of knowledge.

Using a case study approach within a collaborative action research Cooper, Baturo and Grant (2006) uncovered some characteristics of instructional interactions that lead to positive results in student learning. The main result of this study was that highly successful collaborations contained all of the following three levels of pedagogies: i) *Technical* – Practical "tips" related to a particular activity; ii) *Domain* – Teaching strategies proper to a particular topic; iii) *Generic* – Teaching methods valid across math topics, like Krutetskii's: "being flexible", "reversing" and "generalizing" or Hershkowitz's: "use of non-prototypic examples".

## Methodology

The study took place in three public schools in Mexico City, with eight, fifth and sixth grade teachers participating (identified here by PE, MT, AD, OL, MA, CA, MI and YO), with an experience in teaching of between 8 to 27 years. We applied the instruments in the sequence: 1. Classroom observation; 2. Open questionnaire; 3. Second classroom observation and 4. Closed questionnaire (for PE, MT, AD and OL) or an interview (for MA, CA, MI and YO).
The day and subject matter of the classes observed were chosen by the teachers themselves and were videotaped. Their analysis was based on two separate lines. The mathematical content was assessed according to the four core activities of MKT listed before, identified by Ball and Bass (2000). The pedagogical techniques were evaluated according to Askew's four components: tasks, talk, tools and norms.

The open and closed questionnaires were designed taking as a model the questionnaires constructed and used by the group: Learning Mathematics for Teaching, of the School of Education at the University of Michigan (Hill, et al., 2003; Hill and Ball, 2004). The open questionnaire contained twelve items and the closed questionnaire consisted of thirteen multiple choice items, both covering different topics (operations with natural numbers; mental calculation and estimation; fractions and decimals; proportionality and units conversion) and related to the four aspects of Ball and Bass mentioned before. In the results below we reproduce and analyze three items of the Open Questionnaire, addressing the following aspects and topics: OQ2 – Students' errors in additive problems; OQ4 – Illustrations and representations in division of integers; OQ9 – Methods and solutions in estimation; and two of the Closed Questionnaire: CQ5 – Students' thinking in fractions; CQ11 – Evaluating claims or explanations in decimal numbers. The teachers' interviews were based on some of the items of the closed questionnaire, but probing further the answers and knowledge of the teachers. The closed questionnaire and the teachers' interviews were left at the end since they contain the type of knowledge we wanted to find out from the teachers, so we didn't want to give them this information before the classroom observations took place and the open questionnaire was administered.

## Results

*Classroom Observations*

We will describe first some of the classroom observations that took place (the teacher is identified with boldface letters).
**MA:** Used a coin of 20 pesos as unit to talk about percentages. Other coins were related to it through percentages. He directed the lesson through questioning and sometimes requested the students to explain their answers. He often didn't express himself in a precise manner and let the students do the same, which generated some confusion (for example, when asking "How much do I have here?" he expected a percentage answer but the students, not knowing, answer in pesos). In one of the interventions, he wrote on the board: "5 = 1/4 = 25%" trying to clarify things. He evaded the wrong answers given by the students, by asking "Are you sure?" or by re-asking the same question.
**CA:** The content of the lesson was the fraction as a ratio, using a drawing and writing a list of questions on the blackboard about it. A lot of time is spent on just copying this. Even before the students had time to answer them, he wrote another two unrelated ratio questions on the board. It was very hard to detect anything. There were no explanations. The drawing was the only

representation. Only a very few questions were asked orally and when one of the answers was wrong, it was sanctioned with a frown.

**PE:** The content of the lesson was the decimal system. He showed 10 bars representing the numbers 1 through 10, each of different color and size, and then tried to distinguish between their absolute and relative value. He explained for example that the red "has two absolute values: the two and the red." And that "the absolute value is the real value." Then he wrote on the blackboard "M C D U" and puts the red bar in different positions to ask its value now. At the end, they represented in this way, four digit numbers. The time was spent mostly on introducing the material. Again we observed the teacher put across ideas in a very imprecise manner and corrected the wrong answers of his students without any explanation. The "homemade" material used was somewhat ineffective.

**MT:** Also the content was decimal system. From a textbook, he read aloud some rules for the students to write down in their notebooks and then followed them to form numbers with a "homemade" material, which the same teacher has trouble using. He dictated the numbers wrong: "8 units, 40 tens (instead of 4 tens), 500 hundreds (instead of 5 hundreds)…" He even told his students to write the number 503 in "expanded form" as 30500 (instead of $3 + 0\times10 + 5\times100$). This caused some students to write 126809 when the number 908621 was read by him. In general, the inquiries and explanations of the teacher were in great part confusing to the students. The teacher could not detect the errors made by the students.

We observed a second class of this teacher, now on addition of fractions. His explanations were, in general, confusing and many times wrong. When adding two fractions, he illustrated it by representing other fractions with squares. The students added numerators and denominators, guided in part by the teacher. In another exercise on the blackboard, the teacher simplified the fraction 3/4 to 1/2, saying that the 4 you can half it but the 3 you cannot but you can take a third of it (by the same reasoning, he simplified 14/5 to 7/1). In the verbal problem: "… a wood frame of 3/4 thickness is hammered into a wall with 1 and 1/4 nails. How much of the nails got introduced into the wall?" the teacher used the wrong operation getting 8/4. Here we observed a very deficient common math knowledge of the teacher.

**MI:** This class was about units and subunits. He insisted that the students know the table: "Kl (kiloliters), Hl, Dl, l (liters), dl, cl, ml (milliliters)" and how it tells you how to move the point a certain number of places in a conversion. So the teacher centered the learning on rules and memorization. In some problems, he applied the "rule of three" to solve them. After a few exercises, he moved to the table of grams, following the same procedures.

In a second class on fractions, this teacher wrote on the blackboard a "conceptual map of the fraction", connecting it to seven boxes (Reading, Types, Parts, Problems, Equivalence, Conversion and Operations), each of which, was connected to some ideas. For example, the box of Equivalence was connected with i) multiplication makes it bigger and ii) division makes it smaller. Then he illustrated equivalence of fractions with several exercises.

**YO:** He worked a few examples of the addition of fractions, following the usual algorithm (finding a common denominator) and then solved a few problems. There were no explanations. He centered the lesson on how to do the procedures. Errors of the students are not really addressed but just dismissed by saying "you are wrong".

**AD:** This teacher worked on decimal numbers. First he wrote on the blackboard a "conceptual map about the decimal point" (somehow giving its characteristics), which lacks meaning to the students and they only copied it. He then compared a list of decimal numbers, but the exercise was unchallenging because it could be solved ignoring the decimal parts. At the middle of the lesson, the teacher changed to the topic of percentages. Again, we observed some imprecisions and conceptual errors. For example, for the number 2563.50, the teacher expressed that “the fifty are tenths”, and when a student wrote the result of 500 – 391.90 as 109.90, the teacher seemed not to notice it.

**OL:** The subject was the decimal system. He organized a “cashier” game with dice and bills, and the children played on each table. Although he didn’t ignore the students’ errors, he also didn’t do anything to find out more about the source of the problem.

In general we observed many ambiguities on the expositions given by the teachers. These, often confused students. Most noticeable was the emphasis on the mechanical aspects of math, learning by repetition. In general, the teachers gave very few reasons or explanations related to the procedures followed. The orientation was towards “how to do it”. Although, it is hard to assess inside the classroom the breadth of the mathematical knowledge for teaching (since teachers teach what they feel they know and students don’t challenge this knowledge), it was clear that, in some cases, there was a lack even of common math knowledge.

Only one of the eight teachers guided the lectures through questions and requested the students to explain their answers (but only when they were right). The rest of the teachers, asked questions that needed a “yes” or “no” answer, or “the result”, which then the teacher proceeded to explain. The representation models used were mostly concrete materials or drawings of the situations that had only a visual support. The teachers didn’t try to explore the sources of the students’ mistakes, avoiding their wrong answers by repeating them in a questioning mode, or by saying “Are you sure?”, or by repeating the question.

This small sample of eight teachers in Mexico City gives a “statistical sample” of what to expect in general, but it is not even representative of the whole country where we anticipate less prepared teachers. Thus, this shows the need of a comprehensive teachers’ program, putting in evidence what areas should be emphasized.

*Open Questionnaire (some of the items)*

Here we will describe the answers given to three of the items (2, 4 and 9) of the open questionnaire (OQ). The questions presented to the teachers are given first.

**OQ2**. “In a store, the cash register has $254 at the beginning of the sales day and $967 at the end of the day. How much was sold that day?” Some students wrote the answer $1221. How would you explain to them how to solve the problem?

Most of the teachers articulated that they would tell the students that “subtraction is the correct operation” or that “they have to take the difference”. They also said that they would show how to make the subtraction with concrete material. However, there were not explanations of why this was the correct operation instead of a sum, except that one teacher (OL) said that he would tell them that “the result cannot be more”, and another teacher (MI) would represent this problem as: 254 + ? = 967.

**OQ4.** For 100÷15, write several problems which are related to different concepts or interpretations of this division, and represent their solutions accordingly.

Five of the eight teachers (MA, CA, MI, YO and OL) gave a partitive type situation like: “There are 100 chocolates which have to be distributed between 15 children.” Only two (YO and AD) gave a measurement type situation like: “How many pair of socks can I buy with $100 if each one cost $15?” Two didn’t give any problems. PE said “I have no idea” and MT even wrote an incorrect result: “100/15 = 15”.

No representations were given except for the standard algorithm of the division.

**OQ9.** a) Estimate the total of the following bill (explain how you did it):

| | |
|---|---|
| 2 Shirts | $ 388.50 each |
| 6 pairs of socks | $ 24.20 each |
| 2 pants | $ 173.90 each |

b) A student wrote: 1000 + 120 + 400 = 1500. Explain and evaluate what he did.
Four teachers "estimated" the result as: 1400 (MA), 1300 (MI), 1280 (AD) and 1276 (OL). The other four, calculated the exact result (CA with an error).

With respect to part (b), three (CA, MI and AD) said that the student rounded, but AD said that "the student exceeded on his rounding." Two (PE and MT) said that the student made an approximation of each multiplication without being exact. The other three teachers (MA, YO, OL) stated that the student didn't take into account the 20 in the sum and MA added that he would ask the student to review his result.

The open questionnaire was very revealing about the specific teachers' ideas and conceptions explored in each of the items.

*Closed Questionnaire (some of the items)*

Here we will describe the answers given to two of the items (5 and 11) of the closed questionnaire (CQ). The questions presented to the teachers are given first.

**CQ5.** In an activity, several students ordered some fractions, from small to large, in the following way: $\frac{7}{8}, \frac{6}{7}, \frac{5}{4}, \frac{4}{3}$, 1. What criteria are they using to do this?

The bigger the denominator, the bigger the fraction. The unit is the biggest of all the fractions.
The bigger the numerator, the smaller the fraction. The unit is bigger than the proper fractions but smaller than the improper.
The bigger the denominator, the smaller the fraction. The unit is the biggest of all the fractions.
The bigger the numerator, the bigger the fraction. The unit is the biggest of all the fractions.

In this question, only one teacher could identify the right answer (c). All the others chose (b) or (d) as their answers.

**CQ11.** Consider the three decimal numbers: 0.245, 0.2 and 0.0069. Mark with a check ☑, from the following assertions, those which are correct.

The number 0.245 is the biggest because contains more digits different from zero.
The number 0.0069 is the smallest because it gets to the fourth decimal position.
If I multiply 0.245 by 10 we get 2.45 because thousandths are converted into hundredths, hundredths are converted into tenths…
If I divide 0.0069 by 1000 we get 6.9 because we must move the point three positions.
A decimal number divided by 0.1 always gets smaller.

In this item only 40% gave the correct responses. Parts (b), (d) and (e) were the most difficult, in which only about 25% of the teachers gave a correct answer! (Random answers would yield 50%!)

One of the difficulties with the close questionnaire is that random answers produce a "background noise" that can make them somewhat unreliable. In addition, it is a very delicate instrument since in reality, the items might need to be redesigned and recalibrated for different groups of teachers being evaluated.

*Interviews*

The interviews with four of the eight teachers were based on some items of the closed questionnaire, where the interviewer asked the teachers to make explicit their reasoning. Due to the space constraints, we will not show specific results of this instrument, but, like the open questionnaire, it also gives very specific and useful information about teachers' knowledge and their ways of thinking. This however depends strongly on the questioning skills and extent of the knowledge of the interviewer.

*Comparison of Different Instruments*

In the following table, we compare, in a somewhat coarse manner, the results of the different instruments applied. The B represents "one of the best in that instrument" for the group of eight teachers, and W, "one of the worst".

| Teacher: | Classroom: | OQ complete | CQ complete | Interview |
|---|---|---|---|---|
| MA | B | | – | |
| CA | W | | – | W |
| PE | | W | W | – |
| MT | W | W | | – |
| MI | | B | – | B |
| YO | | | – | |
| AD | W | B | B | – |
| OL | B | | | – |

The table shows a strong consistency across the instruments applied. The only teacher that has both, W and B, in different instruments is AD. However, analyzing further this case, although he showed a good content knowledge in both questionnaires, the classroom itself requires more of a mixture of pedagogical and content knowledge, where this teacher showed a deficiency in pedagogical techniques and students' knowledge, which would have helped him decide what proper and challenging tasks are for his students.

The classroom observations give substantial information about the pedagogical knowledge of the teacher, but not so much about his content knowledge, because, the teacher decides the specific subject-matter he wants to discuss in class. In contrast, the interviews and the open questionnaire provide a good look at the math knowledge for teaching, because they can be targeted to specific topics and concepts.

If well designed, a closed questionnaire has the advantage that it can be applied to a large number of teachers, either, giving some comparisons between different groups of teachers or, the changes that a group of teachers might have "before and after". However, it is sensitive to random answers. Its biggest disadvantage is that you cannot observe and examine the teachers' thinking as in an open questionnaire or an interview. These last two can perform very effectively as statistical indicators of the knowledge of the whole population of teachers.

## Conclusions

The classroom observations confirmed our suspicions that teachers, on the pedagogical side, have a reduced knowledge in teaching strategies, representations and purposes of math topics. The whole instruction is geared towards the operational aspect of math, learning by repetition and practice. Also, with respect to their interaction with their students, we observed that it is kept to a minimal level of a few questions and very short responses without argumentation. They ignore or evade wrong answers and thus, they do not address the errors and confusions of their students. We also noticed some shortages and conceptual difficulties on their content knowledge.

It is evident from this study that there should be efforts to improve the teachers' (common and specialized) mathematical knowledge and their ideas on pedagogical knowledge. This should be done very closely to their actual practice; taking into account their deficiencies and helping them make better use of their textbooks and materials.

Finally, one of the most essential elements of MKT, "Profound" Understanding of Fundamental Mathematics, has to be looked at more closely and defined more precisely. Comprehending math concepts or ideas could be achieved at very different deepness levels. It would be practically impossible for a teacher to "master" all the subject matter with a great deal of insight. So it would be important to define in what degree teachers' knowledge is needed in their practice, for each specific concept or idea.

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# COLLABORATIVE EFFORTS BY MATHEMATICS AND SPECIAL EDUCATION TEACHERS FOR THE INCLUSIVE MATHEMATICS CLASS

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*This study investigates effective interventions for students with learning disabilities (LD) and healthy ways to collaborate between special education and mathematics teachers. Twenty-four teachers were interviewed and responded to Likert-scale questions. The results indicate teachers' knowledge, attitudes, and perceptions vary according to their job role. Both groups see inclusion as successful if they consistently employ certain strategies and those appear to increase the academic progress of the student with LD. These strategies are shared in the paper. The two groups have different views about benefits and challenges of inclusion, which suggest the needs of further discussion on the communication part of collaboration.*

## Introduction

Students with learning disabilities have often struggled with acquiring the needed skills to succeed at mathematics at their grade level. There are suggestions and research based practices widely available, but there is a gap between teachers' knowledge and the instruction that provides access to the general education curriculum. Collaboration between special and general educators, as well as training on current best practices, could benefit all students.

Principles and Standards for School Mathematics (NCTM, 2000), No Child Left Behind (NCLB, 2001) and the Individuals with Disabilities Education Act (IDEA, 2004) mandate that all students receive access to age appropriate curriculum, but this is not easily achieved. NCLB was amended in 2007 with this issue in mind (http://www.ed.gov/legislation/FedRegister/finrule/2007-2/040907a.html). The amendment stated that a small number of students may not meet IEP goals, even with appropriate instruction. This amendment was meant for students with severe cognitive disabilities, so its impact on the instruction of students with learning disabilities will be minimal, but this indicates that the government is beginning to recognize the many factors that go in to successful teaching for students with disabilities. It does not, however, lessen the burden on general and special education teachers to meet the goal of improving results and functional outcomes for students with disabilities.

## Theoretical Framework

Students with mathematical disabilities can benefit from many types of interventions. Early intervention can provide benefits, but continued remedial intervention is generally necessary for students with mathematical deficits (Fuchs & Fuchs, 2001; Fuchs et al., 2008; Maccini & Hughes, 1997). It is also imperative to identify the exact nature of the deficit. Interventions for computation may be different from interventions for difficulties in understanding word problems. Students with mathematical disabilities often struggle with counting and are not able to retrieve answers from memory (Fuchs et al., 2008). It is common for students to have deficits in both computation and problem solving, but the remediation should still be specific to the area of need (Fuchs et al., 2008).

Direct instruction, which involves teaching problem solving strategies, rules and principles in an explicit manner, has been shown to be effective for students with learning disabilities and those at risk for academic difficulties (Baker, Gersten, & Lee, 2002). Explicit instruction focuses on a great deal of teacher modeling to help the students learn to ask questions that lead to solving problems (Cardelle-Elawar, 1992).

Effective assistance includes tutors who have been trained in computation remediation. These tutors can teach strategies and help reinforce effective techniques (Baker, Gersten, & Lee, 2002; Fuchs et al., 2008). Peer assisted learning can encourage a low achiever to continue their attempts to master a skill. It also assists the teacher, because it allows more students to be engaged, with one-on-one assistance, in the same amount of time as teacher led instruction (Kunsch, Jitendra, & Sood, 2007). A meta-analysis of peer mediated intervention research indicates more success when the interventions were conducted in general education classrooms (Kunsch, Jitendra, & Sood, 2007). Helping general and special educators learn these specific instructional strategies may benefit the students who are included for mathematics instruction. "For inclusion to be successful, several factors are important: (a) qualified personnel, (b) available support services, (c) adequate space and equipment to meet the needs of all children, and (d) positive teacher attitude toward inclusion" (Leatherman & Niemeyer, 2005, p.23). Collaboration among professionals has to go along with the strategies, training and equipment. Research has shown that collaboration between general and special educators is the key to successful inclusion (Cole & McLeskey, 1997; DeSimone & Parmar, 2006; Lerner, 2006; Proctor & Niemeyer, 2001). When both professionals use their knowledge of specific strategies to modify lessons for students with learning disabilities then they can fill in the gaps between content knowledge and knowledge of effective strategies (van Garderen, Scheuermann, Jackson, & Hampton, 2009). This means helping the general education teacher to do more than offer a seat to a student with a disability. The teacher must provide substantial and specific feedback throughout the lesson (DeSimone & Parmar, 2006).

This study was designed to examine inservice teacher knowledge and attitudes toward inclusion, interventions, and collaboration between special education teachers and mathematics teachers. This study is an effort to investigate perceived effective interventions and healthy ways to collaborate between special education teachers and general education teachers. The specific research questions for this study are as follows: (a) What are the factors that influence teacher attitudes toward inclusion, and (b) What were special education teacher and mathematics teacher perceptions about effective ways to collaborate in the inclusive classroom?

## Method

### *Participants*

The participants of this study were 24 currently employed inservice teachers. They taught in Kindergarten- 8th grade. Their areas of certification included K-12 Intervention specialist, 1-8 Elementary Education, Middle Childhood Mathematics, Language Arts, and Social Studies. Seven teachers were currently teaching special education and seventeen teachers were currently teaching general education classes. The median number of years in the teaching field was 17 years. This group expected to teach students with learning disabilities in the mathematics content area on a daily basis.

### *Data Collection*

Two data sources, questionnaire responses and interview data, were used for this study. Twenty-four practicing teachers were surveyed and interviewed about their views concerning collaboration, knowledge in mathematics and special education, and what would help them better include students with learning disabilities in their mathematics instruction. In the survey, participants were asked to respond to a four point scale with 0 being no knowledge to 3 being very knowledgeable about the area. The survey questions also include open ended questions regarding their perceptions of the benefits to collaboration between special education teachers and mathematics teachers, as well as challenges to collaboration. The survey questions can be found in Table 1.

Structured interviews were used in this study to provide insight into practitioners' knowledge and attitudes toward inclusion and collaboration between mathematics teachers and special education teachers. The interview questions were used to identify possible factors related to the teachers' knowledge and attitudes toward inclusion and collaborative instructional procedures used in the classroom. Twenty open-ended interview questions allowed the participants to discuss the issues encountered implementing interventions and collaboration with other teachers in inclusive classrooms. These questions helped the authors to understand the teacher's attitude toward inclusion, collaboration, and the correct way to implement new interventions in the classroom. The individual interviews that were conducted lasted from 20 minutes to 1 hour.

The participants were asked about their knowledge of mathematics and interventions, including strategies they routinely use to assist students with mathematical learning disabilities; when and how they assist these students; their understanding of the mathematics curriculum; mathematics concepts they feel the most comfortable teaching; kinds of professional development they obtained to improve their teaching of mathematics; their familiarity with the RIT (Response to Intervention Techniques); ways of documenting the intervention techniques they use; and the IEP process. Participants' perceptions about the IEP process, including students with learning disabilities for mathematics with same age peers, and inclusion in mathematics instruction for all the stakeholders were also investigated. Also, participant teacher attitudes toward inclusion and collaboration were investigated in the areas of: collaborating with other teachers to adapt the learning environment for mathematics lessons; ways to collaborate together to adapt the environment; collaborating with other teachers to modify mathematics lessons; and collaboration strategies/techniques they routinely use.

*Data Analysis*

Survey data were analyzed using a spreadsheet to summarize descriptive data including frequencies and percentages. The interviews were analyzed for themes as they evolved from the written transcription. Similar ideas from survey data and interviews were placed into broad categories, and specific categories were defined within each question. After all the information from the data sources was categorized by topics, a content analysis was conducted to extract similar themes and ideas within each teacher's case (Patton, 1990).

## Results

Results indicate that the majority of participants perceive themselves as somewhat knowledgeable about collaboration, utilization of response to intervention techniques, mathematical content, and IEPs (see Table 1).

Figure 1 depicts the perceived benefits and challenges of inclusion that the teachers experienced or viewed. The participants indicated gaining knowledge, sharing ideas, teaching better, and collaboration were the perceived benefits. The mindset/skills of other teachers was the most often listed challenge, followed by time, assessment and ownership. Two participants noted no challenges. The survey results informed the researchers that the participants considered themselves to have the knowledge to teach the subject matter. The fact that they felt sharing ideas was beneficial indicates that this group was open to learning more about effective teaching practices. The mindset and skills of other teachers was listed as a challenge by 75% of the participants. This indicates that perceived equity in knowledge and collaboration skills are very important and should be addressed by schools if they want successful inclusion.

Table 1
*Knowledge, Attitudes, and Perceptions*

| Questions | Mathematics (%) | | | | Special education (%) | | | |
|---|---|---|---|---|---|---|---|---|
| | No | Little | Some | Very | No | Little | Some | Very |
| Assisting students with learning disabilities to access the general education mathematics curriculum for my grade level. | 0 | 12 | 82 | 6 | 0 | 0 | 75 | 25 |
| Collaborating with other teachers to modify mathematics lessons. | 0 | 12 | 76 | 12 | 0 | 25 | 25 | 50 |
| Collaborating with other teachers to adapt the learning environment for mathematics lessons. | 0 | 35 | 59 | 6 | 0 | 25 | 25 | 50 |
| Understanding the mathematics curriculum 75 my grade level. | 0 | 6 | 35 | 59 | 0 | 0 | 75 | 25 |
| Utilize response to intervention techniques. | 6 | 47 | 35 | 6 | 0 | 0 | 75 | 25 |
| Understand the IEP process, in regards to mathematics goals, for students with learning disabilities assigned to my classroom. | 0 | 29 | 65 | 6 | 0 | 0 | 50 | 50 |

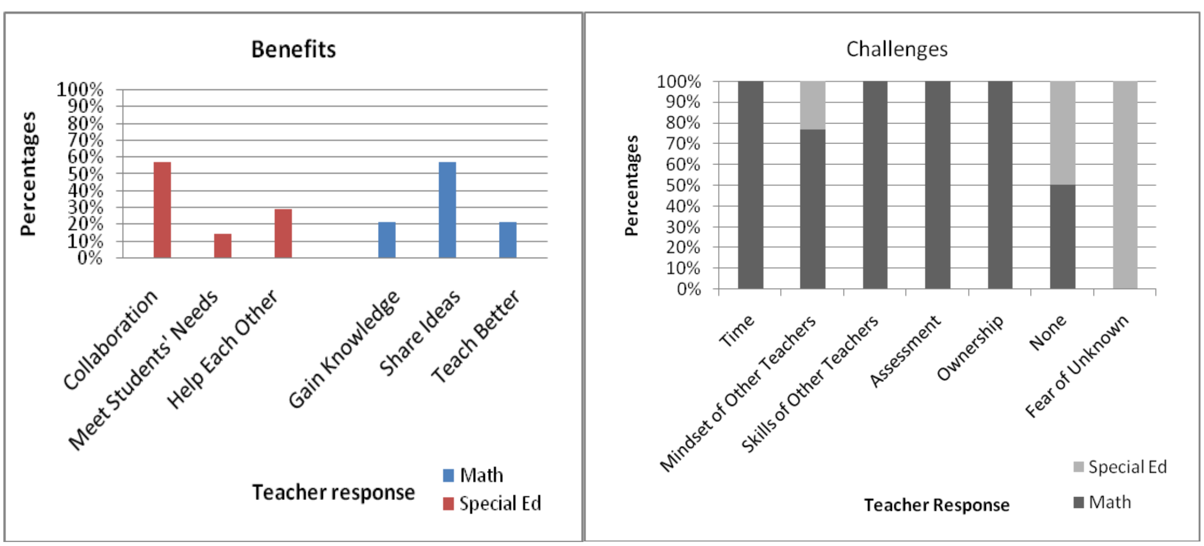


*Figure 1*: Benefits and challenges.

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

*Teacher Knowledge in Mathematics, Interventions, and Inclusion*

Both mathematics and special education teachers felt most comfortable teaching mathematics concepts from their own grade level. Areas that mathematics teachers felt needed improvement were mainly mathematics content areas, achievement, and technology (Smartboard™); special education teachers wanted to learn about assisting students with LD, different strategies, collaboration with a special education teacher, and number sense. Both mathematics and special education teachers wanted to learn more, student-centered, hands-on activities, and algebra.

Surprisingly, 22 of 24 participant teachers were not very knowledgeable about Response to Intervention Techniques (RIT). Mathematics teachers responded that they decide appropriate intervention strategies for the following reason (in the order of frequency): based on student's need, intervention teacher's suggestion, team decision, trial and error, and/or based on data (student achievement). Special education teachers' reasoning is in the following order: use various data, do not use any, trial and error, based on student's need, and/or Ohio Standards.

With regard to including students with disabilities, the participant teachers had different views. Half of the special education teachers felt that including students with learning disabilities for mathematics with same age peers at their school was unsuccessful, and the other half of them felt unsure. Mathematics teachers' responses were divided: not sure, need to include them, and no success.

> I think [inclusion is] unsuccessful at our school cause it rarely happens. Now that special ed kids, IEP kids are to pass the OAT, the Ohio Achievement Test, oh my goodness, the regular teachers are getting really interested in seeing that the special ed kids are successful. So there's more openness and more nervousness from them. The only thing is I'm starting to feel like there is a lot of finger pointing blaming the special ed teacher, me, for the students' lack of abilities and skills and when I receive a student he/she may be behind or deficit 2-3 years already. (Special education Teacher W)

> I do [think inclusion works]. When I first started teaching mathematics I had a pull out class and they didn't have any examples as to what, they didn't have any higher level thinkers in there. So every time you would come up with a question they wouldn't understand. You had no peer interaction, no discussion, nothing to go from. So having them in a regular classroom helps them tremendously because they have that discussion to go off of and basically I don't want to say peer tutoring, but peer examples as to how things are being solved other than just the teachers. (Mathematics Teacher I)

Both mathematics and special education teachers stated that team work is the key for the success of inclusion. Mathematics teachers addressed students' group work (interacting) and peer examples (watching others) as benefits of inclusion.

*Strategies to Assist Students with Mathematical Learning Disabilities*

The participant teachers believed that mathematics teachers should know that students with LD learn differently and teachers should teach them differently. They shared more than ten strategies to teach students with LD. Some strategies were used by both special education and mathematics teachers; some seem to be used only by either special education or mathematics teachers. Common strategies routinely used by mathematics teachers were: modifying lessons (slower pace and lower level content), working (helping) with the kid(s) individually, giving extra time, group work, and/or getting help from volunteer tutors.

> I might pair them up with someone who does get it. Allow them manipulatives and might spend more time with them if no one else needs my assistance at the time and just use different ways to approach the same question. Use visual cues, auditory cues, manipulatives, kinesthetic, actually doing it. (Mathematics Teacher M)

Some strategies were reported only by special education teachers: using textbook, daily facts practice, and reading out loud.

> For mathematics we used a text book and followed the text routinely through each chapter following the standards. We also did a daily facts practice to work on time. When it came closer to the OET tests we did practice questions that were similar to the OET test questions. That is when we let go of the book to focus more on the OET. (Special education Teacher A)

Some strategies were only reported by mathematics teachers: lecturing, reteaching, assistance from volunteer tutors, using different cues, and/or retaking tests. Most special education teachers (and 12 % of mathematics teachers) reported that they use manipulatives to assist students with LD. Some other strategies reported by both groups were: demonstration, intervention based on a specific student need, and physically relocating students (to the front) (closer to the special education teacher).

*Collaboration among Mathematics and Special Education Teachers*

According to the participant teachers, physical convenience or common needs/interest promoted collaboration: among same grade teachers, same subject (mathematics) teachers, and/or special education teachers in the building. More than 50% of participants viewed talking to other teachers as the most routinely used collaboration strategy. Mathematics teachers co-plan and have grade level meetings more frequently than special education teachers. Collaboration occurs quite informally whenever they have time (lunch, before or after school, child's special) and/or during a team meeting.

According to the interview data, mathematics teachers collaborate together to adapt the environment in various ways. The majority of them work together by planning together, holding team meetings, and creating group work. A small number of teachers responded that they do not know, do not change the environment, or do nothing. Also, technology (Smartboard™), hands-on, pull out (reading the test aloud), seating the children in a quiet place or close to the teacher were considered the collaboratively adapted environment by the rest of the mathematics teachers.

> A lot of times we do it, we have a large class teaching, but also we have the kids sitting and we put their tables together so they have smaller groups of where their desks are. We have them work in the hall. We have other areas where we will sit the kids in a quiet place. We have different areas where they can get a lot of manipulatives. They can work there or the computer area, so I guess we kind of do it around the room like that. (Mathematics Teacher M)

Having a wheel chair in the room and using new techniques were unique responses by special education teachers.

The participant teachers suggested the following strategies for enhancing student learning: more interactive approaches; understanding the child; planning together-working together; data driven instruction; smaller class size; being patient, creative, hands-on; more resources, discovery approach, teacher knowledge, multiple approaches, new methods, and being open minded.

Concerning teaching students with LD, the participant teachers wanted to learn more about: ways to include students more, more learning strategies/alternative ways, more hands on, how to reach them more, be open to new ideas, more of the resources, other intervention techniques, assessment, how to motivate them, and what causes a learning disability.

## Discussion/Implication

According to this study, the factors that influence teacher attitudes toward inclusion are many and sometimes depend on which position they have, mathematics teacher or special education teacher. The special education teachers may have the perception that the general education teachers care more about the students currently because of mandated testing. If the special education teachers feel the success or failure of a student still rests only on their shoulders, they may feel unsupported by peers and administration, and therefore determine that inclusion is not successful. The mathematics teacher may feel that inclusion is successful if the students show more skills in the included setting, but if the students are not making acceptable progress, they are also apt to determine that inclusion is not successful. Both groups see inclusion as successful if they consistently employ certain strategies and those appear to increase the academic progress of the student with LD.

Special education teacher and mathematics teacher perceptions about effective ways to collaborate in the inclusive classroom also vary according to their job role. In general, perceived benefits include gaining knowledge, sharing ideas, improved teaching skills and collaboration. The challenges included: the mindset/skills of other teachers, time to plan, ways to assess students with LD, and ownership in terms of the responsibility of teaching the student with LD.

In order for teachers to have more positive attitudes toward inclusion, professional development focusing on children with disabilities and hands-on activities for students with special needs, administrative support, and appropriate support personnel in the classroom are needed (Leatherman & Niemeyer, 2005).

Addressing the perceived challenges and bolstering strengths should be a top priority for administrators seeking improved collaboration and a better whole school attitude toward inclusion. The need for training in the areas mentioned, such as RIT, by the participants is important, but more important is the level of communication between both groups. Both groups list benefits of inclusion, but they list different benefits. Sharing ideas, gaining knowledge and teaching better were perceived benefits for mathematics teachers, while collaboration, helping others and meeting student needs were perceived benefits for special education teachers. An astute administrator would see that each group values slightly different things. While the mathematics teachers hopes to gain knowledge about working with the students with LD the special education teacher hopes to find someone to collaborate and share the responsibility of meeting the students' needs. Time needs to be allotted, not just for planning, but for communicating each professional's needs and strengths. A good partnership means that the teachers learn from each other and understand what skills they can bring to the classroom. After they understand each other they can be more effective at knowing how best to meet students' needs.

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# THE MATHEMATICAL KNOWLEDGE FOR TEACHING (MKT) OF NEW YORK CITY TEACHING FELLOWS

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## Introduction

The New York City Teaching Fellows (NYCTF) Program was started in 2000, in part, "to address the most severe teacher shortage in New York's public school system in decades" (NYCTF, 2008) and to replace uncertified teachers who were concentrated in high-needs schools (Keller, 2000; Stein, 2002). The NYCTF program currently supplies over 60% of new mathematics teachers in New York City Public Schools—more than 300 new mathematics teachers in 2007 alone. Fellows come to the program from a variety of backgrounds, with the majority being either recent college graduates or career changers.

## Objectives and Purposes of the Study

In this paper, we discuss how Fellows' Mathematical Knowledge for Teaching (MKT) (Bass, 2005; Ball and Bass, 2000) interacted with other aspects of the Fellows' teaching. The paper arises from a pattern we saw over the course of some 90 observations. We found that many of the Fellows, in collaboration with their coaches and other mentors, are designing lessons and fostering classroom cultures that are in line with the "process strands" of the New York State Mathematics Core Curriculum (NYSMCC). The NYSMCC promotes having students present alternative solution strategies, using different representations of mathematical ideas, and forming connections between different areas of mathematics, and so on. This approach to teaching puts extra demands on teachers that they are not always equipped for. We found that in the majority of cases the opportunities created by the Fellows were not exploited, due to what we perceive as limitations in their MKT.

## Perspective(s)/Theoretical Framework

In Shulman's (1986) seminal paper he develops the concept of pedagogical content knowledge which he defines as content knowledge but "the particular form of content knowledge that embodies the aspects of content most germane to its teachability [which includes] the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations - in a word, the ways of representing and formulating the subject that make it comprehensible to others." (p.9) Shulman's work was important in expanded the domain of interest in teacher's knowledge beyond pure content knowledge and pedagogical knowledge to include forms and representations of knowledge that are particular to, for example, teachers of mathematics as opposed to mathematicians.

In the aftermath of Shulman's work from researchers such as Ball & Bass (2000), Bass (2005), Ferrini-Mundy & Senk (2006) has attempted to examine the implications of his work for the teaching of mathematics. Ball & Bass (2005) seek to complement the concept of PCK by developing the concept of Mathematical Knowledge for Teaching (MKT). In developing this concept Ball and Bass seek to reflect "the dynamic interplay of content with pedagogy in teachers' real-time problem solving" (p.88). Ball and Bass are particularly interested in the mathematical knowledge that must be brought to bear in order to, in real-time, deal with choosing representations, asking questions, parsing and analyzing student responses etc. They argue that this is specialized mathematical knowledge "not known by many other mathematically trained professionals, for example, research mathematicians. Thus, contrary to popular belief, the purely mathematical part of MKT is not a diminutive subset of what mathematicians know (Bass, 2005, p. 429).

The concept of MKT is treated at length in Ball and Bass (2000) and Bass (2005) and is parsed into four categories namely (1) Common mathematical knowledge; (2) Specialized mathematical knowledge; (3) Knowledge of mathematics and students; and (4) Knowledge of mathematics and teaching. We are particularly interested in the second and fourth categories here as we focus on specific pedagogical paths promoted by the NYSMCC (use of inquiry-based activities, discussion of multiple solution paths, etc.) and carried out by the Fellows in their teaching and how those decisions interplay with the MKT of the teachers. In particular, in analyzing our data we noted a few occasions in which the Teaching Fellows were deficient in their Common mathematical knowledge, but much more common, and of greater interest to us, were instances of deficiency in MKT. Specifically, we found many instances where good faith and committed efforts to teach for understanding made by the Fellows were stymied in the full potential of their implementation because the Fellows lacked the MKT for full and effective implementation of their choices.

## Methods and Modes of Inquiry

The data from this study comes from a large study examining the New York City Teaching Fellows both in their teaching practice and in their growth as professionals as they both certify to teach and work as full time professionals in schools. The primary data sources for the study are regular interviews with eight Fellows and video observations of their teaching (approximately ten times per school year). The observation data is supplemented by post-observation reflections on the class written by the Fellows and post-observation interviews conducted by a researcher with the Fellow. For the larger project this data is further supplemented by student surveys; student focus groups; out of class observations; and interviews with administrators, mentors, and coaches.

### *Observational Data*

We collected observational data in the form of fieldnotes, videotape, and audiotape on average ten times throughout one full school year. The coding scheme that we used to analyze our fieldnotes was produced collaboratively with a larger group of researchers who were also collecting data on mathematics Fellows. Our coding scheme for the fieldnotes included such codes as: teacher math questions, opportunity for meaning making, and student mathematical behavior. For the purposes of this paper we found the code "opportunity for meaning making" particularly useful in identifying occasions when the pedagogical choices of the Fellows proved to be inadequately supported by their MKT. Having identified instances where issues with Fellows' MKT, we chose four vignettes which represented a variety of situations where we can see this interplay of pedagogical decision-making and MKT.

## Results

In the following, we discuss two classroom vignettes in detail and briefly describe conclusions taken from other vignettes.

*Karen: Exploring Addition and Subtraction*

In this vignette we see a teacher, Karen, engaging with several of the state process standards in her work on addition and subtraction of integers with 7th Grade students.

In the vignette we see Karen working with a chips model with black chips representing positive integers and red chips representing negative integers. In interviews, Karen indicated that she was having them work with this model because she wanted them "to really understand" how to add and subtract when there were negative addends involved, and to understand the relationship between addition and subtraction better.

In using the chips Karen is addressing several state process standards: 7.R.1, (use physical objects… as reps) 7.R.2 (explain … using representations) 7.R.4. (explain how different reps express same relationship) 7.CN.1 (understand and make connections among multiple reps of the same math idea).

First Karen demonstrates addition using the chips and the problem 6 + (-9). She draws 6 circles with + signs in each, and 9 circles with - signs in each. She calls on Zahara who suggests "You can remove the zero pairs."

Karen presents a word problem that involves subtracting four from nine, and asks the class "What does subtraction mean?" Kate says "Taking away one number from another number." Karen: "Good." She repeats what Kate has said, and then says "In the word problem it has nine take away four equals five."

At this point, Enrico raises his hand and asks "When you subtract is it the same as when you add with positive and negative numbers?" It's not entirely clear what Enrico means here but, although she answers "Yup," Karen goes on for the rest of this class to insist on interpreting subtraction as a literal "taking away" of the appropriate number of appropriately colored chips. Most of the students in this class comply with her interpretation. However, several students push the idea of subtraction as adding a negative. They do not succeed in persuading Karen to consider it despite her stated desire for them to understand the relationship between addition and subtraction.

Karen then poses the question "What if we have negative six minus negative two?" Dayanna answers "Don't you just draw six negatives? And then you take away … since it's red you take away two." Karen draws a circle around 2 of the red "chips" and then an arrow indicating they are being removed. At this point, Chris asks "I'm confused by this because when you … take more away from a negative number doesn't it become a lower number? … Isn't it supposed to be going that way?" He gestures to the left. There is a long pause. Karen asks Chris if he is thinking of the number line. He and other students say "yeah." At which point Karen says "I want us to the put the number line totally out of our heads." She makes a gesture of casting out the idea from her head. Then she turns to the number line above the board, which has been up there all year. She takes down the number line, to gasps, cries of "no," and a few cheers.

When she has finished taking down the number line, she turns back to Chris and says "And right now, we are going to picture it this way," pointing to the problem on the board. "Okay, Chris? Cause this is what we've been picturing all year and sometimes it gets really confusing when you're thinking about negatives and sometimes positives, okay? I'm going to move on. It will all get clearer when we get through the next few problems."

While this is certainly the most dramatic example of KW refusing to consider the relationship between different representations of a mathematical idea, she showed a similar unwillingness in other observations. In this case, despite her avowed goal for introducing the model as a way "to really understand”, she only worked with the model as an alternative computational method, rather than as a springboard to understanding addition and subtraction differently.

In the next part of the lesson Karen presents the problem: “Eric earned $5. He owes Laura $7. He pays her $5. Show this subtraction on the chip board.” The students are able to solve this immediately, yelling out “two” and “he still owes her two.” Karen ignores several suggestions from students as she goes through this problem. She adds two red and two black chips, and then “takes away” the seven black, leaving the two red chips. Chris proposes starting with the seven red “cuz that’s how in debt he is” and then adding five black to represent what he earned. He is essentially proposing that the problem is -7 + 5 rather than 5 - 7. Karen objects to this, because “the number sentence says to start whatever is here on the left,” pointing to the five.

Again we see an opportunity to explore the meaning of subtraction as something more than “taking away” and we see Karen insist on the “take away” approach.

In this vignette we see Karen, who states that she is interested in her students gaining a deeper understanding of addition and subtraction of integers, employ strategies drawing on several process standards (multiple representations, use of physical objects etc.). However, her implementation is constrained in practice by her insistence on the exclusive use of certain models and by privileging of certain approaches.

*Paul: Exploring Multiple Solutions Paths*

(Unless otherwise noted, the quotes are from Paul, Fieldnotes, January 9th).

There are several learning standards in the NYSMCC that relate to having students present multiple solution strategies, and Paul appears to be committed to this pedagogical path. This places extra demands on his content knowledge – he not only needs to be able to solve the problems he assigns, which he is clearly able to do, but he needs to be able to quickly recognize the validity of solutions he had not foreseen, and to be able to relate different strategies to one another. This requires a different type of knowledge than that gleaned in a typical math content course, where prospective teachers are usually asked to produce a single solution to a problem and while, in a methods course, a prospective teacher will explore several rich problems using multiple solutions paths, it is, of course, not possible that he or she will have a chance to cover the entire curriculum systematically. In this vignette we see Paul taking the pedagogical decision to encourage multiple solution paths, but being unable to take advantage of the alternate solution strategies proposed by the students.

The problem: If you got 6 questions from 7 right, what was your percent score on the exam? Paul’s approach was to set up the proportion 6/7 = n/100 and to solve for n using cross multiplication: “I believe what we were trying to accomplish was, I think I was just trying to get a basic, bare bones cross products, how do you do cross products, how do you set up a proportion?” (Paul, Reflection, January 9th). He reported after the class: “I think they have difficulty understanding … what exactly a proportion is. … They have difficulty seeing the relationship between 2 quantities. Repeatedly throughout the week, I try to have them, before they start setting up a portion to solve, they set up, they take the 2 quantities that are being compared and put them into a ratio. And that always seems be a very hard step for them.” (Paul, Reflection, January 9th).

Although he saw the problem as a proportion problem, he had a student present another solution, even though this added significantly to the amount of time spent on what was supposed to be a quick introductory problem. We interpreted his willingness to take that extra time as evidence of his commitment to respecting alternative solution strategies.

After the students have been working on the problem for a few minutes Paul circulates the room to check on the students work. Of the 8 students who produced work on the problem 6 of them divided 100 by 7 to get the percentage value of one question and then multiplied by 6 to get the percentage score. He notices Jennifer's work and they have the following exchange:

PH: "Can you explain how you got that?"

Jennifer: "So I multiplied what I got – and there are six of them."

PH: "I liked what you did, but you're not getting the answer that I got."

Paul is showing his willingness to let students do the problems in ways that he did not intend.

In her initial work at this point Jennifer has divided 7 into 100 using long division but has made two errors: firstly she got 13.285 . . . rather than 14.285. . . and secondly she has rounded this to 13.2 before multiplying by 6 to get 79.2 rather than 79.71. Paul noticed that this answer must be wrong and has Jennifer check by adding 13.2 to 79.2 which gives 92.4. Jennifer agrees that the answer should be 100 and continues to work on the problem, redoing her calculations and working with another student.

Here Paul is demonstrating how one might use mathematical knowledge for teaching effectively—he is able to recognize quickly that she must be making a mistake and suggest a way for her to find her error.

A few minutes later Jennifer writes her solution on the whiteboard. The long division results in 14.28 (she stops at two decimal places) which she multiplies by 6 and gets the result 85.68. For some reason she then erases her working of the multiplication and simply writes:

```
 14.28
x 6.00
---------
  85.2
```

which is picking up her earlier error again. Paul asks her to explain her work to the class which results in the following exchange:

Jennifer: "What I did, was I divided seven into a hundred. Because it said, there were seven questions on the quiz, and I got I multiplied six times fourteen point twenty eight. Because Umm. It said you only answered six of them. And then I got eighty -five point."

Paul: "Okay."

Jennifer: "Then I got eight-five point." (*inaudible*)

Paul: "So, your answer is … What was your final answer?"

Jennifer: "Eighty five point two."

Paul: "Then your answer is eighty-five point two what?"

*Silence*.

Paul: "What is the question asking?"

A few students: "Percent."

Nobody has noticed Jennifer's mistake. Instead of trying to figure out, as he did before, where the problem is, Paul chooses to move on to his own solution.

Paul: Okay so this is one way to do it. And there are a couple of other ways too. Yesterday we saw two ways to solve percent problems. This is nothing more than a percent problem. Just like what we were doing yesterday. The words are a little different but we can still do it the same way. Let's try another way."

Paul takes the students through a problem they did yesterday and asks them to relate it to the new problem. The students don't make this connection so eventually Paul changes tack and writes on the whiteboard:

Part

--------- =
Whole
and says: “It’s going to be a ratio of some kind. What about the percent? … Look back on your notes from yesterday. Actually turn the page. Percents like what? Percents over one hundred. Like What? Percent over one hundred. Do I know the percent? … We got n over a hundred. Of 6 and 7, what’s the part what’s the whole?”
Jennifer: The part is six the whole is seven
Paul: Why?
Jennifer: Because...
Paul: How many questions are on the test? There are seven questions on the test. That’s the whole test.”
Jennifer: “… and he only did six and that’s the part.”

$$\frac{\text{Part}}{\text{Whole}} = \frac{\%}{100}$$

$$\frac{6}{7} = \frac{n}{100}$$

Paul then guides the students through using cross multiplication to get an answer of 85.7%.
Paul then asks the class:
Paul: ‘Why are they different?’ (referring to 85.7% and 85.2% written on the board)
No one answers. With no one responding, PH responds to the class.
Marietta: The two and the seven.
Paul: That’s how they are different, but why are they different? Why are these two numbers different? [*pause*] I would accept either of these answers, depending on how you did it. Why are they different?”
Student: “Because they want to be.”
*Pause.*
Paul: ‘I’ll save that one for later.’

*Conclusion*

This is a very unhappy resolution to the work on this problem. It seems that Paul is not, in this moment, able to reconcile these two results. Indeed, he tells the students that he would accept both answers “depending on the method you used.” We argue that this is because he is concerned with fidelity to his pedagogical decision to embrace multiple solution paths. Unfortunately the value of the multiple approaches is lost because Paul is unable to reconcile the differences. We argue that with more developed mathematical knowledge for teaching, he would see that these solution paths (as well as the third approach that he attempted at one stage) must result in the same arithmetic operations, and therefore the same answer, and so the difference must be accounted for by either a rounding error or an arithmetic error. We see here in his initial interaction with Jennifer, and we have seen in other observations, that Paul has the mathematical ability to find the error were he to look for it. As it happened, we end up with a set of most likely confused students who are unlikely to develop an appreciation for the connections between the two different approaches, unlikely to value the idea of using multiple approaches to solve problems, and perhaps more convinced than ever of the arbitrariness of mathematics as a discipline.

*Other Vignettes*

In other observations of teachers in the study we see other cases of teachers attempting to employ standards-based approaches but failing through lack of facility in a particular kind of mathematical knowledge. In one case this lack of facility is manifested by a teacher who asks her students to work on an activity designed for students to "discover" a formula for the sum of the interior angles of a convex polygon. The full potential of the task, however, is not realized because she does not seem exhibit the necessary MKT to marshal the various student representations and focus on the important mathematical process of generalizing. There is no doubt that the teacher could do this task herself nor is there any doubt that she understands the relationship between the formula (s – 2) * 180 and the breaking of a convex polygon into triangles. However, her lack of MKT seems to be restricted her ability to see how several different student representations are equivalent to her understanding and how certain student representations are limited in their claims to generality.

In another vignette, we see a teacher trying to make connections between algebra and geometry but he chooses a poor example and runs the risk of leaving his students with the impression that a particular result is a general result.

## Discussion and Conclusions

The need for additional mathematical support shown in the vignettes above is not wholly addressed by the current training and mentoring the Fellows are receiving: both Fellows and coaches tell us that their mentoring is more focused on issues of classroom management and student motivation. We have seen how the developing Pedagogical Content Knowledge (PCK) of the Fellows, as well as their pedagogical decision making in the classroom is affected and complicated by their MKT, and we can begin to determine what kind of mathematical understanding is needed to effectively use these opportunities.

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# MISLEADING STRATEGIES USED IN A NON-STANDARD DIVISION PROBLEM

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*Here we focus on Finnish pre-service elementary teachers' (N = 269) and upper secondary students' (N = 1434) understanding of division. In the questionnaire, we used the following non-standard division problem: "We know, that 498:6 = 83. How could you conclude from this relationship (without using long division algorithm) what is 491:6?" The problem mainly measures adaptive reasoning. Based on the results we conclude that division seems not to be fully understood: only one fifth of the participants produced a completely correct solution. The most central reason for mistakes was insufficient reasoning strategies.*

## Introduction

Teacher education programmes face a major challenge in trying to affect elementary teacher students' views of mathematics, that is, their beliefs, attitudes and knowledge. This paper draws on the work of the research project "Elementary teachers' mathematics" financed by the Academy of Finland (Project No. 8201695), in which data were collected on 269 pre-service elementary teachers at three Finnish universities (Helsinki, Turku, Lapland). Two questionnaires were administered in autumn 2003 to assess the pre-service teachers' knowledge, attitudes and skills in mathematics at the beginning of their mathematics education course. The aim of the questionnaires was to measure their experiences of mathematics, their views of mathematics and their mathematical proficiency in certain topics. As part of the project we also collected comparison data on 1434 upper secondary students (grade 11, average age 17–18 years) from 34 Finnish schools selected at random. In the paper we concentrate on pre-service teachers' and upper secondary students' understanding of division and reasoning strategies used, especially their erroneous strategies.

In the Finnish comprehensive school curriculum (NBE 2004) one of the principal goals already in the second grade is that pupils should master and understand basic calculations. But earlier studies show that also pre-service teachers and upper secondary students have clear weaknesses in understanding of division (e.g. Simon 1993; Campbell, 1996; Merenluoto & Pehkonen, 2002). One of the main reasons for these weaknesses seems to be that pre-service teachers have primitive models of division (e.g., Graeber et al., 1989; Simon, 1993).

## Theoretical Framework

The division task used in this study measures several of the strands of mathematical proficiency mentioned by Kilpatrick (2001), for example, conceptual understanding, procedural fluency and adaptive reasoning. Yet, we view our task as measuring adaptive reasoning above all: to solve the task participants must reflect on and give justification of mathematical arguments, especially the relationships between operations.

### *Understanding Division*

Division is an important but complex arithmetical operation to consider in elementary teacher education. There are many reasons for its complexity: (a) division is taught as the inverse of multiplication, so understanding of division requires good understanding of multiplication; (b) division involving big numbers requires good estimation skills; (c) within the models of equal groups and equal measures two aspects of division can be differentiated: quotitive division (how many sevens there are in 21) and partitive division (21 divided by 7). (e.g., Anghileri et al., 2002)

People can use very different strategies in solving division problems. Some of them are useful and some are misleading. Prior research has identified the following useful strategies (e.g., Heirdsfield et al., 1999): (a) several different counting strategies: skip counting, repeated addition and subtraction, chunks; (b) using a basic fact; (c) holistic strategies.

In a study by Graeber et al. (1989), 129 female pre-service teachers had high scores on solving verbal problems involving the partitive model of division. They were less successful on the quotitive division problems, and these primitive models influence pre-service teachers' choice of operations. Primitive models seem to reflect an understanding whereby a student separate things into equal size groups. In Simon's (1993) study of pre-service elementary teachers the whole-number part of the quotient, the fractional part of the quotient, the remainder, and the products generated in long division did not seem to be connected with a concrete notion of what it means to divide a quantity.

Campbell (1996) studied 21 pre-service elementary teachers' understanding of division with remainder. He conducted clinical interviews with the students, who tried to solve four tasks with abstract contexts. The task we use here has some similarities in contrast to the following task used by Campbell (1996): "Consider the number $6 \cdot 147+1$, which we will refer to as A. If you divide A by 6, what is the remainder? What is the quotient?" (p. 179). In Campbell's study of the 19 participants who tried to solve this task, 15 calculated the dividend although it entailed additional trouble. Of those 15 respondents 9 calculated the dividend and relied upon long division in solving the task. Of those 4 who did not calculate the dividend, only 2 correctly identified the remainder and the quotient.

Zazkis and Campbell (1996) investigated 21 pre-service elementary school teachers' understanding of divisibility and the multiplicative structure of natural numbers in an abstract context. The following is an example of the tasks used: "Consider the numbers 12 358 and 12 368. Is there a number between these two numbers that is divisible by 7 or by 12?" Many pre-service teachers used long division as the procedural activity, but some degree of conceptual understanding was evident as well.

In a study by Silver et al. (1993), a total of 195 sixth, seventh and eighth graders from a large middle school solved three quotient division problems involving remainders with a real-world context (the number of the buses needed). The symbol forms of the word problems were (a) 540:40, (b) 532:40, and (c) 554:40. Of the respondents, 91% used appropriate procedures, and 73% of them applied long division. Only 43% of the participants understood that the result—the number of buses—was an integer.

*Focus of the Paper*

In this paper we focus on the following research question: What kind of erroneous reasoning strategies do pre-service elementary teachers and upper secondary students use in solving a certain non-standard division task? How do these erroneous reasoning strategies used by pre-service elementary teachers and upper secondary students differ from each other?

## Empirical Research

*Research Participants and Data*

The study forms a part of the research project "Elementary Teachers' Mathematics" being carried out in three Finnish universities (Helsinki, Turku, Lapland). Of the 269 pre-service elementary teachers participating in the research, 35% have completed advanced studies in school mathematics in upper secondary school. Two questionnaires were designed, the first measuring the pre-service teachers' mathematical proficiency in certain topics, and the second their attitudes towards mathematics at the beginning of their university studies. The questionnaires were administered at the first lecture in mathematics education studies in all universities in autumn 2003. Students had 60-minutes time for the questionnaires and were not allowed to use calculators. Additional results of the project are described in Kaasila et al. (2008).

The initial proficiency test contained a total of 12 mathematical tasks. The focal content areas were the rational numbers and related operations (in particular, division), because previous research indicates that these are problem areas (e.g. Hannula et al., 2002). All in all, the initial proficiency test focused on content knowledge different from that tested in upper secondary school and on the mathematics component of the matriculation examination.

In conjunction with the project we also collected comparison data with the same questionnaires from upper secondary school. Altogether 50 schools were selected at random from all Finnish upper secondary schools. A letter was sent to the directors of the schools in the sample, in which they were asked to select from their school one group of students in the general course and one in the advanced course in second-year mathematics. We received responses from 34 schools representing a total of 65 student groups. Thus, we obtained in total data from 1434 students.

The non-standard division task we used is the following:

"We know that 498:6 = 83. How could you conclude from this relationship (without using the long division algorithm), what is 491: 6 = ?"

*Data Analysis*

We did not find in the research literature a task similar to the one used in this study. As mentioned earlier, our task shares certain features with one used by Campbell (1996). However, it also differs in a number of respects: First, in the task used by Campbell, the dividend is explicitly mentioned as the 'right hand side' of the division algorithm, whereby respondents have an opportunity to directly identify the quotient and the remainder. In our task, the starting equation is given in the form of division and does not involve a remainder. Second, unlike Campbell, we do not mention in the context of our task the concepts of remainder and quotient. Third, the participants in our study did not have permission to use the long division algorithm or a calculator, which were central aids in Campbell's study.

In the first phase of this study (see Kaasila et al., 2005) we broke the 269 pre-service elementary teachers' solutions down into main categories and subcategories by applying analytic induction. This involves scanning the data for categories of phenomena and for relationships among such categories, developing typologies upon an examination of initial cases, and then modifying them on the basis of subsequent cases (cf. LeCompte, Preissle, & Tesch, 1993).

In the second phase of the study (see Hellinen & Pehkonen, 2008), a deductive approach was used: the 1434 upper secondary students' solutions were categorized using essentially the same classification as in the first phase when analysing pre-service elementary teachers' solutions. A number of categories were identified in addition to those formed in the first phase.

In the third phase we harmonised the categories found in phase one and two by reanalysing a part of the pre-service elementary teachers' solutions. At the end we compared the pre-service elementary teachers' reasoning (or solution) strategies with the upper secondary students' reasoning strategies. For more details see Kaasila et al. (under review).

## Results

The problem was solved totally correctly by one fifth of the pre-service teachers and the upper secondary students. The categories of erroneous strategies used by the pre-service teachers and the upper secondary students are presented in Table 1. More details on results can be found in the paper Kaasila et al. (under review).

Table 1

*Main Categories of Erroneous Strategies Used by the Pre-Service Teachers*

| ***Successful strategies*** | ***PST*** | ***%*** | ***USS*** | ***%*** |
|---|---|---|---|---|
| **All** | **54** | **20** | **221** | **19** |
| ***Erroneous strategies*** | ***PST*** | ***%*** | ***USS*** | ***%*** |
| **Almost correct strategy** | **60** | **22** | **231** | **20** |
| **Thinking limited to integers** | **59** | **22** | **165** | **14** |
| **Clear misconception** | **12** | **5** | **42** | **4** |
| **Other mistakes / irrelevant strategies** | **84** | **31** | **494** | **43** |
| **All** | **215** | **80** | **932** | **81** |

*Note.* (PST, N = 269) and the upper secondary students (USS, N = 1434); the share of successful strategies is given in the same table.

*Erroneous strategies*

Misleading or otherwise erroneous strategies were used by 80% of the pre-service teachers and by 81% of the upper secondary students. We divided these strategies into four main categories, each of which was further divided into subcategories.

*Almost correct strategy*. 22% of the pre-service teachers and 20% of the upper secondary students solved the task almost correctly. The solution of this group indicates a fairly high level of conceptual understanding but all the phases of the solution were not accurately reported, or students made some slippery mistakes. The almost correct strategies can be divided into three subcategories:

In the cases of inaccurately reported reasoning (7% vs. 11%), respondents obtained the correct solution, but some important phases of the solution were not accurately reported or the reasoning used was not indicated at all. In the following, the pre-service teacher did not justify in detail why the remainder was 5.

Example 1. 498–6 = 492. So 492:6 = 82; 491:6 = 81 remainder 5. (3093[1])

The reasoning is essentially correct, but at the end the fractional part was added in the wrong direction (11% vs. 5%):

Example 2. 498–491 = 7, from this it follows that 491:6 must be one unit less than 492:6 with one unit remaining (7–6 = 1), i.e. the answer is 82 remainder 1. (2065)

The most common slippery mistake was an error when calculating the difference between the dividends (4% vs. 4%).

Example 3. Since 6 goes into 498, 83 times and 498–491 = 6, 6 goes into 491 one time fewer than into 498, or 82 times (5802).

*Thinking limited to integers*. 22% of the pre-service teachers and 14% of the upper secondary students were not able to calculate the quotient. These strategies can be divided into three subcategories:

The respondent knew that the answer was not an integer, but he/she was not able to deal with the remainder (12% vs. 9%):

Example 4. The number 491 is 7 units smaller than 498. Therefore 6 should go one time fewer into 491. I can't think of any explanation for the fact that 6 goes into 491 only 81 times. (3016)

In these cases the respondents did not even seem to think that the answer might be something else than an integer (10% vs. 4%).

Example 5. The number 491 is 7 units smaller than 498. Consequently, the answer is 82, but one unit is left over… But perhaps it can be ignored or should the answer be a decimal? (3055)

Here the respondent was aware that division in question was not even, so they thought that there can be no solution to the task (0% vs. 1%).

Example 6. It could not be an integer, because $498–491 \neq 6n, n \in Z$ (5077)

*A clear misconception*. 5% of the pre-service teachers and 4% of the upper secondary students had clear misconceptions in their answers. We divide these answers into two subcategories:

The remainder was considered a decimal (tenths) instead of sixths (3% vs. 1%).

Example 7. 498–491 = 7; 7–6 = 1. The result is 82.1 (1012)

In these cases, the respondents seemed to understand division such a way that the dividend and the quotient change at the same rate (2% vs. 3%).

Example 8. 498–491 = 7; 83–7 = 76. (3057)

*Other mistakes/irrelevant strategies*. 31% of the pre-service teachers and 43% of the upper secondary students obtained no answer at all or presented a solution that was not relevant to the research. These cases are grouped into four subcategories:

The answer was reached by experimenting or in some way without using the connection given in the task (1% vs. 2%). This type of reasoning strategy usually produced erroneous results, but there were also correct ones.

Example 9. 6 goes 50 times to 300, this leaves 191 = > where $30 \cdot 6 = 180$; this then leaves 11, which into which 6 goes almost 2 times; thus 491:6 = > 50+30+2 (almost) ≈ 82 times. (5687)

Correct result without reasoning (3% vs. 6%).

The results were very inaccurate and/or reasoning irrelevant (10% vs. 24%).

The respondents in this category did not produce any result, or anything that made sense (17% vs. 11%).

Example 10. I can't do it without a calculator (3079).

## Discussion

The results indicate that the task was very challenging: only about one fifth of the participants were able to produce a totally correct solution. More than half of the participants either produced no result at all or used misguided strategies. Although division is known to be a difficult operation that has many interpretations, the result is still surprisingly poor. We were surprised that so many pre-service teachers and upper secondary students failed to provide justification with their responses, although it was specifically asked for in the instructions for the task.

We identified three main reasons for mistakes or incomplete solutions: (a) Staying on the integer level: 10% of the pre-service teachers and 4% of the upper secondary student gave their answer as an integer, and it seems that in these cases they did not even think that the answer might be something else than an integer; (b) Inability to handle the remainder: Some of the respondents seemed to understand that the result was not an integer but a fraction, but they could not handle the remainder. For example, they expressed the remainder in the answer in tenths not in sixths (cf. Campbell 1996). It seems that in school dealing with remainders has been a procedural matter, with too little attention focused on the idea that the fractional part of the quotient provides different (yet related) information from the remainder (Simon, 1993); (c) Insufficient reasoning strategies: A little more than a fifth of the participants solved the task almost correctly. In these cases, all the phases of the solution were not accurately reported. The reason for insufficient reasoning strategies may be a lack of language skills because the respondents had great difficulties in providing written explanations of their reasoning (see also Silver et al., 1993).

On the basis of this study we can suggest some guidelines for the content of mathematics courses in teacher education and in school: learners need (a) a concrete, contextualised knowledge of division and (b) the ability to examine division as an abstract mathematical object (cf. Simon, 1993). Above all learners need (c) tasks and situations through which they can develop their adaptive reasoning skills. According to our study, a lack of reasoning skills may be the main factor causing students difficulties when solving non-standard division tasks.

## Endnote

1. The four digit number in the bracket after the example refers to the test participant. When referring to pre-service teachers, the first number is 1, 2, 3 or 4 and when referring to secondary students the first number is 5 or 6.

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# TEACHER-DESIGNED QUESTIONS ABOUT A MATHEMATICAL STORY

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*The purpose of this study was to examine the cognitive level of questions produced by teachers about a mathematical story entitled The Number Devil. The hypothesis was that the mathematical story would inspire higher-level cognitive questions involving exploration of important ides of mathematics as well as philosophical musings about the nature of mathematics. Ten teachers from grades 6 through 12 were asked to read the book and design questions about the important mathematics from each of the 12 chapters. The questions were categorized according to cognitive demand as described by Stein, Smith, Henningsen and Silver (2000). Most of the 455 questions (61%) were of lower-level cognitive demand, procedural in nature, with only 9% of the highest level of cognitive demand, open-ended and conceptual in nature, and only 3% philosophical in nature.*

## Background and Purpose

In a series of articles, Borasi, Siegel, Fonzi, and Smith (1998) and Siegel, Borasi, and Fonzi (1998) described the benefits of reading "mathematical stories" (mathematics-related texts, not technical in nature) in the mathematics classroom. The authors presented evidence that use of such stories can awaken students' interest, stimulate discussion, promote sense-making, provide a variety of opportunities for learning mathematics, and support a constructivist inquiry-based culture in the classroom. By various reading strategies, students are encouraged to ask questions as they read the text, but questions from the teacher also play a role in how the students engage with the text. Indeed, questions that teachers ask are critical in the realization of the vision set forth in the Curriculum and Evaluation Standards for School Mathematics (2000). Regardless of whether the questions are part of a formal assessment or part of the teacher's daily routine of introducing new topics or focusing student attention on a text, questions communicate implicitly what it means to do and know mathematics and establishes expectations (Borasi, 1990; Cooney, Badger, Wilson, 1993).

Teacher-designed questions can vary widely according to the cognitive demands placed on students. In a study by Cooney, Badger, and Wilson (1993), 201 teachers were asked to "write or draw a typical problem that you gave students that you believe tests a deep and thorough understanding of the topic" (p. 241). The problems were categorized according to four levels, depending the requirements of the problem: (1) Simple computation or recognition. (2) Some comprehension (e.g. a one-step word problem). (3) More comprehension (e.g. a multi-step problem). (4) Non-routine reasoning, open-ended. The researchers found that most of the teachers created an item at level 1 or 2. They suggest that reliance on questions that require lower-level cognitive demands, simple reproduction of procedures and generation of an answer, reflect a traditional view of mathematics and mathematics teaching. Reform-based teaching, on the other hand, calls for more non-routine problem solving that places greater cognitive demands on students.

The purpose of this study was to examine the cognitive level of questions produced by teachers about a mathematical story entitled *The Number Devil*. The hypothesis was that the mathematical story would inspire the teachers to design higher-level cognitive questions

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

involving exploration of important ides of mathematics, even have some fun with the mathematics, as well as philosophical musings about the nature of mathematics.

## Theoretical Perspective

In *Implementing Standards Based Mathematics Instruction*, Stein, Smith, Henningsen and Silver (2000) presented a scheme for analyzing tasks according to cognitive demand. In their work, tasks were placed into one of four categories as shown below in Table 1. The first two categories are tasks that represent a low-level cognitive demand, while the second two categories represent tasks with a high-level of cognitive demand. The authors note that "opportunities for student learning are not created simply by putting students into groups, by placing manipulatives in front of them, or by handing them a calculator. Rather, it is the level and kind of thinking in which students engage that determines what they will learn." (page 11). The level and kind of thinking in which students engage depends upon the questions that teachers ask and how cognitively demanding those questions are.

Table 1
*Method of Categorizing the Level of Mathematical Tasks as Described by Stein, Smith, Henningsen and Silver (2000)*

| Category | Description | Characteristics |
|---|---|---|
| 1 | Memorization | Iinvolves reproducing previously learned facts, is not ambiguous, does not involve procedures, has no connection to concepts. |
| 2 | Procedures without connections to understanding, meaning, or concepts | Is algorithmic, not ambiguous, focused on producing correct answer, requires no explanations, focuses solely on describing the procedure that was used, requires limited cognitive effort. |
| 3 | Procedures with connections to understanding, meaning, or concepts | Requires some conceptual understanding of the procedures to complete the task, focuses on use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts, may rely on multiple representations, requires some cognitive effort. |
| 4 | Doing mathematics | Requires complex non-algorithmic thinking, requires exploration, requires understanding of mathematical concepts or processes, requires self-monitoring, requires students to analyze task and possible solutions and access relevant knowledge, requires |

| | | |
|---|---|---|
| | | considerable cognitive effort and may involve some discomfort. |

## Methodology

The teachers who participated in the study were 10 middle-grades and secondary teachers enrolled in a Master of Education program with a specialization in mathematics education at Kennesaw State University. The students were enrolled in a course entitled “MAED 7720: Mathematics in the Humanities.” Over a three-week period the students were asked to read the book “The Number Devil” and complete the following task:

> For each chapter, design four questions that you would ask your students about the most important mathematics in the chapter (to see if they understood the chapter and to get them thinking about the mathematics). Describe how you would answer the questions.

The Number Devil is a 253 page book consisting of 12 chapters, aimed at ages 9 through 12. In each chapter, a middle-grades boy named Robert recounts his night’s dream about “The Number Devil” a trickster who teases him into making observations about numbers. For example, in Chapter One, “The First Night,” the devil shows Robert how you can do almost anything with the number 1, even experience the infinitely large and the infinitely small. On the second night, the devil reveals the power of zero and discusses Roman numerals, place value, and powers. Prime numbers, rational numbers, irrational numbers and roots are the topics of Chapters Three and Four. Triangular numbers are introduced in Chapter Five. Fibonacci numbers are explored in Chapter Six. Pascal’s triangle is introduced in Chapter Seven and used in Chapter Eight to calculate permutations and combinations. Sequences and series are the topics of Chapter Nine. The golden ratio and Euler’s formula are explored in Chapter Ten. Chapters 11 and 12 are the most difficult chapters with discussions of proof and the nature of mathematics and its history.

The questions were assigned to levels 1 through 4 according to the scheme described above by Stein, Smith, Henningsen and Silver (2000). Two additional levels were used: level 5 was a new category for philosophical questions about the nature of mathematics or attitudes towards mathematics; level 6 was created for questions regarding the history of mathematics.

## Results

The teachers provided a total of 479 questions but 24 of the questions contained mistakes, were incomplete, or were difficult to interpret; therefore 455 were analyzed. Samples of questions in each category are shown in Table 2.

Table 2.
*Examples of Questions in Levels 1 through 6*

| Level | Sample questions |
|---|---|
| 1 | 1. What kind of triangle do these 'triangle numbers' (triangular numbers) make when written as dots?<br>2. Prima donnas (the number devil's word for prime numbers) can only be divided by 1 and itself. What types of numbers do these represent?<br>3. What operation is like multiplying in reverse?<br>4. What is meant by rutabaga (the devil's word for square root)?<br>5. Why did the number devil say the Romans had such a difficult time with numbers?<br>6. Give an example of how we used Pascal's Triangle in Algebra 2? |
| 2 | 1. Can you figure the $16^{th}$ triangle number (triangular number)?<br>2. What is the $20^{th}$ Bonacci number (the devil's word for Fibonacci number)?<br>3. Can you write in Roman Numerals the year you were born?<br>4. What is 10!<br>5. In how many different ways could 4 of Robert's classmates be seated? |
| 3 | 1. How does the graph of the function $f(x) = 1/x$ for $x \geq 0$ represent the concepts of "an infinite number of numbers" and "an infinite number of infinitely small numbers."<br>2. Can negative numbers be prima donnas?<br>3. When do you know if you are going to get a remainder and when are you not?<br>4. How does the number devil explain infinity?<br>5. Explain how you would get 1111111 x 1111111 without using a calculator. |
| 4 | 1. Why wouldn't 11111111111 x 11111111111 work as the Number Devil said the other would?<br>2. Why does the sum of two consecutive triangular numbers equal a square number?<br>3. Write your own rule like Bonacci's and create your own set of numbers. Write out the first 13 of them. Do you notice any patterns?<br>4. In your own words, write a definition for infinite.<br>5. The number devil says the following comment, "… proving that no proof exists is a proof in itself…". What do you think he meant by this?<br>6. Why do you think proofs are necessary in mathematics?<br>7. Starting with just one dot, create your own pattern like the coconut triangles (triangular numbers). What patterns can you find? |

| | |
|---|---|
| 5 | 1. Why do you feel that Robert thinks math is a waste of time?<br>2. Why do you think that the number devil changed the names of mathematical concepts into ordinary terms?<br>3. The number devil said that guessing is not allowed in mathematics. What do you think? Should guessing ever be acceptable in mathematics? Explain.<br>4. Can you relate to Robert's frustration to math? Describe a story when you felt like Robert on page twenty-five.<br>5. Why did the Number Devil say that mathematics is an endless story? What do you think?<br>6. The Number Devil talks about another number devil called the Man in the Moon. He talks about proving a certain math rule. However, the Number Devil said another number devil came along and proved him wrong. What does this tell you about math? |
| 6 | 1. Who developed the triangle mentioned by the Number Devil (Pascal's triangle)?<br>2. Who is Fibonacci and what mathematical concept is he known for in mathematics history?<br>3. Where else does the "blankety blank" number (the golden ratio) show up in nature? |

Level 1 questions often connected the text to previously learned facts or vocabulary, or involved simple identification of some fact offered in the text. Questions assigned to level 2 were most often questions that asked the student to reproduce the procedures demonstrated in the text or connect the text to some previously learned procedures. Questions assigned to level 3 involved taking procedures described in the text and requiring the students to explain the procedures or apply the concepts described in the text in new contexts; they also included questions in which students were asked to explain how the number devil or Robert came to certain conclusions. Questions assigned to level 4 required something quite beyond what was provided in the text, including explanations of why certain observations made by the number devil were true or how those observations could be extended to new situations, new sequences or series.

The teachers produced 479 questions but 24 of those questions contained mistakes, were incomplete, or were difficult to interpret. As shown in Table 3, of the remaining 455 questions most were of lower-level cognitive demand (61%) and surprisingly few were of a philosophical nature (3%). Only 33% were of a higher-level cognitive demand and only 9 percent were classified as level 4. Certain teachers tended to questions of higher-level cognitive demand (teacher 4) while others tended to questions of lower-level cognitive demand (teacher 9). Note that most of the philosophical questions and the level 4 questions could be attributed to teacher #3.

Table 3.
*Cognitive Level of Teacher-designed Questions*

| Cognitive level | Teacher #: | | | | | | | | | | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | |
| 1 | 11 | 8 | 8 | 1 | 9 | 7 | 21 | 24 | 37 | 10 | 136 (30%) |
| 2 | 9 | 21 | 9 | 13 | 16 | 24 | 15 | 17 | 5 | 14 | 143 (31%) |
| 3 | 18 | 16 | 8 | 23 | 11 | 9 | 2 | 5 | 2 | 14 | 108 (24%) |
| 4 | 3 | 3 | 10 | 3 | 5 | 2 | 5 | 1 | 1 | 10 | 43 (9%) |
| 5 | 1 | 0 | 6 | 4 | 0 | 0 | 0 | 0 | 1 | 1 | 13 (3%) |
| 6 | 5 | 0 | 5 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 12 (3%) |

## Conclusion and Discussion

The purpose of this study was to examine the cognitive level of questions produced by teachers about a mathematical story entitled *The Number Devil*. The hypothesis was that the mathematical story would inspire higher-level cognitive questions involving exploration of important ides of mathematics as well as philosophical musings about the nature of mathematics. As shown in Table 3, the hypothesis was not supported. As in a previous study by Cooney, Badger, and Wilson (1993), even in the context of a whimsical, imaginative story about a boy who is teased with interesting facts about mathematics, teachers largely produced questions of a lower-level cognitive demand.

According to Cooney, Badger, and Wilson (1993), the tendency for teachers to use questions that involve lower-level cognitive demands could be attributed to their understanding of mathematics as a series of procedures. They state "As long as mathematics is viewed as consisting primarily of a series of steps to be applied in isolated contexts, teachers will view moves toward alternative methods of assessment as peripheral to the 'real curriculum.' As a consequence, a promising vision evaporates into a mirage" ( page 247). In this study, procedures or recall of vocabulary appeared to be the focus of 90% of the questions, despite the opportunities for exploration and more in-depth inquiry. The high occurrence of procedural questions may be a reflection of the teachers' traditional view of mathematics, but it may also be due to the teachers' inexperience with designing questions of higher cognitive demand, compounded with their inexperience with using mathematical stories in the classroom.

Further research is needed to explore ways to raise the cognitive level of teacher-designed questions, as well as how such mathematical stories can promote student questions of a high cognitive level and support mathematical inquiry.

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# THE DEVELOPMENT OF PEDAGOGICAL CONTENT KNOWLEDGE IN COLLABORATIVE HIGH SCHOOL TEACHER COMMUNITIES

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*To support student understanding, math teachers need to develop pedagogical content knowledge (PCK) that contributes to conceptually rich instruction. This analysis tracks the development of PCK in two mathematics teacher workgroups in urban high schools. Using video data, this analysis highlights the differences in the teachers' collaborative discourse as they made sense of students' mathematical difficulties. In the newer group, the teachers elaborated students' mathematical difficulties but their instructional response did not address the scope of difficulties identified. In the more mature group, the teachers' analysis of students' mathematical difficulties was more tightly coupled with instructional responses. This analysis illuminates the ways in which collegial conversations support the development of PCK and highlights professional learning through the discourse of collaborative teacher communities.*

## Objective and Purpose

The improvement of secondary mathematics instruction continues to be a pressing issue in the United States. Several recent studies indicate that math students routinely engage in classroom activity without understanding connections to important mathematics and scientific ideas (e.g., Banilower, Smith, Weiss & Pasley, 2006). In an observational study, Banilower et al. (2006) found that only 14% of all lessons in a national sample had a climate of intellectual rigor. In addition only 16% of lessons incorporated questioning that was likely to move students' thinking forward and two-thirds of all lessons lacked sense-making discourse. Inattention to these kinds of classroom discourses is compounded by teachers' limited understanding of students' conceptions, thus making it difficult to adjust instruction accordingly.

But how do we move teachers toward this kind of conceptually rich instruction? A number of studies point out that this kind of mathematics instruction depends on teachers' pedagogical content knowledge, or PCK (Ball, Thames, & Phelps, 2008; Hill, Rowan, & Ball, 2005; Shulman, 1986). By pedagogical content knowledge, researchers refer to mathematical understanding that includes the most useful forms of representing and communicating content, as well as how students best learn the specific concepts and topics of a subject.

If conceptually rich instruction depends on teachers' PCK, how might we support the development of PCK? One possibility comes out of another line of educational research on equitable teaching. Namely, equitable student outcomes — meaning that students' race or class is not strongly predictive of mathematical attainment — are best achieved when teachers work together with a focus on student learning (Gutièrrez, 1996; Lee & Smith, 2001; McLaughlin & Talbert, 2001). It has been posited by a number of researchers that such teacher collaboration can, among other things, foster the kind of PCK that supports teaching for understanding.

For the past five years, on the *Adaptive Professional Development* project[v], we have engaged in a design experiment aimed at increasing access to and rigor in secondary mathematics through intensive, ongoing professional development work in urban high schools. A primary goal of this project is to support conceptually rich mathematics instruction for diverse students. We work toward this goal by providing professional development that places student learning at the center of teaching practice and supporting teachers' ongoing collaboration around student thinking. We started our work at Septima Clark High School,[vi] and after our initial success there, we added other schools to our network. Before the intervention at Clark, student pass rates in first year mathematics hovered just under 50%. After year one, they increased to 70%. Similar improvements were seen on standardized mathematical achievement tests, indicating that the changes in teachers' practice had an effect on student learning, particularly for poor students and African and African American students, two demographics of particular concern in this setting. After two years of deliberate work on improving their teaching practices, the teachers saw pass rates on 10th grade state mathematics assessments rise for both low-income students (from 21.2% to 40.0% passing) and for African and African American students (from 15.7% to 30.1% passing).

We understand these results to be the outcome of at least two things. First of all, with the support of our project, the teachers' worked together in a more focused way than they had previously, giving them opportunities to coordinate expectations for students. This provided a clearly articulated curriculum, from the broader level of pacing down to the finer details of instructional language. Such consistency seems to have raised the level of expectations across the classrooms while providing students with easier transitions between them. Second, the teachers' intensive engagement in professional development coupled with the building-level teacher team collaboration provided rich opportunities for professional learning. It is this second aspect of the phenomenon that I investigate in this paper. Over the course of the project, our inquiry was organized by the question: *how do teachers develop more equitable mathematics teaching practices through targeted and situated professional development?* Unpacking the relationship between professional development activities and teacher learning will contribute to other efforts toward improvement of secondary mathematics education. In this analysis, I focus on how teacher collaboration contributes to PCK that supports teaching mathematics for understanding.

## Theoretical Framework

The study of teachers' professional learning in the context of their workplace groups requires a framework that can account for learners and their settings. Thus I use a situative lens to study the teachers' learning (Greeno, 2006). As Greeno describes, a situative approach to the study of learning is different from a purely cognitive approach: "Instead of focusing on individual learners, the main focus is on activity systems: complex social organizations containing learners, teachers, curriculum materials, software tools, and the physical environment" (p. 79). In this case, the social organization of the teachers' collaborative work provides a social context for the teachers as learners.

The study of learning demands a conceptualization of both mechanism and telos. A situative perspective highlights the role social interactions and contexts play in shifting learners' understandings. In Lave and Wenger's (1991) terms, the mechanism for learning is the shift from peripheral to central participation in an activity. In previous work, I have operationalized this notion in the context of teacher community by looking at the context of teachers' collegial conversations, investigating the ways that discourse patterns over time provide the means for teachers to become more central participants in these interactions (Horn, 2005, 2007, in press). The telos, or goal, of learning in this case is teaching mathematics for student understanding. Thus we would anticipate a stronger and more specific understanding of student thinking and appropriate instruction as teachers learn.

This project conceptualizes teachers' learning about teaching mathematics for understanding as something that can be traced, in part, through their interactions with colleagues. By investigating the nature of those interactions and what might make them more effective in deepening teachers' understanding, this study contributes to educational research by furthering our understanding of teacher learning in workplace contexts. At the same time, there are important practical implications for this analysis since professional learning communities are becoming a more common way for schools to organize teachers for instructional improvement.

## Methods

*Research Context*

For the larger study, we pursued the question: *How do teachers develop more equitable mathematics teaching practices through targeted and situated professional development?* We have approached this project as a design experiment (Barab, 2006; Brown, 1992), where we have used precepts about student learning and teacher learning to develop interventions with the teachers to refine our understanding of how these work. Through active participation for the five years of the project, we have collected hundreds of hours of field notes of professional development activities, yearly interviews with participating teachers, audio and video recordings of teacher collaboration meetings, video tape recordings of teachers' classrooms, and student achievement data.

As described earlier, our work started in Septima Clark High School in the 2004-2005 academic year. In Fall 2005, the Clark teachers formed a Freshman Team to focus on improving instruction for students in first year math classes. Team members consisted of the subset of the mathematics department who taught at that grade level. They were given a common planning period in addition to the regular preparation time.

Because of their initial success, in Fall 2006, teachers at Lotus High School also formed a Freshman Team using a similar structure and principles. The Lotus team faltered somewhat the following year due to staffing issues and a lack of support, but has re-emerged this academic year (2008-2009) under new leadership and with the hiring of committed mathematics teachers.

Theoretically, we view these collegial groups as providing us with a cross-sectional sample of teacher communities in different stages of development. Both groups meet criteria in the literature for professional learning communities: the teachers expressed shared norms and values; they focused together on student learning; they collaborated; deprivatized their practice; and used reflective dialogue in their conversations (Louis, Marks, & Kruse, 1996). While these criteria provide an important starting point for identifying teacher professional learning communities, the field needs a better understanding of not only how such communities might emerge, but how they might evolve over time. We see the differences in the how the Clark and Lotus teachers' collaborate as one way of understanding the evolution of professional learning communities. The question that guides the present analysis is: *How do high school mathematics teachers' collaborative teams support their understanding of equitable teaching practices?* By looking cross-sectionally at professional communities at different points of evolution, this analysis seeks to hone in on the types of interaction that support the development of PCK, particularly as it relates to teaching mathematics for understanding.

*Participants*

The participants featured in this analysis are the current teacher teams at Lotus and Clark high schools. The Lotus team consisted of 3 teachers: Claire (with 9 years experience), Betty (5 years experience); and Kieran (Claire's student teacher). The Clark team consisted of Rose (over

25 years experience), Darla (10 years), Linda (5 years), Wendy (2 years), Anh (Linda's student teacher) and Trevor (Darla's student teacher). All of the teachers demonstrated a commitment to the project of teaching for understanding and had participated in professional development to support the development of these teaching practices. Because the Clark teachers have collaborated longer than the Lotus teachers, we conceptualize them as a more developed teacher community.

*Data*

Since we seek to understand the development of discourse that supports PCK in mathematics teacher workgroups, we have collected data that allows for longitudinal within group analysis as well as cross-sectional across group analysis to make sense of changes in discourse over time. Our primary data for this analysis are videotapes of the teachers' collaborative work. Currently, we have video taped 2 two-hour collaborative meetings at both Clark and Lotus from the start of the school year. In addition, we plan to tape 2 two-hour collaborative meetings at each site in the middle and end of the school year. This will provide us with 12 hours of video data for the analysis. As secondary data, we have collected weekly fieldnotes of the teachers' meetings to help us situate the video data in terms of the groups' overall trajectory, to determine the typicality of the focal events, and to corroborate and refine our preliminary analysis. In this analysis, I focus on a cross-sectional analysis across the two groups.

*Analytic Methods*

To investigate the teachers' learning in their collaborative groups, we parsed the videotapes using a unit of analysis called *episodes of pedagogical reasoning* (EPRs, [Horn, 2005, 2007, in press]). That is, we looked for units of teacher talk where the teachers reveal their reasoning about pedagogical issues. Usually these begin with questions or assertions about practice that are accompanied by a statement of reason, explanation, or justification. EPRs can be single-turns of talk ("I'm not doing that activity because it takes too much time") or more elaborate, multiparty co-constructions. We locate EPRs based on topic shifts in the conversations. Longer EPRs frequently consist of sub-EPRs, as teachers explore different facets of the same problem.

For the present analysis, I focus on two EPRs, a five-minute episode from Lotus and a three-minute episode from Clark. These EPRs were selected for analytic comparison because the teachers are focusing on students' mathematical confusion about algebraic topics, providing insight into the kinds of pedagogical reasoning that might support PCK. In particular, I trace the kinds of pedagogical content knowledge that the teachers assert during these episodes and how they relate it to their instructional decisions. Investigating how the teachers reason about these confusions and then develop pedagogical responses illuminates the role that professional communities can play in the development of PCK as well as influencing teaching practice.

## Results

*Figuring Out Students' Confusion with Linear Equations at Lotus Overview of Episode*

This five-minute EPR comes out of an hour-long collaboration meeting. The Lotus teachers are working to understand why an activity on linear graphing was not successful in their classes. Prior to this meeting, all of the teachers used the activity in their classes, and they have agreed that it did not lead to the kinds of student understanding that they had hoped for. The original goal of the activity was to help students understand the graphical consequences for changes in slope ($m$) and y-intercept ($b$) in linear equations in the form $y = mx + b$. The first part of the episode is spent diagnosing what students had trouble with in the activity. At the end of the episode, the teachers decide upon an appropriate instructional response.

*Pedagogical Content Knowledge Discussed*

In diagnosing the trouble students had with the graphing activity, the teachers invoke a number of issues that relate and represent their pedagogical content knowledge. First, Betty points out that the students' trouble with the scales on the axes led to graphs that did not highlight the contrasts between the different linear equations, as was fundamental for the success of the activity. Kieran, Claire's student teacher, thinks that the students' trouble came out of their difficulty with the task's instructions. Betty, drawing on a similar activity the group did earlier, hypothesizes that the negative numbers students needed to use in this activity led to difficulty, since their integer arithmetic is not strong. Betty then further suggests that students may have had different scales for graphs they were supposed to compare, which obscured the contrasts in the graphs that were critical to the activity. Kieran and Claire did not see the problem with the integer arithmetic in their students' work. When Betty asks them where their students had trouble, they point to issues of scale: students did not keep the intervals constant, particularly around the graph's origin. Claire adds that some of their students were having trouble plotting points correctly. Betty agrees that some of her students' points were not plotted correctly, but she thinks it was because they used scales that were not appropriate for the numbers they were plotting (e.g., using intervals of 20 when the x values were between 1 and 10). Claire also reports student graphing troubles arising from interval choices, but she tells of students using intervals like 3 and having trouble finding even numbers on the axis, leading to non-linear looking graphs.

At the end of this diagnostic talk, the teachers turn to figuring out their instructional response for the follow-up activity, which is similar in structure to the one they have been analyzing. They invoke an ongoing tension in their decision making, described by Betty as "whether or not we want to do give into them and have them started independent of us and how much we want to make sure and clarify questions." They conclude that, because of the extent of the difficulties students are having, they need to do more to "scaffold" students. They decide to do this by "modeling" the task before having students do it independently in their small groups. Betty illustrates this modeling through a *teaching rehearsal* (Horn, 2005, 2007, In press), an imagined or anticipatory narration or enactment of classroom interaction that teachers use when making sense of practice with their colleagues. Betty uses a teacher voice and gestures her graph on an imaginary white board while she says, "'So I'm gonna draw my axes…' 'Everybody check, make sure they're numbered correctly.' And then like, 'Here are my lines, I'll label them like this.'" Her rationale for the modeling follows the rehearsal, when she explains, "just to kind of do a speeded up version of it where you're not giving accurate points, not doing whatever but to actually do those instructions." Her colleagues agree to this strategy for addressing the students' troubles.

*Relationship between Diagnosis of Student Confusion and Instructional Decisions*

The diagnosis of student confusion takes up much of this episode (approximately 4 of 5 minutes). This diagnosis requires the teachers' PCK, as they talk through the ways students understand and interact with the content of linear equations. A number of sources of trouble in students' understanding are elaborated, ranging from instructional (unclear directions) to students' skills (difficulty with integer arithmetic) and conceptual understanding (choosing appropriate intervals for a given set of data). While the teachers reach some consensus about the sources of the student confusion, their agreed upon response only addresses the first type of problem. By resolving the similar student difficulties by modeling the new activity, the teachers planned response helps with the procedural aspects of the tasks without addressing the underlying mathematical issues. In particular, Betty's suggestion to do a "speeded up version" of the activity as a demonstration before the students work independently does not allow a discussion of some of the deeper issues of linear graphing that they have identified, such as choosing appropriate scales for a given set of data, or keeping scales consistent so that students can make comparisons between graphs.

*Understanding Students' Confusion with Solving Systems of Equations at Clark Overview of episode*

This three-minute EPR comes out of a two-hour collaboration meeting. The teachers are focusing on the problems students have using the "elimination method" of solving systems of linear equations. The elimination method refers to taking *n* equations in *n* variables and systematically finding values for the variables that satisfy all *n* equations by eliminating like terms across the equations. In this introductory unit, students were looking at systems of two equations in two variables, typically *x* and *y*. This required that they eliminate one variable (say *x*) across the equations by creating like terms with opposite coefficients and then solving for the value of the remaining variable (*y*), which could then be substituted in one of the original equations to find the first variable (*x*).

Two of the teachers, Darla and Wendy, have taught the unit on solving systems of equations using the team's shared curriculum, while the other teachers around the table are new to this particular unit. Wendy leads the discussion of student difficulties, interweaving her instructional responses. Her diagnosis is supported and elaborated by Darla.

*Pedagogical Content Knowledge Discussed*

Wendy starts the discussion of student difficulties by using a discourse structure I have called a *teaching replay* (Horn, 2005, 2007, in press). Teaching replays provide blow-by-blow accounts of actual and sometimes ongoing classroom events, often with teachers often narrating or acting out their part as teacher. In this instance, Wendy replays a student's confusion over the elimination method, specifically the combining of like terms when the coefficients are not opposite values ("'Okay, I know that goes away, right? […] So I know the 3y and the -3y go away, but what's 4x + -2x?'"). She offers an instructional suggestion to lessen student confusion by sticking "strictly to addition."

Darla agrees with Wendy's diagnosis and adds that, in the past, her students did not have trouble using the strategy of multiplying through one of the equations by a negative one to create opposite like terms that "make zeroes." This, Darla explains, means that students do not need the support of the lab gear, a time-intensive algebra tile-like manipulative that the teachers use in some of their instruction. However, she cautions that while students could carry out the procedure successfully, they still did not recognize why they needed to use it. She illustrates the student confusion through another replay, one that expands on Wendy's: "'Cause you know how Wendy did the – 'this goes away?' Right? […] 'Okay, why does it go away?' No idea." This replay inserts a teacher response to the student confusion portrayed by Wendy.

Wendy supports Darla's elaboration and adds another example of student confusion, the "common mistake" that like terms "go away." She replays a student-teacher interaction to illustrate prototypical instructional dialogue around this confusion, which Darla follows up with a replay of her own: "'What do you mean it goes away?' They're like, 'It eliminates.' I'm like, 'Why?!' They just have no..." Despite the teachers' consensus that students do not understand the mathematics behind the elimination method, Darla adds that the team is making headway in building the conceptual foundation for future years ("in future years, when kids have the language of making zeroes much more strong […] we should have a much easier time").

Nonetheless, they are currently working against student beliefs that “math is just tricks” and does not require deeper understanding.

*Relationship between Diagnosis of Student Confusion and Instructional Decisions*

In this episode, the teachers’ diagnosis of student confusion is tightly coupled with instructional responses. From the beginning, when Wendy signals students’ troubles with combining like terms, she suggests an instructional response (“stick strictly to addition”). Darla further diagnoses the extent of students’ understanding, noting that they can carry out the procedure of multiplying through by negative one, which she also ties to appropriate instruction (“I don’t think they’ll need lab gear for that”). However, she is careful to distinguish between procedural success and conceptual understanding, pointing to the idea that underlies the students’ confusion —terms eliminate when they are of opposite value and “make zero.”

## Discussion

In both of these conversations, there is a clear focus on student learning. Both groups of teachers work together to make sense of students’ mathematical performance and elaborate on the kinds of understanding that students bring to their work. In both groups, an important facet of PCK is highlighted: namely, the kinds of difficulties students have in learning important topics in algebra. According to many scholars of teacher communities, this focus on student learning is a hallmark characteristic of professional learning communities (Louis, Kruse, & Marks, 199x; McLaughlin & Talbert, 2001).

However, the difference between the two conversations seems significant and consequential for teaching mathematics for understanding. While the teachers in the first conversation (Lotus) diagnose many of students’ troubles with linear graphing, the planned instructional response does not fully account for the range of difficulties they identify. In particular, by focusing on students’ procedural rather than conceptual troubles, the teachers miss an opportunity to help students develop a stronger understanding of some of the critical issues in linear graphing. The teachers in the second conversation (Clark) also diagnose students’ conceptual difficulties with an important algebraic topic, solving systems of equations. However, in contrast to the first conversation, the teachers consistently link an assessment of student understanding to instructional responses.

It is interesting to compare the representation of instructional dialogue in both groups. In earlier work in which I studied the conceptual resources for teachers’ pedagogical reasoning, I found that highly collaborative teacher groups tended to use more rehearsals and replays in their talk, and that these tended to represent both student and teacher sides of classroom interactions (Horn, 2005). Similarly, the Lotus teachers do not make much use of teaching rehearsals or replays in their pedagogical problem solving. When it does occur, it is a one-sided representation of teacher talk. In contrast, teaching replays permeate the Clark conversation from the very beginning. Many of the instructional responses, in fact, are represented through a replay of student-teacher dialogue. Further analysis will have to confirm that the Clark teachers consistently use these kinds of interactive representations in their conversations to represent the interactivity of the classroom. If this proves to be the case, this would contribute to the finding that the interactivity represented in these classroom snippets may signal a more tightly bound understanding of student confusion and instructional responses.

This paper makes several contributions. First, I have illustrated the kinds of pedagogical content knowledge that may emerge in teacher collaborative talk. Second, this analysis elaborates the ways in which a focus on student learning can be more tightly linked to instruction in a professional learning community. Finally, the findings here may signal developmental differences in teacher workplace groups. This last implication supports practitioners, be they teacher leaders or school coaches, seeking to develop professional communities in their own schools.

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# A CASE STUDY OF ONE SECONDARY MATHEMATICS TEACHER'S IN-THE-MOMENT NOTICING OF STUDENT THINKING WHILE TEACHING

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*Given the complex nature of the classroom and the instructional flexibility that teachers are expected to demonstrate, understanding how teachers are paying attention to and making sense of what happens during instruction is an important area of research. This study investigates the in-the-moment noticing of one secondary mathematics teacher using a wearable camera with after-the-fact recording capability. When using the camera in his classroom to capture "interesting moments", the teacher focused almost exclusively on his students' mathematical thinking. We discuss the range of student thinking captured and the function that such noticing might play in this teacher's practice.*

Recent research in mathematics education has drawn attention to the importance of understanding how teachers are attending to and making sense of what is happening in their classrooms (Mason, 2002; Jacobs et al., 2007; Sherin, 2007). A focus on this component of teacher thinking is particularly timely given recent calls for mathematics teachers to be flexible and adaptive to student thinking in their instruction (NCTM, 2000). Furthermore, having teachers focus on student thinking and adjust what they are doing based on formative assessment places high demands on teacher's subject matter and pedagogical content knowledge (Ball & Bass, 2003). Yet, understanding the knowledge needed for teachers to be able to make these types of instructional decisions involves not only being able to systematically articulate the kinds of knowledge that they must have, it requires also developing models of how that knowledge is activated and applied in actual teaching situations (Sherin, Sherin, & Madanes, 2000; Schoenfeld, 1998). Understanding this in-the-moment noticing and sensemaking is therefore an important goal for teacher thinking research.

To suggest that mathematics teachers are adaptive in their instruction is not to deny that they walk into the class having some idea of what they would like to accomplish that day. Numerous researchers have documented the kinds of lesson images and agendas that teachers develop in planning for instruction (e.g., Leinhardt & Greeno, 1986; Morine-Dershimer, 1978). However, once in the classroom, teachers must be alert to how the lesson is progressing and make modifications as necessary. In previous work that we have done to study teacher noticing, we have relied heavily on the use of having teacher's view episodes of instruction on video and discuss what they noticed (Sherin & van Es, 2009). Other research has also tried to make use of video to study teacher noticing (Borko et al., 2008; Santagata, Zannoni, & Stigler, 2007.) This work has led to some interesting findings about noticing and how it can be influenced. For example, Sherin and van Es (2009) demonstrate that teacher noticing can be influenced based on participation in video clubs and Star and Strickland (2007) show that a pre-service course resulted in teachers being able to recall more detailed and varied information from a teaching episode they viewed on video. While this work has provided some insight into how teachers' noticing occurs while watching video, the processes of noticing and making sense might differ substantially while they are in the act of teaching. In more recent work and in this study we have begun to try to investigate teacher noticing in-the-moment of instruction.

## Research Design and Methodology

In this work we use a new digital video technology, first used by Sherin and colleagues (2009), which helps provide online access to what teachers pay attention to during instruction. The Deja View CamWear Model 100 (Reich, Goldberg, and Hudek, 2004) is a compact wearable camera that records from the perspective of the teacher, capturing what is in their field of vision rather than from the perspective of the researcher as is often the case in classroom videography. In addition, the camera exhibits an *after-the-fact* capability that enables users to decide to save a record of their experience *after* an event has occurred. The camera maintains a continuous buffer of 30-second activity and when an event occurs that the user wants to record, pressing the capture button saves the previous 30 seconds into digital clip that can later be downloaded onto a computer.

This after-the-fact feature of the technology allows a teacher to wear the camera during class and decide while in the midst of teaching when a noteworthy event occurs that they would like to save for further reflection. We are using the technology as part of a larger ongoing research project to study math and science teachers' noticing during instruction (see Sherin et al., 2009 for a more detailed discussion of the value of this technology for this purpose). For that project we recruited ten secondary math and science teachers of varying experience from a large urban area in the Midwest to use the technology in their classrooms. Here we focus on one mathematics teacher, Mr. Leavenworth (all names are pseudonyms), who received a BS in Mechanical Engineering and became a teacher through an alternative certification program. He was in his fourth year of teaching, his third at this particular school, and taught courses that included Advanced Algebra/Trigonometry, Advanced Mathematics, and an International Baccalaureate AP Calculus class. He had previous experience using video to reflect on issues of teaching and learning with colleagues. For this study he used the camera for four days in his AP Calculus class during the fall of the 2008-09 school.

Before class on the day of taping, Mr. Leavenworth was fitted with the camera and informally discussed with the researcher his plan for the lesson. Mr. Leavenworth was instructed to "press the record button on the camera when something interesting happens in class, when something seems interesting to you." The prompt was intentionally open-ended to allow the teachers to define "interesting" in their own way while teaching. No limit was given on the number of moments the teacher could capture. During class the researcher also video recorded the entire lesson using a camera positioned in the back of the room.

Following each lesson the researcher interviewed the teacher for 30-40 minutes. They first discussed what it was like to use the camera that day and whether it affected the teacher or students' behavior. The researcher and teacher then watched each of the captured moments until the teacher remembered why he had captured that moment. If he could remember, the teacher then explained what was interesting about that particular moment. After viewing all of the clips the researcher also asked the teacher a few questions about whether he had captured what he had intended and whether he was aware of using any specific criteria to select interesting moments. While there was a standard protocol, the interviews were relatively unstructured and conversational in style.

## Results

In the following we briefly discuss the nature of the clips Mr. Leavenworth collected and highlight three aspects of his noticing that emerged from the data analysis. We describe how:

1) A majority of the moments that Mr. Leavenworth captured related to students' ideas about mathematics.
2) Mr. Leavenworth's noticing is structured so that he is on the look out for specific mathematical ideas that are important for students' understanding during instruction.
3) At certain junctures during instruction when a particular conceptual idea or solution strategy is needed, Mr. Leavenworth notices the absence of those ideas needed at that point in the lesson. In response to this situation he has two different kinds of instructional methods that allow him to introduce the necessary ideas into the classroom.

*Collected Clips*

Mr. Leavenworth's instruction can be described as a switching back and forth between public whole group discussions in which new material is being introduced or a problem set is being discussed and private small group or individual seat work on problems as he circulates the room. During the three days that Mr. Leavenworth used the camera he collected a total of 55 clips of "interesting" activity during both of these classroom activities. However, Mr. Leavenworth seemed to demonstrate a slight preference for capturing moments that occurred during the large group discussion component of his classes. Although only 58% of class time was devoted to this public work, 72% of the moments that he captured occurred during this type of activity. Next we examine his reflections on these moments in the post-interview to understand more about what he found interesting.

*A Focus on Student Thinking*

In the post-interview, Mr. Leavenworth discussed each of the clips he captured and explained what he had found interesting about what was happening at that moment. As a first step in characterizing his explanations we applied the set of codes developed by Sherin et al (2009) to describe teacher's reasons for capturing each moment. Table 1 provides a description of each of the codes and indicates the percent of Mr. Leavenworth's clips coded in that category.
We coded his reflection as Student Thinking when the focus of his comments was on the substance of student's mathematical ideas. A Discourse code was applied when his reflection related to process of how ideas were being communicated in the classroom. We coded his reason for capturing the clip as Teacher Moves when he the focus of his comments is on some instructional decision or action that he made. And Student Engagement referred to those reflections in which he was focused on the quality or quantity of student participation in the class activities.

In his reflections Mr. Leavenworth describes an overwhelming percentage (94%) of his captured moments as "interesting" because of the substance of students' mathematical thinking. Whereas he found only a relatively small percentage "interesting" because of Discourse (8%), Teacher Moves (17%) or Student Engagement (2%). Note that the total percentages add up to greater than 100% because double coding was permitted if he cited multiple reasons for finding a moment "interesting". The strong focus on Student Thinking while capturing moments distinguishes Mr. Leavenworth from the other teachers using this camera who have typically only focused on student thinking about one third of the time (Luna, Russ, Colestock, in press; Sherin et al., 2009).

In examining the kinds of Student Thinking moments that he captured, it becomes evident that he was interested in a range of different kinds of student thinking. In Table 1 we briefly describes a few of the specific moments that he captured in order to anecdotally communicate the range of student thinking phenomenon which drew his attention.

Table 1. *Examples of Student Thinking Moments Captured by Mr. Leavenworth*

**Student justification of solution** : When stating her answer to an activity that involved trying to match up a differential equation with its corresponding slope field a student refers to the order of the function as a useful factor to consider. Mr. Leavenworth captures the moment and explains, *"So it was nice that she mentioned [the order of the function] and something that we can build on tomorrow."*

**Student thinking through a problem**: Moments after Ethan's question Daniel is explaining his idea about how they might adopt the methods they are using for solids of revolution to find the volume of a pyramid, Mr. Leavenworth captures the moment and later reflects, *"I thought it was interesting to kind of see him working through what he thought we should do."*

**Student difficulty solving a problem:** The teacher walks over to help a who student was trying to use his graphing calculator in a way that resembled what they had been doing several weeks earlier but that would not work within the context of the problem that they were working on and Mr. Leavenworth captures this interaction *"because I thought it was interesting that he tried to do that."*

**Insightful mathematical question**: On another occasion a student asks if they would always use pi to calculate the volume of other 3-dimensional solids and in reflecting on this captured moment Mr. Leavenworth remarks, *"I thought it was an interesting insight and it was a good question about how [pi] can keep showing up [when calculating the volume of solids of revolution] but is it always that way."*

*Noticing for the Purpose of Monitoring Instructional Progress*

We learn more about Mr. Leavenworth's almost exclusive focus on students' mathematical ideas by examining his discussion about criteria he used in the moment to decide if an event was interesting. During the post-interviews Mr. Leavenworth cited his goals for the lesson as one of the primary criteria for identifying interesting moments. For example, on day three he says:

**Researcher:** How would you characterize the criteria that you were using to decide which moments to capture?

**Teacher:** … My goals for the lesson are to understand the two parts [of the Fundamental Theorem of Calculus]. . . students need to understand that the antiderivative and the derivative are the inverse of each other and just the idea of limits, how you can manipulate definite integrals, the limits of definite integrals. So when I heard those things [in class] then I would press the button.

Mr. Leavenworth's claim that his goals for the lesson influenced what he found worth capturing on this day is supported by an examination of the moments that he collected during this lesson. Nine of the fourteen clips captured on that day involve student comments or questions that relate to the mathematical goals that he explicitly identified for that day.

A later interview with Mr. Leavenworth indicates one role these goal-fulfilling moments of student mathematical thinking may serve for him instructionally. He describes how, "It was almost like I had a checklist of things that I wanted to come out." It seems that Mr. Leavenworth notices moments of student mathematical thinking that indicate to him how the lesson is proceeding – whether it is "working" and going well. That he captures moments of student misconceptions suggests that he also notices when the lesson is progressing differently than he had expected. In other words, one aspect of his noticing seems to involve detecting alignment or misalignment with the lesson image (Schoenfeld, 1998) that he had developed in planning for the day. Attention to student mathematical thinking allows Mr. Leavenworth to monitor the progress of his instruction as it relates to student understanding of the content.

It is possible that Mr. Leavenworth did not actually use the lesson's mathematical goals as a criterion in the moment of collection but instead created an ad hoc theory that accounts for the clips after observing them all together. However, the fact that he mentions mathematical lesson goals as a primary criterion on several different days suggests that it might be a more stable component of his noticing rather than something that is created by engaging with this camera technology.

*Noticing and Improvised Teaching Episodes*

Mr. Leavenworth comes to class with some idea of what he needs to accomplish and his noticing is at least partly directed towards monitoring whether those mathematical understandings emerge at the appropriate time. However, rather than merely telling students an important mathematical insight Mr. Leavenworth will often, and perhaps even prefers, to use student voices to articulate conceptual understanding or generate approaches to a novel kind of problem. For example, when reflecting on one of his captured moments Mr. Leavenworth describes his plan to use student ideas generated in small groups in order to address an interesting but problematic idea that a student posed earlier during public discussion. "Between Eric, Daniel, and there was a few others, [I think we had]… an alternative to what we had started with." In the class discussion following this episode, Mr. Leavenworth calls on one of the students to introduce his alternative to a student's earlier solution. In this case, the students produced the ideas that Mr. Leavenworth needed in order for the lesson to proceed. However, what can a teacher do if he gets to a point in instruction when he notices the absence of some mathematical idea or solution strategy that is supposed to "come out" but has not?

Below we describe two episodes from the class that demonstrate how Mr. Leavenworth will react during his teaching if he notices the absence of a particular mathematical idea that he thinks is necessary at some point during the lesson. From his discussion of the noticed moments we identify two possibilities for how he generally proceeds when faced with this dilemma: either he provides the key conceptual insight on his own or he extends a lesson segment with an improvised teaching episode to help the students voice the required insight.

The first episode takes place after Mr. Leavenworth uses relatively straightforward examples to introduce part of the Fundamental Theorem of Calculus that exploits the inverse nature of derivatives and antiderivatives (integrals). He then presents the students with an extension problem to work on individually or in small groups, which involves limits of integration that are functions instead of constants. He writes the following question on the overhead:

$$\text{Find } \frac{dy}{dx} \text{ for the function } y = \int_1^{x^2} \cos t \, dt$$

Mr. Leavenworth captures a moment in which one student asks "Do you have to take into account the $x^2$ or can you just ignore it?" Mr. Leavenworth considers this a very good question and asks the other students for their opinions as to the answer. When no students are able to provide a reason for ignoring the function or a suggestion as to how to proceed, Mr. Leavenworth says "It is a really great question, here's what we are going to do…" and instructs them to use the Chain Rule that they learned previously to solve the problem.

In the first episode Mr. Leavenworth poses the new problem to the students and gives them some opportunity to grapple with how to deal with the apparent mathematical difference between it and the previous examples. He indicates in his reflection on the moment that he is pleased with the student's thinking that directs the class's attention to the relevant feature of the problem. However, when the students do not offer up any productive suggestions of how to deal with this difference, he is forced to supply the needed insight himself.

Later in the same class Mr. Leavenworth gives the students another problem in which both the lower and upper limits of the antiderivative are functions:

$$\text{Find } \frac{dy}{dx} \text{ for the function } y = \int_{2x}^{x^2} \frac{1}{2+e^t}\,dt$$

During their private work Mr. Leavenworth captures one student's idea for dealing with this new type of problem and later during the large group discussion he captures a series of moments in which different students offer suggestions. However, we can infer that he did not find any of these ideas instructionally tractable as he does not build upon or develop them further. Instead, after several minutes of trying to elicit student suggestions, Mr. Leavenworth walks to the front board and sketches up a representation in which each of the functions in the limit is represented by runners in a race with the 2x slower runner behind the $x^2$ faster runner. He highlights how "the distance [between the functions] is changing all of the time, so as x gets bigger, x squared is getting way out in front... Is there any reference point that we could use to figure out where these guys or girls are in relation to each other?" When a student suggests they use the starting point Mr. Leavenworth relates the idea of the start as a reference to having a lower limit of integration of 0. He asks if they can make the antiderivative they are working on have a lower bound of 0, and another student suggests that they rewrite their integral as two separate integrals by manipulating the upper and lower limits.

In the second episode Mr. Leavenworth did not simply provide the needed insight, but improvised and actively worked towards having students develop an approach that would work for this problem. His improvisation involved using the metaphor of a race to demonstrate how the functions are increasing at different rates but the starting point can be used to measure the distance to each of the runners. This appears to have activated students' prior knowledge of working with definite integrals sufficiently to have a student suggest the appropriate technique.

In both of these episodes Mr. Leavenworth notices students' mathematical thinking and from that thinking assesses the lesson's progress. These episodes provide evidence that Mr. Leavenworth has multiple practices in his teaching repertoire that he can draw on when his lesson image does not proceed as he has expected. It is not clear whether Mr. Leavenworth is aware of these alternative approaches to handling the teaching situation in which students do not provide the ideas that he seeks. However, by examining the moments he captures with the camera, his post-class reflections, and his instructional moves we are able to distinguish these two improvisational instructional strategies.

## Discussion

Through the use of a wearable camera implementing a selective archiving architecture we have examined one high school mathematics teacher's instruction and noticing in-the-moment during four days of his teaching. By asking the teacher to try to capture moments that occurred during class that he found interesting, we were hoping to gain more immediate access to the teacher's thinking and noticing during instruction than other methodologies used to date have been able to provide. Though we cannot say with certainty that the reflections about each clip provided during the post-interview actually represent exactly what he was noticing and thinking during the moment of instruction, we suspect that trying to characterize the moments that he captures and his reasons for capturing them and triangulating that with other elements of his instruction is a productive route towards better understanding Mr. Leavenworth's noticing.

Based on the data on his own teaching that Mr. Leavenworth collected, he seems to have a strong focus on attending to the mathematical substance of the students' ideas. He came into each lesson with some sort of agenda that can loosely be represented by a series of goals and actions that he wanted to have happen in developing the mathematical content for the day. We also observed a general instructional tendency to have students voice important conceptual ideas that might stem from a desire to actively involve students in the class or beliefs about how students learn. This observation was supported in the way that Mr. Leavenworth characterized his noticing during the post-interviews. The desire to have students involved in voicing important conceptual ideas creates a teaching dilemma generated from the unpredictability of student thinking. We highlight two episodes that illustrate how Mr. Leavenworth might react to a situation in which an idea does not arise at the expected moment by providing the idea himself or by doing some more work to try to get students to see the idea for themselves. We suspect that understanding how and when each of these two kinds of events occur represent a rich sight for research into teacher thinking and noticing and in the future plan to continue to use a similar methodology to further investigate these phenomena.

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# TEACH FOR AMERICA TEACHERS' (TFA) MATHEMATICAL KNOWLEDGE AND BELIEFS

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*This study intended to measure mathematical content knowledge both before and after the first year of teaching in the classroom and taking graduate teacher education courses in the Teach for America (TFA) program, as well as what attitudes toward mathematics TFA teachers held over the first year. To determine TFA teacher mathematical content knowledge and attitudes toward mathematics and teaching, participants were given a mathematical content test and two attitudinal questionnaires at the beginning and at the end of their first year teaching and taking graduate education courses.*

There has been a recent interest in studying the effects of TFA teachers in America's classrooms (Darling-Hammond, Holtzman, Gatlin, & Heilig, 2005; Xu, Hannaway, & Taylor, 2008). Darling-Hammond et al. studied the effects of TFA teachers in elementary school classrooms. Xu et al. claim to have produced the first study examining the effects of TFA teachers at the secondary level. However, there have not been any known studies that specifically focus on the mathematics content knowledge and attitudes toward mathematics and teaching for TFA teachers. This study addresses a much needed focus on secondary TFA teachers' mathematical content and attitudes, two areas much neglected in the literature.

## Research Questions

1. What differences exist between Teach for America (TFA) teachers' mathematical content knowledge and attitudes toward mathematics and teaching in the beginning and at the end of their first year teaching and taking teacher education courses in a graduate program?
2. Is there a relationship between TFA teachers' mathematical content knowledge and their attitudes toward mathematics and teaching before and after their first year teaching and taking teacher education courses in a graduate program?
3. How positive are TFA teachers' attitudes toward mathematics and teaching at the end of their first year teaching and taking teacher education courses in a graduate program?

There was a significant increase in both mathematical content knowledge and attitudes toward mathematics over the TFA teachers' first year teaching. Additionally, several significant correlations were found between attitudes toward mathematics and content proficiency. Finally, it was found that TFA teachers, after a year of teaching, had significantly better attitudes toward mathematics and teaching than neutral.

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# A STUDY OF WEB-BASED INSTRUCTION ON PRE-SERVICE TEACHERS' KNOWLEDGE OF FRACTION OPERATIONS

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*This study describes an intervention to compare the effectiveness of web-based and traditional instruction on pre-service teachers' knowledge of fractions. A sample of 48 pre-service teachers was assigned to an experimental and a control group. The experimental group received web-based instruction on fractions whereas the control group received traditional instruction on the same topic. Pre-post-tests comparisons showed that the experimental group achieved significantly better results in the post-test than the control group.*

## Introduction

Research has shown that the effective use of multimedia or interactive web-based modules can increase student learning (Aberson, Berger, Healy, Kyle, & Romero, 2003; Bliwise, 2005). This study determines whether web-based instruction (WBI) represents an improved method for helping pre-service teachers learn fraction operations in procedural and conceptual knowledge (Hiebert and Lefevre, 1986). The purpose was to compare the effectiveness of web-based instruction (WBI) with the traditional lecture in mathematics Content and Methods for the Elementary School course.

## Method

A sample of 48 pre-service teachers was assigned to an experimental and a control group. The experimental group received web-based instruction on fractions whereas the control group received traditional instruction on the same topic.

## Results and Conclusion

The results of this study suggest that the use of web-based instruction (WBI) in learning fraction operations is more effective. In this study, we found that the computer program for the web-based instruction helps to capture the pre-service teachers' attention because the programs are interactive and learner-centered. Second, we found that the interactive websites used for web-based instruction provide a dynamic and animated tool for improving students' visual and conceptual abilities in learning mathematics concepts.

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# ANALYSIS OF GEOMETRY LEARNING STORIES OF ELEMENTRAY SCHOOL PRE-SERVICE TEACHERS

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Teacher knowledge is one of the most important components of teacher quality because the content knowledge of a teacher strongly impacts the enactment of pedagogical tools of the teacher. Brown and Borko (1992) asserted that pre-service teachers' limited mathematical content knowledge is an obstacle for their training on pedagogical knowledge. Ball and a group of researchers developed *mathematical knowledge for teaching* (MKT) which addresses how a teacher uses mathematics for teaching while emphasizing the importance of mathematics knowledge in the teaching settings (Ball, 2000). Research studies show that using student work to facilitate teacher learning results in teachers' deeper subject matter knowledge (Kazemi & Franke, 2004). Furthermore, content knowledge of teachers is important for every subject including geometry, often a neglected topic in the curriculum. The limited number of research projects focusing on knowledge of geometry for teaching showed that beginning teachers are not equipped with necessary content and pedagogical knowledge for teaching geometry (Jones, 2000, Swafford, Jones & Thornton, 1997). Therefore, the purposes of this study are (i) to investigate the effective geometry learning experiences for pre-service elementary school teachers during elementary methods course and (ii) to design set of geometry activities to use in methods course

This study took place at an elementary methods course in a large southeastern public university in the US. The individual interviews of three participants, classroom observations and classroom artifacts were analyzed by narrative analysis (Labov, 1972) and thematic analysis (Coffey & Atkinson, 1996). As a result of narrative analysis two main kinds of stories emerged: *as a learner* and *as a beginning teacher*. The thematic analysis results yield to three themes: (a) *history of learning geometry*, (b) *what is geometry?* (c) *experiences in methods course*. These results were combined with literature on using student work, and they emerged into a two-phase protocol to address pre-service teachers' needs and perceptions during the methods course in order to enhance their geometry learning. The first phase of the protocol consists of geometry activities for pre-service teachers and the second phase consists of studying elementary school students' geometry work.

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# PROJECT-BASED CURRICULUM: TWO TEACHERS' USE OF STANDARDS-BASED MATHEMATICS CURRICULUM MATERIALS FROM AN ENACTMENT PERSPECTIVE

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*This poster highlights two teachers' curriculum development process as they created and enacted project-based curriculum for math instruction. Different from typical studies focused on the use of Standards-based mathematics curriculum materials, this study presents findings from a curriculum enactment perspective. Pictures, project files, and sample student work will be presented.*

Pinar et al. (1995) stated that "teaching is commonly characterized as the means by which curriculum is implemented" (p. 745). The term "implementation," however, has come to mean many different things. Snyder, Bolin, and Zumwalt (1992) categorized three differing perspectives on curriculum implementation: (a) the fidelity perspective, (b) the mutual adaptation perspective, and (c) the enactment perspective. Researchers who perceive curriculum with a *fidelity* lens are interested in studying the degree to which a planned curriculum is implemented by teachers in ways intended by curriculum writers. The role of the teacher is one of a consumer who should "implement the curriculum as those possessing curriculum knowledge have designed it" (p. 429). Further along the continuum is *mutual adaptation*. Researchers falling within this perspective view curriculum knowledge as either residing in the outside experts who developed the curriculum or as a combination of external curriculum knowledge coupled with practitioners' curriculum knowledge. Most research focused on teachers' uses of *Standards*-based mathematics curriculum materials tend to fall within this perspective (see e.g., Cohen, 1990; Remillard & Bryans, 2004). Finally, researchers viewing curriculum implementation from an *enactment perspective* view the actual or enacted curriculum as their focus. They are interested in how the curriculum is shaped *and* how it is experienced by teachers and students. Curriculum knowledge is viewed as an ongoing process and is not necessarily dependent on an externally created piece of curriculum as the center of the study. Researchers from this viewpoint view the role of the teacher and students as critical as there would be no curriculum without them.

This poster highlights the experiences of one mathematics teacher, one science teacher, and 88 middle school students from a curriculum enactment perspective. Although *Standards*-based mathematics curriculum materials were utilized by teachers throughout the study, the focus remained on the daily process of curriculum enactment. Drawing on Jurow's (2005) description of figured worlds and project-based curriculum, this poster highlights an alternate research perspective on the use of *Standards*-based mathematics curriculum materials for mathematics teaching and learning. Planning notes and reflections from teachers, as well as final projects and reflections from students are included. Finally, this poster offers overall suggestions and reflections on project-based learning in middle school mathematics and science.

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# LINKING COLLEGE PRE-CALCULUS STUDENTS' USES OF GRAPHING CALCULATORS TO THEIR UNDERSTANDING OF MATHEMATICAL SYMBOLS

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*This study examined ways in which students make use of a graphing calculator and how use relates to comfort and understanding with mathematical symbols. Analysis involved examining students' words and actions in problem solving to identify evidence of algebraic insight. Findings suggest that lack of symbol sense can lead students to turn to a graphing calculator as a tool for prompting a way to start a problem, or for providing a guess or confirmation. Certain symbols also lead some students to believe that they cannot use a calculator at all. Implications for teaching with a graphing calculator are included.*

## Introduction

Students often have access to graphing calculators and use them to help solve many types of problems. However, teachers and researchers are often unaware of how and why students use graphing calculators and how their use relates to their mathematical thinking, particularly about mathematical symbols. In this study, I address the research question: How are students' uses and understandings of graphing calculators related to students' uses and understandings of symbols?

Symbols are components of mathematical language that allow a person to communicate, manipulate, and reflect upon abstract mathematical concepts. However, symbolic language is often a cause of great confusion for students (Rubenstein & Thompson, 2001). Expert mathematicians or teachers are able to manipulate and to interpret mathematics through its symbolic representations, whereas students may struggle in this endeavor; they often need to be told what to see and how to reason with mathematical symbols (Bakker, Doorman, & Drijvers, 2003; Kinzel, 1999; Stacey & MacGregor, 1999). Arcavi (1994) explains that working fluently with symbols in mathematics requires developing strong *symbol sense* which includes, for example, understanding when to call on or abandon symbols in problem solving, understanding the need to compare symbols meaning with one's own expectations and intuitions, and knowing how to choose possible symbolic representations. Arcavi sees development of symbol sense as a necessary component of general sense making in mathematics. It is a tool that allows students to read into the meaning of a problem and to check the reasonableness of symbolic expressions.

Difficulties with symbol manipulation in mathematics may be one reason that students turn to graphing calculators for assistance in problem solving. Unlike calculators with computer algebra system (CAS) capabilities, most graphing calculators (e.g., TI-83, TI-84, TI-85, Casio FX-9750) cannot algebraically manipulate symbolic equations to produce useful results. Some work with symbols can be done with a non-CAS calculator; for example, symbolic expressions can be entered and viewed in the *Y*= menu, and values can be stored as a variable and substituted into an expression. However, a large benefit of these tools is that users can explore other representational forms of symbolic expressions, such as graphs, tables, or matrices.

## Framework

Pierce and Stacey (2001) used Arcavi's (1994) notion of symbol sense to develop a conceptual framework for looking at what is needed to take a mathematical problem, work with it using the tools and language of the calculator, and interpret and use the results using regular mathematical notation and forms. Pierce and Stacey define *algebraic insight* as a subset of symbol sense that enables a learner to interact effectively with a computer algebra system (CAS) when solving problems. They suggest that the nature of algebraic insight is the same whether work is done by-hand or with a CAS. Thus, I contend that the framework for assessing algebraic insight can also be appropriate for examining students' problem solving with graphing calculators that do not have symbolic manipulation capabilities. The two components that make up algebraic insight, *algebraic expectation* and the *ability to link representations*, elaborate instances and examples of algebraic insight that may be identifiable when analyzing students' work with graphing calculators. Table 1 shows the elements of the algebraic insight framework.

Table 1
*Algebraic Insight Framework (Pierce & Stacey, 2001)*

| **Aspects** | **Elements** | **Common Instances** |
|---|---|---|
| Algebraic Expectation | 1.1 Recognition of conventions and basic properties | 1.1.1 Know meaning of symbols<br>1.1.2 Know order of operations<br>1.1.3 Know properties of operations |
| | 1.2 Identification of structure | 1.2.1 Identify objects<br>1.2.2 Identify strategic groups of components<br>1.2.3 Recognize simple factors |
| | 1.3 Identification of key features | 1.3.1 Identify form<br>1.3.2 Identify dominant term<br>1.3.3 Link form to solution type |
| Ability to Link re-presentations | 2.1 Linking symbolic and graphic reps | 2.1.1 Link form to shape<br>2.1.2 Link key features to likely position<br>2.1.3 Link key features to intercepts and asymptotes |
| | 2.2 Linking symbolic and numeric reps | 2.2.1 Link number patterns or type to form<br>2.2.2 Link key features to suitable increment for table<br>2.2.3 Link key features to critical intervals of table |

This framework specifically addresses ways to plan, assess, and reflect on students' understanding when working with technology to solve mathematical problems (Pierce & Stacey, 2001). Using this framework assists in the task of identifying ways in which students' uses and understandings of mathematical symbols relate to how and why they use a graphing calculator.

## Methods

The method of inquiry for this research is a multi-case study, where a case represents an individual college pre-calculus student. Students were selected for this study using a survey that assessed familiarity with and use of graphing calculators. All invited participants indicated having at least average familiarity with graphing calculators and using graphing calculators at least one-half of the time on homework, but reported varying levels of success in previous mathematics course. Six students agreed to participate in the study and have the pseudonyms: Jill, Nina, Molly, Beth, Elyse, and Shawn.

The data sources in this study include a collection of work on tasks, video recordings of interviews, and computer recordings of calculator keystrokes for work completed on a graphing calculator. For the latter, I connected TI-84+ graphing calculators to a computer via a TI-Presenter device, and Windows Movie-Maker software captured and recorded videos of students' calculator keystrokes (c.f. McCulloch, 2007).

The findings reported in this paper are one piece of a larger doctoral thesis study. Students participated in three different interview settings during the course of the larger study. The results in this paper come from individual task-based interviews that took place near the beginning of the semester. In these sessions, students worked on secondary school-level algebra problems (i.e., problems to which students should have had prior exposure, but which had not recently been covered in class). As they worked on four different tasks (given in Table 2), students talked aloud about their thoughts and actions. They shared reasons for making use of a graphing calculator and discussed the specific activities that they were employing. If a student did not use the graphing calculator at all, I asked them to discuss why it was not useful on the problem and to consider ways in which they could have used it.

Table 2
*Interview Tasks*

| | |
|---|---|
| Task 1 – Solve a rational equation: | $\frac{x-16}{x^2-3x-12}=0$ |
| Task 2 – Solve a polynomial equation: | $x^3+2x-4=8$ |
| Task 3 – Setup and solve a linear word problem: A theater manager sold 5200 tickets and the receipts totaled $32,200. The adult admission is $8.50, and the children's admission is $6.00. How many adult patrons were there? | |
| Task 4 – Solve a linear inequality*:* | $\lvert 3x-\frac{2}{7}\rvert+1.2>5$ |

Transcriptions of videos of work on graphing calculators made it easier to follow students' work with this tool. I began by assigning basic codes indicating the manner of calculator use to the lines of the transcripts to help identify students' uses of the tool (e.g., graphed, used a table, computed, etc). I then looked in-depth at students' words and actions when using a graphing calculator and identified elements of symbol sense that were both evident and lacking in students' work by using the algebraic insight framework.

## Findings

Findings are organized around the four interview tasks. Each of the following subsections explains specific ways in which students engaged with the graphing calculator in activities. Instances of algebraic insight indicate details about students' symbol sense in their work.

*Task Type: Solve a Rational Equation*

Given the rational equation, $\frac{x-16}{x^2-3x-12}=0$:

- Jill and Molly graphed the numerator and denominator separately;
- Molly looked at a table to find the *y*-value when *x*=0;

- Elyse used the graphing calculator for computation only;
- Jill tried to see if linear or quadratic regression could work;
- Molly typed the left side of the equation into the main calculator screen to see if the calculator would "breakdown" the problem or solve for *x*;
- Beth, Nina, and Shawn did not use the graphing calculator at all.

These activities suggest that some students had the symbol sense to know that it was possible to abandon symbolic manipulation. However, the specific ways in which they used the graphing calculator indicate a lack of understanding of what the calculator could do. For example, Molly started by typing a function into the main screen, hoping that the calculator could "give a break down" of the problem. She explained, "The calculator is not simplifying for *x* here. It's not solving for *x*. There's a way to solve for *x*, isn't there?" It seems that she remembered using a graphing calculator in the past to solve for *x*, but could not remember what to do.

Molly and Jill chose to graph the numerator and denominator as separate graphs. This suggests a lack of algebraic insight for linking the rational form to the shape of the graph, which might not be surprising if they did not have experience with graphing rational functions. The linear form of the numerator and quadratic form of the denominator may have been more familiar and may be forms that they could easily link to shape and know what to expect in the graph. However, after graphing the two functions, neither student knew how to use the graphs to solve the given question. Molly looked at a table and found values when *x* equaled zero, saying, "I'm hoping that the calculator will be able to tell which one of the equations I'm supposed to use." Jill abandoned the graphs and attempted to see if linear regression would be useful instead.

Other students gave reasons for not using the calculator here. Beth explained that it could not tell her what steps to follow, saying, "It doesn't tell the mechanics of the problem that you have to do. It just gives you a number." Elyse expressed similar frustration, saying, "I don't know of a way where you can put an *x* in. The only thing I can think of is if you substituted something for the *x* maybe." Both Elyse and Beth were comfortable using the calculator for numeric calculations, but struggled to understand how to use the tool with algebraic symbols.

*Task Type: Solve a Polynomial Equation*

Given the polynomial equation, $x^3 + 2x - 4 = 8$, students used a graphing calculator in the following ways:

- Beth graphed only the left side of the equation
- Molly graphed the function on the left side and evaluated at $x$=8;
- Nina and Beth set the equation equal to zero and graphed to evaluate at $x$=0;
- Beth, Molly, and Shawn used the calculator for computations;
- Nina and Shawn used a table to determine the *x*-value when *y* equaled zero;
- Shawn set the equation equal to zero and looked at the graph to see how many *x*-intercepts existed;
- Jill and Elyse did not use the graphing calculator at all.

Nina and Shawn exhibited evidence of algebraic insight for linking key features to intercepts and to critical intervals for a table. However, both students only chose to use the graphing calculator after prompting from the researcher. Nina admitted that she would never have chosen to use a table to solve the problem on her own, and Shawn had already found an answer on paper and only used the calculator because he was not completely satisfied with his answer.

Shawn and Nina did not entirely trust the answers they found using the calculator. Shawn noticed that the graph only crossed the *x*-axis one time, and found one answer of $x$=2 from the table, but he had found three different answers on paper. Nina also anticipated finding three answers to the problem and expected that there were more answers than the calculator was telling her. This anticipation for a certain number of answers connects to students' algebraic insight for linking the problem form to the solution type and linking symbol meaning to prior experiences. Shawn also exhibited symbol sense for linking key features to critical intervals of a table when he noticed that two of his answers on paper had been decimal values and the table he was using only counted by integers. Neither student, however, made a clear link between the symbolic and graphical representations, which provided strong evidence that there was only one zero for the function $x^3 + 2x - 12$.

Other students struggled with the meaning of symbols and with identifying the dominant term needed for finding the solution (e.g., looking at $x$=0 instead of $y$=0). Molly misinterpreted the meaning of the equation and felt that she was supposed to substitute the value 8 in for $x$. Her trust in this interpretation allowed her to ignore the fact that the calculator produced a result of $y$=524, even though she had anticipated finding a value for $x$. Nina performed a similar action by evaluating the function at $x$=0 to find the zero. However, the calculator's result of $y$=-12 caused her to realize her mistake and change her activity to find $x$ instead of $y$.

*Task Type: Setup and Solve a Linear Word Problem*

Students used a graphing calculator sparingly on Task 3 in the following ways:

- Jill tried to graph an equation with two variables, but stopped when she could not determine how to enter both variables;
- Molly graphed two equations with the same variable;
- Beth, Nina, and Elyse used computations to make sense of the information;
- Molly, Jill, and Shawn computed values to find an answer;
- Nina used computation to see if her equation made sense.

Molly and Jill used the calculator as a numeric tool that could help them abandon symbolic manipulation for a guess and check strategy. Jill had created two useful symbolic equations with two variables and was using the calculator to guess and check instead of solving the equations simultaneously, while Molly was unable to create an equation. With symbolic forms for reference, Jill was able to continually link her results to the meaning of the symbols in the problem, while Molly lost sight of key information and was not able to reach a solution.

Beth, Nina, and Elyse struggled with the symbol sense to select or create possible symbolic representations, and used the calculator in the hopes of discovering a useful relationship in the given information. For example, Elyse tested to see if all of the tickets could be adult tickets by dividing 32200 by 8.50, but was disappointed to get a decimal answer. She explained, "Maybe if it divided evenly, I may believe it was only adults." Nina and Beth both used a similar calculation, but also divided 32200 by 6.00 to see if this value divided evenly. Beth continued dividing all given values by each other in the hopes of finding a number that might work as an answer to the problem. These students tried to manipulate numbers on the calculator to answer the problem and avoid the need for creating symbolic equations.

*Task Type: Solve a Polynomial Inequality*

Due to time constraints in the study, only four of the participants worked on the following inequality problem: Solve for $x$ given $|3x - \frac{2}{7}| + 1.2 > 5$. Students used the graphing calculator as follows:

- Jill, Beth, Nina, and Elyse used the calculator for computations;
- Beth used the calculator to convert decimals and integers into fractions;

– Elyse used the calculator to check a hypothesis.

Beth answered this problem by plugging one number in and using her calculator to compute a value on the left and seeing that the result was greater than five. She made comments such as, "Seeing that that (on paper) and this (on screen) looks the same, then you think, well I must be doing something right," suggesting that she did not realize that copying the direct computations from the calculator was not the same as doing the problem by hand. In this case, she did not recognize her dependence on the calculator for helping her manipulate the values on paper. The other three students manipulated the problem as though it was a linear equation, sometimes paying attention to the sign inside the absolute value symbol or the direction of the inequality. None of the students demonstrated algebraic insight for linking the form of this problem to a proper solution type, and their actions were restricted to numeric manipulations suggested by the operation signs in the problem. They used the calculator for calculating with fractions and decimals only.

When asked if a graphing calculator could be useful for this problem in other ways than as a computational tool, students responded in a variety of ways. Beth said that she did not know of a way to use it. Nina said she could not use it because she did not know how to handle the inequality sign, while Jill said she did not recall how to input absolute value. Elyse answered, "It's algebra so you can't just plug the whole thing in and get your answer." Thus, three of these students identified particular symbols in the problem (absolute value, inequality sign, and the *x*-variable) as the reason for not using the graphing or table features of the calculator. At the same time, numeric symbolic structures such as fractions and decimals were identified by all four students as important reasons for needing the calculator for computations.

## Discussion

A detailed examination of students' use of graphing calculators and what they said while using them can provide insight into the relationship between students' understanding of symbols and understanding of graphing calculators. By looking closely at specific details surrounding students' graphing calculator use, I identified two themes that address the research question:

*1. Lack of Symbol Sense Caused Students To Use a Graphing Calculator For Help.*

Students had some dependence on the graphing calculator as a tool for abandoning symbolic manipulation and finding an answer or a procedure to follow. At times, students treated the tool as a partner (Goos, Galbraith, Renshaw, & Geiger, 2003) that could help in the following ways: (a) by providing directions or a prompt, (b) by providing an accessible answer, and (c) by providing confirmation. The following paragraphs illustrate these categories of use.

Students often had difficulty trying to decide how to start a problem. In these instances, they sometimes turned to the graphing calculator to prompt an activity. For example, Molly wanted the calculator to tell her how to break down the rational equation in the initial interview. She entered the function into the main screen in the hopes that it would tell her something about the manipulations needed for the problem. She also graphed the numerator and denominator of the rational equation, saying that she was hoping for the calculator to tell her which function to use. She turned to the calculator for directions on how to solve the problems.

In some situations, students tried to avoid working with the symbols on paper and worked on the graphing calculator to try to find an answer. For example, Beth, Nina, and Elyse sought answers from the calculator on the linear word problem when they divided different given numbers in the hopes some value would divide evenly into another. They struggled to create symbolic equations for the problem, and tried to avoid a need for symbols by seeking an easy, familiar looking numeric solution from the calculator.

Some students recognized that a calculator was useful for confirming an idea or checking the reasonableness of an answer. For example, while working on a polynomial equation, Molly, Nina, and Shawn all used graphs and tables to find an answer to the problem and compared the calculator's answers with work they had done on paper. This caused problems when calculator answers did not match their expectations, and caused some students to mistrust the calculator.

*2. Lack of Understanding of a Graphing Calculator's Abilities to Handle Symbolic Forms Kept Students From Using Them or Using Them Correctly.*

Many of the students had difficulty knowing when and how to interact with a graphing calculator when solving symbolic problems. Difficulties were due to both misunderstanding of the technology and misconceptions about the mathematics involved in the problem. One fact that was evident from students' work is that they often did not know many of the features that the technology offered. For example, Beth and Elyse insisted that they could not use the graphing calculator when there was a variable in the equation. Similarly, Jill did not think she could enter an absolute value sign on a calculator, and Nina did not think there was a way to work with inequalities on the calculator. None of the students seemed to be familiar with menu options such as MAXIMUM or ZERO or INTERCEPT when working with graphs. Most of the students chose to use TRACE to find points on a graph instead, which does not provide exact values for answering a question. The students often struggled to see a use for the graphing calculator in problem solving because they were not aware of the powerful options it provided.

## Implications and Conclusions

Students were uncomfortable with and not proficient with using graphing calculators, despite their claims for being so on the initial survey. However, these students still had a certain amount of dependence on the graphing calculator for helping them postpone or abandon symbolic manipulation when it was causing them trouble. The fact that graphing calculators provide students with a way to do mathematics without using algebraic manipulation techniques has been identified in the research as a reason that some teachers give for opposing calculator use, especially at the college level (Hennessy, Fung, & Scanlon, 2001). However, other researchers have found that students' use of a calculator in this way is critical for helping them explore a problem to consider expectations before attempting an analytical solution (Quesada & Maxwell, 1994). Data from this study suggests that graphing calculators could be especially useful with weaker students (such as those taking or retaking college pre-calculus) as a tool for helping students gain more experience with important mathematical symbols and concepts. The teachers in this study did not teach or assess with graphing calculators and, consequently, restricted classroom examples and test questions to easy functions (e.g. no fractional coefficients, quadratic functions that could be factored, etc.). This practice may increase students discomfort with less used symbols such as inequality signs, absolute values, fractions, square roots and high powers of $x$. Many mathematical problems cannot or should not be solved by hand, but the students in this study did not seem aware of this possibility (e.g. the polynomial inequality in Task 2 was best solved using a calculator, but all students expected there to be an accessible pen and paper solution method). Awareness and understanding of how a graphing calculator can serve a student's needs when encountering mathematics inside and outside of the mathematics classroom is an important part of teaching mathematics, especially at the college level.

When students have access to a graphing calculator, and do not know how to use it or do not understand or remember what it is capable of doing, they can use it in creative and inefficient ways. Gray and Tall (1994) suggest that students who did not have a strong understanding of the different uses of symbols may develop different, incorrect techniques for problem solving due to their personal interpretations of the symbols. The same idea may apply to students who do not have a strong understanding of how and when to use a graphing calculator. Teachers need to be aware of some of the non-standard uses that students can create to seek assistance from a graphing calculator as they try to avoid or abandon symbolic manipulation.

For the students in this study, understanding how to work with mathematical symbols on paper had a connection to their choices of how and when to use a graphing calculator. However, students demonstrated limited algebraic insight for linking representations and connecting what they were doing on the calculator to their work with symbols on paper. When teaching with a graphing calculator, teachers must be careful not to treat the tool as a different way of approaching a problem, but instead integrate it into a problem and help students reflect on how the work displayed on the screen relates to the symbols on paper.

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# TRIGONOMETRY, TECHNOLOGY, AND DIDACTIC OBJECTS

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*Students have difficulty constructing coherent understandings of trigonometry and trigonometric functions (Brown, 2005; Weber, 2005). This study conjectured that their weak understandings of angle measure and compartmentalized knowledge of right triangle and unit circle trigonometry are sources of the problem. The response was to devise an instructional sequence to promote these foundational understandings and connections. A critical part of this instruction was the use of dynamic applets. These applets were intended as didactic objects to facilitate meaningful conversations supporting student learning. This report discusses the design and implementation of these applets and their role in promoting discourse that facilitated knowledge construction.*

## Background

Trigonometric functions are a common topic of undergraduate mathematics. It is also the case that various topics of physics, engineering, and other sciences rely on trigonometric understandings (e.g., projectile velocities and wave behavior). However, it is frequently the case that students have difficulty when asked to reason about topics reliant on trigonometric function understandings (Brown, 2005; Thompson, 2008; Weber, 2005).

Two different settings are often used to introduce trigonometric functions in the US mathematics curriculum: right triangle trigonometry and unit circle trigonometry. Student difficulties relative to trigonometric functions may be the result of mathematics curriculum frequently treating these two trigonometries as unrelated (or only slightly related), resulting in students not constructing coherent understandings of trigonometric functions. One potential solution to this lack of coherence is to fully develop student conceptions of angle measure and use this foundation as a base to both trigonometries.

This study reports on both the design and implementation of a lesson intended to develop student conceptions of angle measure such that these conceptions enabled coherence between the two trigonometries. In order to enable coherence between the two trigonometries using angle measure, the focus of the lesson was to support students in conceptualizing angle measurement as the openness of an angle and to develop a method for quantifying this measurement. Furthermore, the classroom exploration was conducted such that it attempted to promote student images of angle measure that included variation of an angle's openness. This was especially important relative to future explorations of periodic motion (e.g., a situation where the argument of a trigonometric function varies).

Central to the lesson design and implementation was the use of technology to aid in the development of student understandings. The lesson included the use of *The Geometer's Sketchpad* (GS) applets (Key Curriculum Press, 2002). These teaching aids were intended to promote discussions that reflected the structure of the mathematical understandings that formed the instructional goals of the lesson. The results presented in this report discuss the design and implementation of the various GS applets and the student discussions these applets generated in the context of student learning.

## Theoretical Perspective

The theoretical foundation for this study rests on the idea that all learning begins and ends with the learner, a main stance of radical constructivism (Glasersfeld, 1995). Building from the stance of radical constructivism, a researcher or teacher must consider each individual's knowing his or her own. That is, each student's knowledge is self-constructed and considered fundamentally unknowable to any other individual.

A useful tool in promoting student construction of knowledge is a *didactic object* (Thompson, 2002). A didactic object is designed as an object to talk about in a way that enables and supports reflective mathematical discourse. As Thompson notes, an object is not considered didactic in and of itself. An object is only didactic when it is conceptualized in a manner that enables reflective mathematical discourse. Reflective mathematical discourse refers to discourse focused on mathematical topics, where the discourse becomes an explicit object of student reflection. Students participating in reflective discourse have the opportunity to construct deeper understandings and cognitive connections.

The use of dynamic applets developed using Geometer's Sketchpad was intended to serve as didactic objects on which the teacher and student could engage in meaningful conversations about foundational trigonometric ideas. The desired student conceptions drove the design and implementation of the applets. The role of the dynamic applets during instruction was to generate discussions and reflections not possible without their use.

The use of dynamic applets as didactic objects was intended to generate discussions that resulted in students engaging in covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) and quantitative reasoning (Smith III & Thompson, 2008). Covariational reasoning (e.g., the coordinating of two varying quantities while attending to the ways in which they change in relation to each other) has been shown to be critical for concepts of calculus (Carlson, et al., 2002). In response, the designed applets were intended to engage students in coordinating and discussing varying angle measures in order to prepare students for trigonometric functions, which formalize the relationship between the covariation of angle measure and a ratio of lengths.

In the description of covariational reasoning by Carlson et al. (2002), an individual is described to be reasoning about quantities and relationships between these quantities. These quantities to be reasoned about are thus implied to be conceptual objects derived from experience that have qualities that we can call mathematical, or can be "mathematized." Quantitative reasoning refers to the type of reasoning that is situation sensitive and places an emphasis on the development of conceptual objects (quantities) that individuals are to reason about. Quantitative reasoning is the mental actions of an individual making sense of a situation, constructing an image of measurable quantities composing the situation (*quantification*), and reasoning about relationships between these quantities (e.g., covariational reasoning).

The design of the applets intended to promote quantitative reasoning by aiding the students in visualizing situations and supporting discussions that focused on the quantities they were to reason about. As an example, the applet presented in Figure 1 offered a visualization of a subtended arc-length as a measurable attribute of an angle's openness. The applet was created with the goal of supporting the understanding of angle measure as a subtended arc-length's fraction of a circle's circumference. The applet was also designed to allow an investigation that included discussing a *varying* angle measure and a *varying* radius of the circle used to measure the angle. Thus, the applet was intended to enable and support discussions that centered on a dynamic, opposed to static or discrete, exploration of the relationships between values of arc-length and circumference (e.g., regardless of the unit of measurement, the ratios are equivalent).

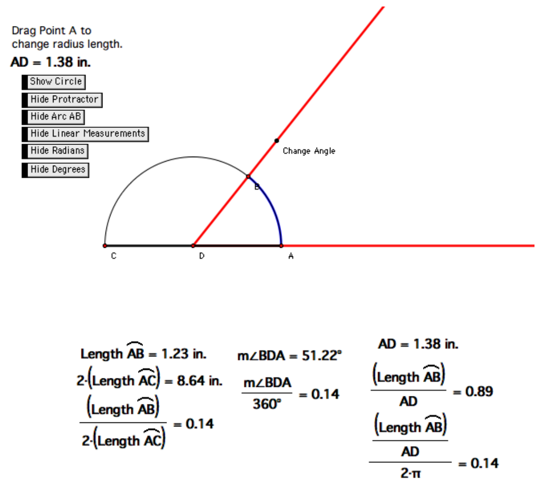

*Figure 17.* Angle measure applet.

## Research Questions

This report is set within a broader investigation that focused on investigating student conceptions of trigonometric functions and topics foundational to trigonometry, where the two main questions driving the research were:

- What understandings of trigonometric functions do students construct during an instructional sequence that is designed on foundations of quantitative and covariational reasoning?
- How do the foundational understandings of angle measure, the radian, and the unit circle influence student understandings of trigonometric functions in the context of the unit circle and right triangles?

An element of this focus was on the role of curriculum relative to student development. This report specifically focuses on the use of computer software in the context of student learning.

## Methodology

This study was conducted with three students from an undergraduate precalculus course at a large public university in the southwest United States in which the researcher (myself) was the instructor. The subjects were chosen on a volunteer basis and monetarily compensated. The precalculus classroom from which the subjects were drawn was part of a design research study where the initial classroom intervention was informed by theory on the processes of covariational reasoning and select literature about mathematical discourse and problem-solving (Carlson & Bloom, 2005; Carlson, et al., 2002; Clark, Moore, & Carlson, accepted). All three subjects were males (Brad, Charles, and Travis). The classroom instruction consisted of direct instruction, whole class discussion, and collaborative activity.

I conducted a three session teaching experiment (Steffe & Thompson, 2000) with the three subjects of the study. The teaching experiment focused on i) developing angle measure in terms of arc-length and circumference, ii) developing the use of a radian as a unit of measurement, iii) and investigating circular motion and the unit circle (e.g., any circle is the unit circle if the unit of measurement is the length of a radii). A witness attended each session and debriefed with the instructor after each instructional session. Each class session was videotaped and digitized. In addition, all student products (white board, activity sheets, homework assignments, etc.) from the teaching sessions were captured digitally for analysis.

The classroom sessions were analyzed following an open coding approach (Strauss & Corbin, 1998). Discrete instances believed to reveal insight into student conceptions were identified and then analyzed in an attempt to determine the mental actions that contributed to the emerging behaviors. The mathematical constructions and interactions that occurred between the subjects and the instructor were examined in an attempt to model and understand the thinking of the subjects. Specifically, this report focuses on the discussions generated through the use of the GS applets in the context of student learning.

## A Brief Conceptual Analysis Of Trigonometry

Before presenting the results of the study, a brief description of the understandings driving the instructional goals is provided. Angle measure is a measure of an angle's openness, which is measured by determining the fraction of circle's circumference that is subtended by the angle. For instance, an angle of measure one degree subtends $1/360^{th}$ of the circumference of any circle centered at the vertex of the angle. Note that the focus of angle measure is on the arc-length subtended by the angle as an attribute that can be measured.

Next, just as the degree is a unit of angle measurement that refers to $1/360^{th}$ of a circle's circumference, the radian is a unit of angle measurement that refers to $1/(2\pi)^{th}$ of a circle's circumference; just as 360 degrees revolve any circle, $2\pi$ lengths of a radius revolve any circle, allowing the length of a radius to be a unit of angle measure. The radian measure of an angle can also be defined as the ratio of an arc-length to the corresponding length of a radius, which gives the fraction of one radius.

With an input quantity of angle measure (arc-length), sine and cosine have an output that is a ratio of two lengths, regardless of the setting. With regards to the unit circle, the output of sine is the ordinate of the terminus of the arc subtended by the angle and cosine is the abscissa of the terminus of the arc subtended by the angle, with both outputs measured as a fraction of one radius. With regards to right triangle trigonometry, sine and cosine are approached in the same manner. Although sine and cosine are often used only to find the lengths of the sides of right triangles, this is simply using one pair of input-output values. Because angle measure corresponds to measuring an arc-length, one can use the hypotenuse of the right triangle as the length of a radius and construct a circle centered at the vertex of the measured angle. Hence, sine and cosine remain the abscissa and ordinate of the terminus of the arc subtended by the angle, with both outputs measured as a fraction of one radius (or hypotenuse).

## Results

What follows is a brief discussion of the results of implementing various GS applets during instruction. As an example of the use of the applet presented in Figure 1, the students were first asked to discuss each measurement of the applet in order to support a discussion focused on the quantification of the situation. When asked about the second value (e.g., the proportion of arc-length to the radius), Brad used the projection of the applet to explain, "How many lengths of AD it takes to get your total arc-length of AB." Charles also added, "How many radians there is in the arc-length." These responses by Charles and Brad imply that they had constructed images of the ratios as representing a proportion of a circle's circumference, images that appear to be aided by referencing the applet.

At this time the dynamic applet was used to focus the students on considering a varying radius and varying angle measure:

1 Instructor: So, what's going to happen…if I increase my radius?
2 Charles: The angle will remain the same, so the amount of radii it will take to
3 make the arc-length will still remain the same.
4 Instructor: Ok, so the three values (*referring to the far right column*), which are
5 going to change and which aren't? Travis, what do you think? Which
6 ones are going to change?
7 Travis: The radius is going to change. (*pause*)
8 Instructor: The radius is going to change, it will increase.
9 Travis: The rest will stay the same.
10 Instructor: The rest will stay the same, right? (*Increasing the radius on the*
11 *applet*) Do we get what we expect?
12 Students: Yes. (*Nodding in agreement*)
13 Instructor: Right, because our angle measure's not changing. Now what if I
14 change the angle?
15 Travis: The…
16 Charles: The, go ahead. (*Pointing to Travis*)
17 Travis: The bottom two will change (*referring to the far right column*).
18 Instructor: Lets say I decrease my angle. What's going to happen to the bottom
19 two?
20 Brad: They'll decrease.
21 Instructor: They'll decrease right. Now how 'bout the three proportions along
22 the bottom there?
23 Brad: They'll all stay equal. (*Instructor decreases angle measure*)

During this discussion, the students were first asked to describe how the values would vary before the applet was used to increase the radius. Both Charles and Travis were able to correctly describe the direction of change of each value (lines 2-3, 7, 9), implying that both of the students had developed images of the quantities of the protractor and angle applet that included their directional covariation relative to an increasing radius. With the conjectures made, the applet was then used to increase the radius in order to verify the students' conjectures (lines 10-11). The instructor then used the applet in a similar manner to investigate a decreasing angle measure. The students were first asked to use the image of the applet to conjecture the result of varying the angle measure (lines 13-14) and then the result of decreasing the angle measure (lines 18-19). This resulted in the students (correctly) referencing the values on the applet and how they would change in relation to a varying angle measure (lines 17, 20, 23). With these conjectures given, the instructor used the applet to verify the responses of the students (line 23). Thus, this applet appears to have supported a visualization of the situation such that the students could conceive of and discuss the values and quantities of interest. Also, the applet supported conjecture-verify sequences of the students imagining and then describing the dynamic situation in relation to the varying quantities. From these sequences, the instructor was able to infer that the students were constructing understandings consistent with the instructional goals.

Drawing on the foundations of angle measure that informed the design of the first applet (Figure 1), a second applet was designed to investigate and discuss circular motion (Figure 2). When "Animate Point" is chosen on the applet, a point moves around the circle counter-clockwise at a constant rate. This was used to promote discussions and connections between the numerical, contextual, and graphical representations of the *covarying* quantities of arc-length and vertical height above the center of the circle. Also, the applet enabled a discussion of these representations relative to the motion of the point. These dynamic connections were conjectured to be critical in developing an understanding of the sine function, as the sine of an angle measure cannot readily be algebraically computed. The use of this applet also occurred before formally introducing the sine function. Thus, the applet was used to generate and reflect on a graph between the covarying quantities before formalizing the relationship as the sine function.

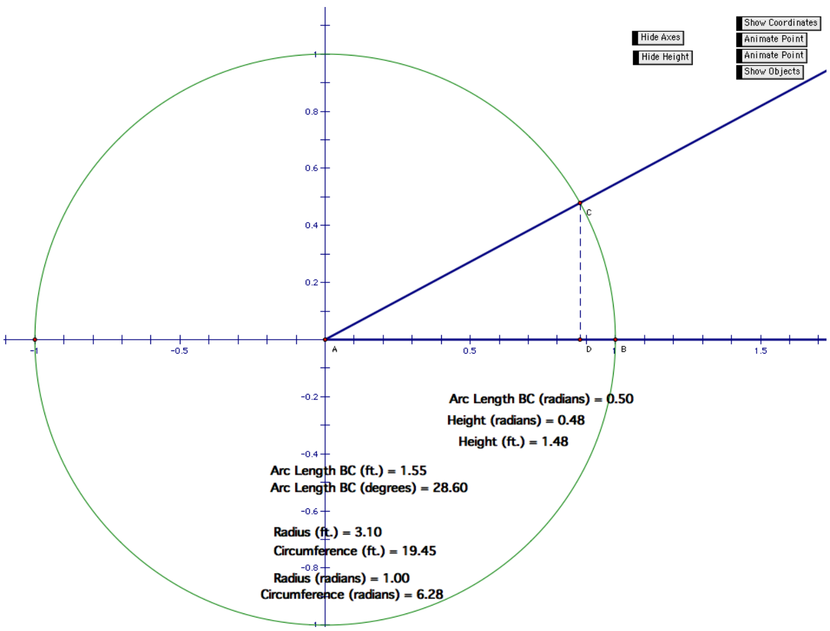

*Figure 2.* Motion on a circle applet.

One result of the use of the applet in Figure 2 was investigating corresponding amounts of change of vertical distance and arc-length. The instructor chose this use of the applet after the students had constructed a non-linear graph but remained unable to correctly describe why the graph should not be composed of linear segments:

Instructor: Ok, so we move along from 0.15 (*radians*) and our height becomes what (*moving an arc-length of* 0.15)?
Brad: 0.15 radians.
Instructor: 0.15. Now, should it be exactly 0.15? Does that make sense?
Brad: No, because you have a curve.
Instructor: So why does us having a curve matter? (*long silence*) So why does us having a curve matter Brad?
Brad: Well because you don't have a, uh, the curve for your arc-length is going to be different because of the curve. Because, uh, you'll have a longer distance over the curve than I think height, over a straight line (*making a reference to the image of the applet*).
Instructor: Ok, so we know it should be less that 0.15 then, right? Because if we move 0.15 along the arc-length, we didn't move completely that on the vertical. What do we expect if I move another 0.15?

Brad: Should be less.
Instructor: If I move a distance of 0.15 radians, what should we have? Well let's see (*moving and arc-length of 0.15 radians*). What was my change in height now?
Brad: 0.14.
Instructor: Was it less? Was that what we expect?
Students: (*nodding heads yes*)
Instructor: What if I move another 0.15? What do we expect?
Brad: It would be less again.
Instructor: It would be less again (*moving an arc-length of 0.15 radians*). What if I move another 0.15?
Brad: Less again.
Instructor: So what's less again? (*moving an arc-length of 0.15 radians*)
Brad: The height, the change in height.

During this interaction, the applet enabled a discussion that focused on the students coordinating arc-length and vertical distance, a coordination the students were having difficulty with previous to the applets use. The instructor first chose to use the applet to represent a change in arc-length of 0.15 radians and question the students on the change of vertical distance. This questioning resulted in Brad describing that the change in height would be less than the change in arc-length. Brad also used the image of the applet to explain that the curve was longer than the linear vertical height (lines 8-11). This explanation by Brad thus appears to have been supported by the applet aiding in his visualization of the situation. After Brad described the result of another change of arc-length, implying he was coordinating changes of arc-length and vertical distance, the instructor made the move to have the students *conjecture* the result of another sequence of increases in arc-length (lines 22 and 24-25). Brad then correctly identified that the change in height would be less than previous (lines 23, 26, and 28), which was verified by using the applet.

Although Brad was the only student to verbally respond in this interaction, both Charles and Travis explained (explanations that were verified using the applet) other sections of the motion in terms of amounts of change of arc-length and amounts of change of vertical distance and how this was reflected in the graph. Thus, the use of this applet supported an investigation and discussion of how a changing arc-length, opposed to static positions, related to changes of vertical distance and how this relationship was represented using the graph. The implementation of the applet was first in a manner that had the students observe and identify changes of vertical distance in relation to changes of arc-length. Then, after the instructor observed the students correctly coordinating the covarying values, the students were asked to conjecture the result of changes of arc-length and these conjectures were verified using the applet.

## Discussion

The two applets described above offered objects the students could conceptualize in a way that supported student discourse and the construction of knowledge. For instance, the second applet (Figure 2) led to a graph, including concavity, emerging out of the students covarying the value of two quantities (e.g., arc-length and vertical distance). Also, the applet allowed the creation of the graph to be related to the *dynamic* contextual situation. The sine function cannot be computationally evaluated with ease; hence it is necessary for students to construct a contextual image of the relationship the sine function defines. Furthermore, the contextual images students construct must include measurable quantities that they can reflect on and reason about. The applets were designed an implemented with this in mind, which included the intention of promoting students constructing and reasoning about *varying* quantities.

What appears to be an important implementation of the applets as didactic objects was encouraging that students predict how the values of quantities would vary before enacting the dynamic features of the applets. This had the possible influence of the students constructing an image of each of the quantities and then *imagining* motion and corresponding variations of each quantity. Then, when the applet was used in a dynamic manner, the students had an opportunity to reflect on their conjectures relative to the values and motion of the applets. This reflection could then result in the students modifying their image of covarying quantities, and hence their conception of trigonometric functions or angle measure.

In conclusion, the use of GS applets allowed a dynamic investigation of trigonometric topics not as readily available without the applets. The design of the applets was focused on creating objects that enabled and sustained discussions that supported the students constructing understandings consistent with the instructional goals. This design included taking advantage of the dynamic abilities of the applets in order to promote discussions that focused on the quantification and covariation of quantities.

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# EXPLORING CHILDREN'S MATHEMATICAL REASONING WHEN PLAYING ONLINE MATHEMATICS GAMES

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*Students have many opportunities for learning mathematics content outside of school. These informal interactions can be rich resources for students and teachers as they weave connections between the mathematics they study in school and the mathematics they experience outside of school. In this study we explored possible connections between formal and informal mathematics by investigating the mathematical reasoning used by students as they played online games. We report on students' interactions with the embedded mathematical content of the online games used in this study. Noting, in particular, the ways in which students engaged in increasingly more sophisticated mathematical reasoning as they progressed through the various levels in the games. These shifts in gameplay were detectable, not only through in-person observations, but also via data mining of online tracking data. Implications for the use and study of children's use of educational games as contexts for informal and formal learning of mathematics are discussed.*

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Mathematics educators have long recognized the importance of helping students make connections between the mathematics that is offered in school and their everyday experiences and interests (see NCTM, 2000). Realizing those connections, however, has been elusive, as researchers have found stark differences in how students use mathematics outside of school and how they perform school tasks (e.g., Nunes, Schliemann, & Carraher, 1993). Yet, contexts in which mathematics is studied play an important role in helping students understand how, when and why particular concepts, procedures, and skills are used, but also see what makes them worth knowing. Grounding mathematics in experiences that are meaningful to students continues to be an important though challenging goal for mathematics education.

Children's natural love for playing games and the growing availability of electronic media, make online games an obvious choice for exploring connections between informal and formal encounters with mathematics. It is no secret that children spend a great deal of time interacting with electronic media. Even before they start school, children interact with all sorts of electronic media. Research reports that children 6 months through 6 years of age spend an average of 2 hours daily with screen media, and that 50 percent of 4-6 year olds play regularly with computer games (Rideout et al., 2003). Although it is widely recognized that children spend a great deal of time interacting with electronic media outside of school, and increasingly with online computer games, the educational value of such experiences are often called into question. Educators in particular often raise concerns as to whether online games can successfully elicit sophisticated mathematical thinking from children (as opposed to for example simple drill and practice).

Although there are many concerns about the educational content of electronic media, the reality is that electronic games, as Gee (2000) suggests, accomplish many educational goals that are too often not met in school classrooms. Minimally, electronic games teach (and entice) players to learn how to play and get better at playing in spite of many persisting difficulties. Gee and others have posed theoretical arguments in favor of the educational potential of electronic games, but there is little empirical evidence to support those claims, and hence a motivation in our project to explore more specifically the question about how children reason when playing with online mathematical games. In addition, we were curious about children's informal interactions with mathematics in an online game environment (we used *Cyberchase* online games as the context for this investigation) and how and whether these interactions were similar to the sorts of strategies and understandings students use when working with regular school mathematics tasks.

## Theoretical Background

Research on how students use mathematics in and out of school often highlights the vast differences between these two contexts (i.e., Lave, 1998). In addition, researchers have also highlighted the tensions that arise when real life mathematics is brought to school (Civil, 2002; Sierpinska, 1995). We join this conversation by taking a different approach. Here we focus on the connections between students' engagement with informal mathematics and the mathematics they study in school. We chose an online game environment as the setting for this investigation because of its prevalence as an out of school activity for many children, but also because of its potential to make visible the sorts of connections we were interested in exploring.

Researchers who study human-computer interaction have sometimes drawn on established theories of human cognition to explain users' thinking while playing games (e.g., Mayer & Moreno, 2003; Moreno, 2006), and have noted similarities that exist between online and offline thinking and behavior (e.g., Gee, 2003). Indeed, research has shown that users' interactions with machines are influenced by the same sorts of social schemas that govern their interactions with other people -- regardless of whether the device in question is an animatronic, talking doll (Strommen, 2003) or a desktop computer (Reeves & Nass, 1996). By the same token, when children play online computer games, we might expect their reasoning to follow the same sorts of paths that they use while figuring out similar educational content in real (offline) life.

The field of computer-assisted instruction (CAI) represents a long history of teaching and assessing knowledge via interactive games (e.g., Price, 1989; Rudestam & Schoenholtz-Read, 2002; Suppes & Macken, 1978). However, unlike the kinds of software traditionally used in CAI, the online *Cyberchase* games that served as the context for this study were not originally designed for assessment. In addition, whereas assessment in CAI frequently focuses on measuring the state of users' knowledge or skills (to determine the types of exercises that the software will provide next; e.g., Corbett & Anderson, 1995; Gunzelmann & Gluck, 2004), we were more interested in observing the evolution of children's problem-solving strategies and mathematical thinking over the course of playing a game. We wondered whether gameplay in the *Cyberchase* games we focused on for this project would reflect children's understanding of educational content and strategies for problem solving that they would have encountered in their mathematics classroom; and if so, what sort of data might we collect to explore this learning?

## Methods

This study is part of a larger research project exploring children's interactions with multiple types of electronic media and their learning of mathematics. The context for this larger project and for the smaller study we report on here is the animated *Cyberchase* television series that airs daily on PBS. *Cyberchase* features three diverse youngsters who are summoned into cyberspace to foil the dastardly Hacker. Each half-hour episode sends the team on a mystery based on a mathematics concept. Through their adventures, the series models mathematical reasoning, problem-solving, and positive attitudes toward mathematics. Its underlying themes are that mathematics is everywhere and is infinitely useful. Nearly five million viewers – 40% of them African- American or Hispanic – tune in each week. A web site *Cyberchase Online* (*http://www.pbskids.org/cyberchase)* complements the series with interactive games and puzzles, based on the same mathematical content as the series. The site is one of the three most-visited sites on PBS Kids Online, with over 1.3 billion page views to date, and average use of more than 1 hour per visit (a marked contrast to the average of 19 minutes for the rest of the PBS Kids site).

Our research team observed 74 third and fourth graders (27 girls and 47 boys) in person as they played three *Cyberchase* online games, regarding decimals, quantity/volume, and proportional reasoning. For example, in the "Railroad Repair" game (*http://pbskids.org/cyberchase/games/decimals/)*, players fill gaps in a train track by using pieces labeled with decimals between .1 and 1.0 (Fig. 1).

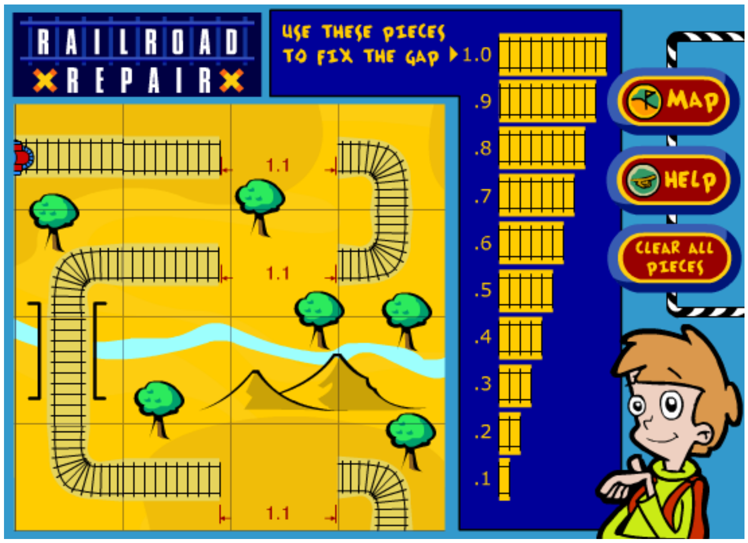


*Figure 1.* Sample screen from *Cyberchase* Railroad Repair game.

In this game multiple correct solutions are possible. However, each length of track can be used only once per screen, thus requiring children to find multiple ways to make sums and plan ahead to make sure that all of the necessary pieces will be available when needed. Children in our study played the games in pairs, to facilitate conversation that could reveal ideas and strategies as they played. Simultaneously, their gameplay was recorded with a custom built tracking software that recorded their mouse clicks and keyboard input automatically. This tracking software was arranged initially for two of the games: Railroad Repair and Sleuths on the Loose. This latter is a game about measurement and proportional reasoning (*http://pbskids.org/cyberchase/games/bodymath/)*. Afterward, children were interviewed explicitly about their strategies for playing each game and solving its mathematical problems.

## Results

Our observations of children's gameplay in Cyberchase online games environments are that children use a range of sophisticated mathematical strategies when solving complex tasks—regardless of whether those tasks are school tasks or out of school ones. Furthermore, the children in our study who used more sophisticated strategies, just as it is reported about children solving rich school mathematics tasks (e.g., Lesh et al., 2000), often did not apply them immediately. Rather, they engaged in cycles of problem solving that began with less sophisticated strategies and then became more sophisticated when necessary.

Consider the following excerpt of conversation from two children in our study while playing the Railroad Repair online game. As they tried to fill gaps in the game's railroad track, they had the following exchange:

> *"I think we're supposed to use the 1 and then the 10."*
> *"Uh-oh. Can we subtract?"*

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

*"This is too confusing."*
*[They clear the pieces from the screen, then start again with a different strategy]*
*"This time, we'll start with the mini-pieces…"*

What is noteworthy here is that when the children had tried something that did not work, and were pretty stuck (above statement: "this is too confusing,"), they changed their approach and tried again. They did not seek outside help, and on their own decided to change and try another strategy. Here we can see several of Gee's (2000) learning principles at work. Most notably what he calls the "psychosocial moratorium" principle, that is, that these learners are able to take risks in a space where there are no serious consequences. After all, they can try and try again without being judged about not solving this problem quickly or having made a mistake (and no real world consequence of having a train potentially derail should they fail to fix the problem).

Another of Gee's principle that is noteworthy here is what he calls the "ongoing learning principle," meaning that as the game progresses the learner must undo their routinized strategy to adapt to the new or changed conditions. This moves the learner through cycles of new learning, automatization, undoing automatization, and new reorganized automatization. In the sample conversation, a strategy that had worked well earlier in the game failed and the players had to adapt and tweak a solidly mastered strategy to the new condition. This is not a small fit, as we know too well from seeing students get stuck using their well learned strategy (say adding) even when that strategy fails or becomes too inefficient in new problem situations (say multiplication situations).

In Railroad Repair, many children began by using a *matching strategy* in which they matched the decimals shown (e.g., a .8 piece of track to fill a .8 gap). When this strategy later proved insufficient (e.g., they ran out of .8 pieces or needed to fill a larger gap), some switched to an *additive strategy* (e.g., combining .6 and .2 to fill a .8 gap). When this strategy, too, proved insufficient (e.g., the .2 piece was needed later), some adopted an *advanced* strategy in which they planned ahead, considered alternate ways to make sums, and reserved pieces they would need later.

Comparisons of observation, interview, and online tracking data revealed that these strategies could be detected via data mining too. Tracking data showed consistent patterns of online responses reflecting each strategy (matching, additive, or advanced), and clusters of errors when children's strategies broke down and they shifted to new ones. Consider, for example, some partial output of the tracking software for one player while playing Railroad Repair. On the first screen, the tracking software shows evidence of the player adopting a matching strategy, picking up a .4 piece (*piecepress*) and placing it (*piecedrop*) to fill a .4 gap. After accidentally putting the piece in the wrong location (row 2 in the example below), the player then places it correctly (row 4):

| **Row #** | **Event** | **Piece** | **Round** | **Successful placement?** | **Elapsed time** |
|---|---|---|---|---|---|
| 1 | piecepress | track4 | 1 | n/a | 7.161 |
| 2 | piecedrop | track4 | 1 | wrong | 7.272 |
| 3 | piecepress | track4 | 1 | n/a | 8.172 |
| 4 | piecedrop | track4 | 1 | success | 10.2 |

On the next screen, the player continues the matching strategy, using a .8 piece to fill a .8 gap (rows 5 – 6 in the example below). However, there is more than one .8 gap on this screen and only one .8 piece. Thus, after using the .8 piece, the player switches to an additive strategy, using two pieces (.7 and .1) to fill the second gap. After accidentally misplacing the .1 piece (rows 9 – 10), the player places it successfully (rows 11 – 12):

| **Row #** | **Event** | **Piece** | **Round** | **Successful placement?** | **Elapsed time** |
|---|---|---|---|---|---|
| 5 | piecepress | track8 | 2 | n/a | 22.09 |
| 6 | piecedrop | track8 | 2 | success | 24.101 |
| 7 | piecepress | track7 | 2 | n/a | 25.329 |
| 8 | piecedrop | track7 | 2 | success | 26.942 |
| 9 | piecepress | track1 | 2 | n/a | 28.503 |
| 10 | piecedrop | track1 | 2 | wrong | 28.711 |
| 11 | piecepress | track1 | 2 | n/a | 29.099 |
| 12 | piecedrop | track1 | 2 | success | 30.91 |

For the next several screens, the player continues to use the additive strategy, until arriving at a screen where this strategy is no longer sufficient. After filling several large gaps, the player combines a .5 piece and a .4 piece to fill a .9 gap (rows 13 – 15), only to find that all of the smaller pieces have been used up, which makes it impossible to fill the remaining small gaps on the screen. Recognizing this, the player hits the "clear" button to clear the screen and start over (row 16). Then, the player starts over by using an advanced strategy, in which the smaller gaps on the screen are filled first (rows 17 – 20), to ensure that the smaller pieces are available when needed. Afterward, the player uses the remaining pieces to fill the larger gaps, which have more flexibility in the variety of ways they can be filled:

| **Row #** | **Event** | **Piece** | **Round** | **Successful placement?** | **Elapsed time** |
|---|---|---|---|---|---|
| 13 | piecedrop | track5 | 5 | success | 280.019 |
| 14 | piecepress | track4 | 5 | n/a | 282.065 |
| 15 | piecedrop | track4 | 5 | success | 283.587 |
| 16 | clear | n/a | 5 | n/a | 285.864 |
| 17 | piecepress | track6 | 5 | n/a | 289.234 |
| 18 | piecedrop | track6 | 5 | success | 290.996 |
| 1 | piecep | tr | 5 | n/a | 291. |

| | | | | | |
|---|---|---|---|---|---|
| 9 | ress | ack5 | | | 982 |
| 2 0 | pieced rop | tr ack5 | 5 | success | 293. 233 |

As the above examples illustrate, we found that children's shifts in strategies were detectable, not only via in-person observations or interviews, but through online tracking data as well. Changes in strategies were often associated with either clusters of errors (indicating the player trying unsuccessfully to use different pieces to fill a gap), use of the "clear" button (indicating the player's recognition that a strategy was not working), and/or simply not having the necessary pieces available to fill gaps that remained on the screen. Thus, we could identify – and differentiate among – instances when children either failed to progress beyond basic strategies, proceeded through more difficult problems via trial and error (without necessarily employing a fundamental change in their thinking), or shifted to more sophisticated strategies over the course of a game.

## Discussion

Taken together, the observation, interview, and tracking data of the Cyberchase games we explored in this study hold implications for researchers and practitioners interested in exploring and bridging connections between computer games and mathematics education. From the standpoint of those interested in children's use of educational games, the parallels between online and offline reasoning highlight the degree to which gameplay is influenced, not only by players' experience and skill level in playing games, but also by their knowledge and skills regarding educational content embedded in such games. This point is not limited to interactive media. For a similar point regarding children's comprehension of educational television programs, see Fisch, 2000, 2004. As in school mathematics, children often do not display the same level of sophistication throughout a game (even if they are capable of relatively sophisticated reasoning). Rather, their mathematical reasoning may begin at a fairly basic level but become more sophisticated over the course of a game, when necessary to respond to the demands of the game.

From the standpoint of math educators, this study offers that online games can provide a good setting for helping students use and make connections between the strategies and reasoning they use when playing online games and those they might use to solve school mathematics tasks. The similarity between online and offline reasoning we discussed suggest that this might be a productive setting to pursue such in and out of school connections.

Additionally, we also found that even in the absence of in-person observations, data mining of online tracking data can provide a window into rich processes of reasoning and problem solving. When recorded and coded appropriately, such data can reflect, not only the outcomes of problem solving, but the process as well. (As a result, we have chosen to include tracking software among the assessments in our current research on children's learning from *Cyberchase* media.)

Yet, our experiences also point toward several challenges that must be overcome if tracking data are to be used effectively as a measure of reasoning. First, as anyone who has analyzed any sort of Web-based tracking data knows, users' clicks produce massive amounts of data. The sample data presented above is taken from a single session – and even that one game produced a spreadsheet containing more than 120 rows of data. When multiplied by the literally thousands of users who might play an online game in a single day, the volume of data can become staggering, posing challenges for both storage and analysis (even if the analysis can be partially automated).

Second, online tracking data must be limited to information that can be collected legally. The Children's Online Privacy Protection Act (COPPA) places strict limitations on the kinds of information that can be collected from children online. To help interpret data on gameplay, reasoning, or problem solving, researchers naturally look to characteristics of the players (e.g., age, gender, level of experience or prior knowledge), but COPPA can make it difficult to gather such information online. Since our project was part of a larger research study, and we had parents' signed consent for their children's participation, we designed the tracking software to record data only for players whose user names matched those in our study. Outside the context of such studies, however, researchers must either find alternate ways to gather demographic data, or do without it.

Third, tracking data are effective only for behavior that players perform clearly and unambiguously on the screen. Whereas the use of tracking data was highly successful for Railroad Repair, it was only partially successful for "Sleuths on the Loose," a game about measurement and proportional reasoning (*http://pbskids.org/cyberchase/games/bodymath/*). In Sleuths on the Loose, we could accurately record and code the answers that children provided, but it was harder to gauge their use of measurement for two reasons. Instead of using the on-screen "ruler" that served as a measuring tool, some children measured via alternate means such as holding their fingers up to the screen; the software could not detect these sorts of offline behavior. In addition, even when children did use the on-screen ruler, tracking data alone was not always a reliable indicator of whether a player was attempting to measure, because some children simply moved the ruler idly around the screen while thinking. Thus, we could tell whether players' answers were correct or incorrect, and identify some instances when players used the on-screen ruler for measurement (by establishing parameters for valid placement of the ruler). However, other cases of measurement could be identified only via in-person observation.

Understanding children's facility with educational content (outside of the confines of school classrooms) is important. As our experience in this study makes clear, educational games can provide a rich environment for studying children's mathematical reasoning in an informal context. However, games and tracking software must be designed carefully in order to not only accomplish educational goals but also produce useful, reliable data. But if they are designed properly, data mining can provide us with deep insight into children's thinking and reasoning -- without our having to peek over children's shoulders at all times.

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# A COMPARISON OF MATHEMATICS TEACHERS' AND STUDENTS' VIEWS ON THE NEW GENERATION HANDHELD TECHNOLOGY

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*Forty-five teachers and 54 middle school students shared their views about the TI-Nspire calculator after their first experience with it. The analysis of teachers' views about both benefits and weaknesses of this technology helped us reflect on why and how teachers come to adopt technologies in their teaching or not. The analysis of the student data empowered us with a student perspective that might have differed from that of the teachers. However, the analysis of our data revealed more similarities than differences.*

## Objectives

The main objective of this study is to analyze mathematics teachers' and students' initial opinions of a novel technology, TI-Nspire. With the advances of this new generation of handheld calculators, many new technological capabilities make novel classroom activities possible. But the main question is whether teachers are willing or ready to use these powerful new calculators in their classrooms. The analysis of teachers' views about both benefits and weaknesses of this technology helped us reflect on why and how teachers come to adopt technologies in their teaching or not. The analysis of the student data empowered us with a student perspective that might have differed from that of the teachers. A comparison of these views enabled us to see the similarities and differences between the views of these students and teachers.

## Perspectives

A large research base confirms the influence teachers' beliefs about teaching and learning mathematics have on their students' learning (Ball, Lubienski, & Mewborn, 2001; Stipek, Givvin, Salmon, & MacGyvers, 2001; Thompson, 1992). Specifically, teachers' beliefs about the roles of graphing calculators have been associated with the use and roles of calculators in mathematics classrooms. Therefore, it is imperative to study teachers' beliefs about the use of calculators in mathematics instruction, particularly with the advent of a new generation of graphing calculators. With this aim in mind, the current study investigated the views of a sample of pre-service and in-service mathematics teachers on the TI-Nspire.

With the advances of this new generation of handheld calculators, many new technological capabilities make novel classroom activities possible. For instance, TI-Nspire brings the dynamic, interactive, and linkage properties of these technologies to new levels. Now some of the linkages among representations become two-way. When using the old generation calculators, the linkage between a graph and its equation was one-way. We needed to enter different values for the coefficients of a function to be able to observe the effects of changes in the symbolic form on the graphical representation. However, manipulating the graph directly to see the effects of that manipulation on the symbolic form as well makes this approach even more powerful. TI-Nspire allows students to dynamically manipulate the graph and observe the immediate effects of that manipulation on the symbolic form.

## Conceptual Framework

Following Shulman's (1986) analysis of teachers' knowledge as a complex structure including content knowledge, pedagogical knowledge, and his introduction of the concept of pedagogical content knowledge, research in this area has become effectively grounded on his framework. With Mishra and Koehler's (2006, Koehler & Mishra, 2005) and Niess' (2005, 2006, 2007) introduction of the concept of the teachers' Technological Pedagogical and Content Knowledge (TPACK), technology-related research in the teachers' professional development and education field has gained a rich new conceptual framework or "analytic lens for studying the development of teacher knowledge about educational technology" (Mishra & Koehler, 2006, p. 1041).

TPACK involves content knowledge, pedagogical knowledge and technological knowledge and their combinations, as depicted in Figure 1. In this study, the content knowledge is 6-12 school mathematics. The pedagogical knowledge includes teaching methods and learning theories. The technological knowledge includes knowing how to operate technological tools (such as graphing calculators) and being able to adapt to ever-changing, novel technologies.

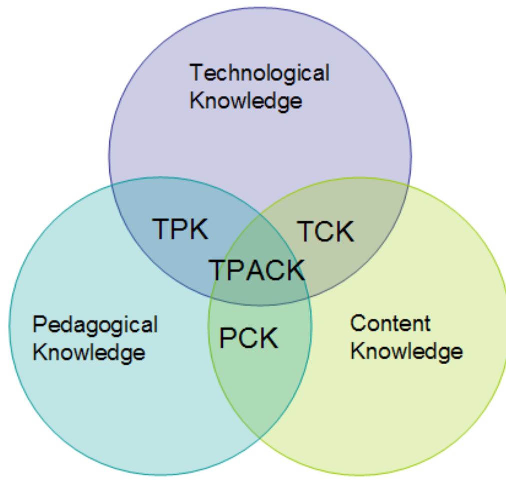


*Figure 1.* Re-creation of Mishra and Koehler's TPACK model.

Looking at the combinations, Pedagogical Content Knowledge (PCK) (Shulman, 1986) focuses on the mutual relationships between content and pedagogy. For example, depending on the specific mathematics content, a certain method could be chosen, or teachers can reflect on how particular pedagogical methods might help (or hinder) students' learning of specific mathematics content. Another combination is Technological Content Knowledge (TCK). In discussing TCK, Mishra and Koehler (2006) note that "teachers need to know not just the subject matter they teach but also the manner in which the subject matter can be changed by the application of technology." (p.1028). On the other hand, Mishra and Koehler believe that "technological pedagogical knowledge (TPK) is knowledge of the existence, components, and capabilities of various technologies as they are used in teaching and learning settings, and conversely, knowing how teaching might change as the result of using particular technologies" (2006, p. 1028). TPACK is at the heart of all of this knowledge and takes all of these factors into consideration (see Figure 1) as "an integrated whole 'Total PACKage'" (Thompson &Mishra, 2007, p. 38).

## Data Collection Methods

Since the new generation calculators bring many new capabilities, it might be difficult to focus on all of them at once. Creating instructional materials focusing on one capability at a time could be easier for students and teachers to make their transition from the old to the new generation. Therefore we created an activity using TI-Nspire, in which users explore the effects of manipulating the graph of a quadratic function on its symbolic representation (Özgün-Koca &

Edwards, 2008). We used this activity with teachers and students to introduce them to TI-Nspire.

We asked 19 pre-service teachers, 26 in-service teachers, and 54 middle school students to reflect on the novel capabilities of this new technology. Our main data collection method for the teacher data was a survey that included open-ended questions that followed an activity using TI-Nspire.

Middle school students experienced the same activity over a three-day period. Our main data collection methods for the student data were surveys that included Likert type and open-ended questions. In addition to that, we videotaped students working with the TI-Nspire in the classroom. Portions of the videos will be shared during the presentation. The data gathered with the Likert type questions were analyzed with descriptive statistics. For the analysis of the qualitative data obtained through the survey, data were coded and analyzed qualitatively to reveal patterns and themes. Data triangulation and peer debriefing were used to ensure the trustworthiness of the data. Results revealed differences and similarities among the middle school students, pre-service teachers, and in-service teachers' views on the use of TI-Nspire in mathematics instruction.

**Results**

We observed that students were eager to learn about this novel technology and they were very motivated. As a response to mid-survey, students stated that they liked both the technical (more features or applications) and physical (larger screen or alpha keys) novelties of the TI-Nspire.

> I liked that the new TI-Nspire calculators have different things that we cannot do on the normal TI-84 plus calculators.

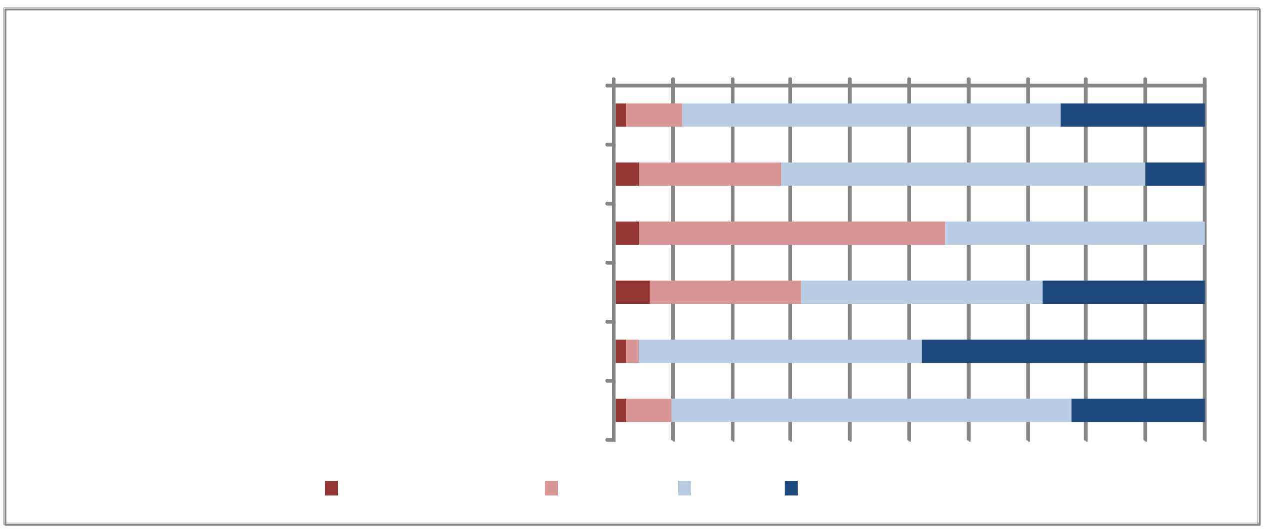

*Figure 2.* Student exit survey.

An overwhelming majority (89%) of the students liked doing mathematics with Nspire. Seventy-two percent of the students agreed that Nspire helped them to better understand the math lesson. While 56% of the students disagreed that Nspire helped them to be more confident during the math class, 44% of them agreed. However 68% of the students agreed that it was difficult to follow the Nspire lesson. Another overwhelming majority (96%) liked being able to move the

*S*
*N*
*C*

graph and observe the equation. Similarly an overwhelming majority (89%) liked multiple representations.

In the following sub section, the TPACK model is used to frame our analysis of teachers' knowledge, but at the same time students' views are discussed at appropriate places. The main objective here is to focus on the technological, pedagogical or content related comments separately.

*Technology in* ***T****PACK*

First, we focused on the technological part of TPACK. Seventy-four percent of the pre-service teachers and 42% of the in-service teachers thought that TI-Nspire was more difficult to use than other calculators they had worked with. Some simply said that it was more difficult to use, but many also mentioned that it might become easier with time and more experience. Some of the reasons for the difficulties they cited included:

> There is so much to learn about the calculator-it can be overwhelming.
>
> [TI-Nspire is more difficult] only because I am used to the TI-83 technology.

In contrast, twenty-one percent of the pre-service teachers and 42% of the in-service teachers thought TI-Nspire was easier to use: "I thought the technology was easy to use. I like how the menus were easy to use."

Sixty-nine percent of the students agreed that it was difficult to follow Nspire math lessons. One stated that "it was very difficult for me to follow these math lessons because I didn't know where any of the buttons were and how to get to places on the calculator." As with the teachers, some students stated that it was only their first experience: "It went kind of fast and was very complicated. With some lessons, I could probably understand it better."

Participants found TI-Nspire different from other calculators that they have used before. They were comparing this novel technology to other technologies using their previous technological knowledge:

> It makes a TI-84+ look like a small scientific calculator (Teacher)
>
> I liked that the new TI-Nspire calculators have different things that we cannot do on the normal TI-84 plus calculators. (Student)

At the same time, some students also mentioned that they would prefer working with TI-84s, because they were more familiar with them: "[TI-Nspires] were really confusing. They are very difficult to graph when you are so used to the TI-84 Silver Edition."

Forty-two percent of the pre-service teachers, 50% of the in-service teachers, and some students mentioned that TI-Nspire had more capabilities and features when compared to other technologies and calculators:

> This is 'better' technology than what I have used. It seems that this calculator has many more functions and abilities than others I have used. There are many features and applications that are great (Teacher)

A few teachers and students compared TI-Nspire to a home computer:

> They are more comparable to a computer with the drop down menu screens (Teacher).
>
> [Nspire has] potential features computer like accurateness and format. Because it acts as accurately as a computer but as portable as a calculator (Student).

Twenty-one percent of the pre-service teachers and 46% of the in-service teachers also mentioned that being able to directly manipulate graphs was a key technological difference: "This calculator allows greater manipulation of equations and graphs. It allows students to see immediately the relationship between vertex forms of the graph as well the graph and other equations." Similarly 96% of the students stated that they liked moving the graph around to see what happened to the equation: “That helped me see how the equation changed with the parabola.”

The novel technological capabilities that students liked were separate buttons for the letters, a bigger screen, and many applications. However, at the same time, they did not like that the buttons were scrunched together.

*Pedagogy in T**P**ACK*

One teacher stated that “this calculator is different because you can move the line and see what it does to slope…Before you had to change the equation to see the line change.” While talking about the technological capabilities of this machine, this teacher made a connection to his or her TPK. In doing so, s/he considered how the new capability of the Nspire might help students who are learning the connection between the symbolic and graphical representations of linear equations. We believe that this teacher was also visualizing the use of Nspire in his or her own classroom. The construction of such a vision has been posited as a necessary condition for the implementation of an innovation (Shaw & Jakubowski, 1991; Simon & Schifter, 1991). A couple of the teachers also used the word “hands-on” when discussing novel features of Nspire and how these new capabilities might help improve the teaching and learning environment.

Interestingly, two pre-service teachers mentioned similarities to a paper and pencil environment: “Anything you want to do can be done by typing exactly what I see on paper into the calculator.” At this point, these teachers saw a resemblance between this new environment that Nspire provided and a very traditional and old teaching and learning environment. This new feature might have attracted their attention, because it was different from the old generation of calculators.

Seventy-two percent of the students agreed that Nspire helped them to better understand the math lesson. Most of the pre-service and in-service teachers were in agreement with the middle school students and believe that TI-Nspire would positively influence student learning. While explaining their reasons for this comment they were using TPK while focusing on their “knowledge of students’ understandings, thinking, and learning with technology” , which was mentioned as a part of TPACK by Niess (p. 197, 2006). The main reasons for this belief appeared to be the ability to create an exploratory learning environment, being able to provide an interactive environment with dynamic linked multiple representations, and being able to increase student motivation.

*Creating an exploratory learning environment*. Thirty-two percent of the pre-service and 23% of the in-service teachers shared that TI-Nspire could be beneficial in creating an environment for discovery. One teacher stated that:

> It allows them to more easily test conjectures. It also allows them to better make connections and see representations. This technology will be as difficult for students as for teachers, but I think a lot of guided activities would have students breezing through its use by the end of the year. I think it allows teachers to develop more discovery/inquiry-based approaches to teaching and learning.

Indeed, we observed this in the classroom, and so did one of the students: “These calculators are very helpful in my learning because they introduce hands on learning and promote good experimenting.”

*Providing an interactive environment with dynamic linked multiple representations.* Twenty-six percent of the pre-service and 46% of the in-service teachers stated that TI-Nspire could bring unique opportunities for students to experience new forms of representations:

> It could help them see how you can bend the parabola and how the coefficients change as it

> gets smaller or larger. This technology is very interesting. I would love to learn to use it so that I could use it in my classroom.

Ninety-one percent of the students agreed that they liked being able to represent a parabola in several different ways (graph, equation, or spreadsheet table): "It was interesting to see how function and graph related to each other so much."

*Student motivation*. Eleven percent of the pre-service and 27% of the in-service teachers mentioned things that have been associated with increased student motivation (Middleton & Spanius, 1999): "I think students would be excited about using these calculators, thus increasing their learning." We observed that students were very eager to use Nspire and 89% of the students stated that they liked doing mathematics with Nspire: "I liked using the new calculators because they were very high-tech and really fun to use. Also it made graphing easy and fun!"

Sixteen percent of the pre-service teachers were not sure or do not think that this technology will help students' learning: "I just worry that this technology may detract students from learning math content, because they get caught up with how to use the calculator and not how the math corresponds to the calculator and display." Some pre-service teachers (21%), but only one in-service teacher agreed that TI-Nspire could be helpful in students' learning, but they fear calculator dependency:

> I believe that this technology will encourage students to ask and find out "why" more often since the work needed will not only be less tedious, but also more fun to do. I think the "why's" of math will be easier for students to learn and therefore their understanding of concepts and connections will increase. I fear that the more advanced technology becomes, the more dependent students will become on it.

*Content Knowledge in TPA**C**K*

The main categories that pre-service teachers discussed were the technological and pedagogical parts of TPACK. It is understandable that when we come across a novel technology, the first thing that we reflect on would be the technology itself. And then the second thing would be how learning and teaching might change with this new technology. Content or curriculum related issues should also be considered, but perhaps not as soon as technological or pedagogical issues. Perhaps as teachers become more knowledgeable with technology, content/curriculum related issues will come more into play.

The activity that we used in this study focused on the effects of the coefficients in the quadratic function (*f(x)=ax^2+bx+c*) on its graph . In a traditional curriculum, the effects of *a* and *c* are studied more commonly than the effects of *b*. With the availability of the new capabilities of TI-Nspire, the way to study the effects of *a* and *c* is enhanced and the effects of *b* may also be studied much more easily.

## Discussion and Educational Importance of the Work

In this section, we first compare pre-service and in-service teachers' views of Nspire. Next we compare teachers' views of with those of students, as well as our observations of the students while they were using Nspire. Finally, we discuss implications for teacher education.

*Comparison of Pre-service and In-service Teachers' Views*

As a result of this analysis, we observed that more in-service teachers found Nspire easier to use as compared to pre-service teachers and the students. Moreover, more in-service teachers than pre-service teachers emphasized the novel capability of Nspire-being able to manipulate the graphical representation physically in the virtual environment. Perhaps having more experience with graphing calculators in classroom instruction and having more knowledge about the graphing calculators' capabilities and limitations might have influenced these differences. Moreover, the larger experience base of the in-service teachers may have helped them to create a more fully developed vision of what using Nspire during instruction might look like in their classrooms. If this analysis is correct, it helps to explain most of the differences between pre-service and in-service teachers that we observed. Finally, when reflecting on the potential effects of the use of Nspire on students' learning, both pre-and in-service teachers articulated similar positive influences of the use of this technology on the teaching and learning environment. Moreover, teachers views about the potential pedagogical issues when teaching with Nspire were very close to the reality we observed in the classroom with students, such as being able to create exploratory environments and an increase in students' motivation.

*Comparison of Teachers' and Students' Views*

Based on our analysis of these data and our observations of students working with Nspire, we saw more similarities than differences in the views of teachers and students. About the only difference we noted was that students were far more likely than teachers to mention the separate buttons for letters of the alphabet as an advantage of the new calculators. We speculate that this might be due to students' past experiences in text messaging.

In most cases the views of students paralleled those of teachers. What was particularly noteworthy is the degree to which the students' views of their work with Nspire tended to confirm the teachers' views of the potential pedagogical benefits of using Nspire. Moreover, our observations of the students' working with the new calculators further confirmed these potential benefits.

*Implications for Teacher Education*

NCTM's vision of school mathematics includes students in grades 9-12 "using technological tools to represent and study the behavior of polynomial, … functions" (NCTM, 2000, p. 297). Moreover, we believe that when students use dynamic geometry tools to manipulate graphs linked to their symbolic algebraic representations, they will be empowered "to convert flexibly among these representations" (NCTM, 2000, p. 360). In doing so, they will experience firsthand "the power of mathematics [that] comes from being able to view and operate on objects from different perspectives" (NCTM, 2000, p. 360). In order for that vision to become a reality, teachers of mathematics, including pre-service teachers, must be able to envision the use of technology in their own classrooms in appropriate ways. We believe that modeling for teachers activities such as we did in this case can help them form new visions of what their classroom might be like. However, while doing so may be a necessary condition, it is clearly not sufficient, as our data vividly show. They would need more experience with the technology in their education and profession.

In methods courses, students could focus on the potential effects of the use of an instructional technology in their teaching. In content courses, they could focus on how the use of this technology might have helped or hindered their learning. Teacher education programs offer pre-service teachers the knowledge and experience that they will need in their profession. However, it is crucial to remember that their belief system and views about their pedagogical implications are also developed during their teacher education program. Therefore, it is essential to identify these beliefs and views in order to inform the teacher education program itself and present opportunities to pre-service teachers to reflect on their beliefs and views on important educational issues.

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# TEACHERS' USE OF ONLINE MATHEMATICS EDUCATION RESOURCES: INSIGHTS ON DIGITAL NATIVES AND PROFICIENT DIGITAL IMMIGRANTS

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*This study investigates how mathematics educators make use of the digital capabilities of online resources both in and out of the classroom. The digitally adept 17 pre- and in-service teachers in a graduate technology in mathematics education course failed to capitalize on the unique abilities afforded by the Internet using resources primarily for lesson planning.*

## Theoretical Background

Today's incoming teachers represent the first generation with personal access to technology in our ever-evolving digital world. Pre-service and novice teachers are in their early to middle twenties, with post-1980 birth years that are suggested for "digital natives" (Palfrey & Gasser, 2008). According to the U.S. Department of Education (2000), as new teachers who have grown up in a technology-rich environment enter the profession, their comfort and skill with technology will lead to increased use of computers for instruction. Most pre-service mathematics teachers enter their mathematics education courses as competent users of the Internet and general-purpose computer software (e.g., word processing and spreadsheets), having gained some experience with these technologies during previous university or K-12 classes (Goos, 2005). Teachers who have been in the profession for 1-6 years have been found to be more comfortable with technology than those who have taught longer (Russell, Bebell, O'Dwyer, & O'Connor, 2003).

A study of over 2000 teachers of grades 4-12 found that mathematics teachers tended to use internet resources very differently and less frequently than other teachers (Becker, 1999). This finding is supported by a similar study from over 4000 grade 4-12 teachers (Becker, 2001). When internet usage was ranked by teachers' subject matter, the 538 mathematics teachers placed second to last with 11% of respondants using the Internet. In the later half of the 1990's, less than one-half of eighth grade mathematics teachers claimed to use computers at all (Technology Counts, 1997).

Policy makers claim that high access to computers in classrooms leads to major improvements in teaching and learning; however, such access has not ensured high use by classroom teachers (Cuban, Kirkpatrick, & Peck, 2001). 82% of teachers surveyed reported using computers for home use at least once per week, yet claimed to utilize computers very seldom for classroom instruction. Describing the incremental changes that occur in pedagogical practices as access to technology increases, the Cuban et al. study showed that while technology can promote student-centered instruction, teacher-centered instruction often remains dominant. Teachers have reported that they used computers primarily for classroom preparation (Technology Counts, 1999). In 2001, Cuban et al. hypothesized that in the absence of major changes to the obstacles facing technology in the classroom such as allocation of time, teacher preparation, reliability and functionality of technology, and increased internet speed, only modest alterations would be made in teaching and learning.

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## Research Questions

The data reported here is part of a larger grounded theory study, which used both quantitative and qualitative data, to explore how and how frequently pre-service and novice mathematics teachers make use of digital resources. The study purposely targets a small group of pre-service and novice teachers, teachers in the first wave of digital natives or exceptionally proficient digital immigrants, to determine their use of online resources. This paper will primarily focus on teachers' use of digital resources by addressing the following series of questions:

1. What types of digital resources do mathematics teachers use?
2. For what educational purposes and how frequently do teachers use digital resources?
3. Do teachers capitalize on the unique capabilities afforded by online technology?
4. Who do teachers see as the end users of mathematics education websites?

## Methodology

The participants of the study included graduate students enrolled in an elective course on technology in mathematics education in a department of learning and instruction at a large research university in New York State. As a part of a technology-infused teacher education program, the course was able to incorporate more than four of the strategies suggested by Kay (2006) for pervasive computer use by teachers. The study participants included 9 pre-service, 7 novice in-service teachers, and 1 veteran teacher (henceforth referred to collectively as teachers), with mathematics emphasis educational backgrounds all living in a 50-mile radius of each other. All grew up in what could be described as middle class homes with reasonable access to technology. The veteran teacher was certified to teach grades K-12. Five of the participants were childhood teachers with a mathematics emphasis, certified (or working toward certification) to teach up to grade 6 or grade 8. Ten were secondary teachers certified (or working toward certification) to teach from grade 5 (or grade 7) up to grade 12. All but three of the teachers were in their early to middle twenties, hence, digital natives. The other three teachers had extensive technology experience and could easily be identified as proficient digital immigrants.

The initial research team consisted of the instructor of the course and a doctoral student in mathematics education, who acted as a participant-observer during the course. As a validating step, the team then recruited another doctoral student in mathematics education who had taken the course two years prior to help review the related literature, the data, and initial research team's conclusions.

Data was collected during the fall semester ranging from August 2008 through mid-January 2009. Employing an inductive process of generative data collection, constant comparisons of data from varying instruments were made. The data reported in this study were collected via researcher-generated surveys, participant-generated surveys, and informal interviews.

At the end of the technology class, the participants were asked to submit 5 to 10 websites they felt were the best mathematics education sites. The list of submissions was compiled into a cumulative list of top websites for the class. Using this cumulative list, the teachers were next asked to complete a website-use survey composed of 19 Likert-scale items to evaluate. For those sites with which they were familiar, they indicated their use of each website as well as who they saw as the end-users of those sites. Following the website use survey, participants created two-question surveys in groups using web-based software as a course assignment. The data compiled in the participant-generated surveys were used to further develop digital consumer profiles, and to ensure reliability of the data found with researcher-generated instruments.

## Results

In addition to enrolling in an elective course entitled *Technology in Mathematics Education*, the data in the following relative frequency tables speak to the general digital consumption and digital facility of the teachers in addition to their use of technology for teaching mathematics.

Digital Consumer Profile: General

| **Do you use/visit the following weekly?** | **Percent** |
|---|---|
| Cellular Telephone | 100 |
| Internet | 100 |
| E-mail | 94.1 |
| Text Messaging | 58.8 |
| Wikis and Online Dictionaries/Encyclopedias | 56.3 |
| Online Social Networking (e.g., MySpace, Facebook) | 52.9 |
| Blogs | 17.6 |
| Social Bookmarking sites (e.g. Del.icio.us) | 5.9 |
| Online Survey/Calendar sites (e.g., Survey Monkey, Google Calendar) | 0.0 |
| Video Sharing sites (e.g., YouTube) | 0.0 |

Digital Consumer Profile: Teaching

| **In my teaching I use (*type of technology*) weekly** | **Percent** |
|---|---|
| Word Processing Software | 100 |
| Internet | 94.1 |
| Equation Editor | 82.4 |
| Graphing Calculator | 70.6 |
| Spreadsheet Software | 47.1 |
| Internet-Based Applets | 46.7 |
| Exam Generator Software | 41.2 |
| Presentation Software | 23.5 |
| Dynamic Geometry Software | 5.9 |
| Downloaded Videos | 5.9 |

The data in both profile tables was collected from items generated by the research team. To ensure the credibility of the data results from the researcher-generated surveys, data are reported that were generated by teacher groups as a part of a class assignment learning to use online survey websites. This information can be found in the *Teacher-Generated Items Related to Teachers' Digital Consumption* table on the next page.

When polled on the best mathematics education sites, the teachers submitted a total of 49 sites. However, when asked to assess the 49 best sites, the teachers only felt familiar enough with 37 of the initial 49 sites submitted to actually evaluate them. Of the 37 sites that were evaluated, only 12 sites were evaluated by at least 25 percent of the participants. The data on the 12 "most used" sites is reported below and includes the relative frequency of teachers who are familiar with the site, the frequency of use for each site, the teachers' ideas about who would be the sites' end users, and the Google pagerank for each site. Note that teachers were able to mark more than one end user, so the relative frequencies reported for end user data represent the percentage of teachers making the recommendation. Google uses a combination of key words and pagerank to return websites on a given search. Using a scale of one to ten, those sites with higher pageranks are more likely to be returned on a search with a given set of key words, than those sites with a lower pagerank given the same key words (Craven, 2009).

Teacher-Generated Items Related to Teachers' Digital Consumption

| Question | Mean | SD |
|---|---|---|
| How often do you use computers for school related activities? (1 = rarely, 5 = very often) | 4.87 | 0.35 |
| How often do you use websites to help create math lessons for your classroom? (1 = rarely, 5 = very often) | 3.56 | 0.89 |
| How likely are you to search websites for existing worksheets rather than create your own? (1 = highly unlikely, 5 = highly likely) | 3.13 | 0.50 |
| Would you use a website if there was a fee for use/membership? (1 = no, 2 = only if paid by district, 3 = maybe, 4 = yes) | 2.75 | 0.58 |
| What do you feel is the most important when choosing technology that will be used in the classroom? (1 = not important, 5 = very important) | | |
| relevant to topic | 4.67 | 1.05 |
| student interaction | 3.60 | 1.24 |
| Fun and interesting | 3.40 | 1.24 |
| state standards | 3.07 | 1.22 |
| national standards | 2.00 | 1.20 |

Most Used Websites for Mathematics Education

| | Website | Percentage of sample familiar with site | Frequency of use (1 = less than once a year, 6 = daily) | | Percentages of Intended Site End User(s) | | | | Google Pagerank (1 = low, 10 = high) |
|---|---|---|---|---|---|---|---|---|---|
| | | | Mean | SD | teacher | student (in-class activity) | student (as a reference) | parent | |
| 1. | Illuminations | 66.7 | 5.00 | 0.82 | 100 | 16.7 | 42.7 | 0.0 | 6 |
| 2. | NCTM | 61.1 | 4.45 | 0.94 | 90.9 | 0.9 | 0.0 | 0.0 | 7 |
| 3. | The Math Forum | 55.6 | 4.00 | 1.00 | 90.0 | 10.0 | 80.0 | 50.0 | 8 |
| 4. | NLVM* | 50.0 | 4.22 | 1.09 | 100 | 55.6 | 66.7 | 44.4 | 6 |
| 5. | Purple Math | 38.9 | 4.14 | 0.90 | 100 | 0.0 | 71.4 | 28.6 | 6 |
| 6. | Regents Prep | 38.9 | 4.67 | 1.03 | 85.7 | 28.6 | 85.7 | 71.4 | 5 |
| 7. | Cool math | 33.3 | 3.83 | 0.75 | 66.7 | 33.3 | 100 | 50.0 | 4 |
| 8. | edHelper | 33.3 | 3.67 | 1.21 | 100 | 0.0 | 33.3 | 33.3 | 6 |
| 9. | Mathbits | 33.3 | 4.17 | 0.98 | 83.3 | 33.3 | 66.7 | 50.0 | 6 |
| 10. | BrainPop | 33.3 | 3.71 | 1.11 | 83.3 | 50.0 | 66.7 | 16.7 | 8 |
| 11. | AAA Math | 27.8 | 4.00 | 1.10 | 100 | 40.0 | 100 | 40.0 | 6 |
| 12. | School Island | 27.8 | 4.80 | 0.84 | 80.0 | 100 | 40.0 | 40.0 | * |

* NLVM represents the National Library of Virtual Manipulatives.
** *Note.* Schoolisland.com is unrankable, (has no pagerank) as is common with commercial, educational tool sites.

To ensure that the data represented mathematics teaching interests across the K-12 landscape, the research team considered the ratio of primary and secondary teachers' responses. For 11 of the 12 most used sites, the ratio of primary and secondary responses was proportional to the ratio of primary and secondary teachers in the study. Purple Math, a site dedicated to algebra, is the only datum on the list that was selected solely by secondary teachers.

The list of the 12 most used websites is comprised of sites that provide a variety of educational resources for mathematics teachers. Several of the sites listed deal with the national and state standards. Illuminations and the NCTM site provide numerous resources that are based on the principles and standards for K-12 mathematics proposed the National Council of Teachers of Mathematics (2000) including activities, applets, lessons, and access to professional development materials. NLVM provides an array of standards-based interactive manipulatives organized by NCTM's content standards and grade bands. Regents Prep provides practice for and addresses issues pertinent to mathematics testing in New York State.

The Math Forum is a comprehensive site that offers a variety of users a wide range of materials related to lesson creation, course material development, homework help, and review of mathematics concepts. Purple Math is a text-dominant algebra review site that includes occasional static images to accompany the text. Cool math is a conglomeration of mathematics lessons, games, and puzzles.

Several sites selected by the participants emphasize the availability of printable materials on their site. With a focus on mathematics and programming, mathbits.com provides printable and multimedia materials including movie clips and instructional guides. EdHelper is a source of worksheets with a limited number of puzzles. AAA Math is a collection of worksheets with built in self-assessment that allow for feedback based on user input. Similarly, schoolisland.com allows for automatic feedback on self-administered worksheets in addition to addressing certain administrative aspects of teaching.

Two sites on the list are accessible by fee. BrainPop, is a multidisciplinary site with a collection of animated educational videos in mathematics as well as other school subjects. School Island is a commercial website that provides teachers with content authoring and course management functions that allow them to create and assess assignments as well as track student progress.

The teachers were asked to report on exactly how they utilized the initially submitted 49 "best" sites. As previously stated, teachers only evaluated those sites with which they were familiar. The information is reported in the *Website Use Survey*.

The four most frequently cited reasons for using a website, Tier 1, primarily addressed teachers' instructional planning. These responses were teacher-centric and focused on the teachers' lesson preparation. The next four responses cited, Tier 2, were more oriented towards the generation of products to be used by the students in class, or were of a more interactive nature. These items included accessing puzzles, games, interactive materials, etc. This tier also included the procurement of worksheets for students.

Three of the following four items, Tier 3, focused on state math assessment, related to state mathematics standards, or national mathematics standards. The final seven items on the list, Tier 4, dealt with components that some view as peripheral to teaching. These sites included information on topics such as student contests, professional development, funding opportunities and historical information related to mathematicians or mathematics, recent developments in mathematics education, and real world data to be used in mathematics lessons.

After considering the data in the *Most Used Websites in Mathematics Education* and *Website Use Survey* tables individually, the data was compared. Of the 12 most used sites, both NCTM (site 2) and The Math Forum (site 3) were identified as meta-resource websites that cut across

Website Use Survey*

| | **I will or have used the sites to: (1 = never, 5 = very often)** | **Mean** | **SD** |
|---|---|---|---|
| Tier 1 | get ideas for lessons | 2.97 | 1.20 |
| | get examples | 2.95 | 1.23 |
| | help develop course materials | 2.91 | 1.27 |
| | help refresh my knowledge/learn more about a topic before I teach it | 2.37 | 1.23 |
| Tier 2 | get mathematical games, like puzzles or riddles for students | 2.29 | 1.19 |
| | get applets, virtual manipulatives, videos, etc. that I will use for interactive demonstrations | 2.27 | 1.40 |
| | get applets, virtual manipulatives, videos, etc. for interactive explorations | 2.26 | 1.39 |
| | get worksheets | 2.23 | 1.25 |
| Tier 3 | provide examples, practice for students on state math assessment instruments | 2.20 | 1.33 |
| | help me learn how what I teach ties in with other subjects/math topics | 2.10 | 1.15 |
| | get information on state math assessment instruments/expectations | 1.94 | 1.24 |
| | learn more about how the standards apply to a lesson/topic | 1.85 | 1.17 |
| Tier 4 | get mathematical support materials like graph paper, coordinate axes, bar graph templates, etc. | 1.65 | 0.94 |
| | get information on important, developing issues in mathematics education. | 1.52 | 1.01 |
| | gather real world data for the students/myself to use in a lesson | 1.50 | 0.83 |
| | get information on conferences/other professional development opportunities | 1.49 | 0.99 |
| | get information on competitions for students | 1.45 | 0.89 |
| | get information on funding opportunities for teachers | 1.43 | 0.88 |
| | gather historical information related to a lesson/person in mathematics | 1.42 | 0.73 |

* *Note*. Responses have been grouped into tiers for ease of reference.

Teacher-Identified End-Users

| | | **End-user** | | | |
|---|---|---|---|---|---|
| | **Primary Focus** | teacher-user | student-user (in-class activity) | student-user (as reference) | parent-user |
| Tier 1 | instructional preparation | 92.3 | 9.8 | 50.2 | 23.1 |
| Tier 2 | student materials, interactivity | 89.1 | 29.8 | 67.2 | 37.5 |
| Tier 3 | state assessment, natl. standards | 97.4 | 36.2 | 46.5 | 28.2 |
| Tier 4 | “peripheral” components | 95.2 | 5.2 | 38.1 | 23.8 |
| *Most Used Websites* (cumulative) | - | 91.1 | 26.8 | 59.0 | 32.2 |

the properties from all four tiers. Sites 1, 2, 3, 5, 7, and 8 from the *Most Used Websites for Mathematics Education* list provide materials that are aligned with the website uses found in Tier 1 of the *Website Use Survey*. Sites 2, 3, 4, 7, 8, 9, 10, 11, and 12 from the list provide materials that align with Tier 2. Sites 1, 2, 4, and 6 from the list provide materials that align with Tier 3. Finally, sites 2 and 3 from the list provide materials that align with Tier 4.

Teachers were asked who would be using the resources found on the list of *Most Used Websites for Mathematics Education.* A weighted average was used to compile the percentage of teachers identifying the four given categories as potential end-users for the 12 sites. Using data analyzed further in the discussion section, the calculated percentages combine information from the *Most Used Websites in Mathematics Education* and the *Website Use Survey* tables. The *Teacher-Identified End-users* table reveals these percentages. Note that the top line of the end-user table represents the cumulative results from all tiers.

## Discussion

### *Teachers: Primary End-users of Mathematics Education Sites*

Teachers overwhelmingly see themselves as the primary end-users for the variety of mathematics websites found in their *Most Used Websites*. They perceive students as using the websites for in-class activities mostly in relation to high stakes testing and standards related material. Students were perceived to use the sites as references far more frequently than as a part of in class activities. Parents were viewed as end-users less frequently than students; however, both peaked in the same tier (2) with those sites that dealt with student materials and interactivity.

### *Digital Capabilities Not Fully Exploited*

The *Website Use Survey* shows teachers use the websites more for instructional planning than accessing any of the unique capabilities these digital resources provide. This confirms the Russell et al. (2003) findings that K-12 educators use technology for the preparation of teaching. A large percentage of the teachers were facile with social networking technology, which includes the features typical of and unique to digital resources like networking, hyperlinks, photo and video sharing, etc. However, these teachers do not make use of websites that offer these types of features in their professional lives. Many of the mathematics education websites provide text with static embedded figures. Notable exceptions include the applets available through Illuminations and NLVM, and the input/feedback worksheets on AAA Math. None of the websites seemed to tap into the power of hyperlinking that is seen on popular websites such as Facebook and Wikipedia.

### *Inconsistencies Related to State Assessment/National Standards*

Teachers saw their students using Tier 3 sites, sites related to standards and state assessment, more than other types of sites for their in-class activities. In addition, the third most frequently used site of the *Most Used Websites* was the New York State Regents Prep site with a mean of 4.67 (SD = 1.03). However, when asked on the *Website Use Survey* about providing examples and practice problems for students on the state mathematics assessment exam or about getting information on state mathematics assessment instruments/expectations, these items received relatively low means of 2.20 (SD = 1.33) and 1.94 (SD = 1.24), respectively.

Further, in the *Teacher-Generated Items* table, the teachers indicated that state standards played a higher role than national standards in their decision to use technology in the classroom. With a mean of 3.07 (SD = 1.22) versus 2.00 (SD = 1.20), the state standards were ranked a full point higher than the national standards. Despite this, there was a marked absence of the New

York State Department of Education site on the initial list of the 49 best websites, even though this site was made available to the teachers via presentation and as a part of the course's homepage as one of the key external links. Moreover the 12 *Most Used Websites* included three national standards-based sites (1, 2, and 4) but only one state site (Regents Prep), a site that deals directly with high stakes testing.

*Limitations of Study/Future Work*

The research team acknowledges the limited scope of the homogeneous population involved. Transferability of the study is high and as such future work should focus on the size of, and heterogeneity of, the sample. This study serves as a first look, a catalyst, for others to investigate how the first wave of digital native mathematics teachers make use of and disseminate digital resources. Despite the fact that many of the obstacles for technology implementation presented by Cuban et al. (2001) have been overcome, we are still seeing the ad hoc applications of technology that have further sustained traditional teaching practices. The importance of this study is to caution teacher educators against making assumptions that future and novice mathematics teachers' digital facility implies they will fully exploit the digital capabilities of online resources.

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# A PROBLEM BASED ON-LINE MATHEMATICS COURSE AND ITS AFFECT ON CRITICAL THINKING, REASONING SKILLS AND ACADEMIC ACHIEVEMENT

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*Teaching students to reason and use high-order thinking skills has been problematic for many years. With innovative thinking and problem solving skills becoming the main stay for success in today's society this study looks at whether critical thinking, reasoning skills, and academic achievement are affected by the use of a hybrid on-line problem-based mathematics course.*

## Objective or Purposes of the Study

In current national and international educational research there is a widespread acceptance that critical thinking development is an important dimension of education (Meadows, 1996; Paul et al., 1995; Perkins, 1993; Gadzella et al., 2006). David H. Jonassen, Chad Carr, and Hsiu-Ping Yueh (1998) believe that rather than using the power of computer technologies to disseminate information, they should be used in all subject domains as tools for engaging learners in reflective, critical thinking about the ideas they are studying.

This study examines students who participate in a hybrid (teacher supervised) on-line technology-based mathematics course with an emphasis on real-world problem solving (TBC). The purpose of this research is to determine if a relationship exists between critical thinking, reasoning skills and student achievement as indicated by the Test of Everyday Reasoning (TER) and Texas Assessment of Knowledge and Skills (TAKS) Test in TBCs. Specifically, we looked at changes in critical thinking skills and reasoning skills, (analysis, evaluation, inference, inductive and deductive reasoning) after using a TBC with a pre-post test model. We also looked at differences in scores according to gender and ethnicity to examine their roles as factors impacting critical thinking and reasoning.

## Theoretical Framework

The National Council of Mathematical Teachers (NCTM) emphasizes the importance of reasoning and sense making in the Process Standards: Problem Solving, Reasoning and Proof, Communication and Representation. All are interlinked and essential in the teaching and learning of mathematics (NCTM, 2000). Yet the report of the Programme for International Student Assessment (2007) suggests that United States students are lagging in their ability to apply mathematics "to analyze and reason as they pose, solve and interpret problems in a variety of situations" (p. 7).We know that globalization and the rise of technology are presenting new economic and workforce challenges (Friedman, 2006). The traditional mathematics curriculum is not sufficient for students entering many fields (Ganter & Barker, 2004). According to the Task Force on the Future of American Innovations 2005, the Committee on Science, Engineering and Public Policy 2006, and Tapping America's Potential 2008, the United States is in danger of losing its leadership position in science, technology, engineering, and mathematics.

In light of what we know about learning, the computer and other technology when used as tools for meaningful problems is a reasonable method for engaging students in problem solving and critical thinking (Muir, 1994; Peck & Dorrricot, 1994). Studies suggest that research interests need to focus on designing computer environments that foster a disposition for critical thinking (Facione, Facione, & Sanchex, 1994; Taube, 1995; Wiburg, 1996). This restructuring of the classroom includes the use of computer to provide active learning, authentic tasks, challenging work, complex problem solving, and higher-order thinking skills (Dalton & Goodrum, 1991; David, 1993).

This study examined students who participated in a technology-based mathematics course with an emphasis on real-world problem solving (TBC). The real-world problems were based on the following problem-based learning (PBL) criteria:

- Problems engage students; they provide a meaningful, authentic context for problem solving (Blumenfeld, et.al.,1991; Van Haneghan et al., 1992), and students have meaningful input into a process or solution (Torp & Sage, 1998).
- Problems allow knowledge to be a tool; students learn what it is for and when and how to use it relative to the context in which it is acquired; this facilitates recall and utility in new problem situations.
- Problems include multiple assessments that will help students build learning on prior knowledge and challenge misconceptions (National Research Council, 2000).
- Problems require students to apply understanding versus apply step-by-step algorithms or unrelated calculations (Wiggins & McTigue, 2005; National Research Council, 2000); the focus is “doing with understanding” (Barron et al., 2000).
- Students learn from others' insights, articulating arguments will help students clarify their own thinking (Wiggins & McTigue, 2005). Support structures help focus students' attention, students learn to research the information they need to know (Torp & Sage, 1998).

The first research question is to determine if students’ reasoning skills, using TER, predicts academic achievement, as shown by TAKS Mathematics Test, when using TBC. Based on the studies of Yeh and Wu (1992) and Frisby (1992), there is a high correlation between critical thinking and academic performance. Yet the findings of the Third International Mathematics and Science (TIMS) study indicate the correlations between critical thinking and school achievement were not considered significant. Therefore, we believed these contradicting results require additional investigations in this area.

The second research question is: How does TBC affect academic achievement as shown by the TAKS Mathematics Test? Due to the above stated research, we believe academic achievement improves significantly with the use of TBC.

The third research question: How does the TBC environment affect students’ abilities to analyze, evaluate, make inferences, and reason inductively and deductively, as indicated by TER? We believe there is evidence to believe that the ability to analyze, evaluate, make inferences and reason inductively and deductively improves significantly with the use of TBC. We looked at overall results as well as test specific results; and when significant, we looked at the effect of gender and ethnicity.

## Method

*Design*

We examined the relationship between critical thinking and reasoning skills as determined by the TER and mathematical achievement as determined by TAKS Test for Mathematics to determine predictability. A regression was used to determine this relationship.

A quasi-experimental pretest-posttest control-group research design was used to test the hypothesis of the study that the use of TBC would affect critical thinking, reasoning skills and academic achievement.

*Participants*

186 students from a large urban/suburban school district in the Greater Dallas area enrolled in algebra or geometry were included in the study. The study involved four schools, a middle school and three high schools. Students ranged from 8$^{th}$ graders to 10$^{th}$ graders, and students from a gifted and talented program, to on-level students. Student's ethnicity varied as well as socio-economic standing. The study groups represented a cross section of students from a suburban/urban school district. Students were divided into two groups, a control and treatment group.

*Instrumentation*

The TER from California Critical Thinking Skills Test series was used to measure the dependent variable, ability to analyze, evaluate, make inference, and use inductive and deductive reasoning. Subsequent studies have been conducted to validate the test's usage (Facione, 1990a; Giancarlo, 1996; Ricketts, 2003). The theoretical framework for the critical thinking part of this study is supported by the Delphi study of Peter Facione (1990a). The TER is designed for test takers in secondary school or the first two years of post secondary education, and for adults of all ages in the general population. The TER is a 35 item multiple choice test that is administered in 50 minutes. The Flesch-Kincaid Readability Level is 6th Grade. No specialized content knowledge is required. Test questions engage the test-takers' reasoning skills using familiar topics and contexts. Different questions progressively invite test-takers to analyze or to interpret information presented in text, charts, or images; to draw accurate and warranted inferences; to evaluate inferences and explain why they represent strong reasoning or weak reasoning; or to explain why a given evaluation of an inference is strong or weak

The content validity refers in general to how the specific items comply with two standards: (a) if the items represent the entire set of possible test items within a specified domain and (b) if 'sensible" methods of test construction are employed. TER complies with (a) standard of representation of all possible test items since every item of TER was carefully selected for its theoretical relationship to the Delphi Critical Thinking conceptualization. In regard to standard (b) the appropriateness of a multiple-choice test format to measure critical thinking must be established. Authors of measurements texts agree that higher order cognitive skills can validly and reliably be measured by well-crafted multiple-choice items (Haldyna, 1994: 28). Most of the critical thinking assessment experts who participated on the Delphi panel (Facione, 1990a) agree to this point. Furthermore, the construct validity of TER is grounded on the results of relative research indicating that it is strongly correlated (0,766) with "California Critical Thinking Skills Test" (CCTST) (Facione et al., 2002; 1990b; 1990c; 2001).

In the test manual (Facione, 2001, p. 16) four separate Kuder-Richardson 20 coefficients are presented as a statistical evaluation of the internal consistency of TER derived from four different samples which range from 0,72 to 0,89 (N=145, KR-20=0,78; N=201, KR-20=0,76; N=582, KR-20=0,72; N=113, KR-20=0,89).

The TAKS Mathematics Tests were used to determine mathematical achievement, another dependent variable. Content validity for this instruments is reviewed each year and includes an annual educator review, revision of all proposed test items before field-testing, and a second annual educator review of data and items after field-testing. In addition, each year panels of recognized experts in the fields of mathematics meet in Austin to critically review the content validity of each of the high school level TAKS assessments to be administered that year. This critical review is referred to as a content validation review and is one of the final activities in a series of quality-control steps designed to ensure that each high school test is of the highest quality possible. For internal consistency, the Kuder-Richardson Formula 20 was used with reliabilities in the.80s and.90s (Texas Education Agency, 2008). The standard error of measurement was calculated "using both the standard deviation and reliability of test scores and represents the amount of variance in a score resulting from factors other than achievement" (Texas Education Agency, 2008).

*Treatment*

The Experiment Group received instruction in a self-enclosed classroom where each student had access to a portable laptop. The students followed the curriculum as provided by the on-line program which was problem-based, modeled problem-solving strategies and heuristic thinking, and used concepts as tools. The curriculum followed the state mandated Texas Essential Knowledge and Skills objectives. Students were tested using TER during the first six week session and tested again using TER during the sixth six week session.

The Control Group also received instruction in a self-enclosed classroom according to the district curriculum that followed the state mandated Texas Essential Knowledge and Skills objectives. Teacher directed lessons were followed by guided practice that was then taken home to be completed as homework.

The TAKS test was given to both groups of students according to their grade. The ninth graders took the TAKS mathematics test for ninth graders and the tenth graders took the TAKS mathematics test for tenth graders in April of 2007.

*Data Analysis Procedure*

When evaluating the TER and the TAKS Test scores for predictability, a regression was used. The data collecting procedures described in the previous section were analyzed using Analysis of Covariance procedures in which the dependent variable was critical thinking and reasoning skills: analysis, evaluation, inference, deductive, and inductive reasoning; and academic achievement. The Control Group served as the covariant. The ninety-five percent confidence level ($p$ <.05) is the criterion for determining statistical significance. The criterion level for educational significance is one-third a standard deviation ($d$ = 0.33) (Cohen, 1988).

## Results

The first research question in study asks: Do students' abilities to analyze, evaluate, make inferences, and use inductive and deductive reasoning as indicated by the TER predict student achievement as shown by the TAKS 9$^{th}$ Grade Mathematics Test when using a TBC. The study tested the following directional research hypothesis: Students' abilities to analyze, evaluate, make inferences and use inductive and deductive reasoning as indicated by the Test of Everyday Reasoning can predict student achievement as shown by the TAKS test when using a TBC. The results obtained when the data relevant to this hypothesis were analyzed using a regression are shown below.

Table 1.
*Regression Analysis Summary for Test of Everyday Reasoning Predicting Texas Assessment of Knowledge and Skills Test*

| Variable | *B* | *SEB* | β |
|---|---|---|---|
| Ninth Grade Mathematic Texas Assessment of Knowledge and Skills Test | .58 | .08 | .54 |

Note: $R^2$ = .29 ($N$ = 144, $p$ < .001)

Students' ability to analyze, evaluate, make inferences, use inductive and deductive reasoning as shown by the TER significantly ($p <$ .01) predicted academic achievement as shown by the TAKS 9th Grade Mathematics Test, β= -.54, $N$(144) = 7.56. The change in TER scores also explained a significant proportion of variance, $R^2$ = .29, $F$(1, 143) = 57.20, $p$ < .01.

The second research question posed by the study: How does TBC affect academic achievement as shown by the TAKS Mathematics Test? The study tested the directional research hypothesis: Student's academic achievement as shown by the TAKS Mathematics Test showed statistically significant improvement after using TBC. The third research question in study asks: How does the TBC environment affect students' abilities to analyze, evaluate, make inferences, and reason inductively and deductively, as indicated by TER? The study tested the following directional research hypothesis: Students' abilities to analyze, evaluate, and make inferences, use inductive and deductive reasoning, shown by the Tests of Everyday Reasoning indicate statistically significant improvement after using TBC. The results obtained when the data relevant to this hypothesis were analyzed are shown below in Table 2 and Table 3.

Table 2.
*Mean Scores, Standard Deviations, and Analysis of Covariance (ANCOVA) for Measures of Mathematical Achievement and Test of Everyday Reasoning*

| | Control | | | | Experimental | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Pretest | | Posttest | | Pretest | | Posttest | | ANCOVA | |
| | *M* | *SD* | *M* | *SD* | *M* | *SD* | *M* | SD | *F*(1, 183) | *d* |
| TAKS Mathematics | 38.21 | 6.31 | 40.78 | 5.51 | 39.92 | 7.84 | 43.12 | 6.33 | 90.42** | .37 |
| TER | 14.85 | 8.10 | 16.79 | 7.72 | 20.02 | 5.86 | 21.65 | 6.23 | 21.14** | .16 |

Note: TAKS = Texas Assessment of Knowledge and Skills Mathematics Test.
TER = Tests of Everyday Reasoning. **$p$ < .001

As shown in Table 2, the Analysis of Covariance yields an F- ratio of 90.43 for Texas Assessment of Knowledge Skills that was statistically significant ($p$ < .001) and had an effect size ($d$ = +0.37) that is considered educational significant. The analysis of covariance also yields an F score of 12.14 for the TER that was statistically significant ($p < .001$) and has an effect size ($d = +0.16$) that is considered small (Cohen, 1988). Both ethnicity and gender were analyzed for statistical significant and none were found.

Table 3.
*Mean Scores, Standard Deviations, and Analysis of Covariance (ANCOVA) for Groups Tested for Measures of Critical Thinking and Reasoning Skills*

| | Control | | | | Experimental | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Pretest | | Posttest | | Pretest | | Posttest | | ANCOVA | |
| | *M* | *SD* | *M* | *SD* | *M* | *SD* | *M* | SD | *F*(1, 174) | □ |
| Analysis | 4.32 | 1.74 | 4.94 | 2.19 | 5.43 | 1.74 | 5.80 | 1.66 | 12.42* | .09 |
| Inference | 6.53 | 3.39 | 8.59 | 3.17 | 8.53 | 2.94 | 9.43 | 2.86 | 28.90** | .37 |
| Evaluation | 4.05 | 2.96 | 4.16 | 3.05 | 6.04 | 2.18 | 6.39 | 2.39 | 1.87 | .02 |
| Deduction | 8.21 | 4.59 | 9.27 | 4.07 | 10.45 | 3.69 | 11.47 | 3.54 | 19.25** | .37 |
| Induction | 6.81 | 3.83 | 7.46 | 3.84 | 9.53 | 2.69 | 10.22 | 3.16 | 10.56* | .15 |

* $p < .01$ ** $p < .001$

A descriptive analysis as part of the Multivariate Analysis of Covariance was conducted to evaluate the hypothesis that scores improved for the TER's subtests, after using TBC. The dependent variable was the change in TER subtest scores before and after the use of TBC. The ANOVA was significant for subtests: Analysis $F(1,174) = 12.42$, $p < .01$, Inference $F(1,174) = 28.90$ $p < .001$, Deduction $F(1,174) = 19.25$, $p < .01$, and Induction $F(1,174) = 10.56$, $p < .01$. The subtest Evaluation did not show significance. The strength of relationship between the treatment subtests and the change in scores, assessed by □ was strong for Inference accounting for 37% of the variance of the dependent variable and Deduction accounting also for 37% of the variance of the dependent variable. The strength of relationship was moderately strong for Analysis accounting for 9% of the variance variable and Induction also accounting for 15% of the variance of the dependent variable. The strength of relationship was low for Evaluation accounting for 2% of the variance variable.

## Conclusions and Discussions

The findings of this study suggest that a TBC environment can stimulate critical thinking and reasoning skills and improve academic achievement. Of the specific skills studied, evaluation, perhaps the most difficult to achieve and measure, needs additional support in a TBC. It is believed by the researchers that more rationalizing through discourse within the context of the problems would affect student's ability to evaluate and function at a high-order thinking level. Nevertheless, this study suggests academic achievement improvement can occur in a TBC environment that is educationally significant, and the authors believe this learning environment deserves further study and emphasis in a growing technological world.

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# USING TECHNOLOGY TO SUPPORT COGNITIVE ACTIVITIES AND TO EXTEND COGNITIVE ABILITIES: A STUDY OF ONLINE MATHEMATICS LEARNING

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*This study investigates secondary students' technology use while solving mathematics problems in computer environments. The students explored their math problems in an online dynamic mathematics environment and performed all solution steps in the computer environment. Their work were recorded by using a screen capturing software and analyzed in terms of using technology as a cognitive tool, more specifically using technology as a partner and as an extension of self. It is documented that the students participated in this study used technology in both ways and that these two types of use are inversely related.*

## Introduction

Recent innovations in technology challenge researchers, math educators, and policy makers with new ways of thinking about mathematics teaching and learning by using technology. In a report prepared for the Ontario Ministry of Education, the role of technology in mathematics classrooms is described as "enabling easier communication, providing opportunities to investigate and explore mathematical concepts, and engaging learners with different representational systems which help them see mathematical ideas in different ways" (Suurtamm & Graves, 2007, p. 50) Their report proposes to connect practice and research in Ontario context and projects onto the global. The focus of their description lies on the use of technology in investigation and exploration of mathematics which is rather challenging in traditional environments. In other words, they describe technology as a tool to extend the current learning opportunities. They also point out to improve students' understanding of mathematical ideas by "different representational systems" such as symbolic and visual representations and the connection between them. The connection between various representations is known as linked representation since early 90s (Kaput, 1992).

Similarly, the Ontario Ministry of Education (2007) delineates the use of technology by suggesting: "students can use calculators and computers to extend their capacity to investigate and analyse mathematical concepts and to reduce the time they might need otherwise spend on purely mechanical activities" (Ontario Ministry of Education, 2007, p. 19). It appears that technology in their description is conceived as a tool to extend students' abilities on the tasks which are challenging or impossible in paper-and-pencil environments. These tasks could be to perform complicated arithmetic operations or to draw complex graphs.

Moreover, the importance of various representations and the connection among these representations are addressed as a goal:

> Representing mathematical ideas and modeling situations generally involve concrete, numeric, graphical, and algebraic representations. Pictorial, geometric representations as well as representations using dynamic software can also be very helpful. Students should be able to recognize the connections between representations, translate one representation into another, and use the different representations appropriately and as needed to solve problems. … When students are able to represent concepts in various ways, they develop flexibility in their thinking about those concepts" (Ontario Ministry of Education, 2007, p. 21).

However, the practice of these recommendations is rather challenging. Some of these challenges are: “limited access to technology, lack of teacher understanding or confidence in using technology, and a lack of technology leadership in some school boards” (Suurtamm & Graves, 2007, p. 55). As seen in the report, the main focus is still on the technology use because limited access to technology could be solved in time. However, teachers’ concern about the use of technology seems to remain a challenge for awhile. Teachers in school experience difficulty in understanding the proper way of using technology and in connecting the technology to the current topics of mathematics.

These recommendations and teacher concerns lie on one side of the coin. What about the other side of the coin? How do students cope with technology and more importantly, how do they use technology in mathematics? Despite the theoretical and practical concerns in integrating technology into mathematics education, students widely use technology in their daily life with an increasing rate. Because these students were born in the information age, they are confident enough in using technology; and even they have no idea about a life without technology, say internet and computer. There is no doubt that they can use technology effectively, and many studies document that they use technology as anticipated (Artigue, 2002; Izydorczak, 2003; Karadag & McDougall, 2008; Kieran, 2007; Kieran & Drijvers, 2006; Lagrange, 1999; Moreno-Armella & Santos-Trigo, 2004; Moyer, Niexgoda, & Stanley, 2005). However, the question is whether or not they use technology in mathematics when they are alone. A further question could be if they use technology as recommended in the literature.

Therefore, we designed this study to track students’ technology use while they are solving mathematics problems. The research question of this study was to explore how secondary school students use technology while solving mathematics problems. The following sections provide a theoretical framework on the use of technology, a detailed description of the study, a detailed presentation of the data and the results, and a discussion of the results and the research.

## Theoretical Framework

Research and theoretical discussions about the use of technology in mathematics education have been mainly focused on the cognitive tools (Galbraith, 2006; Pea, 1985; 1993; Perkins, 1993; Salomon, 1985; 1993; Sweller, 1999; 2003). Salomon (1993) describes the effective use of technology as “a characterization of computer tools (as contrasted with more independently functioning programs and intelligent tutors) as not being autonomous in their operation, thus requiring the active operation by and mental involvement of students” (p. 193). Many scholars believe that computers change “how effectively we do traditional tasks, amplifying or extending our capabilities, with the assumption that these tasks stay fundamentally the same” and some others assert that “a primary role for computers is changing the tasks we do by reorganizing our mental functioning, not only by amplifying it” (Pea, 1985, p. 168). In another paper, Pea (1993) states that “whereas amplification suggests primarily quantitative changes in accomplishments, what humans actually do in their activities changes when the functional organization of that activity is transformed by technologies” (p. 57).

Salomon (1993) identifies two types of computer use in education: performance-oriented tools and pedagogic tools. The performance-oriented tools are used as intelligent partners, and “cognitions become 'distributed' in the sense that the tool and its human partner *think jointly*” (Salomon, 1993, p. 182). He seems to assume that computers have thinking mechanisms such as artificial intelligence and criticizes this type of cognition share. He argues that this way of cognition share will increase students' performance during the partnership and that no cognitive residue and improvement in students' abilities will be obtained when the technology is removed. That is why he calls the outcomes obtained from this type of technology use as the "effects with technology" (Salomon, 1993; 1985).

Furthermore, he describes the pedagogic tool as an "intellectual partnership of the division of labor" (p. 182) so that computer does "tedious, labor and memory intensive lower level processes that often block higher order thinking" (p. 181) and leaves higher order thinking such as drawing conclusion, making conjectures, reasoning, thinking causal relationships, and testing conjectures to the learner. Any labor can be shared but thinking should not be shared with any type of tools. He also argues that performance-oriented use can be regarded if a cognitive residue in the human partner is left. Additionally, he suggests that the performance-oriented tools must allow their human partners to function at a higher level and to enter increasingly higher levels of new partnership with themselves at later stages.

The use of virtual manipulatives falls into the pedagogic tools whereas the use of Computer Algebra Systems (CAS) as a black box could be an example of the performance-oriented tool. However, the use of CAS to create patterns and to make numeric and symbolic relationships explicit has to be considered as an example of the pedagogic tool.

Pea (1993) criticizes the contemporary practice of technology use by arguing “many schools, technology developers, and researchers now use technologies to ‘enhance’ education by making the achievement of traditional objective more efficient” (p. 71). He also states that “objectives for education are not reconceptualized; the computer is conceived as a means for ‘delivering’ key components of instructional activity not for redistributing intelligence and new uses of students’ potentials for activity and participation” (Pea, 1993, p. 71). He seems to suggest more advanced use of computer than performing only complicated numeric and symbolic calculation or drawing graphics. Rather than using computers in labor and time intense purposes only, he suggests incorporating computers into our thinking processes. Dynamic learning tools allowing multiple representations and a linkage among these representations fall into this category because they visualize mathematization and make mathematics explicitly visible for students.

Galbraith (2006) provides a more general framework for the use of technology and identifies four types of technology use in mathematics education: technology as a master, technology as a servant, technology as a partner, and technology as an extension of self. The first category, technology as a master, points out the use of technology as a black box, meaning that students do not know why a certain outcome appears and use what is provided by technology as it is. The second type, technology as a servant, describes the use of technology as a tool to save time and make the things happen easier. The third type of use, technology as a partner, includes the use of technology to support the user’s cognitive activities such as keeping some information in databases and serving it when it is needed and correcting typing or calculation mistakes. The use of technology as extension of self is the fourth type and describes to use technology to extend students’ cognitive abilities and to improve their understanding.

In this study, we followed the framework provided by Galbraith (2006) and focused on the last two types of technology use. These two types of technology use, such as a partner and as an extension of self, fit into the definitions provided by Salomon (1993) and Pea (1993).

## Methodology

In this qualitative research, five Grade 12 students were recruited, and pseudonyms are used instead of their original names. We trained the students on how to use software, Geogebra and

Wink, via email conferencing. The Geogebra is a free, online, and user-friendly software, and the students used Geogebra to explore their mathematics problems. The Wink is also a free and user-friendly software, and the students used this software to record their problem solving processes performed in computer environments.

The students were given four assignments each containing four open-ended problems to be explored in the Geogebra. They recorded their problem solving processes by using Wink. Once they recorded their solutions, they put them on CDs and delivered to the researchers. We used the frame analysis method to analyze the data gathered in the study. The frame analysis method is a microgenetic approach aiming to focus on the development of a task performed in a very short period of time. Since we asked students to set up the software recording rate as two frames per second (24 frames per second in regular video recording), we were able to focus on each half second of their problem solving.

The study summarized in this paper is part of a larger study. The data was analyzed both qualitatively and quantitatively in the main study. However, since the focus of this paper is related to the technology use, we present mostly quantitative results and refer qualitative results in the main study as needed.

## Results

As a first step in the frame analysis method, we described and interpreted students actions performed on the computer screen. We performed two types of analyses on these descriptions. First, we documented the number of their technology use as a partner in each assignment, and then we calculated the period of their technology use as an extension of self. These analyses and the results obtained from analyses are explained in detail in the following sections.

*Use of Technology as a Partner*

The use of technology as a partner means that technology is used to provide some specific information when needed, to correct writing and calculation errors, or to extend the student's memory. We use this term to describe the use of technology to record information such as using a notepad, to keep information in a particular format such as using Geogebra as a picture, and to benefit from specific features of computer and software such as spell check and auto-correction.

We developed a specific index to describe the level of technology use as a partner. We counted the number of times the students were using these features in an assignment and divided this number by the number of the frames of the assignment to create the index. Since the numbers were very small, we multiplied them by 1000 to round them off to the nearest integer. Thus, the index of technology use as a partner is defined as:

$$I = \frac{\text{The number of the technology use as a partner in an assignment}}{\text{Total number of frames in that assignment}} x1000$$

After analyzing the data according to the formula, we tabulated the results to see the full picture of the data (table 1).

Table 1. *The Index of Technology Use as a Partner*

| | Teresa | Ursula | Ru | Andrea | Nancy | Average |
|---|---|---|---|---|---|---|
| Assignment 1 | 5.18 | 7.76 | 0.8 | 6.54 | 6.65 | 5.39 |
| Assignment 2 | 4.46 | 6.24 | 5.68 | 5.39 | 8.26 | 6.01 |
| Assignment 3 | 2.06 | 1.84 | 2.48 | 6.92 | 2.16 | 3.09 |
| Assignment 4 | 4.62 | 2.73 | 3.07 | 6.51 | 6.53 | 4.69 |

The lower values of index denote less use of technology as a partner whereas higher values mean more frequent use. The less use of technology as a partner could be caused by various reasons including using in a more advanced way and being unfamiliar to use technology in mathematical purposes. For example, the averages of the assignments illustrated in the last column reveal that the index values vary from 3.09 (third assignment) to 6.01 (second assignment). It appears that these students used technology as a partner more in the second assignment than the third one. This difference could be explained by the difficulty level of assignments. The third assignment contained more challenging problems and demanded more exploration. This result may suggest that the index of technology use as a partner could be task-dependant.

Teresa’s and Andrea’s patterns seem to be steadier whereas Nancy’s pattern has more fluctuations. It can be interpreted that Teresa’s and Andrea’s number of use of technology as a partner remains similar whereas Nancy’s use varies with assignments. There is a fall in the Ursula’s index values in the third and fourth assignments. This fall could be explained by her familiarity with the medium and becoming less dependant to this type technology use as she gets experienced.

The data in the first assignment illustrates that Ru’s index is considerably lower than others’ indices. Interestingly, Andrea’s index in the third assignment is the second unusual case and considerably higher than others’. In the next section, we will provide a possible explanation for these unusual sharp changes in the patterns.

*Use of Technology as an Extension of Self*

Galbraith (2006) describes the use of “technology as an extension of self” as “the partnership between technology and student merging to a single identity” (p. 286) which is highest intellectual way to use technology. This type of technology use extends the user’s mental thinking and cognitive abilities because technology acts as a part of user’s mind. For example, linked representation (Kaput, 1992) between symbolic and visual representation could be a relevant example for this type of use because manipulations done in one of the representations affect the others.

In our study, the features of Geogebra serve this type of use and extend students’ cognitive abilities to beyond what they usually do in symbolic and paper-and-pencil environments. This type of use is a continuous action, occurs in a period of time, compared to the use as a partner, and it is quite easy to state accurately when the action starts and ends. That is why we measured the time spent and took the percentage of this period of time with respect to the total time spent in that assignment by a particular student to create a new index.

$$P = \frac{\text{time spent while using technology as an extension of self}}{\text{time spent in that certain assignment}} x100$$

As a result, we created a table to see the full picture of the data gathered (table 2).

Table 2. *The Percentage of Technology Use as Extension of Self*

| | **Teresa** | **Ursula** | **Ru** | **Andrea** | **Nancy** | **Average** |
|---|---|---|---|---|---|---|
| Assignment 1 | 7.7 | 10.7 | 16.9 | 12.9 | 17.5 | 13.14 |
| Assignment 2 | 11.5 | 12.7 | 14 | 16.4 | 12 | 13.32 |
| Assignment 3 | 27.4 | 22 | 23.1 | 15.9 | 34.8 | 24.64 |
| Assignment 4 | 25.8 | 34.1 | 31.8 | 24.2 | 33.3 | 29.84 |

The table provides an overview and a general idea about results and reveal that the percentage derived from the data is both person- and assignment-dependant. For example, a particular student having a lower index in a row may have a higher index in the other rows comparing to her friends.

The student who has a high index value in a certain assignment uses technology as an extension of self for a longer period in that particular assignment. The reasons affecting this longer use could be students' familiarity with using technology and confidence in technology use and task's demand for exploration. Students' familiarity in technology use seems to have a significant effect because the average of the index values increases with the number of assignment. In other words, the students used technology longer in later assignments.

It appears that the students all used the technology in a similar way in the second assignment because the pattern remains steady. The reason could be the nature of the problems because the problems in this assignment do not require advanced use of technology. The indices in the first and second assignments are smaller than the others. This result is reasonable because the students were becoming more experienced, and they may have become more advanced users of technology in the third and fourth assignments. The percentages of Teresa and Nancy in the third assignment are higher than others. This means that they started to use the technology earlier than others and at a higher level. Ursula's use seemed low at the beginning but she made a sharp increase in the last assignment. Andrea demonstrates an increase only in the fourth assignment. It appears that assignment four appears is the one in which technology is used as an extension of self, due to students experience and they moved to a stage where they could use technology for more advanced purposes.

When these two tables are compared to each other, it appears that the index of technology use as a partner could be inversely related to the percentage of the technology use as extension of self. This inverse relation seems unsurprising because the more students use technology for advanced purposes, the less they may use technology for novice purposes. That is, as they become confident with technology and familiar with its features, they may prefer using technology for more advanced purposes. This may explain Ru's and Andrea's unusual index values in technology use as a partner which is discussed in the previous section.

## Discussion

Despite the four ways of using technology in education by Galbraith (2006), the use of technology as a cognitive tool has been limited with two ways. These ways of using technology have been described as technology use to support cognitive activities and technology use to extend cognitive abilities (Galbraith, 2006; Pea, 1985; 1993; Salomon, 1985;1993).

In the study, I documented three ways of technology use, two of them being cognitive tools. The first type, which is not related to cognitive activities, is to use technology for writing, deleting, and drawing. This type of use is identified as using technology as a servant (Galbraith, 2006) and was not considered related to the study.

The second type, technology as a partner (Galbraith, 2006), was documented while students were using technology for elementary purposes. These elementary purposes include taking notes, storing notes and images, and spell check feature of the software and served to support the working memory of the students. Sweller (1999; 2003) explains this type of memory support by cognitive load theory. He analyzes the cognitive capacity of working memory and the reasons increasing the cognitive load of the memory and suggests techniques to increase the capacity or to decrease the cognitive load. For example, he suggests that using long-term memory or cognitive tools instead of storing all information in the working memory. The students' use of computer features to store information falls into this category.

The third type of technology use documented in the study was to use technology as an extension of self (Galbraith, 2006). The students used Geogebra to extend their cognitive capacity of imagination, to create mathematical objects, and to manipulate these objects. This type of using technology affected their "nature of understanding mathematics" (Heid, 2005, p. 357) because students perceived graphs and points as living bodies. Instead of describing their properties as a static image or concept, they were describing them as if these objects were moving bodies. Moreno-Armella, Hegedus, & Kaput (2008) identify this effect as an evolution in students' mathematical thinking.

In conclusion, students use technology in various ways. However, we can categorize the ones related to cognitive activities in two main groups based on their roles. These roles include use of technology to support cognitive activities (technology as a partner) and to extend cognitive abilities (technology as an extension of self) and were found inversely related with the other.

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# A COMPARISON OF MATHEMATICAL DISCOURSE IN ONLINE AND FACE-TO-FACE ENVIRONMENTS: INVESTIGATING CONJECTURES

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*This report is from a study that investigated the similarities and differences of student mathematical discourse in online and face-to-face course delivery systems. Here we focus on similarities and differences between patterns of student mathematical discourse while investigating conjectures. Students proposed their mathematical conjectures and solutions similarly in both environments. However, the rigor of their investigations after such proposals contrasted greatly, with the face-to-face students deliberating and further developing their mathematical ideas more often and in more meaningful ways than the online students. We discuss the implications of these findings for online mathematics education.*

In education today, there are generally two ways students take courses: distance education or face-to-face education. Until recently, the portion of students using distance education has been small. This portion has grown tremendously, however, with the advent of the Internet. Vasarhelyi and Graham (1997) found that in 1993 there were only 93 schools devoted to online learning, but in 1997 the number had jumped to 762. Engelbrecht and Harding (2005) estimated that the e-learning market went from $10.3 billion in 2002 to $83.1 billion in 2006, and projected that it will eventually swell to over $212 billion by 2011.

The growth of online education is due in large measure to its advantages over face-to-face education, as it provides "unique alternatives for reaching larger audiences than ever before possible… traditional or non-traditional, full-time or part-time, and international—who perhaps have had limited access to advanced educational opportunities" (Bartley & Golek, 2004, p. 167). Bartley & Golek further reported that online classes, degrees, and certificates are especially valuable to those with demanding work, family, and social schedules. In addition, online education allows students to spend less time in class, less money on travel, and more time learning at home (Beard, Harper & Riley, 2004).

Although there are many advantages to online courses, there are disadvantages as well. According to Coates, Humphreys, Kane, and Vachris (2004), younger undergraduate students who were found to lack the technological skills and discipline necessary to survive and participate in online courses felt inundated with course demands and soon dropped out. They found that the overall student dropout rate was higher for online courses. Piotrowski and Vodanovich (2000) pointed out that piracy issues are a disadvantage to the online environment because when students submit their work online, there is no way to verify if they are the ones who actually did the work. They also found that there were times when technological problems prevented access to the course materials, pointing out that these issues of piracy and course access often became more of a focus for students and teachers than course content.

## Purpose

Researchers have studied the effectiveness of distance education primarily through measuring student success (i.e., midterm test scores, final test scores, final grades). Such studies tend to report that there is no significant difference with regard to student success between online or face-to-face modes of delivering instruction (e.g., Akkoyunlu & Yilmaz Soylu, 2004; Aragon, Johnson, & Shaik, 2002; Brown, Stein, & Forman, 1996; Cooper, 2001; Ellis, Goodyear, Prosserz, & O'Hara, 2006; Russell, 1999; Smith, 2004). Such studies, however, look almost exclusively at courses that are taught quite traditionally. Traditional face-to-face courses typically feature a teacher lecturing to their students. Afterward, the students go home, do the homework, and learn the mathematics with their notes and book. This format is similar to that of traditional online courses which begin with instruction for the student through a video recorded lecture or other media. Then, after they receive the instruction, the students do the homework, and learn the math with their notes and book. Thus, the role of the student remains the same, regardless of the environment. There is little wonder, therefore, that when researchers compared course grades, the differences between online and face-to-face learning were insignificant—students in both environment were learning the material in essentially the same way.

Consistent with postmodern epistemologies such as social constructivism, reform educators seek to organize their classrooms and lessons in ways that facilitate and make explicit students' construction of mathematical knowledge (Ball, 1993; Lampert, 1990). In such classrooms, students are given opportunities to solve meaningful mathematics problems and to discuss with their peers their solutions and interpretations. This discourse is seen as an intrinsic part of the learning process. Social constructivism has had a huge impact on the way educators envision effective face-to-face classrooms, but its impact on the vision of online education is still largely unknown. Although live chat, email, and discussion boards are avenues that some have used to discuss mathematical ideas and resolve conflicts, there is still relatively little known about the extent to which such avenues have the potential to moderate meaningful mathematical discourse. The purpose of this study was to characterize and to compare the discourse of students as they discussed mathematical tasks online and face-to-face. We used these findings as one way to evaluate the online medium as an alternative to the face-to-face environment. This paper focuses on one particular aspect of student discourse—the investigation of conjectures.

## Theoretical Framework

The NCTM *Professional Teaching Standards* (1991) outlined a vision of the role that students are to fulfill as they discuss mathematics in a reform classroom environment:

> The teacher of mathematics should promote classroom discourse in which students—[1] listen to, respond to, and question the teacher and one another; [2] use a variety of tools to reason, make connections, solve problems, and communicate; [3] initiate problems and questions; [4] make conjectures and present solutions; [5] explore examples and counterexamples to investigate a conjecture; [6] try to convince themselves and one another of the validity of particular representations, solutions, conjectures, and answers; [7] rely on mathematical evidence and argument to determine validity. (NCTM, 1991, p. 45).

In this paper, we focus on how and how often students explored conjectures and solutions in online and face-to-face environments. For us, student mathematical discourse consists of the dialog and mannerisms students use as they converse about and solve mathematical tasks. Additionally, student mathematical conjectures consist of any comment from a student submitted to the group for consideration as a proposal for a part of or the entire solution. In order to understand how students investigated conjectures, we first needed a way to recognize how their conjectures were initiated. As students discussed mathematics, whether online or face-to-face, they began by initiating the problems in the task. This initiating was done in a couple of ways. Sometimes students asked for initial thoughts on how to approach the problem. At other times students first worked on the problem individually then came together to discuss their initial work. After this initiation, students had debates over or gave explanations of how the task could be solved. We revered to this initiation as the making of conjectures and to the debating or explaining as investigating conjectures. Eventually the students agreed on a solution, often through yet more investigation. We referred to this latter stage as making and investigating solutions.

Face-to-face students have the opportunity to initiate problems by negotiating the direction they would take in a mathematical task quickly because they are together in the same room at the same time. They can converse about their initial ideas for the solution of the problem by showing their work and working out their differences in real time. They can make, explain, and negotiate conjectures so all can understand and agree. Eventually, a common idea would surface, which the group takes as their solution. Face-to-face students usually carry out this process one problem at a time, in one sitting, until the task is complete.

In the online environment, all the above things can be done, but it all takes more time to develop. Students are not all together at the same time. Therefore, they intermittently attend to the same process the face-to-face group does in order to complete the task; each phase of the process thus necessarily takes longer to complete. In addition, students usually make and independent decision as to how much of the task they want to complete in the first post to the group. They must achieve a balance of covering enough information, but not too much, in order to have a cohesive discussion. If the right amount of material is proposed as a conjecture for discussion, then the next stage of making and investigating conjectures can go well. In addition, students can formulate explanations and arguments to prove their conjectures as they type out their post before they are entered into the conversation. This formulation thus has the potential to be more coherent and less spontaneous than face-to-face discourse.

Our guiding question for this paper is as follows: What are the characteristics of how students make and investigate conjectures and solutions in the online and face-to-face environment and how do these characteristics compare?

## Methodology

This study compared the mathematical discussions of two lab sections of an Introduction to Calculus class at a large university in the Western United States. One lab section was conducted online and the other face-to-face. Each lab section was composed of about 25 students. The students were divided up into groups of five and given tasks to complete. We collected data from the students' work on three tasks that dealt with derivatives. The online students discussed the task questions using a discussion board. The conversations from the discussion board were saved for analysis. The face-to-face students discussed the task questions in a classroom. Their discussions were video recorded and transcribed for analysis.

We first analyzed the online data for evidence of when students initiating and investigating conjectures and solutions. We noticed and recorded a number of patterns during this initial pass through the data. We built a set of codes from these patterns and then returned to the data and coded them according to those patterns. For example, one such pattern was that when a conjecture or solution was presented, the subsequent comments, or investigations, tended to either state agreement or state disagreement, often in the form of a new conjecture based off the old one. The most apparent characteristic of the investigations that agreed with the initial conjectures was that usually there was no mathematical evidence given to back these claims. We therefore, looked at all cases of initiating and investigating conjectures and analyzed them according to the degree to which mathematical evidence was used to back them up. We used a similar approach to analyze the face-to-face data, using the codes derived from the online data and adding to them as needed.

## Results

*Characterization Of Online Students Investigating Conjectures And Solutions*

Approximately 70% of students' mathematics discussions were coded as either initiating or investigating conjectures or solutions, about evenly distributed between initiating and conjecturing. In only 29% of those investigations did students use mathematical evidence to support or refute the preliminary conjecture or solution. There were three primary approaches that students took when they investigated a conjecture or solution: (1) students posted that they agreed with the mathematical idea, (2) students posted that they disagreed with the mathematical idea, or (3) students posted their own conjecture (whether it be the same or different) without agreement or disagreement with the current conjecture. In each approach students either backed up their response with mathematical evidence or they did not. We now describe each of the three primary approaches according to whether students used mathematical evidence.

*Without mathematical evidence.* Forty percent of the investigations students did online stated that they agreed with the initial conjecture/solution without showing mathematical evidence. They would post single comments like, "I got the same answer!" or "Everything is correct for Parts A and B." In essence students agreed but gave no evidence of *why* they agreed.

The next largest category of how students investigated conjectures without mathematical evidence was when students replied to a conjecture with a new conjecture without stating whether they agreed or disagreed. This was done 20% of the time. For example, a response like, "I think the answer is 48," was common in this category. Although the "I think" provides some evidence that this student was aware that their answer of 48 was different from the previous post, the response gave no indication of why or how the student came to the solution. Other responses in this category merely gave a different solution or conjecture, with no mathematical evidence to back it up, but also with no evidence that the student was even aware of the previous post.

About 11% of the time students' posts were explicit about the fact that they disagreed with a previous post, yet did not provide mathematical evidence for the correctness of their response or the incorrectness of the prior post. For example, one post stated, "Hey! I got almost the same answers as you but I got a 6 instead of an 18 in 1b, but I could be wrong." The post ended here, with no discussion of how the student arrived at "6" or how they thought the other student may have arrived at "18".

*With mathematical evidence*. The most frequent investigation approach that included mathematical evidence was when students offered a new conjecture for the group to consider in place of the old one, without stating explicitly whether they agreed or disagreed with the initial conjecture. They did this 18% of the time. For example, students occasionally borrowed new conjectures from other groups, prefacing their posts with statements like, "Here is how someone else went about doing these problems." Such posts often included some mathematical evidence to back them up, but did not indicate whether the borrowed response agreed or not with the groups' current strand (nor, by the way, did such posts indicate to what extent the borrower agreed with the borrowed conjecture).

Ten percent of the time when students investigated conjectures, they used mathematical evidence to demonstrate why they agreed with the initial conjecture. For example, a student might respond to the initial conjectures from the group by stating, "That is the right way to do it," then go on to elaborate on the procedure and reiterate the perceived correct answers. Finally, there was only one instance in the online discourse where a student explicitly disagreed with an initial conjecture and then used mathematical evidence to back up their argument.

*Characterization Of Face-To-Face Students Investigating Conjectures And Solutions*

Approximately 60% of students' face-to-face discourse was coded as initiating or exploring conjectures and solutions, with exploring making up about 60% of these instances. In 53% of these group explorations, the students used mathematical evidence to support or refute the preliminary conjecture or solution. When students investigated a conjecture or solution, they: (1) agreed with the conjecture or solution without discussing the mathematical evidence, (2) gave clarifications to a conjecture or solution, (3) gave additional explanations to further illustrate conjectures and solutions, (4) agreed with a conjecture or solutions and showing mathematical evidence to support, (5) disagreed with showing mathematical evidence, and (6) disagreed without showing mathematical evidence.

*Without mathematical evidence*. Face-to-face students agreed with proposed conjectures and solutions about 44% of the time without using mathematical evidence. It was usually a brief assessment of the conjecture/solution. The students would usually listen to a conjecture and then think about its correctness. The conjectures made were usually mathematically correct. Usually, all that a student wanted to do was to understand the process, agree with it, and move on. Comments made by students that portrayed this type of investigation were: "Okay," "It sounded good to me," "I agree," "Right," or "Yeah."

The students in this environment disagreed with other's conjectures and solutions with out using mathematical evidence only a few times. For example, a group was working on a task that asked them to find the average rate people arrived on a beach from 3:45 to 4:00 p.m. The group was using an average rate formula given them. Kim began by simply stating her answer of 7.75. She did not say how she got it. Josh stated his as well, 14. He also did not say how he got it. Kevin stated another number considerably higher, 3258. Gary asked Kevin how he got that number and Kevin went into a procedural description of how he got it using operations without explanations. In the end, Gary seemed to have been successful in copying down the procedures Kevin used, but was still left with questions like why Kevin used certain values in the denominator of the formula and what they represented in the problem.

*With mathematical evidence*. The most frequent way (22% of the time) students in the face-to-face environment investigated conjectures using mathematical evidence was to make clarifications on proposed conjectures or solutions. There were times when certain students in the group did not understand where another student came up with their conjecture or solution. The students then investigated the conjecture further by making clarification statements or asking clarification questions. An example of this type of investigation began with Jayden making a conjecture about the proper procedure in a specific calculation pertaining to the problem:

Jayden: And then you would have to put 60 over point 5.

Gary: Did you put 60 times 2.5 or did you say point 25?

Jayden: Sixty over point 5. Oh, 16 times point 25.

After Jayden made his conjecture, Gary asked a clarification question. This caused Jayden to reexamine his conjecture and investigate it further. After he did, he realized that he had made an incorrect operation.

The next most frequent way (17%) students investigated conjectures face-to-face using mathematical evidence was to provide an additional explanation of the conjecture either for the benefit of themselves or another member of the group. An example that illustrates this came from when the group had been struggling to determine the average rate. They approached their average rate calculation by finding the average distance traveled divided by the time that elapsed over the distance change. They discussed several intervals to use for the time. Kim made a conjecture that they needed to focus on the meaning of the variables in their formula for average rate and reexamine their computations they used from it. Jayden gave the following additional explanation to show what they were not getting right and what the average rate time interval should be. “So, it’s going to be between 1.5 and 2. We were doing it between zero and 1.5. We need to be doing it between 1.5 and 2 cause that’s the last half-second [of the time interval].” The students were previously working with the incorrect interval and thus obtaining results that did not make sense (i.e., values that were obviously too fast or too slow). Jayden’s additional explanation, which investigated the conjecture to reexamine their computations, called the group’s attention to the error and its resolution.

Students in the face-to-face environment simply agreed with the group’s conjectures and solutions and followed their agreement with mathematical evidence 8% of the time. Students would make comments saying, “Yes, it is correct because…” and then explain why in response to a numerical answer.

With a frequency of 6%, students investigated conjectures by using mathematical evidence to back up a disagreement with the group’s or student’s conjecture or solution. Here is an example where Kim, Jayden, and Ashley were discussing a question regarding the path of a rocket. Kim was asking about the moment when the rocket reaches its highest point during the flight:

Kim: Ok, so is that 2 seconds?

Jayden and Ashley: Eight point five.

Ashley: Wait, that’s for [Question] 2, right?

Jayden: Yeah.

Kim: Why would you say it’s 8.5?

Jayden: Because the velocity is positive until that point. Because this is the graph of the derivative, not, it’s not a normal function. If it were a graph of distance over time, it would be one thing but it’s not showing the distance of the rocket, it’s showing the velocity.

Jayden and Ashley disagreed with Kim’s conjecture. Then Ashley made sure that Kim was on the same question that they were on. Kim asked why they thought the answer was 8.5 (rather than 2) and Jayden used mathematical evidence to back up his assertion.

## Discussion

With respect to mathematical discussion in the online and face-to-face environments, students in both environments investigated mathematical conjectures about the same amount of time. However, the subsequent discussion differed greatly. Table 1 outlines the similarities and differences in how the students investigated conjectures in each environment. Investigating conjectures is a vital component of effective mathematical discourse. Although the frequency of such investigations was similar for each environment, the nature of these investigations were significantly different. The 69% for the online environment and the 60% for the face-to-face environment only reflected the first investigation made by a student about a conjecture given previously. In the face-to-face environment 62% of the time conjectures were investigated they did so with more than one look, which means the idea was picked up by the group and deliberated. This uptake happened only half of the time in the online environment. The amount of mathematical evidence used in these investigations greatly favored the face-to-face environment: 53% to 29%. This means that even though students in both environments investigated their conjectures about the same amount of time, the investigations face-to-face were deeper and more mathematically rich, and more consistent with reform visions of mathematics classrooms (Ball, 1993; Lampert, 1990; NCTM, 1991; Yackel & Cobb, 1996).

Table 1
*Comparison of Online and Face-to-Face Mathematical Discourse—Investigating Conjectures*

| Discourse Aspect | Online Environment Frequency | Face-to-Face Environment Frequency |
|---|---|---|
| Investigating Conjectures | 69% | 60% |
| Further Investigations | 51% | 62% |
| Mathematical Evidence Used | 29% | 53% |

The nature of the environment seemed to significantly influence the degree to which students used mathematical evidence in their investigation of conjectures. The face-to-face students were able to have dynamic conversations and to show each other their mathematics as they made and investigated conjectures and solutions by pointing to their work and verbally discussing it. When students made and investigated conjectures online, they would have to type out all their work and all their discussions. It is not as easy to type out everything as it is to show it and discuss it verbally (see Ellis, 2001; Meyer, 2003; Tiene, 2000 for nuances).

Although the online environment certainly has the potential for productive mathematical discussion, there are some key differences in the ways students used (or did not use) this environment from the face-to-face environment. If students do not receive more guidance from the teacher in the online environment, their mathematical conjectures and solutions will not tend to be rooted well in mathematical evidence. Even with such intervention, the instantaneous and dynamic nature of face-to-face discourse will be difficult to replicate in the online environment. When the nature of mathematical discourse is considered, there are indeed significant differences in the opportunity to learn mathematics in the face-to-face and the online environments.

## Endnote

[1]Names used in this research report are pseudonyms.

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# TRANSFERABILITY OF AN EXISTING FRAMEWORK FOR ANALYZING TEACHERS' LEARNING IN AN ONLINE PROFESSIONAL TEACHING COMMUNITY

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*In order to document the emergence of a community of practice and the collective learning of the teachers in an online context, we consider the transferability of an existing interpretive framework for analyzing the emergence and concurrent learning of professional teaching communities intended for use with face to face interaction to an online context.*

## Introduction

The findings of a number of investigations indicate that teachers' participation in communities of practice (CoPs) or, more specifically, professional teaching communities (PTCs) can be a crucial resource as they attempt to develop instructional practices that place student reasoning a the center of instructional decision making (Cobb & McClain, 2001; Franke & Kazemi, 2001; Gamoran, Secada, & Marrett, 2003; Kazemi & Franke, 2004; Little, 2002; Stein, Silver, & Smith, 1998). As a consequence, designs for supporting teachers' learning that involve guiding the initial emergence and subsequent development of professional teaching communities have become increasingly common.

We are in the beginning phases of a project that seeks to extend the current research on PTCs to the online environment. We do so for two primary reasons. First, we have found that the online environment offers significant affordances that are not present in face-to-face settings. In addition to providing "anytime, anywhere" learning that can easily fit into learners' busy lives, the online environment (1) provides teachers with opportunities to devote significant amounts of time to thinking about the mathematics at hand, ideally in multiple sessions of thinking and reflecting, recording, analyzing, and revisiting the initial thoughts, comments and reflections of their colleagues and (2) automatically records a permanent record of each participant's thinking, solutions, or questions that provides an opportunity for others to comment on, question, and discuss each individual's work. Secondly, and just as important, we feel that the online environment holds great potential for scaling and can "serve" teachers in traditionally marginalized areas: rural and urban schools and districts without access or resources to ongoing, mathematics specific professional development activities.

## Theoretical Perspective

At the core of Wenger's (1998) definition of a CoP is the practice that is the source of coherence in the community. Against the background of that practice, Wenger then discusses three interrelated dimensions that clarify what distinguishes a community of practice from a group: a joint enterprise, mutual engagement, and a shared repertoire. Joint enterprise is a shared purpose that is more than a goal; it "creates among participants relations of mutual accountability that become an integral part of the practice" (Wenger, 1998, p. 78). Mutual engagement includes the social complexities and relationships that are developed in pursuit of a joint enterprise, as well as the norms of participation that are specific to the community. A shared repertoire includes historical events, tools, styles, discourses, actions, stories, artifacts, and concepts that have been produced or appropriated by the community in the course of its existence and have become a part of its practice.

Our overarching goal is to support the emergence of an online professional teaching community that focuses on the practice of “doing mathematics” with an eye to instruction. More specifically, it would be normative for the members of the group to engage in:

(1) Making mathematical thinking, and not just solutions, public
(2) Focusing on understanding each others’ mathematical thinking
(3) Making public reflections about aspects of others’ understandings that are consistent or inconsistent with one’s own
(4) Collaborating on ways to improve mathematical understanding (with a focus on hypothesizing, generalizing, connecting, etc.) and *generating* mathematics
(5) Connecting mathematical activity with classroom instructional practice

These five normative ways of acting are based on our work with Online Asynchronous Collaborations in Mathematics Education (OAC) (Clay & Silverman, 2008, 2009; Silverman & Clay, submitted). Briefly, OAC involves cycles of individual, small group, and whole class interaction and collaboration. We begin with providing a private workspace for students to draft a solution, initial approach, or questions on a set of purposefully selected, open-ended mathematics tasks. At this point, we do not expect that everyone in the class will be able to complete each of these activities, but we do expect each student to attempt the assigned task and either pose a solution method and solution or ask relevant questions or wonderings that would assist in the completion of the activity. Students have approximately four days to post and revise their postings. After the individual work phase, each student’s initial postings are made public and students add comments, respond to questions, or ask for clarifications. Finally, we tie up loose ends in an asynchronous discussion.

## Prior Research Results

We have reported elsewhere that OAC has been effective at supporting mathematical interaction (Clay & Silverman, 2008, 2009). Consider the following excerpts from teachers’ work on *Pig Pens*, a Math Forum Problem of the Week that we have successfully used in OAC. The problem is as follows: Farmer Mead has 36 guinea pigs on her farm. She wants to build one rectangular pen they can all run around in. She must allow one square meter of area (1 $m^2$) for each guinea pig in the pen. The fencing costs one dollar per meter. Find three different rectangular pens that have exactly the area needed for 36 guinea pigs.

> ***Marcy*[2]:** The first thing I noticed about the problem is that it asks for 3 different types of pens, and my first thought is why 3 different pens a rectangle is a rectangle, just make a simple rectangular pen and be done with it. Things I’m still wondering about: Why three pens? While I haven’t attempted to solve the problem yet, I wouldn’t have thought that the shape of the rectangle will make a difference in price, but evidently it does considering the question.

First, note that Marcy’s post was unpolished and provided insight into her conceptions about some of the big mathematical ideas involved in the problem. We contend that such unrefined insight, especially for *each* student, is virtually impossible in a synchronous learning environment. The class, and Noelle in particular, took up Marcy’s post as a focus for discussion:

> **Noelle**: A rectangle isn't a rectangle. There are short and fat rectangles, there are rectangles with even or nearly even sides. Look around your house, your door is one style of a rectangle, but your phonebook is a different type of rectangle. If you will have noticed from the other responses that there is quite a bit of difference between the cheapest pen 6x6= $24 and the most expensive pen 1x36 = $74. That is more than double the price. Then you started to figure it out, you were on the right track. I think you need to revisit perimeter vs area and then it might make a little more sense to you. I agree that you were over thinking the problem. After looking at the others, does it make more sense now?

It is important to note that Noelle answers Marcy's original question about rectangles by careful questioning and not through recapitulating her solution method. By engaging Marcy's current understanding and not imposing her own on the conversation, we believe that Noelle was able to help Marcy with crucial idea behind her troubles. Moreover, the entire class was able to observe and discuss this crucial pedagogical step.

Our second excerpt provides an example of the ability of OAC to position students to learn from the public discourse. Marcy, continuing her initial post from above, noted:

> **Marcy**: If the total area is 36 square meters for the pen then no matter what the length and width are it is still going to be 36 square meters which should not change the price. The dimensions can be w= 4m X L = 9m or they can be w=3m x L=12m or they can be 6m x 6m. Nope this is wrong. I need 4 sides because I know the perimeter of a rectangle = 2L+2w. Ok I will come back to this. Then there is the price. I don't think this is correct but to get the price, ok there are 36 square meters 36(36) which equals 1,296. 1 meter costs $1.00. So how many meters are in 36 square meters?

Derek M. and Betsy S. followed up on Marcy's post:

> **Derek**: Don't feel silly, your conversation sounds very much like one I have heard from my wife. … [Remember:] "a number that is squared" is not the same as "squaring a number." … The question says that you have 36 pigs, and that each one needs "1" Sq. Meter each to run around in.

> **Betsy**: A number that is squared and squaring a number are the same thing. Meters squared is the unit of measurement for area because Length in meters multiplied by the width in meters is meters times meters which becomes meters squared.

> **Derek**: Betsy, You are right, let me rephrase that, when we see a number such as 36 Sq Meters, it has already been squared. The fact that the question mentioned that it needed to be 36 Sq Meters does not mean that you had to square 36 meters. What I was trying to say was that it is easy to confuse the note of Sq Meters or Feet as having to automatically squaring a number. It is important to determine what the number represents. Thank you for pointing this out, it is easy to forget that what runs through my mind and what I write are not always the same.

Derek attempts to address issues that seem to trouble Marcy, though he does so with less clarity than Noelle did. This begins an exchange that diverges from Marcy's learning to Derek's learning and the learning is about mathematical communication rather than content, which will be crucial to teaching. Finally, Derek is able to learn from Betsy and displays his learning to which Betsy answers kindly.

Through all of the communication about her problem, much of which likely resulted in her classmates' learning, Marcy also shows evidence of learning. She is able to learn from Derek and Betsy, not only from their discussion of her work but also from their original posted solutions. We have noticed that on many occasions, students refer to learning from the other posts, even though the posts were not available as they generated their initial solutions. It is also important to notice that Marcy believes that her learning will affect her practice. While these are self-reported, they are evidence that these teachers have begun to reconsider their pedagogical practices; comments such as these are much more common than we expected and plan to further analyze the effects of them.

> **Marcy**: Dave & Betsy, THANK YOU!! so much for your posts. The square meters is what screwed me up. I kept thinking 36 square meters is 36 square meters no matter what the dimensions are. But I now see that it is the perimeter I was pricing out. I actually feel kind of foolish because it seems so simple to me now. This is what happens to me all the time. Dave what you posted helped so much to clarify my confusion and then when I saw Betsy's drawings it really made sense. Thank you Betsy. I will draw everything out from now on.

> **Marcy**: Ok, now that I understand the problem, I would definitely use manipulatives to explain this problem to my students. I think I would like to make a fence with cardboard and put a bunch of little plastic pigs or something small that would represent the pigs inside the different dimensions. Then I would label each side, having at least 4 possible dimensions, I would discuss the multiples of 36 so everyone understood that and I would demonstrate how to find the perimeter having the students put together their fences and add up the 4 sides.

Finally, it is also important to note that Marcy ended up revising her solution:

> **Marcy**: I realized that I did not use the multiples of 36 to figure out the perimeter. I know that Perimeter of a rectangle 2L + 2W Therefore the using the multiples of 36: 1*36 = 1+1+36+36= 74 @ $1.00 per meter = $74.00 12*3=12+12+3+3= $30.00 4*9= 4+4+9+9=$26.00 6*6 = 6+6+6+6=$24.00 6X6 = is the cheapest pen.

During OAC, we consistently get a copious amount of interaction. This allows for making mathematical thinking public, understanding each others' mathematical thinking, reflecting on aspects of others' understandings that are consistent or inconsistent with one's own, collaboratively working to improve mathematical understanding, and connecting mathematical activity with classroom practice. We can say with confidence that individuals have or continue to engage with these practices, as evidenced by the above excerpts and many others. The evidence provided to this point, although promising, has been rather anecdotal and focuses on the individual learning of one participating teacher. In order to document the emergence of a community of practice and the collective learning of the teachers in this online context, we consider the transferability of an existing interpretive framework for analyzing the emergence and concurrent learning of professional teaching communities (Dean, 2005) intended for use with face to face interaction to an online context.

## Purpose of Investigation

The purpose of this study is to examine the feasibility of using the evolution of norms in a social grouping to document the evolution of an online CoP. This is based on an existing framework for investigating the emergence and concurrent learning of professional teaching communities (Dean, 2005). Dean’s framework was developed in order to analyze the *collective* learning of the teachers as they participated in face-to-face professional development work sessions. Pragmatically, the need for an interpretive framework of this type derives from the increasing attention that researchers on teaching are giving to teachers' participation in pedagogical communities. Theoretically, an interpretive framework of this type documents the evolution of communal norms that constitute the immediate social setting of the participating teachers' learning

Methodologically, it is important to clarify that norms are identified by discerning patterns or regularities in the ongoing interactions of the members of the professional teaching community. A norm is therefore not an individualistic notion but is instead a joint or collective accomplishment of the members of a community (Voigt, 1995). A primary consideration when conducting analyses of this type is to be explicit about the types of evidence used when determining that a norm has been established so that other researchers can monitor the analysis. A first, relatively robust type of evidence occurs when a particular way of reasoning or acting that initially has to be justified is itself later used to justify other ways of reasoning or acting (Stephan & Rasmussen, 2002). In such cases, the shift in the role of the way of reasoning or acting within an argument structure from a claim that requires a warrant, to a warrant for a subsequent claim provides direct evidence that it has become normative and beyond justification. A second, robust type of evidence is indicated by Sfard’s (2000) observation that normative ways of acting are not mere arbitrary conventions for members of a community that can be modified at will. Instead, these ways of acting are value-laden in that they are constituted within the community as legitimate or acceptable ways of acting. This observation indicates the importance of searching for instances where a teacher appears to violate a proposed communal norm in order to check whether his or her activity is constituted as legitimate or illegitimate. In the former case, it would be necessary to revise the conjecture that a particular activity was normative whereas, in the latter case, the observation that the teachers’ activity was constituted as a breach of a norm provides evidence in support of the conjecture (cf. Cobb, Stephan, McClain, & Gravemeijer, 2001). Finally, a third and even more direct type of evidence occurs when the members of a professional teaching community talk explicitly about their respective obligations and expectations. Such exchanges typically occur when one or more of the members perceive that a norm has been violated.

## Discussion

The types of evidence we have outlined were developed to document the emergence of norms within a group interacting in a face-to-face setting. Our question is whether they are appropriate for analyzing interaction and the emergence of normative practices in an online setting. Using the examples of the online interactions above as the context, we are exploring potential issues when using the face-to-face interpretive framework discussed above to analyze the emergence of norms in an online community. Following are three of the issues we are currently investigating.

*How Are Online "Whole Group" Discussions Different from Whole Class Face-to-Face Discussions?*

Initially, our concern was claiming something was normative when there was no "whole group" discussion in an online context. For example, when a group of people are in a room and someone makes an utterance, everyone hears it. In contrast, when a group of people are "in" an online discussion and someone makes a posting who knows if they read it? Upon further investigation, we concluded that the technology provides us with information about how many times a participant accessed a particular online discussion. Of course that does not mean the participants have legitimately engaged with the content, but what guarantee do we have of that in a face-to-face discussion? Just because someone is present in the room, does not mean they are engaged in the conversation. This is a particularly important issue when addressing social norms of the community. It is important to note, in the online environment, spending time in a discussion board is a first level of engagement with a particular post. Replying to that task is a second level of engaging with a post. We are currently seeking to understand how these two levels of engagement are correlated with "being present in a room".

*Talking at the Same Time*

We were concerned about the fact that there are often multiple conversations going on at the same time in the online context. That being the case, it further problematizes the notion of a whole group discussion. While participants can be posting to the Discussion Board, there is no guarantee that everyone engages with a particular utterance in a particular thread. In response to this, McConnell (2000) notes that

> the various activities of [a] group are not discrete in the way they often are in a face-to-face group. Members work on multiple issues at the same time. And because the issues are introduced by the participants themselves and are about important concerns and interests they have in relation to the work of the group, there is a different quality to the group members relationship with the issues. (p. 78)

In other words, while there are multiple conversations going on at once, the conversations are, by and large, organic and arise based on the perceived needs and goals of the group. Again, in this case, the significant factor seems to be the social norms, namely that participants are obliged to engage and to do so in a particular way.

*Is There Time?*

We also note that engaging in the multiple conversations legitimately takes time and, as such, this is another challenge to online communities: Is there time to engage with each utterance. This is methodologically important as claims about normative practices are based on posts but also the form and function of the replies to posts. McConnell (2000) notes that "The permanent nature of the group and its work … and the way in which members are able to "rejoin" at any time, means that participants seem to carry the work of the group with them in their everyday lives" (p. 78). Topics that are addressed in an online thread are chosen by the participants and therefore demonstrate the legitimacy of the topic. Thus, time to engage with each utterance is not necessary. Given the permanency of online conversations (i.e., there is a permanent record of all conversations) threads that were previously ignored can gain legitimacy. Therefore, practices consistent with productive social norms are often present in part of the group at the constitution of the group and, as such, there is a need to add weight to patterns in change in the violators, not just the responses to the violators.

## Conclusion

Given the growing number of universities and professional developers that utilize online settings, it is imperative that we have a feasible framework for analyzing participants' learning. This paper is an initial attempt to investigate this issue. Using the background of an investigation of Online Asynchronous Collaborations in Mathematics Education, we have highlighted the issues we are currently exploring while investigating the transferability of an existing framework for analyzing the normative practices of a professional teaching community. Given the outlined issues, we conjecture that Dean's framework is a viable starting point for exploring the emergence and concurrent learning of an online community of practice.

## Endnotes

1. Both authors contributed equally to this collaborative research; order of authorship follows alphabetical convention.
2. All participants' names are pseudonyms.

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# PRE-SERVICE TEACHERS EXPERIENCES WITH ADVANCED DIGITAL TECHNOLOGIES: THE INTERPLAY BETWEEN TECHNOLOGY IN A PRE-SERVICE CLASSROOM AND IN FIELD PLACEMENTS

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## Introduction

This paper reports on a study of a group of 20 pre-service teachers enrolled in a first-semester mathematics teaching methods course. Course activities included participation in two separate field experiences in neighboring school districts. The methods class placed considerable emphasis on the use of advanced digital technologies in the teaching and learning of mathematics, with particularly extensive use of the TI-Nspire. The field experiences varied in the extent to which technology was used from virtually nil in some classrooms to full implementation in others.

The methods course in which the pre-service teachers in this study participated was designed to introduce candidates to inquiry-based learning with open-ended questioning. The course proceeded from the premise that the use of advanced digital technologies can be a useful method for teaching in such a manner. In our experience, pre-service teachers generally fall into three categories with regard to their beliefs regarding the utilization of technology in mathematics classrooms: (a) "naysayers" who would like to minimize use of technology; (b) "Yes, buts" who believe that students should learn concepts first and only then have access to technology; and (c) "Yes, ands" who believe that technology should be used as an integral part of learning concepts. As pre-service teachers are developing a teaching identity it is interesting to observe the influences that push them in one direction or another. It seems to remain the case that most pre-service teachers have achieved their own mathematical success by being proficient in traditional environments (especially at the college level where use of technology is often outrightly discouraged) and, therefore, tend, in our experience, to be either "naysayers" or "yes, buts." Educators generally do not fall naturally into the third position without some coaxing, and the methods course was taught in a manner which might persuade the pre-service teachers to, at least, consider this position.

The data of our study shows that the methods class and the field placements served to challenge the pre-service teachers to clarify where they stand in this categorization and what they imagine the place of technology to be in their future careers as mathematics teachers.

The study was on a small scale with a limited number of participants and a limited number of cooperating schools, but the principal conclusion of the study is that there seems to be a crucial, perhaps, decisive effect that modeling of exemplary practice in the field placement has on candidate attitudes regarding the use of advanced digital technologies in their teaching, that is to say, in creating the possibility that they may become "Yes, ands." Many of the preservice teachers in this study were resistant to the extent of the emphasis on technology in the methods class but, with one exception, students whose field placement was in a school where technology was used extensively developed a positive attitude to technology. Not surprisingly, candidates with positive technology-oriented experiences in the field express stronger desires to incorporate technology into their own teaching. There is evidence that the pre-service teachers' experiences in the classroom primed them for the possibilities of technology but it takes the experiencing of exemplary practice to convince them of the benefits of working to incorporate technology in their own teaching.

A secondary conclusion is that while there was a general improvement in the quality of the lesson plans written by the pre-service teachers as the semester progressed, the lesson plans written by those students with field placements in technology-rich environments showed more sophistication, not just in the use of technology, but in terms of implementing inquiry-based and open-ended instructional approaches.

## Theoretical Context

There is a growing body of research which indicates that digital technologies, including graphing calculators and CAS-enabled calculators, can enhance young students' conceptual and procedural knowledge of mathematics (Dunham, 2000; Thompson & Senk, 2001). Research has also shown the benefits to students of dynamically linked representations (Kaput, 1994; Rich, 1996) whereby upon altering a given representation, every other representation is automatically updated to reflect the same change.

As technology has become more sophisticated and functionalities of hand-held calculators have increased dramatically, there is a need for research that explores the extent to which teachers are able to employ technology effectively (Koehler & Mishra, 2005; Niess, 2005) and the extent to which students are able to work effectively in such technology-rich environments (Edwards, 2004; Meagher & Brown, 2007).

After Shulman’s (1986) analysis of teachers’ knowledge as a complex structure including content knowledge, pedagogical knowledge, and his introduction of the concept of pedagogical content knowledge, the research in this topic has become effectively grounded on his framework. With Mishra and Koehler’s (2006, Koehler & Mishra, 2005) and Niess’ (2005, 2006, 2007) introduction of the concept of the teachers’ Technological Pedagogical Content Knowledge (TPCK), technology-related research in the teachers’ professional development and education field has gained a new rich conceptual framework or “an analytic lens for studying the development of teacher knowledge about educational technology” (Mishra & Koehler, 2006, p. 1041). The TPCK model involves the content knowledge, the pedagogical knowledge, and the technology knowledge required to teach in technologically-rich environments (see Figure 1). Furthermore, the TPCK model discusses the combination of that basis knowledge such as technological content knowledge (TCK) versus technological pedagogical knowledge (TPK). For TPK, Mishra and Koehler (2006) discuss TCK as follows “teachers need to know not just the subject matter they teach but also the manner in which the subject matter can be changed by the application of technology” (p.1028). On the other hand, “technological pedagogical knowledge (TPK) is knowledge of the existence, components, and capabilities of various technologies as they are used in teaching and learning settings, and conversely, knowing how teaching might change as the result of using particular technologies” (Mishra & Koehler, 2006, p. 1028). Finally technological pedagogical content knowledge, according to Mishra and Koehler (2006) is:

> the basis of good teaching with technology and requires an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students’ prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones (p.1029).

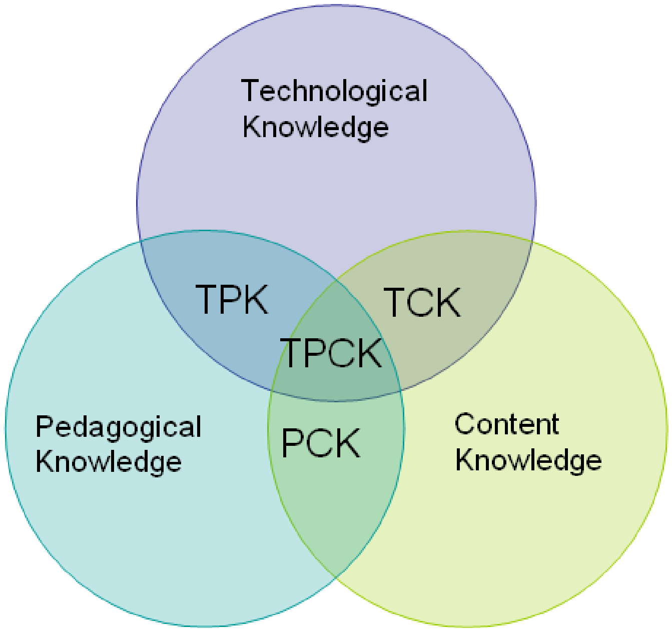


*Figure 1*. Re-creation of Mishra & Koehler's TPCK model.

Clearly, there is much to consider when studying pre- and in-service teachers' knowledge, views, beliefs, attitudes, and decisions about the use of technology in their classroom. Niess (2006, 2007) discussed how teachers' beliefs about teaching mathematics with technology play a crucial role in the development of TPCK. Moreover, Niess (2005) explained that there are four basic components of TPCK:

1. An overarching conception of what it means to teach a particular subject such as mathematics integrating technology in the learning.
2. Knowledge of instructional strategies and representations for teaching particular mathematical topics with technology.
3. Knowledge of students' understandings, thinking, and learning with technology in a subject such as mathematics.
4. Knowledge of curriculum and curriculum materials that integrates technology with learning mathematics(p.197).

Increasingly pre-service teachers are asked to incorporate technology into their teaching (NCTM, 2000). The extent to which they are willing or able to do so is influenced by a number of factors including their own experience with technology, constraints imposed by their teaching placements, and the quality of their training in the technology (Bullock, 2004; Moursund & Bielefeldt, 1999).

## Data Collection

The pre-service teachers (n=20) were engaged in routine activities that comprise a mathematics teaching methods course at a small Midwestern university and were using the TI-Nspire handheld regularly. The course was designed specifically for pre-service secondary school mathematics teachers, with the subjects engaging in activities focused primarily on pedagogical issues (e.g. constructing lesson plans and grading rubrics, creating technology-oriented math activities) and content issues (solving mathematics problems, assessing student work).

Specific activities included:

- Field Experience Reports: On two separate occasions, candidates researched, developed, and implemented mathematics lessons as part of the field teaching component of the class. The first critique focused on student behavior, teacher/candidate interactions, and instructional effectiveness. The second critique focuses on problem-posing and analysis of student mathematical thinking.
- Activity Writeups: The teachers submitted five secondary grades math activities that they constructed (either wholly original or modified from pre-existing materials). The teachers were encouraged to use these materials in their field teaching (if possible).
- Graphing Calculator Teaching Project: Candidates conducted original research dealing with the teaching of a secondary mathematics problem (or set of related problems) using the TI-Nspire graphing calculator. The problem(s) selected for study were subject to instructor approval to be well-suited for study with graphing calculators. The research was to meaningfully include the TI-Nspire in the investigation of the problem(s).

In addition to the routine activities of the methods class, the preservice teachers completed a *Mathematics Technology Attitudes Survey* and three short surveys, each of which consists of a mixture of multiple choice and open-ended items (administered electronically in weeks 4, 8, and 13 of the study). Finally they completed an open-ended exit survey with more general questions than those asked in the Week 4, 8, and 13 surveys.)

## Data Analysis and Discussion

Our analysis of the data focuses on two principal dimensions: (a) the interplay between the effects of the methods class and of the field placement on the pre-service teachers' experiences of and attitudes to technology, and (b) the evolution of the pre-service teachers' lesson plans over the course of the semester.

*The Interplay between the Methods Class and the Field Placement and the Influence of the Field Placement*

Looking at the open-ended exit survey we see a correlation between the pre-service teachers' field placement and their disposition to the use of technology in the future. Despite the emphasis placed on technology in the methods class, half of the students whose field placements had minimal or no technology did not develop a positive attitude to the use of technology in the teaching and learning of mathematics. One teacher who commented that "I found that my field teachers did not use technology in their classroom. I found their teaching methods to be more practical, and I will probably lean more towards their style," had said about the technology that "I found the TI-Nspire to be too complicated and not worth the hassle figuring it all out. I spent more time trying to figure out how to use it than I did learning about math." On the other hand, almost all of the teachers (there was one exception) who had been exposed to exemplary practice in technology-rich environments were eager to incorporate technology in their future practice. One student commented "My two field experiences were on different ends of the technology spectrum. One school barely had any technology and the other school had a lot." These two experiences resulted in this student declaring that "I am now more likely to use technology in my teaching. Technology offers so many advances for students and can relate to many different learning styles." Another student commented that "I will definitely want to use technology" having been in a school where “there was a grant for TI-89s in one class, for laptops in another and … their teacher was proficient in all of these technologies.”

It can be difficult for the students to make the connection between the practices developed and encouraged in their methods classes without exemplary experiences in classroom of how these ideas can be put into practice. The students who made comments such as “I am less likely because of un-user friendly the TI-Nspire was,” were not students who saw the TI-Nspire being used in a classroom. This correlation between the development of a positive disposition toward teaching with technology and exemplary experiences in a technology-rich environment calls for the development of a closer school-university partnership to allow students to make meaningful connections between their methods classes and he reality of classrooms. We see this as further evidence that students are primed with a certain level of TPCK in their Methods class but their ability to consolidate and implement these ideas is sensitive to the field placements both in terms of their willingness to implement good practice in using technology and their ability to implement good practice.

*Lesson Plans*

Lesson plans were written by the students at four different stages in the semester: two before their field placements, one between the first and second field placements and one in conjunction with the second field placement The lesson plans were scored along three dimensions: Implementation of Technology [Active (1); Neutral (0); Passive (-1)]; Implementation of Inquiry Based Methods (Adapted from Northwest Regional Educational Library (NWREL - http://www.nwrel.org/msec/science_inq/answers.html)

[Student initiated (1); Guided inquiry (0); Structured Inquiry (-1)]; and Quality of Problem Solving [Active (1); Neutral (0); Passive (-1)]. See Appendix 1 for the full rubric.

Analysis of the scores showed that the students struggled along all three dimensions but improved in all three areas as the semester progressed.

| | Technology | Inquiry | Problem solving |
|---|---|---|---|
| Lesson 1 | -0.25 | -0.5 | 0.25 |
| Lesson 2 | -0.5 | -0.375 | 0.125 |
| Lesson 3 | -0.25 | -0.125 | 0.5 |
| Lesson 4 | -0.125 | -0.375 | -0.125 |
| Lesson 5 | 0 | -0.25 | 0.375 |

The first two sets of lesson plans were generally poor and reflected the fact that novice pre-service teachers constructed the materials. The second set shows some improvement over the first but are still very teacher-centered. Several of the lesson plans explicitly use language such as "the teacher will lead students …” In most of these lessons, technology still seems like an afterthought for most teacher candidates, an "add on" rather than a tool to drive instruction. The third set of lesson plans, while far from high in quality, show marked improvement over the first two sets. This highlights the growing experience of the students and is reflective of the fact that these lessons were written after the first field placement.

The students’ use of technology was very slow to develop and, before the second field placement, the use of technology in the lesson plans is not very sophisticated. For example, the use of Dynamic Geometry Systems (DGS) in the set of lessons focused on the Pythagorean Theorem, most of the pre-service teachers only require students to draw a particular instance of a problem rather than using DGS to generalize and/or form conjectures. It is further noteworthy that in the set of lesson plans where the use of technology was optional, several students chose not to use technology at all while other suggest the use of technology for "High Level kids only." Those that did employ technology in their lesson plans tend to focus on "what buttons to press" rather than rich opportunities for student discovery.

There is a rather sharp improvement in the student scores on Lesson 5. At one level this is unsurprising as one might expect the students to reach a maturity point late in the semester. We argue however for a more significant factor: the second field placement during which Lesson 5 was written. Support for this thesis is borne out when the scores for the lesson plans are disaggregated into two groups: those that had a minimal technology second field placement (MT) and those that had a technology-rich second field placement (TR):

| | Technology | | Inquiry | | Problem Solving | |
|---|---|---|---|---|---|---|
| | MT | TR | MT | TR | MT | TR |
| Lesson 1 | -0.25 | -0.25 | 0 | -0.5 | 0 | 0.5 |
| Lesson 2 | 0 | -0.75 | -0.25 | -0.5 | 0.25 | 0 |
| Lesson 3 | -0.25 | -0.25 | 0 | -0.25 | 0.5 | 0.5 |
| Lesson 4 | 0 | -0.25 | -0.25 | -0.5 | 0.25 | -0.5 |
| Lesson 5 | -0.25 | 0.5 | -0.5 | 0 | -0.125 | 1 |

The lesson plan scores for both sets of teachers are similar for the first four lessons. There is, however, a quite dramatic difference in Lesson 5. The teachers in technology-rich field placements score much higher in the use of technology, which is to be expected but, significantly, they score higher in Implementation of Inquiry Based Methods and much higher in Quality of Problem Solving. In the presence of technology, they developed more pedagogically sound activities and their TPK and TPCK skills were clearly developing.

A significant feature of the lesson plans written in the technology-rich environment is that the tasks were formulated so that the use of technology was a necessary component of the lesson i.e. the tasks were designed assuming access to and ability to use technology. For example, one pre-service teacher designed a lesson centered around the classic birthday problem which involved the use of repeated simulations on the graphing calculator. This would be virtually impossible to replicate in a single classroom without technology. Another candidate posed a problem to students involving systems of inequalities. In the lesson, students attempted to construct a closed region satisfying the following constraints using TI-Nspire:

1. The system includes at least four linear inequalities
2. The graph of the system generates closed region
3. At least one pair of lines in the system is perpendicular
4. Every line has to have slope (i.e. not zero or undefined slope)

This lesson was markedly richer than this students' previous lesson plans, not just in the use of technology but in terms of inquiry-based teaching and problem solving.

We argue that an important element in the pre-service teachers developing calculator active tasks, perhaps becoming “Yes, ands” is their placement in a technology-rich environment. The modeling of exemplary practice and the mentorship available to them was, we believe, significant in this move. As one pre-service teacher commented on his placement in his Field Report:

> Overall, this class [i.e. the field placement class] had many students eager to work with the new technology. It was exciting to enter the classroom at a point when new and innovative techniques are being introduced and see the results the calculators had not only on the willingness of students to engage in the material, but also on their abilities to construct good solutions to challenging problems. I hope that more students will have access to this new technology in the future to fuel new mathematic interest, just as the Nspires did for the students in Mr. C's class.

It is interesting to compare the analysis of the Lesson Plans with the pre-service teachers responses when they were asked in the surveys to discuss the extent to which they were thinking and working with technology as they design activities. Several pre-service teachers mentioned and complained that the use of TI-Nspire was *required* at Week 4. However, others stated that they liked working with TI-Nspire and it capabilities: "I have been able to incorporate things such as the TI Nspire and GSP into my activity write-ups, and I think that incorporating these types of technology into lessons helps to make them more multifaceted and thus easier for a larger percentage of the students in a classroom to understand." Again here this student was considering students' learning with the help of technology. S/he was reflecting on her TPK.

The pre-service teachers had, of course, seen many activities in the methods class by this stage which showed the possibilities for advanced digital technologies in the class but without a model of exemplary practice the teacher quoted here was struggling with how they themselves could actualize this practice.

## Conclusions and Future Directions

The overall conclusions of this study are that (a) if pre-service teachers' are to develop a positive attitude to the use of advanced digital technologies in their instructional practice they require more than a methods class to develop TPCK and that modeling of exemplary practice in the field placement has a crucial, perhaps, decisive effect on the student's attitude, and (b) that the most significant improvement in the quality of the pre-service teachers' lesson plans, in terms of being inquiry-based and open-ended, came when students had field placements in technology-rich environments.

This significant influence of the field placement suggests further direction for research. Specifically it calls for the development of school university partnerships so that students can engage in the following learning cycle: (i) A class of preservice high school teachers will work with the TI-Nspire to develop lessons/short units designed for technology-rich environments; (ii) experienced inservice will review the lessons/short units and present an initial redesign; (iii) the inservice teachers will teach the lessons, observed by the preservice teachers; (iv) the preservice teachers and inservice teachers will meet together to reflect on and redesign the lesson based on their experiences in the classroom. Engagement in such a cycle allows students to gain the benefit of exemplary practice in task design and use of advanced digital technologies and allows them to focus on this aspect of practice without having to also deal with the early pedagogical aspects which arise from teaching the class themselves i.e. classroom management, questioning etc.

We believe that the implementation of such a cycle in conjunction with a methods class would allow students to develop a more inquiry-based approach to their teaching and to see how the use of advanced digital technologies can facilitate that approach.

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# HIGH SCHOOL TEACHERS' USE OF GRAPHING CALCULATORS: PROFESSED BELIEFS AND OBSERVED PRACTICE

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*This study investigated secondary mathematics teachers' professed beliefs about graphing calculators, the teachers' use of graphing calculators when teaching the concept of function, and the extent to which the professed beliefs explain the teachers' use of graphing calculators. Survey and interview data indicated that the teachers believed that they usually balanced among the various representations of functions. However, classroom observations showed that equations and graphs seemed to dominate more than tables.*

## Purpose of the Study

Research has shown that when teachers use graphing calculators, their roles tend, in general, to shift to those of fellow investigators, facilitators, or consultants, while their teaching strategies tend to involve higher level questioning and more in-depth problem solving, and the classroom discourse grows richer (Simonsen & Dick, 1997). It cannot be overemphasized that some teaching styles are more compatible with the use of graphing calculators than others. Teaching styles that use more open-ended questioning and involve engaging students in discovery activities seem to be more compatible with the use of graphing calculators than those styles that are more teacher-centered. Research also shows that when teachers use graphing calculators there is an increase in cooperative learning where students not only take more responsibility for their own learning but also work together with their peers and learn from each other as well (Harskamp, Suhre, & Van-Streun, 2000).

However, some studies have also shown that teachers who have always taught in teacher-centered classrooms are sometimes uncomfortable with the unpredictability that may arise as a result of introducing graphing calculators, while other teachers are often reluctant to use graphing calculators in creative ways because of their beliefs about what mathematics is and what their role as teachers should be (Milou, 1999; Simmt, 1997). Such teachers tend to confine the graphing calculator to performing computational roles and hence deny their students opportunities to exploit the powerful capabilities of graphing calculators. Yet other studies have shown that there are disagreements in terms of teachers' attitudes and beliefs towards graphing calculators. Some studies have pointed out that there is a link between teachers' philosophical orientation and attitudes and beliefs about graphing calculator use and called for continued investigation of this issue (Aguirre & Speer, 2000).

Even though a substantial amount of research has been done involving graphing calculators, there is still need for more studies to be done in this area. For example, we need to know teachers' perspectives regarding the effects of graphing calculators in exploring various representations of functions as well as their views regarding the extent to which they would let their students explore with the graphing calculators. Since it is true that in general what one reports about himself or herself may not always be consistent with his or her practice, a good study on these issues is one that is designed to investigate the consistency or variance between what teachers report and what they do in their classrooms. This study attempts to address this by (a) investigating secondary mathematics teachers' professed beliefs about graphing calculators,

(b) investigating how these teachers use graphing calculators to teach linear and quadratic functions, and (c) investigating the extent to which the professed beliefs explain the teachers' use of graphing calculators.

## Theoretical Perspectives

This study draws on Vygotsky's (1978) sociocultural theory of learning. According to Vygotsky, education is both a theory of development and a process of enculturation whereby mediated activity helps shape higher human mental functions. The mediator may be a sign system (e.g., language, tabular or graphical representation of a pattern) or a technological tool (e.g., computer, graphic calculator). Vygotsky contends, "if one changes the tools of thinking available to a child, his mind will have a radically different structure" (p. 126). In this study, I take the position that the graphing calculator is an instrument of access to the knowledge, activities and practices of a social group that is the mathematics classroom (Meira, 1998). In this case, using the calculator can be seen as an external activity (using graphs, tables, and numbers to manipulate mathematical concepts), which is then transformed into an internal activity (gaining an understanding of the mathematical concepts).

In addition to Vygotsky's sociocultural theory, I also draw on a theoretical framework developed by Salomon, Perkins, and Globerson (1991) for studying the interaction between technology and user. Salomon and his colleagues define intelligent technologies as those that can be used in partnership in a manner that they can "assume part of the intellectual burden of information processing (such as mental or pen and paper calculations)" on behalf of the user (p. 3). Graphing calculators qualify as intelligent technologies under this definition. In this framework, Salomon, Perkins, and Globerson distinguish between two sets of *principal effects* that arise from this kind of partnership, namely (1) *principal effects with* the technology and (2) *principal effects of* the technology. For purposes of clarity, I will refer to the first set as *planned principal effects,* and the second set as *emergent principal effects.* The work of Goos, Galbraith, Renshaw, and Geiger (2003), which provides metaphors for studying the interaction between calculator and user, is closely related to this partnership framework and so I seek to draw parallels to the metaphors when discussing the principal effects.

Characteristics of *planned principal effects* include elaborate planning (laying out the specifics concerning how the calculator will be used), executing the plan (using the calculator in the desired ways), and interpreting the results. The teacher here predetermines exactly when it will be appropriate to turn to the calculator during problem solving and in what ways this should be done. For example, the teacher may plan when it will be necessary for a particular graph to be displayed, or when he or she will need to enter a certain equation or table of values in the calculator. When there will be need to transform the function in certain ways, the teacher will plan to modify the equation or table in particular ways. Jones (1993) notes that this interaction allows the user to operate at a higher level than otherwise possible, provided he/she constantly monitors the information given by the calculator to ensure that it is consistent with his/her understanding of the problem at hand. Jones (1993) cautions, however, that passing over the entire cognitive responsibility to the calculator, what Goos et al. (2003) refer to as "technology as servant" (p. 78), can be counterproductive and may lead to misconceptions. According to Goos and her colleagues, it is not worthwhile to use technology as "a supplementary tool that amplifies cognitive processes without using it in creative ways to change the nature of activities" (p. 78). They cite using the overhead projection panel as an electronic chalkboard to provide a medium for demonstrating calculator operations to the class as an example of inappropriate use of technology. Goos et al. contend that using technology in this manner only helps reinforce the teacher's preferred teaching methods and this may not be beneficial to students. They suggest that teachers should use the graphing calculator in conjunction with other material resources in ways that further enhance the calculator's capacity for linking multiple representations of concepts. Jones (1993) advises that teachers must develop instructional strategies that promote the formation of intelligent partnerships between their students and the calculator. The relationships between users and calculators in such partnerships should be complementary rather than dependent.

*Emergent principal effects* are characterized by spontaneity, that is, effects that the teacher does not intentionally plan for. These effects then "carry over to other related but not calculator dependent mathematical activities" (Jones, 1993, p. 214). Using the metaphor of technology as partner, Goos et al. (2003) refer to this level of using a graphing calculator as the "cognitive re-organization effects" (p. 79). According to Goos and colleagues, these effects are characterized by using technology to explore new tasks or new approaches to existing tasks and to mediate mathematical discussion in the classroom between students and teacher or between small groups of students. They suggest, "instead of functioning as a transmitter of teacher input, the overhead projection panel can become a medium for students to present and examine alternative mathematical conjectures" (p. 79). Goos' metaphor of "technology as extension of self" is also consistent with emergent principal effects. According to this metaphor, a teacher who attains this level would write unit plans that support integrating technology into the teaching program. That is, the teacher would incorporate technological expertise as a natural part of his or her mathematical and/or pedagogical repertoire. For meaningful emergent principal effects to arise in a classroom, the teacher must be willing to allow his or her students to explore new areas with the calculators and guide the students into discussions that will help them make sense of their findings.

## Research Questions

1) What are secondary mathematics teachers' professed beliefs about using graphing calculators in the teaching and learning of linear and quadratic functions?
2) How do secondary school mathematics teachers use graphing calculators when teaching linear and quadratic functions?
3) What are the relationships between the teachers' professed beliefs about graphing calculators and observed practice?

## Methodology

I conducted the study using a two-phase design. In the first phase, I sent out a survey about graphing calculator use to secondary mathematics teachers ($9^{th}$ - $12^{th}$ grade) in a mid-sized city school district and the neighboring school districts in a northeastern state in the United States. I developed survey instrument using items adapted from Fleener (1995). In addition to demographic information, this instrument consisted of 24 items with Likert-type responses on a five point scale with SA=Strongly Agree, A=Agree, N=Neither agree nor disagree, D=Disagree, and SD=Strongly Disagree. One of the questions on the demographic part asked participants to state how often they used graphing calculators in their classes. I used the responses to this question to categorize the teachers into three groups of frequent users (nearly every lesson), moderate users (once every 2 or 3 lessons), and infrequent users (once every 4 or 5 lessons). I then selected, on a voluntary basis, three teachers from each of the frequent users and moderate users groups to participate in the second phase of the study. I intentionally excluded teachers from the infrequent users group.

The second phase of the study comprised of semi-structured interviews and classroom observations. Data sources included one task-based interview (Goldin, 1999) with each teacher prior to classroom observations, three classroom observations for each teacher with pre-observation (planning) interviews and post observation (debriefing) interviews, copies of teachers' lesson plans, teacher notes, instructional activities and/or tasks. In order to collect focused information during the observations I used the electronic classroom observation (ECOVE) software.

## Results

Analysis of the survey data indicated that on the most part teachers believe that graphing calculators are valuable for students in the study of linear and quadratic functions. The teachers also in general feel confident about their knowledge of graphing calculators and they believe that they make use of graphing calculators whenever opportunities for doing so are available. Additionally, the results from the interviews showed that the teachers believed that they balanced between the various representations of functions.

On the other hand, classroom observations revealed that teachers preferred graphical approaches and algebraic approaches over tabular approaches. For example, in their instructional tasks, the teachers specified algebraic approaches the most (in 36% of all tasks analyzed), followed closely by graphical approaches (in 31% of all tasks analyzed), then tabular representation approach in 17% of the tasks, and another 16% of the tasks not being specific on any of the three representations. This is contrary to what they stated in the interviews about balancing between representations. It should be noted however, that specifying a representation in a task did not restrict the teachers to sticking within that representation alone. The teachers shifted from the specified representation to other representations once instruction began. This shifts mainly resulted from the teachers guiding students through the tasks and so leading the way towards using other representations but on other occasions they came as a result of students being involved in negotiating the problem solving process.

Also revealed from the classroom observation data was the fact that when graphical approach was specified in the tasks the shift to algebraic approach (and vice versa) dominated the shift to tabular approach – about 70% to 30% overall. This goes further to show how much graphical and algebraic representations dominated over tabular representations. A near balance in representations appeared to be achieved when the specified representation was tabular as the percentages of shifts to graphical or algebraic representations differed only slightly – 54% and 46% respectively. However, this apparent balancing between representations was overshadowed by the fact that there were very few tasks in which tabular representation approaches were specified. Moreover, when only verbal representation was specified the teachers switched to algebraic representation more than four times as they did with tabular representation.

## Discussion

The teachers in this study generally believed that they usually balance among the various representations of functions. However, classroom observations seemed to reveal a different story. Analysis of the classroom observation data showed that equations and graphs seemed to dominate more than tables. Most instructional tasks made specific reference to either an equation for which a graph would be drawn and various explorations done on it, or a graph on which various explorations would be done. Only a handful of tasks directly specified use of tables. In cases involving word problems, it was common to see equations being generated then graphs drawn.

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# INTEGRATION OF KNOWLEDGE THROUGH WIKI USE IN AN ONLINE, SYNCHRONOUS ENVIRONMENT

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*In this report we analyze how students integrate information from a wiki to expand, verify, improve, and complete their solution to an open-ended mathematics problem as they work in an online synchronous environment. Significant research has analyzed collaborative problem solving, but virtually no research exists that investigates online collaborative problem solving of open-ended problems. This report uses an established research model to analyze a new type of discursive interaction.*

## Purpose

Policymakers and educators in organizations such as the Consortium for School Networking and the International Society for Technology in Education believe that online learning has both educational and social value. In states like Alabama and Michigan, high school students must complete at least one online course to graduate (Hu, 2009). However, advances in digital technologies and their introduction into society seem to outpace the typical timeframe needed for researchers to investigate meaningfully the potential of new digital technologies for education. For instance, Web 2.0 has ushered in possibilities for collaboration across physical boundaries with such technologies as blogs, file sharing, social networking, and wikis. In mathematics education, research is needed to understand the possibilities and problems of digital technologies to enhance student engagement with and development of mathematical thinking.

This research will investigate how students construct knowledge using a wiki, a Web 2.0 tool. Through a model focused on student discourse, we analyze student integration of material in a wiki to become part of a collaborative solution by a group. We expand the use of the model to include inscriptions. Though we monitor student discourse, teacher intervention in this research is limited largely to task construction. The problems in this research are combinatorics tasks that are not covered in traditional high school mathematics curricula and therefore the students spend significant time in each session constructing problem-solving techniques.

## Theoretical Framework

Our research focuses on the interactions of students in an online environment. We use the interaction analysis model (IAM) introduced by Gunawardena, Lowe, and Anderson (1997) to analyze students' online communications. Several studies (Hou, Chang, & Sung, 2008; Jeong, 2003; Marra, Moore, & Klimczak, 2004; Sing & Khine, 2006) have used this coding scheme to analyze online discussions. Table 1 illustrates the coding scheme we used to analyze how information from a wiki, an interactive webpage that allows for collaborative editing, is integrated into another online environment.

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

Gunawardena, Lowe, and Anderson (1997) developed the interaction analysis model to study the construction of knowledge in an online debate. They used grounded theory to develop the model through content analysis with a focus on knowledge construction by a group and on change in individual understanding through the group interaction. The five phases in Table 1 occur when there is discrepancy or inconsistency that needs to be resolved by a group. Gunawardena, Lowe, and Anderson concluded that the debate format influenced the participants in such a way that they offered mostly P1 comments (191 of 206 postings) and very few of the meta-cognitive skills needed for P4 and P5. They cited that the debate format was a possible limitation to the construction of knowledge and therefore to the collaboration that occurred.

Table 1. *Gunawardena, Lowe, and Anderson (1997) Interaction Analysis Model (IAM)*

| Code | Phase | Operation |
|---|---|---|
| P1 | Sharing or comparing information about discussion topics | Statement of observation or opinion; statement of agreement among participants |
| P2 | Discovery and exploration of dissonance or inconsistency among participants | Identifying areas of disagreement, asking, or answering questions to clarify disagreement |
| P3 | Negotiations of meaning or co-construction of knowledge | Negotiating the meaning of terms and negotiation of the relative weight to be used for various agreement |
| P4 | Testing and modification of proposed synthesis or co-construction | Testing the proposed new knowledge against existing cognitive schema, personal experience, or other sources |
| P5 | Agreement statement(s) or application of newly constructed meaning | Summarizing agreement and meta-cognitive statements that show new knowledge construction |

The process began by asking two questions: 1) Was knowledge constructed within the group by a process of social negotiation? and 2) Did individual participants change their understanding or create new personal constructions of knowledge as a result of interactions within the group?

The use of wikis in education will expand as teachers, students, and administrators become familiar with Web 2.0 tools and as online education becomes more prevalent. Tonkin (2005) offers four different categories for educational wikis: Single-user Wikis, Lab Book, Collaborative Writing, and Knowledge-Based. The wiki used in this research is concentrated in the Knowledge-Based wikis, which are defined by Tonkin as a place where teams can retain their experiences and have quick access to the data.

## Method

The participants in this research are students in the *e*Math project, which connects two different schools in New Jersey: Rutgers Preparatory School (RPS), a suburban independent school, and Long Branch High School (LBHS), an urban public school. The RPS students are in an Advanced Algebra and Trigonometry class and the LBHS students are in a Contemporary Mathematics class. The students have completed Algebra 2 and decided to take a full-year math course at the next level, which is not required. One student at RPS is a junior and all other students in the project are seniors. The teams in the online environment have three or four students with no more than two students from any one school. In this project all of the tasks are open-ended mathematics problems that are not seen in traditional mathematics classes. In the fall semester of 2008, the *e*Math project conducted seven online sessions. This report focuses on the fourth and fifth sessions, October 23rd and 29th, respectively. During the October 23rd session, the students worked on the Pizza Problem with Halves with questions one through three (Figure 1). For the October 29th session, we augmented the task with a fourth question, indicated in Figure 1 below the horizontal line.

**PIZZA PROBLEM WITH HALVES**

A local pizza shop has asked us to help them keep track of pizza orders. Their standard "plain" pizza contains cheese with tomato sauce. In addition, a customer may order toppings on the whole or half of a pizza. A customer can then select from two different toppings: peppers and pepperoni.

1. How many different choices for pizza does a customer have?
2. Work together with your teammates to list all the possible different selections.
3. With your teammates, write a report of your findings for questions 1 and 2 above. It should include a justification that you have accounted for all possible pizzas. You want your report to convince others who are not in your team that your findings are correct. Post your report to the Summary tab.

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4. After the last session, the owner of the pizza shop was confused by the different solutions she received from the different teams. The owner desires that your team urgently respond to these two requests:
   a) From the Wiki tab, read the solution of each of the other teams. If your team agrees with another team's solution, explain why in the Summary. If you do not agree with any other team's solution, present a numbered list of all possible pizzas when choosing from the two toppings.
   b) The owner is considering making available another topping—sausage—and wants your team to develop a numbered list of all possible pizzas when choosing from the three toppings. The owner needs to be certain that your list is complete and, therefore, wants you to convince her that your group's solution is indeed correct.

*Figure 1*. The pizza problem with halves.

The students communicate through an online synchronous environment, the Virtual Math Teams Chat (VMT Chat), developed under a grant from the National Science Foundation by researchers at Drexel University (Stahl, 2006). The VMT Chat (Figure 2) is Java-based and facilitates Internet communication through several interrelated spaces: a whiteboard, chat, Wiki, Summary, Browser and Help pages. The whiteboard and the chat spaces are where a majority of the students' interactions occur. The whiteboard is a shared, dynamic workspace where students can enter textboxes, draw lines and ellipses, and use other tools similar to those in popular word processing software. The other main communicative space is the chat on the right side of the screen where each member of a chat team has a distinct color for their typed entries. In all spaces of the environment, students' contributions become visible to other members in the room once they click outside the entry or press the return key. On the upper left side of Figure 2, the second tab is the Summary where members of a chat team write a collective summary of their process and solution. The Topic tab allows the students to read and review the statement of a task. The Wiki tab contains the webpage that has links to screenshots of each team's summary from the previous session (Figure 3). The Browser and Help tabs are available, but they remain unused as we direct the students to use VMT as their only means of communication and ask them to consult the members of their team for technical questions regarding the environment. A referencing tool, seen in Figure 3, is also available for students to direct teammates' attention to a particular part of the whiteboard, summary, topic, or chat. While the students are working on laptops in their classroom they are arranged around the perimeter of the room in such a pattern as not to be in close proximity to the other member of their team. All interactions between a student's computer and the server are recorded for later review by the researchers.

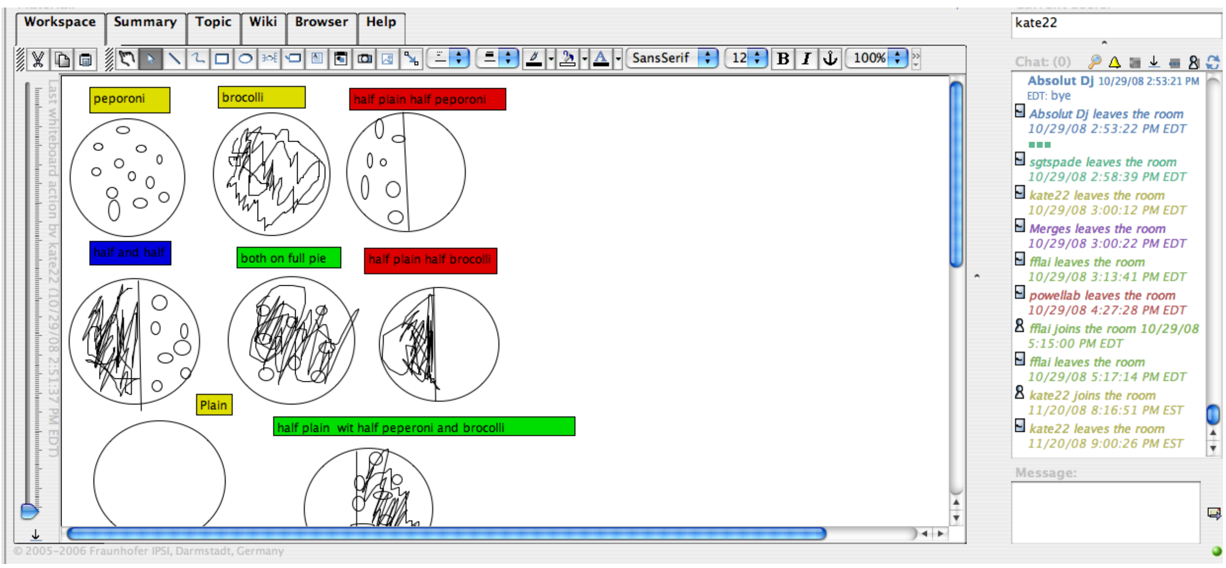


*Figure 2.* A screenshot of VMT Chat dual interactive spaces: dynamic whiteboard and text chat.

Students in each of the four VMT Chat rooms had access to the wiki. Upon clicking on the wiki tab the participants are able to see four links to wiki pages. Each link contains .jpg images of screenshots of the summaries students posted during the October 23rd session. The students were informed that content in the summary tab is a public space to which other teams would have access. The whiteboard remained the team's private workspace.

We observed participants during each research session but did not intervene during the session. We read scripts prior to each session to convey pertinent information to the students in each school. The task design also was an interventional method as we constructed each session's task to further probe students' ideas evidenced in the prior session.

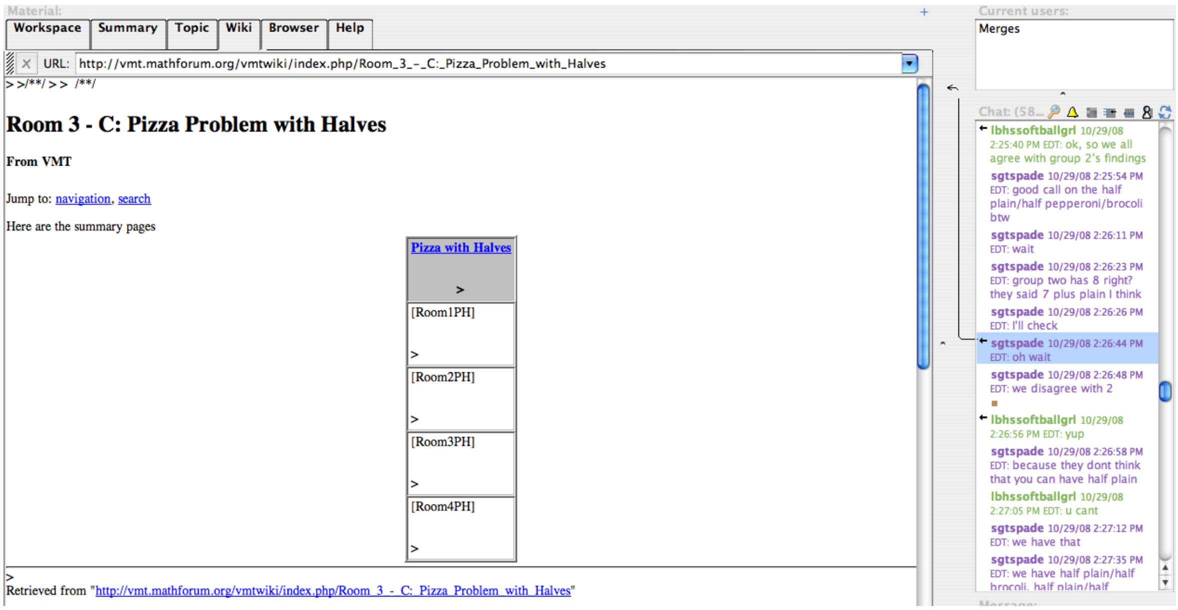


*Figure 3.* Wiki page viewable by participants after clicking on the wiki tab in VMT Chat.

After analyzing the session on October 23rd, we decided that it would be appropriate to implement the wikis during the session on the 29th. Prior sessions' scripts included an emphasis on using the Summary as a space that would be shared with other groups. A portion of the October 13th script stated that "sometimes you will be asked to contribute some work to the Summary tab, which is a more public space. The audience for the work you place in the Summary tab will be the other teams." In both schools, the teacher in the room read the following script at the start of the October 29th session.

Script for eMath Research Session of Wednesday, 29 October 2008
Rutgers Preparatory School and Long Branch High School

1. Before opening your room, click on the wiki link (green symbol with red circle on top) and read the summaries of last sessions for each of the four rooms. The entries in each box of the table will connect you to a screenshot of the solution to the The Pizza with Halves Problem offered by another team. Please review each of the other team's solution before entering your room. When in the room there is a wiki tab if you need to remind yourself of a group summary.

2. Please re-read The Pizza with Halves Problem since it has been altered to include an additional question. This question—number 4— should be the focus of your work today.

*Figure 4.* Script read to students in the classroom before they entered VMT Chat.

For each session and each team, there are four formats in which we can analyze the data: 1) The Concert Chat Player (Figure 5), 2) HTML line-by-line code with an author column to denote to which participant performed the action or chat entry, 3) HTML line-by-line, but each participant has a separate column instead of the one column for all authors, and 4) Excel spreadsheets or Word documents with tables of HTML files (Table 1). The spreadsheets and tables allowed for the addition of columns for coding purposes. The player files can be reviewed at variable speeds of real-time. There are sliding bars to move to a specific time or position on the page as needed. An automated transcriber transforms every recorded interaction with the environment into a table in HTML. The HTML tables can then be copied into Word and Excel tables and spreadsheets so that coding can be added.

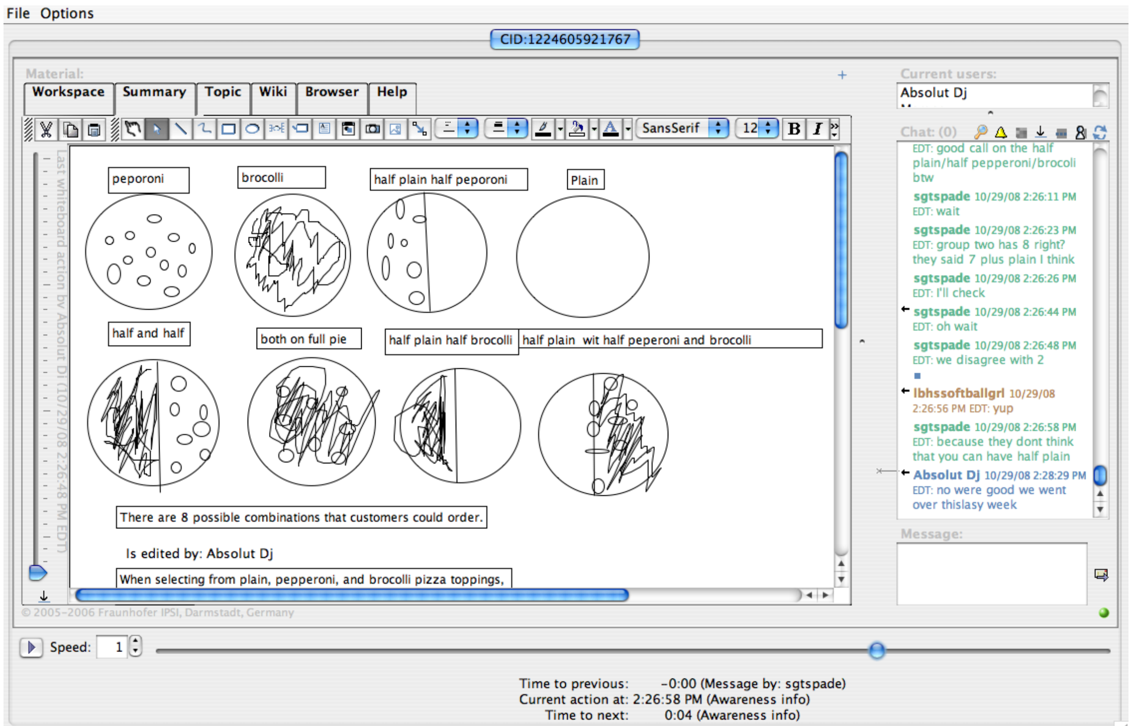


*Figure 5*. Concert Chat player with participant discussion of comparison of summaries in wiki.

## Results

Sgtspade, one of the students, engaged the linking tool as he reviewed each step of the wiki page. His actions as viewed in the HTML table, allowed us to track the pages as they were viewed. It is assumed that it was unintentional referencing because sgtspade never mentions, nor directs, teammates to look at the pages he references. In prior and subsequent sessions, sgtspade and others used the referencing tool and commented in the chat space directed at the point of interest. Nevertheless, he does discuss the specific material viewed on the wiki pages in his chat messages.

A limitation that was found during this research was categorizing. Once in the P4 code and once in the P5 code entries we recorded entries that were images and not typed chat. The VMT environment allows participants to communicate through their inscriptions in the chat and on the whiteboard. The creation of an object, or series of objects, on the whiteboard is a form of communication with the other team members in the chat room. In the case of the P4 stage, Absolut DJ creates a series of ellipses to represent the pizza with two toppings on one half. This action is interpreted as an integration of ideas from observing the wiki as well as dialogue with his team. Absolut DJ is testing this new knowledge and proposing a new idea that was previously not considered to be in the solution set. Later, Absolut DJ writes, "When selecting from plain, pepperoni, and broccoli pizza toppings, there are 8 possible combinations that can make up any one pie. As can be seen from the diagram above, once all possible half and full pie combinations of toppings are accounted for, there are exactly 8 different pizzas that can be ordered." While he summarizes this possible solution with his team, he enters this in the whiteboard as a textbox, which we code as P5. Entering textboxes is a common practice amongst the participants because we request that the students write a team summary. Since the placement of the text on the screen makes it an image as well as text, we alter IAM to include image and text as separate codes in each stage as students might respond differently to images and to text.

Table 2 is an example of the Excel spreadsheet used in the analysis, and shows a discussion that occurred during the October 23rd session. In this data, lbhssoftballgrl introduces one pizza not included in the original answer the team offered at the end of the October 23rd session. After a review of the wiki in the subsequent session the group decided that they had overlooked this pizza. In this chat lbhssoftballgrl asked "can u put two toppings on one half of the pizza?" Sgtspade did not think that it was possible, but he asks what the group thinks. The discussion closes when Absolut DJ says, "na lets keep it out." The students in this discussion consider the original statement and discuss an area of dissonance. It seems plausible that this interaction on October 23rd establishes the idea in the group and makes it easier to adopt as part of the solution on October 29th when it is seen that other groups include this as part of their solution.

Table 2. *Chat of Students During October 23rd Session with IAM Coding in Right Column*

| Chat Index | Time Start Typing | Time of Posting | Author | Content | IAM code |
|---|---|---|---|---|---|
| 69 | 13:41:26 | 13:41:36 | lbhssoftballgrl | can u put two toppings on one half of the pizza? | P1 |
| 70 | 13:41:44 | 13:41:47 | Sgtspade | No i dont think so | P1 |
| 71 | 13:42:43 | 13:42:46 | Sgtspade | That is a good question though | P2 |
| 72 | 13:42:46 | 13:42:52 | Sgtspade | it doesn't really say if we can or not | |
| 73 | 13:43:28 | 13:43:40 | Sgtspade | What do you guys think we should do? | P2 |
| 74 | 13:43:40 | 13:43:55 | lbhssoftballgrl | well it doesnt say we cant, so i think it would be safe to do so | P2 |
| | 13:44:29 | 13:44:43 | Sgtspade | [sgtspade has fully erased the chat message] | |
| 75 | 13:44:54 | 13:45:00 | Absolut DJ | na lets keep it out | P2 |

After initial introductions, during the October 29th session, Lbhssoftball enters, "[OK] so I think there are 8 pies when we really looked at it again right?" The students recognize that an eighth pizza is necessary. While reviewing the wikis of the other three teams, this team realized that they had not considered a pizza that was half plain and half peppers and pepperoni. Viewing the wiki allowed the group to reconsider this pizza and include it in their solution.

During a review of the data eight references to the wiki were found. An example of the referencing is available in Figure 3. Five of the eight references to the wiki were directed at wiki images of the other *e*Math teams. This referencing tool can be used to reinforce integration of ideas into team solutions.

## Discussion

In our data, we found 40 entries that relate to the use of the wiki in the VMT environment and coded them using the IAM framework. We coded ten entries as P1 and ten as P2, six as P3, eight as P4, and six as P5. Our data shows 35% of the codes in the P4 and P5 areas of meta-cognitive skills which differs greatly from the findings of Gunawardena, Lowe, and Anderson (1997) where over 92% of the codes were recorded as P1, sharing and comparing information, but not processing the information and drawing conclusion. It is possible that the technology and construction of the task created a setting for comments that were in a wider range of stages. The nature of a debate through an asynchronous forum is different from a question that asks students to work as a team and compare their solution with that of other groups. The effect of task design could explain the increased percentage of higher-level thinking. Further research in this field is necessary to study the ways technological advancements can be used by teachers to increase student use of meta-cognitive skills in problem solving and analysis of information.

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# LEARNING ABOUT PROOF BY BUILDING CONJECTURES

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*In this paper we propose a geometry technology-based activity appropriate for the middle school level that can be used to improve reasoning and proof writing skills. The activity uses the free, open source software GeoGebra and gives the student practice in conjecture building, an activity found to help students understand and improve proof writing skills. By building and testing conjectures, students can engage in advanced mathematical skills at their own level of expertise.*

The goal of the mathematics teacher is for her/his students to engage in advanced mathematical thinking at any grade level (Tall, 1991; Dreyfus, 1991). Building conjectures in geometry that lead to formal and informal proof is one example of advanced mathematical thinking (Boero, Garuti, Lemut, & Mariotti 1996; Hanna, 2000). NCTM (2000) recommends reasoning and proof be a fundamental part of the mathematics curricula at all levels, from pre-kindergarten through grade 12. Researchers identify proof writing as one of the most challenging aspects of the mathematics curricula in the U.S. and internationally (Senk, 1985; Healy & Hoyles, 2000; Galbraith, 1981). While introducing proof in the early grades might seem difficult, researchers find that even upper elementary students can deal with proof ideas and can be successful in mathematical activities related to proof (Lester, 1975). Educators tend to see proof as something obtainable only for a small minority of students and, therefore, attention to proof writing in the classroom although desired is often neglected (Knuth, 2002). One way to booster proof writing skills may lie in the ability to make conjectures or hypotheses based on empirical results. There is a direct connection between reasoning skills associated with conjecturing and reasoning skills used in proof writing (Boero, Garuti, Lemut, & Mariotti, 1996). Researchers argue that during the process of making a conjecture, the student must work through internal arguments and sort through which are plausible, similar to the activity a mathematician goes through when building a proof (Boero, Garuti, Lemut, & Mariotti, 1996). These researchers propose that the process of building conjectures should be emphasized more in mathematics instruction. The process of making and testing a conjecture is made easier by the increased use and popularity of dynamic geometry software (Clements & Battista, 1992; Hanna, 2000; Edwards & Jones, 2006).

In this paper, we explore the use of a free, open source, interactive geometry software, GeoGebra, to introduce a conjecturing activity at the middle school level. By building the students' reasoning and conjecturing skills, we are laying the groundwork for their future success in formal proof and logic.

## The Role of Technology in Building Conjectures and Constructing Proofs

Researchers have explored the possible benefits and drawbacks of the increased use of technology in the classroom (Harel & Sowder, 2006; Mariotti, 2000; Hadas, Hershkowitz, & Schwartz, 2000; Hanna, 2000). Educators express a concern that technology might give students a disincentive to seek a more formal proof (Harel & Sowder, 2006). However, there is strong evidence that using dynamic geometry software can help students realize a need for formal proofs (Marrades & Gutierrez, 2000; Hadas, Hershkowitz, & Schwartz, 2000). Although the transition from empirical to abstract ways of conjecture and justification can be a slow process, dynamic geometry software, such as Cabri, Geometer Sketchpad, and GeoGebra, have been shown to improve proof writing skills (Marrades & Gutierrez, 2000).

In this paper we use the GeoGebra software package to present a geometry activity that allows students to make and test assertions and prepare for more formal proof writing. We chose this particular software because it is freely available on-line, it is supplemented with a variety of dynamic worksheets, and it is user friendly (www.GeoGebra.org). The unique feature of GeoGebra is the integration of dynamic geometry software and a computer algebra system into a single tool for mathematics education (Hohenwarter & Preiner, 2007; Edwards & Jones, 2006). Students can build a geometric construction and simultaneously observe how changes in a formula in the algebra window are affected by manipulation of the construction and vice versa. Teachers can use GeoGebra to create interactive web pages or dynamic worksheets to develop student activities as we have done here. By participating in the technology based activity below, the student will engage in advanced mathematical thinking that leads to building conjectures.

## A Proof Enriching Activity for the Middle School Classroom

In the first challenge of this activity, students are asked to divide a rectangular garden into two equivalent regions. Students are asked to find as many possible solutions as they can. The dynamic software allows them to easily manipulate the fence line as well as observe changes in the area of each region (see figure 1).

The dynamic geometry software also allows the student to construct a conjecture about a common feature of all presented solutions. In this problem, the generalization is as follows: Every line segment that goes through the center of the rectangle, the point of intersection of the diagonals, will divide the rectangular region into two equivalent areas.

Based on our classroom experience, students first typically present four solutions (see figure 2) and, after further experimentation, may find other possible segments (see figure 3).

The second challenge in this activity describes a rectangular garden that must be built around a fixed rectangular structure. The objective is still to divide the garden with a straight fence into two equivalent regions (see figure 4). The conjecture students' developed in the first challenge assists them in finding a solution to this more advanced problem. The generalization for the second challenge is as follows: Regardless of the position or the size of the enclosed rectangular structure, the line segment that goes through both rectangle centers will divide the region into two equivalent parts. Depending on the level of the students' geometry development, the teacher may chose to extend the activity further by replacing the interior rectangular structure with another geometric figure (e.g. a circle, triangle, or pentagon) and asking similar questions. A further extension could be a 3-dimensional model of this problem, replacing the rectangle with a prism and the line segment with a 2-dimensional plane.

This activity presents an equal challenge regardless of the students' content background and can be easily adapted for students at various levels of geometric development. Activities such as the one presented, show students how mathematicians think through a problem, using reasoning to construct and test conjectures. By building and testing conjectures with technology as a tool, students make further progress toward advanced mathematical thinking.

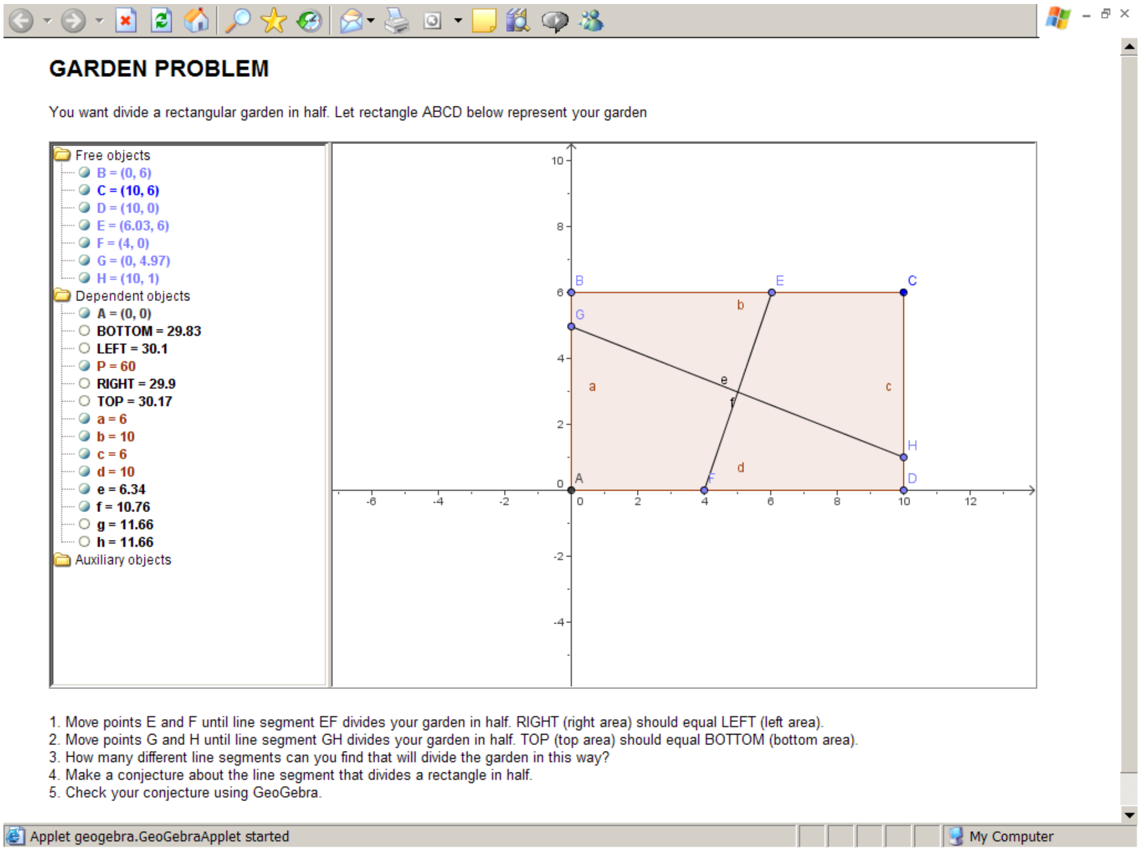


*Figure 1.* A dynamic worksheet created for this activity.

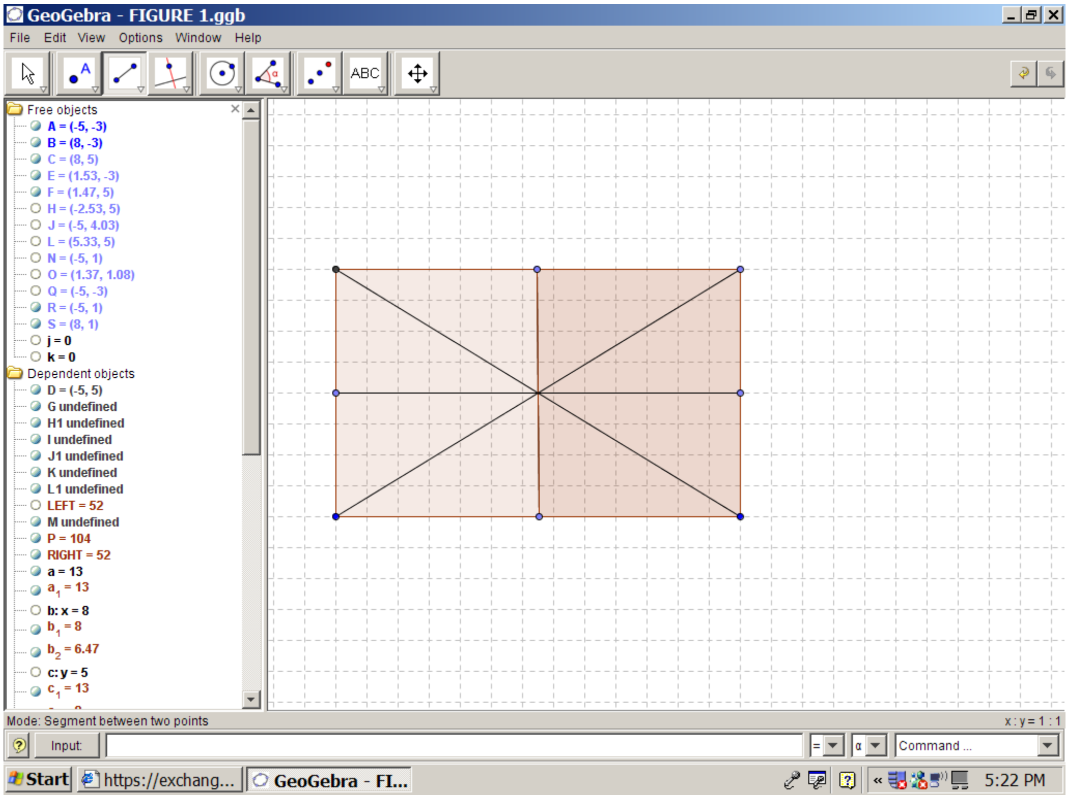


*Figure 2.* Student sample work 1.

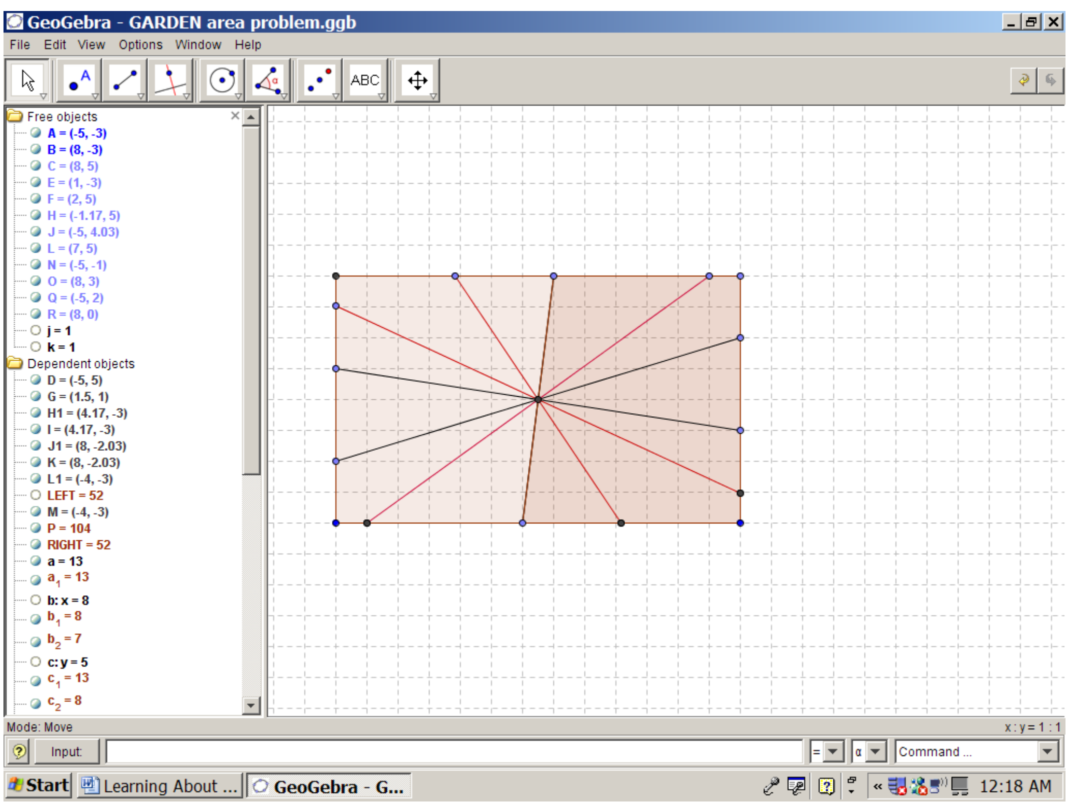


*Figure 3.* Student sample work 2.

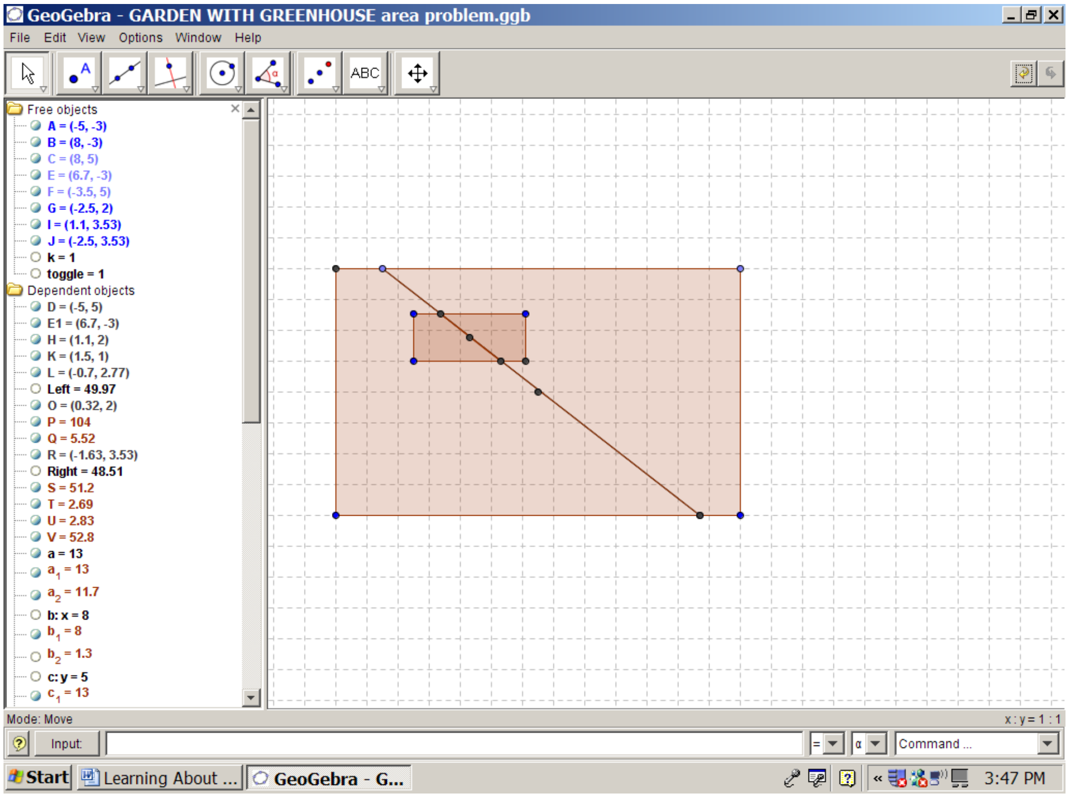

*Figure 4.* Student sample work 4.

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# INCORPORATING WEB TECHNOLOGY INTO THE DEVELOPMENT OF MATHEMATICS TEACHING MATERIALS

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*This study reports thirteen school teachers' efforts to develop standards-based mathematics teaching materials on an interactive Website. The merits of Web technology demonstrated by participating teachers in their development of mathematics teaching materials were addressed. Participating teachers' limitations on incorporating Web technology into the development of mathematics teaching materials were also discussed.*

## Introduction

Considering the enormous growth of the Web and the educational attention to it, school teachers should use the Web in an effective manner. Most Web education literature focuses on Web-based instruction and learning rather than promoting the development of Web itself. In order to develop and optimize school teachers' adaptability in using the Web, all participating teachers in this study would be expected to develop standards-based mathematics teaching materials on their own Websites.

## Theoretical Framework and Research Questions

The theoretical framework of the Learning Cycle Approach recommended by Gabel (2003) was adopted for use in the development of mathematics teaching materials. The Learning Cycle Approach involves three phases: exploration, invention, and application. Two research questions guided this study: (a) what merits of Web technology are demonstrated from the participating teachers' development of mathematics teaching materials; and (b) what limitations are demonstrated from the participating teachers' use of Web technology on their development of mathematics teaching materials?

## Methodology

At the exploration phase, participating teachers were asked to explore a challenging mathematics topic that they intended to work on but lacked sufficient supporting materials from the Web society of mathematics education. At the invention phase, teachers would need to support their chosen topic by inventing standards-based mathematics teaching materials across three or more grade levels. At the application phase, teachers were required to incorporate the merits of Web technology into their newly developed mathematics teaching materials.

## Findings

This study found that participating teachers acquired valuable information to support their invention of teaching materials via Web exploration. They enriched their mathematics teaching materials by linking to various supporting Websites, and shared their own mathematics teaching materials via their own Websites. Several teachers invented unique mathematics teaching materials on their Websites. However, participating teachers did not incorporate the benefit of interactive Web capabilities by soliciting ideas to enrich their teaching materials. Several determined topics did not reflect a high level of difficulty and did not meet with the needs for the

Web society of mathematics education. Dynamic Web communications and collaboration did not occur among participating teachers. Several teachers had difficulty coping with technical issues such as condensing file sizes and editing video clips, and showed limitations in incorporating various technology tools and resources into their development of teaching materials.

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# INCORPORATING ICT IN MATH AND SCIENCE HIGH SCHOOL CLASSROOMS

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*This contribution presents five different teacher ways of incorporating ICT in mathematics high school classrooms. They were instrumented by 15 teachers after they had participated in an on line training course of six months (see "Specialization on Mathematics and Technology", at: http://upn.sems.gob.mx). Observational data were issued from the videos participant teachers did at the end of the course, showing the meaning they had assigned to their math and technological experience acquired along the six mentioned months by accomplishing series of pedagogical activities and tasks using mathematics technology (Zbiek & Hollebrands, 2008).*

## Theoretical Underpinnings, Methodology and Results

Ruthven (2007) proposes examining teacher response to new technologies, based on Kerr's (1991: 121) assertion that "If the technology must find a place in classroom practice, it must be examined in the context of class life, [just] as the teachers life it." (Cited by Ruthven, ibidem: 3).

There were some pedagogical tasks included in the on line course that resulted specially productive in according with Kerr´s point of view: to (a) choose one math topic from the high school curriculum, along with digital materials you would like to use in your class; b) organize the instrumentation of the activity into the classroom; c) video-record that work session; d) upload a seven minute version of that recording to YouTube; and e) finally upload to the training platform (http://upn.sems.gob.mx) a descriptive report of the video's content together with its URL. From teachers' accomplishment of this task series we obtained the data that allowed us to construct the case study we are reporting here, on the incorporation of technology into high school classes. In addition, the on line teacher training course included pedagogical sequences of activities on the use of dynamic software (eg. Logo, Geogebra, Aplusix, Excel, RecCon, FunDer), along with the exploration of a wide range of possibilities set up on the Internet, as the library of virtual materials of Utah University (see: http://nlvm.usu.edu/en/nav/vlibrary.html).

From the obtained data, we found five different ways that teachers used to incorporate mathematics technology in the classroom: a pattern of incorporation that flowed from the classic approach to teaching; a modified version of this pattern that added teacher interaction with the students, basically by means of teacher questioning; instrumentalization of the activity directed by a script; orchestration (Trouche, 2000) of the activity using different instruments or artifacts, plus group negotiation of meaning; and finally, organization of cooperative work centered on student appropriation of technology.

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# MATHEMATICAL HABITS OF MIND

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*The idea of "mathematical habits of mind" has been introduced to emphasize the need to help students think about mathematics "the way mathematicians do." There seems to be considerable interest among mathematics educators and mathematicians in helping students develop mathematical habits of mind. The objectives of this working group are: (a) to discuss various views and aspects of mathematical habits of mind, (b) to explore avenues for research, (c) to encourage research collaborations, and (d) to interest doctoral students in this topic. To facilitate the discussion during the working group meetings, we provide an overview of mathematical habits of mind, including concepts that are closely related to habits of mind—ways of thinking, mathematical practices, knowing-to act in the moment, cognitive disposition, and behavioral schemas. We invite mathematics educators who are interested in habits of mind, and especially those who have conducted research related to habits of mind, to share their work during the first working group meeting. If you would like to give a 10-minute presentation, please contact Kien Lim or Annie Selden in advance.*

## An Overview of Mathematical Habits of Mind

There are several terms and points of view in mathematics education that are somewhat similar or support each other, and might be brought together under the single phrase "mathematical habits of mind." We discuss several of these views that we see as related.

*Habits of mind* were introduced by Cuoco, Goldenberg, and Mark (1996) as an organizing principle for mathematics curricula in which high-school students and college students think about mathematics the way mathematicians do. They asserted:

> The goal is … to help high school students learn and adopt some of the ways that mathematicians *think* about problems. … A curriculum organized around habits of mind tries to close the gap between what the users and makers of mathematics *do* and what they *say*. … It is a curriculum that encourages false starts, calculations, experiments, and special cases. (p. 376)

They identified two broad classes of habits of mind: (a) general habits of mind that cuts across every discipline, and (b) content-specific habits of mind for the discipline of mathematics. General habits of mind include "pattern-sniffing," experimenting, formulating, "tinkering," inventing, visualizing, and conjecturing. Mathematical habits of mind, or mathematical approaches to things, include *talking big thinking small* (e.g., instantiating with examples), *talking small thinking big* (e.g., generalizing, abstracting), thinking in terms of functions, using multiple points of view, mixing deduction and experiment, and pushing the language (e.g., at first assuming the existence of things we want to exist, such as $2^0$).

Habits of mind have two important characteristics: the "thinking" characteristic and the "habituated" characteristic. In addition, habits of mind are reflexively related to classroom practices. Below we discuss various related views of habits of mind.

### *The Thinking Characteristic*

Harel's (2007, 2008) notion of *ways of thinking* underscores the thinking aspect of habits of mind. Harel (personal communication) regards habits of mind as internalized *ways of thinking*. In

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

Harel's view, mathematics consists of two complementary subsets: (a) the first consists of institutionalized *ways of understanding*, which is a collection of established definitions, axioms, theorems, proofs, problems, and solutions that have been accepted by the mathematical community; and (b) the second is a collection of ways of thinking, which are conceptual tools that are useful for the generation of the first subset (Harel, 2008). The distinction between ways of thinking and ways of understanding underscores the importance of mathematical habits of mind, which tend to be neglected in traditional mathematics curricula.

According to Harel's *duality principle* (2007), "Students develop ways of thinking only through the construction of ways of understanding, and the ways of understanding they produce are determined by the ways of thinking they possess" (p. 272). This principle asserts that ways of thinking cannot be improved independently of ways of understanding, and vice versa. Hence, Harel advocates that both ways of understanding and ways of thinking should be incorporated as learning objectives for students.

In their introductory article to a special issue on advanced mathematical thinking that considered symbolizing, mathematizing, algorithmatizing, defining, and reasoning, Selden and Selden (2005) stated:

> Sometimes referred to as "mathematical habits of mind" or "mathematical practices," these [aforementioned specific] ways of thinking about and doing mathematics may be fairly widely regarded as productive, but are often left to the implicit curriculum. (p. 1)

Also, according to Bass (2005), mathematical habits of mind are critical to many aspects of the educational process. He argued that:

> the knowledge, practices, and habits of mind of research mathematicians are not only relevant to school mathematics education, but that this mathematical sensibility and perspective is essential for maintaining the mathematical balance and integrity of the educational process—in curriculum development, teacher education, assessment, etc. (p. 418)

Bass (2008, January) has considered habits of mind as *practices*—things that mathematicians do. Such practices include asking 'natural' questions, seeking patterns or structure, consulting the literature and experts, making connections, using mathematical language with care and precision, seeking and analyzing proofs, generalizing, and exercising aesthetic sensibility and taste. Bass claims that children can, and should, cultivate these practices from their early school years on. By capitalizing on children's curiosity their inquisitive minds can be harnessed. Goldenberg (2009, January) offered some strategies that capitalize on children's phenomenal language-learning ability and abstracting-from-experience ability to develop certain algebraic ideas such as *breaking* [apart] *numbers and rearranging parts* (commutative property, associative property), and *breaking arrays and describing constituent parts* (distributive property). Goldenberg provided evidence to show that children can indeed use "algebra" as a language to describe a process or a pattern and to express what they already know.

For Leikin (2007), "employing habits of mind means inclination and ability to choose effective patterns of intellectual behavior" (p. 2333). With respect to the mental habit of solving problems in different ways, Leikin considers a problem-solving strategy as a habit of mind when it is within one's "personal solution spaces of many problems from different parts of [the] mathematical curriculum" (p. 2336). One goal of mathematical instruction is then to move solutions from students' *potential solution spaces* (containing solutions that are produced with the help of others; i.e., solutions that are within one's *zone of proximal development*) into their *personal solution spaces*.

*The Habituated Characteristic*

The *habituated* character of habits of mind is underscored in Goldenberg's description of habits of mind, which "one acquires so well, makes so natural, and incorporates so fully into one's repertoire, that they become mental habits—one not only can draw upon them easily, but one is likely to do so" (p. 13). Mason and Spence's (1999) notion of *knowing-to act in the moment* accentuates this habituated character. They have differentiated between two types of knowledge. The first type, referred to as *knowing-about*, consists of Ryle's (1949, cited in Mason & Spence) three classes of knowledge: *knowing-that* (factual knowledge), *knowing-how* (procedural skills), and *knowing-why* (personal stories to account for phenomena). The second type, referred to as *knowing-to*, is tacit knowledge that is context/situation dependent and becomes present in the moment when it is required. This distinction is important because "knowing to act when the moment comes requires more than having accumulated knowledge-about . . ." (Mason & Spence, 1999, p. 135).

> Knowing-about … forms the heart of institutionalized education: students can learn and be tested on it. But success in examinations gives little indication of whether that knowledge can be used or called upon when required, which is the essence of knowing-to. (p. 138)

Mason and Spence advocate the practice of reflection as a means to help students improve their knowing-to act in the moment. Students should be encouraged to reflect on (a) what they have done after an action, and (b) what they are doing while enacting it, which were termed by Schön (1983) *reflection-on-action* and *reflection-in-action* respectively. With respect to reflection-in-action, students should routinely ask themselves "What do I know?" and "What do I want?" (Mason & Spence, p. 154).

The habituated character of habits of mind is also reflected in Lim's (2008) notion of *spontaneous anticipation* by a student—when he or she immediately anticipates and carries out an action for a situation based on the first idea that comes to mind. Whereas Cuoco, Goldenberg, and Mark's (1996) notion of habits of mind has a positive connotation, Lim's spontaneous anticipation can be either desirable or undesirable. *Interiorized anticipation* is desirable in that "one spontaneously proceeds with an idea without having to analyze the problem situation because one has interiorized the relevance of the anticipated action to the situation at hand" (p. 45). Interiorized anticipation is similar to Mason and Spence's notion of knowing-to. *Impulsive anticipation*, on the other hand, is undesirable in that "one spontaneously proceeds with an idea that comes to mind, without analyzing the problem situation and without considering the relevance of the anticipated action to the problem situation" (p. 44).

Lim notes that a habit of mind can also be regarded as a *cognitive disposition*—a tendency to act, mentally, in a certain way in response to certain situations. When a person has a particular habit of mind, he or she has a disposition to act according to that habit of mind. Lim (2009, January) uses the term *impulsive disposition* to refer to the proclivity of "doing whatever first comes to mind … or diving into the first approach that comes to mind" (Watson & Mason, 2007, p. 207). Lim (2009, January) offered the following strategies to address impulsive disposition: (a) do not teach algorithms and formulas prematurely; (b) pose problems that necessitate a particular algorithm or concept, that intrigue students, that require students to attend to the meaning of numbers and symbols, and that require students to explain and justify; (c) include contra-problems to promote skepticism; and (d) include superficially-similar-but-structurally-different problems on tests and examinations.

Selden and Selden (2009, January) have conceptualized (small) habits of mind as habitual situation-action pairs or *behavioral schemas*—"a form of (often tacit) procedural knowledge that yields immediate (mental or physical) actions." They are developing this perspective in the context of proving in a design experiment with advanced undergraduate and beginning graduate students (Selden, McKee, & Selden, 2009), and in a teaching experiment with mid-level undergraduate real analysis students. Indeed, the entire proving process might be seen as a sequence of mental or physical actions (that cannot be fully reconstructed from the written proof). The individual actions often appear to be due to the enactment of behavioral schemas (that is, small, simple habits of mind). Here is an example of a common beneficial behavioral schema. The situation is having to prove a universally quantified statement such as, "For all real numbers *x*," and the linked action is writing into the proof something like, "Let *x* be a real number," meaning *x* is arbitrary but fixed. While some students are at first reluctant to write this, doing so can become habitual and automated, that is, become a behavioral schema and eventually just seems to be "the right thing to do." In contrast, a detrimental behavioral schema in proving is focusing on the hypotheses of a theorem too soon, and simply "forging ahead," without first examining the conclusion to see what is to be proved. Selden and Selden think it is likely that some larger, more complex, habits of mind can be decomposed into behavioral schemas. Also, they think this perspective would probably be useful in other kinds of reasoning, such as problem solving, and with K-12 students.

Selden and Selden think that focusing specifically on small habits of mind has two advantages. First, the uses, interactions, and origins of behavioral schemas are relatively easy to examine. For example, behavioral schemas tend to reduce the burden on working memory. Also, the process of enactment of a behavioral schema occurs outside of consciousness, but apparently the triggering situation must be conscious. Thus, such schemas cannot be "chained together" outside of consciousness with only the final action being conscious (Selden & Selden, 2008). For example, one cannot produce the solution to a linear equation without being conscious of the intervening steps. Second, this perspective is not only descriptive but also suggests concrete teaching actions, such as encouraging the writing of the formal-rhetorical parts of a proof at the beginning of the proving process (Selden & Selden, in press). In this way, it is fairly easy for a teacher to devise ways of helping a student strengthen a beneficial, or weaken a detrimental, behavioral schema.

*Relating Habits of Mind and Classroom Practices*

In *Fostering Algebraic Thinking: A Guide for Teachers Grades 6-10*, Driscoll (1999) views habits of mind as ways of thinking, that when used habitually, can lead to successful learning of algebra. He stresses the development of three algebraic habits of mind: (a) *doing/undoing* which involves reversing mathematical processes; (b) *building rules to represent functions* which involves pattern-recognition and generalization; and (c) *abstracting from computation* which involves thinking about computations structurally without being tied to specific numbers, such as recognizing the equivalence of 5% of 7000 and 7% of 5000. He and his colleagues later developed a four-module toolkit for educators to work with teachers to learn how to foster these algebraic habits of mind in their classrooms (see Driscoll et al., 2001). Subsequently in *Fostering Geometric Thinking: A guide for teachers grades 5-10*, Driscoll, DiMatteo, Nikula, and Egan (2007) promote four geometric habits of mind: (a) reasoning with relationships, (b) generalizing geometric ideas, (c) investigating invariants, and (d) sustaining reasoned exploration by trying different approaches and stepping back to reflect while solving a problem. *The Fostering Geometric Thinking Toolkit* was published a year later (see Driscoll et al., 2008).

Cuoco (2008, January) has advocated making mathematical habits of mind a key component of the syllabus because "without explicit attention to mathematical ways of thinking, the goals of 'intellectual sophistication' and 'higher order thinking skills' will remain elusive." He offered some suggestions for helping students cultivate desirable habits of mind: (a) working on problems with students, (b) being explicit about one's own thinking, and (c) making thought experiments an integral part of the learning experience. Rasmussen (2009, January) emphasized the need for teachers to be deliberate about initiating and sustaining particular classroom norms so as to promote certain desirable habits of mind and effect students' beliefs and values.

The RAND Mathematics Study Panel (2003) referred to "mathematical know-how—what successful mathematicians and mathematics users *do*" (p. 29) as *mathematical practices*. They also identified mathematical practices as one of the three foci for a proposed research and development program aimed at improving mathematical proficiency among U.S. school students. The Panel stated:

- "A focus on understanding these practices and how they are learned could greatly enhance our capacity to create significant gains in student achievement, especially among currently low-achieving students who may have had fewer opportunities to develop these practices" (p. 29)
- "These practices are not, for the most part, explicitly addressed in schools. Hence, whether people somehow acquire these practices is part of what differentiates those who are successful with mathematics from those who are not" (p. 32-33)

The Panel recommended the following lines of research: (a) developing an understanding of specific mathematical practices, and their interactions, along the domains of representation, justification, and generalization; (b) examining the use of these mathematical practices in different settings (e.g., in school, at home, at work); and (c) investigating ways for developing these practices in classrooms. Further, the Panel stated that "such [mathematical] practices must be deliberately cultivated and developed, and therefore research and development should be devoted to addressing this challenge." (p. 40)

Many theoretical ideas and pedagogical suggestions related to habits of mind have been raised. However, the research on this topic is still relatively thin. Using Cobb and Yackel's (1996) emergent perspective in which "learning is a constructive process that occurs while participating in and contributing to the practices of the local community" (p. 185), we regard mathematical habits of mind as individual dispositions that are reflexively related to mathematical practices of a classroom community. Cobb and Yackel suggested that "analysis whose primary purpose is psychological should be conducted against the background of an interactionist analysis of the social situation in which the student is acting" (p. 188). Hence, we encourage research on understanding the interaction between individual mathematical habits of mind and classroom mathematical practices, in addition to research on how students develop mathematical habits of mind.

## Purpose of this Working Group

This working group is a follow-up to two panel-discussion sessions on "Helping Students Develop Mathematical Habits of Mind" at two consecutive Joint Mathematics Meetings (JMM) of the American Mathematical Society and the Mathematical Association of America held in 2008 and 2009. The presenters-cum-panelists at the JMM 2008 session in San Diego included Hyman Bass, Al Cuoco, Guershon Harel, and Annie Selden. The presenters-cum-panelists at the JMM 2009 session in Washington DC included Hyman Bass, Paul Goldenberg, Kien Lim, Chris Rasmussen, Annie Selden, and John Selden. Both sessions were well attended and well received by the audience. The second session was in fact an encore of the first session. Based on attendance reactions to these two sessions, there seems to be considerable interest among mathematics educators and mathematicians in this topic. This PME-NA working group can offer a platform for mathematics educators who are interested in this topic to explore research opportunities.

The primary purpose of this working group is to generate interest among mathematics educators for conducting research related to mathematical habits of mind. The second purpose is to encourage research collaborations. The objectives of this working group are:

- To discuss various views and aspects of mathematical habits of mind.
- To explore avenues for future research.
- To facilitate mathematics educators with similar research interests to form research groups.
- To motivate doctoral students who may plan to work on this topic for their dissertations.

## Proposed Activities for this Working Group

Meeting 1

- An overview on mathematical habits of mind.
- Individual presentations, if any, on research related to habits of mind.
- An open forum to discuss theoretical and pedagogical issues related to mathematical habits of mind.
- A brainstorming session to identify worthwhile avenues of research.

Meeting 2

- Small-group breakout sessions to identify research opportunities, formulate research questions, and discuss research designs.

Meeting 3

- Small-group presentations of plans for research.
- Discussion of next steps.

## Anticipated Follow-up

We anticipate that promising avenues for research related to mathematical habits of mind will be identified. The working group may broaden the scope of research for some mathematics educators by integrating their existing research with research on mathematical habits of mind. This working group is likely to continue if there are groups of researchers who plan to conduct collaborative research on this topic. There may be a possibility of eventually having a special issue of a journal dedicated to mathematical habits of mind.

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# LESSON STUDY WORKING GROUP

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The Lesson Study Working Group met at three previous PME/NA conferences: Merida, Mexico (2006), Lake Tahoe (2007) and Morelia, Mexico, (2008). Over 20 researchers attended each meeting. At the sessions participants shared current research around lesson study and discussed future plans and goals. Several projects shared artifacts from their work. The group discussed at length current trends, issues, and problems facing research and implementation of lesson study. Following is a summary of our discussions.

## Lesson Study: Structures, History, and Variation

Lesson study incorporates characteristics of effective professional development programs identified in prior research: it is site-based, practice-oriented, focused on student learning, collaboration-based, and research-oriented (Bell & Gilbert, 2004; Borko, 2004; Cochran-Smith & Lytle, 1999, 2001; Darling-Hammond, 1994; Wang & O'Dell, 2002; Little, 2001; Hawley & Valli, 1999; Wilson & Berne, 1999). What separates lesson study from other instructional improvement approaches is that it places teachers at the center of the professional activity, with their interests and desire to better understand student learning based on their own teaching experiences. The idea is simple: teachers organically come together with a shared question regarding their students' learning, plan a lesson to make student learning visible, and examine and discuss what they observe. Through multiple iterations of the process, teachers have many opportunities to discuss student learning and how their teaching affects it.

After identifying a lesson goal, teachers plan a lesson. The goals can be general at first (e.g., how students understand equivalent fractions), and are increasingly refined and focused throughout the lesson study process to become specific research questions at the end (e.g., strategies students use to compare 2/4 and 3/6). Teachers choose and/or design a teaching approach to make student learning visible, keeping their lesson goal in mind. The main purpose of this step is not to plan a perfect lesson but to test a teaching approach (or investigate a question about teaching) in a live context to study how students learn. As they plan, they anticipate students' possible responses and craft the details of the lesson. Teachers come to know the key aspects of the lesson, to anticipate how students may respond to these aspects, and to explore different thinking and reasoning that may lie behind the possible responses. During planning, teachers also have an opportunity to study curricular materials, which can help teachers' content knowledge development. During the lesson, teachers attend to student thinking and take notes on different student approaches. During the debriefing after the lesson, teachers discuss the data they have collected during the observation**.**

There are other professional development programs that incorporate many of the characteristics of lesson study (e.g., action research, teacher research). However, what sets lesson study apart is the *live* research lesson. The live research lesson creates a unique learning opportunity for teachers. Shared classroom experiences expose teachers' professional knowledge that may otherwise not be shared: teachers notice certain aspects of teaching and learning, and this implicit and organic noticing does not happen in artificially replicated professional development settings.

In Japan, lesson study has been widely used for over a century. Many Japanese educators attribute success in changing their teaching practice to participation in lesson study (Lewis, Perry, &

Murata, 2006; Murata & Takahashi, 2002: Shimizu, et. al., 2005). As a foundational mechanism to support the improvement of teaching, lesson study is used to examine and better understand new educational approaches, curricular content, and instructional sequences introduced in Japan. In many cases, teachers play the central role in making new approaches adoptable and content accessible. Lesson study makes teaching approaches more practical and understandable to teachers through developing deeper understanding of content and student thinking. In this manner, lesson study works effectively to connect theory and practice.

While lesson study is known in the United States (and other parts of the world) as a small, school-based collaboration, typically in the subject area of mathematics, lesson study comes in many different shapes and sizes in Japan. There is small and school-based lesson study as well as large-scale, national-level lesson study (Murata & Takahashi, 2002; Lewis & Tsuchida, 1998; Shimizu, et. al., 2005). Different formats for lesson study meet different needs and interests of the teachers. A typical Japanese teacher has multiple opportunities to participate in lesson study throughout his/her professional career.

## Introducing Lesson Study to the World

Lesson study came to attract the attention of an international audience in the past decade, and in 2002 it was one of the foci for the Ninth Conference of the International Congress on Mathematics Education (ICME). It subsequently spread to many other countries and more than a dozen international conferences and workshops were held around the world in which people shared their experiences and progress with lesson study as they adopted this new form of professional development in their unique cultural contexts (e.g., Conference on Learning Study, 2006; Fujita, et. al., 2004; Lo, 2003; National College for Educational Leadership, 2004; Shimizu, et. al., 2005).

## Issues of Fidelity of Implementation

There are several issues and concerns around *implementing* lesson study in diverse settings. For example, there are unique issues of implementation with preservice or inservice teachers. Other issues include: content knowledge competency, variations in curricula, time and availability to meet, administrative support and cultural differences. As a result of these difficulties in maintaining fidelity of implementation with the Japanese model, lesson study is being *adapted* to meet unique needs in a variety of settings.

The limited depth of mathematical knowledge of some teacher groups attempting to implement lesson study, particularly at the elementary-level, has raised the question of whether lesson study work can be completely teacher-driven. Related to this issue is the role the outside coach or expert should or could play in such a lesson study community. A second issue is implementation within existing, traditional and/or rigidly structured curricula. Unlike the Japanese mathematics curriculum which provides a loosely defined framework for teachers to build off of, many curricula used in other countries are quite structured or scripted and not conducive to the planning cycles used in Lesson Study. A third issue is a lack of administrative support necessary to alter existing curricula, provide financial support, and schedule opportunities to meet and plan. The daily schedule of most *elementary* teachers prevents regular meetings and opportunities to collaborate. A related issue is the limited extrinsic reward available. Lesson Study presupposes blocks of time for teachers to work together. This frequently must be outside school hours and districts are often constrained by contracts that demand stipends and/or release time. Finally there may be fundamental differences in the cultures of teachers from different communities and countries. It was suggested that the fiercely independent nature of some cultures may limit success in building collaborative groups.

## Research on Lesson Study

The body of knowledge about lesson study is growing, but remains somewhat elusive and composed of discrete research endeavors. While the literature suggests that lesson study can facilitate greater reflection and more focused conversations about teaching and learning than are often realized with other types of professional development (Lewis, 2002), as well as specific and authentic conversations about management, student learning, and impact of significant and subtle changes in lesson design (Marble, 2006), there is still much to be learned.

The Lesson Study Working Group is attempting to explore the existing research on lesson study and synthesize the work into a resource for other researchers and educators. Following are abstracts from current contributions.

*Lesson Study: A Partnership Model for Studying Students' Thinking*

Alice Alston, Lou Pedrick, Kim Morris, and Roya Bassu, (Rutgers University, US)

Faculty and staff of the Robert B. Davis Institute for Learning in the Graduate School of Education at Rutgers are engaged with teachers and administrators of partnering districts in implementing school-based professional development using a modified form of Lesson Study. During a semester a group of ten teachers resourced by two university researchers worked together to develop a series of mathematical tasks intended to embody concepts that are basic to their district's curriculum and address specific mathematical goals that they had identified as important for their students. The series of tasks were implemented in six classes including grades 5 through 8 that were taught by members of the group during several weeks at the end of the term. The teacher-researchers, studying the videotapes, observer notes and student work from their session, select, transcribe and analyze critical events from each class that provide evocative examples of the mathematical strategies and representations of their students, These analyses are shared in follow-up discussions and compiled to produce an overall analysis of the development of the mathematical ideas as evidenced in the mathematical activity of the students across the four grades involved in the Lesson Study project. The research of the university educators is based on data that includes notes from the earlier sessions when goals were set and the tasks developed as well as the videotapes of the implementations,, debriefing discussions, and the subsequent analyses and group discussions of the teachers. This analysis focuses on the teachers' reflections and actions for evidence of a shift from surface characteristics of the classroom activity toward a closer attention to students' thinking and subsequent implications for instruction.

*What's Going on Backstage? Revealing the Work of Lesson Study*

Catherine D. Bruce & Mary Ladky, (Trent University, Canada)

This study describes some of the key findings related to teacher activity during a grade 4-10 lesson study project and provides an enhanced model of the lesson study stages, highlighting the backstage work within the cycle and thereby contributing to a more nuanced understanding of the process of lesson study. We explore some of the less documented teacher activity - what we are calling the "backstage work" - identified by the teachers in our lesson study research as they moved between and from stage to stage. During focus group interviews, researchers asked teachers to describe the activity that took place between the formal stages of the cycle. Further, in whole group discussions, 12 teachers built a new model illustrating their activities beyond the explicit four stages of lesson study that formed the backbone of their work. For example, teachers identified five main backstage activities between stage 1 (goal setting) and stage 2 (planning) of the lesson study cycle. These are: 1) searching for research on the Internet, in data bases, and in teacher resources about the topic in focus; 2) Conceptualizing: brainstorming, self talk and informal conversations, going off on valuable tangents; 3) Investigating the use of manipulatives and technological tools with students to expand the teacher and students' repertoire; 4) Keeping up with details such as on-going student assessment which provides insights into student learning and assists in the planning of the lesson; and 5) Team building and developing trust amongst lesson study team members. Other specific activities between stages 2 & 3, 3 & 4, and 4& 1 were also identified by teacher participants and are described in the paper**.** As researchers, we recognise this backstage work as essential to a successful lesson study cycle, representing the ongoing work and commitment of the teacher teams as they support one another through the process.

*Learning from Lesson study: Power Distribution in a Community of Practice*

Dolores Corcoran (St Patrick's College, Dublin City University, IRE)

Lesson study is a collaborative practice that has been a culturally embedded part of school life in Japan for more than one hundred years (Isoda, 2007). Its value as a means of teacher professional development is recognized outside Japan (Stigler and Hiebert, 1999) and the past ten years saw a burgeoning interest in lesson study in other educational systems, particularly in the US (Murata, Chapter One). Recently, lesson study was adopted in diverse school systems such as Chile and Thailand as a means of developing innovative classroom teaching and learning of mathematics (APEC, 2008). Japanese lesson study protocols are most often associated with professional development of practising teachers, but the study on which this research is based set out to trial lesson study in an Irish pre-service teacher education context. Results from the study shed light on aspects of lesson study that emerged during its use as part of a primary teacher education programme in Dublin, Ireland.

*What Japanese Lesson Study Can Teach Us About Formative Assessment Practice*

Michele D. Crockett (University of Illinois at Urbana-Champaign, US)

In U.S. mathematics classrooms, teaching and learning are often viewed as disparate activities, a significant problem of practice. The purpose of this paper is to frame Japanese lesson study as formative assessment and to consider what it can teach us about the formative assessment practices that promote students' understanding of mathematics. I use excerpts from Fernandez and Yoshida's (2004) account of lesson study as illustrations of what formative assessment looks like in actual practice and what such practice entails – specifying clear lesson goals, making substantive observations, and anticipating student thinking. Japanese teachers are *always* acting formatively when engaged in the lesson study process. Student thinking is central to their pedagogical decision-making. Lesson study as a model for practical action offers an opportunity to take seriously the role of assessment in teaching and learning in teachers' professional development experiences.

*Critical Considerations for Educational Reform through Lesson Study*

Brian Doig & Susie Groves (Deakin University, AU); Toshiakira Fujii (Tokyo Gakugei University, Japan)

This paper presents an argument for focussing, *Lesson Study* approaches to teacher professional development, firmly on the type and rôle of the mathematical task (*hatsumon*) used in the mathematics classroom. Drawing on research conducted in Australia and Japan, the authors argue that not all elements of *Lesson Study*, or particularly the research lesson, are equal in the impact that they have on children's learning. Further, it is demonstrated how Japanese educators place a strong emphasis on task selection, and that this effort is largely ignored by non-Japanese adapters of Lesson Study. Finally, the authors suggest that in order to use *Lesson Study* effectively in non-Japanese mathematics classrooms, it is necessary to build on the current practices of teachers that are commensurate with the elements of *Lesson Study*. Examples, from Japanese and Australian classrooms, are presented as illustrations of how the selection of the task is critical to the outcomes of the lesson.

*Approaches to Lesson Study in Prospective Teacher Education*
Maria Lorelei Fernandez (Florida International University, US) & Joseph Zilliox (University of Hawai'I, US)

The work of two mathematics educators each using a lesson study approach with prospective teachers of mathematics is reported. One educator worked with prospective secondary teachers in a Microteaching Lesson Study context and the other worked with prospective elementary teachers during initial field experiences in K-6 schools. The lesson study experiences in both contexts incorporated important features of Japanese lesson study including operationalizing an overarching learning goal driving recursive cycles of collaborative planning, lesson observation by colleagues and other knowledgeable advisors, analytic reflection, and ongoing revision. The prospective teachers exposed their knowledge, beliefs and practices to the scrutiny of peers and other experts, developing and reconsidering their thinking and practices through collaboration on shared teaching experiences. Similarities and differences in the secondary and elementary prospective teachers' experiences and learning in relation to elements comprising the lesson study approaches are discussed. Similarities included trajectories of their lesson plans toward more student-centered teaching, importance of negotiation for their learning, and value of the cooperative nature of the experiences for sharing varying ideas and perspectives. Differences included development of mathematics knowledge, extent of focus on classroom processes and management, use of videotaped lessons, conduct of oral reports of their group lesson study, and participation of knowledgeable advisors.

*Development of the Habits of Mind of a Lesson Study Community*
Lynn C. Hart (Georgia State University, US) & Jane Carriere (City Schools of Decatur, US)

This paper describes implementation of a Lesson Study project with third grade teachers in a small school district to study the development of the critical lenses (habits of mind) necessary for meaningful Lesson Study work. Adapting the Lesson Study process to meet school system needs, two outside facilitators stimulated development of the critical lenses through mathematics explorations and probing/what if questioning. Using a qualitative methodology and the group as the unit of analysis, data were coded for evidence of and change in the lenses. After one year, the 8 participating teachers showed a qualitative difference in two of the three lenses: the student lens and the curriculum developer lens. No change was seen in the researcher lens.

*Lesson Study: A Case of the Investigations Mathematics Curriculum with Practicing Teachers at Fifth Grade*
Penina Kamina (SUNY College at Oneonta, US) & Patricia Tinto ( Syracuse University New York, US)

Practicing teachers are often at the heart of reform initiatives and often with little professional development or support. Teachers wrestle with shedding their old pedagogical beliefs, understanding mathematical content, and learning how to use curricular materials such as *Investigations* (Putnam, 2003). The discrepancy between the implementers' prior experiences, National Council of Teachers of Mathematics (NCTM) principles, and *Investigations'* objectives presented an important problem for study. A qualitative case study research design was used to explore teachers' implementation of *Investigations'* mathematics in fifth-grade classrooms. Data were collected in the form of lesson plans and audiotape and videotape of lesson study meetings. Results of this study showed that teachers that collaborated with each other in lesson-study meetings were quickly able to establish new classroom instructional approaches and implement new curriculum. Their enhanced content knowledge, pedagogical knowledge, and reformed pedagogical beliefs that emerged from participating in lesson study enabled these teachers to be versatile in implementing the *Investigations* curriculum.

*Lesson Study as a Learning Environment for Coaches of Mathematics Teachers*

Andrea Knapp (University of Georgia, US), Megan Bomer (Illinois Central College, US), Cynthia Moore (Illinois State University, US)

This qualitative study focused on the professional development of two Coaches of mathematics teachers and one classroom Teacher as they engaged in the lesson study process. The Coaches progressively designed, taught, and refined standards-based lessons which they co-taught with the classroom Teacher. Participants developed three aspects of mathematical knowledge for teaching (MKT): knowledge of content and teaching (KCT), knowledge of content and students (KCS), and specialized content knowledge (SCK) (Hill, Rowan & Ball, 2005). KCT developed as the Coaches and Teacher collaborated during lesson study to place a stronger emphasis on inquiry in lessons. In particular, Coaches investigated reform curricula and research which enhanced the Teacher's KCT. In addition, the group developed KCS as they listened to students and observed them on videotape. Furthermore, Coaches developed KCS from the Teacher as the Teacher shared with Coaches his knowledge of student difficulties. Finally, the Coaches and Teacher developed SCK by considering mathematical perturbations from the lesson with the Teacher. Thus, lesson study mutually enhanced the teaching abilities of both Coaches and the Teacher whom they supported.

*Lesson Study: The Impact on Teachers' Content Knowledge*

Rachelle D. Meyer & Trena Wilkerson (Baylor University, US)

This multiple case study examined the effects lesson study had on middle school mathematics teachers' content knowledge. Participants for this study consisted of 26 middle school mathematics teachers, from a large urban school district, who formed eight lesson study groups. The researchers sought to examine the experiences and impact lesson study had on the participating teachers' content knowledge in mathematics from the eight case studies. More specifically, the researchers focused on (1) the need for teachers' content understanding while planning the research lesson and (2) the participating teachers' growth in content knowledge.

This qualitative research used seven measures to gather data which consisted of the following: two baseline surveys; transcripts from planning and reflection sessions; observation notes; lesson plans; and a reflective questionnaire. Analysis of the data consisted of both a within and across case comparison. For the with-in case analysis, each case was first treated as a comprehensive case in and of itself. Once the analysis of each case was completed, a cross-case analysis began in order to develop more sophisticated descriptions and more powerful explanations. Data revealed lesson study did improve teachers' content knowledge for three of eight case studies as a result of teacher collaboration.

*Lesson Study in Preservice Elementary Mathematics Methods Courses:Connecting Emerging Practice and Understanding*

Aki Murata & Bindu Pothen (Stanford University, US)

The paper outlines how lesson study is used in preservice elementary mathematics methods courses to support preservice teachers' connections between their emerging practice and understanding. The course structure is described, and week-by-week course activities and assignments are summarized. Lesson study in preservice teacher education program has a potential to support on-going teacher learning by connecting the course experiences with field-based assignments. By continuously focusing on student learning of mathematics, research lesson teaching ties together the various experiences in the course to help preservice teachers develop new understanding of their practice, therefore a professional vision. Pedagogical content knowledge is meaningfully developed in the collaborative learning settings. Short summary of research findings on teacher learning is presented based on quantitative and qualitative data.

*Lesson Study: A Tale of Two Journeys*

Jo Clay Olson (Washington State University, US), Paul White (Australian Catholic University), Len Sparrow (Curtin University, Australia)

Lesson study is accepted in Japan as an effective model for teacher professional development and growth. There is less evidence that such a model is viable in international settings. This paper adds to the pool of evidence as it reports on the experiences of five elementary teachers with a lesson study approach to professional development over a year. Two groups were formed, but the results of their experiences were different. One group accepted the challenges highlighted by the lesson study process. They reflected on and changed their classroom practice in fundamental ways. The other team rejected the challenges and maintained their traditional pedagogy. System-wide requirements, for example state testing, constrained the development of one team while the ability to personalize insights from the lesson study process and critical reflection became a catalyst to personal professional growth for the other.

*The Intersection of Lesson Study and Design Research: A 3-D Visualization Curriculum Development Project*

Jacqueline Sack (Rice University) and Irma Vazquez (University of St. Thomas )

In this paper, we share how two teacher-researchers and a teacher apply the principles of lesson study in our research process for developing a 3-dimensional visualization program for elementary children. We share a common belief that children learn best through social constructivist approaches (Cobb et al., 2001) with explicit opportunities for differentiated instruction (Tomlinson, C. A. & McTighe, J., 2006). While many lesson-study experiences offer teachers opportunities for personal professional development for deepening their pedagogical content knowledge, our focus is to develop and investigate new curricular materials that enable students to move among various visual and verbal representations (van Niekerk, 1997). We utilize a dynamic computer interface, Geocadabra (Lecluse, 2005) that simultaneously integrates several of these representations. Our goal is to extend the body of knowledge on how children think and learn about geometric space, ultimately to publish instructional materials to support children's development of 3-dimensional and 2-dimensional spatial reasoning skills. Our study takes place in a linguistically- and academically-diverse inner-city school, during its after-school program, with third- and fourth-grade students, for one hour each week for each grade level. We illustrate our adaptation of lesson study as our process for lesson design, enactment, reflection, and iterative re-enactment.

*Preparing for Lesson Study: Tools for Success*

Mary Pat Sjostrom (Chaminade University of Honolulu, US) & Melfried Olson
(University of Hawaiʻi at Mānoa, US)

This paper describes the experiences of one group of elementary school teachers as they engaged in a three-year professional development experience culminating in a one-year lesson study program. The partners in this project, university professors, school administrators and teachers, worked together to modify the professional development plan to serve the needs of the teachers and students. Although lesson study was not part of the original plan, it became the focus of year three. This case study illustrates the difficulties encountered in introducing lesson study, and examines the way in which the components of the first two years, notably the Reflective Teaching Model, collaborative problem solving and analysis of student work, helped pave the way for success in lesson study in year three.

*Towards Improving Pedagogical and Content Knowledge Through Lesson Study: Insights gained from working with schools in the U.S.*

Makoto Yoshida (Center for Lesson Study, William Paterson University, US) &
William C. Jackson (Scarsdale Public Schools, US)

An important issue that has surfaced in conducting lesson study in the U.S. is how lesson study can help improve teachers' pedagogical and content knowledge, particularly in elementary and middle schools. Even though more and more teachers and schools are conducting lesson study in the U.S., we have not found the most effective ways to cope with this issue.

Based on their experiences working with elementary and middle schools in the U.S., the authors share ideas and provide examples on how to address this issue. Important insights, such as improving the research lesson planning process through *kyozaikenkyu* (instructional material investigation), learning to build students' conceptual understanding by studying coherent and focused curricula, understanding the importance of process standards, and using knowledgeable others to support the lesson study process are discussed.

*Lesson Study as a Framework for Pre-service Teachers' Early Field-Based Experiences*

Paul W. Yu (Grand Valley State University, US)

The theoretical framework for this study is based on a review of the literature across two different, yet mutually relevant, areas of research in mathematics education, pre-service teachers' field based experiences (Zeichner, 1981) and Japanese lesson study (Stigler & Hiebert, 1999; Takahashi and Yoshida, 2004). Zeichner (1981) discusses two contrasting issues related to pre-service teachers' field based experiences. First, these field-based experiences are perceived to be a necessary component for teacher preparation. In contrast, some scholars question the significance of the experience other than an enculturation into the existing socio-cultural norms of the teaching profession. An emerging framework for in-service teacher improvement is lesson study.

This paper reflects on the use of lesson study as a framework for pre-service teachers' field-based experiences that takes place early in their collegiate coursework. The goal was to use lesson study to give students a different model for professional development, that is (1) collaborative, (2) focused on children's understanding of mathematics, and (3) exposes the pre-service teachers to the nature of mathematics instruction. The paper describes how lesson study was modified to accommodate the difference between pre-service and in-service teachers' experiences, and reflects on the enactment of these modifications in the collegiate course.

## Future Goals for the Working Group

It is clear that the body of knowledge around lesson study is in its infancy and assembling work that is currently being done would be productive. Plans were initiated for putting together an edited book where individuals conducting research in Lesson Study could share their findings, questions, and other issues. The group agreed that this would be useful to others with interest in implementing a Lesson Study program or with questions about the state of research on Lesson Study. Discussion on the format and conceptual organization of the proposed book is on-going and a primary goal of this working group.

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# ADDRESSING EQUITY AND DIVERSITY ISSUES IN MATHEMATICS EDUCATION

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*The Working Group will focus attention both to issues of mathematics teaching and learning and to issues of equity and diversity. This will include topics such as analyzing what counts as mathematics learning, in whose eyes, and how these culturally bound distinctions afford and constrain opportunities for students of color to have access to mathematical trajectories in school and beyond. Further, asking questions about systematic inequities leads to methodologies that allow the researcher to look at multiple levels simultaneously. This work begins to take a multifaceted approach, aimed at multiple levels from the classroom to broader social structures, within a variety of contexts both in and out of school, and at a broad span of relationships including researcher to study participants, teachers to schools, schools to districts, and districts to national policy.*

## Brief History

This Working Group will build on and extend the work of the Diversity in Mathematics Education (DiME) Group, one of the Centers for Learning and Teaching (CLT) funded by the National Science Foundation (NSF). DiME is a group of emerging scholars (new faculty and graduate students) who graduated from, or are still studying at, three major universities (University of Wisconsin-Madison, University of California-Berkley, and UCLA). The Center is dedicated to creating a community of scholars poised to address some critical problems facing mathematics education, specifically with respect to issues of equity (or, more accurately, issues of inequity).

The DiME Group (as well as subsets of that group) has already engaged in important scholarly activities. After two years of a cross-campus collaboration dedicated to studying issues framed by the question of why particular groups of students (ie. poor students, students of color, English learners) fail in school mathematics in comparison to their white (and sometimes Asian) peers, we presented a symposium at AERA 2005 (DiME, 2005). This was followed by the writing of a chapter in the recently published *Handbook of Research on Mathematics Teaching and Learning* which examined issues of culture, race, and power in mathematics education (DiME, 2007). In 2006, a group of four DiME graduates, currently new faculty at universities across the United States, applied to the NSF for a grant to support work on synthesizing research in the area of Professional Development that addresses both issues of mathematics and issues of equity and diversity. The proposal received good reviews, but was not funded. In an effort to expand the community of scholars interested in this work, DiME, at AERA in 2008, sponsored a one day Professional Development session examining equity and diversity issues in Mathematics Education.

The Center has historically held DiME conferences each summer. These conferences provide a place for fellows and faculty to discuss their current work as well as to hear from leaders in the emerging field of equity and diversity issues in mathematics education. This past summer of 2008, the DiME Conference was opened to non-DiME graduate students with similar research interests from other CLTs such as the Center for the Mathematics Education of Latinos/as (CEMELA), as well as graduate students not affiliated with an NSF CLT. This was an initial attempt to bring together a larger group of emerging scholars whose research focuses on issues of equity and diversity in mathematics education. In addition, DiME graduates, as they have moved to other universities have begun to work with scholars and graduate students with connections to other NSF CLTs such as MetroMath and the Urban Case Studies Project in MAC-MTL whose projects also incorporate issues of equity and diversity in mathematics education.

It is important to acknowledge some of the people whose work in the field of diversity and equity in mathematics education has been important to our work. Theoretically we have been building on the work of such scholars as Marta Civil (Civil & Bernier, 2006; González, Andrade, Civil, & Moll, 2001), Megan Franke (Franke, Kazemi, & Battey, 2007),Eric Gutstein (Gutstein, 2006), Danny Martin (Martin, 2000), Judit Moschkovitch (Moschkovich, 2002), and Na'ilah Nasir (Nasir, 2002). We have as well been building on the work of our advisors, Tom Carpenter (Carpenter, Fennema, & Franke, 1996), Geoff Saxe (Saxe, 2002), Alan Schoenfeld (Schoenfeld, 2002), and again Megan Franke (Kazemi & Franke, 2004), as well as many others outside of the field of mathematics education.

A significant strand of the work of the DiME Center for Learning and Teaching included implementing professional development programs grounded in teachers' practice and focusing on equity at each site. The research and professional development efforts of DiME scholars are deeply intertwined, and much of the research thus far produced by members of the DiME Group addresses issues of equity within Professional Development. Additionally, since the majority of the DiME graduates, as new professors, along with a number of current Fellows, are engaged in teaching Mathematics Methods courses, the integration of issues of equity with issues of mathematics teaching and learning in Math Methods has become a site of interest for research. As suggested, these two areas will be the focus for much of our research plans in the near future. These scholars have learned through their work with DiME that collaboration is a critical component to our work and are eager for an opportunity to continue working together as well as to expand the group to include other interested scholars with similar research interests.

## Focal Issues

Under the umbrella of attending to equity and diversity issues in mathematics education, researchers are currently focusing on such issues as teaching and classroom interactions, Professional Development, pre-service teacher education (primarily in Math Methods classes), student learning (including the learning of particular sub-groups of students such as African-American students or English learners), and parent perspectives. Much of the work attempts to contextualize the teaching and learning of mathematics within the local contexts in which it happens, as well as to examine the structures within which this teaching and learning occurs (e.g. large urban, suburban, or rural districts; under-resourced or well-resourced schools; and high-stakes testing environments). How the greater contexts and policies at the national, state, and district level impact the teaching and learning of mathematics at specific local sites is an important issue, as is how issues of culture, race, and power intersect with issues of student achievement and learning in mathematics.

Existing research tends either to focus on professional development in mathematics (e.g., Barnett, 1998; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Kazemi & Franke, 2004;

Lewis, 2000; Saxe, Gearhart, & Nasir, 2001; Schifter, 1998; Schifter & Fosnot, 1993; Sherin & vanEs, 2003), or professional development for equity (e.g., Sleeter, 1992, 1997; Lawrence & Tatum, 1997a). Little research exists, however, which examines professional development or mathematics methods courses that integrate both. The effects of these separate bodies of work, one based on mathematics and one based on equity, limits the impact that teachers can have in actual classrooms. The former can help us uncover the complexities of children's mathematical thinking as well as the ways in which curriculum can support mathematical understanding in a number of domains. The latter has produced a body of literature that has helped to reveal educational inequities as well as demonstrated ways in which inequities in the educational enterprise could be over come.

To bridge these separate bodies of work, the Working Group will focus on analyzing what counts as mathematics learning, in whose eyes, and how these culturally bound distinctions afford and constrain opportunities for students of color to have access to mathematical trajectories in school and beyond. Further, asking questions about systematic inequities leads to methodologies that allow the researcher to look at multiple levels simultaneously. This research begins to take a multifaceted approach, aimed at multiple levels from the classroom to broader social structures, within a variety of contexts both in and out of school, and at a broad span of relationships including researcher to study participants, teachers to schools, schools to districts, and districts to national policy.

Some of the research questions that the Working Group will consider are:

*Teachers and Teaching*

- What are the characteristics, dispositions, etc. of successful mathematics teachers for *all* students across a variety of local contexts and schools? How do they convey a sense of purpose for learning mathematical content through their instruction?
- How do beginning mathematics teachers perceive and negotiate the multiple challenges of the school context? How do they talk about the challenges and supports for their work in achieving and equitable mathematics pedagogy.
- What impediments do teachers face in teaching mathematics for understanding?
- How can mathematics teachers learn to teach mathematics with a culturally relevant approach?
- What does teaching mathematics for social justice look like in a variety of local contexts?
- What are the complexities inherent in teacher learning about equity pedagogy when students come from a variety of cultural and/or linguistic backgrounds all of which may differ from the teacher's background?
- What are dominant discourses of mathematics teachers?

*Students and learning*

- What is the role of student academic and mathematics identity in achievement?
- How do students out-of-school experiences influence their learning of school mathematics?
- What is the role of perceived/historical opportunity on student participation in mathematics?

*Policy*

- How does an environment of high stakes standardized testing affect whether and how teachers teach mathematics for understanding? How does this play out across a variety of

local contexts? How can we support teachers to teach mathematics for understanding in that environment?
- How do we address issues of tracking/ability grouping and in particular the grouping of students by test designation?

## Plan for Working Group

The overarching goal of the group is to facilitate collaboration within the growing community of scholars and practitioners concerned with understanding and addressing the challenges of attending to issues of equity and diversity in mathematics education. The PME Working Group will provide a forum for these scholars to come together with other interested researchers who identify their work as attending to equity and diversity issues within mathematics education in order to develop plans for future research. Our main goal for this year, then is to begin a sustained collaboration around key issues (theoretical and methodological) related to research design and analysis in studies attending to issues of equity and diversity in mathematics education.

In order to support this collaborative research, smaller research groups will be formed from participants in the large Working Group. These groups will be dependent on the research interests of the Working Group participants. For example, a smaller group may discuss research on in-service teachers engaged in Professional Development, or research done within the context of a Mathematics Methods course.

Much of our work is qualitative in nature and we recognize that one way to increase the number of participants is to conduct research across several sites. In order to do this, we need to use the same protocols for data gathering. Our intention is to use some of the PME meeting time to share and/or develop, and revise interview, observation and other research protocols which may be used across a variety of research projects. It is anticipated that one aspect of the sub-group meetings will be to discuss potential funding opportunities. Here we may identify and begin to draft grants proposals to fund research across contexts.

More specifically, we will work across these various areas in the following way.

SESSION 1:

- Presentation and discussion of goals of Working Group.
- Introduction of participants
- Work on such tasks as examining research protocols or sharing existing data. (Sample protocols for classroom observation, video analysis, and interviewing will be available. Plans are in place to invite collaborators to examine existing data sets.)

SESSION 2:

- Sub-group meetings to discuss plans to address research questions such as those set forth in the Focal Issues.
- Further work on examination and refining of protocols and/or data sharing from Session 1.

SESSION 3:

- Sharing and discussion of work from Sessions 1 and 2.
- Planning for further collaboration, including designation of a person who will facilitate each sub-group.
- Developing a tentative agenda for future Working Group meetings

## Anticipated Follow-up Activities

It is anticipated that interested Working Group participants will align themselves with other participants who have similar research programs (as described more fully above). It is also anticipated that Interest Groups will also form along the lines of research protocols. That is, there may be researchers who use video, or classroom observations in their research who are interested in exploring further some video analysis or classroom observation protocols to use in their work. And so it is anticipated that Working Group participants will leave the conference ready to work on developing particular aspects of their research (such as particular data gathering methods) and/or ready to develop a new project in collaboration with other Working Group participants. It is our hope that the work of this group will begin at this Conference in Atlanta in 2009, and we anticipate it will continue for many years.

## Previous Work of the Group

Not applicable.

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# PRESERVICE ELEMENTARY SCHOOL TEACHERS' CONTENT KNOWLEDGE IN MATHEMATICS

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*This working group continues its focus on the study of preservice elementary teachers' content knowledge in mathematics for teaching. Participants continue the considerations of (a) the types of knowledge needed for teachers of mathematics, (b) principles that guide the design of courses to facilitate the development of such knowledge, (c) research studies that have been conducted in these areas, (d) implications of existing research studies, and (e) identification of further research needs. Dialogues (on- and off line), resource sharing, as well as collaboration among members of the study group will be encouraged and facilitated both during the conference and afterwards.*

## Focuses and Aims of the Working Group

*Defining the Types of Knowledge Needed for Teaching Elementary School Mathematics*

One of the goals of this working group is to further discuss and conceptualize the concept and aspects of mathematical content knowledge for teaching. The definition of the knowledge needed to teach mathematics has been the focus of recent discussions in the mathematics education community. Groups are meeting at various conferences (for example: PME-NA 2007, AMTE 2008, AMTE 2009) to gain a better understanding of what preservice elementary school teachers should know in order to become effective teachers and to explore how mathematics educators can assist preservice and in-service elementary school teachers in developing such knowledge.

Hill, Ball, & Shilling (2008) introduced a framework for distinguishing the different types of knowledge included in the construct of mathematical knowledge for teaching (see Figure 1).

*Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.*

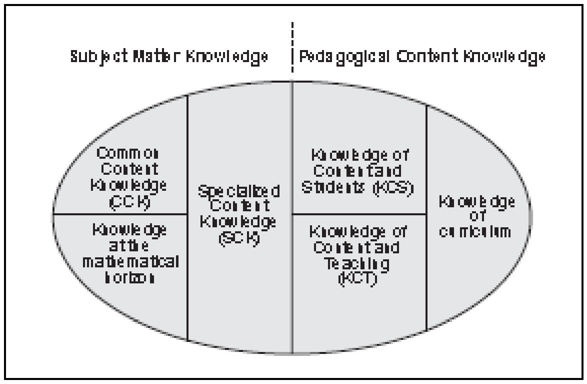


*Figure 1.* Domain map of mathematical knowledge for teaching (Hill, Ball, & Schilling, 2008, p. 377).

This framework distinguishes between subject matter knowledge and pedagogical content knowledge. Subject matter knowledge is subdivided into common content knowledge, specialized content knowledge, and knowledge on the mathematical horizon. Pedagogical content knowledge is subdivided into knowledge of content and students, knowledge of content and teaching and knowledge of curricula. Hill, Rowan & Ball (2005) provide empirical support linking teachers' "mathematical knowledge for teaching" to student achievement gains. In their study, they define mathematical knowledge for teaching as "mathematical knowledge used to carry out the work of teaching mathematics" (p. 373). Examples of such knowledge include "explaining terms and concepts to students, interpreting students' statements and solutions, judging and correcting textbook treatments of particular topics, using representations accurately in the classroom, and providing students with examples of mathematical concepts, algorithms, or proofs" (p. 373).

This framework was discussed among participants of a symposium at AMTE.[10] All participants found the framework to be useful when discussing different types of knowledge they want their preservice teachers (PSTs) to develop. However, the distinctions among the different types of knowledge seemed blurry at times[11]. For example, Hill et al (2008) described common content knowledge (CCK) as "knowledge that is used in the work of teaching in ways *in common with* how it is used in any other professions or occupations that also use mathematics" (p. 377) and specialized content knowledge (SCK) as "the mathematical knowledge that allows teachers to engage in particular *teaching* tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures and examine and understand unusual solution methods or problems" (p. 377). Considering that we want children to not simply be able apply the correct procedures but be able to go beyond that and understand why the procedures work, the distinction between the two types of knowledge is not clear. In recent years, several influential organizations have called for a focus on conceptual understanding (Kilpatrick, Swafford, & Findell, 2001; Lundin & Burton, 1998; National Council of Teachers of Mathematics, 2000) to help students become successful mathematics learners (Thompson, Philipp, Thompson, & Boyd, 1994). Thus, if a goal of instruction is that children

[10]Christine Browning, Meg Moss, Randy Phillip, Eva Thanheiser, Tad Watanabe
[11] Ball, Thames & Phelps (2008) acknowledge the problem of the fuzzy boundary among their six sub-domains.

understand the underlying concepts (i.e. being able to explain why three digit subtraction works) ) it appears that teachers must be able to support children to develop SCK, or at least part of it? Similar issues were raised when comparing other types of knowledge. For example, in the domain map of mathematical knowledge for teaching, knowledge of content and students resides in the PCK area, however, in recent work (Philipp, et. al. 2007), knowledge of content and students was used as a means to develop specialized content knowledge. Thus a connection to children's mathematical thinking may cross the boundaries between SCK and PCK.

*Developing a Framework for Design Principles for Preservice Elementary School Teacher Courses*

Related to the question of how best to prepare elementary school teachers, the working group will discuss the design principles that guide the development of the mathematics content courses for PSTs and their impact on the development of PSTs' knowledge. The participants of a symposium at AMTE found they share the following principles. The working group will continue to explore components of the structure.

1. Mathematical ideas are built on the PSTs' currently held conceptions.
2. "Model" teaching.
3. Connect to other kinds of knowledge
    - Knowledge of content and teaching
    - Knowledge of content and children
    - Knowledge of curriculum

Participants found their attempts to build mathematical ideas from PSTs' currently held conceptions take two different approaches. For some teacher educators this means identifying what conceptions PSTs hold when they enter our classrooms and building on those conceptions. This approach grows out of a rich cognitive-science paradigm focused upon children's prior knowledge in learning situations, a consideration that is equally important in work with adults (Bransford, Brown, & Cocking, 1999). Once mathematics teacher educators (MTEs) know PSTs initial conceptions we need to understand how those conceptions develop (this is discussed in more detail below). Other teacher educators support the development of PSTs' conceptions by limiting the mathematical ideas that can be used in explorations. Only those ideas developed by the classroom community are allowed. Both of these approaches could be employed simultaneously. For example, rules related to area formulas for polygons would have to be developed as a class, but they could be built on the conceptions of area the PSTs bring with them.

For all of us, modeling instruction means that we teach the way we would like our PSTs to conduct their classrooms. This includes engaging PSTs in creating their own knowledge, facilitating small group and classroom discussions, and conducting formative assessment of PSTs' knowledge and development to inform the instruction. In short, teacher educators model practices consistent with those described in the Principles and Standards for School Mathematics (2000). What is less clear is how modeling impacts PCK knowledge and later practice. This open question is a critical component of the larger research agenda developed by the working group.

Many MTEs believe that content knowledge is connected to and supported by other types of knowledge. A goal of MTEs' practice is the development of connections between knowledge. Pedagogy employed by MTEs' in our attempts to achieve this goal include explicating our own teacher moves (connecting to knowledge of content and teaching), using artifacts of children's mathematical thinking (connecting to knowledge of content and children), and explaining our own curriculum decisions (connecting to knowledge of content and curriculum). For the case of modeling, little is known about the impact of MTEs' pedagogy on PSTs' knowledge and connections within that knowledge. Notable exceptions include Philips et al. (2007) and Harkness, D'Ambrosio, and Morrone (2007) . Both sets of authors demonstrate the impact of particular activities on the development of PSTs beliefs and conceptions.

*Discussing "The Big Ideas" of Preservice Elementary Content Courses*

A recurring theme in discussions of MTEs is focused on curriculum. What "big ideas" should be addressed in the preservice elementary courses? Which mathematics content topics should be included? Structures for preservice education vary across universities. Some have as little as a one semester course devoted to content and methods; others have up to 4 semesters for content and an additional methods course. The rest rank somewhere in the middle. No matter how many courses are included in a program, MTEs agree that we cannot teach PSTs all they need to know. This commonly held view brings us back to the question of focus. What is the essential content for the courses we have devoted to mathematics content and pedagogy? Or is the process of teaching mathematics content more important t than the particular content itself?? Given that MTEs focus on helping the preservice elementary teachers identify what they do not know and then assist them in building knowledge, we are working to help PSTs make sense of what it means to understand mathematics (as well as helping their beliefs and attitudes about learning mathematics). Could our main goal be in teaching this process? Thus far it is not clear yet what effects our preservice teacher education classes have on the PSTs' teaching. There seems to be common agreement among MTEs that we want to prepare our PSTs to be life-long learners. How can this be done? In our working group we will focus on further exploring this question.

*Synthesizing the Research on Preservice Elementary Teachers' Content Knowledge and Identifying Areas of Further Study*

This working group seeks to synthesize the research on PSTs' content knowledge and identify areas of further study. This is essential so we understand our PSTs better when they enter our own classrooms. Some research has been conducted on the subject matter knowledge of preservice and in service teachers. Much of this research has focused on the teachers' lack of mathematical content knowledge needed to teach in key areas such as numbers and operations, (Ball, 1988/1989; Graeber, Tirosh, & Glover, 1989; Kastberg, 2007; Lo, Grant, & Flowers, 2008; Ma, 1999; McClain, 2003; Simon & Blume, 1996; Thanheiser, 2005), probability and statistics (Canada, 2007; Gfeller, Nieses, & Lederman, 1999; Jacobbe, 2007, Leavy & O'loughlin, 2006), geometry and rational numbers (Browning, et al, 2007; Jones & Mooney, 2002; Menon, 1998; Quinn, 1997). For example, Thanheiser (2005) found that only 3 of 15 preservice teachers held a conception of place value that allowed them to explain how and why the subtraction algorithms with three-digit numbers work. MTEs need more empirical evidence of what learning opportunities most contribute to more knowledgeable and confident teachers in order to make more informed changes to their programs (Mewborn, 2000). Large scale studies that help MTEs better understand how, when and where preservice elementary teachers might gain more specialized content knowledge are a critical need in the field (Adler, Ball, Krainer, Lin, Novotna, 2004; Mewborn, 2000)

To help teachers develop the content knowledge needed to teach mathematics educators need to understand the PSTs' currently held conceptions. As the authors of *The Mathematical Education of Teachers* suggest, "The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they *do* know" (CMBS, 2001, p. 17). While some research has been conducted on identifying PSTs' conceptions, for example Thanheiser (2005) introduced a framework for PSTs' conceptions of multidigit whole numbers, there is still much to be learned about preservice teachers' conceptions. A potentially fruitful orientation toward such work includes the examination and testing of learning trajectories developed from work with children. While the trajectories may not have the necessary longitudinal dimension to help explain adult development, they certainly contain elements of use to MTEs as illustrated by the work of McClain (2003). Further research on conceptions and their development is essential for the further development of teacher education courses.

*Developing/Continuing Individual or Collaborative Projects to Move the Field Forward, Support Each Other's Research Efforts, and Keeping Each Other Updated of New Studies, Findings, and Progress in Our Own Work*

A central goal of this working group is to support existing collaborations and develop new ones. Subgroups of the organizers of the proposed working group have met previously at professional conferences including PME-NA at Toronto and Mérida and Lake Tahoe to discuss shared research interests. A discussion group was held at PME-NA in Lake Tahoe. These meetings and discussions have led to a joint symposium for NCTM research pre-session at Atlanta in March 2007 and most recently a joint symposium at AMTE in Orlando, 2009. Our research is similar in its focus pertaining to preservice elementary teachers' content knowledge but varies in specific content, use of prior work (conceptual analysis of content, prior work with children, prior work with adults), and theoretical framework.

At PME-NA 2007 we agreed on the need for the construction of a research base for the study of preservice teacher content knowledge. This includes a need to synthesize and summarize existing (completed and current) research and findings and develop a research agenda. This work can then support the development of research questions focused on areas where findings might inform our understanding of the impact of course work on knowledge development and practice. Additional needs were identified as follows:

- Share results with each other and the wider research community.
- Share resources (tasks, videos, materials, etc.) with each other and the wider research community.
- Explicate theoretical frameworks used by MTEs in research and teaching.

Questions were also raised about the implications of research on PSTs' content knowledge. In particular, participants wondered how the research influences the work of mathematics educators in the classroom. How does this research influence the work of mathematics educators in the classroom? Towards the goal of establishing ongoing collaborations a subgroup of the participants of this working group has started to meet regularly (once a month) via the internet (live chat) to discuss various topics relating to preservice elementary teacher education. The organizers seek to expand this group, sustain conversations, and work toward the production of research syntheses, a research agenda, and elaboration of theoretical frameworks.

*Developing Common Resources for Teaching PSTs*

One of the goals of this working group will be to discuss/share/develop common resources for teaching PSTs. For example, many of us use artifacts of children's mathematical thinking in our courses. In addition to discussing our motivation for doing so and how we use these artifacts to improve content knowledge, there is a need to make such resources (and the rationale for them) accessible to a broader audience. One solution for this might be a website but further discussion is needed on how such a website could be established, maintained and what information should be presented on this website.

## Outline of Working Group Sessions

In our first session, we will start with an introduction and overview of the working group. This will be followed by a brief summary of the last working group (PME-NA 2007) and other activities since (working groups at AMTE 2008 & 2009; symposium at NCTM 2007).

Brief introduction/presentations will be given by some the participants during which they will discuss:

- research s/he is conducting at his/her institution,
- theoretical framework and prior studies,
- current findings,
- plans to conduct further research, and
- challenges s/he sees in conducting this line of research.
- design principles for teaching preservice teachers.

Eva Thanheiser will serve as the moderator to facilitate the discussion. Before the end of the session Christine Browning and Eva Thanheiser will briefly introduce the monthly internet chat meetings and invite everyone to participate. These meetings will be hosted by various participants.

In our second session, we will start by forming small groups to work on the following tasks:

- React to/discuss definitions of the type of knowledge needed for teaching elementary school mathematics.
- React to/discuss design principles for preservice teacher courses.
- React to/discuss "the big ideas" of preservice elementary content courses.
- React to/discuss synthesis of the research on preservice elementary teachers' content knowledge and identifying areas of further study.
  - Generate research questions from the identified areas and brainstorm research methods for investigating this question.
  - Identify the resources needed to support such study.
- React to/discuss individual or collaborative projects that can move the field forward, support each other's research effort, and keep each other updated of new studies and findings in this area.
- React to/discuss ideas for developing common resources for teaching PSTs:

Then each group will take turns sharing their work. We hope to stimulate interest for collaboration through these focused discussions.

In our third session we will synthesize the current status of the working group and discuss modes of communication to sustain collaboration throughout the year. A calendar of discussion chats will be established and published through an online forum.

## Follow-Up Activities

Participants will publish an on-line forum for open discussion of topics of interest. In addition monthly online chat meetings will be established. Possible collaborations may include joint research projects, mini-conferences, and a book proposal to the Mathematics Teacher Education series at Springer.

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# WORKING GROUP FOR LEARNING TO REASON PROBABILISTICALLY AND STATISTICALLY THROUGH EXPERIMENTS AND SIMULATIONS

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*Recent foci in the Working Group have been to understand: (1) students' and teachers' reasoning when simulating probability experiments with hands-on materials and computer tools, and (2) connections between probability and statistical concepts such as inference and variability. At PME-NA 31 the group will build on the research agenda that it has been pursuing over the past several years. Group members will discuss recent literature reviews on learning and teaching probability as it relates to statistical reasoning, revisit previous research and discuss designs for cross-national, collaborative research to be conducted in 2010. Emerging research from Working Group members will lead to a set of papers that could comprise a monograph, journal special issue, and/or joint presentations at future conferences.*

## Nature and Topic of the Working Group

This Working Group was formed at PME-NA 20 (Maher, Speiser, Friel, & Konold, 1998) and has convened annually at PME-NA since, except for the joint PME and PME-NA meeting held in 2008 in Mexico. Through shared research, rich and engaging conversations, and analysis of instructional tasks, we continually seek to understand how students learn to reason probabilistically, with particular focus this year on how probabilistic reasoning with simulations can support statistical reasoning.

## Aims of the Working Session

There are several critical aims that guide our work together. In particular, we are examining: (1) mathematical and psychological underpinnings that foster or hinder students' probabilistic reasoning, (2) the influence of experiments and simulations in the building of ideas by learners, particularly with emerging technology tools, (3) learners' interactions with and reasoning about data-based tasks, representations, models, socially situated arguments and generalizations, (4) the development of reasoning across grades, with learners of different cultures, ages, and social backgrounds, and (5) the interplay of statistical and probabilistic reasoning and the complex role of key concepts such as sample spaces and data distributions. Through our work, we have stimulated collaborations across universities and plan to engage in and support additional research related to the complexity of learning to reason probabilistically, particularly when modeling probability situations with tools that enable experiments and simulations.

## Literature Background

The ways in which students reason about the likelihood of an event can be considered in terms of an objective or subjective view of probability (e.g., see Batanero, Henry, & Parzysz, 2005; Borovcnik, Bentz, & Kapadia, 1991). In an *objectivist* perspective, probability is viewed as an inherent property of the event and can be well estimated either through a classical or frequentist approach. A repeated finite set of trials would most likely yield a different experimental estimate of the actual probability and may in fact allow one to change the estimate of the probability based on new data. In a *subjectivist* perspective, probability is viewed as a condition of the information known to the individual assigning the probability and not an objective property of the given event. It is worth remembering that these are perspectives on probability in the real world, and not on the mathematical theory of probability.

The *law of large numbers* is a mathematical theorem used to interpret empirical results in relation to probabilities. This theorem states that for an experiment with fixed probabilities, the likelihood of a large difference between the relative frequency of an event and the events' probability limits to zero as more trials are collected. In various ways it supports the viability of all the above mentioned approaches to probability. From a frequentist approach, an estimated probability should be reasonably close to a probability computed from a classical approach. With a subjectivist perspective, two people may assign different probabilities to the same event based on different *a priori* information, even after they observe the same empirical data. However, after observing large amounts of empirical data, it seems reasonable that two people operating within a subjectivist perspective would assign similar probabilities. As more data is collected, each of their knowledge about the event would include an increasing proportion of shared information.

A frequentist approach to probability, grounded in the law of large numbers, has only recently made its way into curricular aims in schools, which is typically dominated by a classical approach (Jones, 2005). Notably absent is any significant attention to a subjective approach to probability (Jones, Langrall, & Mooney, 2007). Recent curricula recommendations (e.g., NCTM, 2000) encourage teachers to use an empirical introduction to probability by allowing students to experience repeated trials of the same event, either with concrete materials or through computer simulations (e.g., Batanero, Henry, & Parzysz, 2005; Jones, Langrall, & Mooney, 2007; Parzysz, 2003). In these types of curricula, a theoretical model of probability based on a classical approach is not the starting point. Rather, a theoretical model is constructed based on observing that the relative frequencies of an event from a repeated random experiment stabilize as the number of trials or sets of trials (different samples) increases. However, there is general agreement that research on students' probabilistic reasoning has been lacking sufficient study of students' and teachers' understanding of the connection between observations from empirical data and a theoretical model of probability (e.g., Jones, 2005; Jones, Langrall, & Mooney, 2007; Parzysz, 2003).

Mathematical probability concepts are central to understanding theoretical statistics, but the role of probability in developing statistical reasoning is much less clear. Developing sample spaces and computing probabilities via combinatorial arguments is a traditional introduction to probability in the classroom, while the availability of handheld technology has brought a strong frequentist vein of investigations involving simulated trials. And though not a part of most classroom introductions, a view of probability as a measure of information would not go unwelcomed in many statistical applications. Regardless of the nature of a student's understanding of probability, there is still the question of how this understanding impacts a student's statistical reasoning. When a student seeks to infer qualities of an object or a set of objects by gathering data, if the data is gathered via a random experiment, the student is conducting a statistical investigation (Franklin et al, 2005). The level of *understanding* a student has of the inferences that can be made very much depends upon the student's understanding of the ties between data and model in a probabilistic phenomenon.

The working group is aiming to contribute a better understanding to some of these issues, particularly using experiments and simulations to generate empirical data from which students' can reason statistically. Also of importance in this research is how students reason about the process of conducting a probability experiment or constructing a simulation and their understanding of how the experiment or simulation is an appropriate model of the original probability context.

## Summary of Activities from Past Two Meetings of Group

In 2007, twelve researchers (faculty and graduate students) from the United States, Canada, and Mexico met During PME-NA 29 in Lake Tahoe, NV. In 2006, eight researchers (faculty and graduate students) from the United States, Canada, and Mexico met During PME-NA 28 in Mexico. In 2007, members read and discussed the recent literature review on probability learning by Jones, Langrall, and Mooney (2007). In particular we discussed the call for research on understanding more about how students make connections between empirical and theoretical probability. Members shared several simulation environments (ProbLab, Fathom, and Probability Explorer) and discussed a few probability tasks that engaged students in reasoning about theoretical and empirical distributions. We further discussed how these tasks can be interpreted differently by students and teachers. Moreover, discussions focused on issues students face in trying to generate and analyze empirical data to make inferences about an unknown probability distribution. Participants shared tasks that could be used with students and teachers to engage them in conducting probability experiments or simulations. A rich discussion was held concerning the nature of tasks that can be used in further research to help expand the knowledge base on what students understand about probability in contexts where repeated trials are generated and data needs to be analyzed to make a decision or inference. The group expressed interest in pursuing the following questions:

- What are students' intuitions regarding whether real-world phenomena can or cannot be simulated? Are there differences between simulations and modeling tools?
- How do students (and teachers) relate technology simulation models to real-world phenomena?
- How do learners move between empirical data and theoretical models of probability? To what extent do students attend to issues of sample size, variation, sampling distributions, and data collection?
- What metaphors emerge as students engage in probability tasks, and how do these support or hinder the development of probabilistic reasoning?
- What are the key issues in the design of probability tasks in order to promote reasoning about probabilistic events which occur repeatedly in an experiment or simulation? What issues do teachers face in implementing such tasks?
- How can argumentation and justification be used as a tool to increase conceptual understanding of chance and randomness?

Stemming from the work in 2007, several members participated in a research symposium at the NCTM research presession in 2008 focused on different software environments that are designed to promote connections among empirical data and theoretical probability distributions. The papers from that symposium were all submitted to journals. Several group members also attended and participated in the Joint ICMI/IASE Study Conference (http://www.ugr.es/~icmi/iase_study/) held at the Instituto Tecnológico y de Estudios Superiores, in Monterrey, Mexico in July 2008. This study conference was focused on the challenges for teaching and teacher education in statistics education, with several papers specifically addressing issues related to the use of simulation environments in the teaching of statistics.

## Planned Activities for the 2009 Meeting

At PME-NA 31, the group will build on the research agenda listed above. We plan to do the following during the time allotted:

1) Discuss important aspects of conducting a probability experiment or simulation that can provide a venue for students' reasoning about empirical data and its connection to a theoretical model of probability and one's ability to reason statistically;
2) Provide time for members to share recent research endeavors on students' and teachers' engagement in probability experiments or simulation tasks. This will likely include some engagement with simulation tasks and group video analysis;
3) Discuss results of recent research by group members and others in light of summaries and calls for research by Jones, Langrall and Mooney (2007) and the topics of importance raised at the ICMI/IASE 2008 joint conference; and
4) Generate suggestions for further refinement of individual's research and suggestions for additional research by group members that will form the basis of a collection of papers.

We intend for the cumulative research efforts from Working Group members to be compiled as a set of papers for either a monograph or journal special issue. Clearly our proposed activities are closely aligned with Goals of PME-NA, namely "to promote international contacts and the exchange of scientific information in the psychology of mathematics education," "to promote and stimulate interdisciplinary research...," and "to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof."

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# SUPPORTING TEACHERS TO INCREASE RETENTION

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*Several national reports have identified the need to increase the pool of highly qualified mathematics teachers as a way to improve mathematics education. However, providing high quality mathematics education for all students goes beyond the recruitment of knowledgeable teachers. This working group is designed to offer an opportunity to examine the role that professional development and support play in the work and retention of mathematics teachers. Retention will focus on new teachers especially those in urban area and mathematics teachers in hard-to-hire settings. Discussions will concentrate on the study of interventions through professional development and support models. Efforts to deepen our understanding of the complex and multifaceted picture of why teachers leave and why they stay, and how efforts to retain teachers impact their work in the classroom and their decisions to stay or leave will be developed through the sharing of research designs, data collection, and on-going results. This working group will be appropriate for anyone who has work to share or who is thinking about a support for retention project. Throughout, we will address this very complex task both in terms of the opportunities and challenges for the mathematics education researchers to provide quantitative and qualitative input on a major political issue. It is hoped that this working group will enrich the dialogue about a national crisis in mathematics education.*

## Brief History of Working Group

Although teacher retention is a topic being included in many conferences and position papers, this is an initial request for a Working Group to investigate the relationship between Professional Development/Support and the retention of mathematics teachers, thus is not the continuation of any prior Working Group. One of the goals will be to identify the absences in the research so that we may move forward in tackling the complex national issue of mathematics teacher retention.

## Issues of Psychology of Mathematics Education to be Focused On

The study of the relationship between Professional Development and Support Models on the Work and Retention of Mathematics Teachers in grades 7 – 12 merits careful examination. Several national reports have pointed to the need to increase the pool of highly qualified mathematics teachers as a way to improve mathematics education and maintain the United States' economic competitiveness (National Academy of Sciences, 2007; Glenn Commission, 2000). However, providing high quality mathematics education for all students goes beyond the recruitment of mathematically knowledgeable teachers to encompass issues of teacher support, professional development, and retention. Over the past two decades, analyses of teacher employment patterns reveal that new recruits leave their school and teaching a short time after they enter. Ingersoll, using data from the School and Staffing Survey concluded that in 1999-2000, 27% of first year teachers left their schools. Of those, 11 percent left teaching and 16 percent transferred to new schools (Smith & Ingersoll, 2003). Earlier research revealed that teachers who leave first are likely to be those with the highest qualifications (Murnane & other, 1991; Schlecty & Vance, 1981). This "revolving door" is even higher in large urban districts; for example, 25% of the teachers new to Philadelphia in 1999-2000 left after their first year and more than half left within four years (Neild & other, 2003). In Chicago, an analysis of turnover rates in 64 high-poverty, high-minority schools revealed that 23.3 percent of new teachers left in 2001-2002.

Reasons for the lack of retention of new teachers and teacher in high-poverty schools are often related to "working conditions" and lack of support (Ingersoll, 2001; Smith & Ingersoll, 2004; Johnson et al., 2004), though pay also plays a role (Hanushek, Kain, & Rivkin, 2001). This support includes professional and collegial support such as working collaboratively with colleagues, coherent, job-embedded assistance, professional development, having input on key issues and progressively expanding influence and increasing opportunities (Johnson, 2006). Preparation, support, and working conditions are important, because they are essential to teachers' effectiveness on the job and their ability to realize the intrinsic rewards that attract many to teaching and keep them in the profession despite the profession's relatively low pay (Johnson & Birkeland, 2003; Liu, Johnson, & Peske, 2004; Lortie, 1975).

Recently, a status report on teacher development focusing on professional development and support of teachers (Darling-Hammond et al., 2009) summarized findings and put forth recommendations for effective professional development. The basis for the paper included national surveys with self reported data, a meta-analysis of 1,300 research studies and specific studies. The conclusion is that "well designed" professional development can influence teacher practice and student performance. The paper focuses on what is or could be considered as well defined. One strand of the paper is that of effective support for new teachers. Although half of the states require support for new teachers (Education Week, 2008) it was found that rates of participation in teacher induction programs varied by school types with highest rates in schools with least poverty and lowest in schools with high levels of poverty. Beyond the rates of participation and availability of support, there is the question of what is effective support. An on-going large-scale research project was sited which is currently underway that aims to measure impacts in terms of classroom practices, student achievement and teacher mobility. Initial results seem to reflect the difficulty in identifying the impact of support.

Another study is presently in its second year and is studying the support of mathematics across the State of California. This five year study is looking through the lens of 10 support models to trace the knowledge, classroom practices, professional communities of support, leadership and needed support. Initial results are complex but are showing relationships between sustained professional development and support and teacher retention. In one year the attrition is half what it was over the prior five-year period.

But, what is the relationship between the support and the retention? One of the 10 sites from this study observed that success of a retention initiative takes root in a variety of needs: the need to know your District and its teachers – a necessity that often relies on established, long-term relationships between the university and district leaders; the need to offer sustained support as opposed to punctual interventions in order to break the isolation of beginning teachers and create a sustainable community; the need to establish relevance in the professional development activities proposed by engaging participants in deep introspection of their own knowledge gaps; the need to involve all actors of the community to prevent miscommunication from annihilating attempts made towards change; the need to nurture the community created by moving its members forward into roles and responsibilities they are ready to take on; and last but not least, the need to refine even successful models to keep the momentum (Felter & Faughn, 2009).

As is indicated in the comments above, support comes from multiple sources. Another recent study from Peabody College, Vanderbilt University, finds that principals play a critical role in the support of new mathematics teachers (McGraner, 2009).

This working group is designed to offer a comprehensive, multifaceted examination of the on-going preparation, support and retention of grade 7-12 mathematics teachers based on the results of research studies and ideas of the participants. It is hoped that this working group will enrich the dialogue relating the "support gap" and the work and retention of teachers of mathematics. It is also expected that this working group will propose areas ripe for further research.

## Plan for Active Engagement of Participants in Productive Reflection on the Issues

At this point, people from multiple projects presently involved in related research have been contacted and have indicated an interest in actively participating in this working group. Papers from various groups have been given at MAA, The Eleventh National New Teacher Center Symposium, Curtis Center for Mathematics Teachers Conference, AERA, and NCSM. Models of professional development as well as systematic and sustained support vary. For example, "support" models may include intensive professional development, coaching and mentoring, lesson study groups, school site networking and meetings, data driven reflection, access to resources, online networking support, conference attendance, district and/or school administrative support among others. The impact also varies from teacher knowledge, to student learning, to classroom practices, to building of productive professional collaborative communities, to extended learning opportunities, to teachers' reported value of the support.

Can "support" impact teacher retention? If so, what are key dimensions relating "support" and retention?

If the working group is approved, contacts will be made with interested parties. A possible outline will be developed which will rely heavily on group discussions to bring to bear challenges and opportunities for this area of research. The coordinators submitting this proposal represent one project that involves 10 different sites investigating teacher retention across California. In their initial two years of work, they have identified a decrease in attrition through professional development and support. But, understanding the relationship between retention and "support", the work of the classroom, the network of support and the rewards that help retain teachers is complex and requires an open discussion across multiple projects and perspectives. To bring together researchers across related existing and potential projects offers the type of investment that this topic needs to progress and provide bases for ongoing efforts. Dimensions to be shared and discussed include designs for research, data collection, results and interpretations.

The anticipated structure for the working group will emphasize sharing across questions such as: 1) What is effective professional development and support? For instance, does networking or learning communities increase retention? Why? – 2) What are the challenges for teachers? What models or programs address those challenges? – 3) What are important outcomes? What are some unintended outcomes in these efforts? What issues emerge? – 4) How do we establish studies of the relationship between support and outcomes? What are some promising practices? What evidence indicates this? Do these practices support retention?

In each of these questions we will explore opportunities for and challenges to research efforts on retention. Hopefully we will be able to work through questions, models of inquiry, data, research results in light of participant reflections and knowledge. The overall structure will include introductions, the establishment of key questions, brainstorming relative to the key questions, small breakout group discussions, sharing and whole group grappling with reflections and recommendations. Subgroups may be needed as the initial discussion develops. If so, some potential dimension guiding the formation of subgroups include retention and research issues such as self reported data, guidelines for support including use of technology, school and state policies driving support, job-embedded professional development, or necessary time, content and opportunities for support. Or the determination of the groups could focus on retention and the model of professional development, retention and knowledge, retention and technology, retention and on-site support, retention and equity, and/or retention and leadership. No attempt will be made to identify areas of interest in advance. Small group discussions topics will be determined by the participants and draw from the initial group conversations.

This working group will strive to build a network to collaboratively identify potential research, existing results and establish a guide for further paradigms of teacher professional learning that encourages transformations in teaching practices and rewards resulting in improved retention. An end-of-conference result is outlining a plan for future collaboration among the participants, and also refining and deepening our research efforts.

A more detailed layout of the three working sessions could be as follows: In Session 1, current research projects will be shared as a foundation for discussions. From these discussions we will refine our questions, identify absences in the current research, as well as important themes emerging from the participants' brainstorming. In Session 2, we may ask the participants to choose a subgroup based on the themes previously identified, and explore these dimensions more in-depth among their groups before reporting to the whole group. In Session 3, we will strive to develop a working plan for further collaboration and dissemination, as described in the anticipated follow-up activities below.

## Anticipated Follow-up Activities

Follow-up activities will include ongoing networking and hopefully collaboration on further efforts. One example would be the sharing of developing papers and research. Another would be the development of a weblink. It would be hoped that results of 2009 research could form the basis for the Working Group on Retention for the 2010 PME-NA conference and possibly the development of a series of thematic collaborative papers and/or monograph synthesizing our work on mathematics teacher retention and support. The network will also be included in the development of a conference on Supporting Teachers to Increase Retention scheduled to take place in three years.

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# CULTIVATING CONTENT KNOWLEDGE OF PROSPECTIVE TEACHERS IN US AND CHINESE TEACHER PREPARATION PROGRAMS

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*This paper presents initial points of interest outlined by a working group focused on teacher preparation at the US-Sino conference in June 2008. It was at this conference that a group of mathematics and science educators formed a collaborative to begin identifying critical factors related to content knowledge and understanding in teacher preparation programs. Critical points that emerged related to looking more intimately at university methods and content courses and the experiences provided in them and how these experiences foster the growth of prospective teachers to know and understand the content they are to teach. Some members of this working group would like to continue the efforts in this area and create and/or identify a framework that will guide the development of valuable tasks and experiences prospective teachers encounter in teacher preparation programs in the United States and China.*

## History of Working Group

This Working Group, focused on mathematics and science teacher preparation, was recently created at the US-Sino Workshop on Mathematics and Science Education – Common Priorities that Promote Collaborative Research. The conference was held at Middle Tennessee State University (MTSU) in Murfreesboro, TN from June 22-27, 2008 and was co-hosted by MTSU and Northwest Normal University (NWNU in Lanzhou, P.R. China). The conference had approximately 125 people participating – where approximately 50 scholars were from China and 75 from the United States. The purpose of the workshop was to facilitate the formation of research working groups whose membership would consist of scholars from both the United States and P.R. China. Working groups were formed around six central themes – curriculum, assessment, teacher preparation, professional development, integrating technology into teaching and learning, and reaching underserved populations – which are common to mathematics and science education.

This specific US-Sino Teacher Preparation working group is in its early infancy stage. Members met for the first time at the workshop (with no initial interactions) and had to formulate items of interest to mathematics and science education at that particular time. The group generated two key areas of focus for future study and collaboration prior to the closing of the workshop. They included (1) critically looking at university methods and content courses and what they address from a content perspective and (2) the experiences provided in coursework and how they foster the growth of prospective teachers to know and understand the content they will teach in their future classrooms. This was very much in agreement with what Liping Ma, one of the workshop Plenary Speakers, challenged the mathematics and science education communities to do. She indicated the need for teacher preparation programs to look carefully at what prospective teachers know, what they need to know, and how programs prepare them to teach what they know and do not know. Several points were reiterated (from her book) that teacher education is a strategically vital period during which changes can be made and the educational community needs to assume this responsibility (1999).

Working group members met two to three times at the US-Sino workshop, but since that time, there has been minimal group interaction. Thus, there is a need to find an opportunity for the group to convene and exert efforts on collecting sources of information, making progress towards generating tasks, framing the tasks, and organizing and implementing any short-term or long-term research goals that come out of this collaboration. PME and PME-NA were options for the group to assemble where both United States members and Chinese members could meet and interact face-to-face. In addition, the working group believes this collaborative effort supports this year's conference theme of Embracing Diverse Perspectives. We are interested in identifying perspectives of Chinese teacher preparation programs as well as variations in US preparation programs.

### Teaching Mathematics: What Should It Look Like and How Do We Get There?

In Lampert's *When the Problem Is Not the Question and the Solution is Not the Answer: Mathematical Knowing and Teaching* (1990), the manner in which knowing mathematics is viewed in most classrooms is presented. It is said that "… doing mathematic means following the rules laid down by the teacher; knowing mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical truth is determined when the answer is ratified by the teacher." Although this research study was carried out almost 20 years ago, it is unfortunate that it paints a familiar image of today's classrooms and the role of the teacher. This image is one that teacher preparation programs must confront and take steps to change.

One "big step" towards making a transformation was in the publication of *The Mathematical Education of Teachers* (Conference Board of the Mathematical Sciences – CBMS 2001). In this report, eleven general recommendations were identified and elaborated upon for all levels of teaching mathematics. The following five recommendations are of varying importance to the progress of this working group: (a) Prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach (p. 7); (b) Courses on fundamental ideas of school mathematics should focus on a thorough development of basic mathematical ideas. All courses designed for prospective teachers should develop careful reasoning and mathematical "common sense" in analyzing conceptual relationships and in solving problems (p. 8); (c) Along with building mathematical knowledge, mathematics courses for prospective teachers should develop the habits of mind of a mathematical thinker and demonstrate flexible, interactive styles of teaching (p. 8); (d) The mathematical education of teachers should be seen as a partnership between mathematics faculty and mathematics education faculty (p. 9); and (e) There needs to be more collaboration between mathematics faculty and school mathematics teachers (p. 10). While these five recommendations were identified in some manner at the US-Sino conference, common themes or problems (to China and the United States) narrowed the group to focus on content knowledge and preparing teachers to know and understand the content (both mathematics and science).

NCTM recently updated the *Professional standards for teaching mathematics* (1991) in its 2007 publication of *Mathematics teaching today*. This document articulates that teachers of mathematics should have a deep knowledge of

- sound and significant mathematics,

- theories of student intellectual development across the spectrum of diverse learners,
- modes of instruction and assessment, and
- effective communication and motivational strategies (p. 19).

More specifically teachers must possess the skills to problem solve, reason, communicate, connect, and make use of multiple representations. How to help new teachers develop this repertoire of strategies is the challenge presented to preparation programs of today. *Mathematics teaching today* indicates that teacher preparation programs should ensure that teachers of mathematics are fluent in the language of mathematics and have a deep knowledge of mathematical content including:

- mathematical concepts and procedures and the connections among them;
- multiple representations of mathematical concepts and procedures;
- ways to reason mathematically, solve problems, and communicate mathematics effectively at different levels of formality:
- the cultural contexts for mathematics, including the contributions of different cultures toward the development of mathematics and the role of mathematics in culture and society;
- the evolving nature of mathematical practice and instruction resulting from the availability of technology; and
- the relationship of school mathematics to the discipline of mathematics, to other fields of study, and to mathematical applications (p. 119).

This document, along with other reform-oriented documents and reports (e.g. *Knowing and learning mathematics for teaching,* 2001), echo the importance of teacher knowledge and understanding of content to support students in becoming problem solvers and educated consumers of the future.

In order to attempt to achieve these ambitious challenges, prospective teachers need to be provided with opportunities to reflect upon their experiences in coursework and come to recognize the impact it has on their teaching. Artzt and Armour-Thomas (2002) describe this process as developing into a reflective mathematics teacher who is capable of facilitating student-centered teaching. The growth of the reflective practitioner is also of interest to this teacher preparation group.

## Plans for Engagement and Post-conference Activities

This working group would like to focus on problems, situations, and tasks that compel prospective teachers to reflect upon their own level of understanding of content. As was identified in our working group at the US-Sino conference, we would like to construct/organize some sort of framework to guide in the study of such situations. There is the interest to identify specific tasks and implement them in teacher preparation courses, collect data at our various institutions, and then see what components are congruent or incongruent –in the US and in China. A piece of our working session will be to design some plan of implementation to accomplish this undertaking.

Post-conference activities will involve working group members to return to their respective institutions and carry out the developed situations discussed and/or developed at the conference (or some format similar to what we decide upon). When implementation of the organized materials has been accomplished, we plan to re-convene and continue progress in this area. We anticipate executing a critical analysis of successful problems, tasks, and experiences that promote reflection on knowing and learning mathematics that teachers must teach in their classrooms. Some of this may be done “long-distance,” but other elements will require face-to-face interactions at a future working group venue.

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# RESEARCH ADVANCES IN THEORIES OF MATHEMATICS EDUCATION

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## Purpose of Working Group

This working group revolves around the launch of a new book series entitled *Advances in Mathematics Education* by Springer Science, Heidelberg, and in particular on the first book in the series which focuses on Theories of Mathematics Education. This edited book in turn is based on a research forum on Theories of Mathematics Education at PME 29 in Melbourne, 2005, which resulted in two ZDM special issues on theories of mathematics education(issue 6/2005 and issue 1/2006). Since the research forum in Melbourne, numerous advances have taken place in the area of theory development in mathematics education in Europe and in North America. The purpose of this working group on research advances in theories of mathematics education is to integrate, synthesize and present a coherent picture on the state of the art. The working group will attempt to be both summative as well as forward looking by highlighting theories from psychology, philosophy and social sciences that continue to influence theory building, as well as provide participants insights into new developments in feminist, critical and political theories of mathematics education.

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## Theory and Its Role in Mathematics Education

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Theories are like toothbrushes…everyone has their own and no one wants to use anyone else's."(Campbell,2006)

Abstract: The increased recognition of the theory in mathematics education is evident in numerous handbooks, journal articles, and other publications. For example, Silver and Herbst (2007) examined "Theory in Mathematics Education Scholarship" in the *Second Handbook of Research on Mathematics Teaching and Learning* (Lester, 2007) while Cobb (2007) addressed "Putting Philosophy to Work: Coping with Multiple Theoretical Perspectives" in the same handbook. And a central component of both the first and second editions of the *Handbook of International Research in Mathematics Education* (English, 2002; 2008) was "advances in theory development." Needless to say, the comprehensive second edition of the *Handbook of Educational Psychology* (Alexander & Winne, 2006) abounds with analyses of theoretical developments across a variety of disciplines and contexts. Numerous definitions of "theory" appear in the literature (e.g., see Silver & Herbst, in Lester, 2007). It is not our intention to provide a "one-size-fits-all" definition of theory per se as applied to our discipline; rather we consider multiple perspectives on theory and its many roles in improving the teaching and learning of mathematics in varied contexts.

At the 2008 *International Congress on Mathematical Education*, Assude, Boero, Herbst, Lerman, and Radford (2008) referred to theory in mathematics education research as dealing with the teaching and learning of mathematics from two perspectives: a *structural* and a *functional* perspective. From a structural point of view, theory is "an organized and coherent system of concepts and notions in the mathematics education field." The "functional" perspective considers theory as "a system of tools that permit a 'speculation' about some reality." When theory is used as a *tool*, it can serve to: (a) conceive of ways to improve the teaching/learning environment including the curriculum, (b) develop methodology, (c) describe, interpret, explain, and justify classroom observations of student and teacher activity, (d) transform practical problems into research problems, (e) define different steps in the study of a research problem, and (f) generate knowledge. When theory functions as an *object*, one of its goals can be the advancement of theory itself. This can include testing a theory or some ideas or relations in the theory (e.g., in another context or) as a means to produce new theoretical developments.

Silver and Herbst (2007) identified similar roles but proposed the notion of theory as a mediator between problems, practices, and research. For example, as a mediator between research and problems, theory is involved in, among others, generating a researchable problem, interpreting the results, analysing the data, and producing and explaining the research findings. As a mediator between research and practice, theory can provide a norm against which to evaluate classroom practices as well as serve as a tool for research to understand (describe and explain) these practices. Theory that mediates connections between practice and problems can enable the identification of practices that pose problems, facilitate the development of researchable problems, help propose a solution to these problems, and provide critique on solutions proposed by others. Such theory can also play an important role in the development of new practices, such as technology enhanced learning environments.

What we need to do now is explore more ways to effectively harmonize theory, research, and practice (Silver & Herbst, 2007) in a coherent manner so as to push the field forward. This leads to an examination of the extant theoretical paradigms and changes that have occurred over the last two decades. This was briefly discussed at the outset of this chapter.

*Changes in Theoretical Paradigms*

As several scholars have noted over the years, we have a history of shifting frequently our dominant paradigms (Berliner, Calfee, in Alexander & Winne, 2006). Like the broad field of psychology, our discipline "can be perceived through a veil of 'isms'" (Alexander & Winne, 2006, p.982). We have witnessed, among others, shifts from behaviourism, through to stage and level theories, to various forms of constructivism, to situated and distributed cognitions, and more recently, to complexity theories and neuroscience. For the first couple of decades of its life, mathematics education as a discipline drew heavily on theories and methodologies from psychology. According to Lerman (2000), the switch to research on the social dimensions of mathematical learning towards the end of the 1980s resulted in theories that emphasized a view of mathematics as a social product. Social constructivism, which draws on the seminal work of Vygotsky and Wittgenstein (Ernest, 1994) has been a dominant research paradigm for many years. Evidence of the social turn can be found in Lerman's analysis of articles published from 1990 to 2001 in *Educational Studies in Mathematics* (ESM), *Journal for Research in Mathematics Education* (JRME), and the *Proceedings of the International Group for the Psychology of Mathematics Education* (PME), revealed that, while the predominant theories used during this period were traditional psychological and mathematics theories, an expanding range from other fields was evident especially in PME and ESM. Psycho-social theories, including re-emerging ones, increased in ESM and JRME. Likewise, papers drawing on sociological and socio-cultural theories also increased in all three publications together with more papers utilizing linguistics, social linguistics, and semiotics. Lerman's analysis revealed very few papers capitalizing on broader fields of educational theory and research and on neighboring disciplines such as science education and general curriculum studies. This situation appears to be changing in recent years, with interdisciplinary studies emerging in the literature (e.g., see English, 2008); and papers that address the nascent field of neuroscience in mathematics education

Numerous scholars have questioned the reasons behind these paradigm shifts. Is it just the power of fads? Does it only occur in the United States? Is it primarily academic competitiveness (new ideas as more publishable)? One plausible explanation is the diverging, epistemological perspectives about what constitutes mathematical knowledge. Another possible explanation is that mathematics education, unlike "pure" disciplines in the sciences, is heavily influenced by unpredictable cultural, social, and political forces (e.g., Sriraman, 2007). A critical question, however, that has been posed by scholars now and in previous decades is whether our paradigm shifts are genuine. That is, are we replacing one particular theoretical perspective with another that is more valid or more sophisticated for addressing the hard core issues we confront (Kuhn, 1966; Alexander & Winne, 2006)? Or, as Alexander and Winne ask, is it more the case that theoretical perspectives move in and out of favour as they go through various transformations and updates? If so, is it the voice that speaks the loudest that gets heard? Who gets suppressed? The rise of constructivism in its various forms is an example of a paradigm that appeared to drown out many other theoretical voices during the 1990s (Goldin, 2003). In essence, the question we need to consider is whether we are advancing professionally in our theory development.

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## Appreciating *Scientificity* in Qualitative Research

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Abstract: This paper is situated within an educational paradigm that is concerned with the education of itself, its peers and its students. From here, we acknowledge the necessity for knowledge and that *in learning* we discover knowledge either through ourselves, through our peers or through synthesizing a dialectic between the governing bodies of knowledge and an educational system. We might understand that we discover knowledge in an educational setting by processes that are akin to scientific discovery. I propose that we establish knowledge in this very way and in reflecting on our *constructing-knowledge* enterprise, we endeavor to adhere to a *meta-constructionist phenomenology,* which draws upon the learning theory of constructionism (Papert & Harel, 1991) whereby we establish a construction built on a faithful establishment of education and assess the mechanics of the constructed phenomenon through reflexivity and interactivity with the field.

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## Mathematics Education as a Design Science

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Abstract: We propose re-conceptualizing the field of mathematics education research as that of a design science akin to engineering and other emerging interdisciplinary fields which involve the interaction of “subjects”, conceptual systems and technology influenced by social constraints and affordances. Numerous examples from the history and philosophy of science and mathematics and ongoing findings of M&M research are drawn to illustrate our notion of mathematics education research as a design science. Our ideas are intended as a framework and do not constitute a, “grand” theory (see Lester. 2005, this issue). That is, we provide a framework (a system of thinking together with accompanying concepts, language, methodologies, tools, and so on) that provides structure to help mathematics education researchers develop both models and theories, which encourage diversity and emphasize Darwinian processes such as: (a) selection (rigorous testing), (b) communication (so that productive ways of thinking spread throughout relevant communities), and (c) accumulation (so that productive ways of thinking are not lost and get integrated into future developments)

## Teaching Mathematics through Problem Solving: What We Know and Where We Are Going

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Abstract: Problem solving has a long history in school mathematics. In the past several decades, there have been significant advances in the understanding of the complex processes involved in problem solving. There also has been considerable discussion about teaching mathematics with a focus on problem solving. However, teaching mathematics *through* problem solving is a relatively new idea in the history of problem solving in the mathematics curriculum. In fact, because teaching mathematics through problem solving is a rather new conception, it has not been the subject of much research.

Contemporary discussions of goals for mathematics education emphasize the importance of thinking, understanding, reasoning, and problem solving, with an emphasis on connections, applications, and communication. This view stands in contrast to a more conventional view of mathematics, involving the memorization and recitation of facts, rules, and procedures, with an emphasis on the application of well-rehearsed procedures to solve routine problems. Because teaching mathematics through problem solving has been considered an instructional approach better aligned with the contemporary views of school mathematics, it is receiving increasingly strong support from researchers, educators, and teachers. Although less is known about the actual mechanisms students use to learn and make sense of mathematics through problem solving, there is widespread agreement that teaching through problem solving holds the promise of fostering student learning.

While there is no universal agreement about what teaching mathematics through problem solving should really look like, there are some commonly accepted features of teaching mathematics through problem solving. Teaching through problem solving starts with a problem. Students learn and understand important aspects of a mathematical concept or idea by exploring the problem situation. The problems tend to be open-ended and allow for multiple correct answers and multiple solution approaches. Students play a very active role in their learning—exploring problem situations with teacher guidance and "inventing" their own solution strategies. In fact, the students' own exploration of the problem is an essential component in teaching with this method. In students' problem solving, they can use any approach they can think of, draw on any piece of knowledge they have learned, and justify any of their ideas that they feel are convincing. While students work on the problem individually, teachers talk to individual students in order to understand their progress and provide individual guidance. After students have used at least one strategy to solve the problem or have attempted to use a strategy to solve the problem, students are given opportunities to share their various strategies with each other. Thus, students' learning and understanding of mathematics can be enhanced by considering one another's ideas and debating the validity of alternative approaches. During the process of discussing and comparing alternative solutions, the students' original solutions are supported, challenged, and discussed. Students listen to the ideas of other students and compare other students' thoughts with their own. Such interactions help students clarify their ideas and acquire different perspectives of the concept or idea they are learning. In other words, students have ownership of the knowledge because they devise their own strategies to construct the solutions. At the end, teachers make concise summaries and lead students to understand key aspects of the concept based on the problem and its multiple solutions.

Theoretically, this approach makes sense. Empirically, there is lacking of data confirming the promise of teaching through problem solving. In particular, we need to seek answers to a number of important research questions, such as, (1) Does classroom instruction using a problem-solving approach have any positive impact on students' learning of mathematics? If so, what is the magnitude of the impact? (2) How does classroom instruction using this approach impact students' learning of mathematics? (3) What actually happens inside the classroom when a problem-solving approach is used effectively or ineffectively? (4) What do the findings from research suggest about the feasibility of teaching mathematics *through* problem solving in classroom?

In this paper, I will explore these research questions through reviewing two lines of research. The FIRST line of research includes those recently conducted studies on *NSF-funded* curricular programs that teach mathematics through problem solving and that have been implemented by teachers in classroom. The *NSF-funded* curricula are problem-based curricula, and the intent is to teach mathematics and to build students' understanding of important mathematical ideas through explorations of real-world situations and problems.

The SECOND line of research includes studies based on innovative materials developed by researchers in specific content areas. Unlike the first line of research, in this second line, researchers usually focus on teaching grade-specific mathematical topics using a problem-solving approach. These studies are important because they provide insights into the ways teachers teach specific content topics through problem solving in classroom.

## Networking Strategies for Connecting Theoretical Approaches

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Abstract: One of the characteristics of the research community in mathematics education seems to be the large diversity of different theories, research paradigms and theoretical frameworks. This diversity has become an important issue to discuss at many conferences and in many publications. Is diversity a problem or a resource or a barrier for further research? How shall the scientific community deal with this diversity? Internationally different approaches have been developed to cope with this diversity (see Sriraman / English in monograph 1). Under a European perspective the approach of networking strategies, which aim to connect different theoretical approaches using several strategies, has been developed. This perspective bases its work on the assumption that the variety of different theoretical approaches and perspectives in mathematics education research is a *rich resource* upon which the scientific community should build more consequently. This perspectives calls for the connections of different theories and rejects isolationistic tendencies of separating different theoretical approaches. This approach does not intend to develop one grand unified theory, but intends to network local theories, which deal with background theories but use diverging conceptual systems for describing the same phenomena.

Different networking strategies are presented in a landscape, linearly ordered according to their degree of integration. These networking strategies such as comparing or contrasting, combining or coordinating can contribute to the development of theories and their connectivity and offer hence an interesting research strategy for the didactics of mathematics as scientific discipline.

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## Feminist Perspectives and Mathematics Education

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Abstract: Feminism has many faces. There is, however, a unifying dimension to all the theoretical shades of feminism. Feminism is considered a movement for attaining the right of women to be equal to men in all aspects of life – social, political, legal, and educational. The impact of feminist thinking on mathematics education research is the focus of this presentation. In her article entitled *Feminist pedagogy and mathematics*, which is to be reproduced in the first monograph in the Springer series, *Advances in mathematics education*, Judith Jacobs (1994) concluded that:

> ... previous research and intervention programs designed to promote females … have been based on the assumption of male as the norm, the model of the successful mathematics student or mathematician who is to be emulated if the non-successful are to

> succeed. Little research and work has begun from the assumption that females have strengths, experiences and learning styles that can succeed in mathematics.

Jacobs (1994) provided a theoretical framework for a feminist pedagogy for which the assumption was “that being a woman is the norm for females” (p. 16), and the teacher had the responsibility “to capitalize on females’ strengths and interests in order to facilitate their success in mathematics” (p. 16). Jacobs believed that all students would benefit from this approach which, she claimed, “in no way denies the power or beauty of mathematics” (p. 16).
Leder (in press) has provided a commentary on Jacobs’ (1994) chapter which is also to be included in the Springer monograph). Leder summarised the main points raised by Jacobs (1994) including the caution not to essentialise women as a group in response to research generalisations about differences between women and men.

Drawing on a number of earlier as well as more contemporary sources, Leder also discussed a range of feminist perspectives and their influence on research into gender issues in mathematics education. The feminist links evident in chapters found in the influential edited collection by Rogers and Kaiser (1995) are highlighted and include:

- Kaiser and Rogers’ (1995) five stages of the mathematics curriculum beginning with “womanless” mathematics” and ending with “mathematics reconstructed”; and
- Becker’s (1995) chapter in which Belenky et al.’s (1986) ‘women’s ways of knowing’ are extended to the knowing of mathematics.

Whilst liberal feminism receives much criticism from many feminist theorists, it would appear to underpin and dominate many research endeavours, particularly those which do not specifically identify with any feminism. Many researchers continue to call for the monitoring of all large scale studies involving achievement and/or participation data for gender differences, and caution not to ignore gender as a factor in smaller, more focussed studies. As is evident in Australia today, educational disadvantages once considered to have been addressed can resurface. Recent data revealing the re-opening of gender gaps favouring males will be presented.

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## Problem Solving Heuristics, Affect, Representations and Discrete Mathematics

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Abstract: It has been suggested that activities in discrete mathematics allow a kind of new beginning for students and teachers. Students who have been “turned off” by traditional school mathematics, and teachers who have long ago routinized their instruction, can find in the domain of discrete mathematics opportunities for mathematical discovery and interesting, non-routine problem solving. Sometimes formerly low-achieving students demonstrate mathematical abilities their teachers did not know they had. To take maximum advantage of these possibilities, it is important to know what kinds of thinking during problem solving can be naturally evoked by discrete mathematical situations—so that in developing a curriculum, the objectives can include pathways to desired mathematical reasoning processes. This article discusses some of these ways of thinking, with special attention to the idea of “modeling the general on the particular.” Some comments are also offered on the global ideas of Moreno-Armella & Sriraman (2005) pertaining to the development of representational systems. The discussion focuses on the co-evolution of symbols and their referents, and the shared interpretation of mathematical symbols in a community of practice. Some future directions are suggested.

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# VISUAL EXPLORATIVE APPROACHES TO LEARNING MATHEMATICS

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*This discussion group focuses on visual explorative approaches to learning mathematics. We address several issues in the discussion such as technology use in mathematics education and its evolution from static to dynamic in conjunction with the visual characteristics of new learning form of mathematics. Among various representations used in mathematics and mathematics education, visual representations of mathematical concepts, the effects of the implementation of visual techniques, and more importantly the importance of visual exploration of mathematics and its effects to mathematics education will be discussed.*

## Background

The focus of this discussion group will be situated at the intersection of the technology use in mathematics education, representation systems in mathematics education, and developing a conceptual understanding in mathematics. An enormous corpus of literature has been accumulated on these topics, and many researchers discussed the topics in the various national and international meetings such as PME, PMENA, and ICMI (Arcavi, 1999; Duval, 1999; Hitt, 1999; Hoyles, 2008; Kaput, 1999; Kaput & Hegedus, 2000; Leatham, & McGehee, 2004; McDougall, 1999; Moreno-Armella, 1999; Presmeg, 1999; Radford, 1999; Santos-Trigo, 1999; Thompson, 1999). Yet, our main focus will be on the visual exploration of mathematics and on the visual versus algebraic understanding of mathematics, particularly in the technology supported learning environments. We will also explore the distinctions between visualization of mathematics, visual exploration of mathematics, and visual understanding of mathematics. We believe that the result of this conceptual exploration will lead us to improve our understanding of the theories on representational systems (Kaput, 1992) and distributed cognition between human and technology (Pea, 1993).

Interestingly, the importance of exploration and visualization was emphasized in the past PMENA conferences. For example, in a paper presented in 1999 PMENA, the author points out the relationship between technology and representation:

> Technology, in all its forms, modifies, substantially, the process of knowledge production. Learning involves the construction of representations. It is through the construction of representations of an observed phenomena, (or of a mathematical concept) that we make sense of the (mathematical) world. *Representations become mediational tools for understanding* (Moreno-Armella, 1999, p. 99)

Similarly, in another PMENA meeting, the authors explore what we know about technology use and its effects in teaching and learning mathematics:

> Much of what we know about the use of technology in the teaching and learning of mathematics is anectodal and might be referred to as “possibility” research…. What do we really know regarding teaching and learning mathematics with technology? What frameworks, methodologies and collaborations will support the research that will produce this knowledge?(Leatham & Peterson, 2005, p. 1)

Some authors propose new perspectives for the use of technology rather than digital interpretations of traditional paper-and-pencil techniques. They suggest using dynamic, interactive, and collaborative features of technology:

> The advance of dynamic technological environments allows us to combine multiple individual cognitive acts of reference. This is possible since individuals can project their intentions and expressivity through the notations they create and share. They can also realize and generalize the structure of the mathematics through co-active collaboration with these environments. This can be made possible through the advances in representation infrastructures (dynamic mathematics software) and communication infrastructures (social and digital networks). (Moreno-Armella, Hegedus, & Kaput, 2008, p. 110)

*Various Types of Visualizations*

The literature documents at least three distinct meaning for the term visual mathematics such as (1) for studying advanced visual objects, (2) for visualizing algebraic rules, and (3) for exploring mathematics visually. The first one, studying advanced visual objects such as fractals and 3D functions, is a focus where mathematicians use technology to extend their imaginations and to make their ideas visible whereas the other two are related to mathematics education.

Virtual manipulatives (NLVM, 1999; SAMAP, 2006), learning objects (Reis & Karadag, 2004; Sorkin, Tupper, and Harmeyer, 2004), and animations (Tchoshanov, n.d.) are the examples to visualize mathematical concepts, rules, and relationships. The idea behind creating this group of mathematical objects can be described as to make mathematical rules visible in order to improve students' understanding. For example, the example from Tchoshanov's (n.d.) collection visualizes a very well-known algebraic identity (figure 1). He visualizes the identity of $(a+b)^2 = a^2 + 2ab + b^2$ by using animations.

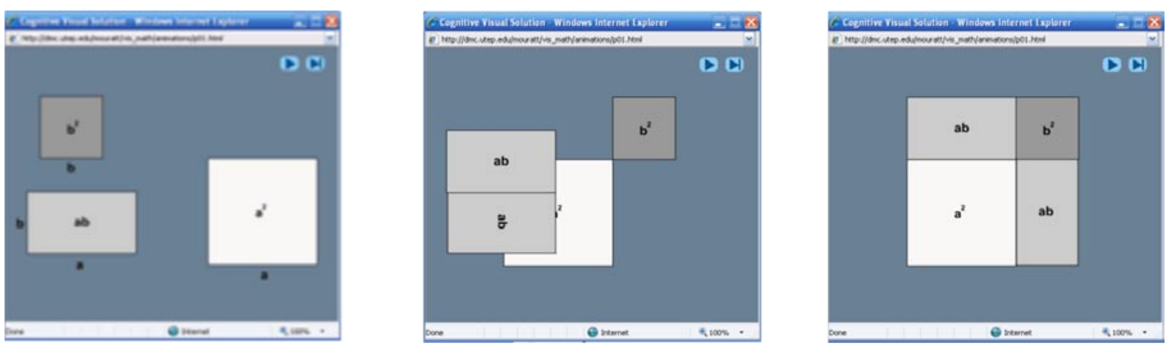


*Figure 18.* Screenshots from Tchoshanov's animations.

Similarly, virtual manipulatives developed by Utah State University team serve for the same purpose. For example, they use virtual algebra tiles to illustrate distributive law of algebra (figure 2). The figure illustrates a geometric representation to explain the rule for the example x (y + z) = xy + 2x.

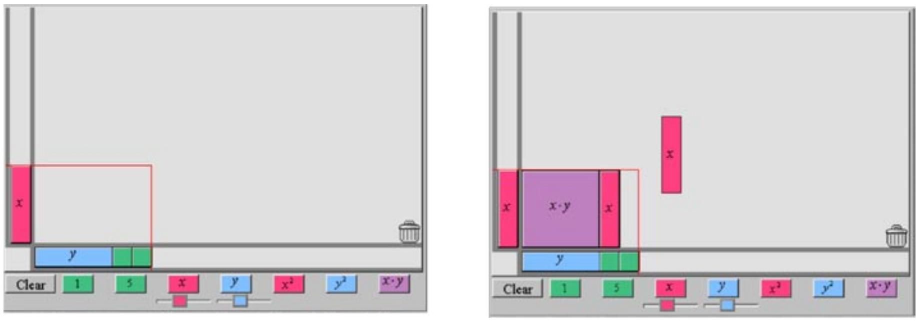


*Figure 19.* Algebra tiles as virtual manipulatives.

However, these virtual manipulatives could also be used to explore mathematics visually. For example, student may develop an insight for patterning while using Towers of Hanoi example (figure 3). This example provides a scenario to encourage students explore the problems illustrated. Students, with or without guidance, are expected to realize the pattern while performing the task.

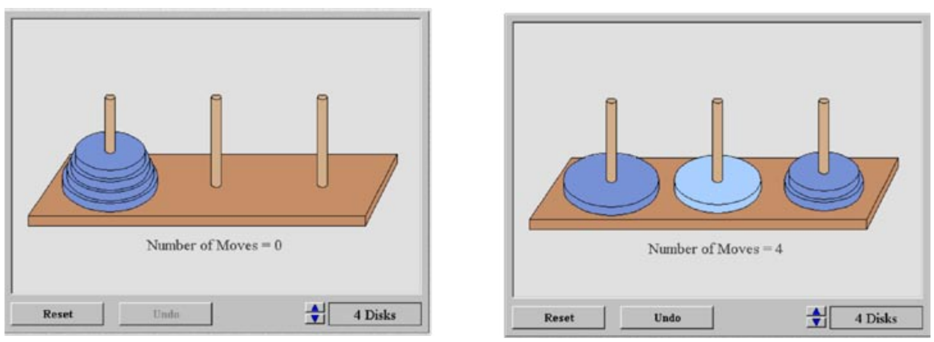


*Figure 20.* The Towers of Hanoi as virtual manipulatives.

Another exploratory environment to encourage students to improve their patterning skills is the Math Towers. The Math Towers provides scenarios such as billiard boards to engage students to develop some patterning relationships (figure 4). The scenarios illustrated in the Math Towers and the Towers of Hanoi aim to engage students to be part of a virtual environment and to explore mathematics visually.


*Figure 21.* The Math Towers.

Moreover, we have dynamic learning environments to create dynamic worksheets. For example, Geogebra, a free online dynamic software, allows us to create mathematical objects and to explore these objects visually and dynamically. In a study, we created a dynamic worksheet illustrating the relationship between unit circle and trigonometric functions (figure 5). We asked students to manipulate these mathematical objects and explore the relationship between them (they were not told that the functions were trigonometric).

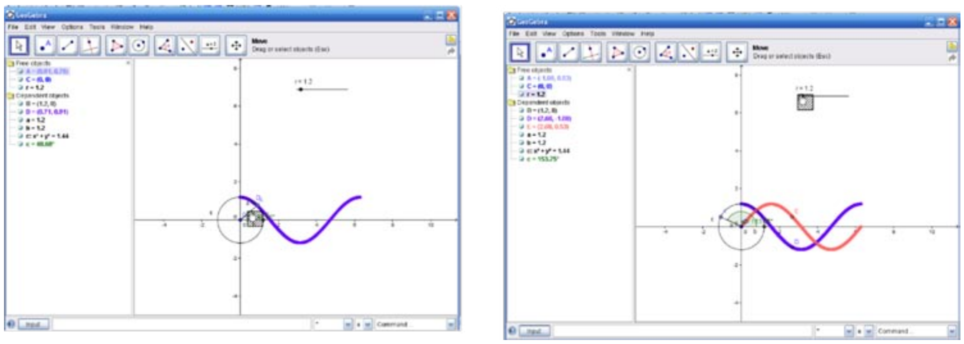

*Figure 22.* Visual exploration in Geogebra.

As seen in the examples described here, there are different types of visualizations. The cognitive collaboration between human and tools seem to be quite distinct in these examples although we call them all as visualization. Thus, it is important to identify this distinction and to develop theories explaining various visualization types.

More importantly, some scholars argue that mathematics education evolves through integration of technology and needs a groundbreaking change to complete its evolution (Galbraith, 2006; Moreno-Armella, Hegedus, & Kaput, 2008). Moreno-Armella, Hegedus, and Kaput (2008) describe symbolic structures as “an environment that enable us to think deeper” (p. 100) and argue that “the nature of mathematical symbols have evolved in recent years from static, inert inscriptions to dynamic objects or diagrams that are constructible, manipulable and interactive” (p. 103). According to their perspective, students may think, reflect on their thoughts, and construct new mathematical knowledge in the dynamic learning environments. The dynamic learning mathematical environments enable students act mathematically, such as seeking visual patterns, define these patterns, and explore the properties of these patterns as they usually do symbolically in paper-and-pencil environments.

## The Rationale and Goals of Discussion Group

The goals of this discussion group are to explore and discuss the dynamic and visual features of technology in mathematics education, to develop awareness on the potential effects of visual exploration in conceptual understanding of mathematics, and to set up a research agenda on the study of visual exploration in mathematics education.

The discussion group will review the past and current use of technology as visual and dynamic cognitive tools and extend the theory of distributed cognition over the participatory mathematical activities. This discussion will deepen our understanding of the effects of visual explorative activities in learning mathematics and seek possible strategies to integrate web 2.0 technologies with dynamic learning tools.

*Objectives*

- To perform a comprehensive review of various representational systems used in mathematics in junction with a linkage among them
- To explore various perspectives of visual learning in mathematics
- To explore the features of dynamic learning systems in conceptual understanding of mathematics
- To identify possible effects of visual and dynamic learning environments in mathematics education
- To explore the possible scenarios for the future of the mathematics education supported with contemporary technologies such as Web 2.0, dynamic learning environments, and visual representation

*Questions to frame our discussion*

- Which representation types are used in mathematics education?
- What types of visualizations do we use and how could they be identified?
- Which representation is more compatible with the nature of human learning?
- Could algebra be a barrier to learn mathematics?
- How can we observe or track the effects of visual versus symbolic representations in learning mathematics and their effects on concept development?
- How could the Web 2.0 technologies and dynamic learning environments be integrated to engage students for visual exploration of mathematical concepts?

## Format for the Discussion Group Meeting

*Day 1: Identifying the Current Situation and Contemporary Perspectives*

The organizers of the group will begin the discussion with a brief introduction of the topic and some quotes from literature. Then, we will ask the participants to reflect on these quotes in small groups and share their reflections with whole group. The second hour of the discussion will focus on the technology, representation, and visualization. We will engage the participants to brainstorm on the expectations of technology in math education, on the various forms of representations used in mathematics, and on the meanings of visualization in small groups and share their reflections with whole group.

- Introducing group members
- Introducing discussion group topics and goals
- Reviewing selected quotes form previous work and engaging participants to reflect on these quotes
- Brainstorming on the expectation of technology in mathematics education
- Brainstorming on the various forms of representations used in mathematics
- Brainstorming on the meanings of visualization

The quotes will be the evolution of the mathematics education (Moreno-Armella, Hegedus, & Kaput, 2008) and the metaphor of the technology use (Galbraith, 2006).

*Day 2: Projecting on the Future*

We will start the meeting by briefing previous day's discussion and by outlining themes emerged from the discussion. Then, we will demonstrate a couple of dynamic worksheets created by Geogebra and ask participants to develop scenarios in their small groups on how to use these dynamic worksheets in classrooms and to share their scenarios with the whole group. During the last half hour, we will encourage the participants to discuss the opportunities of integrating Web 2.0 technologies (i.e. wikis) with these dynamic worksheets.

*Day 3: Summarizing the Discussions and Setting Goals for the Future Collaboration Opportunities*

- Summarizing previous discussions
- Reminding the metaphor by Galbraith (2006) and engaging to brainstorm on the mathematics education for the next fifty years
- Forming an international working group on the topic
- Setting an agenda for the future

**Possible Future Agenda Items**

- Set up a working group for dynamic and visual learning
- Create possible research questions
- Seek possible strategies to disseminate the themes emerged through discussion

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